Large N matrix models for 3d $\mathcal{N} = 2$ theories: twisted index, free energy and black holes

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ABSTRACT: We provide general formulae for the topologically twisted index of a general three-dimensional $\mathcal{N} \geq 2$ gauge theory with an M-theory or massive type IIA dual in the large N limit. The index is defined as the supersymmetric path integral of the theory on $S^2 \times S^1$ in the presence of background magnetic fluxes for the R- and global symmetries and it is conjectured to reproduce the entropy of magnetically charged static BPS AdS$_4$ black holes. For a class of theories with an M-theory dual, we show that the logarithm of the index scales indeed as $N^{3/2}$ (and $N^{5/3}$ in the massive type IIA case). We find an intriguing relation with the (apparently unrelated) large N limit of the partition function on S^3. We also provide a universal formula for extracting the index from the large N partition function on S^3 and its derivatives and point out its analogy with the attractor mechanism for AdS black holes.

Contents

1   Introduction

In this paper we study the large N behavior of the topologically twisted index introduced in [1] for three-dimensional $\mathcal{N} \geq 2$ gauge theories. It is defined as the partition function of the theory on $S^2 \times S^1$ with a topological twist along S^2 [2, 3] and it is a function of magnetic charges and chemical potentials for the flavor symmetries. The large N limit of the index for the ABJM theory was successfully used in [4] to provide the first microscopic counting of the microstates of an AdS$_4$ black hole. Here we extend the analysis of [4] to a larger class of $\mathcal{N} \geq 2$ theories with an M-theory or massive type IIA dual, containing bi-fundamental, adjoint and (anti-)fundamental chiral matter. Most of the theories proposed in the literature are obtained by adding Chen-Simons terms [5–12] or by flavoring [13–15] four-dimensional quivers describing D3-branes probing CY$_3$ singularities. We refer to these theories as having a four-dimensional parent. They all have an M-theory phase where the index is expected to scale as $N^{3/2}$. The main motivation for studying the large N limit of the index for these theories comes indeed form the attempt to extend the result of [4] to a larger class of black holes, and we hope to report on the subject soon. However, the matrix model computing the index reveals an interesting structure at large N which deserves attention by itself. In particular, we will point out analogies and relations with other matrix models appeared in the literature on three-dimensional $\mathcal{N} \geq 2$ gauge theories.

The index can be evaluated using supersymmetric localization and it reduces to a matrix model. It can be written as the contour integral

$$Z(\mathfrak{n}, y) = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} Z_{\text{int}}(\mathfrak{m}, x; \mathfrak{n}, y) \qquad (1.1)$$

of a meromorphic differential form in variables x parameterizing the Cartan subgroup and subalgebra of the gauge group, summed over the lattice of magnetic charges $\mathfrak{m}$ of the group. The index depends on complex fugacities y and magnetic charges $\mathfrak{n}$ for the flavor symmetries. As a difference with other well known matrix models arising from supersymmetric localization in three dimensions, like the partition function on S^3 [16–18] or the superconformal index [19], in the large N limit all the gauge magnetic fluxes contribute to the integral making difficult its evaluation. Here we use the strategy employed in [4] to explicitly resum the integrand and consider the contour integral of the sum

$$Z_{\text{resummed}}(x; \mathfrak{n}, y) = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} Z_{\text{int}}(\mathfrak{m}, x; \mathfrak{n}, y) \qquad (1.2)$$

which is a complicated rational function of x. We set up, as in [4], an auxiliary large N problem devoted to find the positions of the poles of Z_{resummed} in the plane x. We write a set of algebraic equations for the position of the poles, which we call *Bethe Ansatz Equations* (BAEs), and we write a *Bethe functional* whose derivative reproduces the BAEs. The method for solving the BAEs is similar to that used in [20, 21] for the large N limit of the partition function on S^3 in the M-theory limit and the one used for the partition function on S^5 of five-dimensional theories [22–24]. We take an ansatz for the eigenvalues where the imaginary parts grow in the large N limit as some power of N. The solution of

the BAEs in the large N limit is then used to evaluate $Z(\mathfrak{n}, y)$ using the residue theorem. In this last step we need to take into account (exponentially small) corrections to the large N limit of the BAEs which contribute to the index due to the singular logarithmic behavior of its integrand.

We focus on the limit where N is much greater than the Chern-Simons couplings k_a. For the class of quivers we are considering, this limit corresponds to an M-theory description when $\sum_a k_a = 0$ and a massive type IIA one when $\sum_a k_a \neq 0$. We recover the known scalings $N^{3/2}$ and $N^{5/3}$ for the M-theory and massive type IIA phase, respectively. Similarly to [21], we find that, in order to have a consistent $N^{3/2}$ scaling of the index in the M-theory phase, we need to impose some constraints on the quiver. In particular, quivers with a chiral 4d parent are not allowed, as in [21]. They are instead allowed in the massive type IIA phase.

In the course of our analysis, we find a number of interesting general results.

First, we find a simple universal formula for computing the index from the Bethe potential, $\overline{\mathcal{V}}(\Delta_I)$, as a function of the chemical potentials,

$$\mathbb{Re} \log Z = -\frac{2}{\pi} \overline{\mathcal{V}}(\Delta_I) - \sum_I \left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi} \right) \frac{\partial \overline{\mathcal{V}}(\Delta_I)}{\partial \Delta_I} \right] . \tag{1.3}$$

We call this the *index theorem.* It allows to avoid the many technicalities involved in taking the residues and including exponentially small corrections to the index. By comparing the index theorem with the attractor formula for the entropy of asymptotically AdS$_4$ black holes, we are also led to conjecture a relation between the Bethe potential and the prepotential of the dimensionally truncated gauged supergravity describing the compactification on AdS$_4 \times Y_7$, with Y_7 a Sasaki-Einstein manifold. This relation is discussed in section 7.

Secondly, we find an explicit relation between the Bethe potential and the S^3 free energy of the same $\mathcal{N} \geq 2$ gauge theory. Although the two matrix models are quite different at finite N, the BAEs and the functional form of the Bethe potential in the large N limit are *identical* to the matrix model equations of motion and free energy functional for the path integral on S^3 found in [21]. This result implies that the index can be extracted from the free energy on S^3 and its derivatives in the large N limit. It also implies a relation with the volume functional of (Sasakian deformations of) the internal manifold Y_7. These relations deserve a better understanding.

In this paper we give the general rules for constructing the Bethe potential and the index for a generic Yang-Mills-Chen-Simons theory with bi-fundamental, adjoint and fundamental fields and few explicit examples of their application. Many other examples can be found in an upcoming paper by one of the authors [25], including models for well-known homogeneous Sasaki-Einstein manifolds, $N^{0,1,0}, Q^{1,1,1}, V^{5,2}$, and various nontrivial checks of dualities. We leave the most interesting part of the story concerning applications to the microscopic counting for AdS$_4$ black holes for the future.

The paper is organized as follows. In section 2 we review the definition of the topologically twisted index and the strategy for determining its large N limit used in [4]. In section 3 we give the general rules for constructing the Bethe potential and the index for a generic Yang-Mills-Chen-Simons theory with bi-fundamentals, adjoint and fundamental

fields with $N^{3/2}$ scaling. In section 4 we prove the identity of the Bethe potential and the S^3 free energy at large N. In section 5 we derive the index theorem that allows to express the index at large N in terms of the Bethe potential and its derivatives. In section 6 we discuss the rules for a $N^{5/3}$ scaling. In section 7 we give a discussion of some open issues and point out analogies with the attractor mechanism for AdS black holes. Appendices A and C contains the explicit derivations of the rules for $N^{3/2}$ and $N^{5/3}$ scalings, respectively. Appendix B contains an explicit example, based on the quiver for the Suspended Pinch Point.

2 The topologically twisted index

The topologically twisted index of an $\mathcal{N} \geq 2$ gauge theory in three dimensions is defined as the partition function on $S^2 \times S^1$ with a topological twist along S^2 [1]. It depends on a choice of fugacities y for the global symmetries and magnetic charges $\mathfrak{n}$ on S^2 parameterizing the twist. The index can be computed using localization and it is given by a matrix integral over the zero-mode gauge variables and it is summed over a lattice of gauge magnetic charges on S^2. Explicitly, for a theory with gauge group G of rank r and a set of chiral multiplets transforming in representations $\mathfrak{R}_I$ of G, the index is given by [1]

$$Z(\mathfrak{n}, y) = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} \prod_{\text{Cartan}} \left(\frac{dx}{2\pi i x} x^{k\mathfrak{m}} \right) \prod_{\alpha \in G} (1 - x^{\alpha}) \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \left(\frac{x^{\rho_I/2} y_I}{1 - x^{\rho_I} y_I} \right)^{\rho_I(\mathfrak{m}) - \mathfrak{n}_I + 1},$$
(2.1)

where α are the roots of G and ρ_I are the weights of the representation $\mathfrak{R}_I$. In this formula,[1] $x = e^{i(A_t + i\beta\sigma)}$ parameterizes the gauge zero modes, where A_t is a Wilson line on S^1 and runs over the maximal torus of G while σ is the real scalar in the vector multiplet and runs over the corresponding Cartan subalgebra. $\mathfrak{m}$ are gauge magnetic fluxes living in the co-root lattice $\Gamma_{\mathfrak{h}}$ of G (up to gauge transformations). The index is integrated over x and summed over $\mathfrak{m}$. k is the Chern-Simons coupling for the group G, and there can be a different one for each Abelian and simple factor in G. Supersymmetric localization selects a particular contour of integration and the final result can be formulated in terms of the Jeffrey-Kirwan residue [1].

The index depends on a choice of fugacities y_I for the flavor group and a choice of integer magnetic charges $\mathfrak{n}_I$ for the R-symmetry of the theory. In an $\mathcal{N} \geq 2$ theory, the R-symmetry can mix with the global symmetries and we can also write

$$\mathfrak{n}_I = q_I + \mathfrak{p}_I,$$
(2.2)

where q_I is a reference R-symmetry and $\mathfrak{p}_I$ magnetic charges under the flavor symmetries of the theory. Both y_I and $\mathfrak{n}_I$ are thus parameterized by the global symmetries of the theory. Each monomial term W in the superpotential imposes a constraint

$$\prod_{I \in W} y_I = 1, \qquad \sum_{I \in W} \mathfrak{n}_I = 2,$$
(2.3)

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[1] β is the radius of S^1.

where the product and sum are restricted to the fields entering in W. Each Abelian gauge group in three dimensions is associated with a topological U(1) symmetry. The contribution of a topological symmetry with fugacity $\xi = e^{iz}$ and magnetic flux $\mathfrak{t}$ to the index is given by

$$Z^{\text{top}} = x^{\mathfrak{t}} \xi^{\mathfrak{m}}, \tag{2.4}$$

where x is the gauge variable of the corresponding U(1) gauge field.

In this paper we are interested in the large N limit of the topologically twisted index for theories with unitary gauge groups and matter transforming in the fundamentals, bi-fundamentals and adjoint representation. As in [4], we evaluate the matrix model in two steps. We first perform the summation over magnetic fluxes introducing a large cut-off M.[2] The result of this summation produces terms in the integrand of the form

$$\prod_{i=1}^{N} \frac{\left(e^{iB_i^{(a)}}\right)^M}{e^{iB_i^{(a)}} - 1}, \tag{2.5}$$

where we defined

$$e^{i\,\text{sign}(k_a)B_i^{(a)}} = \xi^{(a)} (x_i^{(a)})^{k_a} \prod_{\substack{\text{bi-fundamentals} \\ (a,b) \text{ and } (b,a)}} \prod_{j=1}^{N} \frac{\sqrt{\frac{x_i^{(a)}}{x_j^{(b)}} y_{(a,b)}}}{1 - \frac{x_i^{(a)}}{x_j^{(b)}} y_{(a,b)}} \frac{1 - \frac{x_j^{(b)}}{x_i^{(a)}} y_{(b,a)}}{\sqrt{\frac{x_j^{(b)}}{x_i^{(a)}} y_{(b,a)}}}$$

$$\times \prod_{\substack{\text{fundamentals} \\ a}} \frac{\sqrt{x_i^{(a)} y_a}}{1 - x_i^{(a)} y_a} \prod_{\substack{\text{anti-fundamentals} \\ a}} \frac{1 - \frac{1}{x_i^{(a)}} \tilde{y}_a}{\sqrt{\frac{1}{x_i^{(a)}} \tilde{y}_a}}, \tag{2.6}$$

and adjoints are identified with bi-fundamentals connecting the same gauge group $(a = b)$. In this way the contributions from the residues at the origin have been moved to the solutions of the "Bethe Ansatz Equations" (BAEs)

$$e^{i\,\text{sign}(k_a)B_i^{(a)}} = 1. \tag{2.7}$$

It is convenient to use the variables $u_i^{(a)}$ and Δ_I, defined modulo 2π,[3]

$$x_i^{(a)} = e^{iu_i^{(a)}}, \qquad y_I = e^{i\Delta_I}, \qquad \xi^{(a)} = e^{i\Delta_m^{(a)}}, \tag{2.8}$$

and take the logarithm of the Bethe ansatz equations

$$0 = \log\left[\text{r.h.s. of } (2.6)\right] - 2\pi i n_i^{(a)}, \tag{2.9}$$

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[2]According to the rules in [1], the residues to take in (2.1) depend on the sign of the Chern-Simons couplings. We can choose a set of co-vectors in the Jeffrey-Kirwan prescription such that the contribution comes from residues with $\mathfrak{m}_a \leq 0$ for $k_a > 0$, residues with $\mathfrak{m}_a \geq 0$ for $k_a < 0$ and residues in the origin. We can then take a large positive integer M and perform the summations in eq. (2.1), with $\mathfrak{m}_a \leq M - 1\,(k_a > 0)$ and $\mathfrak{m}_a \geq 1 - M\,(k_a < 0)$.

[3]Notice that the index is a holomorphic function of y_I and ξ. There is no loss of generality in restrict to the case of purely imaginary chemical potentials Δ in (2.8).

where $n_i^{(a)}$ are integers that parameterize the angular ambiguities. The BAEs (2.9) can be obtained as critical points of a "Bethe potential" $\mathcal{V}(u_i^{(a)})$.

We then need to solve these auxiliary equations in the large N limit. Once the distribution of poles in the integrand in the large N limit has been found, we can finally evaluate the index by computing the residue of the resummed integrand of (2.1) at the solutions of (2.9). In the final expression, the dependence on M disappears.

3 The large N limit of the index

We are interested in the properties of the topologically twisted index in the large N limit of theories with an M-theory dual. We focus on quiver Chern-Simons-Yang-Mills gauge theories with gauge group

$$\mathcal{G} = \prod_{a=1}^{|G|} \mathrm{U}(N)_a \,, \tag{3.1}$$

and bi-fundamental, adjoint and fundamental chiral multiplets. Most of the conjectured theories living on M2-branes probing CY_4 singularities are of this form. Moreover, many of them are obtained by adding Chern-Simons terms and fundamental flavors to quivers appeared in the four-dimensional literature as describing D3-branes probing CY_3 singularities. We refer to these theories as quivers *with a 4d parent*. In order to have a CY_4 moduli space, the Chern-Simons couplings must satisfy

$$\sum_{a=1}^{|G|} k_a = 0 \,. \tag{3.2}$$

The M-theory phase of these theories is obtained for $N \gg k_a$ and this is the limit we consider here. We expect the index to scale as $N^{3/2}$.

As in [4], we consider the following ansatz for the large N saddle-point eigenvalue distribution:

$$u_i^{(a)} = i N^{1/2} t_i + v_i^{(a)} \,. \tag{3.3}$$

Notice that the imaginary parts of all the $u_i^{(a)}$ are equal. In the large N limit, we define the continuous functions $t_i = t(i/N)$ and $v_i^{(a)} = v^{(a)}(i/N)$ and we introduce the density of eigenvalues

$$\rho(t) = \frac{1}{N} \frac{di}{dt} \,, \tag{3.4}$$

normalized as $\int dt \, \rho(t) = 1$.

The large N limit of the Bethe potential is performed in details in appendix A.1, generalizing the analysis in [4]. Here, we report the final result and some of the crucial subtleties. We need to require the cancellations of long-range forces in the BAEs, as originally observed in a similar context in [21], and this imposes some constraints on the quiver. Once these are satisfied, the Bethe potential $\mathcal{V}$ becomes a local functional of $\rho(t)$ and $v_i^{(a)}(t)$ and it scales as $N^{3/2}$. The same constraints guarantee that the index itself scales as $N^{3/2}$.

3.1 Cancellation of long-range forces

As in [21], when bi-fundamentals are present, we need to cancel long-range forces in the BAEs. These are detected by considering the force exerted by the eigenvalue $u_j^{(b)}$ on the eigenvalue $u_i^{(a)}$ in (2.9). They can grow with large powers of N and need to be canceled by imposing constraint on the quiver and matter content if necessary. Since $u_j^{(b)} - u_i^{(a)} \sim \sqrt{N}$ for $i \neq j$, when the long-range forces vanish, the BAE and the Bethe potential get only contributions from $i \sim j$ and they become local functionals of $\rho(t)$ and $v_i^{(a)}(t)$.

Let us consider the effects of such long-range forces in the Bethe potential $\mathcal{V}$. A single bi-fundamental field connecting gauge groups a and b contributes terms of the form

$$\sum_{i<j} \frac{\left(u_i^{(a)} - u_j^{(b)}\right)^2}{4} - \sum_{i<j} \frac{\left(u_j^{(a)} - u_i^{(b)}\right)^2}{4}, \tag{3.5}$$

to the Bethe potential [see eq. (A.27)]. In the large N limit, they are of order $N^{5/2}$. In order to cancel these terms, we are then forced, as in [21], to consider quivers where for each bi-fundamental connecting a and b there is also a bi-fundamental connecting b and a. The contribution of the two bi-fundamentals then cancel out [see eq. (A.19) and eq. (A.21)].

From a pair of bi-fundamentals, we get another contribution to the Bethe potential of the form [see eq. (A.24)]

$$-\frac{1}{2}\left[\left(\Delta_{(a,b)} - \pi\right) + \left(\Delta_{(b,a)} - \pi\right)\right] \sum_{j \neq i}^{N} \left(u_i^{(a)} - u_j^{(b)}\right) \operatorname{sign}(i-j). \tag{3.6}$$

This term can be canceled by the contribution of the angular ambiguities in (2.9) to the Bethe potential $\mathcal{V}$

$$2\pi \sum_{i=1}^{N} n_i^{(a)} u_i^{(a)}, \tag{3.7}$$

provided that,[4]

$$\sum_{I \in a} (\pi - \Delta_I) \in 2\pi\mathbb{Z}, \tag{3.8}$$

where the sum is taken over all bi-fundamental fields with one leg in the node a.[5] Since for any reasonable quiver the number of arrows entering a node is the same as the number of arrow leaving it, this equation is obviously equivalent to $\sum_{I \in a} \Delta_I \in 2\pi\mathbb{Z}$ and can be also written as

$$\prod_{I \in a} y_I = 1. \tag{3.9}$$

This condition implies that the sum of the charges under all global symmetries of the bi-fundamental fields at each node must vanish. For quivers with a 4d parent, this is

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[4]This is actually true only when N is odd. For even N we are left with a common factor $\pi \sum_{i=1}^{N} u_i^{(a)}$ which can be reabsorbed in the definition of $\xi^{(a)}$.

[5]Adjoint fields are supposed to be counted twice.

equivalent to the absence of anomalies for the global symmetries of the 4d theory. Taking the product over all the nodes in a quiver, we also get

$$\text{Tr}\, J = 0\,, \tag{3.10}$$

for any global symmetry of the theory, where the trace is taken over all the bi-fundamental fermions.

There are also contributions to the Bethe potential of $\mathcal{O}(N^2)$. The Chern-Simons terms give indeed

$$\sum_a k_a \sum_i^N \frac{\left(u_i^{(a)}\right)^2}{2}\,. \tag{3.11}$$

However, the $\mathcal{O}(N^2)$ term cancels out when the condition (3.2) is satisfied. Finally, there is a $O(N^2)$ contributions of the fundamental fields given by (A.34). This vanishes if the *total* number of fundamental and anti-fundamental fields in the quiver is the same.

We turn next to the large N limit of the index. The vector multiplet contributes a term of $\mathcal{O}(N^{5/2})$ [see eq. (A.36)]

$$i \sum_{i<j}^N \left(u_i^{(a)} - u_j^{(a)} + \pi \right)\,. \tag{3.12}$$

The contribution of $\mathcal{O}(N^{5/2})$ of a chiral multiplet is [see eq. (A.43)]:

$$i \sum_{I \in a} \frac{(\mathfrak{n}_I - 1)}{2} \sum_{i<j}^N \left(u_i^{(a)} - u_j^{(a)} + \pi \right)\,. \tag{3.13}$$

To have a cancellation between terms of $\mathcal{O}(N^{5/2})$ and $\mathcal{O}(N^2)$ for each node a we must have

$$2 + \sum_{I \in a} (\mathfrak{n}_I - 1) = 0\,. \tag{3.14}$$

For a quiver with a 4d parent, this condition is equivalent to the absence of anomalies for the R-symmetry. If we sum over all the nodes we also obtain the following constraint

$$|G| + \sum_I (\mathfrak{n}_I - 1) = 0\,. \tag{3.15}$$

The above equation is equivalent to $\text{Tr}\, R = 0$ for any trial R-symmetry, where the trace is taken over all the bi-fundamental fermions and gauginos.

Summarizing, we can have a $N^{3/2}$ scaling for the index if for each bi-fundamental connecting a and b there is also a bi-fundamental connecting b and a, the total number of fundamental and anti-fundamental fields in the quiver is equal, eq. (3.9) and eq. (3.14) are fulfilled. All these conditions are automatically satisfied for quivers with a toric vector-like 4d parent and also for other interesting models like [26]. However, they rule out many interesting chiral quivers appeared in the literature on M2-branes. We note a striking analogy with the conditions imposed in [21].

3.2 Bethe potential at large N

In this section we give the general rules for constructing the Bethe potential for any $\mathcal{N} \geq 2$ quiver gauge theory which respects the constraints (3.9) and (3.14):

1. Each group a with CS level k_a and chemical potential for the topological symmetry $\Delta_m^{(a)}$ contributes the term

$$-ik_a N^{3/2} \int dt\, \rho(t)\, t\, v_a(t) - i\Delta_m^{(a)} N^{3/2} \int dt\, \rho(t)\, t\,. \qquad (3.16)$$

2. A pair of bi-fundamental fields, one with chemical potential $\Delta_{(a,b)}$ and transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ of $\mathrm{U}(N)_a \times \mathrm{U}(N)_b$ and the other with chemical potential $\Delta_{(b,a)}$ and transforming in the $(\overline{\mathbf{N}}, \mathbf{N})$ of $\mathrm{U}(N)_a \times \mathrm{U}(N)_b$, contributes

$$iN^{3/2} \int dt\, \rho(t)^2 \left[g_+ \left(\delta v(t) + \Delta_{(b,a)} \right) - g_- \left(\delta v(t) - \Delta_{(a,b)} \right) \right]\,, \qquad (3.17)$$

where $\delta v(t) \equiv v_b(t) - v_a(t)$. Here, we introduced the polynomial functions

$$g_\pm(u) = \frac{u^3}{6} \mp \frac{\pi}{2} u^2 + \frac{\pi^2}{3} u\,, \qquad g'_\pm(u) = \frac{u^2}{2} \mp \pi u + \frac{\pi^2}{3}\,, \qquad (3.18)$$

and we assumed them to be in the range

$$0 < \delta v + \Delta_{(b,a)} < 2\pi\,, \qquad -2\pi < \delta v - \Delta_{(a,b)} < 0\,, \qquad (3.19)$$

which can be adjusted by choosing a specific determination for the Δ that are defined modulo 2π. We will also assume, and this is certainly true if δv assumes the value zero, that

$$0 < \Delta_I < 2\pi\,. \qquad (3.20)$$

3. An adjoint field with chemical potential $\Delta_{(a,a)}$, contributes

$$ig_+(\Delta_{(a,a)}) N^{3/2} \int dt\, \rho(t)^2\,. \qquad (3.21)$$

4. A field X_a with chemical potential Δ_a transforming in the fundamental of $\mathrm{U}(N)_a$, contributes

$$-\frac{i}{2} N^{3/2} \int dt\, \rho(t)\, |t| \left[v_a(t) + \left(\Delta_a - \pi \right) \right]\,, \qquad (3.22)$$

while an anti-fundamental field with chemical potential $\tilde{\Delta}_a$ contributes[6]

$$\frac{i}{2} N^{3/2} \int dt\, \rho(t)\, |t| \left[v_a(t) - \left(\tilde{\Delta}_a - \pi \right) \right]\,. \qquad (3.23)$$

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[6]We also assume $0 < v_a(t) + \Delta_a < 2\pi$ and $0 < -v_a(t) + \tilde{\Delta}_a < 2\pi$.

Adding all the previous contributions for all gauge groups and matter fields, we get a local functional $\mathcal{V}(\rho(t), v_a(t), \Delta_I)$ that we need to extremize with respect to the continuous functions $\rho(t)$ and $v_a(t)$ with the constraint $\int dt\rho(t) = 1$. Equivalently we can introduce a Lagrange multiplier μ and extremize

$$\mathcal{V}\left(\rho(t), v_a(t), \Delta_I\right) - \mu\left(\int dt\rho(t) - 1\right). \tag{3.24}$$

This gives the large N limit distribution of poles in the index matrix model.

The solutions of the BAEs have a typical piece-wise structure. Eq. (3.24) is the right functional to extremize when the conditions (3.19) are satisfied. This gives a central region where $\rho(t)$ and $v_a(t)$ vary with continuity as functions of t. When one of the $\delta v(t)$ associated with a pair of bi-fundamental hits the boundaries of the inequalities (3.19), it remains frozen to a constant value $\delta v = -\Delta_{(b,a)}$ (mod 2π) or $\delta v = \Delta_{(b,a)}$ (mod 2π) for larger (or smaller) values of t. This creates "tail" regions where one or more δv are frozen and the functional (3.24) is extremized with respect to the remaining variables. In the tails, the derivative of (3.24) with respect to the frozen variable is not zero and it is compensated by subleading terms that we omitted. To be precise, the equations of motion [see eq. (A.14)] includes subleading terms

$$\frac{\partial \mathcal{V}}{\partial(\delta v)} + iN\rho\left[\mathrm{Li}_1\left(e^{i\left(\delta v + \Delta_{(b,a)}\right)}\right) - \mathrm{Li}_1\left(e^{i\left(\delta v - \Delta_{(a,b)}\right)}\right)\right] = 0, \tag{3.25}$$

which are negligible except on the tails, where δv has exponentially small correction to the large N constant value

$$\delta v = -\Delta_{(b,a)} + e^{-N^{1/2}Y_{(b,a)}}, \qquad \delta v = \Delta_{(a,b)} - e^{-N^{1/2}Y_{(a,b)}}, \qquad (\text{mod } 2\pi). \tag{3.26}$$

The quantities Y are determined by equation (3.25) and contribute to the large N limit of the index.

3.2.1 The ABJM example

As an example, we briefly review here the solution to the BAEs for the ABJM model found in [4]. A more complicated example, for a $\mathrm{U}(N)^3$ quiver is discussed in appendix B. The reader can find many other examples in [25]. ABJM is a Chern-Simons-matter theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, with two pairs of bi-fundamental fields A_i and B_i transforming in the representation $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$ of the gauge group, respectively, and superpotential

$$W = \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right). \tag{3.27}$$

We assign chemical potentials $\Delta_{1,2} \in [0, 2\pi]$ to A_i and $\Delta_{3,4} \in [0, 2\pi]$ to B_i. Invariance of the superpotential under the global symmetries requires that $\sum_i \Delta_i \in 2\pi\mathbb{Z}$ (or equivalently $\prod_i y_i = 1$). Conditions (3.9) and (3.14) are then automatically satisfied. The Bethe

potential, for $k = 1$,[7] reads

$$\mathcal{V} = iN^{3/2} \int dt \left\{ t\, \rho(t)\, \delta v(t) + \rho(t)^2 \left[\sum_{a=3,4} g_+ \left(\delta v(t) + \Delta_a \right) - \sum_{a=1,2} g_- \left(\delta v(t) - \Delta_a \right) \right] \right\}. \tag{3.28}$$

The solution for $\sum_i \Delta_i = 2\pi$ and $\Delta_1 \leq \Delta_2$, $\Delta_3 \leq \Delta_4$ is as follows [4]. We have a central region where

$$\rho = \frac{2\pi\mu + t(\Delta_3\Delta_4 - \Delta_1\Delta_2)}{(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_3)(\Delta_1 + \Delta_4)(\Delta_2 + \Delta_4)}$$

$$\delta v = \frac{\mu(\Delta_1\Delta_2 - \Delta_3\Delta_4) + t\sum_{a<b<c} \Delta_a\Delta_b\Delta_c}{2\pi\mu + t(\Delta_3\Delta_4 - \Delta_1\Delta_2)} \qquad -\frac{\mu}{\Delta_4} < t < \frac{\mu}{\Delta_2}. \tag{3.29}$$

When δv hits $-\Delta_3$ on the left the solution becomes

$$\rho = \frac{\mu + t\Delta_3}{(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_3)(\Delta_4 - \Delta_3)}, \quad \delta v = -\Delta_3, \qquad -\frac{\mu}{\Delta_3} < t < -\frac{\mu}{\Delta_4}, \tag{3.30}$$

with the exponentially small correction $Y_3 = (-t\Delta_4 - \mu)/(\Delta_4 - \Delta_3)$, while when δv hits Δ_1 on the right the solution becomes

$$\rho = \frac{\mu - t\Delta_1}{(\Delta_1 + \Delta_3)(\Delta_1 + \Delta_4)(\Delta_2 - \Delta_1)}, \quad \delta v = \Delta_1, \qquad \frac{\mu}{\Delta_2} < t < \frac{\mu}{\Delta_1}, \tag{3.31}$$

with $Y_1 = (t\Delta_2 - \mu)/(\Delta_2 - \Delta_1)$. Finally, the on-shell Bethe potential is

$$\mathcal{V} = \frac{2i}{3}\mu N^{3/2} = \frac{2iN^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}. \tag{3.32}$$

There is also a solution for $\sum_i \Delta_i = 6\pi$ which, however, is obtained by the previous one by a discrete symmetry $\Delta_i \to 2\pi - \Delta_i$ $\left(y_i \to y_i^{-1} \right)$.

3.3 The index at large N

We now turn to the large N limit of the index for an $\mathcal{N} \geq 2$ quiver gauge theory without long-range forces. Here, we give the rules for constructing the index once we know the large N solution $\rho(t), v_a(t)$ of the BAE, which is obtained by extremizing (3.24). The final result scales as $N^{3/2}$.

1. For each group a, the contribution of the Vandermonde determinant is

$$-\frac{\pi^2}{3} N^{3/2} \int dt\, \rho(t)^2. \tag{3.33}$$

2. A $U(1)_a$ topological symmetry with flux $\mathfrak{t}_a$ contributes

$$-\mathfrak{t}_a N^{3/2} \int dt\, \rho(t)\, t. \tag{3.34}$$

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[7]There is a similar solution for $k > 1$ with $\mathcal{V} \to \mathcal{V}\sqrt{k}$. However, we also need to take into account that, for $k > 1$, there are further identifications among the Δ_I due to discrete $\mathbb{Z}_k$ symmetries of the quiver.

3. A pair of bi-fundamental fields, one with magnetic flux $\mathfrak{n}_{(a,b)}$ and chemical potential $\Delta_{(a,b)}$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ of $\mathrm{U}(N)_a \times \mathrm{U}(N)_b$ and the other with magnetic flux $\mathfrak{n}_{(b,a)}$ and chemical potential $\Delta_{(b,a)}$ transforming in the $(\overline{\mathbf{N}}, \mathbf{N})$ of $\mathrm{U}(N)_a \times \mathrm{U}(N)_b$, contributes

$$- N^{3/2} \int dt\, \rho(t)^2 \left[\left(\mathfrak{n}_{(b,a)} - 1 \right) g'_+ \left(\delta v(t) + \Delta_{(b,a)} \right) + \left(\mathfrak{n}_{(a,b)} - 1 \right) g'_- \left(\delta v(t) - \Delta_{(a,b)} \right) \right] . \tag{3.35}$$

4. An adjoint field with magnetic flux $\mathfrak{n}_{(a,a)}$ and chemical potential $\Delta_{(a,a)}$, contributes

$$- \left(\mathfrak{n}_{(a,a)} - 1 \right) g'_+ \left(\Delta_{(a,a)} \right) N^{3/2} \int dt\, \rho(t)^2 . \tag{3.36}$$

5. A field X_a with magnetic flux $\mathfrak{n}_a$ transforming in the fundamental of $\mathrm{U}(N)_a$, contributes

$$\frac{1}{2}(\mathfrak{n}_a - 1) N^{3/2} \int dt\, \rho(t)|t| , \tag{3.37}$$

while an anti-fundamental field with magnetic flux $\tilde{\mathfrak{n}}_a$ contributes

$$\frac{1}{2}(\tilde{\mathfrak{n}}_a - 1) N^{3/2} \int dt\, \rho(t)|t| . \tag{3.38}$$

6. The tails, where δv has a constant value, as in (3.25), contribute

$$- \mathfrak{n}_{(b,a)} N^{3/2} \int_{\delta v \approx -\Delta_{(b,a)} (\mathrm{mod}\ 2\pi)} dt\, \rho(t) Y_{(b,a)} - \mathfrak{n}_{(a,b)} N^{3/2} \int_{\delta v \approx \Delta_{(a,b)} (\mathrm{mod}\ 2\pi)} dt\, \rho(t) Y_{(a,b)} , \tag{3.39}$$

where the integral are taken on the tails regions.

As an example, for ABJM, using the above solution of the BAEs, one obtains the simple expression [4]

$$\mathbb{Re} \log Z = -\frac{N^{3/2}}{3} \sqrt{2\Delta_1 \Delta_2 \Delta_3 \Delta_4} \sum_a \frac{\mathfrak{n}_a}{\Delta_a} . \tag{3.40}$$

4 Bethe potential versus free energy on S^3

We would like to emphasize a remarkable connection of the large N limit of the Bethe potential, which for us is an auxiliary quantity, with the large N limit of the free energy F on S^3 of the same $\mathcal{N} \geq 2$ theory.

Recall that the free energy F on S^3 of an $\mathcal{N} = 2$ theory is a function of trial R-charges Δ_I for the chiral fields [17, 18]. They parameterize the curvature coupling of the supersymmetric Lagrangian on S^3. The S^3 free energy can be computed using localization and reduced to a matrix model [16]. The large N limit of the free energy, for $N \gg k_a$, has been computed in [20, 21, 27–29] and scales as $N^{3/2}$. For example, the free energy for ABJM with $k = 1$ reads [21]

$$F = \frac{4\pi N^{3/2}}{3} \sqrt{2\Delta_1 \Delta_2 \Delta_3 \Delta_4} . \tag{4.1}$$

Bethe potential	S^3 free energy
k_a	$-k_a$
μ	$\frac{\mu}{2}$
$v_a(t)$	$\frac{v_a(t)}{2}$
$\rho(t)$	$4\rho(t)$
Δ_I	$\pi\Delta_I$
Δ_m	$-\pi\Delta_m$
$\mathcal{V}$	$4\pi i F$
$\mathcal{V}\big|_{\text{BAEs}}$	$\frac{i\pi}{2}F\big|_{\text{On-shell}}$

Table 1. The large N Bethe potential versus the S^3 free energy of [21].

We notice a striking similarity with (3.32). This is not a coincidence and generalizes to other theories. Indeed, remarkably, although the finite N matrix models are quite different, for any $\mathcal{N} = 2$ theory, the large N limit of the Bethe potential becomes exactly equal to the large N limit of the free energy F on S^3. We can indeed compare the rules for constructing the Bethe potential with the rules for constructing the large N limit of F, which have been derived in [21]. By comparing the rules in section 3.2 with the rules given in section 2.2 of [21], we observe that they are indeed the same up to a normalization. For reader's convenience the map is explicitly given in table 1. The conditions for cancellation of long-range forces (and therefore the allowed models) are also remarkably similar.

It might be surprising that our chemical potentials for global symmetries are mapped to R-charges in the free energy. However, remember that our Δ_I are angular variables. The invariance of the superpotential under the global symmetries implies that

$$\prod_{I\in\text{matter fields}} y_I = 1\,, \tag{4.2}$$

in each term of the superpotential, which is equivalent to

$$\sum_{I\in\text{matter fields}} \Delta_I = 2\pi\ell \qquad \ell \in \mathbb{Z}\,, \tag{4.3}$$

where now Δ_I are the index chemical potentials. Under the assumption $0 < \Delta_I < 2\pi$, few values of ℓ are actually allowed. In the ABJM model reviewed above, only $\ell = 1$ and $\ell = 3$ give sensible results, with $\ell = 3$ related to $\ell = 1$ by a discrete symmetry of the model. We found a similar result in all the examples we have checked, and we do believe indeed that a solution of the BAE only exists when

$$\sum_{I\in\text{matter fields}} \Delta_I = 2\pi\,, \tag{4.4}$$

for each term of the superpotential, up to solutions related by discrete symmetries. Δ_I/π then behaves at all effects like a trial R-symmetry of the theory and we can compare the index chemical potentials in $\mathcal{V}$ with the R-charges in F.

5   An index theorem for the twisted matrix model

Under mild assumptions, the index at large N can be actually extracted from the Bethe potential with a simple formula.

Theorem 1. *The index of any $\mathcal{N} \geq 2$ quiver gauge theory which respects the constraints (3.9) and (3.14), and satisfies in addition (4.4), can be written as*

$$\mathrm{Re}\log Z = -\frac{2}{\pi}\overline{\mathcal{V}}(\Delta_I) - \sum_I \left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi}\right)\frac{\partial\overline{\mathcal{V}}(\Delta_I)}{\partial\Delta_I}\right], \tag{5.1}$$

where $\overline{\mathcal{V}}$ is the extremal value of the functional (3.24)

$$\overline{\mathcal{V}}(\Delta_I) \equiv -i\mathcal{V}\Big|_{\mathrm{BAEs}} = \frac{2}{3}\mu N^{3/2}, \tag{5.2}$$

and μ is the Lagrange multiplier appearing in (3.24).[8]

Proof. We first replace the explicit factors of π, appearing in eqs. (3.17)–(3.23), with a formal variable $\boldsymbol{\pi}$. Note that the "on-shell" Bethe potential $\overline{\mathcal{V}}$ is a homogeneous function of Δ_I and $\boldsymbol{\pi}$ and therefore it satisfies

$$\overline{\mathcal{V}}(\lambda\Delta_I, \lambda\boldsymbol{\pi}) = \lambda^2\,\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi}) \quad\Rightarrow\quad \frac{\partial\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})}{\partial\boldsymbol{\pi}} = \frac{1}{\boldsymbol{\pi}}\left[2\,\overline{\mathcal{V}}(\Delta_I) - \sum_I \Delta_I\frac{\partial\overline{\mathcal{V}}(\Delta_I)}{\partial\Delta_I}\right]. \tag{5.3}$$

Now, we consider a pair of bi-fundamental fields which contribute to the Bethe potential according to (3.17). The derivative of $\mathcal{V}(\Delta_I, \boldsymbol{\pi})$ with respect to $\Delta_{(b,a)}$ and $\Delta_{(a,b)}$ is given by

$$\sum_{I=(b,a),(a,b)}\mathfrak{n}_I\frac{\partial\mathcal{V}(\Delta_I, \boldsymbol{\pi})}{\partial\Delta_I}$$

$$= iN^{3/2}\int dt\,\rho(t)^2\left[\mathfrak{n}_{(b,a)}g'_+\left(\delta v(t) + \Delta_{(b,a)}\right) + \mathfrak{n}_{(a,b)}g'_-\left(\delta v(t) - \Delta_{(a,b)}\right)\right]$$

$$+ \sum_{I=(b,a),(a,b)}\mathfrak{n}_I\underbrace{\frac{\partial\mathcal{V}}{\partial\rho}\frac{\partial\rho}{\partial\Delta_I}}_{\text{vanishing on-shell}} + \sum_{I=(b,a),(a,b)}\mathfrak{n}_I\underbrace{\frac{\partial\mathcal{V}}{\partial(\delta v)}\frac{\partial(\delta v)}{\partial\Delta_I}}_{\text{tails contribution}}. \tag{5.4}$$

The expression in the first line is precisely part of the contribution of a pair of bi-fundamentals (3.35) to the index. In the tails, using (3.25), we find

$$\frac{\partial(\delta v)}{\partial\Delta_{(b,a)}} = -1, \qquad \frac{\partial(\delta v)}{\partial\Delta_{(a,b)}} = 1, \qquad \frac{\partial\mathcal{V}}{\partial(\delta v)} = -iY_{(b,a)}\rho, \qquad \frac{\partial\mathcal{V}}{\partial(\delta v)} = iY_{(a,b)}\rho. \tag{5.5}$$

Therefore, the last term in eq. (5.4) can be simplified to

$$iN^{3/2}\mathfrak{n}_{(b,a)}\int_{\delta v\approx-\Delta_{(b,a)}} dt\,\rho(t)\,Y_{(b,a)} + iN^{3/2}\mathfrak{n}_{(a,b)}\int_{\delta v\approx\Delta_{(a,b)}} dt\,\rho(t)\,Y_{(a,b)}. \tag{5.6}$$

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[8]The second identity in (5.2) is a consequence of a virial theorem for matrix models (see appendix B of [29]).

This precisely gives the tail contribution (3.39) to the index. Next, we take the derivative of the Bethe potential with respect to $\boldsymbol{\pi}$. It can be written as

$$\frac{\partial \mathcal{V}}{\partial \boldsymbol{\pi}} = -iN^{3/2} \int dt\, \rho(t)^2 \left[g'_+ \left(\delta v(t) + \Delta_{(b,a)} \right) + g'_- \left(\delta v(t) - \Delta_{(a,b)} \right) \right]$$
$$+ iN^{3/2} \int dt\, \rho(t)^2 \left[\frac{2\boldsymbol{\pi}^2}{3} - \frac{\boldsymbol{\pi}}{3} \left(\Delta_{(b,a)} + \Delta_{(a,b)} \right) \right] . \tag{5.7}$$

The expression in the first line completes the contribution of a pair of bi-fundamentals (3.35) to the index. The expression in the second line, after summing over all the bi-fundamental pairs, can be written as

$$\sum_{\text{pairs}} \left[\frac{2\boldsymbol{\pi}^2}{3} - \frac{\boldsymbol{\pi}}{3} \left(\Delta_{(b,a)} + \Delta_{(a,b)} \right) \right] = \frac{\boldsymbol{\pi}}{3} \sum_I \left(\boldsymbol{\pi} - \Delta_I \right) = \frac{\boldsymbol{\pi}^2}{3} |G| , \tag{5.8}$$

which is precisely the contribution of the gauge fields (3.33) to the index. Here, we used the condition

$$\boldsymbol{\pi} |G| + \sum_I \left(\Delta_I - \boldsymbol{\pi} \right) = 0 . \tag{5.9}$$

This condition follows from the fact that, assuming (4.4) for each superpotential term, Δ_I/π behaves as a trial R-symmetry, so that (3.14) yields

$$2 + \sum_{I \in a} \left(\frac{\Delta_I}{\pi} - 1 \right) = 0 , \tag{5.10}$$

which, summed over all the nodes, since each bi-fundamental field belongs precisely to two nodes, gives (5.9). Condition (5.9) is indeed equivalent to $\text{Tr} R = 0$, where the trace is taken over the bi-fundamental fermions and gauginos in the quiver and R is an R-symmetry. Combining everything as in the right hand side of eq. (5.1) we obtain the contribution of gauge and bi-fundamental fields to the index. The proof for all the other matter fields and the topological symmetry is straightforward. $\qquad\square$

If we ignore the linear relation among the chemical potentials, we can always use a set of Δ_I such that $\overline{\mathcal{V}}$ is a homogeneous function of degree two of the Δ_I alone.[9] In this case, the index theorem simplifies to

$$\mathbb{R}e \log Z = -\sum_I \mathfrak{n}_I \frac{\partial \overline{\mathcal{V}}(\Delta_I)}{\partial \Delta_I} . \tag{5.11}$$

6 Theories with $N^{5/3}$ scaling of the index

Chern-Simons quivers of the form (3.1) have a rich parameter space. If condition (3.2) is satisfied and $N \gg k_a$, they have an M-theory weakly coupled dual. In the t'Hooft limit,

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[9]This is what happens in (3.32) for ABJM. Recall that $\sum_i \Delta_i = 2\pi$ so that the four Δ_i are not linearly independent.

$N, k_a \gg 1$ with $N/k_a = \lambda_a$ fixed and large, they have a type IIA weakly coupled dual. When instead

$$\sum_{a=1}^{|G|} k_a \neq 0 \,, \tag{6.1}$$

they probe massive type IIA [30]. There is an interesting limit, given (6.1), where again $N \gg k_a$. The limit is no more an M-theory phase [31], but rather an extreme phase of massive type IIA. Supergravity duals of this type of phases have been found in [31–38]. The free energy scales as $N^{5/3}$ [31]. We now show that also the topologically twisted index scales in the same way. As it happens for the S^3 matrix model [21, 36], we find a consistent large N limit whenever the constraints (3.10) and (3.15) are satisfied.

The ansatz for the eigenvalue distribution is now, as in [21, 36],

$$u^{(a)}(t) = N^{1/3}(it + v(t)) \,. \tag{6.2}$$

The scaling is again dictated by the competition between the Chern-Simons terms, now with (6.1), and the gauge and bi-fundamental contributions.

6.1 Long-range forces

Since the eigenvalue distribution is the same for all gauge groups, the long-range forces (3.5) cancel automatically. We see that, differently from before, we can have a consistent large N limit also in the case of chiral quivers. We also need to cancel the long-range forces (3.6). They compensate each other if condition (3.9) is satisfied. Since the eigenvalues are the same for all groups, it is actually enough to sum over nodes and we obtain the milder constraint (3.10) on the flavor charges:

$$\mathrm{Tr}\, J = 0 \,, \tag{6.3}$$

where the trace is taken over bi-fundamental fermions in the quiver.

We obtain similar conditions by looking at the scaling of the twisted index. As in section 3, vector multiplets and chiral bi-fundamental multiplets contribute terms (3.12) and (3.13) which are of order $\mathcal{O}(N^{7/3})$. They compensate each other if condition (3.14) is satisfied. Since the eigenvalues are the same for all groups, it is again enough to sum over nodes and we obtain the constraint (3.15) on the flavor magnetic fluxes:

$$\mathrm{Tr}\, R = |G| + \sum_I (\mathfrak{n}_I - 1) = 0 \,, \tag{6.4}$$

where the trace is taken over bi-fundamental fermions and gauginos in the quiver.

Conditions $\mathrm{Tr}\, R = \mathrm{Tr}\, J = 0$ are certainly satisfied for all quivers with a four-dimensional parent, even the chiral ones.

6.2 Bethe potential at large N

1. Each group a with CS level k_a contributes

$$-ik_a N^{5/3} \int dt\, \rho(t)\, t\, v(t) + \frac{k_a}{2} N^{5/3} \int dt\, \rho(t)\, \left(t^2 - v(t)^2\right) \,. \tag{6.5}$$

2. A bi-fundamental field with chemical potential Δ_I contributes

$$i\, g_+ \left(\Delta_I\right) N^{5/3} \int dt \, \frac{\rho(t)^2}{1 - iv'(t)} \, . \tag{6.6}$$

3. A fundamental field contributes

$$-\frac{1}{4} N^{5/3} \int dt \, \rho(t) \, \text{sign}(t) \left[it + v(t)\right]^2 \, , \tag{6.7}$$

while an anti-fundamental field contributes

$$\frac{1}{4} N^{5/3} \int dt \, \rho(t) \, \text{sign}(t) \left[it + v(t)\right]^2 \, . \tag{6.8}$$

6.3 The index at large N

1. For each group a, the contribution of the Vandermonde determinant is

$$-\frac{\pi^2}{3} N^{5/3} \int dt \, \frac{\rho(t)^2}{1 - iv'(t)} \, . \tag{6.9}$$

2. A chiral bi-fundamental field, with chemical potential Δ_I and magnetic flux $\mathfrak{n}_I$ contributes

$$-(\mathfrak{n}_I - 1) \, g'_+ \left(\Delta_I\right) N^{5/3} \int dt \, \frac{\rho(t)^2}{1 - iv'(t)} \, . \tag{6.10}$$

Fundamental fields do not contribute to the index explicitly.

Notice that the relation with the S^3 free energy discussed in section 4 and the *index theorem* of section 5 also hold for this class of quiver gauge theories.[10]

7 Discussion and conclusions

In this paper we have studied the large N behavior of the topologically twisted index for $\mathcal{N} = 2$ gauge theories in three dimensions. We have focused on theories with a conjectured M-theory or massive type IIA dual and examined the corresponding field theory phases, where holography predicts a $N^{3/2}$ or $N^{5/3}$ scaling for the path integral, respectively. We correctly reproduced this scaling for a class of $\mathcal{N} = 2$ theories and we also uncovered some surprising relations with apparently different physical quantities.

The first surprise comes from the identification of the *Bethe potential* $\overline{\mathcal{V}}$ with the S^3 *free energy* F of the same $\mathcal{N} = 2$ gauge theory. Recall that, in our approach, the BAE and the Bethe potential are auxiliary quantities determining the position of the poles in the matrix model in the large N limit. $\overline{\mathcal{V}}$ depends on the chemical potentials for the flavor symmetries, satisfying (4.3), while F depends on trial R-charges, which parameterize the curvature couplings of the theory on S^3. Both quantities, $\mathcal{V}$ and F are determined in terms of a matrix model (auxiliary in the case of $\mathcal{V}$). The two matrix models, and the corresponding equations of motion are different for finite N, but, quite remarkably become

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[10]The coefficient $2/3$ in front of μ in eq. (5.2) must be replaced by $3/5$.

indistinguishable in the large N limit. Also the conditions to be imposed on the quiver for the existence of a $N^{3/2}$ or $N^{5/3}$ scaling are the same. Although the structure of the long-range forces and the mechanism for their cancellation is different, they rule out quivers with chiral bi-fundamentals in the M-theory phase and impose the same conditions on flavor symmetries.

This identification leads to a relation of the Bethe potential $\overline{\mathcal{V}}$ with the volume functional of Sasaki-Einstein manifolds. The exact R-symmetry of a superconformal $\mathcal{N} = 2$ gauge theory can be found by extremizing $F(\Delta_I)$ with respect to the trial R-charges Δ_I [21], but $F(\Delta_I)$ makes sense for arbitrary Δ_I. The functional $F(\Delta_I)$ has a well-defined geometrical meaning for theories with an $AdS_4 \times Y_7$ dual, where Y_7 is a Sasaki-Einstein manifold. The value of F upon extremization is related to the (square root of the) volume of Y_7. More generally, at least for a class of quivers corresponding to $\mathcal{N} = 3$ and toric cones $C(Y_7)$, the value of $F(\Delta_I)$, as a function of the trial R-symmetry parameterized by Δ_I, has been matched with the (square root of the) volume of a family of Sasakian deformation of Y_7, as a function of the Reeb vector. For toric theories, the volume can be parameterized in terms of a set of charges Δ_I, that encode how the R-symmetry varies with the Reeb vector, and it has been conjectured in [39–41] to be a homogeneous quartic function of the Δ_I, in agreement with the homogeneity properties of $\overline{\mathcal{V}}$ and F.

A second intriguing relation comes from the index theorem (5.1). The original reason for studying the large N limit of the topologically twisted index comes from the counting of AdS_4 black holes microstates. The entropy of magnetically charged black holes asymptotic to $AdS_4 \times S^7$ was successfully compared with the large N limit of the index in [4], when extremized with respect to the chemical potential Δ_I. We expect that a similar relation holds for magnetically charged BPS black holes asymptotic to $AdS_4 \times Y_7$, for a generic Sasaki-Einstein manifold. Given the very small number of black holes known, this statement is difficult to check. Assuming, however, that it is true, we can compare the on-shell index of a superconformal $\mathcal{N} = 2$ gauge theory dual to $AdS_4 \times Y_7$ twisted by a set of magnetic charges $\mathfrak{n}_I$ with the entropy of a black hole in $AdS_4 \times Y_7$ supported by the same magnetic charges. The entropy of such black hole is determined in supergravity by the attractor mechanism [42]. The black hole can be written as a solution of the $\mathcal{N} = 2$ gauged supergravity obtained by truncating the KK spectrum on Y_7 to a consistent set of modes, which contains vector and hypermultiplets [43–47]. In a gauge supergravity with only vectors, the entropy of the black hole can be obtained by extremizing with respect to X^I the quantity [44]

$$\mathcal{I}(X^I) = -\sum_I \mathfrak{n}^I \frac{\partial \mathcal{F}}{\partial X^I} \,, \tag{7.1}$$

where $\mathcal{F}(X^I)$ is the supergravity prepotential and X^I a set of covariantly-constant homogeneous holomorphic sections. Here, we are working in the gauge $\sum_I g_I X^I = 1$, where g_I are the electric gaugings of the theory and we assume that there are no magnetic ones. The presence of hypermultiplets just add algebraic constraints [48, 49]. Comparison of the attractor equation (7.1) with the index theorem (5.11) suggests the identification of Δ_I with X^I and a proportionality between $\overline{\mathcal{V}}(\Delta_I)$ and $\mathcal{F}(X^I)$, valid also before extremization.

This proportionality certainly holds for ABJM since the prepotential is

$$\mathcal{F} = i\sqrt{X^0 X^1 X^2 X^3}\,, \tag{7.2}$$

which can be clearly mapped to (3.32). It would be quite interesting to see how to formulate this identification in more general theories with hypermultiplets.

We thus see an intriguing chain of identifications

$$\textit{Bethe potential } \overline{\mathcal{V}} \qquad \Longleftrightarrow \qquad S^3 \textit{ free-energy } F \qquad \Longleftrightarrow \qquad \textit{prepotential } \mathcal{F}$$

of functionals depending on chemical potentials, trial R-charges and bulk scalar fields, respectively. A relation between the free energy F and the prepotential of the compactified theory was already suggested in [41]. This chain of identifications certainly calls for further investigation.[11]

The main motivation of our analysis comes certainly from the attempt to extend the result of [4] to a larger class of black holes. The difficulty of doing so is mainly the exiguous number of existing black holes solutions with an M-theory lift. Few numerical examples are known in Sasaki-Einstein compactifications [48], mostly having Betti multiplets as massless vectors. Some interesting examples involves chiral quivers and are therefore outside the range of our technical abilities at the moment. It is curious that apparently well-defined chiral quivers, which passed quite nontrivial checks [52], have an ill-defined large N limit both for the S^3 free energy and the topologically twisted index in the M-theory phase. It would be quite interesting to know whether this is just a technical problem and another saddle-point with $N^{3/2}$ scaling exists, or the models are really ruled out.

It would be also quite interesting to find new examples of AdS$_4$ M-theory and massive type IIA black holes directly in eleven or ten dimensions (see, for example, [53]) or in some other consistent truncations of eleven dimensional supergravity where to test our results.

We hope to come back to all these questions quite soon.

Acknowledgments

A   Derivation of general rules for theories with $N^{3/2}$ scaling of the index

In this appendix we give a detail derivation of the rules presented in the main text for finding the Bethe potential and the index at large N. The result is a straightforward generalization of [4].

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[11] As well as the relation to other extremization problems and generalizations [50, 51].

We consider the following large N saddle-point eigenvalue distribution ansatz

$$u_i^{(a)} = iN^\alpha t_i + v_i^{(a)} \,. \tag{A.1}$$

Notice that the imaginary parts of the $u_i^{(a)}$ are equal. We also define

$$\delta v_i = v_i^{(b)} - v_i^{(a)} \,. \tag{A.2}$$

In the large N limit, we define the continuous functions $t_i = t(i/N)$ and $v_i^{(a)} = v^{(a)}(i/N)$ and we introduce the density of eigenvalues

$$\rho(t) = \frac{1}{N} \frac{di}{dt} \,, \tag{A.3}$$

normalized as $\int dt\, \rho(t) = 1$. Furthermore, we impose the additional constraint

$$\sum_{a=1}^{|G|} k_a = 0 \,, \tag{A.4}$$

corresponding to quivers dual to M-theory on $AdS_4 \times Y_7$ and $N^{3/2}$ scaling.

A.1 Bethe potential at large N

We may write the Bethe ansatz equations as

$$0 = \log\left[\text{r.h.s. of } (2.6)\right] - 2\pi i n_i^{(a)} \,, \tag{A.5}$$

where $n_i^{(a)}$ are integers that parameterize the angular ambiguities. We define the "Bethe potential" as the function whose critical points gives the BAEs (A.5). In the large N limit the Bethe potential $\mathcal{V}$ will be the sum of various contributions,

$$\mathcal{V} = \mathcal{V}^{\text{CS}} + \mathcal{V}^{\text{bi-fund}} + \mathcal{V}^{\text{adjoint}} + \mathcal{V}^{\text{(anti-)fund}} \,. \tag{A.6}$$

α will be determined to be $1/2$ by the competition between Chern-Simons terms and matter contribution.

A.1.1 Chern-Simons contribution

Each group a with CS level k_a and topological chemical potential $\Delta_m^{(a)}$, contributes to the finite N Bethe potential as

$$\mathcal{V}^{\text{CS}} = \sum_{i=1}^{N} \left[-\frac{k_a}{2} \left(u_i^{(a)}\right)^2 - \Delta_m^{(a)} u_i^{(a)} \right] \,. \tag{A.7}$$

Given the large N saddle-point eigenvalue distribution (A.1), we find

$$\mathcal{V}^{\text{CS}} = \frac{k_a}{2} N^{2\alpha} \sum_{i=1}^{N} t_i^2 - iN^\alpha \sum_{i=1}^{N} \left(k_a t_i v_i^{(a)} + \Delta_m^{(a)} t_i \right) \,. \tag{A.8}$$

Summing over nodes the first term vanishes (since $\sum_{a=1}^{|G|} k_a = 0$). Taking the continuum limit, we obtain

$$\mathcal{V}^{\text{CS}} = -ik_a N^{1+\alpha} \int dt\, \rho(t)\, t\, v_a(t) - iN^{1+\alpha} \Delta_m^{(a)} \int dt\, \rho(t)\, t \,. \tag{A.9}$$

A.1.2 Bi-fundamental contribution

For a pair of bi-fundamental fields, one with chemical potential $\Delta_{(a,b)}$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ of $U(N)_a \times U(N)_b$ and one with chemical potential $\Delta_{(b,a)}$ transforming in the $(\overline{\mathbf{N}}, \mathbf{N})$ of $U(N)_a \times U(N)_b$, the finite N contribution to the Bethe potential is given by

$$
\mathcal{V}^{\text{bi-fund}} = \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \sum_{i,j=1}^{N} \left[\text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) - \text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} - \Delta_{(a,b)}\right)} \right) \right]
$$

$$
- \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \sum_{i,j=1}^{N} \left[\frac{\left(\Delta_{(b,a)} - \pi\right) + \left(\Delta_{(a,b)} - \pi\right)}{2} \left(u_j^{(b)} - u_i^{(a)} \right) \right] , \qquad \text{(A.10)}
$$

up to constants that do not depend on $u_j^{(b)}$, $u_i^{(a)}$.

We would like to remind the reader that all angular variables are defined modulo 2π. Part of the ambiguity in Δ_I can be fixed by requiring that

$$
0 < \delta v + \Delta_{(b,a)} < 2\pi , \qquad\qquad -2\pi < \delta v - \Delta_{(a,b)} < 0 . \qquad \text{(A.11)}
$$

The remaining ambiguity of simultaneous shifts $\delta v \rightarrow \delta v + 2\pi$, $\Delta_{(a,b)} \rightarrow \Delta_{(a,b)} + 2\pi$, $\Delta_{(b,a)} \rightarrow \Delta_{(b,a)} - 2\pi$ can also be fixed by requiring that $\delta v(t)$ takes the value 0 somewhere, if it vanishes at all, which we assume. We then have

$$
0 < \Delta_I < 2\pi . \qquad \text{(A.12)}
$$

To compute $\mathcal{V}^{\text{bi-fund}}$, we break

$$
\sum_{i,j=1}^{N} \text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right)
$$

$$
= \sum_{i>j} \text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) + \sum_{i<j} \text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right)
$$

$$
+ \sum_{i=1}^{N} \text{Li}_2 \left(e^{i\left(u_i^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) . \qquad \text{(A.13)}
$$

The crucial point here is that the last term is naively of $\mathcal{O}(N)$ and thus subleading; however, we should keep it since its derivative is not subleading on part of the solution when δv hits $\Delta_{(a,b)}$ or $-\Delta_{(b,a)}$. Therefore, we keep

$$
N \int dt\, \rho(t) \left[\text{Li}_2 \left(e^{i\left(\delta v(t) + \Delta_{(b,a)}\right)} \right) - \text{Li}_2 \left(e^{i\left(\delta v(t) - \Delta_{(a,b)}\right)} \right) \right] . \qquad \text{(A.14)}
$$

This will be important in the *tail contribution* to the Bethe potential. The second term in (A.13) is

$$
\sum_{i<j} \text{Li}_2 \left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) = N^2 \int dt\, \rho(t) \int_t dt'\, \rho(t')\, \text{Li}_2 \left(e^{i\left(u_b(t') - u_a(t) + \Delta_{(b,a)}\right)} \right) .
$$

$$
\text{(A.15)}
$$

We first write the dilogarithm function as a power series, i.e.,

$$\text{Li}_2(e^{iu}) = \sum_{k=1}^{\infty} \frac{e^{iku}}{k^2}. \tag{A.16}$$

Then, we consider the integral

$$
\begin{aligned}
I_k &= \int_t dt'\, \rho(t')\, e^{ik\left(u_b(t') - u_a(t) + \Delta_{(b,a)}\right)} \\
&= \int_t dt'\, e^{-kN^\alpha(t'-t)} \sum_{j=0}^{\infty} \frac{(t'-t)^j}{j!} \partial_x^j \left[\rho(x)\, e^{ik\left(v_b(x) - v_a(t) + \Delta_{(b,a)}\right)} \right]_{x=t},
\end{aligned}
$$

where in the second equality we have Taylor-expanded the integrand around the lower bound. Doing the integration over t' we see that the leading contribution is for $j = 0$, thus

$$I_k = \frac{\rho(t)\, e^{ik\left(v_b(t) - v_a(t) + \Delta_{(b,a)}\right)}}{kN^\alpha} + \mathcal{O}(N^{-2\alpha}). \tag{A.17}$$

Substituting we find

$$\sum_{i<j} \text{Li}_2\left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) = N^{2-\alpha} \int dt\, \text{Li}_3\left(e^{i\left(\delta v(t) + \Delta_{(b,a)}\right)} \right) \rho(t)^2 + \mathcal{O}(N^{2-2\alpha}). \tag{A.18}$$

Next, we need to compute the first term in (A.13). In order for the integral to be localized at the boundary, we need to invert the integrand. Since $0 < \text{Re}\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)} \right) < 2\pi$:

$$
\begin{aligned}
\text{Li}_2\left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)}\right)} \right) = {} & -\text{Li}_2\left(e^{i\left(u_i^{(a)} - u_j^{(b)} - \Delta_{(b,a)}\right)} \right) + \frac{\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)} \right)^2}{2} \\
& - \pi \left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)} \right) + \frac{\pi^2}{3}.
\end{aligned}
\tag{A.19}
$$

The summation $\sum_{i>j}$ of the first term in the latter expression is similar to (A.18) but with $-\text{Li}_3\left(e^{-i\left(\delta v(t) + \Delta_{(b,a)}\right)} \right)$ instead of Li_3. The two contributions may then be combined, using (D.5),

$$
\begin{aligned}
N^{2-\alpha} \int dt \left[\text{Li}_3\left(e^{i\left(\delta v(t) + \Delta_{(b,a)}\right)} \right) - \text{Li}_3\left(e^{-i\left(\delta v(t) + \Delta_{(b,a)}\right)} \right) \right] \rho(t)^2 & \\
= iN^{2-\alpha} \int dt\, g_+\left(\delta v(t) + \Delta_{(b,a)} \right) \rho(t)^2, &
\end{aligned}
\tag{A.20}
$$

where we have introduced the polynomial function $g_+(u)$ defined in eq. (3.18).

The second term in the first line of (A.10) can be treated similarly. We now have $-2\pi < \text{Re}(u_j^{(b)} - u_i^{(a)} - \Delta_{(a,b)}) < 0$ and

$$
\begin{aligned}
-\text{Li}_2\left(e^{i\left(u_j^{(b)} - u_i^{(a)} - \Delta_{(a,b)}\right)} \right) = {} & \text{Li}_2\left(e^{i\left(u_i^{(a)} - u_j^{(b)} + \Delta_{(a,b)}\right)} \right) - \frac{\left(u_j^{(b)} - u_i^{(a)} - \Delta_{(a,b)} \right)^2}{2} \\
& - \pi \left(u_j^{(b)} - u_i^{(a)} - \Delta_{(a,b)} \right) - \frac{\pi^2}{3}.
\end{aligned}
\tag{A.21}
$$

As before, the result of the summation $\sum_{i>j}$ together with that of $\sum_{i<j}$ yields a cubic polynomial expression

$$- iN^{2-\alpha} \int dt \, g_- \left(\delta v(t) - \Delta_{(a,b)} \right) \rho(t)^2 \,,$$ (A.22)

where $g_-(u)$ is defined in eq. (3.18).

The left over terms from (A.19) and (A.21), throwing away the constants which do not affect the critical points, are

$$\left[\left(\Delta_{(a,b)} - \pi \right) + \left(\Delta_{(b,a)} - \pi \right) \right] \sum_{i>j} \left(u_j^{(b)} - u_i^{(a)} \right) \,,$$ (A.23)

which, combined with the second line in (A.10), gives

$$\frac{1}{2} \left[\left(\Delta_{(a,b)} - \pi \right) + \left(\Delta_{(b,a)} - \pi \right) \right] \sum_{i \neq j} \left(u_j^{(b)} - u_i^{(a)} \right) \text{sign}(i - j) \,.$$ (A.24)

This term can be precisely canceled by

$$- 2\pi \sum_{i=1}^{N} \left(n_i^{(b)} u_i^{(b)} - n_i^{(a)} u_i^{(a)} \right) \,,$$ (A.25)

provided that $\sum_{I \in a} \Delta_I \in 2\pi \mathbb{Z}$.[12]

Notice that a single bi-fundamental chiral multiplet, with chemical potential $\Delta_{(b,a)}$, transforming in the representation $(\overline{\mathbf{N}}, \mathbf{N})$ of $\text{U}(N)_a \times \text{U}(N)_b$ contributes to the Bethe potential as

$$\sum_{i,j=1}^{N} \left[\text{Li}_2 \left(e^{i \left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)} \right)} \right) - \frac{\left(u_j^{(b)} - u_i^{(a)} + \Delta_{(b,a)} \right)^2}{4} \right] \,.$$ (A.26)

Using eq. (A.19) we find the following long-range terms

$$\sum_{i<j} \frac{\left(u_i^{(a)} - u_j^{(b)} \right)^2}{4} - \sum_{i<j} \frac{\left(u_j^{(a)} - u_i^{(b)} \right)^2}{4} \,.$$ (A.27)

In the large N limit, they are of order $N^{5/2}$ and cannot be canceled for *chiral quivers*.

To find a nontrivial saddle-point the leading terms of order $N^{1+\alpha}$ and $N^{2-\alpha}$ have to be of the same order, so we need $\alpha = 1/2$. Putting everything together we arrive at the final expression for the large N contribution of the bi-fundamental fields to the Bethe potential

$$\mathcal{V}^{\text{bi-fund}} = iN^{3/2} \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \int dt \, \rho(t)^2 \left[g_+ \left(\delta v(t) + \Delta_{(b,a)} \right) - g_- \left(\delta v(t) - \Delta_{(a,b)} \right) \right] \,.$$

(A.28)

---

[12]When $N \in 2\mathbb{Z}_{\geq 0} + 1$. For even N one can include an extra $(-1)^{\mathfrak{m}}$ in the twisted partition function, which can be reabsorbed in the definition of the topological fugacity ξ, to compensate the overall factor of π.

In the sum over pairs of bi-fundamental fields (b, a) and (a, b), adjoint fields should be counted once and should come with an explicit factor of $1/2$. Keeping this in mind and setting

$$v_b = v_a \,, \qquad\qquad \Delta_{(b,a)} = \Delta_{(a,b)} = \Delta_{(a,a)} \,, \qquad\qquad \text{(A.29)}$$

we find the contribution of fields transforming in the adjoint of the ath gauge group with chemical potential $\Delta_{(a,a)}$ to the large N Bethe potential,

$$\mathcal{V}^{\text{adjoint}} = i N^{3/2} \sum_{\substack{\text{adjoint} \\ (a,a)}} g_+ \left(\Delta_{(a,a)} \right) \int dt \, \rho(t)^2 \,. \qquad\qquad \text{(A.30)}$$

A.1.3 Fundamental and anti-fundamental contribution

The fundamental and anti-fundamental fields contribute to the large N Bethe potential as[13]

$$\mathcal{V}^{\text{(anti-)fund}} = \sum_{i=1}^{N} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} \text{Li}_2 \left(e^{i\left(-u_i^{(a)} + \tilde{\Delta}_a\right)} \right) - \sum_{\substack{\text{fundamental} \\ a}} \text{Li}_2 \left(e^{i\left(-u_i^{(a)} - \Delta_a\right)} \right) \right]$$
$$+ \frac{1}{2} \sum_{i=1}^{N} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} (\tilde{\Delta}_a - \pi) u_i^{(a)} + \sum_{\substack{\text{fundamental} \\ a}} (\Delta_a - \pi) u_i^{(a)} \right]$$
$$- \frac{1}{4} \sum_{i=1}^{N} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} \left(u_i^{(a)} \right)^2 - \sum_{\substack{\text{fundamental} \\ a}} \left(u_i^{(a)} \right)^2 \right] \,. \qquad \text{(A.31)}$$

Let us denote the total number of (anti-)fundamental fields by $(\tilde{n}_a) \, n_a$. Substituting in $\mathcal{V}^{\text{(anti-)fund}}$ the ansatz (A.1) and taking the continuum limit, the first line contributes

$$- \frac{(\tilde{n}_a - n_a)}{2} N^2 \int_{t>0} dt \, \rho(t) \, t^2$$
$$+ i N^{3/2} \int_{t>0} dt \, \rho(t) \, t \left\{ \sum_{\substack{\text{anti-fundamental} \\ a}} \left[v_a(t) - (\tilde{\Delta}_a - \pi) \right] - \sum_{\substack{\text{fundamental} \\ a}} \left[v_a(t) + (\Delta_a - \pi) \right] \right\} \,, \qquad \text{(A.32)}$$

while the second and the third lines give

$$\frac{(\tilde{n}_a - n_a)}{4} N^2 \int dt \, \rho(t) \, t^2$$
$$- \frac{i}{2} N^{3/2} \int dt \, \rho(t) \, t \left\{ \sum_{\substack{\text{anti-fundamental} \\ a}} \left[v_a(t) - (\tilde{\Delta}_a - \pi) \right] - \sum_{\substack{\text{fundamental} \\ a}} \left[v_a(t) + (\Delta_a - \pi) \right] \right\} \,. \qquad \text{(A.33)}$$

---

[13] Up to a factor $-\pi(\tilde{n}_a - n_a) u_i/2$ that cancels at this order for total number of fundamentals equal to total number of anti-fundamentals, which we will need to assume for consistency.

Combining eq. (A.32) and eq. (A.33), we obtain

$$
\begin{aligned}
\mathcal{V}^{(\text{anti-})\text{fund}} = & -\frac{(\tilde{n}_a - n_a)}{4} N^2 \int dt\, \rho(t)\, t\, |t| \\
& + \frac{i}{2} N^{3/2} \sum_{\substack{\text{anti-fundamental} \\ a}} \int dt\, \rho(t)\, |t| \left[v_a(t) - (\tilde{\Delta}_a - \pi) \right] \\
& - \frac{i}{2} N^{3/2} \sum_{\substack{\text{fundamental} \\ a}} \int dt\, \rho(t)\, |t| \left[v_a(t) + (\Delta_a - \pi) \right].
\end{aligned}
\tag{A.34}
$$

Summing over nodes the first term vanishes, demanding that

$$
\sum_{a=1}^{|G|} (\tilde{n}_a - n_a) = 0.
\tag{A.35}
$$

We see that we need to consider quivers where the *total* number of fundamentals equal the *total* number of anti-fundamentals. For each single node this number can be different.

A.2 The index at large N

We are interested in the large N limit of the logarithm of the twisted partition function.

A.2.1 Gauge vector contribution

Given the expression for the matrix model in section 2, the Vandermonde determinant contributes to the logarithm of the index as

$$
\begin{aligned}
\log \prod_{i \neq j} \left(1 - \frac{x_i^{(a)}}{x_j^{(a)}} \right) &= \log \prod_{i<j} \left(1 - \frac{x_j^{(a)}}{x_i^{(a)}} \right)^2 \left(-\frac{x_i^{(a)}}{x_j^{(a)}} \right) \\
&= i \sum_{i<j}^{N} \left(u_i^{(a)} - u_j^{(a)} + \pi \right) - 2 \sum_{i<j}^{N} \mathrm{Li}_1 \left(e^{i\left(u_j^{(a)} - u_i^{(a)} \right)} \right).
\end{aligned}
\tag{A.36}
$$

The first term is of $\mathcal{O}(N^2)$ and, therefore, a source of the long-range forces and will be canceled by the contribution coming from the chiral multiplets. The second term is treated as in appendix A.1.2, and gives

$$
\mathbb{Re} \log Z^{\text{gauge}} = -\frac{\pi^2}{3} N^{3/2} \int dt\, \rho(t)^2 + \mathcal{O}(N).
\tag{A.37}
$$

A.2.2 Topological symmetry contribution

A $\mathrm{U}(1)_a$ topological symmetry with flux $\mathfrak{t}_a$ contributes as

$$
i \sum_{i=1}^{N} u_i^{(a)} \mathfrak{t}_a.
\tag{A.38}
$$

In the continuum limit, we get

$$
\mathbb{Re} \log Z^{\text{top}} = -\mathfrak{t}_a N^{3/2} \int dt\, \rho(t)\, t + \mathcal{O}(N).
\tag{A.39}
$$

A.2.3 Bi-fundamental contribution

We can rewrite the contribution to the twisted index of a bi-fundamental chiral multiplet transforming in the $(\overline{\mathbf{N}}, \mathbf{N})$ of $U(N)_a \times U(N)_b$, with magnetic flux $\mathfrak{n}_{(b,a)}$ and chemical potential $\Delta_{(b,a)}$ as:[14]

$$
\prod_{i=1}^{N} \left(\frac{x_i^{(b)}}{x_i^{(a)}} \right)^{\frac{1}{2}\left(\mathfrak{n}_{(b,a)}-1\right)} \left(1 - y_{(b,a)} \frac{x_i^{(b)}}{x_i^{(a)}} \right)^{\mathfrak{n}_{(b,a)}-1} \times
$$
$$
\times \prod_{i<j}^{N} (-1)^{\mathfrak{n}_{(a,b)}-1} \left(\frac{x_i^{(a)} x_i^{(b)}}{x_j^{(a)} x_j^{(b)}} \right)^{\frac{1}{2}\left(\mathfrak{n}_{(b,a)}-1\right)} \left(1 - y_{(b,a)} \frac{x_j^{(b)}}{x_i^{(a)}} \right)^{\mathfrak{n}_{(b,a)}-1} \left(1 - y_{(b,a)}^{-1} \frac{x_j^{(a)}}{x_i^{(b)}} \right)^{\mathfrak{n}_{(b,a)}-1}.
$$
(A.40)

The first term in $\prod_i$ is subleading and the second term only contributes in the tail where $\delta v \approx -\Delta_{(b,a)}$,

$$
N \left(\mathfrak{n}_{(b,a)} - 1 \right) \int dt \, \rho(t) \log \left(1 - e^{i\left(\delta v + \Delta_{(b,a)}\right)} \right) \tag{A.41}
$$
$$
= -N^{3/2} \left(\mathfrak{n}_{(b,a)} - 1 \right) \int_{\delta v \approx -\Delta_{(b,a)}} dt \, \rho(t) \, Y_{(b,a)}(t) + \mathcal{O}(N) \tag{A.42}
$$

The first two terms in $\prod_{i<j}$ give a long-range force contribution to the index

$$
\frac{i}{2} \left(\mathfrak{n}_{(b,a)} - 1 \right) \sum_{i<j} \left[\left(u_i^{(a)} - u_j^{(a)} + \pi \right) + \left(u_i^{(b)} - u_j^{(b)} + \pi \right) \right], \tag{A.43}
$$

while the last two terms result in

$$
- N^{3/2} \left(\mathfrak{n}_{(b,a)} - 1 \right) \int dt \, \rho(t)^2 \left[\mathrm{Li}_2 \left(e^{i\left(\delta v + \Delta_{(b,a)}\right)} \right) + \mathrm{Li}_2 \left(e^{-i\left(\delta v + \Delta_{(b,a)}\right)} \right) \right] + \mathcal{O}(N)
$$
$$
= -N^{3/2} \left(\mathfrak{n}_{(b,a)} - 1 \right) \int dt \, \rho(t)^2 g_+' \left(\delta v(t) + \Delta_{(b,a)} \right) + \mathcal{O}(N). \tag{A.44}
$$

A bi-fundamental field transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ of $U(N)_a \times U(N)_b$, with magnetic flux $\mathfrak{n}_{(a,b)}$ and chemical potential $\Delta_{(a,b)}$ gives the same contribution with the replacement $a \leftrightarrow b$ and $\delta v \to -\delta v$.

The long-range force contribution of bi-fundamental fields at node a cancels with the gauge contribution in (A.36), provided that

$$
2 + \sum_{I \in a} (\mathfrak{n}_I - 1) = 0, \tag{A.45}
$$

where the sum is taken over all chiral bi-fundamentals I with an endpoint in a.

In picking the residues, we need to insert a Jacobian in the twisted index and evaluate everything else at the pole. The matrix $\mathbb{B}$ appearing in the Jacobian is $2N \times 2N$ with

---

[14]The phases can be neglected, as we will be interested in $\log |Z|$.

block form

$$\mathbb{B} = \frac{\partial\big(e^{iB_j^{(a)}}, e^{iB_j^{(b)}}\big)}{\partial(\log x_l^{(a)}, \log x_l^{(b)})} = \begin{pmatrix} x_l^{(a)} \dfrac{\partial e^{iB_j^{(a)}}}{\partial x_l^{(a)}} & x_l^{(b)} \dfrac{\partial e^{iB_j^{(a)}}}{\partial x_l^{(b)}} \\[4mm] x_l^{(a)} \dfrac{\partial e^{iB_j^{(b)}}}{\partial x_l^{(a)}} & x_l^{(b)} \dfrac{\partial e^{iB_j^{(b)}}}{\partial x_l^{(b)}} \end{pmatrix}_{2N \times 2N} , \qquad (A.46)$$

and only contributes in the tails regions,[15]

$$-\log \det \mathbb{B} = -N^{3/2} \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \int_{\delta v \approx -\Delta_{(b,a)}} dt\, \rho(t)\, Y_{(b,a)}(t)$$
$$+ \int_{\delta v \approx \Delta_{(a,b)}} dt\, \rho(t)\, Y_{(a,b)}(t) + \mathcal{O}(N \log N)\,.$$

Summarizing, pairs of bi-fundamental fields contribute to the logarithm of the index as

$$\mathbb{Re}\log Z_{\text{bulk}}^{\text{bi-fund}} = -N^{3/2} \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \int dt\, \rho(t)^2 \big[(\mathfrak{n}_{(b,a)} - 1)\, g_+' \left(\delta v(t) + \Delta_{(b,a)}\right)$$
$$+ (\mathfrak{n}_{(a,b)} - 1)\, g_-' \left(\delta v(t) - \Delta_{(a,b)}\right) \big]\,. \qquad (A.47)$$

The tails contribution is also given by

$$\mathbb{Re}\log Z_{\text{talis}}^{\text{bi-fund}} = -N^{3/2} \sum_{\substack{\text{bi-fundamentals} \\ (b,a) \text{ and } (a,b)}} \mathfrak{n}_{(b,a)} \int_{\delta v \approx -\Delta_{(b,a)}} dt\, \rho(t)\, Y_{(b,a)}(t)$$
$$+ \mathfrak{n}_{(a,b)} \int_{\delta v \approx \Delta_{(a,b)}} dt\, \rho(t)\, Y_{(a,b)}(t)\,. \qquad (A.48)$$

A field transforming in the adjoint of the ath gauge group with magnetic flux $\mathfrak{n}_{(a,a)}$ and chemical potential $\Delta_{(a,a)}$ only contributes to the bulk index. To find its contribution we need to include an explicit factor of $1/2$ in the expression (A.47) and take

$$v_b = v_a\,, \qquad \Delta_{(b,a)} = \Delta_{(a,b)} = \Delta_{(a,a)}\,, \qquad \mathfrak{n}_{(b,a)} = \mathfrak{n}_{(a,b)} = \mathfrak{n}_{(a,a)}\,. \qquad (A.49)$$

A.2.4 Fundamental and anti-fundamental contribution

The fundamental and anti-fundamental fields contribute to the logarithm of the index as

$$\log \prod_{i=1}^{N} \sum_{\substack{\text{anti-fundamental} \\ a}} \left(x_i^{(a)}\right)^{\frac{1}{2}(\tilde{\mathfrak{n}}_a - 1)} \left[1 - \tilde{y}_a \left(x_i^{(a)}\right)^{-1}\right]^{\tilde{\mathfrak{n}}_a - 1}$$
$$\times \sum_{\substack{\text{fundamental} \\ a}} \left(x_i^{(a)}\right)^{\frac{1}{2}(\mathfrak{n}_a - 1)} \left[1 - y_a^{-1} \left(x_i^{(a)}\right)^{-1}\right]^{\mathfrak{n}_a - 1}\,. \qquad (A.50)$$

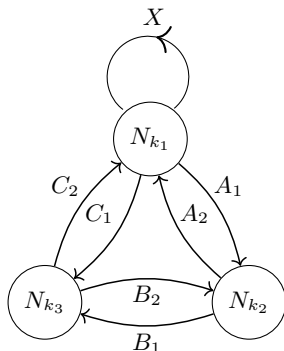[15]We refer the reader to [4] for a detailed analysis of the Jacobian at large N.

Figure 1. The SPP Chern-Simons-matter quiver.

Using the scaling ansatz (A.1), in the continuum limit we get

$$
\log \prod_{i=1}^{N} \sum_{\substack{\text{anti-fundamental} \\ a}} \left(x_i^{(a)} \right)^{\frac{1}{2}(\tilde{\mathfrak{n}}_a - 1)} \sum_{\substack{\text{fundamental} \\ a}} \left(x_i^{(a)} \right)^{\frac{1}{2}(\mathfrak{n}_a - 1)}
$$

$$
= -\frac{1}{2} N^{3/2} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} (\tilde{\mathfrak{n}}_a - 1) + \sum_{\substack{\text{fundamental} \\ a}} (\mathfrak{n}_a - 1) \right] \int dt \, \rho(t) \, t + \mathcal{O}(N) , \quad \text{(A.51)}
$$

and

$$
\log \prod_{i=1}^{N} \sum_{\substack{\text{anti-fundamental} \\ a}} \left[1 - \tilde{y}_a \left(x_i^{(a)} \right)^{-1} \right]^{\tilde{\mathfrak{n}}_a - 1} \sum_{\substack{\text{fundamental} \\ a}} \left[1 - y_a^{-1} \left(x_i^{(a)} \right)^{-1} \right]^{\mathfrak{n}_a - 1}
$$

$$
= N^{3/2} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} (\tilde{\mathfrak{n}}_a - 1) + \sum_{\substack{\text{fundamental} \\ a}} (\mathfrak{n}_a - 1) \right] \int_{t>0} dt \, \rho(t) \, t + \mathcal{O}(N) . \quad \text{(A.52)}
$$

Putting the above equations together we find:

$$
\mathbb{R}e \log Z^{(\text{anti-})\text{fund}} = \frac{1}{2} N^{3/2} \left[\sum_{\substack{\text{anti-fundamental} \\ a}} (\tilde{\mathfrak{n}}_a - 1) + \sum_{\substack{\text{fundamental} \\ a}} (\mathfrak{n}_a - 1) \right] \int dt \, \rho(t) \, |t| .
$$

$$
\text{(A.53)}
$$

B An explicit example: the SPP theory

We now consider, as an example, the quiver gauge theory which describes the dynamics of N M2-branes at the suspended pinch point (SPP) singularity (see figure 1). The Chern-Simons levels are $(2k, -k, -k)$ and the superpotential coupling is given by

$$
W = \text{Tr} \left[X \left(A_1 A_2 - C_1 C_2 \right) - A_2 A_1 B_1 B_2 + C_2 C_1 B_2 B_1 \right] . \quad \text{(B.1)}
$$

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The marginality condition on the superpotential (4.4) impose constraints on the chemical potential of the various fields

$$\Delta_A + \Delta_B = \pi\,, \qquad \Delta_B + \Delta_C = \pi\,, \qquad 2\Delta_A + \Delta_X = 2\pi\,, \qquad (B.2)$$

where we have used the symmetry of the quiver to set $\Delta_{A_1} = \Delta_{A_2} = \Delta_A$, and so on. Hence,

$$\Delta_B = \Delta\,, \qquad \Delta_X = 2\Delta\,, \qquad \Delta_A = \Delta_C = \pi - \Delta\,, \qquad (B.3)$$

and

$$\mathfrak{n}_B = \mathfrak{n}\,, \qquad \mathfrak{n}_X = 2\mathfrak{n}\,, \qquad \mathfrak{n}_A = \mathfrak{n}_C = 1 - \mathfrak{n}\,, \qquad (B.4)$$

where $\mathfrak{n}_I$ denotes the flavor magnetic flux of the field I. We assume $0 \leq \Delta \leq 2\pi$ and we enforced condition (4.4). One can check that all other solutions are related to the one we are presenting by a discrete symmetry of the quiver.[16]

B.1 The BAEs at large N

The theory under consideration is invariant under

$$A \leftrightarrow C\,, \qquad \mathrm{U}(N)^{(2)} \leftrightarrow \mathrm{U}(N)^{(3)}\,. \qquad (B.7)$$

Let us assume that the saddle-point solution is also invariant under this $\mathbb{Z}_2$ symmetry. Thus, we can choose

$$v_i^{(1)} = v_i\,, \qquad v_i^{(2)} = v_i^{(3)} = w_i\,. \qquad (B.8)$$

Given the rules of section 3.2, the Bethe potential reads

$$\frac{\mathcal{V}}{iN^{3/2}} = 2k \int dt\, t\, \rho(t)\, \delta v(t) + \int dt\, \rho(t)^2\, \Delta \left[(\pi - \Delta)(2\pi - \Delta) - 2\delta v^2\right]$$
$$- \mu \left[\int dt\, \rho(t) - 1\right] - \frac{2i}{N^{1/2}} \int dt\, \rho(t) \left[\pm \mathrm{Li}_2\left(e^{i[\delta v(t) \pm (\pi - \Delta)]}\right)\right]\,, \qquad (B.9)$$

where we defined

$$\delta v(t) = w(t) - v(t)\,, \qquad (B.10)$$

and we included the subleading terms giving rise to the equation of motion (3.25).[17] The eigenvalue density distribution $\rho(t)$, which maximizes the Bethe potential, is a piece-wise function supported on $[t_\ll, t_\gg]$. We define the inner interval as

$$t_< \text{ s.t. } \delta v(t_<) = -(\pi - \Delta)\,, \qquad t_> \text{ s.t. } \delta v(t_>) = \pi - \Delta\,. \qquad (B.11)$$

---

[16]There is a solution for

$$\Delta_A + \Delta_B = 3\pi\,, \qquad \Delta_B + \Delta_C = 3\pi\,, \qquad 2\Delta_A + \Delta_X = 4\pi\,. \qquad (B.5)$$

which is obtained, using the invariance of Z under $y_I \to 1/y_I$, from (B.12)–(B.16) by performing the substitutions

$$\mu \to -\mu\,, \qquad k \to -k\,, \qquad \Delta \to \pi - \Delta\,, \qquad Y^\pm \to -Y^\pm\,. \qquad (B.6)$$

[17]Notice that these terms are subleading in the equation of motion for ρ, since Li_2 is finite when its argument is one, while affect the equation of motion for δv since Li_1 is singular.

Schematically, we have:

The transition points are at

$$t_{\ll} = -\frac{\mu}{2k(\pi - \Delta)}, \quad t_{<} = -\frac{\mu}{k(2\pi - \Delta)}, \quad t_{>} = \frac{\mu}{k(2\pi - \Delta)}, \quad t_{\gg} = \frac{\mu}{2k(\pi - \Delta)}. \quad \text{(B.12)}$$

In the *left tail* we have

$$\rho = \frac{1}{2\Delta^2}\left(\frac{\mu}{\pi - \Delta} + 2kt\right), \qquad \delta v = -(\pi - \Delta)$$

$$Y^- = -\frac{\mu + k(2\pi - \Delta)t}{\Delta} \qquad\qquad\qquad\qquad t_{\ll} < t < t_{<}. \quad \text{(B.13)}$$

In the *inner interval* we have

$$\rho = \frac{\mu}{2(\pi - \Delta)(2\pi - \Delta)\Delta}, \qquad \delta v = \frac{k(\pi - \Delta)(2\pi - \Delta)t}{\mu} \qquad\qquad t_{<} < t < t_{>} \quad \text{(B.14)}$$

and $\delta v' > 0$. In the *right tail* we have

$$\rho = \frac{1}{2\Delta^2}\left(\frac{\mu}{\pi - \Delta} - 2kt\right), \qquad \delta v = \pi - \Delta$$

$$Y^+ = -\frac{\mu - k(2\pi - \Delta)t}{\Delta} \qquad\qquad\qquad\qquad t_{>} < t < t_{\gg}. \quad \text{(B.15)}$$

Finally, the normalization fixes

$$\mu = 2k^{1/2}(\pi - \Delta)(2\pi - \Delta)\sqrt{\frac{\Delta}{4\pi - 3\Delta}}. \quad \text{(B.16)}$$

$\mu > 0$ implies the following inequality

$$0 < \Delta < \pi. \quad \text{(B.17)}$$

For $k > 1$ there can be discrete $\mathbb{Z}_k$ identifications among the chemical potential which can affect the final result. We have not been too careful about them here.

B.2 The index at large N

The rules of the large N twisted index imply that the free energy functional is

$$\mathbb{R}\mathrm{e}\log Z = -N^{3/2}\int dt\, \rho(t)^2\left[\Delta(4\pi - 3\Delta) + \mathfrak{n}\left(3\Delta^2 - 6\pi\Delta + 2\pi^2 - 2\delta v^2\right)\right]$$

$$- N^{3/2}2(1 - \mathfrak{n})\int_{\delta v \approx -(\pi - \Delta)} dt\, \rho(t)\, Y^-(t) - N^{3/2}2(1 - \mathfrak{n})\int_{\delta v \approx (\pi - \Delta)} dt\, \rho(t)\, Y^+(t). \quad \text{(B.18)}$$

We should take the solution to the BAEs, plug it back into the functional (B.18) and compute the integral. Doing so, we obtain the following expression for the logarithm of the index:

$$
\mathbb{Re}\log Z = -\frac{4}{3}\frac{k^{1/2}N^{3/2}}{(4\pi - 3\Delta)^{3/2}\sqrt{\Delta}}
$$
$$
\times \left[\Delta\left(7\Delta^2 - 18\pi\Delta + 12\pi^2\right) + \mathfrak{n}\left(-6\Delta^3 + 19\pi\Delta^2 - 18\pi^2\Delta + 4\pi^3\right)\right] . \quad \text{(B.19)}
$$

Notice that

$$
\mathbb{Re}\log Z = -\frac{2}{\pi}\,\overline{\mathcal{V}}(\Delta) - \left[\left(\mathfrak{n} - \frac{\Delta}{\pi}\right)\frac{\partial\overline{\mathcal{V}}(\Delta)}{\partial\Delta}\right] , \quad \text{(B.20)}
$$

where

$$
\overline{\mathcal{V}}(\Delta) = \frac{4}{3}k^{1/2}N^{3/2}(\pi - \Delta)(2\pi - \Delta)\sqrt{\frac{\Delta}{4\pi - 3\Delta}} , \quad \text{(B.21)}
$$

as expected from the index theorem.

C Derivation of general rules for theories with $N^{5/3}$ scaling of the index

We assume that in the large N limit the eigenvalues corresponding to all the gauge groups are the same to leading order in N and they behave as

$$
u^{(a)}(t) = N^{\alpha}(it + v(t)) , \quad \text{(C.1)}
$$

for some $0 < \alpha < 1$. We also assume that $\sum_{a=1}^{|G|} k_a \neq 0$ in the following discussion, as appropriate for theories with a massive type IIA behavior. The long-range force analysis is identical to appendix A. Here we discuss the $N^{5/3}$ contributions.

C.1 Bethe potential at large N

Each group a with CS level k_a contributes to the finite N Bethe potential as

$$
\mathcal{V}^{\mathrm{CS}} = -\frac{k_a}{2}\sum_{i=1}^{N}\left(u_i^{(a)}\right)^2 . \quad \text{(C.2)}
$$

Using the scaling ansatz (C.1), we find

$$
-ik_a N^{2\alpha+1}\int dt\,\rho(t)\,t\,v(t) + \frac{k_a}{2}N^{2\alpha+1}\int dt\,\rho(t)\left(t^2 - v(t)^2\right) . \quad \text{(C.3)}
$$

To obtain the large N behavior of a bi-fundamental field between $\mathrm{U}(N)_a \times \mathrm{U}(N)_b$ we follow the same strategy as in section A.1.2. For example, consider

$$
\sum_{i<j}^{N}\mathrm{Li}_2\left(e^{i\left(u_j^{(b)} - u_i^{(a)} + \Delta_I\right)}\right) . \quad \text{(C.4)}
$$

We first write the dilogarithm function as a power series, i.e.,

$$
\mathrm{Li}_2(e^{iu}) = \sum_{k=1}^{\infty}\frac{e^{iku}}{k^2} , \quad \text{(C.5)}
$$

and then consider the integral

$$I_k = \int_t dt'\rho(t')e^{i(u_b(t')-u_a(t)+\Delta_I)}$$

$$= \int_t e^{-kN^\alpha(t'-t)}\sum_{j=0}^\infty \frac{(t'-t)^j}{j!}\partial_x^j\left[\rho(x)e^{ik[N^\alpha(v(x)-v(t))+\Delta_I]}\right]_{x=t}. \tag{C.6}$$

Performing the integral in t' we find

$$\int_t dt' e^{-kN^\alpha(t'-t)}\frac{(t'-t)^j}{j!} = (kN^\alpha)^{-j-1}. \tag{C.7}$$

Next, we extract the leading contribution of the left over term, i.e.,

$$\partial_x^j\left[\rho(x)e^{ik[N^\alpha(v(x)-v(t))+\Delta_I]}\right]_{x=t} \sim (ikN^\alpha)^j\left[v'(x)^j\rho(x)e^{ik[N^\alpha(v(x)-v(t))+\Delta_I]}\right]_{x=t}$$

$$= (ikN^\alpha)^j\,v'(t)^j\rho(t)e^{ik\Delta_I}. \tag{C.8}$$

Bringing the pieces together we find

$$I_k = \frac{e^{ik\Delta_I}}{k}\rho(t)N^{-\alpha}\sum_{j=0}^\infty\left[iv'(t)\right]^j = \frac{e^{ik\Delta_I}}{k}N^{-\alpha}\frac{\rho(t)}{1-iv'(t)}. \tag{C.9}$$

Thus,

$$\sum_{i<j}^N\mathrm{Li}_2\left(e^{i\left(u_j^{(b)}-u_i^{(a)}+\Delta_I\right)}\right) = N^{2-\alpha}\int dt\,\mathrm{Li}_3\left(e^{i\Delta_I}\right)\frac{\rho(t)^2}{1-iv'(t)}. \tag{C.10}$$

Following the same steps as before, we get

$$\mathcal{V}^{\text{bi-fund}} = ig_+\left(\Delta_I\right)N^{2-\alpha}\int dt\,\frac{\rho(t)^2}{1-iv'(t)}. \tag{C.11}$$

To have a nontrivial saddle-point, we need $\alpha = 1/3$ which ensure that the Chern-Simons terms and the matter contributions scale with the same power of N.

The contribution of (anti-)fundamental fields to the Bethe potential is given by [see section A.1.3],

$$\mathcal{V}^{(\text{anti-})\text{fund}} = \frac{(\tilde{n}_a - n_a)}{4}N^{5/3}\int dt\,\rho(t)\,\mathrm{sign}(t)\,[it+v(t)]^2. \tag{C.12}$$

Notice that, when the *total* number of fundamental and anti-fundamental fields in the quiver are equal, this contribution vanishes.

C.2 The index at large N

The contribution of the Vandermonde determinant to the index can be found using the same techniques presented in appendix A. We need to compute

$$-2\sum_{i<j}^N\mathrm{Li}_1\left(e^{i\left(u_j^{(a)}-u_i^{(a)}\right)}\right). \tag{C.13}$$

After a small calculation we find,

$$\mathbb{Re} \log Z^{\text{gauge}} = -\frac{\pi^2}{3} N^{5/3} \int dt\, \frac{\rho(t)^2}{1 - iv'(t)}\,. \tag{C.14}$$

We now consider a bi-fundamental field with chemical potential Δ_I and flavor magnetic flux $\mathfrak{n}_I$. Using the same methods given in appendix A, we obtain

$$\mathbb{Re} \log Z^{\text{bi-fund}}_{\text{bulk}} = -(\mathfrak{n}_I - 1)\, g'_+ (\Delta_I)\, N^{5/3} \int dt\, \frac{\rho(t)^2}{1 - iv'(t)}\,. \tag{C.15}$$

The contribution of (anti-)fundamental fields to the index, at large N, is subleading and they just contribute through the Bethe potential.

D Polylogarithms

In this appendix we review the Polylogarithms and their properties which we used in the paper. The polylogarithm function $\text{Li}_n(z)$ is defined by a power series

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}\,, \tag{D.1}$$

in the complex plane over the open unit disk, and by analytic continuation outside the disk. For $z = 1$ the polylogarithm reduces to the Riemann zeta function

$$\text{Li}_n(1) = \zeta(n)\,, \qquad \text{for} \quad \mathbb{Re}\, n > 1\,. \tag{D.2}$$

The polylogarithm for $n = 0$ and $n = 1$ is

$$\text{Li}_0(z) = \frac{z}{1 - z}\,, \qquad\qquad \text{Li}_1(z) = -\log(1 - z)\,. \tag{D.3}$$

Notice that $\text{Li}_0(z)$ and $\text{Li}_1(z)$ diverge at $z = 1$. For $n \geq 1$, the functions have a branch point at $z = 1$ and we shall take the principal determination with a cut $[1, +\infty)$ along the real axis. The polylogarithms fulfill the following relations

$$\partial_u \text{Li}_n(e^{iu}) = i\, \text{Li}_{n-1}(e^{iu})\,, \qquad\qquad \text{Li}_n(e^{iu}) = i \int_{+i\infty}^{u} \text{Li}_{n-1}(e^{iu'})\, du'\,. \tag{D.4}$$

The functions $\text{Li}_n(e^{iu})$ are periodic under $u \to u + 2\pi$ and have branch cut discontinuities along the vertical line $[0, -i\infty)$ and its images. For $0 < \mathbb{Re}\, u < 2\pi$, polylogarithms satisfy the following inversion formulæ[18]

$$
\begin{aligned}
&\text{Li}_0(e^{iu}) + \text{Li}_0(e^{-iu}) = -1 \\[4pt]
&\text{Li}_1(e^{iu}) - \text{Li}_1(e^{-iu}) = -iu + i\pi \\[4pt]
&\text{Li}_2(e^{iu}) + \text{Li}_2(e^{-iu}) = \frac{u^2}{2} - \pi u + \frac{\pi^2}{3} \\[4pt]
&\text{Li}_3(e^{iu}) - \text{Li}_3(e^{-iu}) = \frac{i}{6}u^3 - \frac{i\pi}{2}u^2 + \frac{i\pi^2}{3}u\,.
\end{aligned}
\tag{D.5}
$$

One can find the formulæ in the other regions by periodicity.

---

[18]The inversion formulæ in the domain $-2\pi < \mathbb{Re}\, u < 0$ are obtaind by sending $u \to -u$.

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