Large induced forests in sparse graphs

Noga Alon*, Dhruv Mubayi†, Robin Thomas‡

February 22, 2002

Abstract

For a graph G, let $a(G)$ denote the maximum size of a subset of vertices that induces a forest. Suppose that G is connected with n vertices, e edges, and maximum degree Δ. Our results include:
(a) if $\Delta \leq 3$, and $G \neq K_4$, then $a(G) \geq n - e/4 - 1/4$ and this is sharp for all permissible $e \equiv 3 \pmod 4$,
(b) if $\Delta \geq 3$, then $a(G) \geq \alpha(G) + (n - \alpha(G))/(\Delta - 1)^2$.

Several problems remain open.

1 Introduction

For a (simple, undirected) graph $G = (V, E)$, we say that an $S \subseteq V$ is an *acyclic set* if the induced subgraph $G[S]$ is a forest. We let $a(G)$ denote the maximum size of an acyclic set in G. In [4], the minimum possible value of $a(G)$ is determined, where G ranges over all

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel, *email: noga@math.tau.ac.il.* Research supported in part by a USA Israeli BSF grant.

†School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, *email: mubayi@math.gatech.edu.* Research supported in part by NSF under Grant No. DMS–9970325.

‡School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, *email: thomas@math.gatech.edu.* Research supported in part by NSF under Grant No. DMS-9970514, and by NSA under Contract No. MDA904-98-1-0517.

1991 Mathematics Subject Classification: 05C35

Keywords: *Induced forests, d-degenerate graph*

graphs on n vertices and e edges, for every n and e. In particular, the results imply that if the average degree of G is at most $d \geq 2$, then $a(G) \geq \frac{2n}{d+1}$. This is sharp whenever $d+1$ divides n as shown by a disjoint union of cliques of order $d+1$. For bipartite graphs, one can do better, since trivially $a(G) \geq n/2$. Recently, using probabilistic techniques, the first author has shown that this trivial bound can be improved, but only slightly.

Theorem 1.1. [3] *There exists an absolute positive constant b such that for every bipartite graph G with n vertices and average degree at most d, where $d \geq 1$,*

$$a(G) \geq \left(\frac{1}{2} + \frac{1}{e^{bd^2}}\right) n.$$

Moreover, there exists an absolute constant $b' > 0$ such that for every $d \geq 1$ and all sufficiently large n there exists a bipartite graph with n vertices and average degree at most d such that

$$a(G) \leq \left(\frac{1}{2} + \frac{1}{e^{b'\sqrt{d}}}\right) n.$$

Theorem 1.1 was motivated by the following conjecture of Albertson and Haas [2], which remains open.

Conjecture 1.2. *If G is an n vertex planar bipartite graph, then $a(G) \geq 5n/8$.*

Conjecture 1.2, if true, is sharp as shown by the following example.

Example 1.3. The cube Q_3 is the graph with $V(Q_3) = \{v_1, v_1', \ldots, v_4, v_4'\}$, and edges $v_i v_{i+1}, v_i' v_{i+1}', v_i v_i'$, where $1 \leq i \leq 4$ and subscripts are taken modulo 4. It is easy to see that $a(Q_3) = 5$.

In this paper, we prove results that refine Theorem 1.1 for sparse bipartite graphs, and also apply to the larger class of triangle-free graphs. We also obtain bounds for $a(G)$ in terms of the *independence number* $\alpha(G)$ of G.

Given a graph G, let $N_G(v)$ or simply $N(v)$ denote the set of neighbors of vertex v. For sets S, A of vertices, $N(S) = \bigcup_{v \in S} N(v)$ and $N_A(S) = N(S) \cap A$. Let $\dot{K}_4$ denote the five vertex graph obtained from K_4 by subdividing an edge.

Definition 1.4. *Let $\mathcal{F}(t,k)$ denote the family of connected graphs with maximum degree 3 consisting of t disjoint triangles and k disjoint copies of $\dot{K}_4$ such that the multigraph obtained by contracting each triangle and each copy of $\dot{K}_4$ to a single vertex is a tree of order $t+k$. Notice that if H_1 and H_2 are copies of K_3 or $\dot{K}_4$, then G has at most one edge between H_1 and H_2. Let $\mathcal{F} = \bigcup_{t,k} \mathcal{F}(t,k)$, where the union is taken over all nonnegative t, k with $t + k > 0$. (See the figure for an example of a graph in $\mathcal{F}(2,3)$.)*

Figure: A graph in $\mathcal{F}(2,3)$

Theorem 1.5. *Let $G = (V, E)$ be a graph with maximum degree 3 and $K_4 \not\subseteq G$. If exactly c components of G are from $\mathcal{F}$, then*

$$a(G) \geq |V| - \frac{|E|}{4} - \frac{c}{4}.$$

A graph $G \in \mathcal{F}(t, k)$ has $n = 3t + 5k$ vertices, $e = 3t + 7k + (t + k - 1) = 4t + 8k - 1$ edges, and every acyclic set in G has size at most $2t + 3k$. Thus $a(G) \leq 2t + 3k = n - e/4 - 1/4$ and hence Theorem 1.5 is sharp for every member of $\mathcal{F}$. Since every element in $\mathcal{F}$ contains triangles, Theorem 1.5 and Example 1.3 immediately yield

Corollary 1.6. *If G is an n vertex triangle-free graph with maximum degree 3, then $a(G) \geq 5n/8$ and this is sharp whenever n is divisible by 8.*

As mentioned in the introduction, n vertex graphs with maximum degree Δ always have an acyclic set of size at least $2n/(\Delta + 1)$. We observe that for triangle-free graphs the factor $2/(\Delta + 1)$ above can be improved to $\Theta(\log \Delta/\Delta)$.

For bipartite graphs, we obtain better bounds through the following result that relates $a(G)$ to the independence number $\alpha(G)$ of G.

Theorem 1.7. *Let G be a connected n vertex graph with maximum degree $\Delta \geq 3$. Then*

$$a(G) \geq \alpha(G) + \frac{n - \alpha(G)}{(\Delta - 1)^2}.$$

In section 2 we present a preliminary result to Theorem 1.5 that applies to triangle-free graphs, and also exhibit some examples with no large acyclic sets. In section 3 we present the proof of Theorem 1.5, in section 4 we prove Theorem 1.7, and in section 5 we summarize our results.

A *cycle* of length k or k-*cycle* is the graph with vertices $v_1, \ldots, v_k$ and edges $v_i v_{i+1}$, for $1 \leq i \leq k$, where indices are taken modulo k. We simply write $v_1 v_2 \ldots v_k$ to denote a k-cycle.

2 Triangle-free graphs

In this section we prove a special case of Theorem 1.5 that is independently interesting.

Lemma 2.1. *If G is a triangle-free graph with n vertices and e edges, then $a(G) \geq n - e/4$.*

Proof. We suppose that G is a minimal counterexample with respect to the number of vertices, and will obtain a contradiction. If G is not connected, then by minimality, we can apply the result to each component. Hence we may assume that G is connected. If G has a vertex v with $deg(v) \geq 4$ or $deg(v) = 1$, then let $G' = G - v$. Now G' has a large acyclic set $S' \subseteq V(G')$. In the first case, set $S = S'$, and in the second case, set $S = S' \cup \{v\}$. Then S is an acyclic set in G of size at least $n - e/4$, a contradiction. If G is 2-regular, then G is a cycle and $a(G) = n - 1 \geq n - e/4$. If uv is an edge, and $deg(u) = 2, deg(v) = 3$, then let $G' = G - u - v$. By minimality, there is a large acyclic set $S' \subseteq V(G')$; we let $S = S' \cup \{u\}$. Then $|S| \geq (n-2) - (e-4)/4 + 1 = n - e/4$. Hence we may assume that G is 3-regular.

Claim: For every pair $uv, uv' \in E(G)$, there exists a vertex w such that $uvwv'$ is a 4-cycle.
Proof of Claim: Let u' be the other neighbor of u, and let $G_1 = G - u - u' \cup \{vv'\}$. If G_1 is triangle-free, then by minimality of G we obtain an acyclic set $S_1 \subseteq V(G_1)$ of size at least $n - 2 - (e-4)/4$. Then $S = S_1 \cup \{u\}$ has size at least $n - e/4$. Furthermore, S is acyclic, since any cycle in $G[S]$ containing u must traverse the vertices v, u, v' in this order, and this would yield a cycle in $G_1[S_1]$ (with the edges vu, uv', replaced by vv'). This contradiction implies that G_1 contains a triangle of the form vwv'. □

Consider a vertex w in G with neighbors x, y, z. If x, y, z have another common neighbor w', then let $G_2 = G - \{w, w', x, y, z\}$. By minimality, G_2 has an acyclic set S_2 of size at least $n - 5 - (e-9)/4$. The set $S = S_2 \cup \{w, w', x\}$ in G is acyclic and has size at least $n - e/4$, a contradiction. Hence by the claim we may assume that there exist a, b, c, with $a \leftrightarrow \{x, y\}$, $b \leftrightarrow \{y, z\}$, and $c \leftrightarrow \{x, z\}$. Let $G_3 = G - \{w, x, y, z, a, b, c\}$. By minimality, G_3 has an acyclic set S_3 of size at least $n - 7 - (e-12)/4$. The set $S = S_3 \cup \{w, x, y, z\}$ in G is acyclic and has size at least $n - e/4$, a contradiction. □

As mentioned earlier, Lemma 2.1 is sharp for $e \leq 3n/2$ and $e \equiv 0 \pmod{12}$, as shown by disjoint copies of Q_3. For 4-regular graphs it gives $a(G) \geq n/2$, but the best example we can find has $a(G) = 4n/7$. A *vertex expansion* in a graph G is the replacement of a vertex $v \in V(G)$ by an independent set Q of new vertices, such that the neighborhood of each vertex of Q is $N_G(v)$.

Example 2.2. Let $G = (V, E)$ be the graph obtained from the 7-cycle $v_1 \dots v_7$ by expanding each vertex to an independent set of size 2. Thus G is 4-regular with $|V| = 14$ and $|E| = 28$. For $1 \le i \le 7$, let $V_i = \{x_i, y_i\}$ be the independent set obtained by expanding v_i. Suppose that S is an acyclic set in V, and let $S_i = S \cap V_i$. The crucial observation is that if $|S_i| = 2$, then $|S_{i-1}| + |S_{i+1}| \le 1$, where subscripts are taken modulo 7. If exactly three of the S_i's have size two, then at least two other S_j's must have size zero, giving $|S| \le 8$. If exactly two of the S_i's have size two, then at at least one other S_j has size zero, giving $|S| \le 8$ again. Thus $a(G) \le (4/7)|V|$, and in fact it is easy to see that equality holds. □

For 5-regular graphs, Lemma 2.1 gives $a(G) \ge 3n/8$, but the best example we can find has $a(G) = n/2$.

Example 2.3. Let $G = (V, E)$ be the graph with $V = \{1, \dots, 14\}$ and all edges ij where $j - i = 1, 4, 7, 10, 13 \pmod{14}$. Thus $|V| = 14$ and G is triangle-free and 5-regular. It can be shown through a tedious case analysis (which we omit here) that every acyclic set S in V has size at most seven, thus giving $a(G) \le |V|/2$. Since $\{1, 2, 4, 5, 7, 10, 13\}$ is acyclic, $a(G) = |V|/2$. □

Remark 2.4. It is well-known (see [6, 5]) that there are triangle-free graphs on n vertices with maximum degree Δ and independence number at most $O(n \log \Delta/\Delta)$. Since every forest contains an independent set of at least half its size, these graphs also have no acyclic set of size greater than $O(n \log \Delta/\Delta)$. Moreover, this result is asymptotically sharp since in [1, 7], it is proved that every triangle-free graph on n vertices and maximum degree Δ has an independent set of size at least $\Omega(n \log \Delta/\Delta)$.

3 Proof of Theorem 1.5

In this section we complete the proof of Theorem 1.5.

Proof of Theorem 1.5: We suppose that G is a minimal counterexample with respect to the number of vertices, and will obtain a contradiction. If G is not connected, then by minimality, we can apply the result to each component. Hence we may assume that G is connected. We have already verified the theorem for graphs in $\mathcal{F}$, so we may assume that $G \notin \mathcal{F}$ and $c = 0$. Suppose that G contains a copy H of $\dot{K}_4$, and v is the vertex of degree two in H. Since $G \notin \mathcal{F}$, $|N_G(v)| = 3$. Let $G' = G - H$. By minimality of G we obtain a large acyclic set S' in G'. Note that G' is connected, and $G' \notin \mathcal{F}$, since otherwise $G \in \mathcal{F}$. Form S by adding to S' any three vertices in H that do not create a triangle. Then

$$a(G) \geq |S| = |S'| + 3 \geq (n-5) - \frac{e-8}{4} + 3 = n - \frac{e}{4},$$

a contradiction. Hence we may assume that G is $\dot{K}_4$-free. If G is triangle-free, then Lemma 2.1 gives a contradiction, so we may assume that xyz is a triangle in G. Let $T = \{x, y, z\}$ and $N = N_G(T) - T$.

Claim: $G[N]$ is a clique.

Proof of Claim: Suppose to the contrary that $y', z' \in N$ with $y \leftrightarrow y', z \leftrightarrow z'$ and $y' \not\leftrightarrow z'$. Let $deg(x) = 2$. Then by minimality of G we obtain a large acyclic set S' in $G' = G - T$. Let $S = S' \cup \{x, y\}$. Then

$$a(G) \geq |S| = |S'| + 2 \geq n - 3 - \frac{e-5}{4} - \frac{c'}{4} + 2, \tag{1}$$

where c' is the number of components of G' from $\mathcal{F}$ (note that $c' \leq 2$ since G is connected). This yields the contradiction $a(G) \geq n - e/4$ unless $c' = 2$, but in this case $G \in \mathcal{F}$ which we have already excluded. We may therefore assume that $deg(x) = 3$.

Form G_1 from $G - T$ by adding the edge $y'z'$ and let c_1 be the number of components in G_1 from $\mathcal{F}$. If H is a copy of $K_4 \subseteq G_1$, then H consists of y', z' and two other vertices in G_1. By minimality of G, the graph $G - T - V(H)$ has an acyclic set of size at least $n - 7 - (e-11)/4 - 1/4$. We form S by adding to this set any five vertices that form an acyclic set within $V(H) \cup T$. It is easy to see that $S \geq n - e/4$. This contradiction allows us to assume that G_1 is K_4-free.

By minimality of G, there is a large acyclic set S_1 in G_1. Set $S = S_1 \cup \{y, z\}$. Since $y'z'$ is an edge in G_1, $c_1 < 3$. The set S is acyclic, since a cycle in S would yield a cycle in S_1 (with $y'yzz'$ replaced by $y'z'$). If $c_1 \leq 1$, then by (1), with $S' = S_1$ and $c' = c_1$, the set S has size at least $n - e/4$, a contradiction. We may therefore assume that $c_1 = 2$. Let $G' = G - T$. By minimality of G, there is a large acyclic set S' in G'. Let x' be the other neighbor of x. Since x' and $\{y', z'\}$ lie in different components of G', adding x, y to S' yields an acyclic set S in G. Because $G \notin \mathcal{F}$, we deduce that $c' \leq 2$, and hence $|S| \geq |S'| + 2 \geq n - 3 - (e-6)/4 - 2/4 + 2 = n - e/4$, a contradiction. □

Because $\Delta(G) \leq 3$, we have $|N| \leq 3$. If $|N| = 1$ and T has two vertices, say x and y, with degree 2 and 3 respectively, then let $G' = G - T$. By minimality of G we obtain an acyclic set S' in G' of size at least $n-3-(e-4)/4$. The set $S = S' \cup \{x, y\}$ is acyclic and has size at least $n - e/4$, a contradiction. The remaining case when $|N| = 1$ is if all vertices of T have degree 3. In this case, since G is connected, $G = K_4$ which the hypothesis excludes.

If $|N| = 2$, and all vertices of T have degree three, then the claim implies that the induced subgraph $G[T \cup N]$ forms a copy of $\dot{K}_4$ which we have already excluded. Hence we may assume that $deg(x) = 2$. Then $G' = G - T$ has a large acyclic set S'. Add x, y to S' to form S. Because G' is connected, $c' \leq 1$ and (1) yields the contradiction $a(G) \geq n - e/4$.

If $|N| = 3$, then the claim implies that G consists of two disjoint triangles with a matching of size three between them. In this case $a(G) = 4 \geq 6 - 9/4$, a contradiction. □

4 From independent sets to forests

In a graph G with maximum degree Δ, we can obtain an acyclic set of size

$$\alpha(G) + \frac{n - \alpha(G)}{\Delta(\Delta - 1) + 1} \tag{2}$$

by considering a maximum independent set I, and successively adding to it vertices whose pairwise distance is at least three. The result of this section improves the factor $\Delta(\Delta-1)+1$ in (2) to $(\Delta - 1)^2$. For small values of Δ, this improvement is significant. Indeed, the result applied to bipartite graphs when $\Delta = 3$ is sharp.

Proof of Theorem 1.7: Let B be an independent set in $G = (V, E)$ with $\alpha(G)$ vertices, and let $A = V - B$. We will iteratively construct a sequence $a_1, \ldots, a_t$ of vertices in A with the following properties:

$$N(a_i) \cap \{a_{i+1}, \ldots, a_t\} = \emptyset \qquad \text{and} \tag{3}$$

$$|N(a_i) \cap (\cup_{j=i+1}^{t} N(a_j)) \cap B| \leq 1 \qquad \text{for each } i. \tag{4}$$

Set $S = \{a_1, \ldots, a_t\} \cup B$. We will show that either S has the required size, or we can augment it by one to have the required size. By (3) any cycle C in $G[S]$ alternates between vertices in A and vertices in B. Let l be the smallest integer for which a_l is on C. By (4), a_l has at most one neighbor from B that lies on C. Hence we conclude that S is acyclic.

Let $D_0 = \emptyset$. We iteratively construct a sequence of sets $D_1, \ldots, D_t$, and put $A_i = D_1 \cup D_2 \cup \ldots \cup D_i$. Let $R_0 = B$, and for $i \geq 1$, let $R_i = N_B(A_i)$. Assume that we have already constructed $D_0, \ldots, D_i$ for some $i \geq 0$. If $A_i = A$, then let $t = i$, and stop. Otherwise, choose $a_{i+1} \in A - A_i$ such that a_{i+1} is adjacent to a vertex $x_{i+1} \in A_i \cup R_i$

(such a vertex exists, since G is connected, and $A_i \neq A$). If $N_B(a_{i+1}) \subseteq \{x_{i+1}\}$, then let $Z_{i+1} = N_B(a_{i+1})$; otherwise choose $z_{i+1} \in N_B(a_{i+1}) - \{x_{i+1}\}$ so that, if possible, a_{i+1} is the only common neighbor of x_{i+1} and z_{i+1}, and put $Z_{i+1} = N_B(a_{i+1}) - \{z_{i+1}\}$. Let

$$D_{i+1} = (N_A(a_{i+1}) \cup N_A(Z_{i+1}) \cup \{a_{i+1}\}) - A_i.$$

The definition of D_{i+1} and a_{i+1} ensures that conditions (3) and (4) are satisfied.

Claim: For $i = 0$, $|D_{i+1}| \leq (\Delta - 1)^2 + 1$ and for $i \geq 1$, $|D_{i+1}| \leq (\Delta - 1)^2$. Moreover, if equality holds above for $i \geq 0$, then there exists a $w \in D_{i+1} - \{a_{i+1}\}$ such that the vertices w, a_{i+1} are not adjacent and have at most one common neighbor in B.

Proof of Claim: We only prove the case $i \geq 1$, noting that the analysis for $|D_1|$ follows similarly. Set $k = |N_B(a_{i+1})|$. If $k = 0$, then $|D_{i+1}| \leq \Delta - 1 + 1 < (\Delta - 1)^2$, because a_{i+1} is adjacent to $x_{i+1} \in A_i$. Thus we may assume that $k \geq 1$. If $Z_{i+1} = N_B(a_{i+1})$, then $Z_{i+1} = N_B(a_{i+1}) = \{x_{i+1}\}$, $k = 1$, and

$$|D_{i+1}| \leq |N_A(a_{i+1}) - A_i| + |N_A(x_{i+1}) - A_i| \leq (\Delta - 1) + (\Delta - 1) \leq (\Delta - 1)^2,$$

since x_{i+1} is adjacent to a_{i+1} and also to a vertex in A_i. If equality holds, then pick $w \in N_A(x_{i+1}) - A_i - \{a_{i+1}\}$; w has the required properties, since $k = 1$, and $w \not\leftrightarrow a_{i+1}$.

We may therefore assume that $Z_{i+1} \subsetneq N_B(a_{i+1})$. In this case,

$$|D_{i+1}| \leq |N_A(a_{i+1}) - A_i| + |N_B(Z_{i+1}) - A_i| + 1 \leq (\Delta - k) + (k-1)(\Delta - 1) - 1 + 1 \leq (\Delta - 1)^2,$$

because $|Z_{i+1}| \leq k - 1$ and each vertex in Z_{i+1} is adjacent to at most $\Delta - 1$ vertices of $A - A_i$ other than a_{i+1}. The term -1 arises because either $x_{i+1} \in A_i$, or $x_{i+1} \in Z_{i+1}$ is adjacent to a vertex in A_i. If equality holds, then $k = \Delta$. This implies that $N_A(a_{i+1}) = \emptyset$ and $x_{i+1} \in B$. Pick $w \in N_A(x_{i+1}) - \{a_{i+1}\}$. By the conditions for equality, w and a_{i+1} have no common neighbor in Z_{i+1}. The choice of z_{i+1} implies that x_{i+1} is the only common neighbor of w and a_{i+1} in all of B. □

As indicated above by the choice of t, we continue this procedure till we have accounted for all of G. By the claim, this yields

$$n - \alpha(G) = |A| = A_t = \sum_{i=1}^{t} |D_i| \leq (\Delta - 1)^2 + 1 + (t - 1)(\Delta - 1)^2. \tag{5}$$

Solving for t gives $t \geq |A|/(\Delta - 1)^2$ unless equality holds everywhere in (5). But in this case, consider the vertex w from the claim obtained when $i = t - 1$. We add $w = a_{t+1}$ to our acyclic set to augment it by one. The conditions for equality stated in the claim yield (3) and (4) with t replaced by $t+1$. Hence $\{a_1, \dots, a_t, a_{t+1}\} \cup B$ is acyclic and of the required size. $\square$

Corollary 4.1. *Suppose that G is an n vertex bipartite graph with maximum degree $\Delta \geq 3$. Then*

$$a(G) \geq \left(\frac{1}{2} + \frac{1}{2(\Delta-1)^2}\right) n \tag{6}$$

and this is sharp for $\Delta = 3, n \equiv 0$ (mod 8).

Proof. Since $\alpha(G) \geq n/2$ when G is bipartite, (6) follows immediately from Theorem 1.7. The cube Q_3 shows that this is sharp for $\Delta = 3$. $\square$

We end this section by constructing n vertex Δ-regular bipartite graphs with $a(G) \leq n/2 + O(n/\Delta^2)$.

Definition 4.2. *For integers $a, b \geq 1$, let $G_{a,b}$ be the bipartite graph with parts X, Y each of size ab, with $X = \{x_{i,j} : 1 \leq i \leq a, 1 \leq j \leq b\}$ and $Y = \{y_{i,j} : 1 \leq i \leq a, 1 \leq j \leq b\}$. Vertices $x_{i,j}$ and $y_{i',j'}$ are adjacent if and only if either $i = i'$ or $j = j'$. For $1 \leq i \leq a$ and $1 \leq j \leq b$, let $R_i = \{x_{i,1}, y_{i,1}, \dots, x_{i,b}, y_{i,b}\}$ and $C_j = \{x_{1,j}, y_{1,j}, \dots, x_{a,j}, y_{a,j}\}$. These are the rows and columns of $G_{a,b}$.*

Theorem 4.3. $a(G_{a,b}) \leq ab + 1$.

Proof. We proceed by induction on $a+b$. We may assume by symmetry that $b \geq a$. If $a = 1$, then $G_{a,b} \cong K_{b,b}$ for which the result trivially holds. This completes the cases $a + b \leq 3$, and we may therefore assume that $a \geq 2$ and $a + b \geq 4$. Consider a subgraph H of $G_{a,b}$ with $ab + 2$ vertices. If $|V(H) \cap R_i| \leq b$ for some i, then let H' be the restriction of H to $G_{a,b} - R_i$. Since $|V(H')| \geq ab + 2 - b = (a-1)b + 2$, and $G_{a,b} - R_i \cong G_{a-1,b}$, we obtain a cycle in H' by induction. Hence we conclude that $|V(H) \cap R_i| \geq b+1$ for all i, and similarly that $|V(H) \cap C_j| \geq a+1$ for all j.

Let r_i be the number of edges of H induced by $V(H) \cap R_i$ and c_j be the number of edges of H induced by $V(H) \cap C_j$. It is easy to see that $|V(H) \cap R_i| \geq b + 1$ implies $r_i \geq b$, and similarly that $|V(H) \cap C_j| \geq a + 1$ implies $c_j \geq a$. Call an edge *vertical* if it has the form $x_{l,m}y_{l,m}$ for some l, m; if an edge is not vertical, call it *diagonal.* Let $e = |E(H)|$ and let t be the number of vertical edges in H. If $t \geq a+1$, then two vertical edges from H lie in the same row, and this results in a 4-cycle in H. Hence we may assume that $t \leq a$.

Each vertical edge of H is in the induced subgraph of one row and of one column. Each diagonal edge of H is in the induced subgraph of one row or one column, but not both. These observations yield

$$ab + ba \leq \sum_i r_i + \sum_j c_j = (e - t) + 2t.$$

Solving for e gives $e \geq 2ab - t \geq 2ab - a \geq ab + 2 = |V(H)|$, which implies that H is not acyclic. $\square$

Taking disjoint copies of $G_{\lfloor(\Delta+1)/2\rfloor,\lceil(\Delta+1)/2\rceil}$ and disjoint copies of $K_{\Delta,\Delta}$ immediately yields

Corollary 4.4. *For integers Δ, n, where $\lfloor(\Delta+1)^2/2\rfloor$ divides n, there exists an n vertex Δ-regular bipartite graph with $a(G) = n/2 + n/(\lfloor(\Delta+1)^2/2\rfloor)$. If 2Δ divides n, then there exists an n vertex Δ-regular bipartite graph with $a(G) = n/2 + n/(2\Delta)$.*

Remark 4.5. *The graphs $G_{a,b}$ also provide our best constructions for 4-regular and 5-regular bipartite graphs with no large acyclic sets. In particular, Theorem 4.3 immediately yields $a(G_{2,3}) = 7$ and $a(G_{3,3}) = 10$.*

5 Summary of Results

In this section, we summarize our results. To do this accurately, we first define some classes of n vertex graphs. Let $\mathcal{G}_{n,d}$ denote the family of d-regular graphs, $\mathcal{G}^-_{n,d}$ denote the family of graphs with maximum degree d. Let $\mathcal{T}_{n,d}$ denote the family of triangle-free d-regular graphs, $\mathcal{T}^-_{n,d}$ denote the family of triangle-free graphs with maximum degree d. Let $\mathcal{B}_{n,d}$ denote the family of bipartite d-regular graphs, $\mathcal{B}^-_{n,d}$ denote the family of bipartite graphs with maximum degree d.

Given a finite family of graphs $\mathcal{F}$, let $a(\mathcal{F})$ denote the minimum of $a(G)$ over all $G \in \mathcal{F}$. Considering vertex disjoint copies of graphs, one can easily see that

$$a(\mathcal{G}_{n_1,d}) + a(\mathcal{G}_{n_2,d}) \geq a(\mathcal{G}_{n_1+n_2,d}).$$

This, and the obvious lower bound $a(G) \geq n/d^2$ imply that the limit

$$\gamma_d := \lim_{n\to\infty} a(\mathcal{G}_{n,d})/n$$

exists and is not equal to zero (Fekete's Lemma, see, e.g., [8]). The same is true for

$$\gamma_d^- := \lim_{n\to\infty} a(\mathcal{G}_{n,d}^-)/n,$$

$$\tau_d := \lim_{n\to\infty} a(\mathcal{T}_{n,d})/n, \qquad \tau_d^- := \lim_{n\to\infty} a(\mathcal{T}_{n,d}^-)/n,$$

$$\beta_d := \lim_{n\to\infty} a(\mathcal{B}_{n,d})/n, \qquad \beta_d^- := \lim_{n\to\infty} a(\mathcal{B}_{n,d}^-)/n.$$

Table of Results

<table>
<tr><td>$d =$</td><td>2</td><td>3</td><td>4</td><td>5</td><td>...</td></tr>
<tr><td>γ_d, γ_d^-</td><td colspan="5">$\frac{2}{d+1}$ [4]</td></tr>
<tr><td>τ_d, τ_d^-</td><td rowspan="2">$\frac{3}{4}$</td><td rowspan="2">$\frac{5}{8}$
Lem. 2.1
Ex. 1.3</td><td>$\geq \frac{1}{2}$
Lem. 2.1
$\leq \frac{4}{7}$
Ex. 2.2</td><td>$\geq \frac{3}{8}$
Lem. 2.1
$\leq \frac{1}{2}$
Ex. 2.3</td><td>$\geq \Omega\left(\frac{\log d}{d}\right)$ [1]
$\tau_d^- = \Theta\left(\frac{\log d}{d}\right)$
Rem. 2.4</td></tr>
<tr><td>β_d, β_d^-</td><td>$\geq \frac{5}{9}$
Cor. 4.1
$\leq \frac{7}{12}$
Rem. 4.5</td><td>$\geq \frac{17}{32}$
Cor. 4.1
$\leq \frac{5}{9}$
Rem. 4.5</td><td>$\geq \frac{1}{2} + \frac{1}{2(d-1)^2}$
Cor. 4.1
$\leq \frac{1}{2} + \frac{1}{\lfloor (d+1)^2/2 \rfloor}$
Cor. 4.4</td></tr>
</table>

6 Acknowledgments

The second author thanks Christopher Heckman and Jeff Steif for useful discussions.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, J. Combinatorial Theory, Ser. A 29 (1980), 354-360.

[2] M. Albertson and R. Haas, A problem raised at the DIMACS Graph Coloring Week, 1998.

[3] N. Alon, Induced acyclic subgraphs in sparse bipartite graphs, preprint.

[4] N. Alon, J. Kahn and P. D. Seymour, Large induced degenerate subgraphs in graphs, Graphs and Combinatorics, 3 (1987), 203–211.

[5] B. Bollobás, Chromatic number, girth and maximal degree, Discrete Math. 24 (1978), 311-314.

[6] A. V. Kostochka and N. P. Mazurova, An inequality in the theory of graph coloring (in Russian), Metody Diskret. Analiz. 30 (1977), 23-29.

[7] J. B. Shearer, A note on the independence number of triangle-free graphs, Discrete Math. 46 (1983), 83-87.

[8] J. H. van Lint and R. M. Wilson, "A course in Combinatorics," Cambridge Univ. Press, Cambridge, UK, 1992, page 85.