Large N Free Energy of 3d $\mathcal{N} = 4$ SCFTs and AdS$_4$/CFT$_3$

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ABSTRACT: We provide a non-trivial check of the AdS$_4$/CFT$_3$ correspondence recently proposed in [1] by verifying the GKPW relation in the large N limit. The CFT free energy is obtained from the previous works [2, 3] on the S^3 partition function for 3-dimensional $\mathcal{N} = 4$ SCFT $T[SU(N)]$. This is matched with the computation of the type IIB action on the corresponding gravity background. We unexpectedly find that the leading behavior of the free energy at large N is $\frac{1}{2}N^2 \ln N$. We also extend our results to richer $T_\rho^{\hat{\rho}}[SU(N)]$ theories and argue that $\frac{1}{2}N^2 \ln N$ is the maximal free energy at large N in this class of gauge theories.

Contents

1 Introduction

In this paper we study a class of 3d $\mathcal{N} = 4$ SCFT $T_{\hat{\rho}}^{\rho}[SU(N)]$ introduced in [4], where $\rho, \hat{\rho}$ are partitions of N. This theory is a 1/2 BPS domain wall theory inside the 4d $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory, and plays crucial roles in the generalizations [5, 6] of the AGT correspondence, as well as the connection with the 3d $SL(2)$ Chern-Simons theory [7–9]. The $T_{\hat{\rho}}^{\rho}[SU(N)]$ theories also appear as the basic building blocks for the 3d mirror of the 4d $\mathcal{N} = 2$ Gaiotto theories [10] compactified on S^1 [3, 11].

The type IIB supergravity dual for $T_{\hat{\rho}}^{\rho}[SU(N)]$ theories has recently been constructed in [1]. In this paper we provide further quantitative consistency checks of this AdS4/CFT3 correspondence by verifying the GKPW relation [12, 13] in the leading large N limit.

On the CFT side, we take the large N limit of the S^3 partition functions of [2, 3], evaluated at the conformal point. On the gravity side, we evaluate the gravity action in the gravity background of [1]. We find that in both cases the leading contribution of the free energy in the large N limit scales as

$$F \sim N^2 \ln N + \mathcal{O}(N^2) \, .$$

More detailed statements will be given momentarily in section 2.2. As we will see, on the CFT side $N^2 \ln N$ comes from the asymptotic behavior of the Barnes G-function. On the gravity side, a factor of N^2 comes from the local scaling of the supergravity Lagrangian, and an extra $\ln N$ comes from the size of the geometry.

The organization of this paper is as follows. We first summarize the notations and the main results (section 2). We then give the derivations the results in gauge theory (section 3) and gravity (section 4). We also include two appendices.

2 Summary of the results

2.1 Review of $T_{\hat{\rho}}^{\rho}[SU(N)]$ theories

Let us first briefly summarize the basics of $T_{\hat{\rho}}^{\rho}[SU(N)]$ theories needed for the understanding of this paper (see [4] for details).

As stated in the introduction, the $T_{\hat{\rho}}^{\rho}[SU(N)]$ theory is specified by two partitions ρ and $\hat{\rho}$ of N:

$$
\rho = \Big[\overbrace{l^{(1)}, l^{(1)}, \ldots, l^{(1)}}^{N_5^{(1)}}, \ \overbrace{l^{(2)}, l^{(2)}, \ldots, l^{(2)}}^{N_5^{(2)}}, \ \ldots, \ \overbrace{l^{(p)}, l^{(p)}, \ldots, l^{(p)}}^{N_5^{(p)}} \Big],
$$
$$
\hat{\rho} = \Big[\overbrace{\hat{l}^{(1)}, \hat{l}^{(1)}, \ldots, \hat{l}^{(1)}}^{\hat{N}_5^{(1)}}, \ \overbrace{\hat{l}^{(2)}, \hat{l}^{(2)}, \ldots, \hat{l}^{(2)}}^{\hat{N}_5^{(2)}}, \ \ldots, \ \overbrace{\hat{l}^{(\hat{p})}, \hat{l}^{(\hat{p})}, \ldots, \hat{l}^{(\hat{p})}}^{\hat{N}_5^{(\hat{p})}} \Big],
\tag{2.1}
$$

where $l^{(a-1)} > l^{(a)}, \hat{l}^{(a-1)} > \hat{l}^{(a)}$ for all a and

$$
\sum_{a=1}^{p} l^{(a)} N_5^{(a)} = \sum_{a=1}^{\hat{p}} \hat{l}^{(a)} \hat{N}_5^{(a)} = N .
\tag{2.2}
$$

As the notation suggests, $N_5^{(a)}$ and $\hat{N}_5^{(a)}$ represent the 5-branes charges of the supergravity solution (see section 4).

To construct the 3d theory, it is useful to use the brane configurations of [14]. Namely, we consider a D3-D5-NS5-brane configuration with N D3-branes suspended between NS5-branes on the left and D5-branes on the right, where $l^{(a)}$ D3-branes ($\hat{l}^{(a)}$ D3-branes) end on the i-th D5-brane (NS5-brane). We can identify the 3d theory after suitable exchanges of D5 and NS5-branes. The result is a 3d $\mathcal{N} = 4$ quiver gauge theory. This theory has a non-trivial irreducible IR fixed point only when [4]

$$
\rho^T > \hat{\rho} \quad \Leftrightarrow \quad \hat{\rho}^T > \rho,
\tag{2.3}
$$

where $\rho > \hat{\rho}$ for $\rho = [n_1, n_2, \ldots]$ and $\hat{\rho} = [m_1, m_2, \ldots]$ is defined by

$$
\sum_{i=1}^{k} n_i > \sum_{i=1}^{k} m_i
\tag{2.4}
$$

for all k. When the inequality is saturated for some value of i, the quiver breaks into pieces and the IR fixed point consists of products of irreducible theories.

The global symmetry of $T_\rho^{\hat\rho}[SU(N)]$ is given by $G_\rho \times G_{\hat\rho}$, where G_ρ is a subgroup of $SU(N)$ commuting with the embedding ρ

$$G_\rho = S(U(N_5^{(1)}) \times \cdots \times U(N_5^{(p)})) \,. \tag{2.5}$$

G_ρ is a symmetry of the Lagrangian, and acts non-trivially on the Higgs branch, whereas $G_{\hat\rho}$ is a quantum mechanical symmetry acting on the Coulomb branch.[1] We can weakly gauge these symmetries to introduce a set of real mass parameters and FI parameters, which we collectively denote by m and $\hat m$, respectively. The two global symmetries are related by 3d mirror symmetry [15] exchanging Higgs and Coulomb branches, together with real mass and FI parameters. This is simply the S-duality of the D3-D5-NS5 system, and in particular, $T_{\hat\rho}^\rho[SU(N)]$ is the mirror of $T_{\hat\rho}^\rho[SU(N)]$.

2.2 Large N free energy

We will verify the GKPW relation in the large N limit:

$$Z_{\rm CFT} = e^{-S_{\rm gravity}} \,, \quad \text{i.e.} \quad F_{\rm CFT} = S_{\rm gravity} \,, \tag{2.6}$$

where $Z_{\rm CFT}$ is a CFT partition function on S^3, $F_{\rm CFT} := -\ln Z_{\rm CFT}$ is the free energy, and $S_{\rm gravity}$ is the action for the type IIB supergravity holographic dual to the CFT.

Our findings are summarized as follows.

- The simplest prototypical example is the $T[SU(N)]$ theory, which is a $T_{\hat\rho}^\rho[SU(N))]$ theory with

$$\rho = \hat\rho = \big[\, \overbrace{1,1,\ldots,1}^{N} \,\big] \,. \tag{2.7}$$

  In this case we find

$$F_{\rm CFT} = S_{\rm gravity} = \frac{1}{2} N^2 \ln N + \mathcal{O}(N^2) \,. \tag{2.8}$$

- More generally we consider the case $\hat p = 1$, i.e.,

$$\rho = \Big[\, \overbrace{l^{(1)}, l^{(1)}, \ldots, l^{(1)}}^{N_5^{(1)}},\ \overbrace{l^{(2)}, l^{(2)}, \ldots, l^{(2)}}^{N_5^{(2)}},\ \ldots\ ,\ \overbrace{l^{(p)}, l^{(p)}, \ldots, l^{(p)}}^{N_5^{(p)}} \,\Big] \,,$$
$$\hat\rho = \Big[\, \overbrace{\hat l, \hat l, \ldots, \hat l}^{\hat N_5} \,\Big] \,. \tag{2.9}$$

  We take the scaling limit

$$N_5^{(a)} = N^{1-\kappa_a} \gamma_a, \quad l^{(a)} = N^{\kappa_a} \lambda^{(a)}, \quad \hat N_5 = N\hat\gamma \,, \tag{2.10}$$

---

[1]The Cartan of this symmetry is the shift of the dual photon, and is present in the Lagrangian.

where we take N large, while keeping $\kappa_a, \lambda^{(a)}, \gamma_a, \hat{\gamma}$ finite. We require

$$\kappa_{a-1} \geq \kappa_a, \quad 0 \leq \kappa_a < 1, \quad \text{for all } a . \tag{2.11}$$

The first condition is necessary for ρ to be partition, and the second ensures that the $N_5^{(i)}$ becomes large, hence justifying the validity of the supergravity solution. We also have from (2.2) the constraint

$$\sum_{a=1}^{p} \gamma_a \lambda^{(a)} = \hat{\gamma} \hat{l} = 1 . \tag{2.12}$$

In this more general case we find (CFT analysis will be provided for $\hat{l} = 1$, and gravity analysis for general $\hat{l}$):

$$F_{\text{CFT}} = S_{\text{gravity}} = \frac{1}{2} N^2 \ln N \left[(1 - \kappa_1) + \sum_{i=2}^{p} \left(\sum_{a=i}^{p} \gamma_a \lambda^{(a)} \right)^2 (\kappa_{i-1} - \kappa_i) \right] + \mathcal{O}(N^2). \tag{2.13}$$

In particular when all $\kappa_a = 0$, i.e. when all $l^{(a)}$ are finite, the leading large N behavior coincides with that in (2.8).

Note the number inside the bracket in (2.13) is a non-negative number smaller than 1 due to (2.11). Motivated by this result we conjecture

$$F_{T_{\hat{\rho}}^{\rho}[SU(N)]} \leq F_{T[SU(N)]} . \tag{2.14}$$

for all $\rho, \hat{\rho}$ satisfying (2.3). It would be interesting to see if some of the above inequalities could be explained in terms of the F-theorem [16, 17] and the RG flows between the fixed points. The rest of this paper will be devoted to the derivation of (2.8) and (2.13).

3 CFT analysis

In this section we analyze the CFT free energy F_{CFT}.

3.1 The S^3 partition function

Let us begin with the $T[SU(N)]$ theory (2.7). The partition function of this theory was computed by localization [18] to be [2, 3] (see also [19]):[2]

$$Z_{S^3}[T[SU(N)]](m, \hat{m}) = \frac{\sum_{w \in \mathfrak{S}_N} (-1)^w e^{2\pi i m \cdot w(\hat{m})}}{\Delta(m) \Delta(\hat{m})}, \tag{3.1}$$

where $(-1)^w$ is a sign of a permutation $w \in \mathfrak{S}_N$, $m = (m_1, \ldots, m_N)$ with $\sum_i m_i = 0$ ($\hat{m} = (\hat{m}_1, \ldots, \hat{m}_N)$ with $\sum_i \hat{m}_i = 0$) are the FI parameters[3] (real mass parameters), and

$$m \cdot w(\hat{m}) := \sum_i m_i \hat{m}_{w(i)} ,$$

---

[2]The expression in [3] contains an extra factor of $1/N!$. However, this factor does not alter the leading behavior of the free energy and hence will be dropped in this paper.

[3]FI parameters are actually the differences of $\hat{m}_i$. However, we will loosely refer to $\hat{m}$ as FI parameters.
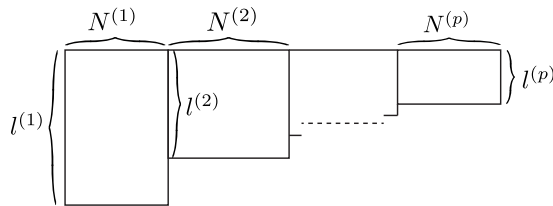
Figure 1. We decompose the young diagram corresponding to ρ into p blocks, see (2.1).

and Δ is the (sinh) Vandermonde determinant

$$\Delta(m) := \prod_{i<j} 2\sinh \pi(m_i - m_j), \quad \Delta(\hat{m}) := \prod_{i<j} 2\sinh \pi(\hat{m}_i - \hat{m}_j). \tag{3.2}$$

For more general $T_{\hat{\rho}}^{\rho}[SU(N)]$ theories, the partition function takes a similar form as in (3.1) [3]:

$$Z_{S^3}[T_{\hat{\rho}}^{\rho}[SU(N)]](m,\hat{m}) = \frac{\sum_{w\in\mathfrak{S}_N}(-1)^w e^{2\pi\imath m_\rho \cdot w(\hat{m}_{\hat{\rho}})}}{\Delta_\rho(m)\Delta_{\hat{\rho}}(\hat{m})}. \tag{3.3}$$

Here $m_\rho, \hat{m}_{\hat{\rho}}$ are N-vectors, and each of their components is associated with a box of the Young diagram (also denoted by $\rho, \hat{\rho}$) corresponding to the partitions $\rho, \hat{\rho}$. For later purposes let us describe them by dividing the boxes of ρ into p blocks, where the a-th block is a rectangle with rows of length $N_5^{(a)}$ and columns of length $l^{(a)}$ (recall (2.1), and see figure 1). A box of ρ could then be labeled by a triple (a,i,α) with $1 \le a \le p, 1 \le i \le N_5^{(a)}, 1 \le \alpha \le l^{(a)}$, where a is the label for the block and i (α) is the label for the column (row) inside the a-th block. The same applies to $\hat{\rho}$. In this notation, we have

$$(m_\rho)_{(a,i,\alpha)} = \imath(w_{l^{(a)}})_\alpha + m_{a,i}, \quad (\hat{m}_{\hat{\rho}})_{(a,i,\alpha)} = \imath(w_{\hat{l}^{(a)}})_\alpha + \hat{m}_{a,i}, \tag{3.4}$$

where w_N is a Weyl vector of the $SU(N)$ Lie algebra defined by

$$w_N = \left(\frac{N-1}{2}, \frac{N-3}{2}, \dots, -\frac{N-1}{2}\right). \tag{3.5}$$

Also, $\Delta_\rho(m)$ and $\Delta_{\hat{\rho}}(\hat{m})$ are defined by

$$\begin{aligned}
\Delta_\rho(m) &= \prod_p \prod_{q<r} 2\sinh \pi((m_\rho)_{[p,q]} - (m_\rho)_{[p,r]}), \\
\Delta_{\hat{\rho}}(\hat{m}) &= \prod_p \prod_{q<r} 2\sinh \pi((\hat{m}_{\hat{\rho}})_{[p,q]} - (\hat{m}_{\hat{\rho}})_{[p,r]}),
\end{aligned} \tag{3.6}$$

where $[p,q]$ represents a box inside $\rho, \hat{\rho}$ at row p and column q. Note that the $(m_\rho)_{[p,q]}$ are simply a relabeling of the $(m_\rho)_{(a,i,\alpha)}$ introduced previously.

Several remarks are now in order. First, the partition function (3.3) is manifestly invariant under the simultaneous exchange of $\rho, \hat{\rho}$ and $m, \hat{m}$. This is a manifestation of the 3d mirror symmetry.

Second, (3.3) vanishes unless $\rho^T \geq \hat{\rho}$ [3]. This is consistent with the condition (2.3) for the existence of a non-trivial IR SCFT. This condition has a counterpart in the gravity dual [1].

Third, the expression (3.1) is either real or pure imaginary, however there is an ambiguity of the phase of the S^3 partition function and we will hereafter concentrate on the absolute value of the S^3 partition function.

3.2 $T[SU(N)]$

Let us study the large N behavior of our partition functions.

For clarity, let us begin with the $T[SU(N)]$ theories, whose partition function is given in (3.1). When the parameters m and $\hat{m}$ are generic and kept finite in the limit,[4] we have $\sum_{w \in \mathfrak{S}_N} \sim \mathcal{O}(N!)$, whose logarithm contributes $\mathcal{O}(N \ln N)$ to the F_{CFT}. The remaining contributions come from the two sinh Vandermonde determinants, each of which involves $\binom{N}{2} \sim \mathcal{O}(N^2)$ terms. This gives

$$F_{\text{CFT}} \sim \mathcal{O}(N^2) . \tag{3.7}$$

This is not surprising since after all our theories are standard gauge theories.

However, the scaling behavior could change if we consider non-generic values of m and $\hat{m}$. This is exactly happens to our CFT case, where we need to take the limit $m, \hat{m} \to 0$ of (3.1):

$$Z_{\text{CFT}} = \lim_{m,\hat{m} \to 0} \left| Z_{S^3} \right| . \tag{3.8}$$

We choose to take the limit in two steps. First, let us take the $\hat{m} \to 0$ limit of (3.3) with $\hat{\rho} = [1, \ldots, 1]$. This is conveniently done by setting $\hat{m} = \epsilon w_N$ and by taking $\epsilon \to 0$, where w_N is defined in (3.5). Using the Weyl denominator formula, we have

$$\sum_{w \in \mathfrak{S}_N} (-1)^w e^{2\pi\imath\epsilon w_N \cdot m_\rho} = \prod_{\alpha > 0} 2\imath \sin\left(\pi\epsilon\alpha \cdot m_\rho\right) = \prod_{j<k} 2\imath \sin\left(\pi\epsilon(m_j - m_k)\right) . \tag{3.9}$$

In the limit $\epsilon_2 \to 0$, this cancels the factor $\Delta(\epsilon w_N) = \prod_{j<k} 2 \sinh \pi(\epsilon(j-k))$, giving

$$\left| \prod_{j<k} \frac{\imath}{(j-k)} \frac{\prod_{j<k}(m_\rho)_j - (m_\rho)_k}{\Delta_\rho(m)} \right| = \frac{1}{G_2(N+1)} \left| \frac{\prod_{j<k}(m_\rho)_j - (m_\rho)_k}{\Delta_\rho(m)} \right| . \tag{3.10}$$

We next need to take the limit $m \to 0$. This is easy for our case, $\rho = [1, \ldots, 1]$;

$$\frac{\prod_{i<j}(m_\rho)_i - (m_\rho)_j}{\Delta_\rho(m)} = \prod_{i<j} \frac{m_i - m_j}{2 \sinh \pi(m_i - m_j)} \to (2\pi)^{-\frac{N(N-1)}{2}},$$

which gives

$$Z_{\text{CFT}} = \frac{1}{(N-1)!(N-2)! \ldots 2!1!} \left(\frac{1}{2\pi}\right)^{\frac{N(N-1)}{2}} = \frac{1}{G(N+1)} \left(\frac{1}{2\pi}\right)^{\frac{N(N-1)}{2}} , \tag{3.11}$$

---

[4]By generic we mean that there are no cancellations in the sum in the numerator of (3.1).

where $G_2(x)$ is the Barnes G-function defined in appendix A. From the asymptotics of $G_2(x)$ (A.3), we have

$$F_{\text{CFT}} = \frac{N^2}{2} \ln N + \left[-\frac{3}{4} - \frac{1}{2} \ln \left(\frac{1}{2\pi} \right) \right] N^2 + \mathcal{O}(N \ln N), \qquad (3.12)$$

which gives (2.8).

3.3 $T_{\hat{\rho}}^{\rho}[SU(N)]$

Let us consider the more general case given in (2.9).

As long as $\hat{\rho} = [1, \dots, 1]$ the argument of the previous subsection works up until (3.10). In (3.10) we already have a factor of $G_2(N + 1)$. Just as in the $T[SU(N)]$ case, this contributes

$$\frac{1}{2} N^2 \ln N, \qquad (3.13)$$

to the free energy. Next, let us send the FI parameters to zero in (3.10). The denominator $\Delta_\rho(m)$ goes to zero in the limit, but the same is true for the numerators, yielding the finite answer. We obtain powers of 2π in this process from the limit of $\Delta_\rho(m)$, however this only gives a subleading contribution of order N^2.

There are still contributions from the numerator $\prod_{i<j} [(m_\rho)_i - (m_\rho)_j]$, which we have not yet taken into account. In the notation of the previous section the limit of this contribution is[5]

$$(m_\rho)_{(a,i,\alpha)} - (m_\rho)_{(b,j,\beta)} = \imath \left[(w_{l^{(a)}})_\alpha - (w_{l^{(b)}})_\beta \right] = \imath (\alpha - \beta),$$

where $1 \le \alpha \le l^{(a)}, 1 \le \beta \le l^{(b)}$ and $\alpha \ne \beta$.

When the two boxes are in the same block, this contributes a factor

$$\left(N_5^{(a)} \right)^2 \ln \left[(l^{(a)} - 1)!(l^{(a)} - 2)! \dots 1! \right],$$

where the factor $\left(N_5^{(a)} \right)^2$ accounts for the degeneracy from the column labels i. This contributes, under the scaling (2.10),

$$-\frac{1}{2} \left[\kappa_a (\lambda^{(a)} \gamma_a)^2 \right] N^2 \ln N + \mathcal{O}(N^2), \qquad (3.14)$$

to the free energy. When the two boxes are in the different blocks a, b with $l^{(a)} \ge l^{(b)}, \kappa_a \ge \kappa_b$, the contribution to the free energy is

$$-2 \left(N_5^{(a)} N_5^{(b)} \right) \ln \left[\left(\frac{l^{(a)} + l^{(b)}}{2} - 1 \right)! \left(\frac{l^{(a)} + l^{(b)}}{2} - 2 \right)! \dots \left(\frac{l^{(a)} - l^{(b)}}{2} \right)! \right].$$

The expression inside the bracket gives

$$\ln \left[G_2 \left(\frac{l^{(a)} + l^{(b)}}{2} + 1 \right) \right] - \ln \left[G_2 \left(\frac{l^{(a)} - l^{(b)}}{2} + 1 \right) \right] \sim \frac{1}{2} l^{(a)} l^{(b)} \ln l^{(a)}.$$

---

[5]A small modification is needed for the formula (3.3) when $\alpha - \beta$ is odd. This does not, however, affect the leading behavior of the free energy.

Thus the contribution amounts to

$$-2\left(N_5^{(a)}N_5^{(b)}\right)\frac{1}{2}l^{(a)}l^{(b)}\ln l^{(a)} = -2\frac{1}{2}\left[(\lambda^{(a)}\gamma_a\lambda^{(b)}\gamma_b)\kappa_a\right]N^2\ln N\,. \qquad (3.15)$$

Collecting all the contributions (3.13), (3.14) and (3.15), we have

$$F_{\text{CFT}} = \frac{1}{2}N^2\ln N\left[1 - \sum_{a=1}^{p}(\lambda^{(a)}\gamma_a)^2\kappa_a - 2\sum_{a\neq b,\, l^{(a)}>l^{(b)}}(\lambda^{(a)}\gamma_a\lambda^{(b)}\gamma_b)\kappa_a\right] + \mathcal{O}(N^2) \qquad (3.16)$$

From (2.12) we can show that this coincides with (2.13).

In all of the examples above, the leading contribution to the partition function comes from the Barnes G-functions. It is curious to note that the same function appears in the formula for the volumes of Lie group $SU(N)$ [20], and hence in the measure for the $SU(N)$ gauge theory. This is probably not a coincidence, since in the correspondence in [3] the S^3 partition function of $T_{\hat{\rho}}^\rho[SU(N)]$ theory is identified with an overlap of wavefunctions of a 1d quantum mechanics, which is obtained from a dimensional reduction of the 2d Yang-Mills theory. The measure of 2d Yang-Mills contains a volume factor for the gauge group $U(N)$. The same $N^2\ln N$ type behavior appears in a number of different contexts, such as Gaussian matrix models, $c=1$, topological string on the conifold or more recently in the weak coupling expansion of the ABJM theory [21].

4 Gravity analysis

In this section we analyze the type IIB supergravity action S_{gravity} in the holographic dual.

4.1 Summary of the gravity solution

First we summarize the holographic duals of the $T_\rho^{\hat{\rho}}[SU(N)]$ theories constructed in [1] (see also [22] for related work), which is based on earlier solutions found in [23, 24].

The geometry of the type IIB backgrounds is an $\text{AdS}_4 \times S^2 \times S^2$ fibration over a two-dimensional Riemann surface Σ. We will parameterize Σ by an infinite strip, although it will turn out that Σ has finite volume and is really compact. Next we introduce complex coordinates on Σ as $z, \bar{z}$. We will also make use of the real coordinates defined by writing $z = x + \imath y$. After fixing Σ, the solution is then determined by two real harmonic functions, h_1 and h_2, on Σ.
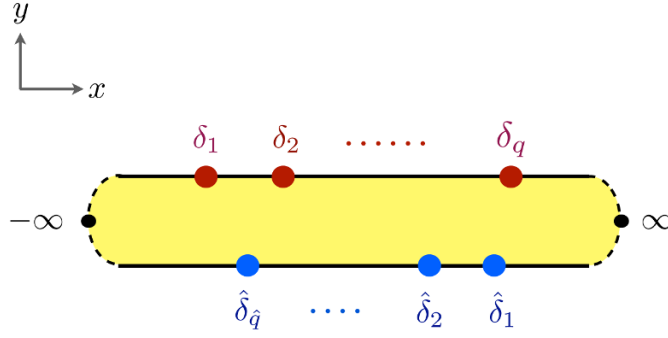
The metric can be written as

$$ds^2 = f_4^2 ds_{AdS_4}^2 + f_1^2 ds_{S_1^2}^2 + f_2^2 ds_{S_2^2}^2 + 4\rho^2 dz d\bar{z}\,, \qquad (4.1)$$

where the warp factors are given by

$$f_4^8 = 16\frac{N_1 N_2}{W^2}\,, \quad f_1^8 = 16h_1^8\frac{N_2 W^2}{N_1^3}\,, \quad f_2^8 = 16h_2^8\frac{N_1 W^2}{N_2^3}\,, \quad \rho^8 = \frac{N_1 N_2 W^2}{h_1^4 h_2^4}\,, \qquad (4.2)$$

and we defined the auxiliary functions

$$W = \partial\bar{\partial}(h_1 h_2)\,, \qquad N_j = 2h_1 h_2|\partial h_j|^2 - h_j^2 W\,. \qquad (4.3)$$

Figure 2. The infinite strip with logarithmic singularities corresponding to stacks of five-branes. The upper (red) singularities correspond to D5-branes, while the lower (blue) singularities correspond to NS5-branes. The geometry smoothly caps off into an S^6 as $x \to \pm\infty$.

This geometry is supported by non-vanishing "matter" fields, which include the dilaton field

$$e^{2\phi} = \sqrt{\frac{N_2}{N_1}}, \tag{4.4}$$

in addition to non-vanishing 3-form and 5-form fluxes which are given in appendix B.

We now turn to the specific solutions corresponding to $T_\rho^{\hat{\rho}}[SU(N)]$. The classical supergravity solutions describing the near horizon limit of D3-branes suspended between p stacks of D5-branes and $\hat{p}$ stacks of NS5-branes is given by the two harmonic functions:

$$
\begin{aligned}
h_1 &= -\sum_{a=1}^{p} \frac{\alpha'}{4} N_5^{(a)} \ln\left[\tanh\left(\frac{\imath\pi}{4} - \frac{z - \delta_a}{2}\right)\right] + c.c.\,, \\
h_2 &= -\sum_{b=1}^{\hat{p}} \frac{\alpha'}{4} \hat{N}_5^{(b)} \ln\left[\tanh\left(\frac{z - \hat{\delta}_b}{2}\right)\right] + c.c.\,,
\end{aligned}
\tag{4.5}
$$

with $-\infty < x < \infty$ and $0 \leq y \leq \pi/2$. Here $\delta_1 < \delta_2 < \ldots < \delta_p$ are the positions of D5-brane singularities on the upper boundary of the strip ($y = \pi/2$), whereas $\hat{\delta}_1 > \hat{\delta}_2 > \ldots > \hat{\delta}_{\hat{p}}$ are the positions of NS5-brane singularities on the lower boundary ($y = 0$) (see figure 2). The points at $x = \pm\infty$ are regular interior points of the ten-dimensional geometry.

The coefficients of the logarithms determine the number of 5-branes located at the singularities. The number of D5-branes located at δ_a is denoted by $N_5^{(a)}$, while the number of NS5-branes located at $\hat{\delta}_{(b)}$ is denoted by $\hat{N}_5^{(b)}$. Unbroken supersymmetry requires that there are only branes (or only anti-branes) of each kind. Thus all the $N_5^{(a)}$ must have the same sign, and likewise for all the $\hat{N}_5^{(b)}$. This positivity condition is also necessary for smoothness away from the 5-brane singularities. The net number of D3-branes ending on the a-th D5-brane stack is denoted by $N_3^{(a)}$, while the number of D3-branes ending on the b-th NS5-brane stack is denoted by $\hat{N}_3^{(b)}$. These quantities are determined by the locations

δ_a and $\hat{\delta}_b$, of the 5-brane stacks as

$$N_3^{(a)} = N_5^{(a)} \sum_{b=1}^{\hat{p}} \hat{N}_5^{(b)} \frac{2}{\pi} \arctan(e^{\hat{\delta}_b - \delta_a}) \,,$$
$$\hat{N}_3^{(b)} = \hat{N}_5^{(b)} \sum_{a=1}^{p} N_5^{(a)} \frac{2}{\pi} \arctan(e^{\hat{\delta}_b - \delta_a}) \,. \tag{4.6}$$

We define the total number of D3-branes as $N \equiv \sum_{a=1}^{p} N_3^{(a)} = \sum_{b=1}^{\hat{p}} \hat{N}_3^{(b)}$, and the linking numbers by $l^{(a)} = N_3^{(a)}/N_5^{(a)}$ and $\hat{l}^{(b)} = \hat{N}_3^{(b)}/\hat{N}_5^{(b)}$.[6] These parameters, $N, N_5^{(a)}, \hat{N}_5^{(a)}, l^{(a)}, \hat{l}^{(a)}$, are identified with the same parameters of the same names in section 2 under the holographic duality.

4.2 The gravity action

The type IIB action in Einstein frame is[7]

$$S_{\text{IIB}} = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \bigg\{ R - \frac{4}{2}\partial_M \phi \partial^M \phi - \frac{1}{2}e^{4\phi}\partial_M \chi \partial^M \chi - \frac{1}{2}e^{-2\phi}|H_{(3)}|^2$$
$$- \frac{1}{2}e^{2\phi}|F_{(3)} + \chi H_{(3)}|^2 - \frac{1}{4}|\hat{F}_{(5)}|^2 \bigg\} + \frac{1}{4\kappa_{10}^2} \int d^{10}x \, C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \,, \tag{4.7}$$

where one imposes the self-duality condition $\hat{F}_{(5)} = *F_{(5)}$ as a supplementary equation. The coupling κ_{10} is related to the string scale α' by $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$.

Due to the presence of the self-duality condition, the action (4.7) cannot be directly used to compute the on-shell value of the action. One way to deal with this is to relax the requirement of Lorentz invariance of the action. In this case an action principle could be obtained along the lines of [25]. As suggested in [26], perhaps the easiest way to implement this for the full type IIB supergravity action is to make a T-duality transformation of the type IIA action. A simpler method is to first dimensionally reduce the theory to 4-dimensions. After carrying out the dimensional reduction, one can then truncate the theory to the 4-dimensional graviton. To see this is consistent, one may check that the solutions of [23, 24] can be extended by replacing the AdS$_4$ space with any space which obeys the same Einstein equations. Thus truncating to the 4-dimensional graviton is a consistent truncation.[8]

---

[6]The relations between the integer brane charges and the supergravity parameters are not easily inverted. To express the latter in terms of the brane charges one must solve a system of transcendental equations.

[7]We use the convention $|F_{(a)}|^2 = \frac{1}{a!} F_{(a) M_1 M_2 \dots M_a} F_{(a)}^{M_1 M_2 \dots M_a}$.

[8]To see this more explicitly, first consider the 10-dimensional metric $ds^2 = f_4^2 ds_{(4)}^2 + f_1^2 ds_{S_1^2}^2 + f_2^2 ds_{S_2^2}^2 + 4\rho dz d\bar{z}^2$, where $ds_{(4)}^2$ is an arbitrary 4-dimensional metric. This is a solution to the type IIB supergravity equations of motion as long as the 4-dimensional Ricci tensor satisfies $R_{(4)\mu\nu} = -3g_{(4)\mu\nu}$. One can then write the 10-dimensional Ricci scalar as $R = f_4^{-2} R_{(4)} + \dots$, where the omitted terms do not depend on $ds_{(4)}^2$. The action then takes the form $S = -\frac{1}{2\kappa_{10}^2} \int d^{10}x (f_4 f_1 f_2)^2 4\rho^2 \sqrt{g_{(4)}} (R_{(4)} + \dots)$, where again the omitted terms do not depend on $ds_{(4)}^2$. Requiring the variation with respect to $ds_{(4)}^2$ to now reproduce the correct equation of motion yields the effective action (4.8).

The effective action for this mode is given by

$$S_{\text{eff}} = -\frac{1}{2\kappa_{10}^2}\text{vol}_6 \int_{\text{AdS}_4} d^4x \sqrt{g_{(4)}}(R_{(4)} + 6)\,, \tag{4.8}$$

where the cosmological constant has been chosen so that the unit AdS_4 space is a solution. The subscript (4) reminds us that $g_{(4)}$ is the 4-dimensional metric and $R_{(4)}$ is the associated Ricci scalar. The quantity vol_6 follows from the initial dimensional reduction and is the volume of the internal space dressed appropriately with the warp factor of AdS_4

$$\text{vol}_6 = (4\pi)^2 \int_\Sigma d^2x (f_4 f_1 f_2)^2 4\rho^2 = 32(4\pi)^2 \int_\Sigma d^2x(-W)h_1 h_2\,. \tag{4.9}$$

The specific solution we are interested in is AdS_4 with Ricci scalar $R_{(4)} = -12$. Thus the on-shell action becomes simply

$$S_{\text{eff}} = -\frac{1}{(2\pi)^7(\alpha')^4}\text{vol}_6 \left(\frac{4}{3}\pi^2\right)(-6)\,, \tag{4.10}$$

where we have used the regularized volume of AdS_4, $\text{vol}_{AdS_4} = (4/3)\pi^2$, which may be computed using holographic regularization [27–30] (see for example section 5 of [31]).

4.3 $T[SU(N)]$

Let us first consider the gravity dual for $T[SU(N)]$. The harmonic functions are:

$$\begin{aligned}
h_1 &= -\frac{\alpha' N}{4} \ln\left[\tanh\left(\frac{\imath\pi}{4} - \frac{z-\delta}{2}\right)\right] + c.c.\,, \\
h_2 &= -\frac{\alpha' N}{4} \ln\left[\tanh\left(\frac{z+\delta}{2}\right)\right] + c.c.\,,
\end{aligned} \tag{4.11}$$

with

$$\delta = -\frac{1}{2}\ln\left[\tan\left(\frac{\pi}{2N}\right)\right]\,, \tag{4.12}$$

where we have used a translation to set $\hat{\delta} = -\delta$. There is one stack of N D5-branes at the position $z = \imath\frac{\pi}{2} - \frac{1}{2}\ln[\tan(\frac{\pi}{2N})]$ and one stack of N NS5-branes at $z = \frac{1}{2}\ln[\tan(\frac{\pi}{2N})]$ with N D3-branes stretched between them.

We now wish to take the large N limit of this configuration. It will turn out that locally the Lagrangian density will scale with a factor of N^2 at leading order in N. Secondly, as N goes to infinity, the positions δ of the 5-brane stacks are sent to infinity in opposite directions (see figure 3). This leaves a large region of geometry between $-\delta$ and δ of size $\ln N$, which will reproduce the $\ln N$ behavior of the partition function. Thus one can understand the leading behavior of the $T[SU(N)]$ partition function as coming from the geometry located between the two stacks of 5-branes.

To make this more explicit and also compute the exact numerical coefficient, we now work out the large N expansion. First we re-scale the x coordinate so that $z = \delta x + \imath y$
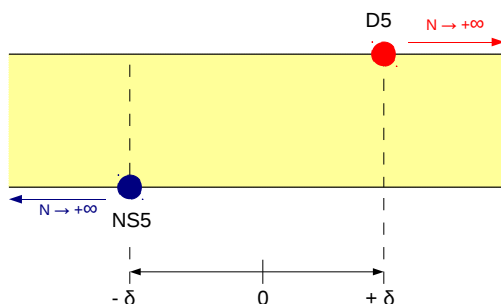
Figure 3. Geometry of the $T[SU(N)]$ dual background represented by the strip with two 5-brane singularities at positions $\pm\delta \sim \pm\frac{1}{2}\log N$. In the large N limit the stacks go to $\pm$ infinity.

and then expand the harmonic functions h_1 and h_2 around large N. At leading order we obtain

$$
\begin{aligned}
h_1 &= \alpha' \sin(y) N\, e^{\delta(x-1)} + \dots && \text{if } x < 1\,, \\
&= \alpha' \sin(y) N\, e^{\delta(1-x)} + \dots && \text{if } x > 1\,, \\
h_2 &= \alpha' \cos(y) N\, e^{\delta(1+x)} + \dots && \text{if } x < -1\,, \\
&= \alpha' \cos(y) N\, e^{-\delta(1+x)} + \dots && \text{if } x > -1\,.
\end{aligned}
\tag{4.13}
$$

From (4.13) we find that the only contribution to the action at this order comes from the central region $-1 < x < 1$. In this region W is given by $W = -\frac{1}{2}e^{-2\delta}N^2(\alpha')^2 \sin(2y)$. Computing the volume of the internal space, (4.9), and plugging into the expression for the effective action, (4.10), we find

$$
\begin{aligned}
S_{\text{eff}} &= \frac{4N^4 \delta e^{-4\delta}}{\pi^2} + \dots \\
&= \frac{1}{2}N^2 \ln N + \mathcal{O}(N^2)\,.
\end{aligned}
\tag{4.14}
$$

This reproduces exactly the leading order behavior of the CFT partition function (3.12). Finally we note that including higher order terms in the expansions of the harmonic functions will give additional contributions of order N^2.

Since we have explicit D5-brane and NS5-brane singularities in the geometry, one may worry about the validity of our approximation. We shall argue that the corrections due to the 5-brane singularities are at most of order N^2 and do not contribute to the leading $N^2 \ln N$ behavior. To do so, we first examine the geometry in the central region in the

large N limit. The metric factors are given by

$$f_4^2 = \sqrt{2}\alpha' N e^{-\delta}[(2 - \cos(2y))(2 + \cos(2y))]^{\frac{1}{4}} \,,$$

$$f_1^2 = 2\sqrt{2}\alpha' N e^{-\delta} \sin(y)^2 \left[\frac{2 + \cos(2y)}{(2 - \cos(2y))^3}\right]^{\frac{1}{4}} \,,$$

$$f_2^2 = 2\sqrt{2}\alpha' N e^{-\delta} \cos(y)^2 \left[\frac{2 - \cos(2y)}{(2 + \cos(2y))^3}\right]^{\frac{1}{4}} \,, \tag{4.15}$$

$$4\rho^2 = 2\sqrt{2}\alpha' N e^{-\delta}[(2 - \cos(2y))(2 + \cos(2y))]^{\frac{1}{4}} \,,$$

while the dilaton and fluxes are given by (see appendix B)

$$e^\phi = e^{-\delta x}\left(\frac{2 + \cos(2y)}{2 - \cos(2y)}\right)^{\frac{1}{4}} \,,$$

$$b_1 = 8\alpha' N e^{-\delta(1+x)} \frac{\sin^3(y)}{2 - \cos(2y)} \,,$$

$$b_2 = -8\alpha' N e^{\delta(x-1)} \frac{\cos^3(y)}{2 + \cos(2y)} \,, \tag{4.16}$$

$$j_1 = -e^{-2\delta} N^2 (\alpha')^2 (3x\delta + \cos(2y)) \,.$$

It is interesting to note that this is exactly the limiting geometry of Janus found in [32] for the case of an infinite jump in the coupling.[9] The radius L of the Janus space is related to N by $L^2 = 2\sqrt{2}\alpha' N e^{-\delta}$. In the case we consider here, the Σ space comes with a natural cutoff at $|x| = \delta$, while for Janus the space is unbounded.

We now consider curvature corrections. Using the above formulas for the metric factor and dilaton, the string frame Ricci scalar in the central region, $-1 \leq x \leq 1$, is given by

$$\alpha' R = \frac{1}{\pi 2^{1/2}}\left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} \frac{419 - 60\cos(4y) + \cos(8y)}{(7 - \cos(4y))^2(2 + \cos(2y))^{1/2}} \,. \tag{4.17}$$

Due to the large N limit, throughout most of the region we have $\alpha' R \ll 1$. However, due to the presence of D5-branes, as one approaches $x = 1$, $\alpha' R$ is of order one and one expects higher curvature corrections to play a role. Since these corrections are localized only in the region near $x = 1$, we expect that they do not receive the $\ln N$ enhancement and therefore contribute only at order N^2. A similar argument can be made when one examines the geometry near the D5-branes using (4.11) before taking the large N limit.

Due to the presence of N5-branes, the second issue for our calculation is to understand if the string coupling, g_s, is small so that string loop corrections can be ignored. The dilaton in the central region, $-1 < x < 1$, is given by

$$g_s = e^{2\phi} = \left(\frac{2N}{\pi}\right)^{-x} \sqrt{\frac{2 + \cos(2y)}{2 - \cos(2y)}} \,. \tag{4.18}$$

We observe that the dilaton is small in the region $0 < x < 1$ but is big in the region $-1 < x < 0$. We first focus our attention on the region $0 < x < 1$. In the large N limit,

---

[9]The supersymmetric Janus solution is dual to $\mathcal{N} = 4$ super-Yang-Mills with a jumping coupling.

the string coupling is small except in the neighborhood of $x = 0$, where it is of order one. Thus we expect string loop corrections to be important, but again we argue that since they are localized near $x = 0$, they will give contributions at most of order N^2.

For the region $-1 < x < 0$, we find that the string coupling is generically large and one might expect string loop corrections to modify the leading $N^2 \ln N$ behavior. From this point of view, the exact match between gravity and CFT partition functions is surprising and we do not have a good a priori argument for why string loop corrections do not modify the $N^2 \ln N$ behavior. One possible explanation can be given in terms of a local S-duality transformation in this region. To be more precise, we divide the manifold into three regions $-1 < x < -\epsilon$, $-\epsilon < x < \epsilon$ and $\epsilon < x < 1$ with $\epsilon \ll 1$. In the first region, we make an S-duality transformation, while in the third region the theory is already weakly coupled. The middle region then has to interpolate between two different S-duality frames and we do not know how to compute the action there. However, since the $\ln N$ enhancement requires the entire internal space and patching only needs to occur locally in the region near $x = 0$, one might hope that the middle region does not receive the $\ln N$ enhancement. Of course this argument is only heuristic and it would be interesting to either make it more precise or determine the exact mechanism for why the loop corrections are suppressed. Similar situations arise when one examines the geometry near the NS5-branes using (4.11) before taking the large N limit.

4.4 $T^\rho_{\hat\rho}[SU(N)]$

We now consider more general partitions which take the form (2.9). In this case, there is a single NS5-brane stack and the charge relations, (4.6), can be easily inverted to express the phases δ_a and $\hat\delta$ in terms of the partitions ρ and $\hat\rho$:

$$\delta_a - \hat\delta = -\ln\left[\tan\left(\frac{\pi}{2}\frac{l^{(a)}}{\hat{N}_5}\right)\right] \, . \tag{4.19}$$

To analyze the large N behavior, we proceed analogously to the $T[SU(N)]$ case and consider the limit where $\hat\delta \to -\infty$ and the $\delta_a \to \infty$. In this case, we approximate the harmonic functions by the following expressions

$$
\begin{aligned}
h_1 &= \alpha' \sin(y) \sum_{a=1}^{p} N_5^{(a)} e^{x-\delta_a} + \ldots \quad \text{if } x < \delta_1 \, , \\
&= \alpha' \sin(y) \sum_{a=i}^{p} N_5^{(a)} e^{x-\delta_a} + \ldots \quad \text{if } \delta_i < x < \delta_{i+1} \, , \\
h_2 &= \alpha' \cos(y) \hat{N}_5 e^{-(x-\hat\delta)} + \ldots \qquad \text{if } x > \hat\delta \, ,
\end{aligned}
\tag{4.20}
$$

while the regions with $x > \delta_p$ and $x < \hat{\delta}$ will give only subleading contributions. In this approximation we find that $W = -h_1 h_2$ so that

$$
\begin{aligned}
-W h_1 h_2 &= \frac{1}{4}(\alpha')^4 \hat{N}_5^2 \left(\sum_{a=1}^{p} N_5^{(a)} e^{-(\delta_a - \hat{\delta})} \right)^2 \sin^2(2y) \quad \text{if } \hat{\delta} < x < \delta_1 , \\
&= \frac{1}{4}(\alpha')^4 \hat{N}_5^2 \left(\sum_{a=i}^{p} N_5^{(a)} e^{-(\delta_a - \hat{\delta})} \right)^2 \sin^2(2y) \quad \text{if } \delta_i < x < \delta_{i+1} .
\end{aligned}
\tag{4.21}
$$

Using this in (4.9) we find

$$
\begin{aligned}
\text{vol}_6 &= 32(4\pi)^2 \int_{\hat{\delta}}^{\delta_p} dx \int_0^{\frac{\pi}{2}} dy \, (-W h_1 h_2) \\
&= 32\pi^3 (\alpha')^4 \hat{N}_5^2 \sum_{i=1}^{p} \left(\sum_{a=i}^{p} N_5^{(a)} e^{-(\delta_a - \hat{\delta})} \right)^2 (\delta_i - \delta_{i-1})
\end{aligned}
\tag{4.22}
$$

where we define $\delta_0 \equiv \hat{\delta}$. Plugging into (4.10) and combining all of the numerical factors, we obtain

$$
S_{\text{eff}} = \frac{2}{\pi^2} \hat{N}_5^2 \sum_{i=1}^{p} \left(\sum_{a=i}^{p} N_5^{(a)} e^{-(\delta_a - \hat{\delta})} \right)^2 (\delta_i - \delta_{i-1}) + \dots \ .
\tag{4.23}
$$

We now consider the scaling behavior defined by (2.10). The idea is to introduce separations between the δ_a which are of order $\ln N$. In this case each region between a given δ_a and δ_{a+1} will contribute to the action at order $N^2 \ln N$. In terms of this scaling the action becomes

$$
\begin{aligned}
S_{\text{eff}} &= \frac{1}{2} N^2 \left[\left(\sum_{a=1}^{p} \gamma_a \lambda^{(a)} \right)^2 \ln \left(\frac{2}{\pi} \frac{\hat{\gamma}}{l^{(1)}} N \right) + \sum_{i=2}^{p} \left(\sum_{a=i}^{p} \gamma_a \lambda^{(a)} \right)^2 \ln \left(\frac{l^{(i-1)}}{l^{(i)}} \right) \right] + \mathcal{O}(N^2) , \\
&= \frac{1}{2} N^2 \ln N \left[(1 - \kappa_1) + \sum_{i=2}^{p} \left(\sum_{a=i}^{p} \gamma_a \lambda^{(a)} \right)^2 (\kappa_{i-1} - \kappa_i) \right] + \mathcal{O}(N^2) ,
\end{aligned}
\tag{4.24}
$$

which coincides with (2.13).

4.5 Subleading terms

So far we have concentrated on the leading $N^2 \ln N$ contributions to the free energy and it is a natural question to ask about the subleading N^2 contributions. Comparing the CFT and gravity partition functions, we find that the subleading N^2 contributions do not match.[10] However, this is not surprising since the gravity solution contains 5-brane singularities around which supergravity approximation breaks down. Additionally, there are regions in the bulk of Σ where the string coupling becomes large. It would be interesting to interpret

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[10]We have checked this numerically for $T[SU(N)]$ using the full expressions for the harmonic functions (4.11).

and if possible match the subleading contributions to the CFT partition function with higher curvature corrections, coming from both string and loop corrections, on the gravity side. For the $T[SU(N)]$ theory, we note that near the D5-brane singularity, the Ricci scalar, (4.17) does not depend on N and so all powers of R will contribute at order N^2. Similarly, one may check that other contractions of the Riemann tensor will also contribute at order N^2. Thus even at order N^2, the CFT partition function contains information about all orders of the higher curvature corrections.

Acknowledgments

A Barnes G-function

Let us briefly summarize the properties of the Barnes G-function. Barnes G-function $G_2(z)$ satisfies

$$G_2(z+1) = \Gamma(z)G_2(z), \quad G_2(1) = 1 . \tag{A.1}$$

From the definition it follows that

$$G_2(N) = (N-2)!(N-3)!\cdots 1!, \quad N = 2, 3, \cdots . \tag{A.2}$$

Its asymptotic expansion is given by

$$\ln G_2(N+1) = \frac{N^2}{2}\ln N - \frac{3}{4}N^2 + \mathcal{O}(N) . \tag{A.3}$$

B Flux formulas

The NS-NS and R-R three forms can be written as

$$\underline{\text{3-forms:}} \quad H_{(3)} = \omega^{45} \wedge db_1 \quad \text{and} \quad F_{(3)} = \omega^{67} \wedge db_2 , \tag{B.1}$$

where ω^{45} and ω^{67} are the volume forms of the unit-radius spheres S_1^2 and S_2^2, while the gauge potentials b_1 and b_2 are given by

$$b_1 = 2\imath h_1 \frac{h_1 h_2 (\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_1} + 2h_2^D ,$$
$$b_2 = 2\imath h_2 \frac{h_1 h_2 (\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_2} - 2h_1^D . \tag{B.2}$$

In this expression one needs the dual harmonic functions, defined by

$$h_1 = -\imath(\mathcal{A}_1 - \bar{\mathcal{A}}_1) \qquad \rightarrow \qquad h_1^D = \mathcal{A}_1 + \bar{\mathcal{A}}_1 \,,$$
$$h_2 = \mathcal{A}_2 + \bar{\mathcal{A}}_2 \qquad \rightarrow \qquad h_2^D = \imath(\mathcal{A}_2 - \bar{\mathcal{A}}_2) \,. \tag{B.3}$$

for holomorphic harmonic functions $\mathcal{A}_1, \mathcal{A}_2$. The constant ambiguity in the definition of the dual functions is related to changes of the background fields under large gauge transformations. The expression for the gauge-invariant self-dual 5-form is given by:

$$\underline{\text{5-form:}} \qquad F_{(5)} = -4\, f_4^4\, \omega^{0123} \wedge \mathcal{F} + 4\, f_1^2 f_2^2\, \omega^{45} \wedge \omega^{67} \wedge (*_2 \mathcal{F}) \,, \tag{B.4}$$

where ω^{0123} is the volume form of the unit-radius AdS$_4$, $\mathcal{F}$ is a 1-form on $\hat{\rho}$ with the property that $f_4^4 \mathcal{F}$ is closed, and $*_2$ denotes Poincaré duality with respect to the $\hat{\rho}$ metric. The explicit expression for $\mathcal{F}$ is given by

$$f_4^4 \mathcal{F} = dj_1 \qquad \text{with} \qquad j_1 = 3\mathcal{C} + 3\bar{\mathcal{C}} - 3\mathcal{D} + \imath \frac{h_1 h_2}{W} (\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2) \,, \tag{B.5}$$

where $\mathcal{C}$ and $\mathcal{D}$ are defined by $\partial \mathcal{C} = \mathcal{A}_1 \partial \mathcal{A}_2 - \mathcal{A}_2 \partial \mathcal{A}_1$ and $\mathcal{D} = \bar{\mathcal{A}}_1 \mathcal{A}_2 + \mathcal{A}_1 \bar{\mathcal{A}}_2$.

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