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LARGE FREE SET

BY

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Introduction. A set $A \subseteq X$ is said to be *free* for a set mapping F from X into the power set $P(X)$ of X provided $x \notin F(y)$ for any distinct x, y in A. Every set map F on the reals with $F(x)$ nowhere dense for each x in R admits a countably infinite free set [3] and indeed an everywhere dense free set [2] (see also MR 88k:03101). The aim of this note is to generalize the above result to the generalized linear continuum (endowed with the lexicographic order topology), which generalizes well-known facts about the real line [4].

The existence of large free set in the generalized linear continuum has never been published, but the nonexistence of free set has been studied in [6].

Definition. Throughout this paper α is an infinite cardinal. C_α is the lexicographically ordered set of all dyadic sequences $(x(\mu))_{\mu<\alpha}$ such that $x(\beta) = 1$, $x(\delta) = 0$ for some β, $\delta < \alpha$ and if $x(\eta) = 0$ for some $\eta < \alpha$, then there exists $\eta < \sigma < \alpha$ with $x(\sigma) = 0$. R_α is the set of all dyadic sequences $(x(\mu))_{\mu<\alpha}$ such that $x(\eta) = 1$ for some $\eta < \alpha$, and $x(\mu) = 0$ for all $\eta < \mu < \alpha$.

A *nowhere dense set map* on C_α is a set map on C_α with $F(x)$ nowhere dense for each x in C_α.

Properties of C_α and R_α

1. $|C_\alpha| = 2^\alpha$ and $|R_\alpha| = \sum_{\beta<\alpha} 2^\beta$.
2. For regular α, C_α is not the union of α many nowhere dense subsets of C_α.
3. For regular α, $|R_\alpha|$ is the least cardinal of an everywhere dense set in C_α [5].

A paper by Harzheim [4] contains an exhaustive examination of C_α and R_α.

The following theorem generalizes a theorem of P. Erdős [3] to higher cardinals.

Theorem 1. *Suppose α is a regular infinite cardinal and C_α is not the* *union of* $|R_\alpha|$ *many nowhere dense subsets of* C_α. *Then every nowhere dense set mapping* F *on* C_α *admits a free set of size* α.

Proof. Let $|R_\alpha| = \Theta$. Then there exists a sequence $(x_\eta)_{\eta<\Theta^+}$ of distinct elements of C_α such that $x_\eta \notin \bigcup_{\mu<\eta} \overline{F(x_\mu)}$ for all $1 \le \eta < \Theta^+$, where $\overline{F(x_\mu)}$ is the closure of $F(x_\mu)$ in C_α. Let $E = \{x_\eta : \eta < \Theta^+\}$. We show first that if N is a subset of E containing no free pair, then $|N| \le \Theta$. To prove this, suppose that $|N| = \Theta^+$. Then N can be written in the form $N = \{y_\eta : \eta < \Theta^+\}$ with $y_\eta \notin \bigcup_{\mu<\eta} \overline{F(y_\mu)}$ for all $1 \le \eta < \Theta^+$. Let $\mathcal{I}$ be the collection of all open intervals having endpoints in R_α. For each $\eta < \Theta^+$, $y_{\eta+1} \notin \overline{F(y_\eta)}$, and consequently there exists an interval $I_\eta \in \mathcal{I}$ such that $y_{\eta+1} \in I_\eta$ and $I_\eta \cap F(y_\eta) = \emptyset$. Since $|\mathcal{I}| = \Theta$, there exist an $I \in \mathcal{I}$ and a set L of size Θ^+ such that $I = I_\eta$ for all $\eta \in L$. This implies that for all $\eta \in L$, $y_{\eta+1} \in I$ and $I \cap F(y_\eta) = \emptyset$. For a fixed element δ in L, $y_{\delta+1} \notin F(y_\eta)$ for all $\eta \in L$, and because of $\{y_{\delta+1}, y_\eta\} \subset N$ we obtain $y_\eta \in F(y_{\delta+1})$ for all $\eta \in L$, which contradicts the choice of $\{y_\eta : \eta < \Theta^+\}$. Thus $|N| \le \Theta$. Now, if A is the set of all free pairs of elements of E and $B = [E]^2 - A$, then $\{A, B\}$ is a partition of $[E]^2$ and consequently by the partition relation [7], "$\Theta^+ \to (\Theta^+, \tau)$, where $\tau = \min\{\mu : \Theta^\mu > \Theta\}$" (this notation means that if the set $[E]^2$ of 2-element subsets of E of size Θ^+ is decomposed as $A \cup B$, then there is a set $P \subseteq E$ such that either $|P| = \Theta^+$ and $[P]^2 \subseteq A$, or else $|P| = \tau$ and $[P]^2 \subseteq B$), there is a free set of size τ. By [1, Satz 6], $\tau \ge \alpha$, which completes the proof.

From Theorem 1 and the facts following the definition of C_α, we obtain the following results.

COROLLARY 1. *If* $|R_\alpha| = \alpha$, *then every nowhere dense set map on* C_α *admits a free set of size* α.

COROLLARY 2. *If* GCH *holds*, *then every nowhere dense set map on* C_α *admits a free set of size* α.

COROLLARY 3. *Every nowhere dense set map on the reals admits a countably infinite free set*. (This is a theorem of P. Erdős [3, Th. 6].)

Remark. Corollary 2 is best possible in the sense that there is a nowhere dense set map on C_α not admitting a free set of size α^+ [6, Ths. 3.7, 3.8].

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