LARGE SAMPLE VARIANCES OF MAXIMUM LIKELIHOOD ESTIMATORS OF VARIANCE COMPONENTS IN THE 3-WAY NESTED CLASSIFICATION, RANDOM MODEL, WITH UNBALANCED DATA

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Abstract

Explicit expressions are presented for the elements of the information matrix of the variance components in a 3-way nested classification, random model, with normality and unbalanced data.

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Summary

Explicit expressions are presented for the elements of the information matrix of the variance components in a 3-way nested classification, random model, with normality and unbalanced data.

1. Introduction and Model

Searle [1970] developed a general method for obtaining, under normality conditions, the elements of the information matrix of the variance components of mixed models, with unbalanced data; in particular he displayed the results for the 2-way nested classification. This paper presents analogous results for the 3-way nested classification, random model, for the general case of unbalanced data.

The linear model for an observation is taken to be

$$y_{ijkl} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + e_{ijkl} \tag{1}$$

where y_{ijkl} is the l-th response within the k-th level of the γ-factor within the j-th level of the β-factor within the i-th level of the α-factor. This nesting is indicated in the observation identifiers which are taken to be

$$i = 1,2,\ldots,a;\ j = 1,2,\ldots,b_i;\ k = 1,2,\ldots,c_{ij};$$

and $l = 1,2,\ldots,n_{ijk}$.

The α's, β's, γ's and e's are assumed to be normally and independently distributed with zero means and variances σ^2_α, σ^2_β, σ^2_γ and σ^2_e respectively.

There is a total of $n_{...} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \sum_{k=1}^{c_{ij}} n_{ijk}$ observations. Suppose they are written as a vector $\underline{y}$ in lexicon order, i.e., ordered by l within k within j within i . Then, similar to the 2-way nested classification of Searle [1970], $\underline{V}$, the variance-covariance matrix of $\underline{y}$ can be written as

$$\underline{V} = \sum_{i=1}^{a}{}^{+} \underline{V}_i , \tag{2}$$

where $\sum^{+}$ denotes the operation of a direct sum of matrices. Each $\underline{V}_i$ in (2) can be written in partitioned matrix form as

$$\underline{V}_i = \left\{\underline{V}_{ij,ij'}\right\} \quad \text{for} \quad j,j' = 1,2,\ldots,b_i , \tag{3}$$

where, for j = j', the partitioned form of $\underline{V}_{ij,ij}$ is

$$\underline{V}_{ij,ij} = \left\{\delta_{kk'}\left(\sigma^2_e\underline{I} + \sigma^2_\gamma\underline{J}_{n_{ijk} \times n_{ijk}}\right) + \left(\sigma^2_\alpha + \sigma^2_\beta\right)\underline{J}_{n_{ijk} \times n_{ijk}}\right\} \text{ for } k,k' = 1,2,\ldots,c_{ij} \tag{4}$$

and for j $\neq$ j', the partitioned form of $\underline{V}_{ij,ij'}$ is

$$\underline{V}_{ij,ij'} = \left\{\sigma^2_\alpha\underline{J}_{n_{ijk} \times n_{ij'k'}}\right\} \text{ for } k = 1,2,\ldots,c_{ij} \text{ ; } k' = 1,2,\ldots,c_{ij'} . \tag{5}$$

In (4) and (5) $\delta_{kk'}$ is the Kronecker delta and $\underline{J}_{n_{ijk} \times n_{ij'k'}}$ is a matrix of order $n_{ijk} \times n_{ij'k'}$ having every element unity.

2. Method

When $\underline{\sigma}^2$ is the vector of variance components in a linear model, i.e., $\underline{\sigma}^2 = \left(\sigma^2_1, \sigma^2_2,\ldots,\sigma^2_f\right)$, and $\tilde{\underline{\sigma}}^2$ the corresponding vector of maximum likelihood estimators, Searle [1970] has shown that the large sample variance-covariance matrix of $\tilde{\underline{\sigma}}^2$, namely var($\tilde{\underline{\sigma}}^2$), can be written as

$$\text{var}(\tilde{\underline{\sigma}}^2) = 2\underline{T}^{-1} = 2\left\{t_{\sigma_r^2\sigma_s^2}\right\}^{-1} \text{ for } r,s = 1,2,\ldots,f. \tag{6}$$

A typical element of $\underline{T}$ is

$$t_{\sigma_r^2\sigma_s^2} = \text{trace}\left(\underline{V}^{-1}\frac{\partial\underline{V}}{\partial\sigma_r^2}\ \underline{V}^{-1}\frac{\partial\underline{V}}{\partial\sigma_s^2}\right) \tag{7}$$

where $\frac{\partial\underline{V}}{\partial\sigma_r^2}$ is the matrix of partial derivatives of the elements of $\underline{V}$ with respect to σ_r^2.

After deriving (7), Searle [1970] obtains explicit expressions for the elements of $\underline{T}$ for the 2-way nested classification with unbalanced data. Direct extension of the methods used there yields the analogous results for the 3-way nested classification, given in section 3 that follows. The algebraic manipulations involved in deriving these results are detailed in a lengthy appendix to this paper which is available on request from the authors. Development of the results relies heavily on lemma 2 of Urquhart [1962] (quoted in Searle [1970]), and on properties of direct sum matrices, for deriving $\underline{V}^{-1}$ of (7) from $\underline{V}$ defined by equations (2)-(5). Repetitive use is also made of the fact that $\underline{J}_{n\times n} = \underline{1}_n\underline{1}_n'$ where $\underline{1}_n$ is an n-vector of 1's, and that

$$(\underline{A} + \lambda\underline{u}\underline{v}')^{-1} = \underline{A}^{-1} - \frac{\lambda\underline{A}^{-1}\underline{u}\underline{v}'\underline{A}^{-1}}{1 + \lambda\underline{u}'\underline{A}^{-1}\underline{v}}$$

for which existence conditions hold for the matrices defined by (2) through (5).

3. Results

Presentation of the results for the 3-way nested classification is simplified by using the following notation, which is an extension of that used by Searle [1970].

$$\underset{\sim}{\sigma}^2 = \left(\sigma^2_\alpha,\ \sigma^2_\beta,\ \sigma^2_\gamma,\ \sigma^2_e\right) = (\alpha,\ \beta,\ \gamma,\ e), \tag{8}$$

$$m_{ijk} = n_{ijk}\sigma^2_\gamma + \sigma^2_e\ , \tag{9}$$

$$A_{ijpq} = \sum_{k=1}^{c_{ij}} \frac{(n_{ijk})^p}{(m_{ijk})^q}\ , \tag{10}$$

$$p_{ij} = 1 + \sigma^2_\beta\ A_{ij11}\ , \tag{11}$$

and
$$q_i = 1 + \sigma^2_\alpha \sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij}\right)\ . \tag{12}$$

Then $\underset{\sim}{T}$ of (6) is the 4×4 symmetric matrix

$$\underset{\sim}{T} = \begin{bmatrix} t_{\alpha\alpha} & t_{\alpha\beta} & t_{\alpha\gamma} & t_{\alpha e} \\ t_{\alpha\beta} & t_{\beta\beta} & t_{\beta\gamma} & t_{\beta e} \\ t_{\alpha\gamma} & t_{\beta\gamma} & t_{\gamma\gamma} & t_{\gamma e} \\ t_{\alpha e} & t_{\beta e} & t_{\gamma e} & t_{ee} \end{bmatrix} ,$$

and its 10 different elements are as follows:

$$t_{\alpha\alpha} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right) \right]^2 / q_i^2 \quad , \tag{13}$$

$$t_{\alpha\beta} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \quad , \tag{14}$$

$$t_{\alpha\gamma} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2 \right] / q_i^2 \quad , \tag{15}$$

$$t_{\alpha e} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} A_{ij12}/p_{ij}^2 \right] / q_i^2 \quad , \tag{16}$$

$$t_{\beta\beta} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \left\{ \left(A_{ij11}/p_{ij} \right)^2 - 2\sigma_\alpha^2 A_{ij11}^3 / q_i p_{ij}^3 + \sigma_\alpha^4 \left(A_{ij11}/p_{ij} \right)^2 \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \right\} , \tag{17}$$

$$t_{\beta\gamma} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \left\{ A_{ij22}/p_{ij}^2 - 2\sigma_\alpha^2 A_{ij11} A_{ij22} / q_i p_{ij}^3 + \sigma_\alpha^4 \left(A_{ij22}/p_{ij}^2 \right) \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \right\} , \tag{18}$$

$$t_{\beta e} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \left\{ A_{ij12}/p_{ij}^2 - 2\sigma_\alpha^2 A_{ij11} A_{ij12} / q_i p_{ij}^3 + \sigma_\alpha^4 \left(A_{ij12}/p_{ij}^2 \right) \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \right\} , \tag{19}$$

$$t_{\gamma\gamma} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ A_{ij22} - 2\sigma_\alpha^2 A_{ij33}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij33}/p_{ij}$$

$$+ 2\sigma_\alpha^2 \sigma_\beta^2 A_{ij22}^2/q_i p_{ij}^3 + \sigma_\beta^4 \left(A_{ij22}/p_{ij}\right)^2$$

$$+ \sigma_\alpha^4 \left(A_{ij22}/p_{ij}^2\right) \Big[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2 \Big] / q_i^2 \Big\} \,, \qquad (20)$$

$$t_{\gamma e} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ A_{ij12} - 2\sigma_\alpha^2 A_{ij23}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij23}/p_{ij}$$

$$+ 2\sigma_\alpha^2 \sigma_\beta^2 A_{ij12} A_{ij22}/q_i p_{ij}^3 + \sigma_\beta^4 A_{ij12} A_{ij22}/p_{ij}^2$$

$$+ \sigma_\alpha^4 \left(A_{ij12}/p_{ij}^2\right) \Big[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2 \Big] / q_i^2 \Big\} \,, \qquad (21)$$

and

$$t_{ee} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ A_{ij02} - 2\sigma_\alpha^2 A_{ij13}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij13}/p_{ij}$$

$$+ 2\sigma_\alpha^2 \sigma_\beta^2 A_{ij12}^2/q_i p_{ij}^3 + \sigma_\beta^4 \left(A_{ij12}/p_{ij}\right)^2$$

$$+ \sigma_\alpha^4 \left(A_{ij12}/p_{ij}^2\right) \Big[\sum_{j=1}^{b_i} A_{ij12}/p_{ij}^2 \Big] / q_i^2 \Big\}$$

$$+ (n_{\cdots} - c_{\cdot\cdot})/\sigma_e^4 \,. \qquad (22)$$

4. Validation

The above results have been partially validated in 2 major ways; by ensuring that they reduce both to those for the 2-way nested classification and to those for the balanced data case. The first way is to set either $\sigma^2_\alpha = 0$, or $\sigma^2_\beta = 0$, or $\sigma^2_\gamma = 0$ and appropriately adjust the model and the factor subscripts so that the model reduces to the 2-way nested classification. With obvious adjustments to equations (8) through (12) in all 3 cases the results (11) through (20) then reduce to those given by Searle [1970]. For example, if $\sigma^2_\alpha = 0$ then $t_{\beta\beta}$ given in (17) corresponds to $t_{\alpha\alpha}$ of (27) given in Searle [1970].

The second validation was to consider the balanced data wherein $n_{ijk} = n$ for all i, j, k, $c_{ij} = c$ for all i and j, and $b_i = b$ for all i . Results (13) through (21) then lead to the customary results for balanced data as indicated, for example, in Mahamunulu [1963].

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