LARGE GENUS ASYMPTOTICS FOR LENGTHS OF SEPARATING CLOSED GEODESICS ON RANDOM SURFACES

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ABSTRACT. In this paper, we investigate basic geometric quantities of a random hyperbolic surface of genus g with respect to the Weil-Petersson measure on the moduli space \mathcal{M}_g . We show that as g goes to infinity, a generic surface $X \in \mathcal{M}_g$ satisfies asymptotically:

- (1) the separating systole of X is approximately $2\log g$;
- (2) there is a half-collar of width approximately $\frac{\log g}{2}$ around any separating systolic curve on X;
- (3) the length of the shortest separating closed multi-geodesics on X is approximately $2 \log g$.

As applications, we also discuss the asymptotic behavior of the extremal separating systole, the non-simple systole and the expected length of the shortest separating closed multi-geodesics as g goes to infinity.

1. INTRODUCTION

The overall behavior of geometric quantities such as systole, diameter, eigenvalues of Laplacian, Cheeger constant, *etc.*, for all closed hyperbolic surfaces of a given genus g, is a classical object of study. While there are many results and conjectures about the maximal/minimal values of these quantities, as functions on the moduli space \mathcal{M}_g , Mirzakhani initiated a new approach in [Mir13] to the subject: based on her celebrated thesis works [Mir07a, Mir07b], she obtained asymptotic results on certain statistical information about these quantities, viewed as random variables with respect to the probability measure $\operatorname{Prob}_{WP}^g$ on \mathcal{M}_g given by the Weil-Petersson metric. One may see the book [Wol10] of Wolpert and the recent survey [Wri20] of Wright for more details.

1.1. Separating systole. It was shown by Mirzakhani [Mir13, Corollary 4.3] that if we consider the *systole*

 $\ell_{\rm sys}(X) := \min \left\{ \ell_{\gamma}(X) \, ; \, \gamma \subset X \text{ is a simple closed geodesic} \right\}$

as a function on \mathcal{M}_g , where the variable $X \in \mathcal{M}_g$ is a closed hyperbolic surface of genus g and $\ell_{\gamma}(X)$ is the length of γ , then the expected value of $\frac{1}{\ell_{\text{sys}}}$ is bounded from above and below by two positive constants independent of g. Meanwhile, [Mir13] also contains results on the *separating systole*

 $\ell_{\rm sys}^{\rm sep}(X) := \min \left\{ \ell_{\gamma}(X) \, ; \, \gamma \subset X \text{ is a separating simple closed geodesic} \right\},$

implying that $\ell_{\text{sys}}^{\text{sep}}$ behaves drastically differently from ℓ_{sys} . The function $\ell_{\text{sys}}^{\text{sep}}$ is unbounded on \mathcal{M}_g : if X carries a pants decomposition consisting of

arbitrarily short non-separating closed geodesics, the classical Collar Lemma (e.g. see [Kee74]) implies that the length of any separating closed geodesic is arbitrarily large because it has nonempty intersection with certain curves in the pants decomposition. In this paper we study the asymptotic behavior of the shortest simple separating closed geodesics as the genus g goes to infinity. First we recall the following result. One may also see [Mir10, Theorem 4.2] of Mirzakhani's 2010 ICM report for a weaker version.

Theorem (Mirzakhani, Mir13, Theorem 4.4]). Let 0 < a < 2. Then

$$\operatorname{Prob}_{\operatorname{WP}}^g \left(X \in \mathcal{M}_g \, ; \, \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) < a \log g \right) = O\left((\log g)^3 g^{\frac{a}{2} - 1} \right).$$

This result in particular implies that for any $\epsilon > 0$,

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) > (2 - \epsilon) \log g \right) = 1.$$

Let $\omega : \{2, 3, \dots\} \to \mathbb{R}^{>0}$ be any function satisfying

(1)
$$\lim_{g \to \infty} \omega(g) = +\infty \text{ and } \lim_{g \to \infty} \frac{\omega(g)}{\log \log g} = 0.$$

The main part of this article is to show

Theorem 1. Let $\omega(g)$ be a function satisfying (1). Consider the following two conditions defined for all $X \in \mathcal{M}_q$:

- (a). $|\ell_{\text{sys}}^{\text{sep}}(X) (2\log g 4\log\log g)| \le \omega(g);$
- (b). $\ell_{\text{sys}}^{\text{sep}}(X)$ is achieved by a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$.

Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; X \text{ satisfies } (a) \text{ and } (b)) = 1$$

The result in particular implies that for any $\epsilon > 0$,

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (2 - \epsilon) \log g < \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) < 2 \log g \right) = 1.$$

Although $\ell_{\text{sys}}^{\text{sep}}$ is unbounded on \mathcal{M}_g as introduced above, the very recent joint work [PWX21] of H. Parlier with the second and third named authors of this paper shows that the expected value $\mathbb{E}_{\text{WP}}^g[\ell_{\text{sys}}^{\text{sep}}] \sim 2 \log g$ as $g \to \infty$ (*c.f.* Section 10).

Remark. We remark that the seemingly cumbersome upper and lower bounds $2 \log g - 4 \log \log g \pm \omega(g)$ of $\ell_{\text{sys}}^{\text{sep}}(X)$ in the theorem above is related to the expected number of multi-geodesics of length less than L on $X \in \mathcal{M}_g$ bounding a one-holed torus or a three-holed sphere, which is roughly $\frac{L^2 e^{\frac{L}{2}}}{g}$. One may see the remark following Lemma 29 for more details.

In the subsequent subsections we discuss applications of Theorem 1 or the proof of Theorem 1.

RANDOM SURFACES

1.2. Long half collar and extremal length. A collar of a simple closed geodesic γ in a hyperbolic surface X is an embedded symmetric hyperbolic cylinder centered at γ , bounded by two equidistant curves from γ , whereas a half-collar of γ is an embedded hyperbolic cylinder bounded by one equidistant curve along with γ itself. For fixed g, if $X \in \mathcal{M}_g$ has a very long separating systolic curve γ , then the width of the maximal collar of γ is very small, because the area of X is fixed. On the other hand, as g goes to infinity, as an application of Theorem 1, we show that in an asymptotic sense, for a generic point $X \in \mathcal{M}_g$, there is an arbitrarily long half-collar around a separating systolic curve. More precisely,

Theorem 2. Given $\epsilon > 0$, consider condition (b) from Theorem 1 and the following condition defined for all $X \in \mathcal{M}_q$:

(c). There is a half-collar around γ in the $S_{g-1,1}$ -part of X with width $\frac{1}{2}\log g - (\frac{3}{2} + \epsilon)\log\log g$.

Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; X \text{ satisfies } (b) \text{ and } (c)) = 1.$$

Note that one cannot replace "half-collar" by "collar" in the above theorem. In fact, since a geodesic $\gamma \subset X$ realizing $\ell_{\text{sys}}^{\text{sep}}(X)$ is arbitrarily long and bounds a one-holed torus for a generic point $X \in \mathcal{M}_g$ by Theorem 1, the maximal embedded half-collar about γ in the one-holed torus must be arbitrarily thin because the area of a one-holed torus is equal to 2π .

The theory of *extremal length* was developed by Ahlfors and Beurling (*e.g.* see [Ahl10, Chapter 4]). One may also see [Ker80, Section 3] of Kerckhoff for its deep connection to the geometry of Teichmüller space. Here we deduce from Theorem 2 a consequence about extremal lengths of separating curves. Let $\text{Ext}_{\gamma}(X)$ denote the extremal length of the family of curves homotopic to γ (see Subsection 8.2 for the precise definition) and $\text{Ext}_{\text{sys}}^{\text{sep}}(X)$ denote the separating extremal length systole of X, defined as the infimum of $\text{Ext}_{\gamma}(X)$ over all separating simple closed geodesics γ on X. It is known by Maskit [Mas85] that $\ell_{\gamma}(X) \leq \pi \text{Ext}_{\gamma}(X)$, hence $\ell_{\text{sys}}^{\text{sep}}(X) \leq \pi \text{Ext}_{\text{sys}}^{\text{sep}}(X)$. Conversely, [Mas85] also provided an upper bound for $\text{Ext}_{\gamma}(X)$ (see Lemma 57), implying that hyperbolic and extremal lengths are comparable for short curves, but not for long ones in general. As an application of Theorem 2, we show:

Theorem 3. Given any $\epsilon > 0$, we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \frac{\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X)}{\ell_{\operatorname{sys}}^{\operatorname{sep}}(X)} < \frac{2+\epsilon}{\pi} \right) = 1,$$

as a consequence of this result and Theorem 1,

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g; \frac{(2-\epsilon)}{\pi} \log g < \operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X) < \frac{(4+\epsilon)}{\pi} \log g \right) = 1.$$

Remark. For $X \in \mathcal{M}_g$, the extremal length systole $\operatorname{Ext}_{\operatorname{sys}}(X)$ of X is defined as the infimum of $\operatorname{Ext}_{\gamma}(X)$ over all simple closed geodesics γ on X. For any systolic curve $\gamma \subset X$, it is known that the maximal collar of γ has width bounded from below by a uniform positive constant independent of g (see e.g. [Wu19, Lemma 4.6] if the systole length ≥ 1 and use the classical Collar Lemma if the systole length < 1). Then by Maskit [Mas85] (or see Lemma 57) it is not hard to see that $\operatorname{Ext}_{\operatorname{sys}}(X)$ is uniformly comparable to $\ell_{\operatorname{sys}}(X)$. Thus, as the genus g goes to infinity, the asymptotic behavior of $\operatorname{Ext}_{\operatorname{sys}}$ on \mathcal{M}_g is similar as the behavior of $\ell_{\operatorname{sys}}$ on \mathcal{M}_g , which has already been studied by Mirzakhani [Mir13] and Mirzakhani-Petri [MP19].

1.3. Shortest non-simple closed geodesic. A shortest non-simple closed geodesic on a closed hyperbolic surface is always a figure eight geodesic (*e.g.* see [Bus10, Theorem 4.24]), and has length at least $4 \operatorname{arcsinh}(1)$ (*e.g.* see [Bus10, 4.2.2]). The *non-simple systole* $\ell_{\text{sys}}^{ns}(X)$ of a hyperbolic surface X is defined as

 $\ell_{\rm sys}^{\rm ns}(X) := \min \left\{ \ell_{\gamma}(X) \, ; \, \gamma \subset X \text{ is a non-simple closed geodesic} \right\}.$

As another application of Theorem 1, we show that as g goes to infinity, asymptotically on a generic point $X \in \mathcal{M}_g$ the non-simple systole behaves roughly like log g. More precisely,

Theorem 4. Given any $\epsilon > 0$, then we have

 $\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (1 - \epsilon) \log g < \ell_{\operatorname{sys}}^{\operatorname{ns}}(X) < 2 \log g \right) = 1.$

1.4. Shortest separating closed multi-geodesics. The union of disjoint non-separating simple closed curves may also separate a closed surface. The following geometric quantity was used by Schoen-Wolpert-Yau [SWY80] to study the eigenvalues of the Laplacian operator on hyperbolic surfaces.

Definition 5. For any $X \in \mathcal{M}_q$, we define

$$\mathcal{L}_1(X) := \min \left\{ \ell_{\gamma}(X) \, ; \begin{array}{l} \gamma = \gamma_1 + \dots + \gamma_k \text{ is a simple closed} \\ \text{multi-geodesics separating } X \end{array} \right\}$$

As a byproduct of the proof of Theorem 1, we show a similar result on \mathcal{L}_1 as follows.

Theorem 6. Let $\omega(g)$ be a function satisfying (1). Consider the following two conditions defined for all $X \in \mathcal{M}_g$:

- (e). $|\mathcal{L}_1(X) (2\log g 4\log\log g)| \le \omega(g);$
- (f). $\mathcal{L}_1(X)$ is achieved by either a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$ or three simple closed geodesics separating X into $S_{0,3} \cup S_{g-2,3}$.

Then we have

$$\lim_{q \to \infty} \operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; X \text{ satisfies } (e) \text{ and } (f)) = 1.$$

Now we consider the expected value of \mathcal{L}_1 over \mathcal{M}_g . Unlike the unboundness of $\ell_{\text{sys}}^{\text{sep}}$ on \mathcal{M}_g we first show that $\sup_{X \in \mathcal{M}_g} \mathcal{L}_1(X) \leq C \log g$ for some universal constant C > 0 independent of g (see Proposition 61). And then we apply Theorem 6 to show that

Theorem 7. The expected value $\mathbb{E}^g_{WP}[\mathcal{L}_1]$ of \mathcal{L}_1 on \mathcal{M}_g satisfies

$$\lim_{g \to \infty} \frac{\mathbb{E}_{\mathrm{WP}}^g[\mathcal{L}_1]}{\log g} = 2$$

As another byproduct of the proof of Theorem 1 we show the following useful property. First we make the following definition generalizing \mathcal{L}_1 .

Definition 8. For any integer $m \in [1, g - 1]$ and $X \in \mathcal{M}_g$, we define

$$\mathcal{L}_{1,m}(X) := \min_{\Gamma} \ell_{\Gamma}(X)$$

where the minimum runs over all simple closed multi-geodesics Γ separating X into $S_{g_1,k} \cup S_{g_2,k}$ with $|\chi(S_{g_1,k})| \ge |\chi(S_{g_2,k})| \ge m$.

By definition we know that

$$\mathcal{L}_{1,1}(X) = \mathcal{L}_1(X)$$

and

$$\mathcal{L}_{1,m-1}(X) \le \mathcal{L}_{1,m}(X), \quad \forall m \in [2,g-1].$$

Proposition 9. Let $\omega(g)$ be a function satisfying (1). Then we have that for any fixed $m \geq 1$ independent of g,

$$\lim_{a \to \infty} \operatorname{Prob}_{\operatorname{WP}}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,m}(X) \ge 2m \log g - (6m - 2) \log \log g - \omega(g) \right) = 1$$

If m = 1, this is part of Theorem 6.

Remark. As in [Mir13], for all $1 \le m \le g-1$ the *m*-th geometric Cheeger constant $H_m(X)$ of X is defined as

$$H_m(X) := \inf_{\gamma} \frac{\ell_{\gamma}(X)}{2\pi m}$$

where γ is a simple closed multi-geodesics on X with $X \setminus \gamma = X_1 \cup X_2$, and X_1 and X_2 are connected subsurfaces of X such that $|\chi(X_1)| = m \le |\chi(X_2)|$.

The geometric Cheeger constant H(X) of X is defined as

$$H(X) := \min_{1 \le m \le g-1} H_m(X).$$

Mirzakhani in [Mir13] showed that

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, H(X) > \frac{\log 2}{2\pi} \right) = 1.$$

As a direct consequence of Theorem 6, we obtain the following result on the asymptotic behavior of the first geometric Cheeger constant H_1 on \mathcal{M}_g .

Corollary 10. For any $\epsilon > 0$, we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (1 - \epsilon) \cdot \frac{\log g}{\pi} < H_1(X) < \frac{\log g}{\pi} \right) = 1.$$

It would be also *interesting* to study the asymptotic behavior of H_m on \mathcal{M}_g when $2 \leq m \leq (g-1)$ as g goes to infinity. One may see the last section for more discussions.

Strategy on the proof of Theorem 1. We conclude this introduction by a brief outline of the proof of Theorem 1. We divide the statements into two parts. An easier part is that $\mathcal{L}_1(X) \geq 2\log g - 4\log\log g - \omega(g)$ with high probability, or equivalently,

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \le 2 \log g - 4 \log \log g - \omega(g) \right) = 0.$$

The proof uses the techniques of Mirzakhani in [Mir13, Section 4.3], and is based on expectation estimates of functions on \mathcal{M}_g of the form $N_{g_0,n_0}(X,L)$, which counts simple closed multi-geodesics on X with length at most L bounding subsurfaces of type S_{g_0,n_0} .

The second part, where the main novelty of this paper lies, is the assertion that with high probability, a random $X \in \mathcal{M}_g$ contains a simple closed geodesic γ with length at most $L(g) := 2 \log g - 4 \log \log g + \omega(g)$ bounding a one-holed torus, or equivalently,

$$\lim_{q \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, N_{1,1}(X, L(g)) = 0 \right) = 0.$$

We consider instead the function $N_{1,1}^*(X, L(g))$, which counts those γ 's whose intersection with any other γ is "simple" in a certain sense, and show the stronger statement that the above limit holds if $N_{1,1}$ is replaced by $N_{1,1}^*$. To achieve this, first Chebyshev's Inequality (see Equation (16)) tells that one may bound it from above by the following terms where $Z^*(X, L)$ has certain intersections involved:

$$\begin{aligned} &\operatorname{Prob}_{\mathrm{WP}}^{g}\left(N_{1,1}^{*}(X,L)=0\right) \leq \frac{1}{\mathbb{E}_{\mathrm{WP}}^{g}[N_{1,1}^{*}(X,L)]} \\ &+ \frac{\mathbb{E}_{\mathrm{WP}}^{g}[Y^{*}(X,L)] - \mathbb{E}_{\mathrm{WP}}^{g}[N_{1,1}^{*}(X,L)]^{2}}{\mathbb{E}_{\mathrm{WP}}^{g}[N_{1,1}^{*}(X,L)]^{2}} + \frac{\mathbb{E}_{\mathrm{WP}}^{g}[Z^{*}(X,L)]}{\mathbb{E}_{\mathrm{WP}}^{g}[N_{1,1}^{*}(X,L)]^{2}} \end{aligned}$$

As $g \to \infty$, we combine the ideas in the aforementioned works [Mir07a, Mir13, MZ15] to show that the first two terms of the RHS above converge to 0. The most delicate part of the proof, inspired by the work [MP19], is to show the third term of the RHS above also converges to 0 as $g \to \infty$: we estimate the expected number of intersecting pairs of γ 's by resolving the intersections, then apply a new usage of Mirzakhani's generalized Mc-Shane identity [Mir07a] on 4-holed spheres and 2-holed tori to control the multiplicity occurring in the resolution procedure.

Plan of the paper. In Sections 2, 3, 4 and 5, we review the backgrounds, introduce some notations, and prove a few technical lemmas. We then prove the lower bound part and the upper bound part of Theorem 1 and 6 in Sections 6 and 7, respectively. In Section 8, we prove Theorem 2 and 3. Theorem 4 and 7 will be proved in Section 9. We will pose several advanced questions in Section 10.

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CONTENTS

1.	Introduction	1
2.	Preliminaries	7
3.	Union of two subsurfaces with geodesic boundaries	11

4.	Weil-Petersson volume	16
ч.		10
5.	Mirzakhani's generalized McShane identity	23
6.	Lower bound	27
7.	Upper bound	32
8.	Half-collars and separating extremal length systole	53
9.	Non-simple systole and expected value of \mathcal{L}_1	57
10.	. Further questions	61
References		62

2. Preliminaries

In this section, we set our notations and review the relevant background material about moduli spaces of Riemann surfaces, Weil-Petersson metric and Mirzakhani's Integration Formula.

2.1. Weil-Petersson metric. We denote by $S_{g,n}$ an oriented surface of genus g with n punctures such that $2g + n \geq 3$, and let $\mathcal{T}_{g,n}$ denote the Teichmüller space of $S_{g,n}$, formed by equivalence classes of complete hyperbolic surfaces of finite area marked by $S_{g,n}$. The tangent space $T_X \mathcal{T}_{g,n}$ at a point $X \cong (S_{g,n}, \sigma(z)|dz|^2)$ is identified with the space of finite area harmonic Beltrami differentials on X, i.e. forms on X expressible as $\mu = \overline{\psi}/\sigma$ where $\psi \in Q(X)$ is a holomorphic quadratic differential on X. Let z = x + iyand $dA = \sigma(z)dxdy$ be the volume form. The Weil-Petersson metric is the Hermitian metric on $\mathcal{T}_{g,n}$ arising from the Petersson scalar product

$$\langle \varphi, \psi \rangle = \int_X \frac{\varphi \cdot \overline{\psi}}{\sigma^2} dA$$

via duality. We will concern ourselves primarily with its Riemannian part g_{WP} . Throughout this paper we denote by $\text{Teich}(S_{g,n})$ the Teichmüller space endowed with the Weil-Petersson metric. By definition it is easy to see that the mapping class group $\text{Mod}_{g,n} := \text{Diff}^+(S_{g,n})/\text{Diff}^0(S_{g,n})$ acts on Teich $(S_{g,n})$ as isometries. Thus, the Weil-Petersson metric descends to a metric, also called the Weil-Petersson metric, on the moduli space of Riemann surfaces $\mathcal{M}_{g,n}$ which is defined as $\mathcal{T}_{g,n}/\text{Mod}_{g,n}$. Throughout this paper we also denote by $\mathcal{M}_{g,n}$ the moduli space endowed with the Weil-Petersson metric and write $\mathcal{M}_g = \mathcal{M}_{g,0}$ for simplicity. Given $\mathbf{L} = (L_1, \cdots, L_n) \in \mathbb{R}^n_{\geq 0}$, the weighted Teichmüller space $\mathcal{T}_{g,n}(\mathbf{L})$ parametrizes hyperbolic surfaces X marked by $S_{g,n}$ such that for each $i = 1, \cdots, n$,

- if $L_i = 0$, the *i*th puncture of X is a cusp;
- if $L_i > 0$, one can attach a circle to the i^{th} puncture of X to form a geodesic boundary loop of length L_i .

The weighted moduli space $\mathcal{M}_{g,n}(\mathbf{L}) := \mathcal{T}_{g,n}(\mathbf{L}) / \operatorname{Mod}_{g,n}$ then parametrizes unmarked such surface. The Weil-Petersson volume form is also well-defined on $\mathcal{M}_{g,n}(\mathbf{L})$ and its total volume, denoted by $V_{g,n}(\mathbf{L})$, is finite. 2.2. The Fenchel-Nielsen coordinates. Recall that for any surface $S_{g,n}$, a pants decomposition \mathcal{P} of $S_{g,n}$ is a set of (3g+n-3) disjoint simple closed curves $\{\alpha_i\}_{i=1}^{3g+n-3}$ such that the complement $S_{g,n} \setminus \bigcup_{i=1}^{3g+n-3} \alpha_i$ of $S_{g,n}$ consists of disjoint union of three-holed spheres. For each $\alpha_i \in \mathcal{P}$, there are two natural real positive functions on $\mathcal{T}_{g,n}$: the geodesic length function $\ell_{\alpha_i}(X)$ and the twist function $\tau_{\alpha_i}(X)$ along α_i . Associated to \mathcal{P} , the Fenchel-Nielsen coordinates, given by $X \mapsto (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X))_{i=1}^{3g+n-3}$, is a global coordinate for $\mathcal{T}_{g,n}$. Wolpert in [Wol82] showed that the Weil-Petersson sympletic structure has a simple form in Fenchel-Nielsen coordinates:

Theorem 11 (Wolpert). The Weil-Petersson sympletic form ω_{WP} on $\mathcal{T}_{g,n}$ is given by

$$\omega_{\rm WP} = \sum_{i=1}^{3g+n-3} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

In the sequel, we mainly work with the Weil-Petersson volume form

$$d\text{vol}_{\text{WP}} := \frac{1}{(3g+n-3)!} \underbrace{\omega_{\text{WP}} \wedge \dots \wedge \omega_{\text{WP}}}_{3g+n-3 \text{ copies}}$$

It is a $\operatorname{Mod}_{g,n}$ -invariant measure on $\mathcal{T}_{g,n}$, hence is the lift of a measure on $\mathcal{M}_{g,n}$, which we still denote by $d\operatorname{vol}_{WP}$. The total volume of $\mathcal{M}_{g,n}$ is known to be finite and is denoted by $V_{g,n}$.

Our main objects of study are geometric quantities on \mathcal{M}_g . Following [Mir13], we view such a quantity $f : \mathcal{M}_g \to \mathbb{R}$ as a random variable on \mathcal{M}_g with respect to the probability measure $\operatorname{Prob}_{WP}^g$ defined by normalizing $d\operatorname{vol}_{WP}$, and let $\mathbb{E}_{WP}^g[f]$ denote the expectation. Namely,

$$\operatorname{Prob}_{\operatorname{WP}}^{g}(\mathcal{A}) := \frac{1}{V_g} \int_{\mathcal{M}_g} \mathbf{1}_{\mathcal{A}} dX, \quad \mathbb{E}_{\operatorname{WP}}^{g}[f] := \frac{1}{V_g} \int_{\mathcal{M}_g} f(X) dX,$$

where $\mathcal{A} \subset \mathcal{M}_g$ is any Borel subset, $\mathbf{1}_{\mathcal{A}} : \mathcal{M}_g \to \{0, 1\}$ is its characteristic function, and we always write $d\mathrm{vol}_{\mathrm{WP}}(X)$ as dX for short. In this paper, we view certain geometric quantities as random variables on \mathcal{M}_g , and study their asymptotic behaviors as $g \to \infty$. One may also see [DGZZ22, GMST21, GPY11, MT21, MP19] for related interesting topics.

2.3. Mirzakhani's Integration Formula. In [Mir07a], Mirzakhani gave a formula to integrate geometric functions over moduli spaces, which is essential in the study of random surfaces with respect to Weil-Petersson metric. Then in the same paper she calculated the volume of moduli spaces together with her generalized McShane identity. In [Mir13], applying this formula, she gave many estimations for some geometry variables in probability meaning. Here we give the version stated in [Mir13], which is a little more general than the one in [Mir07a].

Given a homotopy class γ of closed curves on a topological surface $S_{g,n}$ and $X \in \mathcal{T}_{g,n}$, we denote by $\ell_{\gamma}(X)$ the hyperbolic length of the unique closed geodesic in the homotopy class γ on X. We also write $\ell(\gamma)$ for simplicity if the surface X is clear from the context. Let $\Gamma = (\gamma_1, \dots, \gamma_k)$ be an ordered k-tuple of disjoint homotopy classes of nontrivial, non-peripheral, simple closed curves on $S_{g,n}$. Denote the orbit of Γ under the $\operatorname{Mod}_{g,n}$ -action by

$$\mathcal{O}_{\Gamma} = \{(h \cdot \gamma_1, \cdots, h \cdot \gamma_k); h \in \mathrm{Mod}_{g,n}\}.$$

Given a function $F: \mathbb{R}_{\geq 0}^k \to \mathbb{R}$, we consider the function on $\mathcal{M}_{g,n}$ given by

$$F^{\Gamma}: \mathcal{M}_{g,n} \to \mathbb{R}$$
$$X \mapsto \sum_{(\alpha_1, \cdots, \alpha_k) \in \mathcal{O}_{\Gamma}} F(\ell_{\alpha_1}(X), \cdots, \ell_{\alpha_k}(X))$$

provided that the sum converges.

Remark. Although ℓ_{γ} is only defined on $\mathcal{T}_{g,n}$, the function F^{Γ} is well-defined on $\mathcal{M}_{g,n}$.

Assume $S_{g,n} - \cup \gamma_j = \bigcup_{i=1}^s S_{g_i,n_i}$. For any given $\boldsymbol{x} = (x_1, \cdots, x_k) \in \mathbb{R}^k_{\geq 0}$, we consider the moduli space $\mathcal{M}(S_{g,n}(\Gamma); \ell_{\Gamma} = \boldsymbol{x})$ of (possibly disconnected) hyperbolic Riemann surfaces homeomorphic to $S_{g,n} - \cup \gamma_j$ with $\ell_{\gamma_i^1} = \ell_{\gamma_i^2} = x_i$ for $i = 1, \cdots, k$, where γ_i^1 and γ_i^2 are the two boundary components of $S_{g,n} - \cup \gamma_j$ given by γ_i . We consider the volume

$$V_{g,n}(\Gamma, \boldsymbol{x}) = \operatorname{Vol}_{WP} \left(\mathcal{M}(S_{g,n}(\Gamma); \ell_{\Gamma} = \boldsymbol{x}) \right).$$

In general

$$V_{g,n}(\Gamma,oldsymbol{x}) = \prod_{i=1}^s V_{g_i,n_i}(oldsymbol{x}^{(i)})$$

where $\boldsymbol{x}^{(i)}$ is the list of those coordinates x_j of \boldsymbol{x} such that γ_j is a boundary component of S_{g_i,n_i} . And $V_{g_i,n_i}(\boldsymbol{x}^{(i)})$ is the Weil-Petersson volume of the moduli space $\mathcal{M}_{g_i,n_i}(\boldsymbol{x}^{(i)})$. Mirzakhani used Theorem 11 of Wolpert to get the following integration formula. One may see [Mir07a, Theorem 7.1] or [MP19, Theorem 2.2] or [Wri20, Theorem 4.1].

Theorem 12. For any $\Gamma = (\gamma_1, \dots, \gamma_k)$, the integral of F^{Γ} over $\mathcal{M}_{g,n}$ with respect to Weil-Petersson metric is given by

$$\int_{\mathcal{M}_{g,n}} F^{\Gamma}(X) dX = C_{\Gamma} \int_{\mathbb{R}^{k}_{\geq 0}} F(x_{1}, \cdots, x_{k}) V_{g,n}(\Gamma, \boldsymbol{x}) \boldsymbol{x} \cdot d\boldsymbol{x}$$

where $\boldsymbol{x} \cdot d\boldsymbol{x} = x_1 \cdots x_k dx_1 \wedge \cdots \wedge dx_k$ and the constant $C_{\Gamma} \in (0,1]$ only depends on Γ . Moreover, $C_{\Gamma} = 2^{-k}$ if each γ_i in Γ separates off a one-holed torus.

Remark. An explicit expression for C_{Γ} is provided by Wright in [Wri20, Section 4].

Remark. Given an unordered multi-curve $\gamma = \sum_{i=1}^{k} c_i \gamma_i$ where $\gamma'_i s$ are distinct disjoint homotopy classes of nontrivial, non-peripheral, simple closed curves on $S_{g,n}$, when F is a symmetric function, we can define

$$F_{\gamma}: \mathcal{M}_{g,n} \to \mathbb{R}$$

$$X \mapsto \sum_{\sum_{i=1}^{k} c_{i}\alpha_{i} \in \operatorname{Mod}_{g,n} \cdot \gamma} F(c_{1}\ell_{\alpha_{1}}(X), \cdots, c_{k}\ell_{\alpha_{k}}(X)).$$

It is easy to check that

$$F^{\Gamma}(X) = |\operatorname{Sym}(\gamma)| \cdot F_{\gamma}(X)$$

where $\Gamma = (c_1 \gamma_1, \cdots, c_k \gamma_k)$ and $\operatorname{Sym}(\gamma)$ is the symmetry group of γ defined by

$$\operatorname{Sym}(\gamma) = \operatorname{Stab}(\gamma) / \bigcap_{i=1}^{k} \operatorname{Stab}(\gamma_i).$$

Actually we consider the integration of F_{γ} for most times in this paper.

2.4. Counting functions. In this subsection we introduce some notations that will be used in the whole paper here.

On a topological surface $S_{g,n}$ with $\chi(S_{g,n}) = 2 - 2g - n < 0$, let $\gamma = \sum_{i=1}^{k} \gamma_i$ be a simple closed multi-curves where $\gamma'_i s$ are disjoint homotopy classes of nontrivial, non-peripheral, simple closed curves on $S_{g,n}$. For any $X \in \mathcal{T}_{g,n}$, we put

$$\ell_{\gamma}(X) := \sum_{i=1}^{k} \ell_{\gamma_i}(X).$$

We sometimes write $\ell_{\gamma}(X)$ as $\ell(\gamma)$ if the surface X is clear from the context.

Consider the orbit of γ under the mapping class group $\operatorname{Mod}_{q,n}$ action, which we denote by

$$\mathcal{O}_{\gamma} = \{h \cdot \gamma \, ; \, h \in \mathrm{Mod}_{g,n}\}$$

where $h \cdot \gamma = h \cdot \sum_{i=1}^{k} \gamma_i = \sum_{i=1}^{k} h \cdot \gamma_i$. For any $X \in \mathcal{T}_{g,n}$ and L > 0, we can define the counting function

$$N_{\gamma}(X,L) := \#\{\alpha \in \mathcal{O}_{\gamma} ; \ell_{\alpha}(X) \le L\}.$$

Moreover, although ℓ_{γ} is only defined for $\mathcal{T}_{g,n}$, the counting function $N_{\gamma}(\cdot, L)$ is well-defined on $\mathcal{M}_{g,n}$.

Note that the orbit \mathcal{O}_{γ} of a simple closed multi-curve γ is determined by the topology of $S_{g,n} - \gamma$. We also use the following notations for some special types of multi-curves γ .

When α consists of n_0 simple closed curves separating S_g into $S_{g_0,n_0} \cup$ $S_{g-g_0-n_0+1,n_0}$ (e.g. see Figure 1 for the case that $n_0 = 1$ and $g_0 = 1$), we write

$$N_{q_0,n_0}(X,L) := N_{\alpha}(X,L).$$

When γ consists of n_0 simple closed curves separating S_g into q+1 pieces $S_{g_0,n_0} \cup S_{g_1,n_1} \cup S_{g_2,n_2} \cup \dots \cup S_{g_q,n_q}$ with $n_1 + \dots + n_q = n_0$ and $g_0 + g_1 + \dots + g_q = n_0$ $\cdots + g_q + n_0 - q = g$ (e.g. see Figure 1), we write

$$N_{g_0,n_0}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L) := N_{\gamma}(X,L).$$

In particularly, $N_{g_0,n_0}(X,L) = N_{g_0,n_0}^{(g-g_0-n_0+1,n_0)}(X,L)$.

We will also use some other less common counting functions in this paper and will introduce them when required.

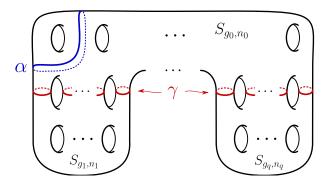


Figure 1

3. Union of two subsurfaces with geodesic boundaries

In this section, we present some hyperbolic-geometric constructions and lemmas used in Section 7 below.

Construction. Fix a closed hyperbolic surface $X \in \mathcal{M}_g$ and let X_1, X_2 be two distinct connected, compact subsurfaces of X with geodesic boundaries, such that $X_1 \cap X_2 \neq \emptyset$ and neither of them contains the other. Then the union $X_1 \cup X_2$ is a subsurface whose boundary is only piecewise geodesic. We can construct from it a new subsurface, with geodesic boundary, by deforming each of its boundary components $\xi \subset \partial(X_1 \cup X_2)$ as follows:

- if ξ is homotopically both nontrivial and distinct from any other component of $\partial(X_1 \cup X_2)$, we deform $X_1 \cup X_2$ by shrinking ξ to the unique simple closed geodesic homotopic to it;
- if ξ is homotopically trivial, we fill into $X_1 \cup X_2$ the disc bounded by ξ ;
- if ξ is homotopic to some other component ξ' of $\partial(X_1 \cup X_2)$, we fill into $X_1 \cup X_2$ the annulus bounded by ξ and ξ' .

Denoted the resulting compact subsurface with geodesic boundary by X_3 .

By construction, it is clear that

$$\ell(\partial X_3) \le \ell(\partial X_1) + \ell(\partial X_2).$$

We will mainly apply this construction to the situation where X_1 and X_2 are both one-holed torus (that is, of type $S_{1,1}$). We introduce the following notation for this case:

Definition 13. Suppose $X \in \mathcal{M}_g$. For a simple closed geodesic $\alpha \subset X$ bounding a one-holed torus, let X_{α} denote the one-holed torus bounded by α . For two such geodesics α, β with $\alpha \neq \beta, \alpha \cap \beta \neq \emptyset$, let $X_{\alpha\beta}$ denote the subsurface X_3 of X constructed above for $X_1 = X_{\alpha}$ and $X_2 = X_{\beta}$. See Figure 2.

Remark. The first example in Figure 2 illustrates the case where β is obtained from α by *n*-times Dehn twist along another simple closed curve. In this case, $X_{\alpha\beta}$ is always of type $S_{1,2}$. Note that $X_{\beta} \setminus X_{\alpha}$ is a disjoint union of strips homotopic to each other in this case. So one can construct a

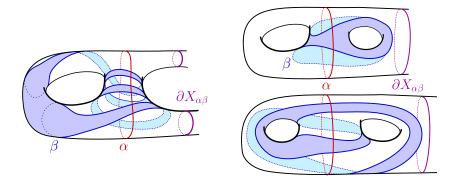


FIGURE 2. Examples of (α, β) and $X_{\alpha\beta}$. The one-holed torus X_{β} is colored. $X_{\alpha\beta}$ is of type $S_{1,2}$ in the first example and of type $S_{2,1}$ in both examples on the right.

pair (α, β) with $|\chi(X_{\alpha\beta})|$ arbitrarily large by modifying these strips, making them not homotopic.

We now return to the general case and establish a basic property for X_3 :

Lemma 14. Let X_1 , X_2 and X_3 be as above. Then we have

 $X_1 \cup X_2 \subset X_3,$

and the complement $X_3 \setminus (X_1 \cup X_2)$ is a disjoint union of topological discs and cylinders.

Proof. We begin with the observation that $X_0 := X_1 \cup X_2$ is a subsurface of X with *concave* piecewise geodesic boundary, where the concavity means that for each junction point $p \in \partial X_0$ of two geodesic pieces of ∂X_0 , the inner angle $\angle_p X_0$ of X_0 at p is greater than π (see Figure 3). This is because $\angle_p X_0$ is formed by overlapping the two π -angles given by X_1 and X_2 at p.

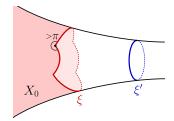


FIGURE 3

By the construction of X_3 , in order to prove the required statements, we only need to show that if ξ is a component of ∂X_0 which is homotopically nontrivial and consists of at least two geodesic pieces, then ξ and the simple closed geodesic ξ' homotopic to ξ together bound an annulus outside of X_0 , as Figure 3 shows.

Suppose by contradiction that ξ violates this property. Then we are in one of the following cases:

Case 1. ξ' is contained in $X_0 \setminus \xi$ (see Figure 4). Applying the Gauss-

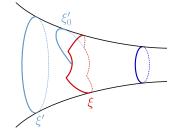


FIGURE 4

Bonnet formula to the annulus A bounded by ξ and ξ' in this case, we get

$$-\operatorname{Area}(A) + \sum_{p \in J(\xi)} (\pi - \angle_p X_0) = 2\pi \chi(A) = 0,$$

where $J(\xi)$ denote the set of junction points of the geodesic pieces of ξ . This is a contradiction because the LHS is negative.

Case 2. Otherwise, we have $\xi' \cap \xi \neq \emptyset$. In this case, ξ' contains an arc ξ'_0 in $X_0 \setminus \xi$ joining two points q_1, q_2 of ξ (see Figure 4). These two points separate ξ into two arcs and one of them, denoted by ξ_0 , bounds a disc D together with ξ'_0 because ξ' is homotopic to ξ . Applying Gauss-Bonnet to D, we get

$$-\operatorname{Area}(D) + (\pi - \angle_{q_1} D) + (\pi - \angle_{q_2} D) + \sum_{p \in J(\xi_0)} (\pi - \angle_p X_0) = 2\pi \chi(D) = 2\pi,$$

which also leads to a contradiction.

We proceed to give bounds on the Euler characteristic of X_3 :

Lemma 15. Let X_1 , X_2 and X_3 be as above. Then we have

$$|\chi(X_3)| \ge 1 + \max\{|\chi(X_1)|, |\chi(X_2)|\}$$

and

$$|\chi(X_3)| \le |\chi(X_1)| + |\chi(X_2)| + \frac{\ell(\partial X_1) + \ell(\partial X_2)}{2\pi}$$

Proof. By Gauss-Bonnet formula and the assumption that neither X_1 nor X_2 contains the other, we have

$$\begin{aligned} |\chi(X_3)| &= \frac{1}{2\pi} \operatorname{Area}(X_3) \\ &> \frac{1}{2\pi} \max\{\operatorname{Area}(X_1), \operatorname{Area}(X_2)\} \\ &= \max\{|\chi(X_1)|, |\chi(X_2)|\}, \end{aligned}$$

which is equivalent to the required lower bound of $|\chi(X_3)|$ because Euler characteristics are integers.

To prove the upper bound, let ξ_1, \dots, ξ_r be the boundary components of $X_1 \cup X_2$ which are piecewise geodesics with at least two pieces. Let Idenote the set of indices $i \in \{1, \dots, r\}$ such that ξ_i is homotopically trivial, J denote the set of indices $j \in \{1, \dots, r\}$ such that ξ_j is homotopic to a component of ∂X_3 , and K denote the set of indices $k \in \{1, \dots, r\}$ such that ξ_k is homotopic to a geodesic in the interior of X_3 .

By Lemma 14, $X_3 \setminus (X_1 \cup X_2)$ is a disjoint union of topological discs $\{D_i\}_{i \in I}$, cylinders $\{C_j\}_{j \in J}$ and cylinders $\{C'_p\}_{p \in P}$, where ∂D_i is exactly ξ_i , ∂C_j is the union of ξ_j and some boundary component of X_3 , and $\partial C'_p$ is the union of two elements $\xi_{k_p^1}$ and $\xi_{k_p^2}$ of $\{\xi_k\}_{k \in K}$. Each element of $\{\xi_1, \dots, \xi_r\}$ appears in $\{\partial D_i\}_{i \in I}$ or $\{\partial C_j\}_{j \in J}$ or $\{\partial C'_p\}_{p \in P}$ exactly once.

By Isoperimetric Inequality for topological discs and cylinders on hyperbolic surfaces (e.g. see [Bus10] or [WX22]), we have

$$\operatorname{Area}(D_i) \leq \ell(\partial D_i) = \ell(\xi_i),$$

$$2\operatorname{Area}(C_j) = \operatorname{Area}(2C_j) \leq \ell(\partial(2C_j)) = 2\ell(\xi_j),$$

$$\operatorname{Area}(C'_p) \leq \ell(\partial C'_p) = \ell(\xi_{k_p^1}) + \ell(\xi_{k_p^2}),$$

where $2C_i$ denote the double of C_i along its geodesic boundary component in ∂X_3 . Therefore,

$$Area(X_3) = Area(X_1 \cup X_2) + \sum_{i \in I} Area(D_i) + \sum_{j \in J} Area(C_j) + \sum_{p \in P} Area(C'_p)$$

$$\leq Area(X_1 \cup X_2) + \ell(\xi_1) + \dots + \ell(\xi_r)$$

$$\leq Area(X_1) + Area(X_2) + \ell(\partial(X_1 \cup X_2))$$

$$\leq Area(X_1) + Area(X_2) + \ell(\partial X_1) + \ell(\partial X_2).$$

This gives the required upper bound of $|\chi(X_3)|$ again by Gauss-Bonnet. \Box

In the case where X_1 and X_2 are one-holed torus, Lemma 15 implies:

Lemma 16. On $X \in \mathcal{M}_g$, let α, β be two simple closed geodesics bounding one-holed torus with $\ell(\alpha) \leq L, \ell(\beta) \leq L$ and $\alpha \neq \beta, \alpha \cap \beta \neq \emptyset$. Then we have

(1) The genus of $X_{\alpha\beta}$ is at least 1, and the Euler characteristic $\chi(X_{\alpha\beta})$ satisfies

$$2 \le |\chi(X_{\alpha\beta})| \le \frac{1}{\pi}L + 2.$$

(2) If $|\chi(X_{\alpha\beta})| = 2$ and $g \ge 3$, then $X_{\alpha\beta}$ is of type $S_{1,2}$.

Proof. Statement (1) follows from Lemma 15. Statement (2) is because the only surfaces S_{g_0,n_0} such that $|\chi(S_{g_0,n_0})| = 2$ and $g_0 \ge 1$ are $S_{1,2}$ and $S_{2,0}$, whereas X cannot have a subsurface of type $S_{2,0}$ if $g \ge 3$.

Finally, we show that in the case where $X_{\alpha\beta}$ is of type $S_{1,2}$, under some additional assumptions, one can reduce $X_{\alpha\beta}$ to a 4-holed sphere:

Lemma 17. On $X \in \mathcal{M}_g$, let α, β be two simple closed geodesics bounding one-holed torus with the following properties for some L > 0:

- $\alpha \neq \beta, \ \alpha \cap \beta \neq \emptyset;$
- $X_{\alpha\beta}$ is of type $S_{1,2}$;
- $\ell(\alpha) \leq L, \ \ell(\beta) \leq L;$
- $\ell(\partial X_{\alpha\beta}) \ge \frac{5}{3}L.$

RANDOM SURFACES

Then α and β have exactly 4 intersection points, and the intersection $\check{X}_{\alpha} \cap \check{X}_{\beta}$ (where ' \circ ' denotes the interior) contains a unique simple closed geodesic δ (see Figure 5).

Note that since a one-holed torus with geodesic boundary is cut by any simple closed geodesic in its interior into a pair-of-pants, the geodesic δ given by the lemma cuts $X_{\alpha\beta}$ into a 4-holed sphere containing both α and β .

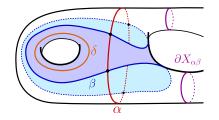


FIGURE 5. Simple closed geodesic $\delta \subset X_{\alpha} \cap X_{\beta}$.

Proof. We first show

$$\#(\alpha \cap \beta) = 4.$$

Since α and β are both separating and hence represent the zero homology class, $\#(\alpha \cap \beta)$ is a positive even number. Moreover, if $\#(\alpha \cap \beta) = 2$, then $\beta \cap X_{\alpha}$ is a single simple geodesic arc splitting X_{α} into at least two pieces (namely, the connected components of $X_{\alpha} \cap \mathring{X}_{\beta}$ and $X_{\alpha} \setminus X_{\beta}$), which is a contradiction because a simple geodesic arc in a one-holed torus joining boundary points cannot separate the one-holed torus.

Thus, if $\#(\alpha \cap \beta)$ is not 4, then it is at least 6. In this case, $\beta \setminus X_{\alpha}$ consists of at least 3 segment. We denote the shortest two among these segments by β_1 and β_2 , whose total length satisfy

$$\ell(\beta_1) + \ell(\beta_2) \le \frac{2}{3}\ell(\beta) \le \frac{2}{3}L.$$

Since β_1 and β_2 are disjoint geodesic arcs in the pair of pants $X_{\alpha\beta} \setminus X_{\alpha}$ with endpoints in the same boundary component α , they are homotopic to each other relative to α . Therefore, $\partial X_{\alpha\beta}$ is homotopic to the two closed piecewise geodesics formed by β_1 , β_2 along with two disjoint segments of α , and hence

$$\ell(\partial X_{\alpha\beta}) < \ell(\alpha) + \ell(\beta_1) + \ell(\beta_2) \le L + \frac{2}{3}L = \frac{5}{3}L,$$

contradicting the assumption $\ell(X_{\alpha\beta}) \geq \frac{5}{3}L$. This proves $\#(\alpha \cap \beta) = 4$.

As a consequence, β is split by α into $\check{4}$ segments. Since the two segments β_1, β_2 outside of \mathring{X}_{α} are homotopic relative to α as above, $X_{\beta} \setminus \mathring{X}_{\alpha}$ is homeomorphic to a disk. We now consider the other two segments, which are in X_{α} , and denote them by β'_1, β'_2 .

It is a basic fact that given a one-holed torus Y with geodesic boundary, any two disjoint simple geodesic arcs $a_1, a_2 \subset Y$ with endpoints in ∂Y belong to one of the following cases (see Figure 6):

(1) If a_1 and a_2 are homotopic relative to ∂Y , then they split Y into two pieces, namely a topological cylinder and a topological disk;

(2) Otherwise, a_1 and a_2 split Y into a single topological disk.

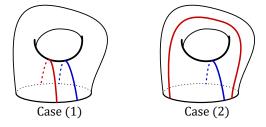


FIGURE 6. Two arcs in a one-holed torus.

Since β'_1 and β'_2 separate X_{α} , they are in Case (1). Thus, among the two pieces of X_{α} split out by β , namely $X_{\alpha} \cap X_{\beta}$ and $X_{\alpha} \setminus \mathring{X}_{\beta}$, one is a cylinder and the other is a disk. But we have shown above that $X_{\beta} \setminus \mathring{X}_{\alpha}$ is a disk, and the argument implies that $X_{\alpha} \setminus \mathring{X}_{\beta}$ is a disk as well if we switch the roles of α and β . Therefore, we conclude that $X_{\alpha} \cap X_{\beta}$ is a cylinder as shown in Figure 5. This cylinder contains a unique simple closed geodesic δ , namely the one homotopic to its boundary loops. And δ is in the interior of the cylinder since it is contained in both X_{α} and X_{β} , as required. \Box

Remark. By construction, $\alpha \cup \beta$ is homotopic to $\partial X_{\alpha\beta} \cup 2\delta$, where 2δ means two copies of δ 's. We will use this observation later in Subsection 7.5.

Remark. The second statement of Lemma 17 actually holds true for any intersecting pair (α, β) of simple closed geodesics bounding one-holed torus such that $X_{\alpha\beta}$ is of type $S_{1,2}$ (*c.f.* the first example in Figure 2). The proof is more complicated and not necessary for our purpose.

4. Weil-Petersson volume

In this section we give some results on the Weil-Petersson volumes of moduli spaces. All of these are already known or generalizations of known results. We denote $V_{g,n}(x_1, \dots, x_n)$ to be the Weil-Petersson volume of $\mathcal{M}_{g,n}(x_1, \dots, x_n)$ and $V_{g,n} = V_{g,n}(0, \dots, 0)$. One may also see [Agg21, Gru01, LX14, Mir07a, Mir07b, Mir13, MZ15, Pen92, ST01, Zog08, AM22] for the asymptotic behavior of $V_{g,n}$ and its deep connection to the intersection theory of $\mathcal{M}_{g,n}$.

First we recall several results of Mirzakhani and her coauthors.

Theorem 18. (1) [Mir07a, Theorem 1.1] The volume $V_{g,n}(x_1, \dots, x_n)$ is a polynomial in x_1^2, \dots, x_n^2 with degree 3g-3+n. Namely we have

$$V_{g,n}(x_1,\cdots,x_n) = \sum_{\alpha; |\alpha| \le 3g-3+n} C_{\alpha} \cdot x^{2\alpha}$$

where $C_{\alpha} > 0$ lies in $\pi^{6g-6+2n-|2\alpha|} \cdot \mathbb{Q}$. Here $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^{2\alpha} = x_1^{2\alpha_1} \cdots x_n^{2\alpha_n}$. (2) [Mir07a, Table 1]

$$V_{0,3}(x, y, z) = 1,$$

$$V_{1,1}(x) = \frac{1}{48}(x^2 + 4\pi^2)$$

Remark. We remark here that $V_{1,1}(x)$ is only one-half of the quantity originally given in [Mir07a] or [Mir08]. The reason is that there exists an involution for $S_{1,1}$. One may also see [DGZZ21, Remark 3.3] for more details.

Lemma 19. (1) [Mir13, Lemma 3.2]

$$V_{g,n} \le V_{g,n}(x_1,\cdots,x_n) \le e^{\frac{x_1+\cdots+x_n}{2}}V_{g,n}.$$

(2) [Mir13, Lemma 3.2] For any $g, n \ge 0$

$$V_{g-1,n+4} \le V_{g,n+2}$$

and

$$b_0 \leq \frac{V_{g,n+1}}{(2g-2+n)V_{g,n}} \leq b_1$$

for some universal constants $b_0, b_1 > 0$ independent of g, n. (3) [Mir13, Theorem 3.5] For fixed $n \ge 0$, as $g \to \infty$ we have

$$\begin{split} \frac{V_{g,n+1}}{2gV_{g,n}} &= 4\pi^2 + O\left(\frac{1}{g}\right),\\ \frac{V_{g,n}}{V_{g-1,n+2}} &= 1 + O\left(\frac{1}{g}\right). \end{split}$$

Where the implied constants depend on n but not on g.

Remark. For Part (3), one may also see the following Theorem 21 of Mirzakhani-Zograf.

Lemma 20. [Mir13, Corollary 3.7] For fixed $b, k, r \ge 0$ and $C < C_0 = 2 \log 2$,

$$\sum_{\substack{g_1 + g_2 = g + 1 - k \\ r + 1 \le g_1 \le g_2}} e^{Cg_1} \cdot g_1^b \cdot V_{g_1,k} \cdot V_{g_2,k} \asymp \frac{V_g}{g^{2r+k}}$$

as $g \to \infty$. The implied constants depend on b, k, r, C but not on g. Here $A \simeq B$ means $c_1 A \leq B \leq c_2 A$ for two constants $c_1, c_2 > 0$ independent of g.

Theorem 21. [MZ15, Theorem 1.2] There exists a universal constant $\alpha > 0$ such that for any given $n \ge 0$,

$$V_{g,n} = \alpha \frac{1}{\sqrt{g}} (2g - 3 + n)! (4\pi^2)^{2g - 3 + n} \left(1 + O\left(\frac{1}{g}\right) \right)$$

as $g \to \infty$. The implied constant depend on n but not on g.

Remark. It is conjectured by Zograf in [Zog08] that $\alpha = \frac{1}{\sqrt{\pi}}$, which is still open.

The following result is motivated by [MP19, Proposition 3.1] whose error term in the form of multiplication is replaced by the following summation. One may also see the very recent work of Anantharaman and Monk [AM22] for sharper results.

Lemma 22. Let $g, n \ge 1$ and $x_1, \dots, x_n \ge 0$, then there exists a constant c = c(n) > 0 independent of g, x_1, \dots, x_n such that

$$\prod_{i=1}^{n} \frac{\sinh(x_i/2)}{x_i/2} \left(1 - c(n) \frac{\sum_{i=1}^{n} x_i^2}{g}\right) \le \frac{V_{g,n}(x_1, \cdots, x_n)}{V_{g,n}} \le \prod_{i=1}^{n} \frac{\sinh(x_i/2)}{x_i/2}$$

Proof. By Theorem 18 we know that $V_{g,n}(2x_1, \dots, 2x_n)$ is a polynomial of x_1^2, \dots, x_n^2 with degree 3g - 3 + n. As in [Mir13, (3.1)] we write

$$V_{g,n}(2x_1,\cdots,2x_n) = \sum_{|\mathbf{d}| \le 3g-3+n} [\tau_{d_1},\cdots,\tau_{d_n}]_{g,n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!}$$

where $d = (d_1, \dots, d_n)$ with $d_i \ge 0$ and $|d| = d_1 + \dots + d_n$. In [Mir13, page 286], Mirzakhani gave the following bound for $[\tau_{d_1}, \dots, \tau_{d_n}]_{g,n}$. Given $n \ge 1$, we have

$$0 \le 1 - \frac{[\tau_{d_1}, \cdots, \tau_{d_n}]_{g,n}}{V_{g,n}} \le c_0 \frac{|\boldsymbol{d}|^2}{g}$$

where c_0 is independent of g and d (but may depend on n). So we have

$$\frac{V_{g,n}(2x_1,\cdots,2x_n)}{V_{g,n}} \le \sum_{|\boldsymbol{d}|\le 3g-3+n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!}$$

and

$$\frac{V_{g,n}(2x_1,\cdots,2x_n)}{V_{g,n}} \ge \sum_{|\mathbf{d}| \le 3g-3+n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} -\frac{c_0}{g} \sum_{|\mathbf{d}| \le 3g-3+n} |\mathbf{d}|^2 \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!}$$

Recall that

$$\prod_{i=1}^{n} \frac{\sinh(x_i)}{x_i} = \sum_{d_1, \cdots, d_n=0}^{\infty} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!}$$

So we get the upper bound

$$\frac{V_{g,n}(2x_1,\cdots,2x_n)}{V_{g,n}} \le \prod_{i=1}^n \frac{\sinh(x_i)}{x_i}$$

For the lower bound, first we have

$$x_1^2 \prod_{i=1}^n \frac{\sinh(x_i)}{x_i} = \left(\sum_{d_1=1}^\infty \frac{x_1^{2d_1}}{(2d_1-1)!} \right) \left(\sum_{d_2,\cdots,d_n=0}^\infty \frac{x_2^{2d_2}}{(2d_2+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} \right)$$

$$= \sum_{d_1,\cdots,d_n=0}^\infty \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} 2d_1(2d_1+1).$$

So

$$\left(\sum_{i=1}^{n} x_i^2\right) \prod_{i=1}^{n} \frac{\sinh(x_i)}{x_i} = \sum_{d_1, \cdots, d_n=0}^{\infty} \left(\frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} \sum_{i=1}^{n} 2d_i(2d_i+1)\right).$$

Then by Cauchy-Schwarz inequality we have

$$\sum_{|\mathbf{d}| \le 3g-3+n} |\mathbf{d}|^2 \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} \le \frac{n}{4} (x_1^2 + \dots + x_n^2) \prod_{i=1}^n \frac{\sinh(x_i)}{x_i}.$$

Recall that the Stirling formula says that

$$k! \sim \sqrt{2\pi k} (\frac{k}{e})^k$$

which implies that for large k > 0,

$$k! \ge (\frac{k}{e})^k.$$

Hence, we have

$$\sum_{|\mathbf{d}|>3g-3+n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!}$$

$$\leq \sum_{k>3g-3+n} \frac{1}{k!} \sum_{|\mathbf{d}|=k} \frac{k!}{d_1! \cdots d_n!} (x_1^2)^{d_1} \cdots (x_n^2)^{d_n}$$

$$= \sum_{k>3g-3+n} \frac{1}{k!} (x_1^2 + \cdots + x_n^2)^k$$

$$\leq \sum_{k>3g-3+n} \left(\frac{e \cdot (x_1^2 + \cdots + x_n^2)}{k}\right)^k.$$

If $\frac{e \cdot (x_1^2 + \dots + x_n^2)}{3g - 2 + n} \le 0.5$, we have

$$\sum_{|\mathbf{d}|>3g-3+n} \frac{x_1^{2d_1}}{(2d_1+1)!} \cdots \frac{x_n^{2d_n}}{(2d_n+1)!} \leq 2\left(\frac{e \cdot (x_1^2 + \dots + x_n^2)}{3g-2+n}\right)^{3g-2+n}$$
$$\leq 4\frac{x_1^2 + \dots + x_n^2}{g} \prod_{i=1}^n \frac{\sinh(x_i)}{x_i}.$$

Then we get when
$$\frac{e \cdot (x_1^2 + \dots + x_n^2)}{3g - 2 + n} \le 0.5$$
,
 $\frac{V_{g,n}(2x_1, \dots, 2x_n)}{V_{g,n}} \ge \prod_{i=1}^n \frac{\sinh(x_i)}{x_i} - \sum_{|d| > 3g - 3 + n} \frac{x_1^{2d_1}}{(2d_1 + 1)!} \cdots \frac{x_n^{2d_n}}{(2d_n + 1)!}$
 $- \frac{c_0}{g} \sum_{|d| \le 3g - 3 + n} |d|^2 \frac{x_1^{2d_1}}{(2d_1 + 1)!} \cdots \frac{x_n^{2d_n}}{(2d_n + 1)!}$
 $\ge \prod_{i=1}^n \frac{\sinh(x_i)}{x_i} \left(1 - (\frac{n}{4}c_0 + 4)\frac{x_1^2 + \dots + x_n^2}{g}\right).$

If $\frac{e(x_1^2+\dots+x_n^2)}{3g-2+n} > 0.5$, then $e \cdot \frac{x_1^2+\dots+x_n^2}{g} > 1$ and the lower bound is trivial in this case.

Remark. In the proof above,

(1) for the lower bound, the $x'_i s$ may depend on g but n is fixed;

(2) for the upper bound, both the $x'_i s$ and n may depend on g as $g \to \infty$.

One may observe from Lemma 19, 20 and Theorem 21 that the asymptotic behavior of $V_{g,n}$ is related to the Euler characteristic $\chi(S_{g,n}) = -2g + 2 - n$. We use the following quantity W_r to approximate any $V_{g,n}$ with 2g-2+n=r:

$$W_r := \begin{cases} V_{\frac{r}{2}+1,0} & \text{if } r \text{ is even,} \\ V_{\frac{r+1}{2},1} & \text{if } r \text{ is odd.} \end{cases}$$

Now we provide the following properties for W_r which will be applied later.

Lemma 23. (1) For any $g, n \ge 0$, we have

$$V_{g,n} \le c \cdot W_{2g-2+n}$$

for some universal constant c > 0. (2) For any $r \ge 1$ and $m_0 \le \frac{1}{2}r$, we have

r

$$\sum_{n=m_0}^{\lfloor \frac{r}{2} \rfloor} W_m W_{r-m} \le c(m_0) \frac{1}{r^{m_0}} W_r$$

for some constant $c(m_0) > 0$ only depending on m_0 .

Proof. For (1), first by Part (2) of Lemma 19 we know that there exists a pair (g', n') with $0 \le n' \le 3$ and 2g' - 2 + n' = 2g - 2 + n such that

$$V_{g,n} \leq V_{g',n'}$$
.

Again by Part (3) of Lemma 19 or Theorem 21 we know that there is a universal constant c > 0 such that

$$V_{g',2} \le cV_{g'+1}$$
 and $V_{g',3} \le cV_{g'+1,1}$.

So for odd n > 0 we have

$$V_{g,n} \le V_{g',n'} \le cV_{g+\frac{n-1}{2},1} = cW_{2g-2+n}$$

and for even $n \ge 0$ we also have

$$V_{g,n} \le V_{g',n'} \le cV_{g+\frac{n}{2}} = cW_{2g-2+n}$$

which completes the proof of (1).

For (2), we only show it for the case that both m_0 and r are odd. The proofs of other cases are similar. We leave them as an exercise to the readers. First by Part (3) of Lemma 19, there is a universal constant c > 0 such that for odd m,

$$W_m \le c \frac{1}{m} V_{\frac{m+3}{2}}$$

Recall that Part (3) of Lemma 19 implies that for some universal constant c' > 0,

$$\frac{V_{g+1}}{V_{g,1}} \le c' \cdot g$$

Then it follows by Lemma 20 that there exist two constants $c'(m_0), c(m_0) > 0$ only depending on m_0 such that

$$\sum_{m=m_0}^{\left[\frac{r}{2}\right]} W_m W_{r-m} \leq \sum_{\substack{m=m_0+1\\m \text{ even}}}^{\left[\frac{r}{2}\right]} \frac{c}{r-m} V_{\frac{m}{2}+1} V_{\frac{r-m+3}{2}} + \sum_{\substack{m=m_0\\m \text{ odd}}}^{\left[\frac{r}{2}\right]} \frac{c}{m} V_{\frac{m+3}{2}} V_{\frac{r-m}{2}+1}$$

$$\leq \frac{c}{r} \sum_{k=\frac{m_0+3}{2}}^{\left[\frac{r}{4}\right]+1} V_k V_{\frac{r+5}{2}-k} + \frac{c}{m_0} \sum_{k=\frac{m_0+3}{2}}^{\left[\frac{r}{4}\right]+1} V_k V_{\frac{r+5}{2}-k}$$

$$\leq c'(m_0) \frac{1}{r^{m_0+1}} V_{\frac{r+3}{2}}$$

$$\leq c(m_0) \frac{1}{r^{m_0}} V_{\frac{r+1}{2},1}$$

$$= c(m_0) \frac{1}{r^{m_0}} W_r,$$

as required.

The following lemma is a generalization of [MP19, lemma 3.2] and [GMST21, lemma 6.3]. Here we allow the n_i 's and q to depend on g as $g \to \infty$.

Lemma 24. Assume $q \ge 1, n_1, \dots, n_q \ge 0, r \ge 2$. Then there exists two universal constants c, D > 0 such that

$$\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q} \le c \left(\frac{D}{r}\right)^{q-1} W_r$$

where the sum is taken over all $\{g_i\}_{i=1}^q \subset \mathbb{N}$ such that $2g_i - 2 + n_i \geq 1$ for all $i = 1, \dots, q$, and $\sum_{i=1}^q (2g_i - 2 + n_i) = r$.

Proof. Given a $\{g_i\}$ in the summation, let $g'_i \ge 0$ and $0 \le n'_i \le 3$ be such that $2g'_i - 2 + n'_i = 2g_i - 2 + n_i$ for each *i*. By Lemma 19 we know that

$$V_{g_i,n_i} \le V_{g'_i,n'_i}.$$

And by Theorem 21, we have

$$V_{g'_i,n'_i} \le \alpha_0 \frac{\sqrt{2}}{\sqrt{2g'_i - 3 + n'_i}} (2g'_i - 3 + n'_i)! (4\pi^2)^{2g'_i - 3 + n'_i}$$
$$= \alpha_0 \frac{\sqrt{2}}{\sqrt{2g_i - 3 + n_i}} (2g_i - 3 + n_i)! (4\pi^2)^{2g_i - 3 + n_i}$$

and

$$W_r \ge \alpha_1 \frac{\sqrt{2}}{\sqrt{r-1}} (r-1)! (4\pi^2)^{r-1}$$

for universal constants $\alpha_0 > \alpha_1 > 0$.

Recall that the Stirling's formula says that as $k \to \infty$,

$$k! \sim \sqrt{2\pi k} (\frac{k}{e})^k.$$

So there exist two universal constants $a_0 > a_1 > 0$ such that

$$a_1\sqrt{2\pi}(\frac{k}{e})^k \le \frac{k!}{\sqrt{k}} \le a_0\sqrt{2\pi}(\frac{k}{e})^k.$$

Now we have

(2)
$$\frac{\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q}}{W_r} \\ \leq \frac{\sum_{\{g_i\}} \prod_{i=1}^q 2\sqrt{\pi}a_0\alpha_0(\frac{2g_i-3+n_i}{e})^{2g_i-3+n_i}(4\pi^2)^{2g_i-3+n_i}}{2\sqrt{\pi}a_1\alpha_1(\frac{r-1}{e})^{r-1}(4\pi^2)^{r-1}} \\ = \frac{1}{2\sqrt{\pi}a_1\alpha_1\frac{e}{4\pi^2}} (2\sqrt{\pi}a_0\alpha_0\frac{e}{4\pi^2})^q \frac{\sum_{\{g_i\}} \prod_{i=1}^q (2g_i-3+n_i)^{2g_i-3+n_i}}{(r-1)^{r-1}}$$

For each $i = 1, \dots, q$, we have $2g_i - 3 + n_i \ge 0$. Now assume exactly j of numbers $(2g_i - 3 + n_i)$ are non-zero. The number of such $\{g_i\}$ (such that $\sum_{i=1}^{q} (2g_i - 3 + n_i) = r - q$) is bounded from above by

$$\binom{q}{j}\binom{r-q-1}{j-1}$$

where $\binom{q}{j} = \frac{q!}{j!(q-j)!}$ is the binomial coefficient.

Recall the following elementary fact: if $\sum_{i=1}^{j} x_i = S$ and $x_i \ge 1$ for all i, then $\prod_{i=1}^{j} x_i^{x_i}$ reaches the maximum value when j-1 of the $x'_i s$ are 1. As a result, we have

$$\prod_{i=1}^{j} x_i^{x_i} \le (S - j + 1)^{S - j + 1}.$$

Thus for each such $\{g_i\}$ we have

$$\prod_{i=1}^{q} (2g_i - 3 + n_i)^{2g_i - 3 + n_i} \le 1^1 \cdots 1^1 \cdot (r - q - j + 1)^{r - q - j + 1}.$$

So we have

$$\frac{\sum_{\{g_i\}} \prod_{i=1}^{q} (2g_i - 3 + n_i)^{2g_i - 3 + n_i}}{(r - 1)^{r - 1}} \\
\leq \frac{1}{(r - 1)^{r - 1}} \sum_{j=0}^{q} {q \choose j} {r - q - 1 \choose j - 1} (r - q - j + 1)^{r - q - j + 1} \\
\leq \sum_{j=0}^{q} {q \choose j} \frac{(r - q - 1)^{j - 1} (r - q - j + 1)^{r - q - j + 1}}{(r - 1)^{r - 1}} \\
\leq \sum_{j=0}^{q} {q \choose j} \frac{1}{(r - 1)^{q - 1}} \\
= \frac{2^q}{(r - 1)^{q - 1}}.$$

Then combining (2) we get

$$\frac{\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q}}{W_r} \le 2 \frac{a_0 \alpha_0}{a_1 \alpha_1} \left(\frac{a_0 \alpha_0 \frac{e}{\pi^{3/2}}}{r-1}\right)^{q-1},$$

as required.

We finish this section by the following useful property.

Proposition 25. Given $m \ge 1$, there exists a constant c(m) > 0 only depending on m such that for any $g \ge m + 1$, $q \ge 1$ and $n_1, ..., n_q \ge 1$, we have

$$\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q} \le c(m) \frac{1}{g^m} V_g,$$

where the sum is taken over all $\{g_i\}_{i=1}^q \subset \mathbb{N}$ such that $2g_i - 2 + n_i \geq 1$ for all $i = 1, \dots, q$, and $\sum_{i=1}^q (2g_i - 2 + n_i) = 2g - 2 - m$.

Proof. If g is bounded from above, then the nonnegative integers $m, q, n_1, \dots n_q$ are all bounded from above, and hence the inequality is trivial. It suffices to show it for large enough g. First by Lemma 24 we know that

$$\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q} \le c \left(\frac{D}{2g - 2 - m}\right)^{q-1} W_{2g - 2 - m}$$

By Part (3) of Lemma 19 or Theorem 21 we know that

$$rac{V_g}{V_{g-1}} symp g^2 \quad and \quad rac{V_{g,1}}{V_g} symp g$$

where we say $f \simeq h$ if there exists a uniform constant $C \ge 1$ such that $\frac{1}{C} \le \frac{f}{h} \le C$. This in particular implies that there exists a constant c'(m) > 0 only depending on m such that

$$W_{2g-2-m} \le c'(m)\frac{1}{g^m}V_g.$$

Therefore, we have that for large enough g > 0,

$$\sum_{\{g_i\}} V_{g_1,n_1} \cdots V_{g_q,n_q} \leq c'(m) (\frac{D}{g})^{q-1} \frac{1}{g^m} V_g$$
$$\leq c(m) \frac{1}{g^m} V_g$$

for some constant c(m) > 0 only depending on m, as required.

5. Mirzakhani's generalized McShane identity

In [Mir07a] Mirzakhani generalized McShane's identity [McS98] as follows, and then calculated the Weil-Petersson volume of moduli spaces by applying her integration formula (see Theorem 12).

Theorem 26. [Mir07a, Theorem 1.3] For $X \in \mathcal{M}_{g,n}(L_1, \dots, L_n)$ with n geodesic boundaries β_1, \dots, β_n of length L_1, \dots, L_n , we have

$$\sum_{\{\gamma_1,\gamma_2\}} \mathcal{D}(L_1,\ell(\gamma_1),\ell(\gamma_2)) + \sum_{i=2}^n \sum_{\gamma} \mathcal{R}(L_1,L_i,\ell(\gamma)) = L_1$$

where the first sum is over all unordered pairs of simple closed geodesics $\{\gamma_1, \gamma_2\}$ bounding a pair of pants with β_1 , and the second sum is over all simple closed geodesics γ bounding a pair of pants with β_1 and β_i . Here \mathcal{D} and \mathcal{R} are given by

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}} \right),$$
$$\mathcal{R}(x, y, z) = x - \log \left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})} \right)$$

We will use this identity in subsection 7.5 to control the number of certain types of closed geodesics in a surface. Here we provide the following elementary properties for $\mathcal{D}(x, y, z)$ and $\mathcal{R}(x, y, z)$.

Lemma 27. Assume that x, y, z > 0, then the following properties hold.

- (1) $\mathcal{R}(x, y, z) \ge 0$ and $\mathcal{D}(x, y, z) \ge 0$.
- (2) $\mathcal{R}(x, y, z)$ is decreasing with respect to z and increasing with respect to y. $\mathcal{D}(x, y, z)$ is decreasing with respect to y and z and increasing with respect to x.
- (3) We have

$$\frac{x}{\mathcal{R}(x,y,z)} \le 100(1+x)(1+e^{\frac{z}{2}}e^{-\frac{x+y}{2}}),$$

and

$$\frac{x}{\mathcal{D}(x,y,z)} \le 100(1+x)(1+e^{\frac{y+z}{2}}e^{-\frac{x}{2}}).$$

Moreover, if x + y > z, we have

$$\frac{x}{\mathcal{R}(x,y,z)} \le 500 + 500 \frac{x}{x+y-z}.$$

Proof. Part (1) is easy to check. Actually \mathcal{D} and \mathcal{R} given in [Mir07a] are lengths of certain segments for x, y, z > 0.

For Part (2), a direct computation shows that

$$\frac{d}{dz} \left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})} \right) \\
= \frac{\frac{1}{2} \sinh \frac{x+z}{2} (\cosh \frac{y}{2} + \cosh \frac{x-z}{2}) + \frac{1}{2} \sinh \frac{x-z}{2} (\cosh \frac{y}{2} + \cosh \frac{x+z}{2})}{(\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2}))^2} \\
= \frac{\sinh \frac{x}{2} \cosh \frac{z}{2} \cosh \frac{y}{2} + \frac{1}{2} \sinh x}{(\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2}))^2} \\
> 0$$

where we have used the elementary equations

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b,$$
$$\sinh a + \sinh b = 2 \sinh \frac{a+b}{2} \cosh \frac{a-b}{2}.$$

So $\mathcal{R}(x, y, z)$ is decreasing with respect to z. The other parts of (2) are obvious.

For Part (3), first as for $\mathcal{R}(x, y, z)$ we have

$$\begin{aligned} (3) \qquad \mathcal{R}(x,y,z) &= \log\left(e^x \frac{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x-z}{2}\right)}{\cosh\left(\frac{y}{2}\right) + \cosh\left(\frac{x+z}{2}\right)}\right) \\ &= \log\left(e^x \frac{e^{\frac{y}{2}} + e^{\frac{-y}{2}} + e^{\frac{x-z}{2}} + e^{\frac{x-z}{2}}}{e^{\frac{y}{2}} + e^{\frac{-y}{2}} + e^{\frac{x+z}{2}} + e^{\frac{x-z}{2}}}\right) \\ &= \log\left(e^x \frac{e^y e^{\frac{x+z}{2}} + e^{\frac{x+z}{2}} + e^{xe\frac{y}{2}} + e^{ze\frac{y}{2}}}{e^y e^{\frac{x+z}{2}} + e^{\frac{x+z}{2}} + e^{x+z}e^{\frac{y}{2}} + e^{\frac{y}{2}}}\right) \\ &= \log\left(1 + \frac{(e^x - 1)(e^y + 1)e^{\frac{x+z}{2}}}{e^y e^{\frac{x+z}{2}} + e^{\frac{x+z}{2}} + e^{x+z}e^{\frac{y}{2}} + e^{\frac{y}{2}}}\right) \\ &\geq \log\left(1 + \frac{(e^x - 1)(e^y + 1)e^{\frac{x+z}{2}}}{e^y e^{\frac{x+z}{2}} + e^{\frac{x+z}{2}} + 2e^{x+z}e^{\frac{y}{2}}}\right) \\ &= \log\left(1 + \frac{e^x - 1}{1 + e^{\frac{x+z}{2}}\frac{2e^{\frac{y}{2}}}{e^y + 1}}\right) \\ &= \log\left(1 + \frac{e^x - 1}{1 + e^{\frac{x+z}{2}}\frac{1}{\cosh\frac{y}{2}}}\right). \end{aligned}$$

Then we treat the following cases separately: **Case 1:** $\frac{e^x - 1}{1 + e^{\frac{x+z}{2}} \frac{1}{\cosh \frac{y}{2}}} \ge 1$. Then we have $e^x \ge 2$ and by (3)

(4)
$$\frac{x}{\mathcal{R}(x,y,z)} \le \frac{x}{\log 2} \le 2x.$$

Case 2: $\frac{e^x - 1}{1 + e^{\frac{x+z}{2}} \frac{1}{\cosh \frac{y}{2}}} < 1$. Recall that $\log(1+t) \ge \frac{t}{2}$ for $0 < t \le 1$. Then by (3) we have

(5)
$$\frac{x}{\mathcal{R}(x,y,z)} \leq \frac{x}{\frac{1}{2}\frac{e^{x}-1}{1+e^{\frac{x+z}{2}}\frac{1}{\cosh\frac{y}{2}}}} \\ = \frac{2x}{e^{x}-1} + e^{\frac{z}{2}}\frac{x}{\sinh\frac{x}{2}}\frac{1}{\cosh\frac{y}{2}} \\ \leq 2 + e^{\frac{z}{2}}\frac{x}{\sinh\frac{x}{2}}\frac{1}{\cosh\frac{y}{2}} \\ \leq 2 + 100(1+x)e^{\frac{z}{2}}e^{-\frac{x+y}{2}}.$$

So combining (4) and (5) we have

$$\frac{x}{\mathcal{R}(x,y,z)} \le 100(1+x)(1+e^{\frac{z}{2}}e^{-\frac{x+y}{2}}).$$

Now assume x + y > z and consider the following subcases of Case 1.

Case 1a: $\frac{e^x - 1}{1 + e^{\frac{x+z}{2}} \frac{1}{\cosh \frac{y}{2}}} \ge 1$ (which implies $e^x \ge 2$) and $1 \ge e^{\frac{x+z}{2}} \frac{1}{\cosh \frac{y}{2}}$.

Then by (3) we have

(6)
$$\frac{x}{\mathcal{R}(x,y,z)} \leq \frac{x}{\log\left(\frac{e^x+1}{2}\right)} \leq 100.$$

Case 1b: $\frac{e^{x}-1}{1+e^{\frac{x+z}{2}}\frac{1}{\cosh\frac{y}{2}}} \ge 1$ (which implies $e^{x} \ge 2$) and $e^{\frac{x+z}{2}}\frac{1}{\cosh\frac{y}{2}} \ge 1$.

Then by (3) we have

(7)
$$\frac{x}{\mathcal{R}(x,y,z)} \leq \frac{x}{\log\left(1 + \frac{e^x - 1}{2e^{\frac{x+z}{2}} \frac{1}{\cosh\frac{y}{2}}}\right)} \\ \leq \frac{x}{\log\left(1 + \frac{e^x - 1}{4e^x}e^{\frac{x+y-z}{2}}\right)} \\ \leq 100 \frac{x}{x+y-z}$$

where in the last inequality we apply the elementary inequality $1 + ae^x \ge e^{ax}$ for any $a \in (0, 1)$ and x > 0.

So combining (5), (6) and (7) we have

$$\frac{x}{\mathcal{R}(x,y,z)} \leq 100 + 100 \frac{x}{x+y-z} + 2 + 100(1+x)e^{\frac{z}{2}}e^{-\frac{x+y}{2}} \\
\leq 202 + 200 \frac{x}{x+y-z}.$$

As for $\mathcal{D}(x, y, z)$, we have

$$\mathcal{D}(x, y, z) = 2\log\left(1 + \frac{2\sinh\frac{x}{2}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}\right).$$

Case 1c: $\frac{2\sinh\frac{x}{2}}{e^{-\frac{x}{2}}+e^{\frac{y+z}{2}}} \leq 1$. Then by the fact that $\log(1+t) \geq \frac{t}{2}$ for $0 < t \leq 1$, we have

(8)
$$\frac{x}{\mathcal{D}(x,y,z)} \leq \frac{x}{2 \cdot \frac{1}{2} \frac{2\sinh\frac{x}{2}}{e^{-\frac{x}{2}} + e^{\frac{y+z}{2}}}} = xe^{-\frac{x}{2}} \frac{1}{2\sinh\frac{x}{2}} + xe^{\frac{y+z}{2}} \frac{1}{2\sinh\frac{x}{2}}$$

Case 1d: $\frac{2\sinh\frac{x}{2}}{e^{-\frac{x}{2}}+e^{\frac{y+z}{2}}} > 1$. Then we have

(9)
$$\frac{x}{\mathcal{D}(x,y,z)} \le \frac{1}{2\log 2}x.$$

So combining (8) and (9) we have

$$\frac{x}{\mathcal{D}(x,y,z)} \le 100(1+x)(1+e^{\frac{y+z}{2}}e^{-\frac{x}{2}})$$

(one may prove this inequality for $0 < x \le 1$ and x > 1 respectively), as required.

RANDOM SURFACES

6. Lower bound

In this section, we will show the easier part of Theorems 1 and 6, namely the lower bound. More precisely, we show that

Proposition 28. Let $\omega(g)$ be a function satisfying (1). Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) \ge 2 \log g - 4 \log \log g - \omega(g) \right) = 1$$

and

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \ge 2 \log g - 4 \log \log g - \omega(g) \right) = 1.$$

Since $\ell_{\text{sys}}^{\text{sep}}(X) \geq \mathcal{L}_1(X)$, it suffices to prove the second limit. We follow the method in [Mir13, Section 4.3] for this part.

Let L > 0 and assume $\mathcal{L}_1(X) \leq L$. Then there exists a simple closed multi-geodesic of length $\leq L$ separating X into $S_{g_0,k} \cup S_{g-g_0-k+1,k}$ for some (g_0,k) with $|\chi(S_{g_0,k})| \leq \frac{1}{2}|\chi(S_g)| = g-1$. That is,

$$\sum_{(g_0,k); 1 \le 2g_0 - 2 + k \le g - 1} N_{g_0,k}(X,L) \ge 1.$$

So we have

(10)
$$\operatorname{Prob}_{WP}^{g} \left(X \in \mathcal{M}_{g} ; \mathcal{L}_{1}(X) \leq L \right)$$
$$\leq \operatorname{Prob}_{WP}^{g} \left(\sum_{(g_{0},k); 1 \leq 2g_{0}-2+k \leq g-1} N_{g_{0},k}(X,L) \geq 1 \right)$$
$$\leq \sum_{(g_{0},k); 1 \leq 2g_{0}-2+k \leq g-1} \mathbb{E}_{WP}^{g} [N_{g_{0},k}(X,L)],$$

where the last equality uses the fact that $\mathbb{P}(N \ge 1) \le \mathbb{E}(N)$ for any $\mathbb{Z}_{\ge 0}$ -valued random variable N.

By Mirzakhani's Integration Formula (see Theorem 12), we have

(11)
$$\mathbb{E}_{WP}^{g}[N_{g_{0},k}(X,L)] = \frac{1}{V_{g}} \frac{2^{-M}}{|\operatorname{Sym}|} \int_{\mathbb{R}_{\geq 0}^{k}} \mathbf{1}_{[0,L]}(x_{1} + \dots + x_{k}) \\ \times V_{g_{0},k}(x_{1},\dots,x_{k}) V_{g-g_{0}-k+1,k}(x_{1},\dots,x_{k}) x_{1}\dots x_{k} dx_{1}\dots dx_{k}$$

where M = 1 if $(g_0, k) = (1, 1)$ and M = 0 otherwise, and $|\text{Sym}| \ge k!$ in general and |Sym| = k! if g > 2 and $(g_0, k) = (1, 1)$ or (0, 3).

Then we split the proof of Proposition 28 by calculating the quantity $\mathbb{E}^{g}_{WP}[N_{q_0,k}(X,L)]$ for three different cases.

Lemma 29. For $(g_0, k) = (1, 1)$ or (0, 3), we have

$$\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)] = \frac{1}{384\pi^2} L^2 e^{\frac{L}{2}} \frac{1}{g} \left(1 + O\left(\frac{1}{g}\right)\right) \left(1 + O\left(\frac{1}{L}\right)\right) \left(1 + O\left(\frac{L^2}{g}\right)\right)$$

and

$$\mathbb{E}_{\rm WP}^{g}[N_{0,3}(X,L)] = \frac{1}{48\pi^2} L^2 e^{\frac{L}{2}} \frac{1}{g} \left(1 + O\left(\frac{1}{g}\right)\right) \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^2}{g})\right),$$

where the implied constants are independent of L and g.

Proof. First we consider the case $(g_0, k) = (1, 1)$. By using Theorem 18 and Lemma 19, 22 and Equation (11), we have

$$\begin{split} \mathbb{E}_{WP}^{g}[N_{1,1}(X,L)] &= \frac{1}{V_{g}} \frac{1}{2} \int_{0}^{L} V_{1,1}(x) V_{g-1,1}(x) x dx \\ &= \frac{1}{2V_{g}} \int_{0}^{L} \frac{1}{48} (x^{2} + 4\pi^{2}) x \frac{\sinh(x/2)}{x/2} dx \times V_{g-1,1} \left(1 + O(\frac{L^{2}}{g})\right) \\ &= \frac{1}{48} L^{2} e^{\frac{L}{2}} \frac{V_{g-1,1}}{V_{g}} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \\ &= \frac{1}{384\pi^{2}} L^{2} e^{\frac{L}{2}} \frac{1}{g} \left(1 + O(\frac{1}{g})\right) \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right). \end{split}$$

Similarly for $(g_0, k) = (0, 3)$, we have

$$\begin{split} \mathbb{E}_{WP}^{g}[N_{0,3}(X,L)] &= \frac{1}{V_{g}} \frac{1}{3!} \int_{0 \le x+y+z \le L} \frac{\sinh(x/2)}{x/2} \frac{\sinh(y/2)}{y/2} \frac{\sinh(z/2)}{z/2} \\ & xyz dx dy dz \times V_{g-2,3} \left(1 + O(\frac{L^{2}}{g})\right) \\ &= \frac{1}{6} L^{2} e^{\frac{L}{2}} \frac{V_{g-2,3}}{V_{g}} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \\ &= \frac{1}{48\pi^{2}} L^{2} e^{\frac{L}{2}} \frac{1}{g} \left(1 + O(\frac{1}{g})\right) \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right), \end{split}$$
s required.

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Remark. The dominating term $L^2 e^{\frac{L}{2}} \frac{1}{g}$ in both expressions in Lemma 29 is where the upper and lower bounds $2\log g - 4\log \log g \pm \omega(g)$ in Theorems 1 and 6 come from. In fact, a function L(g) in the variable $g \in \{2, 3, \dots\}$ has the form $2\log g - 4\log \log g + \omega(g)$, with $\omega(g)$ satisfying the assumption of Theorems 1, if and only if

$$\lim_{g \to \infty} L(g)^2 e^{\frac{L(g)}{2}} \frac{1}{g} \to +\infty, \quad L(g)^2 e^{\frac{L(g)}{2}} \frac{1}{g} = O\left((\log g)^\epsilon\right)$$

for any $\epsilon > 0$. Similarly, L(g) has the form $2\log g - 4\log\log g - \omega(g)$ if and only if $\lim_{g\to\infty} L(g)^2 e^{\frac{L(g)}{2}} \frac{1}{g} \to 0$ and $L(g)^2 e^{\frac{L(g)}{2}} \frac{1}{g} \ge C(\log g)^{-\epsilon}$ when g is large enough for any $C, \epsilon > 0$.

Lemma 30. For any given positive integer m, there exists a constant c(m) > c(m)0 independent of L and g such that

$$\sum_{|\chi(S_{g_0,k})|=m} \mathbb{E}^g_{WP}[N_{g_0,k}(X,L)] \le c(m)(1+L^{3m-1})e^{\frac{L}{2}}\frac{1}{g^m}$$

where the summation is over all pairs (g_0, k) satisfying $g_0 \ge 0, k \ge 1$, and

$$|\chi(S_{g_0,k})| = m.$$

Proof. Assume $2g_0 - 2 + k = |\chi(S_{g_0,k})| = m$. Then both g_0 and k are bounded from above by m + 2. By Theorem 18 of Mirzakhani we know that $V_{q_0,k}(x_1, \dots, x_k)$ is a polynomial of degree $6g_0 - 6 + 2k$ with coefficients bounded by some constant only depending on m. So when $0 \leq x_1 + \cdots + x_k \leq L$ we have

$$V_{g_0,k}(x_1,\cdots,x_k) \le c'(m)(1+L^{6g_0-6+2k})$$

for some constant c'(m) > 0 only depending on m. Then by Lemma 19, 22 and Equation (11) we have

Since g_0, k are bounded above by m + 2, there exists a constant c(m) > 0 only depending on m such that

$$\sum_{|\chi(S_{g_0,k})|=m} \mathbb{E}^g_{\mathrm{WP}}[N_{g_0,k}(X,L)] \le c(m)(1+L^{3m-1})e^{\frac{L}{2}}\frac{1}{g^m}$$

as desired.

Lemma 31. For any given positive integer m, there exists a constant c(m) > 0 independent of L and g such that

$$\sum_{m \le |\chi(S_{g_0,k})| \le g-1} \mathbb{E}_{\mathrm{WP}}^g [N_{g_0,k}(X,L)] \le c(m) e^{2L} \frac{1}{g^m}$$

where the summation is over all pairs (g_0, k) satisfying $g_0 \ge 0, k \ge 1$, and

$$m \le |\chi(S_{g_0,k})| \le g - 1.$$

Proof. First by Part (1) of Lemma 19 we know that

$$V_{g_0,k}(x_1,\cdots,x_k) \le e^{\frac{x_1+\cdots+x_k}{2}}V_{g_0,k}$$

and

$$V_{g-g_0-k+1,k}(x_1,\cdots,x_k) \le e^{\frac{x_1+\cdots+x_k}{2}}V_{g-g_0-k+1,k}.$$

Then by (11) we have

$$\mathbb{E}_{WP}^{g}[N_{g_{0},k}(X,L)] \leq \frac{1}{V_{g}} \frac{1}{k!} \int_{0 \leq \sum x_{i} \leq L} e^{x_{1} + \dots + x_{k}} x_{1} \cdots x_{k} dx_{1} \cdots dx_{k} V_{g_{0},k} V_{g-g_{0}-k+1,k} \\ \leq \frac{1}{k!} \frac{V_{g_{0},k} V_{g-g_{0}-k+1,k}}{V_{g}} e^{L} \int_{0 \leq \sum x_{i} \leq L} x_{1} \cdots x_{k} dx_{1} \cdots dx_{k} \\ = \frac{1}{k!} \frac{L^{2k}}{(2k)!} e^{L} \frac{V_{g_{0},k} V_{g-g_{0}-k+1,k}}{V_{g}}.$$

Recall that Part (2) of Lemma 19 says that for any $g,n\geq 0$

$$V_{g-1,n+4} \le V_{g,n+2}.$$

So we have

 $V_{g_0,k} \leq V_{g_0 + \frac{k-k'}{2},k'} \quad and \quad V_{g-g_0-k+1,k} \leq V_{g-g_0-k+1 + \frac{k-k'}{2},k'}$

where $k' \in \{1, 2, 3\}$ with even $k - k' \ge 0$. For any fixed integer k > 0, we consider the summation over g_0 with $m \le |\chi(S_{g_0,k})| \le g - 1$. By Lemma 20 we have

$$\sum_{g_0; \, m \le |\chi(S_{g_0,k})| \le g-1} \mathbb{E}^g_{\mathrm{WP}}[N_{g_0,k}(X,L)] \le c(m) \frac{1}{k!} \frac{L^{2k}}{(2k)!} e^L \frac{1}{g^m}$$

for some constant c(m)>0 only depending on m. Then the total summation satisfies that

$$\sum_{\substack{m \le |\chi(S_{g_0,k})| \le g-1}} \mathbb{E}_{WP}^g[N_{g_0,k}(X,L)] \le \sum_{\substack{k \ge 1}} c(m) \frac{1}{k!} \frac{L^{2k}}{(2k)!} e^L \frac{1}{g^m}$$
$$\le c(m) e^{2L} \frac{1}{g^m}$$

because $\sum_{k\geq 1} \frac{1}{k!} \frac{L^{2k}}{(2k)!} \leq e^L$. This finishes the proof.

Now we are ready to prove Proposition 28.

Proof of Proposition 28. First by Equation (10), Lemma 29, 30 and 31, we have that for large g > 0,

$$\begin{aligned} \operatorname{Prob}_{\mathrm{WP}}^{g} \left(X \in \mathcal{M}_{g} \, ; \, \mathcal{L}_{1}(X) \leq L \right) &\leq \sum_{(g_{0},k); \, 1 \leq |\chi(S_{g_{0},k})| \leq g-1} \mathbb{E}_{\mathrm{WP}}^{g} [N_{g_{0},k}(X,L)] \\ &= \left(\mathbb{E}_{\mathrm{WP}}^{g} [N_{1,1}(X,L)] + \mathbb{E}_{\mathrm{WP}}^{g} [N_{0,3}(X,L)] \right) \\ &+ \sum_{m=2}^{10} \sum_{|\chi(S_{g_{0},k})| = m} \mathbb{E}_{\mathrm{WP}}^{g} [N_{g_{0},k}(X,L)] + \sum_{11 \leq |\chi(S_{g_{0},k})| \leq g-1} \mathbb{E}_{\mathrm{WP}}^{g} [N_{g_{0},k}(X,L)] \\ &\leq cL^{2} e^{\frac{L}{2}} \frac{1}{g} + \sum_{m=2}^{10} cL^{3m-1} e^{\frac{L}{2}} \frac{1}{g^{m}} + ce^{2L} \frac{1}{g^{11}} \end{aligned}$$

for some uniform constant c > 0. Now for

$$L = 2\log g - 4\log\log g - \omega(g),$$

we have

$$\begin{split} L^2 e^{\frac{L}{2}} \frac{1}{g} &= O(e^{-\frac{\omega(g)}{2}}), \\ \sum_{m=2}^{10} L^{3m-1} e^{\frac{L}{2}} \frac{1}{g^m} &= O(\frac{(\log g)^{29}}{g}), \end{split}$$

and

$$\frac{e^{2L}}{g^{11}} = O(\frac{1}{g^7}).$$

Recall that $\omega(g) \to \infty$ as $g \to \infty$. Hence we get

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \le 2 \log g - 4 \log \log g - \omega(g) \right) = 0,$$

which implies that

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \ge 2 \log g - 4 \log \log g - \omega(g) \right) = 1,$$

as desired.

Actually the argument above also leads to Proposition 9, which will be applied later. First we recall the following definition generalizing \mathcal{L}_1 in the Introduction. For any integer $m \in [1, g - 1]$ and $X \in \mathcal{M}_g$,

$$\mathcal{L}_{1,m}(X) := \min_{\Gamma} \ell_{\Gamma}(X)$$

where the minimum runs over all simple closed multi-geodesics Γ separating X into $S_{g_1,k} \cup S_{g_2,k}$ with

$$|\chi(S_{g_1,k})| \ge |\chi(S_{g_2,k})| \ge m.$$

Now we are ready to prove Proposition 9.

Proposition 32 (=Proposition 9). Let $\omega(g)$ be a function satisfying (1). Then we have that for any fixed $m \ge 1$ independent of g,

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,m}(X) \ge 2m \log g - (6m - 2) \log \log g - \omega(g) \right) = 1.$$

Proof. The proof is almost the same as the proof of Proposition 28. First we have that for large g > 0,

$$\operatorname{Prob}_{WP}^{g} \left(X \in \mathcal{M}_{g} ; \mathcal{L}_{1,m}(X) \leq L \right) \leq \sum_{(g_{0},k); m \leq |\chi(S_{g_{0},k})| \leq g-1} \mathbb{E}_{WP}^{g} [N_{g_{0},k}(X,L)]$$
$$= \sum_{|\chi(S_{g_{0},k})|=m} \mathbb{E}_{WP}^{g} [N_{g_{0},k}(X,L)] + \sum_{m+1 \leq |\chi(S_{g_{0},k})| \leq 10m} \mathbb{E}_{WP}^{g} [N_{g_{0},k}(X,L)]$$
$$+ \sum_{10m+1 \leq |\chi(S_{g_{0},k})| \leq g-1} \mathbb{E}_{WP}^{g} [N_{g_{0},k}(X,L)].$$

Now for

$$L = 2m \log g - (6m - 2) \log \log g - \omega(g),$$

by Lemma 29, 30 and 31 we have

$$\sum_{|\chi(S_{g_0,k})|=m} \mathbb{E}^g_{\mathrm{WP}}[N_{g_0,k}(X,L)] = O(L^{3m-1}e^{\frac{L}{2}}\frac{1}{g^m}) = O(e^{-\frac{\omega(g)}{2}}),$$

$$\sum_{m+1 \le |\chi(S_{g_0,k})| \le 10m} \mathbb{E}^g_{\mathrm{WP}}[N_{g_0,k}(X,L)] = O(\sum_{j=m+1}^{10m} L^{3j-1}e^{\frac{L}{2}} \frac{1}{g^j}) = O(\frac{(\log g)^{30m-1}}{g}),$$

and

$$\sum_{10m+1 \leq |\chi(S_{g_0,k})| \leq g-1} \mathbb{E}^g_{\mathrm{WP}}[N_{g_0,k}(X,L)] = O(\frac{e^{2L}}{g^{10m+1}}) = O(\frac{1}{g^{6m+1}}).$$

Recall that $\omega(g) \to \infty$ as $g \to \infty$. Hence we get

 $\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,m}(X) \le 2m \log g - (6m - 2) \log \log g - \omega(g) \right) = 0$

which implies that

 $\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,m}(X) \ge 2m \log g - (6m - 2) \log \log g - \omega(g) \right) = 1$ \Box as desired.

Remark. The m = 1 case of Proposition 9 is exactly Proposition 28.

7. Upper bound

In this section, we will show the upper bound in Theorem 1 and 6. We begin with the following definition.

Definition 33. Assume $\omega(g)$ is a function satisfying (1). For any $X \in \mathcal{M}_g$, we say $X \in \mathcal{A}(\omega(g))$ if there exists a simple closed geodesic γ on X such that

(1) γ separates X into $S_{1,1} \cup S_{g-1,1}$; (2) the length $\ell_{\gamma}(X) \leq 2\log g - 4\log\log g + \omega(g)$.

Now we are ready to state the upper bound of Theorem 1 which is also the essential part of this paper.

Theorem 34. Let $\omega(q)$ be a function satisfying (1). Then we have

 $\lim_{a \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, X \in \mathcal{A}(\omega(g)) \right) = 1.$

7.1. Proofs of Theorem 1 and 6. We postpone the proof of Theorem 34 to the next subsections and give here the proof of Theorem 1 and 6 assuming Theorem 34.

Theorem 35 (=Theorem 1). Let $\omega(g)$ be a function satisfying (1). Consider the following two conditions defined for all $X \in \mathcal{M}_q$:

- (a). $|\ell_{\text{sys}}^{\text{sep}}(X) (2\log g 4\log \log g)| \le \omega(g);$ (b). $\ell_{\text{sys}}^{\text{sep}}(X)$ is achieved by a simple closed geodesic separating X into $S_{1,1} \cup S_{q-1,1}$.

Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g ; X \text{ satisfies } (a) \text{ and } (b) \right) = 1.$$

Proof. Let m = 2 in Proposition 9 we get

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,2}(X) > 3.9 \log g \right) = 1.$$

Set

$$\mathcal{A}'(\omega(g)) := \{ X \in \mathcal{M}_g \, ; \, \ell_{\mathrm{sys}}^{\mathrm{sep}}(X) \ge 2 \log g - 4 \log \log g - \omega(g) \}$$

and

$$\mathcal{A}''(g) := \{ X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,2}(X) > 3.9 \log g \}.$$

Then for any $X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}''(g)$ and large enough g > 0, the quantity $\ell_{\rm sys}^{\rm sep}(X)$ is realized by a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$. For any $X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}'(\omega(g))$, we have

$$|\ell_{\rm sys}^{\rm sep}(X) - (2\log g - 4\log\log g)| \le \omega(g).$$

Thus, it follows by Proposition 28, Proposition 9 for m = 2 and Theorem 34 that as $g \to \infty$, $\operatorname{Prob}_{WP}^g(A(\omega(g)))$, $\operatorname{Prob}_{WP}^g(A'(\omega(g)))$ and $\operatorname{Prob}_{WP}^g(A''(\omega(g)))$ all tend to 1. Therefore, we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}'(\omega(g)) \cap \mathcal{A}''(g) \right) = 1,$$

uired. \Box

as required.

Theorem 36 (=Theorem 6). Let $\omega(g)$ be a function satisfying (1). Consider the following two conditions defined for all $X \in \mathcal{M}_q$:

(e). $|\mathcal{L}_1(X) - (2\log g - 4\log\log g)| \le \omega(g);$

(f). $\mathcal{L}_1(X)$ is achieved by either a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$ or three simple closed geodesics separating X into $S_{0,3} \cup S_{g-2,3}$.

Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; X \text{ satisfies } (e) \text{ and } (f)) = 1.$$

Proof. The proof is similar as the proof of Theorem 1. Let m = 2 in Proposition 9 we get

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,2}(X) > 3.9 \log g \right) = 1.$$

Set

$$\mathcal{A}'(\omega(g)) := \{ X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \ge 2 \log g - 4 \log \log g - \omega(g) \}$$

and

$$\mathcal{A}''(g) := \{ X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,2}(X) > 3.9 \log g \}.$$

Then for any $X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}''(g)$ and large enough g > 0, the quantity $\mathcal{L}_1(X)$ is realized by either a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$ or three simple closed geodesic separating X into $S_{0,3} \cup S_{g-2,3}$. For any $X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}'(\omega(g))$, we have

$$|\mathcal{L}_1(X) - (2\log g - 4\log\log g)| \le \omega(g).$$

Thus, it follows by Proposition 28, Proposition 9 for m = 2 and Theorem 34 that

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, X \in \mathcal{A}(\omega(g)) \cap \mathcal{A}'(\omega(g)) \cap \mathcal{A}''(g) \right) = 1,$$

as required.

Remark. It is interesting to study whether $\mathcal{L}_1(X)$ is realized just by a simple closed geodesic separating X into $S_{1,1} \cup S_{g-1,1}$ on a generic point $X \in \mathcal{M}_g$. Or does the following limit hold:

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) = \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) \right) = 1?$$

Set

(12)
$$L = L(g) = 2\log g - 4\log\log g + \omega(g)$$

where $\omega(g)$ is given as above in (1). In the following arguments we always assume that g is large enough. So L is also large enough.

In order to prove Theorem 34, it suffices to show that

(13)
$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, N_{1,1}(X,L) = 0 \right) = 0.$$

For each $X \in \mathcal{M}_g$, we denote $\mathcal{N}_{1,1}(X, L)$ to be the set of simple closed geodesics on X which separate X into $S_{1,1} \cup S_{g-1,1}$ and has length $\leq L$. Then

$$N_{1,1}(X,L) = \# \mathcal{N}_{1,1}(X,L).$$

Instead of $\mathcal{N}_{1,1}(X, L)$, we consider the subset $\mathcal{N}_{1,1}^*(X, L)$ which is defined as follows.

Definition 37. We denote

$$\mathcal{N}_{1,1}^*(X,L) := \left\{ \begin{array}{l} \forall \alpha \neq \gamma \in \mathcal{N}_{1,1}(X,L), \\ \alpha \in \mathcal{N}_{1,1}(X,L) \, ; \text{ either } \alpha \cap \gamma = \emptyset \text{ or} \\ X_{\alpha\gamma} \text{ is of type } S_{1,2} \end{array} \right\}$$

and

$$N_{1,1}^*(X,L) := \# \mathcal{N}_{1,1}^*(X,L),$$

where $X_{\alpha\gamma}$ is defined in section 3.

Since $N_{1,1}^*(X,L) \leq N_{1,1}(X,L)$, we clearly have that $\operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; N_{1,1}(X,L) = 0) \leq \operatorname{Prob}_{WP}^g (X \in \mathcal{M}_g; N_{1,1}^*(X,L) = 0).$ We will show the following limit which implies (13).

(14)
$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, N_{1,1}^*(X,L) = 0 \right) = 0.$$

Remark. The purpose to study $N_{1,1}^*(X, L)$ instead of $N_{1,1}(X, L)$ is to simplify certain estimations. Actually the following method also works for $N_{1,1}(X, L)$ by adding more detailed discussions.

7.2. Bounding probability by expectation. For any nonnegative integervalued random variable N, by Cauchy-Schwarz inequality we have

$$\mathbb{E}[N]^2 = \mathbb{E}\left[N \cdot \mathbf{1}_{\{N>0\}}\right]^2 \le \mathbb{E}[N^2] \cdot \mathbb{E}\left[\mathbf{1}_{\{N>0\}}^2\right] = \mathbb{E}[N^2] \cdot \mathbb{P}(N>0).$$

So we have

$$\mathbb{P}(N>0) \ge \frac{\mathbb{E}[N]^2}{\mathbb{E}[N^2]}.$$

Then since the variance $\operatorname{Var}[N] = \mathbb{E}[N^2] - \mathbb{E}[N]^2$ is nonnegative, we have

$$\mathbb{P}(N=0) \le \frac{\mathbb{E}[N^2] - \mathbb{E}[N]^2}{\mathbb{E}[N^2]} \le \frac{\mathbb{E}[N^2] - \mathbb{E}[N]^2}{\mathbb{E}[N]^2}.$$

Applying this to $N_{1,1}^*(X, L)$, we get (15)

$$\operatorname{Prob}_{\operatorname{WP}}^{g}\left(N_{1,1}^{*}(X,L)=0\right) \leq \frac{\mathbb{E}_{\operatorname{WP}}^{g}[(N_{1,1}^{*}(X,L))^{2}] - \mathbb{E}_{\operatorname{WP}}^{g}[N_{1,1}^{*}(X,L)]^{2}}{\mathbb{E}_{\operatorname{WP}}^{g}[N_{1,1}^{*}(X,L)]^{2}}.$$

In order to control the RHS above, the most essential part is to study $(N_{1,1}^*(X,L))^2$. We decompose it into three different parts as follows. We define

Definition 38.

$$\mathcal{Y}^*(X,L) := \left\{ (\alpha,\beta) \in \mathcal{N}_{1,1}^*(X,L) \times \mathcal{N}_{1,1}^*(X,L) ; \alpha \neq \beta, \alpha \cap \beta = \emptyset \right\},\$$
$$\mathcal{Z}^*(X,L) := \left\{ (\alpha,\beta) \in \mathcal{N}_{1,1}^*(X,L) \times \mathcal{N}_{1,1}^*(X,L) ; \alpha \neq \beta, \alpha \cap \beta \neq \emptyset \right\}.$$

Denote

$$Y^*(X,L) := \#\mathcal{Y}^*(X,L),$$

$$Z^*(X,L) := \#\mathcal{Z}^*(X,L).$$

Then we have

$$N_{1,1}^*(X,L)^2 = N_{1,1}^*(X,L) + Y^*(X,L) + Z^*(X,L).$$

Inserting this decomposition into the RHS of (15) we get

(16)
$$\operatorname{Prob}_{WP}^{g}\left(N_{1,1}^{*}(X,L)=0\right) \leq \frac{1}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]} + \frac{\mathbb{E}_{WP}^{g}[Y^{*}(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}} + \frac{\mathbb{E}_{WP}^{g}[Z^{*}(X,L)]}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}}.$$

In the following subsections we will show that each of the three terms on the RHS of (16) goes to 0 as $g \to \infty$ for $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$, which in particular implies Theorem 34. More precisely,

(A)
$$\lim_{g \to \infty} \frac{1}{\mathbb{E}_{WP}^g[N_{1,1}^*(X,L)]} = 0$$

(B)
$$\lim_{g \to \infty} \frac{\mathbb{E}_{WP}^g[Y^*(X,L)] - \mathbb{E}_{WP}^g[N_{1,1}^*(X,L)]^2}{\mathbb{E}_{WP}^g[N_{1,1}^*(X,L)]^2} = 0$$

and

(C)
$$\lim_{g \to \infty} \frac{\mathbb{E}_{WP}^{g}[Z^{*}(X,L)]}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}} = 0.$$

Remark. The proofs of (A) and (B) are similar: we use $\mathbb{E}_{WP}^{g}[N_{1,1}]$ and $\mathbb{E}_{WP}^{g}[Y]$ to approximate $\mathbb{E}_{WP}^{g}[N_{1,1}^{*}]$ and $\mathbb{E}_{WP}^{g}[Y^{*}]$ respectively (we will define Y(X, L) in Subsection 7.4). For the proof of (C), we will control the number of certain types of simple closed geodesics by using Mirzakhani's generalized McShane identity.

We will prove (\mathbf{A}) , (\mathbf{B}) and (\mathbf{C}) in the following subsections.

7.3. **Proof of** (A). Recall that $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$ goes to ∞ as $g \to \infty$. By Lemma 29 we have

$$\lim_{g \to \infty} \mathbb{E}^g_{\mathrm{WP}}[N_{1,1}(X,L)] = \infty.$$

We will show that $\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]$ is close to $\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]$ for large g > 0. More precisely,

Proposition 39. With the notations as above, we have

 $\lim_{g \to \infty} \left(\mathbb{E}^g_{\mathrm{WP}}[N_{1,1}(X,L)] - \mathbb{E}^g_{\mathrm{WP}}[N^*_{1,1}(X,L)] \right) = 0.$

In particular, Equation (A) holds.

We split the proof into several parts. We always assume that q > 0 is large enough.

By definition, $N_{1,1}(X,L) - N_{1,1}^*(X,L) \ge 0$. Assume that

 $\gamma \in \mathcal{N}_{1,1}(X,L) \setminus \mathcal{N}_{1,1}^*(X,L).$

By definition of $N_{1,1}^*(X, L)$ and Lemma 16 we know that there exists a simple closed geodesic $\alpha \in \mathcal{N}_{1,1}(X,L)$ with $\alpha \neq \gamma$ such that

$$\gamma \cap \alpha \neq \emptyset \quad \text{and} \quad |\chi(X_{\gamma\alpha})| \ge 3$$

Assume that $X_{\gamma\alpha}$ is of type $S_{g_0,k}$. Then $\partial X_{\gamma\alpha}$ is a simple closed multigeodesic that split off an $S_{g_0,k}$ from X. By Lemma 16 we know that

$$g_0 \ge 1$$
 and $3 \le 2g_0 - 2 + k \le g - 1$.

And we have

$$\ell(\partial X_{\gamma\alpha}) \le \ell(\alpha) + \ell(\gamma) \le 2L.$$

Note that by Lemma 14 we have

$$X_{\gamma} \subset X_{\gamma\alpha}.$$

Now we define a counting function as follows:

Definition 40. Define the counting function $\hat{N}_{g_0,n_0}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L_1,L_2)$ to be the number of pairs (γ_1, γ_2) satisfying

- γ_2 is a simple closed multi-geodesics in X consisting of n_0 geodesics that split off an S_{g_0,n_0} from X, and its complement $X \setminus S_{g_0,n_0}$ consists of q components $S_{g_1,n_1}, \cdots, S_{g_q,n_q}$ for some $q \ge 1$; • γ_1 is a simple closed geodesic in that S_{g_0,n_0} and splits off a one-holed
- torus from that S_{g_0,n_0} ;
- $\ell(\gamma_1) \leq L_1$ and $\ell(\gamma_2) \leq L_2$.

(see Figure 7.)

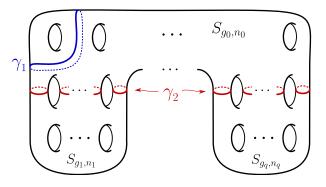


FIGURE 7

Note that the map

$$\gamma \mapsto (\gamma, \partial X_{\gamma \alpha})$$

is injective and $\gamma \cap \partial X_{\gamma\alpha} = \emptyset$, then we have

(17)
$$N_{1,1}(X,L) - N_{1,1}^*(X,L) \le \sum \hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)$$

RANDOM SURFACES

where the summation takes over all possible $(g_0, k), q \ge 1$, and $(g_1, n_1), \cdots, (g_q, n_q)$ such that

- $g_0 \ge 1, \ 3 \le 2g_0 2 + k \le g 1;$ $n_i \ge 1, \ 2g_i 2 + n_i \ge 1, \ \forall 1 \le i \le q;$ $n_1 + \dots + n_q = k, \ g_0 + g_1 + \dots + g_q + k q = g.$

For such a counting function, by Mirzakhani's Integration Formula (see Theorem 12), we have

$$\int_{\mathcal{M}_{g}} \hat{N}_{g_{0},k}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}(X,L,2L)dX$$

$$= \frac{C_{\Gamma}}{|\operatorname{Sym}|} \int_{\mathbb{R}^{k+1}_{\geq 0}} \mathbf{1}_{[0,L]}(y) \mathbf{1}_{[0,2L]} \left(\sum_{i=1}^{q} (x_{i,1} + \dots + x_{i,n_{i}})\right)$$

$$V_{1,1}(y) V_{g_{0}-1,k+1}(y,x_{1,1},\dots,x_{q,n_{q}})$$

$$V_{g_{1},n_{1}}(x_{1,1},\dots,x_{1,n_{1}}) \cdots V_{g_{q},n_{q}}(x_{q,1},\dots,x_{q,n_{q}})$$

$$yx_{1,1} \cdots x_{q,n_{q}} dy dx_{1,1} \cdots dx_{q,n_{q}}.$$

From Theorem 18 of Mirzakhani we know that

$$V_{1,1}(y) = \frac{1}{48}(y^2 + 4\pi^2).$$

Recall that $C_{\Gamma} \leq 1$ and it is clear that the symmetry satisfies

$$|\operatorname{Sym}| \ge n_1! \cdots n_q!$$

By Lemma 22, we have

$$V_{g,n}(x_1, \cdots, x_n) \le \prod_{i=1}^n \frac{\sinh(x_i/2)}{x_i/2} V_{g,n},$$

and we also have that for x > 0,

$$\frac{\sinh(x/2)}{x/2} \le \frac{e^{x/2}}{x}.$$

Set the condition

Cond :=
$$\left\{ 0 \le y \le L, \ 0 \le x_{i,j}, \ \sum_{i=1}^{q} \sum_{j=1}^{n_i} x_{i,j} \le 2L \right\}.$$

Put all these equations together we get

(18)
$$\int_{\mathcal{M}_{g}} \hat{N}_{g_{0},k}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}(X,L,2L)dX \leq \frac{1}{n_{1}!\cdots n_{q}!} V_{g_{1},n_{1}}\cdots V_{g_{q},n_{q}}$$
$$\times \int_{\text{Cond}} \left(\frac{1}{48}(y^{2}+4\pi^{2})ye^{(x_{1,1}+\cdots+x_{q},n_{q})/2}V_{g_{0}-1,k+1}(y,x_{1,1},\cdots,x_{q},n_{q})\right)$$
$$dydx_{1,1}\cdots dx_{q,n_{q}}.$$

Next we control the summation $\sum \mathbb{E}_{WP}^{g}[\hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)]$ for two different cases, and then combine them to obtain Proposition 39.

Lemma 41. Given an integer $m \ge 2$ independent of g, then there exists a constant c(m) > 0 only depending on m such that

$$\sum_{|\chi(S_{g_0,k})|=m} \mathbb{E}^g_{WP}[\hat{N}^{(g_1,n_1),\cdots,(g_q,n_q)}_{g_0,k}(X,L,2L)] \le c(m)(1+L^{3m-1})e^L \frac{1}{g^m}$$

where the summation takes over all possible (g_0, k) , $q \ge 1$ and (g_1, n_1) , \cdots , (g_q, n_q) such that $g_0 \ge 1$ and $2g_0 - 2 + k = m$.

Proof. Since $|\chi(S_{g_0,k})| = m$, we have that all the nonnegative integers $g_0, k, q, n_1, ..., n_q$ are all bounded from above by a constant only depending on m. By Theorem 18 of Mirzakhani, we know that $V_{g_0-1,k+1}(y, x_{1,1}, \cdots, x_{q,n_q})$ is a polynomial of degree $6g_0 - 10 + 2k$. Thus there exists a constant $c_1(m) > 0$ only depending on m such that

(19)
$$V_{g_0-1,k+1}(y, x_{1,1}, \cdots, x_{q,n_q}) \le c_1(m)(1 + L^{6g_0-10+2k}).$$

For the integral in the RHS of (18), there exists a uniform constant c > 0, and two constants c'(m), c''(m) > 0 only depending on m such that

$$\int_{\text{Cond}} \frac{1}{48} (y^2 + 4\pi^2) y e^{(x_{1,1} + \dots + x_{q,n_q})/2} dy dx_{1,1} \cdots dx_{q,n_q}$$

$$\leq c \cdot (1 + L^4) \int_{\text{Cond}} e^{(x_{1,1} + \dots + x_{q,n_q})/2} dx_{1,1} \cdots dx_{q,n_q}$$

$$\leq c'(m) (1 + L^4) (1 + L^{\sum_{i=1}^q n_i - 1}) e^L$$

$$\leq c''(m) (1 + L^{k+3}) e^L.$$

Which together with (18) and (19) imply that there exists a constant c'''(m) > 0 only depending on m such that

$$\int_{\mathcal{M}_g} \hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L) \leq c'''(m)(1+L^{6g_0-7+3k})e^L V_{g_1,n_1}\cdots V_{g_q,n_q}$$
$$= c'''(m)(1+L^{3m-1})e^L V_{g_1,n_1}\cdots V_{g_q,n_q}.$$

By Proposition 25 we know that there exists a constant $c_2(m) > 0$ only depending on m such that

$$\sum_{g_1,\cdots,g_q} V_{g_1,n_1}\cdots V_{g_q,n_q} \le c_2(m)\frac{1}{g^m}V_g.$$

So we have that there exists a constant $c_3(m) > 0$ only depending on m such that

$$\sum_{g_1,\cdots,g_q} \int_{\mathcal{M}_g} \hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L) \le c_3(m)(1+L^{3m-1})e^L \frac{1}{g^m} V_g.$$

Recall that the nonnegative integers $g_0, k, q, n_1, \dots, n_q$ are all bounded from above by a constant only depending on m. Therefore there exists a constant c(m) > 0 only depending on m such that

$$\sum_{|\chi(S_{g_0,k})|=m} \mathbb{E}_{WP}^g[\hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)]) \le c(m)(1+L^{3m-1})e^L\frac{1}{g^m},$$

as required.

Lemma 42. Given an integer $m \ge 2$ independent of g, then there exists a constant c(m) > 0 only depending on m such that

$$\sum_{m+1 \le |\chi(S_{g_0,k})| \le g-1} \mathbb{E}_{WP}^g[\hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)] \le c(m)(1+L^3)e^{\frac{9}{2}L}\frac{1}{g^m}$$

where the summation takes over all possible (g_0, k) , $q \ge 1$ and (g_1, n_1) , \cdots , (g_q, n_q) such that $g_0 \ge 1$ and $m + 1 \le 2g_0 - 2 + k \le g - 1$.

Proof. First by Lemma 19 we know that

$$V_{g_0-1,k+1}(y,x_{1,1},\cdots,x_{q,n_q}) \leq \left(e^{y/2} \cdot \prod_{i=1}^q \prod_{j=1}^{n_i} e^{x_{i,j}/2}\right) \cdot V_{g_0-1,k+1}$$
$$= \left(e^{y/2} \cdot e^{\sum x_{i,j}/2}\right) \cdot V_{g_0-1,k+1}.$$

Then by (18) we have

$$\int_{\mathcal{M}_{g}} \hat{N}_{g_{0},k}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}(X,L,2L) \leq \frac{1}{n_{1}!\cdots n_{q}!} V_{g_{0}-1,k+1} V_{g_{1},n_{1}}\cdots V_{g_{q},n_{q}} \\
\int_{\text{Cond}} \frac{y}{48} (y^{2} + 4\pi^{2}) e^{y/2} e^{\sum x_{i,j}} \\
dy dx_{1,1}\cdots dx_{q,n_{q}}.$$

For the integral in the RHS above, there exists a universal constant c > 0 such that for large enough g and L,

$$\int_{\text{Cond}} \frac{y}{48} (y^2 + 4\pi^2) e^{y/2} e^{\sum x_{i,j}} dy dx_{1,1} \cdots dx_{q,n_q}$$

$$= \int_0^L \frac{y}{48} (y^2 + 4\pi^2) e^{y/2} dy$$

$$\int_{\sum x_{i,j} \le 2L, x_{i,j} \ge 0} e^{\sum x_{i,j}} dx_{1,1} \cdots dx_{q,n_q}$$

$$\le c(1 + L^3) e^{L/2} e^{2L} \int_{\sum x_{i,j} \le 2L, x_{i,j} \ge 0} dx_{1,1} \cdots dx_{q,n_q}$$

$$= c(1 + L^3) e^{\frac{5}{2}L} \frac{(2L)^k}{k!}.$$

So we have

$$\int_{\mathcal{M}_g} \hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L) \le c(1+L^3)e^{\frac{5}{2}L}\frac{(2L)^k}{k!n_1!\cdots n_q!}V_{g_0-1,k+1}V_{g_1,n_1}\cdots V_{g_q,n_q}.$$

Similar as in proof of Lemma 41, it follows by Lemma 24 that

$$\sum_{g_1, \cdots, g_q} V_{g_1, n_1} \cdots V_{g_q, n_q} \le c \left(\frac{D}{2g - 2g_0 - k}\right)^{q-1} W_{2g - 2g_0 - k}$$

Recall that for fixed k, we always have

$$\sum_{n_1+..+n_q=k, \ n_i \ge 0} \frac{k!}{n_1!...n_q!} = q^k.$$

So we have that for large enough g > 0,

$$\sum_{(g_0,k)} \sum_{q} \sum_{n_1,\cdots,n_q} \sum_{g_1,\cdots,g_q} \int_{\mathcal{M}_g} \hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)$$

$$\leq \sum_{(g_0,k)} \sum_{q} c(1+L^3) e^{\frac{5}{2}L} (\frac{D}{g})^{q-1} \frac{(2L)^k}{k!} \frac{q^k}{k!} V_{g_0-1,k+1} W_{2g-2g_0-k}$$

$$\leq \sum_{(g_0,k)} \sum_{q} c(1+L^3) e^{\frac{5}{2}L} (\frac{D}{g})^{q-1} e^{2L} e^q V_{g_0-1,k+1} W_{2g-2g_0-k}$$

$$\leq \sum_{(g_0,k)} c(1+L^3) e^{\frac{9}{2}L} V_{g_0-1,k+1} W_{2g-2g_0-k}$$

Recall that Part (1) of Lemma 23 tells that $V_{g,n} \leq cW_{2g-2+n}$ for a universal constant c > 0. Then it follows by Part (2) of Lemma 23 that there exist two constants c'(m), c(m) > 0 only depending on m such that

$$\begin{split} &\sum_{m+1 \leq |\chi(S_{g_0,k})| \leq g-1} \mathbb{E}_{WP}^{g} [\hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)] \cdot V_g \\ &\leq \sum_{k} \sum_{g_0: \ m+1 \leq 2g_0-2+k \leq g-1} c(1+L^3) e^{\frac{9}{2}L} V_{g_0-1,k+1} W_{2g-2g_0-k} \\ &\leq \sum_{k} \sum_{g_0: \ m+1 \leq 2g_0-2+k \leq g-1} c(1+L^3) e^{\frac{9}{2}L} W_{2g_0-3+k} W_{2g-2g_0-k} \\ &= \sum_{k} \sum_{g_0: \ m \leq 2g_0-3+k \leq g-2} c(1+L^3) e^{\frac{9}{2}L} W_{2g_0-3+k} W_{2g-2g_0-k} \\ &\leq \sum_{k} c'(m)(1+L^3) e^{\frac{9}{2}L} \frac{1}{g^m} W_{2g-3} \\ &= \sum_{k} c'(m)(1+L^3) e^{\frac{9}{2}L} \frac{1}{g^m} V_g \\ &\leq c(m)(1+L^3) e^{\frac{9}{2}L} \frac{1}{g^m} V_g \end{split}$$

where in the last inequality we apply the facts that $k \leq g - 1$ and $V_g \approx gV_{g-1,1}$ (see Part (2) and (3) of Lemma 19). That is,

$$\sum_{m+1 \le |\chi(S_{g_0,k})| \le g-1} \mathbb{E}_{\mathrm{WP}}^g [\hat{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,2L)] \le c(m)(1+L^3)e^{\frac{9}{2}L}\frac{1}{g^m},$$

as required.

Now we are ready to prove Proposition 39.

Proof of Proposition 39. First since $\mathcal{N}_{1,1}^*(X,L) \subset \mathcal{N}_{1,1}(X,L)$,

(20)
$$\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)] \ge 0.$$

It suffices to show the other side. By Equation (17), Lemma 41 and 42 we have

$$\begin{split} & \mathbb{E}^{g}_{\mathrm{WP}}[N_{1,1}(X,L)] - \mathbb{E}^{g}_{\mathrm{WP}}[N^{*}_{1,1}(X,L)] \\ & \leq \sum \mathbb{E}^{g}_{\mathrm{WP}}[\hat{N}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}_{g_{0},k}(X,L,2L)] \\ & = \sum_{3 \leq |\chi(S_{g_{0},k})| \leq 100} \mathbb{E}^{g}_{\mathrm{WP}}[\hat{N}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}_{g_{0},k}(X,L,2L)] \\ & + \sum_{|\chi(S_{g_{0},k})| > 100} \mathbb{E}^{g}_{\mathrm{WP}}[\hat{N}^{(g_{1},n_{1}),\cdots,(g_{q},n_{q})}_{g_{0},k}(X,L,2L)] \\ & \leq \sum_{m=3}^{100} c(m)(1+L^{3m-1})e^{L}\frac{1}{g^{m}} + c(100)(1+L^{3})e^{\frac{9}{2}L}\frac{1}{g^{100}}. \end{split}$$

Recall that $L = L(g) = 2\log g - 4\log \log g + \omega(g)$. As $g \to \infty$ we have that

$$\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)] = O\left(\frac{(\log g)^{4}}{g}e^{\omega(g)}\right) \to 0 \text{ as } g \to \infty.$$

Which together with (20) imply that

$$\lim_{g \to \infty} \left(\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)] \right) = 0.$$

By Lemma 29, we know that $\lim_{g\to\infty} \mathbb{E}^g_{WP}[N_{1,1}(X,L)] = \infty$. So

$$\lim_{g \to \infty} \frac{\mathbb{E}_{\text{WP}}^g[N_{1,1}^*(X,L)]}{\mathbb{E}_{\text{WP}}^g[N_{1,1}(X,L)]} = 1.$$

For (A), as shown above and by Lemma 29 we have

$$\frac{1}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]} \sim \frac{1}{\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]} \sim \frac{1}{\frac{1}{384\pi^{2}}L^{2}e^{\frac{L}{2}}\frac{1}{g}} = O(e^{-\frac{\omega(g)}{2}}) \to 0$$

as $g \to \infty$, which proves (A).

7.4. **Proof of** (B). In this subsection we show (B), whose proof is similar to the one of (A). First we define

Definition 43.

$$\mathcal{Y}(X,L) := \{ (\alpha,\beta) \in \mathcal{N}_{1,1}(X,L) \times \mathcal{N}_{1,1}(X,L) ; \alpha \neq \beta, \alpha \cap \beta = \emptyset \}$$

and

$$Y(X,L):=\#\mathcal{Y}(X,L)=\sum_{\alpha\neq\beta,\alpha\cap\beta=\emptyset}\mathbf{1}_{\mathcal{N}_{1,1}(X,L)}(\alpha)\mathbf{1}_{\mathcal{N}_{1,1}(X,L)}(\beta).$$

Lemma 44. As $g \to \infty$, we have

$$\mathbb{E}_{WP}^{g}[Y(X,L)] = \frac{1}{(384\pi^{2})^{2}} L^{4} e^{L} \frac{1}{g^{2}} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \left(1 + O\left(\frac{1}{g}\right)\right)$$

$$= \mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \left(1 + O\left(\frac{1}{g}\right)\right)$$

where the implied constants are independent of L and g. As a consequence, for $L(g) := 2 \log g - 4 \log \log g + \omega(g)$, we have

$$\lim_{g \to 0} \left(\mathbb{E}_{WP}^{g} [Y(X, L(g))] - \mathbb{E}_{WP}^{g} [N_{1,1}(X, L(g))]^2 \right) = 0.$$

Proof. By Mirzakhani's Integration Formula (see Theorem 12),

$$\begin{split} & \int_{\mathcal{M}_g} Y(X,L) dX \\ = & 2^{-M} \int_{\mathbb{R}^{2}_{\geq 0}} \mathbf{1}_{[0,L]}(x) \mathbf{1}_{[0,L]}(y) V_{1,1}(x) V_{1,1}(y) V_{g-2,2}(x,y) xy dx dy \\ = & \frac{1}{4} \int_{[0,L]^2} \frac{1}{48} x (x^2 + 4\pi^2) \frac{1}{48} y (y^2 + 4\pi^2) V_{g-2,2}(x,y) dx dy. \end{split}$$

By Lemma 19 we know that

$$\frac{V_{g-2,2}}{V_g} = \frac{1}{(8\pi^2 g)^2} \left(1 + O\left(\frac{1}{g}\right) \right).$$

Thus, it follows by Lemma 22 that

$$V_{g-2,2}(x,y) = \frac{\sinh(x/2)}{x/2} \frac{\sinh(y/2)}{y/2} V_{g-2,2} \left(1 + O(\frac{L^2}{g})\right)$$

= $\frac{\sinh(x/2)}{x/2} \frac{\sinh(y/2)}{y/2} \frac{1}{64\pi^4 g^2} V_g \left(1 + O(\frac{L^2}{g})\right) \left(1 + O\left(\frac{1}{g}\right)\right)$

So we have

$$\mathbb{E}_{WP}^{g}[Y(X,L)] = \frac{1}{(384\pi^{2})^{2}} L^{4} e^{L} \frac{1}{g^{2}} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \left(1 + O\left(\frac{1}{g}\right)\right).$$

By Lemma 29 we have

$$\mathbb{E}_{WP}^{g}[Y(X,L)] = \mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2} \left(1 + O(\frac{1}{L})\right) \left(1 + O(\frac{L^{2}}{g})\right) \left(1 + O\left(\frac{1}{g}\right)\right),$$
as required.

Recall that $L = L(g) = 2\log g - 4\log\log g + \omega(g)$. Lemma 44 implies that as $g \to \infty$,

(21)
$$\frac{\mathbb{E}_{WP}^{g}[Y(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2}}{\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2}} = O\left(\frac{1}{L} + \frac{L^{2}}{g} + \frac{1}{g}\right) \to 0.$$

We will show that $\mathbb{E}^g_{WP}[Y^*]$ is an approximation of $\mathbb{E}^g_{WP}[Y]$. More precisely,

Proposition 45. With the notations as above, we have

$$\lim_{g \to \infty} (\mathbb{E}^g_{\mathrm{WP}}[Y(X,L)] - \mathbb{E}^g_{\mathrm{WP}}[Y^*(X,L)]) = 0$$

Moreover, Equation (B) holds.

Proof. First by definition of Y and Y^* we know that

$$Y^*(X,L) \le Y(X,L).$$

So we have

(22)
$$\mathbb{E}^g_{WP}[Y(X,L)] - \mathbb{E}^g_{WP}[Y^*(X,L)] \ge 0.$$

It suffices to show the other side. The proof is similar to the proof of Proposition 39.

For any ordered pair $(\alpha, \beta) \in \mathcal{Y}(X, L) \setminus \mathcal{Y}^*(X, L)$, we have $\alpha \in \mathcal{N}_{1,1}(X, L) \setminus \mathcal{N}_{1,1}^*(X, L)$ or $\beta \in \mathcal{N}_{1,1}(X, L) \setminus \mathcal{N}_{1,1}^*(X, L)$. Without loss of generality we assume $\alpha \in \mathcal{N}_{1,1}(X, L) \setminus \mathcal{N}_{1,1}^*(X, L)$. Then by definition of $\mathcal{N}_{1,1}(X, L)$ and $\mathcal{N}_{1,1}^*(X, L)$, it follows by Lemma 16 that there exists a simple closed geodesic $\alpha' \in \mathcal{N}_{1,1}(X, L)$ with $\alpha' \neq \alpha$ such that

$$\alpha \cap \alpha' \neq \emptyset$$
 and $|\chi(X_{\alpha\alpha'})| \ge 3.$

The relation between X_{β} and $X_{\alpha\alpha'}$ can be divided into the following three cases. (see Figure 8.)

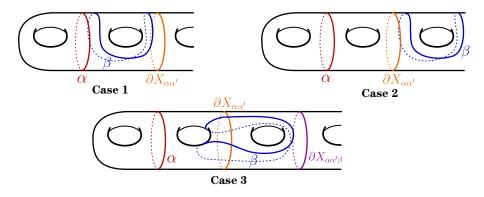


FIGURE 8. Relation between X_{β} and $X_{\alpha\alpha'}$ in the three cases.

Case 1. $X_{\beta} \subset X_{\alpha\alpha'}$. For this case we have

$$\beta \cap \partial X_{\alpha \alpha'} = \emptyset.$$

 $(\beta \text{ won't be part of } \partial X_{\alpha\alpha'} \text{ since } X_{\beta} \text{ is a one-holed torus and } X_{\alpha\alpha'} \text{ is not.})$ So α, β and $\partial X_{\alpha\alpha'}$ are pairwisely disjoint. Assume $X_{\alpha\alpha'}$ is of type $S_{g_0,k}$. Note that X_{α}, X_{β} are two disjoint one-holed torus in $X_{\alpha\alpha'}$, so $g_0 \geq 2$. By Lemma 16, we have

$$3 \le |\chi(X_{\alpha\alpha'})| \le g - 1,$$

and

$$\ell(\alpha) \leq L, \ \ell(\beta) \leq L, \ \ell(\partial X_{\alpha\alpha'}) \leq 2L.$$

Similar to what we have done in the proof of Proposition 39, we define a counting function as follows:

Definition 46. Define the counting function $\dot{N}_{g_0,n_0}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L_1,L_2,L_3)$ to be the number of pairs $(\gamma_1,\gamma_2,\gamma_3)$ satisfying

- γ_3 is a simple closed multi-geodesics in X consisting of n_0 geodesics that split off an S_{g_0,n_0} from X and the complement $X \setminus S_{g_0,n_0}$ consist of q components $S_{g_1,n_1}, \dots, S_{g_q,n_q}$ for some $q \ge 1$;
- γ_1 and γ_2 are two disjoint simple closed geodesics in that S_{g_0,n_0} , and split off two disjoint one-holed torus in that S_{g_0,n_0} ;
- $\ell(\gamma_1) \leq L_1, \, \ell(\gamma_2) \leq L_2, \, \ell(\gamma_3) \leq L_3.$

Since the map

$$(\alpha, \beta) \mapsto (\alpha, \beta, \partial X_{\alpha\alpha'})$$

is injective, the number of pairs $(\alpha, \beta) \in \mathcal{Y}(X, L) \setminus \mathcal{Y}^*(X, L)$ satisfying Case 1 is bounded from above by

(23)
$$Q_1 := \sum \dot{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,L,2L)$$

where the summation takes over all possible $(g_0, k), q \ge 1$ and $(g_1, n_1), \cdots, (g_q, n_q)$ such that

- $g_0 \ge 2, \ 3 \le 2g_0 2 + k \le g 1;$
- $n_i \ge 1, 2g_i 2 + n_i \ge 1, \forall 1 \le i \le q;$
- $n_1 + \dots + n_q = k, g_0 + g_1 + \dots + g_q + k q = g.$

Case 2: $X_{\beta} \cap X_{\alpha\alpha'} = \emptyset$.

For this case we have that α, β and $\partial X_{\alpha\alpha'}$ are pairwisely disjoint. Assume $X_{\alpha\alpha'}$ is of type $S_{q_0,k}$. By Lemma 16, we have

$$g_0 \ge 1, \ 3 \le |\chi(X_{\alpha\alpha'})| \le g - 1,$$

and

$$\ell(\alpha) \le L, \ \ell(\beta) \le L, \ \ell(\partial X_{\alpha\alpha'}) \le 2L.$$

Similar to what we have done in the proof of Proposition 39, we define a counting function as follows:

Definition 47. Define the counting function $\ddot{N}_{g_0,n_0}^{(g_1,n_1),\cdots,(g_q,n_q)}(X, L_1, L_2, L_3)$ to be the number of pairs $(\gamma_1, \gamma_2, \gamma_3)$ satisfying

- γ_3 is a simple closed multi-geodesics in X consisting of n_0 geodesics that split off an S_{g_0,n_0} from X and the complement $X \setminus S_{g_0,n_0}$ consist of q components $S_{g_1,n_1}, \dots, S_{g_q,n_q}$ for some $q \ge 1$;
- γ_1 is a simple closed geodesic in that S_{g_0,n_0} , and splits off a one-holed torus in that S_{g_0,n_0} ;
- γ_2 is a simple closed geodesic in that S_{g_1,n_1} , and splits off a one-holed torus in that S_{g_1,n_1} ;
- $\ell(\gamma_1) \leq L_1, \ \ell(\gamma_2) \leq L_2, \ \ell(\gamma_3) \leq L_3.$

Since the map

$$(\alpha, \beta) \mapsto (\alpha, \beta, \partial X_{\alpha\alpha'})$$

is injective, the number of pairs $(\alpha, \beta) \in \mathcal{Y}(X, L) \setminus \mathcal{Y}^*(X, L)$ satisfying Case 2 is bounded from above by

(24)
$$Q_2 := \sum \ddot{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,L,2L)$$

where the summation takes over all possible $(g_0, k), q \ge 1$ and $(g_1, n_1), \cdots, (g_q, n_q)$ such that

•
$$g_0 \ge 1, \ 3 \le 2g_0 - 2 + k \le g - 1;$$

RANDOM SURFACES

- $n_i \ge 1, 2g_i 2 + n_i \ge 1, \forall 1 \le i \le q;$
- $n_1 + \dots + n_q = k, g_0 + g_1 + \dots + g_q + k q = g;$
- $g_1 \ge 1$.

Case 3: $X_{\beta} \cap X_{\alpha\alpha'} \neq \emptyset$ and $X_{\beta} \nsubseteq X_{\alpha\alpha'}$.

For this case we have that β and $\partial X_{\alpha\alpha'}$ are not disjoint. We consider the subsurface with geodesic boundary $X_{\alpha\alpha'\beta} \subset X$ constructed from $X_{\alpha\alpha'}$ and X_{β} in the way described in Section 3 (*i.e.* $X_1 = X_{\alpha\alpha'}, X_2 = X_{\beta}$ and $X_3 = X_{\alpha\alpha'\beta}$ in the notation of Section 3). Then α, β and $\partial X_{\alpha\alpha'\beta}$ are pairwisely disjoint. Assume $X_{\alpha\alpha'\beta}$ is of type $S_{g_0,k}$. Note that X_{α}, X_{β} are two disjoint one-holed torus in $X_{\alpha\alpha'\beta}$, so $g_0 \geq 2$. Recall that $L = L(g) = 2\log g - 4\log\log g + \omega(g)$. Thus, by Lemma 16 we have that for large enough g > 0,

$$3 \le |\chi(X_{\alpha\alpha'})| \le \frac{1}{2}g_{\Xi}$$

and

$$\ell(\partial X_{\alpha\alpha'}) \le 2L$$

Then again by Lemma 15, we have for large enough g > 0,

$$4 \le |\chi(X_{\alpha\alpha'\beta})| \le g - 1.$$

and

$$\ell(\alpha) \le L, \ \ell(\beta) \le L, \ \ell(\partial X_{\alpha\alpha'\beta}) \le 3L.$$

Since the map

$$(\alpha,\beta)\mapsto(\alpha,\beta,\partial X_{\alpha\alpha'})$$

is injective, the number of $(\alpha, \beta) \in \mathcal{Y}(X, L) \setminus \mathcal{Y}^*(X, L)$ satisfying Case 3 is bounded from above by

(25)
$$Q_3 := \sum \dot{N}_{g_0,k}^{(g_1,n_1),\cdots,(g_q,n_q)}(X,L,L,3L)$$

where the summation takes over all possible $(g_0, k), q \ge 1$ and $(g_1, n_1), \cdots, (g_q, n_q)$ such that

- $g_0 \ge 2, 4 \le 2g_0 2 + k \le g 1;$
- $n_i \ge 1, 2g_i 2 + n_i \ge 1, \forall 1 \le i \le q;$
- $n_1 + \dots + n_q = k, g_0 + g_1 + \dots + g_q + k q = g.$

Then by the discussion above, we have

(26)
$$Y(X,L) - Y^*(X,L) \le 2(Q_1 + Q_2 + Q_3)$$

where the coefficient 2 comes from the reason that we have assumed $\alpha \in \mathcal{N}_{1,1}(X,L) \setminus \mathcal{N}^*_{1,1}(X,L)$; indeed if $\beta \in \mathcal{N}_{1,1}(X,L) \setminus \mathcal{N}^*_{1,1}(X,L)$ and $\alpha \in \mathcal{N}^*_{1,1}(X,L)$ one may have the same upper bound $(Q_1 + Q_2 + Q_3)$.

Then we have the following estimations of $\mathbb{E}_{WP}^{g}[Q_1]$, $\mathbb{E}_{WP}^{g}[Q_2]$ and $\mathbb{E}_{WP}^{g}[Q_3]$, whose proofs are exactly the same as the proofs of Lemma 41 and 42, and we omit the details. For $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$, we have that as $g \to \infty$,

(27)
$$\mathbb{E}^{g}_{WP}[Q_{1}] \leq cL^{8}e^{L}\frac{1}{g^{3}} = O(\frac{(\log g)^{4}}{g}e^{\omega(g)}) \to 0,$$

(28)
$$\mathbb{E}^{g}_{WP}[Q_{2}] \leq cL^{10}e^{\frac{3L}{2}}\frac{1}{g^{4}} = O(\frac{(\log g)^{4}}{g}e^{\frac{3}{2}\omega(g)}) \to 0,$$

and

(29)
$$\mathbb{E}_{WP}^{g}[Q_{3}] \leq cL^{11}e^{\frac{3L}{2}}\frac{1}{g^{4}} = O(\frac{(\log g)^{5}}{g}e^{\frac{3}{2}\omega(g)}) \to 0.$$

Therefore we have that as $g \to \infty$,

(30)
$$0 \le \mathbb{E}_{WP}^{g}[Y(X,L)] - \mathbb{E}_{WP}^{g}[Y^{*}(X,L)] = O(\frac{(\log g)^{5}}{g}e^{\frac{3}{2}\omega(g)}) \to 0.$$

For (\mathbf{B}) , first we rewrite

$$\frac{\mathbb{E}_{WP}^{g}[Y^{*}(X,L)] - \mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}} \\
= \frac{\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2}}{\mathbb{E}_{WP}^{g}[N_{1,1}^{*}(X,L)]^{2}} \times \frac{\mathbb{E}_{WP}^{g}[Y(X,L)]}{\mathbb{E}_{WP}^{g}[N_{1,1}(X,L)]^{2}} \\
\times \left(1 - \frac{\mathbb{E}_{WP}^{g}[Y(X,L)] - \mathbb{E}_{WP}^{g}[Y^{*}(X,L)]}{\mathbb{E}_{WP}^{g}[Y(X,L)]}\right) - 1$$

As $g \to \infty$, it follows by Lemma 29, 39 and 44 that

$$\lim_{g \to \infty} \frac{\mathbb{E}_{WP}^g [N_{1,1}(X,L)]^2}{\mathbb{E}_{WP}^g [N_{1,1}^*(X,L)]^2} = 1 \text{ and } \lim_{g \to \infty} \frac{\mathbb{E}_{WP}^g [Y(X,L)]}{\mathbb{E}_{WP}^g [N_{1,1}(X,L)]^2} = 1.$$

Which together with (30) imply that

$$\lim_{g \to \infty} \frac{\mathbb{E}_{\mathrm{WP}}^g [Y^*(X,L)] - \mathbb{E}_{\mathrm{WP}}^g [N^*_{1,1}(X,L)]^2}{\mathbb{E}_{\mathrm{WP}}^g [N^*_{1,1}(X,L)]^2} = 0.$$

The proof of (B) is complete.

7.5. **Proof of** (C). In this subsection we show (C), where we will apply Mirzakhani's generalized McShane identity for certain counting problem for $S_{0,4}$ and $S_{1,2}$. Our main result in the subsection is as follows.

Proposition 48. With the notations as above, there exists a universal constant c > 0 such that

$$\mathbb{E}^g_{\mathrm{WP}}[Z^*(X,L)] \le cL^3 e^L \frac{1}{g^2}.$$

Moreover, Equation (C) holds.

Consider an ordered pair $(\alpha, \beta) \in \mathcal{Z}^*(X, L)$, that is, $\alpha, \beta \in \mathcal{N}_{1,1}^*(X, L)$ with $\alpha \neq \beta$ and $\alpha \cap \beta \neq \emptyset$. By definition of $\mathcal{N}_{1,1}^*$, we know that $X_{\alpha\beta}$ is of type $S_{1,2}$. Unfortunately, one can not apply Mirzakhani's Integration Formula (see Theorem 12) to the pair $(\alpha, \beta, \partial X_{\alpha\beta})$ because $\alpha \cap \beta \neq \emptyset$. We consider the following map

$$(\alpha,\beta)\mapsto (\alpha,\partial X_{\alpha\beta}).$$

(

This map may not be injective. However, one can control the multiplicity of this map. To do this, it is sufficient to count the number of such $\beta's$ with lengths $\leq L$ in a given $S_{1,2}$ with geodesic boundary. We also need a similar counting result in a given $S_{0,4}$ for some technical reason. More precisely, we have the following lemma.

Lemma 49. Let Y be a hyperbolic surface with geodesic boundary belonging to either of the following cases:

- (a) Y is of type $S_{1,2}$, with boundary components γ_1, γ_2 ;
- (b) Y is of type $S_{0,4}$, with boundary components $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

For any L > 0, let $N(Y, \gamma_1, \gamma_2, L)$ be the number of simple closed geodesics on Y of length $\leq L$ that form a pair of pants together with γ_1 and γ_2 . Then

$$N(Y, \gamma_1, \gamma_2, L) \le \frac{\ell(\gamma_1)}{\mathcal{R}(\ell(\gamma_1), \ell(\gamma_2), L)},$$

where $\mathcal{R}(x, y, z)$ is the function given in Mirzakhani's generalized McShane identity (see Theorem 26).

Proof. We only treat Case (a) here. The proof for (b) is similar.

By Lemma 27 we know that $\mathcal{D} > 0$ and $\mathcal{R} > 0$. So by Mirzakhani's generalized McShane identity (see Theorem 26) we have

$$\ell(\gamma_1) = \sum_{\{\alpha_1,\alpha_2\}} \mathcal{D}(\ell(\gamma_1), \ell(\alpha_1), \ell(\alpha_2)) + \sum_{i=2}^n \sum_{\alpha} \mathcal{R}(\ell(\gamma_1), \ell(\gamma_i), \ell(\alpha))$$

$$\geq \sum_{\alpha'} \mathcal{R}(\ell(\gamma_1), \ell(\gamma_2), \ell(\alpha'))$$

where α' runs over all simple closed geodesics of lengths $\leq L$ and bounding a pair of pants together with the union $\gamma_1 \cup \gamma_2$. By Lemma 27 we know that $\mathcal{R}(x, y, z)$ is decreasing with respect to z. Thus,

$$\ell(\gamma_1) \ge N(Y, \gamma_1, \gamma_2, L) \cdot \mathcal{R}(\ell(\gamma_1), \ell(\gamma_2), L),$$

as required.

Remark. If one considers the number of closed geodesics that are not necessarily simple, it follows from [Bus10, Lemma 6.6.4] that

$$N(Y, \gamma_1, \gamma_2, L) \le ce^L$$

for a universal constant c. In view of Lemma 27, the lemma above implies the better estimate

$$N(Y, \gamma_1, \gamma_2, L) \le c(\ell(\gamma_1), \ell(\gamma_2))e^{\frac{L}{2}}.$$

In contrast, by [Mir08], one may expect that

$$N(Y, \gamma_1, \gamma_2, L) \le c_1(Y)L^4 \quad \text{if } Y \cong S_{1,2},$$

$$N(Y, \gamma_1, \gamma_2, L) \le c_2(Y)L^2 \quad \text{if } Y \cong S_{0,4}$$

for L large enough, where the constants depend on the hyperbolic surface Y. It is not that easy to give explicit expression for c_1 and c_2 which only depend on the lengths of the boundary geodesics.

Now we return to the proof of Proposition 48. Note that the complement of $S_{1,2}$ in X may have several possibilities:

$$X \setminus S_{1,2} = S_{g-2,2} \text{ or } S_{k,1} \cup S_{g-k-1,1}$$

for some $1 \le k \le \frac{1}{2}(g-1)$. We divide $\mathcal{Z}^*(X, L)$ into several parts as follows.

 \Box

Definition 50.

$$\mathcal{Z}^{*0}(X,L) := \{ (\alpha,\beta) \in \mathcal{Z}^*(X,L) ; X \setminus X_{\alpha\beta} \text{ is of type } S_{g-2,2} \}$$

and for any $1 \le k \le \frac{1}{2}(g-1)$,

$$\mathcal{Z}^{*k}(X,L) := \{ (\alpha,\beta) \in \mathcal{Z}^*(X,L) ; X \setminus X_{\alpha\beta} \text{ is of type } S_{k,1} \cup S_{g-k-1,1} \}.$$

On the other hand, recall that for an ordered pair $(\alpha, \beta) \in \mathcal{Z}^*(X, L)$, since $\ell(\alpha) \leq L$ and $\ell(\beta) \leq L$, we have

$$\ell(\partial X_{\alpha\beta}) \le 2L.$$

We divide $Z^*(X, L)$ into two parts

$$Z^*(X,L) = Z_1^*(X,L) + Z_2^*(X,L)$$

where $Z_1^*(X, L)$ and $Z_2^*(X, L)$ are defined as follow.

Definition 51.

$$\begin{aligned} \mathcal{Z}_1^*(X,L) &:= \left\{ (\alpha,\beta) \in \mathcal{Z}^*(X,L) \, ; \, \ell(\partial X_{\alpha\beta}) \le 1.9L \right\}, \\ Z_1^*(X,L) &:= \# \mathcal{Z}_1^*(X,L). \end{aligned}$$

$$\mathcal{Z}_{2}^{*}(X,L) := \left\{ (\alpha,\beta) \in \mathcal{Z}^{*}(X,L) ; \ell(\partial X_{\alpha\beta}) > 1.9L \right\},$$
$$Z_{2}^{*}(X,L) := \# \mathcal{Z}_{2}^{*}(X,L).$$

 $Z_2^*(X,L) := \# Z_2^*(X,L).$ For i = 1, 2 and $1 \le k \le \frac{1}{2}(g-1)$, we also define

$$\begin{aligned} \mathcal{Z}_{i}^{*0}(X,L) &:= \mathcal{Z}^{*0}(X,L) \cap \mathcal{Z}_{i}^{*}(X,L), \\ Z_{i}^{*0}(X,L) &:= \# \mathcal{Z}_{i}^{*0}(X,L), \end{aligned}$$

and

$$\mathcal{Z}_i^{*k}(X,L) := \mathcal{Z}^{*k}(X,L) \cap \mathcal{Z}_i^{*}(X,L),$$
$$Z_i^{*k}(X,L) := \# \mathcal{Z}_i^{*k}(X,L).$$

Remark. The value 1.9 is not crucial and can be replaced by any number in the interval $(\frac{5}{3}, 2)$, where $\frac{5}{3}$ comes from Lemma 17.

We divide the proof of Proposition 48 into the following two lemmas.

Lemma 52. Let $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$ as before. Then we have as $g \to \infty$,

$$\mathbb{E}^{g}_{WP}[Z_{1}^{*}(X,L)] \le cL^{6}e^{0.95L}\frac{1}{g^{2}}$$

for a universal constant c > 0.

Proof. By Lemma 49 we have

$$Z_{1}^{*0}(X,L) = \sum_{\substack{\alpha \neq \beta, \alpha \cap \beta \neq \emptyset, \\ \ell(\partial X_{\alpha\beta}) \leq 1.9L, \\ X \setminus X_{\alpha\beta} = S_{g-2,2}}} \mathbf{1}_{\mathcal{N}_{1,1}^{*}(X,L)}(\alpha) \mathbf{1}_{\mathcal{N}_{1,1}^{*}(X,L)}(\beta)$$

$$\leq \sum_{(\alpha,\gamma_{1},\gamma_{2})} \mathbf{1}_{[0,L]}(\ell(\alpha)) \mathbf{1}_{[0,1.9L]}(\ell(\gamma_{1}) + \ell(\gamma_{2})) N(Y,\gamma_{1},\gamma_{2},L)$$

RANDOM SURFACES

$$\leq \sum_{(\alpha,\gamma_1,\gamma_2)} \mathbf{1}_{[0,L]}(\ell(\alpha)) \mathbf{1}_{[0,1.9L]}(\ell(\gamma_1) + \ell(\gamma_2)) \frac{\ell(\gamma_1)}{\mathcal{R}(\ell(\gamma_1),\ell(\gamma_2),L)}$$

where $(\alpha, \gamma_1, \gamma_2)$ runs over all ordered triples of simple closed geodesics such that $\gamma_1 \cup \gamma_2$ splits off a subsurface Y of type $S_{1,2}$ with complement $S_{g-2,2}$, while α splits off a one-holed torus in that $S_{1,2}$ (see the first picture in Figure 9).

Similarly, for all $1 \le k \le \frac{1}{2}(g-1)$ we have

$$Z_1^{*k}(X,L) \le \sum_{(\alpha,\gamma_1,\gamma_2)} \mathbf{1}_{[0,L]}(\ell(\alpha)) \mathbf{1}_{[0,1.9L]}(\ell(\gamma_1) + \ell(\gamma_2)) \frac{\ell(\gamma_1)}{\mathcal{R}(\ell(\gamma_1),\ell(\gamma_2),L)}$$

where $(\alpha, \gamma_1, \gamma_2)$ runs over all ordered triples of simple closed geodesics such that $\gamma_1 \cup \gamma_2$ splits off an $S_{1,2}$ with complement $S_{k,1} \cup S_{g-k-1,1}$, and α splits off a one-holed torus in that $S_{1,2}$ (see the second picture in Figure 9).

Then one may apply Mirzakhani's Integration Formula (see Theorem 12) to get

$$\int_{\mathcal{M}_{g}} Z_{1}^{*0}(X,L) dX \leq \int_{0 \le z \le L} \int_{0 \le x + y \le 1.9L; \, x, y \ge 0} \frac{x}{\mathcal{R}(x,y,L)} V_{1,1}(z) V_{0,3}(x,y,z) V_{g-2,2}(x,y) xyz dx dy dz$$

and

$$\int_{\mathcal{M}_g} Z_1^{*k}(X,L) dX \leq \int_{0 \le z \le L} \int_{0 \le x + y \le 1.9L; \, x, y \ge 0} \frac{x}{\mathcal{R}(x,y,L)} \\ V_{1,1}(z) V_{0,3}(x,y,z) V_{k,1}(x) V_{g-k-1,1}(y) x y z dx dy dz$$

for all $1 \le k \le \frac{1}{2}(g-1)$.

By Theorem 18 we know that

$$V_{1,1}(z) = \frac{1}{48}(z^2 + 4\pi^2)$$
 and $V_{0,3}(x, y, z) = 1.$

By Lemma 22 we know that

$$V_{g-2,2}(x,y) \le \frac{\sinh(\frac{x}{2})\sinh(\frac{y}{2})}{\frac{x}{2}\frac{y}{2}}V_{g-2,2} \le \frac{e^{\frac{x+y}{2}}}{xy}V_{g-2,2},$$
$$V_{k,1}(x) \le \frac{\sinh(\frac{x}{2})}{\frac{x}{2}}V_{k,1} \le \frac{e^{\frac{x}{2}}}{x}V_{k,1},$$
$$V_{g-k-1,1}(y) \le \frac{\sinh(\frac{y}{2})}{\frac{y}{2}}V_{g-k-1,1} \le \frac{e^{\frac{y}{2}}}{y}V_{g-k-1,1}.$$

By Lemma 27 we know that

$$\frac{x}{\mathcal{R}(x,y,L)} \le 100(1+x)(1+e^{-\frac{x+y}{2}}e^{\frac{L}{2}}).$$

Put all the inequalities above together we have

$$\int_{\mathcal{M}_g} Z_1^*(X,L) dX = \int_{\mathcal{M}_g} \left(Z_1^{*0}(X,L) + \sum_{1 \le k \le \frac{1}{2}(g-1)} Z_1^{*k}(X,L) \right) dX$$

$$\leq \frac{100}{48} \left(V_{g-2,2} + \sum_{1 \leq k \leq \frac{1}{2}(g-1)} V_{k,1} V_{g-k-1,1} \right) \\ \int_{0 \leq z \leq L} \int_{0 \leq x+y \leq 1.9L; \, x, y \geq 0} z(z^2 + 4\pi^2) e^{\frac{x+y}{2}} (1+x)(1+e^{-\frac{x+y}{2}}e^{\frac{L}{2}}) dx dy dz \\ \leq c \cdot \left((1+L^6)e^{0.95L} + (1+L^7)e^{\frac{L}{2}} \right) \\ \left(V_{g-2,2} + \sum_{1 \leq k \leq \frac{1}{2}(g-1)} V_{k,1} V_{g-k-1,1} \right)$$

for some universal constant c > 0. And by Lemma 19 and 20 we know that

$$V_{g-2,2} = \frac{1}{(8\pi^2 g)^2} V_g (1 + O\left(\frac{1}{g}\right))$$

and

$$\sum_{\leq k \leq \frac{1}{2}(g-1)} V_{k,1} V_{g-k-1,1} = O\left(\frac{V_g}{g^3}\right).$$

So as $g \to \infty$, we have

$$\mathbb{E}_{WP}^{g}[Z_{1}^{*}(X,L)] \leq cL^{6}e^{0.95L}\frac{1}{g^{2}}$$

for some universal constant c > 0, as required.

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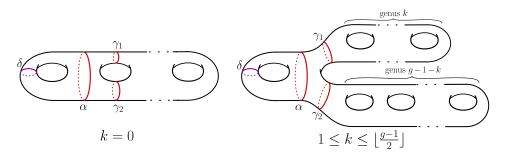


FIGURE 9

Now we estimate the expectation of $Z_2^*(X, L)$.

Lemma 53. Let $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$ as before. Then we have as $g \to \infty$,

$$\mathbb{E}^g_{\mathrm{WP}}[Z_2^*(X,L)] \le cL^3 e^L \frac{1}{g^2}$$

for a universal constant c > 0.

Proof. Assume that $(\alpha, \beta) \in \mathbb{Z}_2^*(X, L)$. Denote the boundary of $X_{\alpha\beta}$ to be the two simple closed geodesics γ_1 and γ_2 . Then we have

$$1.9L < \ell(\gamma_1) + \ell(\gamma_2) \le 2L.$$

By Lemma 17 we know that α and β have exactly 4 intersection points, and the intersection $X_{\alpha} \cap X_{\beta}$ contains a simple closed geodesic δ which is disjoint

with $\alpha, \beta, \gamma_1, \gamma_2$. (see Figure 5.) Since $\alpha \cup \beta$ is homotopic to $\gamma_1 \cup \gamma_2 \cup 2\delta$ (see the remark after Lemma 17), we have

$$\ell(\gamma_1) + \ell(\gamma_2) + 2\ell(\delta) \le \ell(\alpha) + \ell(\beta) \le 2L.$$

Now similar to how we deal with $Z_1^*(X, L)$, by Lemma 49 we have

$$Z_{2}^{*0}(X,L) = \sum_{\substack{\alpha \neq \beta, \alpha \cap \beta \neq \emptyset, \\ 1.9L < (\partial X_{\alpha\beta}) \le 2L, \\ X \setminus X_{\alpha\beta} = S_{g-2,2}}} \mathbf{1}_{\mathcal{N}_{1,1}^{*}(X,L)}(\alpha) \mathbf{1}_{\mathcal{N}_{1,1}^{*}(X,L)}(\beta)$$

$$\leq \sum_{\substack{(\alpha,\gamma_{1},\gamma_{2},\delta) \\ \mathbf{1}_{[0,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2}) + 2\ell(\delta)) \cdot N(Y_{\gamma_{1}\gamma_{2}\delta}, \gamma_{1}, \gamma_{2}, L)}$$

$$\leq \sum_{\substack{(\alpha,\gamma_{1},\gamma_{2},\delta) \\ (\alpha,\gamma_{1},\gamma_{2},\delta)}} \mathbf{1}_{[1.9L,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2})) \mathbf{1}_{[0,L]}(\ell(\alpha))$$

$$= \sum_{\substack{(\alpha,\gamma_{1},\gamma_{2},\delta) \\ (\alpha,\gamma_{1},\gamma_{2},\delta)}} \mathbf{1}_{[1.9L,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2})) \mathbf{1}_{[0,L]}(\ell(\alpha))$$

$$= \mathbf{1}_{[0,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2}) + 2\ell(\delta)) \frac{\ell(\gamma_{1})}{\mathcal{R}(\ell(\gamma_{1}),\ell(\gamma_{2}),L)}$$

where $(\alpha, \gamma_1, \gamma_2, \delta)$ runs over all ordered quadruples of simple closed geodesics such that $\gamma_1 \cup \gamma_2$ splits off an $S_{1,2}$ with complement $S_{g-2,2}$, α splits off a one-holed torus from that $S_{1,2}$, and δ is in that one-holed torus (see the first picture in Figure 9). We let $Y_{\gamma_1\gamma_2\delta}$ denote the hyperbolic surface of type $S_{0,4}$ split off by γ_1, γ_2 and δ .

Similarly, for all $1 \le k \le \frac{1}{2}(g-1)$ we have

$$Z_{2}^{*k}(X,L) \leq \sum_{(\alpha,\gamma_{1},\gamma_{2},\delta)} \mathbf{1}_{[1.9L,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2})) \mathbf{1}_{[0,L]}(\ell(\alpha)) \\ \mathbf{1}_{[0,2L]}(\ell(\gamma_{1}) + \ell(\gamma_{2}) + 2\ell(\delta)) \frac{\ell(\gamma_{1})}{\mathcal{R}(\ell(\gamma_{1}),\ell(\gamma_{2}),L)}$$

where $(\alpha, \gamma_1, \gamma_2, \delta)$ runs over all ordered quadruples of simple closed geodesics such that $\gamma_1 \cup \gamma_2$ splits off an $S_{1,2}$ with complement $S_{k,1} \cup S_{g-k-1,1}$, α splits off a one-holed torus from that $S_{1,2}$, and δ is in that one-holed torus (see the second picture in Figure 9).

When

$$L < 1.9L < \ell(\gamma_1) + \ell(\gamma_2) \le 2L,$$

it follows by Lemma 27 that

$$\frac{\ell(\gamma_1)}{\mathcal{R}(\ell(\gamma_1), \ell(\gamma_2), L)} \le 500 + 500 \frac{\ell(\gamma_1)}{0.9L} < 2000.$$

So we have

$$Z_2^{*0}(X,L) \le 2000 \sum_{(\alpha,\gamma_1,\gamma_2,\delta)} \mathbf{1}_{[0,L]}(\ell(\alpha)) \mathbf{1}_{[1.9L,2L]}(\ell(\gamma_1) + \ell(\gamma_2) + 2\ell(\delta))$$

and

$$Z_2^{*k}(X,L) \le 2000 \sum_{(\alpha,\gamma_1,\gamma_2,\delta)} \mathbf{1}_{[0,L]}(\ell(\alpha)) \mathbf{1}_{[1.9L,2L]}(\ell(\gamma_1) + \ell(\gamma_2) + 2\ell(\delta)).$$

By Theorem 18 we know that

$$V_{0,3}(x, y, z) = 1.$$

By Lemma 22 we know that

$$V_{g-2,2}(x,y) \le \frac{\sinh(\frac{x}{2})\sinh(\frac{y}{2})}{\frac{x}{2}\frac{y}{2}}V_{g-2,2} \le \frac{e^{\frac{x+y}{2}}}{xy}V_{g-2,2}.$$

Then one may apply Mirzakhani's Integration Formula (see Theorem 12) to get

$$\int_{\mathcal{M}_{g}} Z_{2}^{*0}(X,L) dX \\
\leq \int_{0 \leq z \leq L} \int_{1.9L \leq x+y+2w \leq 2L; \, x,y,w \geq 0} 2000 \\
V_{0,3}(z,w,w) V_{0,3}(x,y,z) V_{g-2,2}(x,y) xy zw dx dy dz dw \\
\leq 2000 V_{g-2,2} \int_{0 \leq z \leq L} \int_{1.9L \leq x+y+2w \leq 2L; \, x,y,w \geq 0} e^{\frac{x+y}{2}} zw dx dy dz dw \\
\leq c V_{g-2,2} L^{3} e^{L}$$

for some universal constant c > 0.

Similarly, for all $1 \le k \le \frac{1}{2}(g-1)$ we have

$$\int_{\mathcal{M}_{g}} Z_{2}^{*k}(X,L) dX$$

$$\leq \int_{0 \leq z \leq L} \int_{1.9L \leq x+y+2w \leq 2L; \, x,y,w \geq 0} 2000$$

$$V_{0,3}(z,w,w) V_{0,3}(x,y,z) V_{k,1}(x) V_{g-k-1,1}(y) xy zw dx dy dz dw$$

$$\leq 2000 V_{k,1} V_{g-k-1,1} \int_{0 \leq z \leq L} \int_{1.9L \leq x+y+2w \leq 2L; \, x,y,w \geq 0} e^{\frac{x+y}{2}} zw dx dy dz dw$$

$$\leq c V_{k,1} V_{g-k-1,1} L^{3} e^{L}$$

for some universal constant c > 0.

And by Lemma 19 and 20 we know that

$$V_{g-2,2} = \frac{1}{(8\pi^2 g)^2} V_g (1 + O\left(\frac{1}{g}\right))$$

and

$$\sum_{1 \le k \le \frac{1}{2}(g-1)} V_{k,1} V_{g-k-1,1} = O\left(\frac{V_g}{g^3}\right).$$

Therefore we have

$$\mathbb{E}_{WP}^{g}[Z_{2}^{*}(X,L)] = \mathbb{E}_{WP}^{g}[Z_{2}^{*0}(X,L)] + \sum_{\substack{1 \le k \le \frac{1}{2}(g-1)}} \mathbb{E}_{WP}^{g}[Z_{2}^{*k}(X,L)]$$

$$\leq cL^{3}e^{L} \cdot \frac{V_{g-2,2} + \sum_{\substack{1 \le k \le \frac{1}{2}(g-1)}} V_{k,1}V_{g-k-1,1}}{V_{g}}$$

$$\leq cL^3e^L\frac{1}{g^2}$$

for some universal constant c > 0, as required.

Now are are ready to prove Proposition 48.

Proof of Proposition 48. Recall that $L = L(g) = 2 \log g - 4 \log \log g + \omega(g)$. Thus, it follows by Lemma 52 and 53 that there exists a universal constant c > 0 such that

$$\begin{split} \mathbb{E}^{g}_{\text{WP}}[Z^{*}(X,L)] &= \mathbb{E}^{g}_{\text{WP}}[Z^{*}_{1}(X,L)] + \mathbb{E}^{g}_{\text{WP}}[Z^{*}_{2}(X,L)] \\ &\leq c \left(L^{6}e^{0.95L} + L^{3}e^{L}\right) \frac{1}{g^{2}} \\ &\leq cL^{3}e^{L}\frac{1}{a^{2}}. \end{split}$$

For (C), by Lemma 29 we know that as $g \to \infty$,

$$\mathbb{E}^{g}_{\mathrm{WP}}[N_{1,1}(X,L)] \sim \frac{1}{384\pi^2} L^2 e^{\frac{L}{2}} \frac{1}{g}.$$

Thus, we have

$$\frac{\mathbb{E}_{\rm WP}^g[Z^*(X,L)]}{\mathbb{E}_{\rm WP}^g[N^*_{1,1}(X,L)]^2} = O(\frac{1}{L}) \to 0$$

as $g \to \infty$, which proves (C).

Now we finish the proof of Theorem 34.

Proof of Theorem 34. By the definition of $N_{1,1}(X, L)$ we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^{g} \left(X \in \mathcal{M}_{g} ; X \in \mathcal{A}(\omega(g)) \right) = \lim_{g \to \infty} \operatorname{Prob}_{WP}^{g} \left(N_{1,1}(X,L) \ge 1 \right)$$
$$= 1 - \lim_{g \to \infty} \operatorname{Prob}_{WP}^{g} \left(N_{1,1}(X,L) = 0 \right).$$

By Proposition 39, 45, 48 and Equation (16) we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(N_{1,1}^*(X,L) = 0 \right) = 0$$

Since $N_{1,1}^*(X, L) \le N_{1,1}(X, L)$, we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(N_{1,1}(X,L) = 0 \right) = 0,$$

as required.

8. HALF-COLLARS AND SEPARATING EXTREMAL LENGTH SYSTOLE In this section, we prove Theorem 2 and 3. 53

8.1. Half-collars. While a *collar* of a simple closed geodesic γ on a complete hyperbolic surface means an equidistant neighborhood U of γ homeomorphic to a cylinder, we will mainly consider a half of U cut out by γ , which we call a *half-collar*. More precisely:

Definition 54. Given l, w > 0, let $C_{l,w}$ denote the hyperbolic cylinder such that one of the two boundary components is a simple closed geodesic of length l, while every point on the other component has distance w from the geodesic component (see Figure 10). Given a hyperbolic surface X

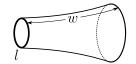


FIGURE 10. The cylinder $C_{l,w}$.

and a simple closed geodesic $\gamma \subset X$, a half-collar of width w around γ is a subsurface $C \subset X$ isometric to $C_{\ell_{\gamma}(X),w}$ such that γ is the geodesic boundary component of C.

The following result is standard:

Lemma 55. Let X be a compact hyperbolic surface of type $S_{g,1}$ with geodesic boundary $\gamma := \partial X \cong \mathbb{S}^1$. Given w > 0, the following conditions are equivalent:

- (i) there is no half-collar of width w around γ ;
- (ii) there is a simple geodesic arc $a \subset X$ of length $\leq 2w$ with endpoints in γ .

Proof. We first prove the implication "(ii) \Rightarrow (i)" by showing that if (i) fails then (ii) fails as well. So suppose there is a half-collar C of width w around γ and let a be any geodesic arc with endpoints in γ . Since a cannot be entirely contained in C (otherwise it would give rise to a geodesic bigon, which is impossible), there are disjoint sub-arcs $a_1, a_2 \subset a$, each of which joins a point of γ with a point in the non-geodesic boundary component of C. It follows that $\ell(a) > \ell(a_1) + \ell(a_2) \ge 2w$, hence (ii) fails. We have thus shown "(ii) \Rightarrow (i)".

As for the implication "(i) \Rightarrow (ii)", consider the ϵ -neighborhood $U_{\epsilon} := \{x \in X ; d(x, \gamma) < \epsilon\}$ of γ in X. When ϵ is small enough, the closure \overline{U}_{ϵ} is homeomorphic to a cylinder with boundary, hence is a half-collar of width ϵ . As ϵ grows larger, there is a critical value ϵ_0 such that $\overline{U}_{\epsilon_0}$ stops to be a cylinder for the first time, which is characterized by the existence of a point $x_0 \in X$ with $d(x, \gamma) = \epsilon_0$ such that $\overline{U}_{\epsilon_0}$ touches itself at x_0 . One can then draw two geodesic segments of length ϵ_0 from x_0 to γ which fit together to form a simple geodesic arc of length $2\epsilon_0$.

Now, Condition (i) just means $\epsilon_0 \leq w$. In this case, the arc that we just constructed implies that (ii) holds. This shows "(i) \Rightarrow (ii)".

We can now prove Theorem 2 by using Theorem 1 and Proposition 9.

Theorem 56 (=Theorem 2). Given $\epsilon > 0$, consider the following conditions defined for all $X \in \mathcal{M}_q$:

- (c). There is a half-collar around γ in the $S_{g-1,1}$ -part of X with width $\frac{1}{2}\log g (\frac{3}{2} + \epsilon)\log\log g;$
- (d). $\tilde{\ell}_{sys}^{sep}(X)$ is achieved by a simple closed geodesic γ separating X into $S_{1,1} \cup S_{g-1,1}$;

Then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g ; X \text{ satisfies } (c) \text{ and } (d) \right) = 1.$$

Proof. Fix the function $\omega(g)$ as in Theorem 1 and let \mathcal{A}_g and \mathcal{B}_g denote the subsets of \mathcal{M}_g as follow.

$$\mathcal{A}_g := \left\{ \begin{aligned} &|\ell_{\rm sys}^{\rm sep}(X) - (2\log g - 4\log\log g)| \le \omega(g) \\ &X \in \mathcal{M}_g \; ; \; \text{and} \; \ell_{\rm sys}^{\rm sep}(X) \; \text{is achieved only by simple} \\ & \text{closed geodesics bounding one-holed torus} \end{aligned} \right\}$$

$$\mathcal{B}_g := \{ X \in \mathcal{M}_g \, ; \, \mathcal{L}_{1,2}(X) > 4 \log g - 10 \log \log g - \omega(g) \}.$$

Fix $\epsilon > 0$ and suppose $g \ge 3$ satisfies

(31)
$$\omega(g) < 2\epsilon \log \log g.$$

We claim that every $X \in \mathcal{A}_g \cap \mathcal{B}_g$ satisfies the condition stated in Theorem 2. That is, for any simple closed geodesic γ achieving $\ell_{\text{sys}}^{\text{sep}}(X)$, which separates X into $S_{1,1} \cup S_{g-1,1}$ because $X \in \mathcal{A}_g$, there is a half-collar around γ in the $S_{q-1,1}$ -part of X with width $\frac{1}{2} \log g - (\frac{3}{2} + \epsilon) \log \log g$.

Suppose by contradiction that the claim is false. Then by Lemma 55, there exists an $X \in \mathcal{A}_g \cap \mathcal{B}_g$, a simple closed geodesic $\gamma \subset X$ achieving $\ell_{\text{sys}}^{\text{sep}}(X)$, and a simple geodesic arc *a* in the $S_{g-1,1}$ -part of X with endpoints on γ , such that

$$\ell(a) \le \log g - (3+2\epsilon) \log \log g$$

In this situation, there are simple closed geodesics γ_1 and γ_2 homotopic to the two closed piecewise geodesics formed by a and the two arcs of γ split out by a, respectively (see Figure 11), such that γ_1 , γ_2 and γ together bound

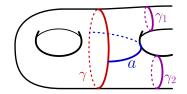


FIGURE 11. From an arc to a pair-of-pants

a pair of pants outside of the one-holed torus X_{γ} .

Since each of γ_1 and γ_2 is shorter than the corresponding closed piecewise geodesic, we have

(32)
$$\ell(\gamma_1) + \ell(\gamma_2) \le \ell(\gamma) + 2\ell(a) = \ell_{\text{sys}}^{\text{sep}}(X) + 2\ell(a)$$
$$\le 2\log g - 4\log\log g + \omega(g) + 2(\log g - (3+2\epsilon)\log\log g)$$
$$= 4\log g - (10+4\epsilon)\log\log g + \omega(g).$$

But on the other hand, by definition of $\mathcal{L}_{1,2}(X)$ (see the definition in the Introduction) and the assumption $X \in \mathcal{B}_g$, we have

$$\ell(\gamma_1) + \ell(\gamma_2) \ge \mathcal{L}_{1,2}(X) > 4\log g - 10\log\log g - \omega(g).$$

This leads to a contradiction because by (31), the lower bound of $\ell(\gamma_1) + \ell(\gamma_2)$ here is greater than the upper bound in (32). We have thus shown the claim.

As $g \to \infty$, since $\operatorname{Prob}_{WP}^g(\mathcal{A}_g)$ and $\operatorname{Prob}_{WP}^g(\mathcal{B}_g)$ both tend to 1 by Theorem 1 and Proposition 9, we have $\operatorname{Prob}_{WP}^g(\mathcal{A}_g \cap \mathcal{B}_g) \to 1$ as well. In view of the above claim, this implies the required statement. \Box

8.2. Extremal length. Given a Riemann surface U and a set Γ of rectifiable curves on U, the *extremal length* $\text{Ext}_{\Gamma}(U)$ of Γ is defined as (*e.g.* see [Ahl10, Chapter 4] and [Ker80, Section 3])

$$\operatorname{Ext}_{\Gamma}(\mathbf{U}) := \sup_{\sigma} \frac{\inf_{\alpha \in \Gamma} \ell_{\sigma}(\alpha)^2}{A_{\sigma}(\mathbf{U})},$$

where the supremum is over all Borel-measurable conformal metrics σ on U, and $\ell_{\sigma}(\alpha)$ and $A_{\sigma}(U)$ denote the length of α and the area of U under σ , respectively. In particular, given a closed hyperbolic surface $X \in \mathcal{M}_g$ and a simple closed geodesic $\gamma \subset X$, we denote

$$\operatorname{Ext}_{\gamma}(\mathbf{X}) := \operatorname{Ext}_{\Gamma_{\gamma}}(\mathbf{X})$$

for the set Γ_{γ} of all rectifiable closed curves on X homotopic to γ . We then define the *separating extremal length systole* $\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X)$ of X as

$$\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X) := \inf_{\gamma} \operatorname{Ext}_{\gamma}(X)$$

where the infimum is over all separating simple closed geodesics on X.

Maskit [Mas85] established some basic relations between the extremal length $\text{Ext}_{\gamma}(X)$ and the hyperbolic length $\ell_{\gamma}(X)$. The following lemma is a reformulation of [Mas85, Prop. 1]:

Lemma 57. Let $X \in \mathcal{M}_g$. For any simple closed geodesic $\gamma \subset X$, we have $\ell_{\gamma}(X) \leq \pi \operatorname{Ext}_{\gamma}(X)$.

Conversely, if there exists a half-collar around γ with width w, then

$$\ell_{\gamma}(X) \geq 2(\arctan(e^w) - \frac{\pi}{4}) \operatorname{Ext}_{\gamma}(X).$$

Proof. The first inequality is exactly Inequality (2) in [Mas85, Prop. 1]. On the other hand, Inequality (1) in [Mas85, Prop. 1] implies that if we identify the universal cover of X with the upper half-plane \mathbb{H}^2 in such a way that γ lifts to $i\mathbb{R}_+$, and assume that γ has a half-collar C which lifts to

$$\left\{z \ ; \ \frac{\pi}{2} - \theta \le \arg(z) \le \frac{\pi}{2}\right\}$$

for some $\theta \in (0, \frac{\pi}{2})$, then $\ell_{\gamma}(X) \geq \theta \operatorname{Ext}_{\gamma}(X)$. By an elementary hyperbolicgeometric calculation, the width w of C is related to θ by $\cosh(w) = \frac{1}{\cos\theta}$, which is equivalent to $2\left(\arctan(e^w) - \frac{\pi}{4}\right) = \theta$ (this can be seen by using the trigonometric identity $\tan(\phi) + \cot(\phi) = 2\csc(2\phi)$). The second required inequality follows.

We can now deduce Theorem 3 from Theorem 1.

Theorem 58 (=Theorem 3). Given any $\epsilon > 0$, then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \frac{\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X)}{\ell_{\operatorname{sys}}^{\operatorname{sep}}(X)} < \frac{2+\epsilon}{\pi} \right) = 1$$

As a consequence of Theorem 1, we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, \frac{(2-\epsilon)}{\pi} \log g < \operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X) < \frac{(4+\epsilon)}{\pi} \log g \right) = 1.$$

Proof. Let \mathcal{A}_g denote the subset of \mathcal{M}_g consisting of those $X \in \mathcal{M}_g$ satisfying the conditions in Theorem 1 and 2. The sequence (\mathcal{A}_g) has the property that given any w > 0, every $X \in \mathcal{A}_g$ with g large enough contains a half-collar of width w around some separating simple closed geodesic γ with $\ell_{\gamma}(X) = \ell_{\text{sys}}^{\text{sep}}(X)$.

Now fix $\epsilon > 0$ and let $w_{\epsilon} > 0$ be large enough such that

$$\frac{1}{2(\arctan(e^{w_{\epsilon}}) - \frac{\pi}{4})} \le \frac{2+\epsilon}{\pi}$$

For every $X \in \mathcal{A}_g$ with g large enough, letting $\gamma \subset X$ be the separating simple closed geodesic described in the property above, which achieves $\ell_{\text{sys}}^{\text{sep}}(X)$ and has a half-collar of width w_{ϵ} . By Lemma 57, we have

$$\frac{\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X)}{\ell_{\operatorname{sys}}^{\operatorname{sep}}(X)} = \frac{\operatorname{Ext}_{\operatorname{sys}}^{\operatorname{sep}}(X)}{\ell_{\gamma}(X)} \\
\leq \frac{\operatorname{Ext}_{\gamma}(X)}{\ell_{\gamma}(X)} \\
\leq \frac{1}{2(\operatorname{arctan}(e^{w_{\epsilon}}) - \frac{\pi}{4})} \\
\leq \frac{2 + \epsilon}{\pi}.$$

Therefore, the first statement of the theorem follows from Theorem 1. The second statement is then a consequence of the first statement, the fact that

$$\ell_{\rm sys}^{\rm sep}(X) \le \pi {\rm Ext}_{\rm sys}^{\rm sep}(X)$$

which follows from Lemma 57, and the fact that

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (2 - \epsilon) \log g < \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) < (2 + \epsilon) \log g \right) = 0,$$

which follows from Theorem 1. The proof is now complete.

9. Non-simple systole and expected value of \mathcal{L}_1

In this section consider the non-simple systole and the expected value of \mathcal{L}_1 over \mathcal{M}_g for large genus. First we provide the following elementary property needed in the proofs of Theorem 4 and 7.

Proposition 59. Let $X \in \mathcal{M}_g$ and $\gamma \subset X$ be a self-intersecting closed geodesic. Then there exists a connected subsurface $X(\gamma)$ of X such that

- (1) $\gamma \subset X(\gamma);$
- (2) the boundary $\partial X(\gamma)$ is a simple closed multi-geodesic with

$$\ell\left(\partial X(\gamma)\right) \le 2\ell(\gamma);$$

(3) Area $(X(\gamma)) \leq 3\ell(\gamma)$.

Proof. The construction for $X(\gamma)$ is the same as the one for X_3 at the beginning of Section 3, only with the role of $X_1 \cup X_2$ replaced by γ . Namely, we get $X(\gamma)$ by deforming the boundary components of γ (or more precisely, boundary components of an ε -neighborhood of γ for small enough ε) into simple closed geodesics in the way described there. Property (2) is clear from the construction.

By construction γ (or its small ε -neighborhood) is freely homotopic to a subset of $X(\gamma)$. Since $X(\gamma)$ is a surface with geodesic boundary, the unique closed geodesic representing this free homotopy class should be contained in $X(\gamma)$. So property (1) holds.

For (3), we write

$$X(\gamma) \setminus \gamma = (\sqcup D_i) \sqcup (\sqcup C_j)$$

where the D_i 's and the C_j 's are topological discs and cylinders, respectively, all disjoint from each other. By the classical Isoperimetric Inequality (*e.g.* see [Bus10, WX22]), we know that

Area
$$(D_i) \leq \ell(\partial D_i)$$
 and Area $(C_i) \leq \ell(\partial C_i)$.

Thus, we have

$$\begin{aligned} \operatorname{Area}(X(\gamma)) &= \operatorname{Area}(X(\gamma) \setminus \gamma) = \sum_{i} \operatorname{Area}(D_{i}) + \sum_{j} \operatorname{Area}(C_{j}) \\ &\leq \sum_{i} \ell(\partial D_{i}) + \sum_{j} \ell(\partial C_{j}) \\ &\leq \ell(\partial X(\gamma)) + \ell(\gamma) \\ &\leq 3\ell_{\gamma}(X), \end{aligned}$$

as required.

Remark. The multi-geodesic $\partial X(\gamma)$ is empty if γ is filling in X.

Now we are ready to prove Theorem 4.

Theorem 60 (=Theorem 4). Given any $\epsilon > 0$, then we have

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (1 - \epsilon) \log g < \ell_{\operatorname{sys}}^{\operatorname{ns}}(X) < 2 \log g \right) = 1.$$

Proof. Fix the function $\omega(g)$ as in Theorem 1 and let \mathcal{C}_g and \mathcal{D}_g denote the subsets of \mathcal{M}_g as follow.

$$\mathcal{C}_g := \begin{cases} |\ell_{\text{sys}}^{\text{sep}}(X) - (2\log g - 4\log\log g)| \le \omega(g) \\ X \in \mathcal{M}_g; \text{ and } \ell_{\text{sys}}^{\text{sep}}(X) \text{ is achieved only by simple} \\ \text{closed geodesics bounding one-holed torus} \end{cases}$$

$$\mathcal{D}_g := \{ X \in \mathcal{M}_g \, ; \, \mathcal{L}_1(X) \ge 2 \log g - 4 \log \log g - \omega(g) \}.$$

By Theorem 1 and 6 we know that

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{C}_g \cap \mathcal{D}_g \right) = 1.$$

So it suffices to show that given any $\epsilon > 0$, as $g \to \infty$, for any $X \in \mathcal{C}_g \cap \mathcal{D}_g$ we have

$$(1-\epsilon)\log g < \ell_{\rm sys}^{\rm ns}(X) < 2\log g$$

We first show the upper bound: $\ell_{\text{sys}}^{\text{ns}}(X) < 2 \log g$. Since $X \in C_g$, one may let $\gamma \subset X$ be a shortest simple separating closed geodesic bounding a one-holed torus $X_{1,1}$. Let $w(\gamma) > 0$ be the maximal value such that the half collar $C(w(\gamma))$ is embedded in $X_{1,1}$ where

$$C(w(\gamma)) := \{ z \in X_{1,1} ; \operatorname{dist}(z,\gamma) < w(\gamma) \}.$$

It is known that the hyperbolic metric ds^2 on $C(w(\gamma))$ satisfies that

$$ds^{2} = d\rho^{2} + \left(\ell_{\text{sys}}^{\text{sep}}(X)\cosh\rho\right)^{2} dt^{2}$$

where $\rho \in [0, w(\gamma))$ is the distance to γ and $t \in [0, 1]$. Since $\operatorname{Area}(X_{1,1}) = 2\pi$ and $C(w(\gamma))$ is embedded in $X_{1,1}$, we have

$$\operatorname{Area}(C(w(\gamma))) = \ell_{\operatorname{sys}}^{\operatorname{sep}}(X) \sinh(w(\gamma)) < 2\pi.$$

By the choice of $w(\gamma)$ we know that one component α of the boundary of the half collar $C(w(\gamma))$ is a non-simple closed curve. By continuity we know that the length $\ell(\alpha)$ satisfies that

$$\ell(\alpha) = \ell_{\text{sys}}^{\text{sep}}(X) \cosh(w(\gamma))$$

$$= \sqrt{\ell_{\text{sys}}^{\text{sep}}(X)^2 + \ell_{\text{sys}}^{\text{sep}}(X)^2 \sinh^2(w(\gamma))}$$

$$< \sqrt{\ell_{\text{sys}}^{\text{sep}}(X)^2 + (2\pi)^2}$$

$$< 2 \log g$$

for large g because $X \in C_g$. Then the unique closed geodesic in $X_{1,1}$ representing α gives the upper bound.

Now we show the lower bound: $(1 - \epsilon) \log g < \ell_{\text{sys}}^{\text{ns}}(X)$. First by above one may assume that $\gamma' \subset X$ is a shortest non-simple closed geodesic with $\ell_{\gamma'}(X) = \ell_{\text{sys}}^{\text{ns}}(X) < 2 \log g$ for large g. Then by Proposition 59 there exists a subsurface $X(\gamma')$ of X such that

$$\ell(\partial(X(\gamma'))) \le 2\ell_{\rm sys}^{\rm ns}(X) < 4\log g \text{ and } \operatorname{Area}(X(\gamma')) \le 3\ell_{\rm sys}^{\rm ns}(X) < 6\log g.$$

Recall that by Gauss-Bonnet we know that $\operatorname{Area}(X) = 4\pi(g-1)$. So for large g, we have that $X(\gamma')$ is a proper subsurface of X. Clearly the boundary $\partial X(\gamma')$ consists of multi simple closed geodesics which separate X. Hence, we have

$$\mathcal{L}_1(X) \le \ell(\partial(X(\gamma'))) \le 2\ell_{\rm sys}^{\rm ns}(X).$$

Then the lower bound follows because $X \in \mathcal{D}_g$.

Before proving Theorem 7, unlike the unboundness of $\ell_{\rm sys}^{\rm sep}$ we first show that

Proposition 61. There exists a universal constant C > 0 independent of g such that

$$\sup_{X \in \mathcal{M}_g} \mathcal{L}_1(X) \le C \log g.$$

Proof. Since each $X_g \in \mathcal{M}_g$ admits a pants decomposition such that each geodesic has length no more than the Bers' constant depending on g (see *e.g.* [Bus10, Theorem 5.1.2]), $\sup_{X \in \mathcal{M}_g} \mathcal{L}_1(X) < \infty$ has a upper bound only depending on g. So it suffices to consider the case that g is large. For any

 $X \in \mathcal{M}_g$, by [Sab08, Theorem 1.3] of Sabourau we know that there exists a separating closed geodesic $\gamma \subset X$ which may not be simple such that

$$\ell_{\gamma}(X) \le C' \log g$$

for some universal constant C' > 0. If γ is simple, we are done. Now we assume that γ is non-simple, by Proposition 59 there exists a subsurface $X(\gamma)$ of X such that

$$\ell(\partial X(\gamma)) \le 2C' \log g$$
 and $\operatorname{Area}(X(\gamma)) \le 3C' \log g$.

Recall that by Gauss-Bonnet we know that $\operatorname{Area}(X) = 4\pi(g-1)$. So for large g, we have that $X(\gamma)$ is a proper subsurface of X. Clearly the boundary $\partial X(\gamma)$ consists of multi simple closed geodesics which separate X. Hence, we have

$$\mathcal{L}_1(X) \le \ell(\partial X(\gamma)) \le 2C' \log g$$

which completes the proof by setting C = 2C' > 0.

Remark. It is known by work of Buser-Sarnak [BS94] that for all $g \geq 2$, there exists a hyperbolic surface $X_g \in \mathcal{M}_g$ such that $\ell_{sys}(X_g) \geq K \log g$ for some uniform constant K > 0 independent of g (one may also see [BMP14] for more details). This together with the proposition above imply that

$$\sup_{X\in\mathcal{M}_g}\mathcal{L}_1(X)\asymp\log g.$$

Now we are ready to prove Theorem 7.

Theorem 62 (=Theorem 7). The expected value $\mathbb{E}^{g}_{WP}[\mathcal{L}_{1}]$ of \mathcal{L}_{1} on \mathcal{M}_{g} satisfies

$$\lim_{g \to \infty} \frac{\mathbb{E}_{WP}^g[\mathcal{L}_1]}{\log g} = 2$$

Proof. First we set

$$\mathcal{B}(\omega(g)) = \{ X \in \mathcal{M}_g; |\mathcal{L}_1(X) - (2\log g - 4\log\log g)| \le \omega(g) \}.$$

By Theorem 6 we know that

$$\lim_{g \to \infty} \frac{\operatorname{Vol}(\mathcal{B}(\omega(g)))}{V_g} = 1.$$

For the lower bound, we have

$$\begin{split} \frac{\mathbb{E}_{\mathrm{WP}}^{g}[\mathcal{L}_{1}]}{\log g} &\geq \frac{1}{V_{g}} \int_{\mathcal{B}(\omega(g))} \frac{\mathcal{L}_{1}(X)}{\log g} dX \\ &\geq \frac{2\log g - 4\log\log g - \omega(g)}{\log g} \cdot \frac{\mathrm{Vol}(\mathcal{B}(\omega(g)))}{V_{g}} \end{split}$$

which implies that

$$\liminf_{g \to \infty} \frac{\mathbb{E}_{\mathrm{WP}}^g[\mathcal{L}_1]}{\log g} \ge 2.$$

For the upper bound, it follows by Proposition 61 that

$$\frac{\mathbb{E}_{\mathrm{WP}}^{g}[\mathcal{L}_{1}]}{\log g} = \frac{1}{V_{g}} \int_{\mathcal{B}(\omega(g))} \frac{\mathcal{L}_{1}(X)}{\log g} dX + \frac{1}{V_{g}} \int_{\mathcal{M}_{g} \setminus \mathcal{B}(\omega(g))} \frac{\mathcal{L}_{1}(X)}{\log g} dX$$

$$\leq \frac{2\log g - 4\log\log g + \omega(g)}{\log g} \cdot \frac{\operatorname{Vol}(\mathcal{B}(\omega(g)))}{V_g} + C \cdot \frac{\operatorname{Vol}(\mathcal{M}_g \setminus \mathcal{B}(\omega(g)))}{V_g}$$

Letting $g \to \infty$, we get

$$\limsup_{g \to \infty} \frac{\mathbb{E}_{\mathrm{WP}}^g[\mathcal{L}_1]}{\log g} \le 2$$

as required.

10. Further questions

In this last section we propose several questions related to the results in this article.

10.1. Shortest separating simple closed multi-geodesics. By Theorem 1 we know that on a generic $X \in \mathcal{M}_g$, a separating systolic closed geodesic of X separates X into $S_{1,1} \cup S_{g-1,1}$. However, by Theorem 6 we only know that on a generic $X \in \mathcal{M}_g$, the shortest separating simple closed multi-geodesics of X separates X into either $S_{1,1} \cup S_{g-1,1}$ or $S_{0,3} \cup S_{g-2,3}$. A natural question is to determine the weights of these two cases. Or more precisely,

Question 63. On a generic $X \in \mathcal{M}_g$, is $\mathcal{L}_1(X)$ achieved by a separating systole as $g \to \infty$?

10.2. Expectation of $\ell_{\text{sys}}^{\text{sep}}$. Theorem 7 tells that as $g \to \infty$, the expected value $\mathbb{E}_{\text{WP}}^g[\mathcal{L}_1]$ behaves like $2 \log g$. The two ingredients in the proof are Theorem 6 and Proposition 61, the latter one says that $\sup_{X \in \mathcal{M}_g} \mathcal{L}_1(X) \leq C \log g$ for some universal constant C > 0. For $\ell_{\text{sys}}^{\text{sep}}$, although we still have the first ingredient, namely Theorem 1, the second is missing because it is known that $\sup_{X \in \mathcal{M}_g} \ell_{\text{sys}}^{\text{sep}}(X) = \infty$. So we raise the following question:

Question 64. Does the following limit hold:

$$\lim_{g \to \infty} \frac{\mathbb{E}_{WP}^g[\ell_{sys}^{sep}]}{\log g} = 2?$$

Remark. Very recently joint with H. Parlier, the second and third named authors of this article in [PWX21] give an affirmative answer to Question 64 above.

10.3. Geometric Cheeger constants. Recall as in the Introduction, for all $1 \leq m \leq g-1$ the *m*-th geometric Cheeger constant $H_m(X)$ of X is defined as

$$H_m(X) := \inf_{\gamma} \frac{\ell_{\gamma}(X)}{2\pi m}$$

where γ is a simple closed multi-geodesics on X with $X \setminus \gamma = X_1 \cup X_2$, and X_1 and X_2 are connected subsurfaces of X such that $|\chi(X_1)| = m \leq |\chi(X_2)|$. As a direct consequence of Theorem 6, the first geometric Cheeger constant H_1 on \mathcal{M}_q asymptotically behaves as

$$\lim_{g \to \infty} \operatorname{Prob}_{WP}^g \left(X \in \mathcal{M}_g \, ; \, (1 - \epsilon) \cdot \frac{\log g}{\pi} < H_1(X) < \frac{\log g}{\pi} \right) = 1$$

for any $\epsilon > 0$. A natural question is to study general H_m .

Question 65. For $m \in [1, g-1]$, what is the asymptotic behavior of H_m on \mathcal{M}_q as $g \to \infty$?

This question is related to [Wri20, Problem 10.5] of Wright on the asymptotic behavior of the classical Cheeger constant h(X) of X (which is recently solved by Budzinski-Curien-Petri [BCP22]), because H(X) := $\min_{1 \le m \le g-1} H_m(X)$ serves as a natural upper bound for h(X). For fixed m > 0 independent of g, the question above may be reduced to the following explicit one:

Question 66. Let $\omega(g)$ be a function as (1) and m > 0 be fixed. Then does the following limit hold: as $g \to \infty$,

 $\operatorname{Prob}_{\operatorname{WP}}^{g} \left(X \in \mathcal{M}_{g} \, ; \, |\mathcal{L}_{1,m}(X) - (2m \log g - (6m - 2) \log \log g)| \le \omega(g) \right) \to 1?$

Theorem 6 answers Question 65 and 66 for m = 1, By Proposition 9 it suffices to study the upper bound.

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