

Large-scale instability in interacting dark energy and dark matter fluids

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If dark energy interacts with dark matter, this gives a new approach to the coincidence problem. But interacting dark energy models can suffer from pathologies. We consider the case where the dark energy is modelled as a fluid with constant equation of state parameter w . Non-interacting constant- w models are well behaved in the background and in the perturbed universe. But the combination of constant w and a simple interaction with dark matter leads to an instability in the dark sector perturbations at early times: the curvature perturbation blows up on super-Hubble scales. Our results underline how important it is to carefully analyze the relativistic perturbations when considering models of coupled dark energy. The instability that we find has been missed in some previous work where the perturbations were not consistently treated. The unstable mode dominates even if adiabatic initial conditions are used. The instability also arises regardless of how weak the coupling is. This non-adiabatic instability is different from previously discovered adiabatic instabilities on small scales in the strong-coupling regime.

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I. INTRODUCTION

In the standard cosmological model, dark energy and dark matter are the dominant sources in the evolution of the late universe. They are currently only indirectly detected via their gravitational effects, and this produces an important degeneracy [1]. In particular, there could be a coupling between dark energy and dark matter without violating observational constraints. A coupling in the dark sector could help to explain why the dark energy only comes to dominate after galaxy formation. But some of these models may be ruled out by instabilities that are not apparent in the background solution.

Various forms of coupling have been considered (see e.g. [2, 3, 4] and references therein). A general coupling may be described in the background by the energy balance equations of cold dark matter (c) and dark energy (x),

$$\rho'_c = -3\mathcal{H}\rho_c - aQ, \quad (1)$$

$$\rho'_x = -3\mathcal{H}(1 + w_x)\rho_x + aQ, \quad (2)$$

where $w_x = P_x/\rho_x$, $\mathcal{H} = d \ln a / d\tau$ and τ is conformal time, with $ds^2 = a^2(-d\tau^2 + d\vec{x}^2)$. Here Q is the rate of energy density transfer, so that $Q > 0$ (< 0) implies that the direction of energy transfer is dark matter \rightarrow dark energy (dark energy \rightarrow dark matter).

The density evolution in the dark sector deviates from the standard case. We can use effective equation of state parameters for the dark sector to describe the equivalent uncoupled model in the background: writing $\rho'_c + 3\mathcal{H}(1 + w_{c,\text{eff}})\rho_c = 0$ and $\rho'_x + 3\mathcal{H}(1 + w_{x,\text{eff}})\rho_x = 0$, we have

$$w_{c,\text{eff}} = \frac{aQ}{3\mathcal{H}\rho_c}, \quad w_{x,\text{eff}} = w_x - \frac{aQ}{3\mathcal{H}\rho_x}. \quad (3)$$

When $Q > 0$, we have $w_{c,\text{eff}} > 0$, so that dark matter redshifts faster than a^{-3} , while $w_{x,\text{eff}} < w_x$, so that dark energy has more accelerating power. The opposite holds for $Q < 0$. When $Q > 0$, the coupled dark energy can behave like an uncoupled “phantom” model, i.e., with $w_{x,\text{eff}} < -1$, but without the usual problems associated with phantom dark energy [5].

In order to avoid stringent “fifth-force” constraints, we assume that baryons (b) and photons (γ) are not coupled to dark energy and are separately conserved, and we assume the same for neutrinos (ν). So the balance equation for fluid A is

$$\rho'_A = -3\mathcal{H}(1 + w_A)\rho_A + aQ_A, \quad (4)$$

with $Q_b = Q_\gamma = Q_\nu = 0$ and $Q_c = -Q = -Q_x \neq 0$. The Friedman equation is

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2(\rho_\gamma + \rho_\nu + \rho_b + \rho_c + \rho_x). \quad (5)$$

Once a form for Q is given, the background dynamics are fully determined by the above equations, and typically the analysis focuses on the possibility of accelerating attractor solutions (for recent work with further references, see e.g. [2, 4]). The models may also be tested against geometric observational constraints (see e.g. [2]).

In the perturbed universe, there are subtleties and complications that do not arise for the background dynamics.

- Firstly, one needs a covariant form for the dark sector energy-momentum transfer that holds in an inhomogeneous universe, and reduces to the background form in a Friedman-Robertson-Walker (FRW) universe. For example, if one uses the ansatz $Q = Q_0 a^n$, then the background dynamics can be determined and the parameters Q_0, n can be constrained by geometric observations. However, there is no covariant form for such ad hoc ansatzes, and therefore one is unable to compute the perturbations – no consistent cosmological model can be constructed on the basis of such ansatzes.
- Secondly, one needs to ensure that dark energy perturbations are stable, i.e., $c_{sx}^2 > 0$ where c_{sx} is the dark energy sound speed (the speed at which fluctuations propagate). For a scalar field model of dark energy, $c_{sx}^2 = 1$ follows without assumptions [6]. But for fluid models as used here, we need to impose $c_{sx}^2 > 0$ by hand, so that the dark energy fluid is effectively non-adiabatic.

The sound speed problem applies equally to uncoupled dark energy, but since the coupling itself can introduce non-adiabatic modes, the issue is even more important in the coupled case.

Here we consider a dark energy fluid, with $w_x = \text{const}$, that is coupled to dark matter via a covariant energy-momentum transfer four-vector Q^μ , which reduces in the background to $\pm a^{-1} Q \delta_0^\mu$, where Q is a simple function of energy density. We show that the gauge-invariant curvature perturbation has a super-Hubble instability in the early radiation era, no matter how small the coupling is, and even if adiabatic initial conditions are used. This rules out these models. It appears that constant- w_x fluid models of dark energy, even with the imposition of $c_{sx}^2 = 1$, are unstable to couplings with the dark matter. The non-adiabatic large-scale instability that we find is different from the small-scale instabilities in the strong coupling and adiabatic regime that have been previously discussed [7, 8].

In order to avoid the large-scale instability, w_x must increase sufficiently in the early radiation and matter eras. In other words, the simple constant- w_x fluid model cannot be extended to the early radiation era. This is in contrast to the case of uncoupled dark energy, where constant- w_x fluid models are well behaved in the primordial universe. A dynamically evolving quintessence field will likely avoid the instability we find, since w_x typically does not remain constant back to the early radiation era.

The plan of the paper is as follows. In Sec. II, we present the density and velocity perturbation equations for a general model of dark energy and an arbitrary form of coupling to dark matter. We pay special attention to the dark energy sound speed and pressure fluctuations. In Sec. III we present a simple physically motivated coupling, and we derive analytical solutions for the perturbations in the early radiation era, in the case $w_x = \text{const}$. These solutions reveal the non-adiabatic super-Hubble instability, which is confirmed by numerical solutions using a modified version of CAMB [9]. In Sec. IV, we extend the analysis to another simple class of background couplings, and we show that the same instability persists. We conclude in Sec. V.

II. DENSITY AND VELOCITY PERTURBATION EQUATIONS

The general perturbation equations for coupled fluids are given in [10] and various subsequent papers. We follow broadly the notation of [11] and specialize to the case of dark sector coupling. We pay particular attention to the covariant form of the coupling and the correct treatment of momentum transfer (which vanishes in the background).

Scalar perturbations of the flat FRW metric are given in general by

$$ds^2 = a^2 \left\{ - (1 + 2\phi) d\tau^2 + 2\partial_i B d\tau dx^i + \left[(1 - 2\psi) \delta_{ij} + 2\partial_i \partial_j E \right] dx^i dx^j \right\}. \quad (6)$$

The background four-velocity is $\bar{u}^\mu = a^{-1} \delta_0^\mu$, and the A -fluid four-velocity is

$$u_A^\mu = a^{-1} \left(1 - \phi, \partial^i v_A \right), \quad u_\mu^A = a \left(-1 - \phi, \partial_i [v_A + B] \right), \quad (7)$$

where v_A is the peculiar velocity potential. The volume expansion rate, which generalizes the Newtonian relation $\theta = \vec{\nabla} \cdot \vec{v}$, is [12]

$$\theta_A = -k^2 (v_A + B). \quad (8)$$

Energy-momentum tensors

We choose u_A^μ as the energy-frame four-velocity, i.e., there is zero momentum flux relative to u_A^μ , so that $T_{A\nu}^\mu u_A^\nu = -\rho_A u_A^\mu$. Then the A -fluid energy-momentum tensor is

$$T_{A\nu}^\mu = (\rho_A + P_A) u_A^\mu u_\nu^A + P_A \delta^\mu_\nu + \pi_{A\nu}^\mu, \quad (9)$$

where $\rho_A = \bar{\rho}_A + \delta\rho_A$ and, $P_A = \bar{P}_A + \delta P_A$. The anisotropic stress $\pi_{A\nu}^\mu$ is given by

$$\pi_{A\nu}^0 = 0, \quad \pi_{Aj}^i = \left(\partial^i \partial_j - \frac{1}{3} \delta^i_j \nabla^2 \right) \pi_A. \quad (10)$$

The total (conserved) energy-momentum tensor is $T^\mu_\nu = \sum T_{A\nu}^\mu$, so that

$$(\rho + P)u^\mu u_\nu + P\delta^\mu_\nu + \pi^\mu_\nu + q^\mu u_\nu + q_\nu u^\mu = \sum_A (\rho_A + P_A)u_A^\mu u_\nu^A + \sum_A P_A + \sum_A \pi_{A\nu}^\mu. \quad (11)$$

Here q^μ is the total momentum flux relative to the total four-velocity u^μ . In general this four-velocity has the form

$$u^\mu = a^{-1} (1 - \phi, \partial^i v). \quad (12)$$

The choice of v depends on how the total four-velocity is defined.

It follows from Eqs. (11) and (12) that $\rho = \sum \rho_A$, $P = \sum P_A$, $\pi^\mu_\nu = \sum \pi_{A\nu}^\mu$, and the total momentum flux is $q^i = a^{-1} \sum (\rho_A + P_A) \partial^i v_A - a^{-1} (\rho + P) \partial^i v$. Thus the total energy frame ($q^i = 0$) is defined by

$$(\rho + P)v = \sum (\rho_A + P_A)v_A. \quad (13)$$

This is the choice of v that we will use from now on.

Energy-momentum balance

The covariant form of energy-momentum transfer is [10, 13]

$$\nabla_\nu T_A^{\mu\nu} = Q_A^\mu, \quad \sum_A Q_A^\mu = 0. \quad (14)$$

A general energy-momentum transfer can be split relative to the total four-velocity as [10, 11]

$$Q_A^\mu = Q_A u^\mu + F_A^\mu, \quad Q_A = \bar{Q}_A + \delta Q_A, \quad u_\mu F_A^\mu = 0, \quad (15)$$

where Q_A is the energy density transfer rate and F_A^μ is the momentum density transfer rate, relative to u^μ . Then it follows that $F_A^\mu = a^{-1} (0, \partial^i f_A)$, where f_A is a momentum transfer potential, and

$$Q_0^A = -a [Q_A (1 + \phi) + \delta Q_A], \quad (16)$$

$$Q_i^A = a \partial_i [f_A + Q_A (v + B)]. \quad (17)$$

The perturbed energy transfer includes a metric perturbation term $Q_A \phi$, in addition to the perturbation δQ_A . The perturbed momentum transfer is made up of two parts: the momentum transfer potential $Q_A (v + B)$ that arises from energy transport along the total velocity, and the intrinsic momentum transfer potential f_A . In the background, the energy-momentum transfer four-vectors have the form

$$Q_c^\mu = a^{-1} (Q_c, \vec{0}) = a^{-1} (-Q, \vec{0}) = -Q_x^\mu, \quad (18)$$

so that there is no momentum transfer.

Total energy-momentum conservation implies

$$0 = \sum Q_A = \sum \delta Q_A = \sum f_A. \quad (19)$$

For each A -fluid, Eq. (14) gives the perturbed energy and momentum balance equations (in Fourier space),

$$\delta\rho'_A + 3\mathcal{H}(\delta\rho_A + \delta P_A) - 3(\rho_A + P_A)\psi' - k^2(\rho_A + P_A)(v_A + E') = aQ_A\phi + a\delta Q_A, \quad (20)$$

$$[(\rho_A + P_A)(v_A + B)]' + 4\mathcal{H}(\rho_A + P_A)(v_A + B) + (\rho_A + P_A)\phi + \delta P_A - \frac{2k^2}{3a^2}\pi_A = aQ_A(v + B) + af_A. \quad (21)$$

Sound speed and pressure perturbations

The sound speed c_{sA} of a fluid or scalar field, labelled by A , is the propagation speed of pressure fluctuations in the A rest frame [10, 14, 15]:

$$c_{sA}^2 = \left. \frac{\delta P_A}{\delta \rho_A} \right|_{\text{rf}}. \quad (22)$$

For a scalar field φ , the rest frame is defined by the hypersurfaces $\varphi = \text{const}$, orthogonal to the rest-frame four-velocity $u_\mu^\varphi \propto \nabla_\mu \varphi$. Thus the kinetic energy density in the rest frame is $-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi = \varphi'^2/(2a^2)$, while $\delta\varphi = 0$ in the rest frame, so that $\delta V = 0$, where $V(\varphi)$ is the potential. The density and pressure perturbations are consequently equal in the rest frame: $\delta\rho_\varphi = \delta(\frac{1}{2}a^{-2}\varphi'^2 + V) = a^{-2}\varphi'\delta\varphi' = \delta(\frac{1}{2}a^{-2}\varphi'^2 - V) = \delta p_\varphi$. The sound speed is therefore equal to the speed of light, independent of the form of $V(\varphi)$:

$$\delta\varphi|_{\text{rf}} = 0 \quad \Rightarrow \quad c_{s\varphi}^2 = 1. \quad (23)$$

We can define the ‘‘adiabatic sound speed’’ for any medium via

$$c_{aA}^2 = \frac{P'_A}{\rho'_A} = w_A + \frac{w'_A}{\rho'_A/\rho_A}. \quad (24)$$

For a barotropic fluid, $c_s^2 = c_a^2$, and if $w = \text{const}$, then $c_a^2 = w$. By contrast, for a scalar field, $c_{s\varphi}^2 \neq c_{a\varphi}^2 \neq w_\varphi$.

The fluid model for dark energy with constant w_x is at face value a barotropic adiabatic model. But if we treat the dark energy strictly as an adiabatic fluid, then the sound speed c_{sx} would be imaginary ($c_{sx}^2 = c_{ax}^2 = w_x < 0$), leading to instabilities in the dark energy. In order to fix this problem, it is necessary to impose $c_{sx}^2 > 0$ by hand [15], and it is natural to adopt the scalar field value Eq. (23). Thus

$$c_{sx}^2 = 1, \quad c_{ax}^2 = w_x = \text{const} < 0. \quad (25)$$

This is what is done in the CAMB [9] and CMBFAST [16] codes.

In the perturbation equations (20) and (21), we need to relate δP_A to $\delta\rho_A$ via Eq. (22). The A rest frame (the zero momentum gauge or comoving orthogonal gauge) is the comoving ($v_A|_{\text{rf}} = 0$) orthogonal ($B|_{\text{rf}} = 0$) frame, so that

$$T_{A0}^i|_{\text{rf}} = 0 = T_{Ai}^0|_{\text{rf}}. \quad (26)$$

We make a gauge transformation, $x^\mu \rightarrow x^\mu + (\delta\tau_A, \partial^i \delta x_A)$, from the rest frame gauge to a general gauge:

$$v_A + B = (v_A + B)|_{\text{rf}} + \delta\tau_A, \quad \delta P_A = \delta P_A|_{\text{rf}} - P'_A \delta\tau_A, \quad \delta\rho_A = \delta\rho_A|_{\text{rf}} - \rho'_A \delta\tau_A. \quad (27)$$

Thus $\delta\tau_A = v_A + B$, and substituting into the pressure and density fluctuations, we obtain

$$\delta P_A = c_{aA}^2 \delta\rho_A + (c_{sA}^2 - c_{aA}^2) [\delta\rho_A + \rho'_A (v_A + B)] = c_{aA}^2 \delta\rho_A + \delta P_{\text{nad}A}, \quad (28)$$

where $\delta P_{\text{nad}A}$ is the intrinsic non-adiabatic pressure perturbation in the A -fluid.

Our result applies to both coupled and uncoupled fluids, where the difference enters via the term ρ'_A . This recovers the expression for the uncoupled case in [14, 15]. For the coupled case, the background coupling Q_A enters δP_A explicitly:

$$\delta P_A = c_{sA}^2 \delta\rho_A + (c_{sA}^2 - c_{aA}^2) \left[3\mathcal{H}(1 + w_A)\rho_A - aQ_A \right] \frac{\theta_A}{k^2}. \quad (29)$$

This corrects the expression used in [17], which omits the Q_A term. As a consequence, there are errors in the equations in [17, 18] for δ'_x and θ'_x . We will discuss the implications of this in Sec. V.

General equations

From the above equations we can derive evolution equations for the dimensionless density perturbation $\delta_A = \delta\rho_A/\rho_A$ and for the velocity perturbation θ_A (which has dimension of k):

$$\delta'_A + 3\mathcal{H}(c_{sA}^2 - w_A)\delta_A + (1 + w_A)\theta_A + 3\mathcal{H}[3\mathcal{H}(1 + w_A)(c_{sA}^2 - w_A) + w'_A]\frac{\theta_A}{k^2} - 3(1 + w_A)\psi' + (1 + w_A)k^2(B - E') = \frac{aQ_A}{\rho_A} \left[\phi - \delta_A + 3\mathcal{H}(c_{sA}^2 - w_A)\frac{\theta_A}{k^2} \right] + \frac{a}{\rho_A} \delta Q_A, \quad (30)$$

$$\theta'_A + \mathcal{H}(1 - 3c_{sA}^2)\theta_A - \frac{c_{sA}^2}{(1 + w_A)}k^2\delta_A + \frac{2}{3a^2(1 + w_A)\rho_A}k^4\pi_A - k^2\phi = \frac{aQ_A}{(1 + w_A)\rho_A}[\theta - (1 + c_{sA}^2)\theta_A] - \frac{a}{(1 + w_A)\rho_A}k^2f_A. \quad (31)$$

The curvature perturbations on constant- ρ_A surfaces and the total curvature perturbation (on constant- ρ surfaces), are given by the gauge-invariant quantities

$$\zeta_A = -\psi - \mathcal{H}\frac{\delta\rho_A}{\rho'_A}, \quad \zeta = -\psi - \mathcal{H}\frac{\delta\rho}{\rho'} = \sum \frac{\rho'_A}{\rho'} \zeta_A. \quad (32)$$

The total energy conservation equation leads to

$$\zeta' = -\frac{\mathcal{H}}{(\rho + P)}\delta P_{\text{nad}}. \quad (33)$$

The gauge-invariant relative entropy perturbation for any two fluids is

$$S_{AB} = 3\mathcal{H} \left(\frac{\delta\rho_B}{\rho'_B} - \frac{\delta\rho_A}{\rho'_A} \right) = 3(\zeta_A - \zeta_B). \quad (34)$$

III. A COVARIANT MODEL OF DARK SECTOR COUPLING

In order to apply the equations of the previous section, we need to choose a model of the dark sector coupling via a covariant choice of the transfer four-vector $Q_c^\mu = -Q_x^\mu$. For example, a coupling model motivated by scalar-tensor theory has [19]

$$Q_c^\mu = -Q_x^\mu = \beta(\varphi)T_{c\nu}^\nu \nabla^\mu \varphi, \quad (35)$$

where φ is the scalar field dark energy and β is a coupling function. Using this form, the perturbed energy transfer $\delta Q_c = -\delta Q_x$ and momentum transfer $f_c = -f_x$ can be calculated unambiguously. Note that $\nabla^\mu \varphi$ is parallel to the dark energy four-velocity u_x^μ , i.e. $Q_c^\mu = -Q_x^\mu \propto u_x^\mu$.

For more phenomenological models of coupling, especially in the context of fluid dark energy, it is not always clear what the covariant form of the transfer should be. For example, consider the background transfer models [20]

$$Q = \frac{\mathcal{H}}{a}(\alpha_c \rho_c + \alpha_x \rho_x), \quad (36)$$

where α_c and α_x are dimensionless constants. One problem with these models is ambiguity under perturbation: What is the covariant form of $Q_c^\mu = -Q_x^\mu$ that reduces to Eq. (36) in the background? This has not been made explicit in previous work [17, 18], as pointed out in [7].

A further problem with Eq. (36) is the explicit presence of the universal expansion rate \mathcal{H} . This is designed for mathematical simplicity rather than physical motivation: one does not expect the dark sector coupling at each event to depend on the global behaviour of the universe, but to depend only on purely local quantities. A generous interpretation is that the \mathcal{H} factor is an approximation to the temperature-dependence of the interaction rate.

Here we propose a covariant model of coupling that avoids these problems:

$$Q_c^\mu = -Q_x^\mu = \Gamma T_{c\nu}^\nu u_c^\mu = -\Gamma \rho_c u_c^\mu, \quad (37)$$

where Γ is a constant interaction rate, ρ_c is the dark matter density in the inhomogeneous universe, and u_c^μ is the dark matter four-velocity. The notable features of this phenomenological coupling model are:

- (1) The interaction rate Γ is ‘local’, i.e. it is determined by local interactions and not by the universal expansion rate.
- (2) In the rest frame of the dark matter, there is no momentum transfer. (By contrast, for Eq. (35) the momentum transfer vanishes in the dark energy rest frame.)
- (3) The case $\Gamma > 0$ corresponds in the dark matter frame to the decay of dark matter into dark energy. This opens the possibility of an alternative approach to the coincidence problem: instead of trying to achieve a constant nonzero ratio Ω_c/Ω_x on the basis of primordially existing dark energy, one could try to build models where the dark energy accumulates via the decay of dark matter, and dominates in the late universe because the decay rate Γ is small.

Background dynamics

In the background, the coupling (37) reduces to Eq. (18), with

$$Q = \Gamma \rho_c. \quad (38)$$

When $\Gamma > 0$, this coincides with a special case of a model in which superheavy dark matter particles decay to a quintessence scalar field [21]. It also has the same form as simple models to describe the decay of dark matter into radiation [22], or a curvaton field into radiation [11]. The background dynamics with Eq. (38) have been analysed in [4] for the case of scalar field dark energy. The growth factor and weak lensing have been investigated for the case $w_x(a) = w_0 + w_a(1 - a)$ in [23].

We impose the condition $w_x > -1$ so as to avoid a phantom fluid model of dark energy. In order to have a close-to-standard matter dominated era for structure formation, and in order to be consistent with the observed angular diameter distance to last scattering, it is necessary that $|w_{c,\text{eff}}|$ is small, i.e.

$$\frac{|Q|}{\rho_c} \lesssim 0.1 H_0, \quad (39)$$

where $H_0 = \mathcal{H}_0/a_0 = \mathcal{H}_0$ is today’s Hubble rate.

For this coupling model, Eq. (3) implies that

$$w_{c,\text{eff}} = \frac{a\Gamma}{3\mathcal{H}}, \quad w_{x,\text{eff}} - w_x = -\frac{a\Gamma}{3\mathcal{H}} \frac{\rho_c}{\rho_x}, \quad (40)$$

where $|\Gamma| \ll H_0$ by Eq. (39). Since $w_x > -1$, it follows that the total effective equation of state satisfies $w_{\text{tot}} > -1$, so that H is a decreasing function and therefore $a|\Gamma|/\mathcal{H}$ decreases as we look backward into the past. Thus $w_{c,\text{eff}} \ll 1$ for all times up to the present, i.e., the dark matter effectively does not see the coupling for all times from today to the past:

$$\rho_c = \rho_{c0} a^{-3}. \quad (41)$$

If $|\Gamma|/H_0 < 3(1 + w_x)\Omega_{x0}/\Omega_{c0}$, then the dark sector coupling is negligible at late times, by Eqs. (40) and (41), and we have $\rho_x = \rho_{x0} a^{-3(1+w_x)}$. This is valid to the past until the coupling term $|a\Gamma\rho_c|$ is equal to the redshift term $|3\mathcal{H}(1 + w_x)\rho_x|$ in Eq. (2). At earlier times, the coupling term will dominate for a small enough a , regardless of how small $|\Gamma|$ is.

In the radiation era,

$$\mathcal{H} = \tau^{-1}, \quad a^2 = H_0^2 \Omega_{r0} \tau^2, \quad (42)$$

and the energy balance equations (1) and (2) lead to a simple solution for early times in the case $w_x < -2/3$:

$$a\Gamma \frac{\rho_c}{\rho_x} = (3w_x + 2)\tau^{-1}, \quad w_x < -\frac{2}{3}. \quad (43)$$

From now on we assume that $w_x < -2/3$, which is consistent with observations. Equations (41) and (43) imply that

$$\rho'_x = -\mathcal{H}\rho_x \Rightarrow w_{x,\text{eff}} = -\frac{2}{3} \quad \text{and} \quad \rho_x \propto a^{-1} \propto \tau^{-1}. \quad (44)$$

Equation (41) shows that $\rho_c > 0$ for all times. Therefore by Eq.(43), ρ_x becomes negative¹ when $\Gamma > 0$. This is the case that corresponds to the decay of dark matter to dark energy. The rigidity of the assumption that w_x is constant leads to this problem with the decaying dark matter case. When dark energy is modelled as a scalar field [4], it remains positive for all times and there is no such problem in the case $\Gamma > 0$.

These analytical approximations are confirmed by numerical integration, as illustrated in Fig. 1.

Dark sector perturbations

For the model described by the covariant energy-momentum transfer four-vector $Q_c^\mu = -Q_x^\mu = -\Gamma\rho_c u_c^\mu$, we need to determine $\delta Q_c = -\delta Q_x$ and $f_c = -f_x$ from the conditions imposed by energy-momentum balance. Using Eqs. (7) and (37), we find the components of $Q_\mu^c = -Q_\mu^x$:

$$Q_\mu^c = -Q_\mu^x = a\Gamma\rho_c \left[1 + \phi + \delta_c, \partial_i(v_c + B) \right]. \quad (45)$$

Comparing with Eqs. (16) and (17), it follows that

$$\delta Q_c = -\Gamma\rho_c\delta_c = -\delta Q_x, \quad f_c = \Gamma\rho_c(v - v_c) = -f_x, \quad (46)$$

where v is the total energy frame velocity, defined by Eq. (13).

The density and velocity perturbation equations (30) and (31) for the dark sector, with $\pi_c = 0 = \pi_x$, $w_c = 0 = w'_x$ and $c_{sx}^2 = 1$, can then be given, in longitudinal (Newtonian) gauge ($B = E = 0$):

$$\delta'_x + 3\mathcal{H}(1 - w_x)\delta_x + (1 + w_x)\theta_x + 9\mathcal{H}^2(1 - w_x^2)\frac{\theta_x}{k^2} - 3(1 + w_x)\psi' = a\Gamma\frac{\rho_c}{\rho_x} \left[\delta_c - \delta_x + 3\mathcal{H}(1 - w_x)\frac{\theta_x}{k^2} + \phi \right], \quad (47)$$

$$\theta'_x - 2\mathcal{H}\theta_x - \frac{k^2}{(1 + w_x)}\delta_x - k^2\phi = \frac{a\Gamma}{(1 + w_x)}\frac{\rho_c}{\rho_x}(\theta_c - 2\theta_x), \quad (48)$$

and

$$\delta'_c + \theta_c - 3\psi' = -a\Gamma\phi, \quad (49)$$

$$\theta'_c + \mathcal{H}\theta_c - k^2\phi = 0. \quad (50)$$

Note that the dark matter velocity perturbation equation (50) is the same as in the uncoupled case. (In particular, this means that in synchronous gauge, we can consistently set $\theta_c = 0$, as is done in the standard, uncoupled case.) This is due to the fact that there is no momentum transfer in the dark matter frame.

We are now in a position to see qualitatively why there is a large-scale instability in the dark sector perturbations during the early radiation era. The coupling term Q_x in δP_x , Eq. (29), leads to a driving term

$$-2\frac{a\Gamma}{(1 + w_x)}\frac{\rho_c}{\rho_x}\theta_x = \frac{-2(3w_x + 2)}{1 + w_x}\mathcal{H}\theta_x \quad (51)$$

on the right hand side of Eq. (48). Here the multiplier of $\mathcal{H}\theta_x$ is a positive number (since $w_x < -2/3$) – and it becomes very large if w_x is close to -1 . This causes rapid growth of θ_x . Qualitatively, this is the source of the instability: in the presence of energy-momentum transfer in the perturbed dark fluids, momentum balance requires a run-away growth of the dark energy velocity. The precise form of the instability is computed analytically below.

Radiation era

Tight coupling between photons and baryons means that (a) the only nonzero momentum transfer is in the dark sector, and (b) the only nonzero anisotropic stress is that of the neutrinos (which have decoupled). The perturbed

¹ Note that we can avoid $\rho_x < 0$ for $\Gamma > 0$ if $w_x > -2/3$ or $w_x < -1$, but we exclude these cases.

Einstein equations reduce to

$$3\tau^{-1}\psi' + k^2\psi + 3\tau^{-2}\phi = -4\pi Ga^2\delta\rho, \quad (52)$$

$$k^2(\psi' + \tau^{-1}\phi) = 4\pi Ga^2(\rho + P)\theta, \quad (53)$$

$$\psi'' + 2\tau^{-1}\psi' - \tau^{-2}\psi + \tau^{-1}\phi' + \frac{k^2}{3}(\psi - \phi) = 4\pi Ga^2\delta P, \quad (54)$$

$$\psi - \phi = 8\pi G\pi_\nu. \quad (55)$$

The perturbed balance equations in the dark sector are given by Eqs. (47)–(50), with background coefficients determined by Eq. (42). For the photon-baryon sector,

$$\delta'_\gamma = -\frac{4}{3}\theta_\gamma + 4\psi', \quad \delta'_b = -\theta_b + 3\psi', \quad (56)$$

$$\theta'_\gamma = \frac{1}{4}k^2\delta_\gamma + k^2\phi, \quad \theta'_b = -\mathcal{H}\theta_b + c_{sb}^2k^2\delta_b + k^2\phi, \quad (57)$$

and for neutrinos [12],

$$\delta'_\nu = -\frac{4}{3}\theta_\nu + 4\psi', \quad \theta'_\nu = \frac{1}{4}k^2\delta_\nu + k^2\phi - k^2\sigma_\nu, \quad \sigma'_\nu = \frac{4}{15}\theta_\nu, \quad (58)$$

where $\sigma_\nu := 2k^2\pi_\nu/[3a^2(\rho_\nu + P_\nu)]$, and we have neglected the neutrino octopole, i.e., we work to leading order in $k\tau$.

Adiabatic initial conditions

Now we look for a solution in the radiation era, in the super-Hubble scale limit, $k\tau \ll 1$. We find that we can set adiabatic initial conditions to lowest order in $k\tau$:

$$\phi = A_\phi = \text{const}, \quad \psi = \left(1 + \frac{2}{5}R_\nu\right)\phi, \quad R_\nu := \frac{\rho_\nu}{\rho_\nu + \rho_\gamma}, \quad (59)$$

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c = -2\phi, \quad \theta_\gamma = \theta_\nu = \theta_b = \theta_c = \frac{1}{2}(k\tau)k\phi, \quad \sigma_\nu = \frac{1}{15}(k\tau)^2\phi, \quad (60)$$

$$\delta_x = \frac{1}{4}\delta_\gamma, \quad \theta_x = \theta_\gamma. \quad (61)$$

The expression for δ_x follows from $\zeta_x - \zeta_\gamma = 0$ [see Eq. (34)], using the early-time attractor solution Eq. (44).

However, at higher order in $k\tau$, this solution is *not* adiabatic. In the standard case with uncoupled dark energy, this is not an issue – since the deviation from adiabaticity is decaying and suppressed [24]. However, for the coupled dark energy model considered here, the situation is dramatically different – because there is a strongly growing non-adiabatic mode on super-Hubble scales. Even with adiabatic conditions in the limit $k\tau \rightarrow 0$, the terms of higher order in $k\tau$ contain the non-adiabatic mode, and this mode will dominate since it is strongly growing (see below). Note that this mode is regular, and it is stimulated by the dark sector coupling, as explained via Eq. (51).

The detailed analysis of all perturbative modes is given elsewhere [25]. There we also show numerically how the total curvature perturbation starts off constant with the initial conditions Eqs. (59)–(61), but begins to grow dramatically after a short time (well before equality).

The dominant non-adiabatic mode

In order to find the non-adiabatic mode, we assume a leading-order power-law form for the perturbations:

$$\psi = A_\psi(k\tau)^{n_\psi}, \quad \phi = A_\phi(k\tau)^{n_\phi}, \quad \delta_A = B_A(k\tau)^{n_A}, \quad \theta_A = C_A(k\tau)^{s_A}, \quad \sigma_\nu = D_\nu(k\tau)^{n_\sigma}, \quad (62)$$

where n_ψ and n_ϕ are not zero ($n_\psi = 0 = n_\phi$ is the adiabatic case). The perturbed Einstein and balance equations (47)–(58) may then be solved, to leading order in $k\tau$, in terms of ψ :

$$\phi = J\psi, \quad (63)$$

$$\delta_\gamma = \delta_\nu = 4\psi, \quad (64)$$

$$\delta_b = \delta_c = \frac{3}{4}\delta_\gamma = 3\psi, \quad (65)$$

$$\theta_\gamma = \theta_\nu = \theta_b = \frac{(J+1)}{(n_\psi+1)}(k\tau)k\psi, \quad (66)$$

$$\sigma_\nu = \frac{4}{15(n_\psi+2)}(k\tau)\frac{\theta_\nu}{k}, \quad (67)$$

$$\theta_c = \frac{(n_\psi+1)}{(n_\psi+2)}\frac{J}{(J+1)}\theta_\gamma, \quad (68)$$

$$\delta_x = \frac{-2k^3(3w_x+2)[n_\psi^2+(J+1)n_\psi-(J+2)]}{\Gamma H_0^2 \Omega_{c0}(n_\psi-1)}(k\tau)^{-3}\psi, \quad (69)$$

$$\theta_x = -\frac{(n_\psi+2)}{3(1-w_x)}(k\tau)k\delta_x. \quad (70)$$

The gauge-invariant curvature perturbation, Eq. (32), is given in terms of ψ as

$$\zeta = -\frac{1}{2}(n_\psi+J+2)\psi. \quad (71)$$

The solution is thus fully determined up to an arbitrary normalization of the amplitude parameter A_ψ . The stress anisotropy parameter J is given by

$$J := \frac{A_\phi}{A_\psi} = 1 - \frac{16R_\nu}{5(n_\psi+2)(n_\psi+1)+8R_\nu}. \quad (72)$$

The power-law index n_ψ is determined in terms of w_x as

$$n_\psi = n_\pm = \frac{-(1+2w_x) \pm \sqrt{3w_x^2-2}}{1+w_x}. \quad (73)$$

The fastest growing mode is the n_+ -mode. Equation (69) shows that the modes are regular (i.e., well-behaved as $k\tau \rightarrow 0$) provided that $\text{Re}n_\pm \geq 3$. This leads to the conditions,

$$n_+ \text{ regular if } -1 < w_x \leq -\frac{4}{5}, \quad n_- \text{ regular if } -\frac{9}{11} < w_x \leq -\frac{4}{5} \quad (74)$$

The only explicit dependence on the coupling rate Γ in the early-time solutions is in the δ_x solution, Eq. (69). The uncoupled limit $\Gamma = 0$ leads to a singularity in that equation. This reflects the fact that the solutions are not valid in the limit $\Gamma = 0$. We cannot recover the $\Gamma = 0$ limit since we have used in an essential way that $\Gamma \neq 0$, see Eqs. (43) and (44). The most important difference is that $n_\psi = 0$ in the uncoupled case, so that ψ and ϕ are constant, and in addition $\zeta = \text{const}$. By contrast, when $\Gamma \neq 0$, we see that n_+ is typically very large:

$$w_x \sim -1 \Rightarrow n_+ \sim \frac{2}{1+w_x} \gg 1. \quad (75)$$

This large exponent signals the blow-up of ψ , and therefore of all perturbations, including the gauge-invariant curvature perturbation, Eq. (71), on super-Hubble scales in the early radiation era. The instability is stronger the closer w_x is to -1 .

The instability occurs no matter how weak the coupling is. A smaller value of $|\Gamma|$ simply moves the blow-up to earlier times. This is in contrast with the strong-coupling instabilities discussed in [7, 8]. Furthermore, the instability is non-adiabatic, in accordance with Eq. (33), since the curvature perturbation blows up. Again, this is in contrast to the adiabatic instabilities of [7, 8]. The origin of this large-scale non-adiabatic instability is not simply the fact that the dark energy fluid is non-adiabatic, i.e., $c_{ax}^2 \neq c_{sx}^2$. In the uncoupled case, the same non-adiabatic fluid behaviour is also present, but there is no instability. The coupling plays an essential role in driving the large-scale non-adiabatic instability.

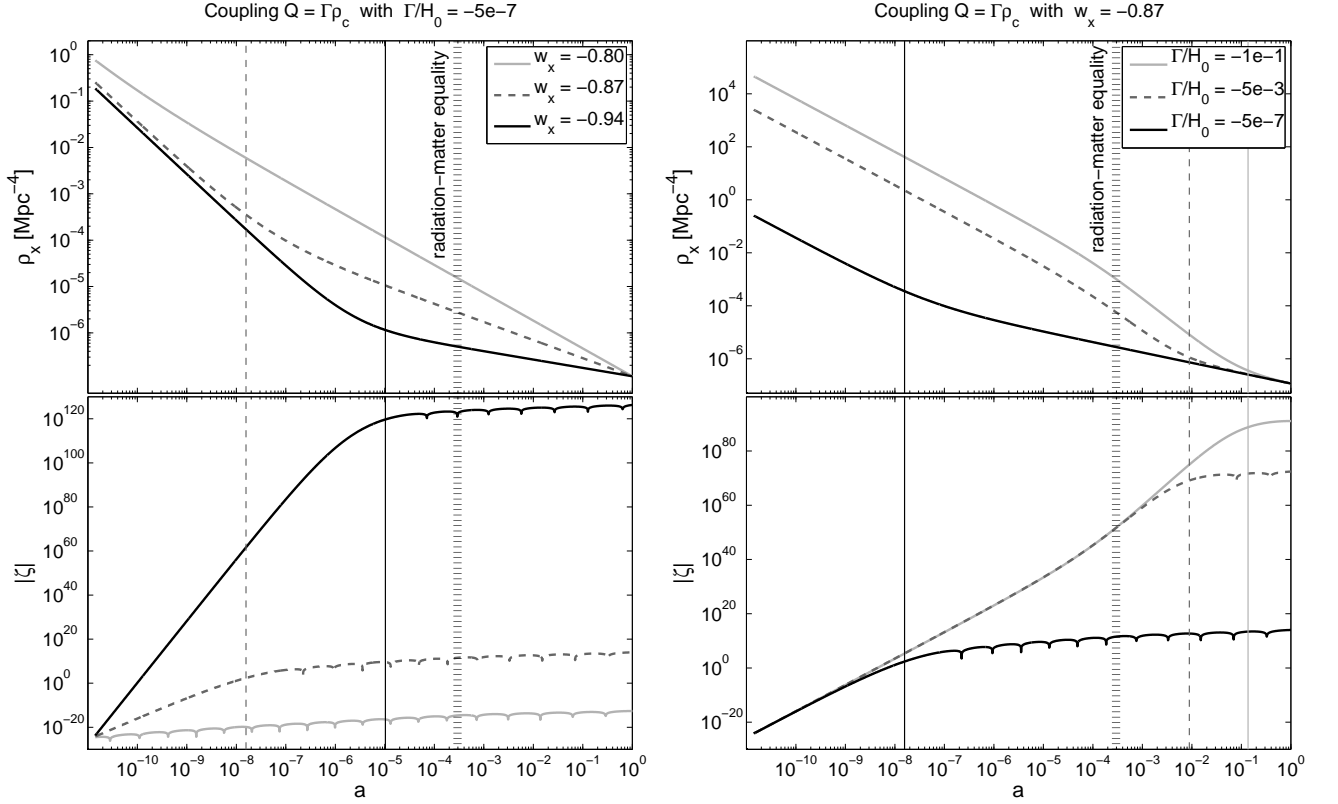


FIG. 1: The background evolution of dark energy density ρ_x (top panels) and the evolution of the gauge-invariant curvature perturbation ζ for a super-Hubble scale $k = 7 \times 10^{-5} \text{Mpc}^{-1}$ (bottom panels), as functions of scale factor a , for the model with coupling given by Eq. (37). In panels on the left, we vary w_x with Γ fixed, while the right-hand panels have fixed w_x and varying Γ . For each case, the vertical lines (in matching style) indicate the moment when $|3\mathcal{H}(1+w_x)\rho_x|$ and $|a\Gamma\rho_c|$ are equal in Eq. (2). To the right of these lines the background evolves as in the uncoupled case, i.e. $\rho_x \propto a^{-3(1+w_x)}$. To the left, the coupling modifies the background evolution to $\rho_x \propto a^{-1}$, Eq. (44). (Note that in the left panels for $w_x = -0.80$ this happens very far in the past, not shown in the figure.) All the curves show the full numerical solution obtained with our modified version of CAMB, with the initial amplitude of ψ set to 10^{-25} . The analytical solution for the blow-up of ζ , Eq. (71), is practically indistinguishable from the numerical solution at early times.

The analytical demonstration of the large-scale instability at early times is confirmed by numerical solutions, using a modified version of CAMB. Examples are shown in Fig. 1. As long as the coupling modifies the background evolution, the curvature perturbation is extremely rapidly growing: $\zeta \propto a^{n_+}$ where n_+ is given by Eq. (73) during radiation domination. When the background starts to behave as uncoupled, the blow-up of ζ ends and it begins to oscillate about zero with large (and mildly) increasing amplitude.

Equation (73) shows that for large enough w_x , there are oscillations super-imposed on the super-Hubble blow-up mode: $-\sqrt{2/3} < w_x \leq -4/5$, where the upper limit comes from Eq. (74). See Fig. 1 for an example (left panel, with $w_x = -0.8$).

Figure 1 also shows that if $|\Gamma|$ is large enough to modify the background after radiation-matter equality, then $|\zeta|$ follows a matter-dominated attractor solution:

$$\zeta \propto a^{\hat{n}_+} \quad \text{where } \hat{n}_+ \sim \frac{3}{2}n_+ \text{ for } w_x \sim -1. \quad (76)$$

The background solution in the matter era that corresponds to Eq. (43) is

$$a\Gamma \frac{\rho_c}{\rho_x} = 3(2w_x + 1)\tau^{-1}, \quad w_x < -\frac{1}{2}. \quad (77)$$

The scale-dependence of the instability is illustrated in Fig. 2. The important point is that the instability cannot be removed by simply re-scaling A_ψ : even if we can match the large-scale CMB power with a small enough A_ψ , the full CMB and matter power spectra will exhibit a strong scale-dependence in violation of observations.

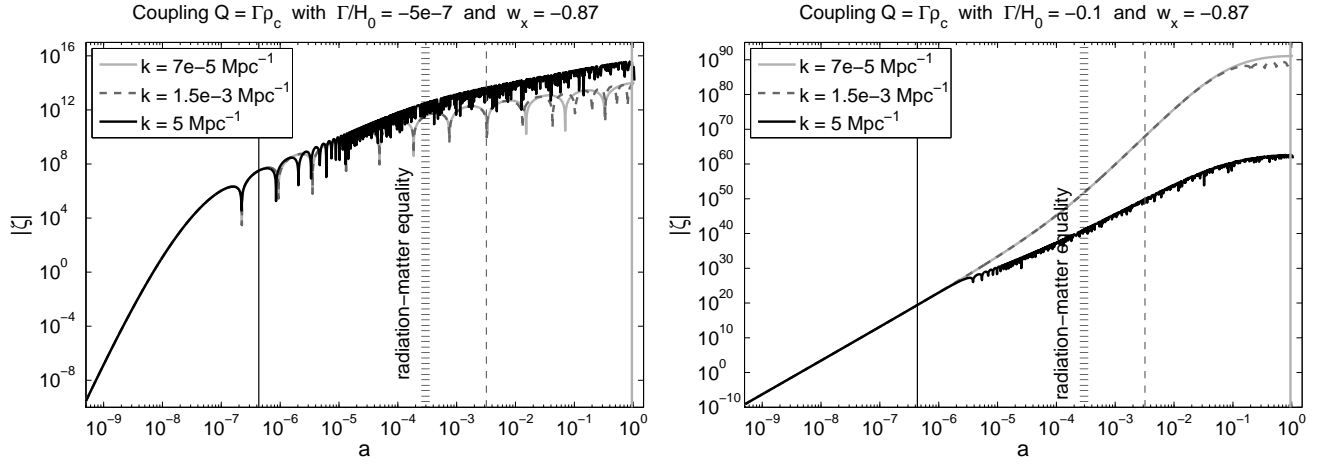


FIG. 2: The evolution of the gauge-invariant curvature perturbation ζ for three different scales as a function of scale factor a , for the model with coupling given by Eq. (37). In the panel on the left, the coupling $|\Gamma|$ is very small while in the right-hand panel, $|\Gamma|$ is larger. Vertical lines indicate the moment when each mode enters the horizon ($k\tau \sim 1$). The largest scale ($k = 7 \times 10^{-5} \text{ Mpc}^{-1}$) stays super-Hubble all the way up to today. The intermediate scale ($k = 1.5 \times 10^{-3} \text{ Mpc}^{-1}$) enters the horizon during matter domination, and the smallest scale ($k = 5 \text{ Mpc}^{-1}$) enters deep in the radiation era.

Note also that the analytical derivation of the instability does not depend on the sign of Γ , i.e., there is a blow-up for positive and negative Γ . However, as shown by Eq. (42), when $\Gamma > 0$, there is always a time τ_* when ρ_x goes through zero and is negative for $\tau < \tau_*$. The perturbation equations are singular at τ_* .

IV. EXTENSION TO OTHER COUPLING MODELS

The instability in the model with coupling $Q_c^\mu = -Q_x^\mu = -\Gamma\rho_c u_c^\mu$ is not peculiar to the particular form of the coupling. The other models discussed in Sec. III also suffer from this instability. We consider two background couplings, which are special cases of Eq. (36): $aQ = \alpha\mathcal{H}\rho_c$ and $aQ = \beta\mathcal{H}(\rho_c + \rho_x)$.

Model with background coupling $aQ = \alpha\mathcal{H}\rho_c$

For this coupling the background balance equations may be solved exactly:

$$\rho_c = \rho_{c0} a^{-(3+\alpha)}, \quad (78)$$

$$\rho_x = \rho_{x0} a^{-3(1+w_x)} + \left(\frac{\alpha}{\alpha - 3w_x} \right) \rho_{c0} a^{-3} (a^{-3w_x} - a^{-\alpha}), \quad (79)$$

where we assume that $\alpha > 3w_x$ (otherwise the coupling strength $|\alpha|$ would be too large). The dark matter density is always positive. For the dark energy density to cross zero and become negative, there must be a solution $a_n < 1$ to $a_n^{3w_x - \alpha} = 1 + \Omega_{x0}(\alpha - 3w_x)/(\Omega_{c0}\alpha)$. The left hand side is always > 1 , since $3w_x - \alpha < 0$, whereas the right hand side is > 1 only for $\alpha > 0$. Thus for $\alpha > 0$, i.e. for the case of dark matter decaying into dark energy, the dark energy density always becomes negative in the past. In the early universe, $a \ll 1$, the exact solution implies

$$\frac{\rho_x}{\rho_c} \rightarrow \frac{\alpha}{3w_x - \alpha}, \quad \alpha > 3w_x. \quad (80)$$

To analyze perturbations in this model, we need a covariant form of energy-momentum transfer four-vector that reduces to $Q = \alpha\mathcal{H}\rho_c/a$ in the background. We propose to use the same form as Eq. (37):

$$aQ_c^\mu = -aQ_x^\mu = -\alpha\mathcal{H}\rho_c u_c^\mu. \quad (81)$$

Here we are following the implicit assumption made in other work that αH is an approximation to an interaction rate that varies with time but not in space, so that there is no perturbation of H in δQ_c^μ .

With this covariant form of energy-momentum transfer, the momentum transfer is

$$af_c = \alpha \mathcal{H} \rho_c (v - v_c) = -af_x, \quad (82)$$

and the dark sector density and velocity perturbations are given by Eqs. (47)–(50), with Γ replaced by $\alpha \mathcal{H}/a$.

The early radiation solution to leading order in $k\tau$ is qualitatively similar to the solution for the Γ model in the previous section, with differences arising because the interaction rate αH varies with time, as opposed to the constant rate Γ . The solutions for the dark energy perturbations become:

$$\theta_x = \frac{2\sqrt{\Omega_{r0}}(3w_x - \alpha)}{3\alpha\Omega_{c0}(1+w_x)} \frac{[n_+^2 + (J+1)n_+ - (J+2)]}{(n_+ + 1)} \frac{k^2}{H_0} \psi, \quad (83)$$

$$\delta_x = -\frac{3(\alpha+3)(1-w_x)}{(n_+ + 2 - \alpha)} (k\tau)^{-1} \frac{\theta_x}{k}. \quad (84)$$

The key indicator of instability, i.e. the power-law index for the fastest growing mode, n_+ , takes a more complicated form:

$$n_+ = \frac{3\alpha + (\alpha - 6)w_x + \sqrt{(\alpha^2 + 16\alpha + 40)w_x^2 - 2(\alpha^2 + 6\alpha + 8)w_x + (\alpha^2 - 4\alpha - 20)}}{2(1 + w_x)}. \quad (85)$$

For example, if $w_x = -0.87$ and $\beta = -0.003$, then $n_+ = 38.95$. Unlike the Γ -model expression Eq. (73), here n_+ depends explicitly on the interaction rate parameter α . Note that, as for the Γ -model, the limit $\alpha = 0$ is not admitted in Eqs. (83)–(85), since the derivation uses $\alpha \neq 0$ in an essential way. For small $|\alpha|$ and w_x close to -1 ,

$$n_+ \sim \frac{6}{1 + w_x} \gg 1. \quad (86)$$

This is triple the corresponding index for the Γ model. The analytical form for the early-time instability is confirmed by numerical integration.

Model with background coupling $aQ = \beta \mathcal{H}(\rho_c + \rho_x)$

The background coupling Q is proportional to the total dark sector density $\rho_c + \rho_x$, which obeys an energy conservation equation. The balance equations lead to an exact solution [20] for ρ_x/ρ_c , with 3 cases according to the sign of $-\beta + 3w_x/4$. The non-negative cases are not relevant since they violate $|\beta| \ll 1$. For the remaining case,

$$\frac{\rho_x}{\rho_c} = \frac{(B + 2\beta - 3w_x)}{2\beta} \left(\frac{1 - ba^{B+2\beta-3w_x}}{1 + ba^{B+2\beta-3w_x}} \right) + \frac{3w_x - 2\beta}{2\beta}, \quad \beta > \frac{3w_x}{4}, \quad (87)$$

where

$$B := \sqrt{3w_x(3w_x - 4\beta)} - 2\beta + 3w_x, \quad (88)$$

and $b := [B - 2\beta\Omega_{x0}/\Omega_{c0}]/[B - 6w_x + 2\beta(2 + \Omega_{x0}/\Omega_{c0})]$. It follows that

$$\frac{\rho_x}{\rho_c} \rightarrow \frac{1}{2\beta} \times \begin{cases} B & a \rightarrow 0 \\ B - 2\sqrt{3w_x(3w_x - 4\beta)} & a \rightarrow \infty \end{cases} \quad (89)$$

If $\beta > 0$, then ρ_x/ρ_c is negative in the early universe, $a \rightarrow 0$, and also in the future, $a \rightarrow \infty$. Therefore the $\beta > 0$ case of this model is unphysical. For $\beta < 0$, if we fix Ω_{x0}/Ω_{c0} at a value greater than the late attractor in Eq. (89), then in the past ρ_x/ρ_c becomes negative. For a physical model, we thus require $3w_x/4 < \beta < 0$ and Ω_{x0}/Ω_{c0} less than the late-time attractor.

Perturbation of this coupling model is more complicated because it is determined by the total density, and therefore there is more ambiguity in the appropriate choice of four-velocity in the definition of $Q_c^\mu = -Q_x^\mu$. In previous work [17, 18], this issue was not explicitly discussed, and no form for $Q_c^\mu = -Q_x^\mu$ was given (this was also pointed out in [7]). It appears that the dark sector perturbation equations in [17] do not conform to momentum balance. They also neglect the coupling term in the expression for δP_x , i.e., the Q_x term in Eq. (29), as we pointed out in Sec. II. It turns out that this second error is decisive for the instability, whereas the error in $f_c = -f_x$ only leads to small corrections.

Using the correct form, Eq. (29), for δP_x , the evolution equation for dark energy density perturbations (which is independent of f_x) becomes

$$\begin{aligned} \delta'_x + 3\mathcal{H}(1 - w_x)\delta_x + (1 + w_x)\theta_x + 9\mathcal{H}^2(1 - w_x^2)\frac{\theta_x}{k^2} - 3(1 + w_x)\psi' \\ = \beta\mathcal{H} \left[\left(1 + \frac{\rho_c}{\rho_x}\right) \left\{ \phi + 3\mathcal{H}(1 - w_x)\frac{\theta_x}{k^2} \right\} + \frac{\rho_c}{\rho_x}(\delta_c - \delta_x) \right] \end{aligned} \quad (90)$$

In [17, 18], Eq. (7), the right hand side has $-\delta_c - \rho_x\delta_x/\rho_c$ instead of $\delta_c - \delta_x$. We find an instability (see below), whereas there is no instability in the results of [17]. Their omission of the coupling term in δP_x has inadvertently removed the instability.

The dark matter density perturbations obey

$$\delta'_c + \theta_c - 3\psi' = -\beta\mathcal{H} \left[\left(1 + \frac{\rho_x}{\rho_c}\right) \phi + \frac{\rho_x}{\rho_c}(\delta_x - \delta_c) \right]. \quad (91)$$

The equations for δ'_x and δ'_c are independent of the momentum transfer $f_c = -f_x$. In order to compute the velocity perturbations, we need the momentum transfer. If we follow the previous models and choose the energy-momentum transfer four-vector to be aligned with the dark matter four-velocity (so that there is no momentum transfer in the dark matter frame), then

$$aQ_c^\mu = -aQ_x^\mu = -\beta\mathcal{H}(\rho_c + \rho_x)u_c^\mu. \quad (92)$$

It follows that

$$af_c = \beta\mathcal{H}(\rho_c + \rho_x)(v - v_c) = -af_x, \quad (93)$$

and the velocity perturbation equations become

$$\theta'_x - 2\mathcal{H}\theta_x - \frac{k^2}{(1 + w_x)}\delta_x - k^2\phi = \frac{\beta\mathcal{H}}{(1 + w_x)} \left(1 + \frac{\rho_c}{\rho_x}\right) [\theta_c - 2\theta_x], \quad (94)$$

$$\theta'_c + \mathcal{H}\theta_c - k^2\phi = 0. \quad (95)$$

The θ'_c equation agrees with [17, 18], but their θ'_x equation has $+\theta_x$ in place of our $\theta_c - 2\theta_x$ on the right-hand side. Any other choice for the four-velocity along Q_c^μ will lead to a nonzero right-hand side in Eq. (95). It appears that no consistent choice of Q_c^μ can recover the equations of [17, 18].

We find that the problem of not accounting correctly for momentum transfer has a minor effect on the instability. The key driver for the instability is the coupling term in δP_x , leading to the correct forms Eq. (90) and (94) for the δ'_x and θ'_x equations.

Using Eqs. (90)–(95), we can find the early radiation solution. The solution is simplified if $|\beta|$ is small enough, and we find that:

$$\theta_x = \frac{4\beta\sqrt{\Omega_{r0}}}{3B\Omega_{c0}(1 + w_x)} \frac{[n_+^2 + (J + 1)n_+ - (J + 2)]}{(n_+ + 1)} \frac{k^2}{H_0} \psi, \quad (96)$$

$$\delta_x = \left[(1 + w_x)(n_+ - 2) + \frac{2\beta(B + 2\beta)}{B} \right] (k\tau)^{-1} \frac{\theta_x}{k}. \quad (97)$$

The power-law index is given by

$$\begin{aligned} n_+ = \frac{1}{2(1 + w_x)} \left\{ -M(1 + w_x) - 2N \right. \\ \left. + \sqrt{[M(1 + w_x) + 2N]^2 - 4(1 + w_x) \left[(M + 2)(-2 - 2w_x + 2N) - 3(1 - w_x)(N - 3 - 3w_x) \right]} \right\}, \end{aligned} \quad (98)$$

where $M = -3w_x + 2\beta^2/B$ and $N = \beta + 2\beta^2/B$. For example, if $w_x = -0.87$ and $\beta = -0.003$, then $n_+ = 38.95$, as in the previous model. For $|\beta| \ll 1$ and $w_x \sim -1$, we have

$$n_+ \sim \frac{6}{1 + w_x} \gg 1. \quad (99)$$

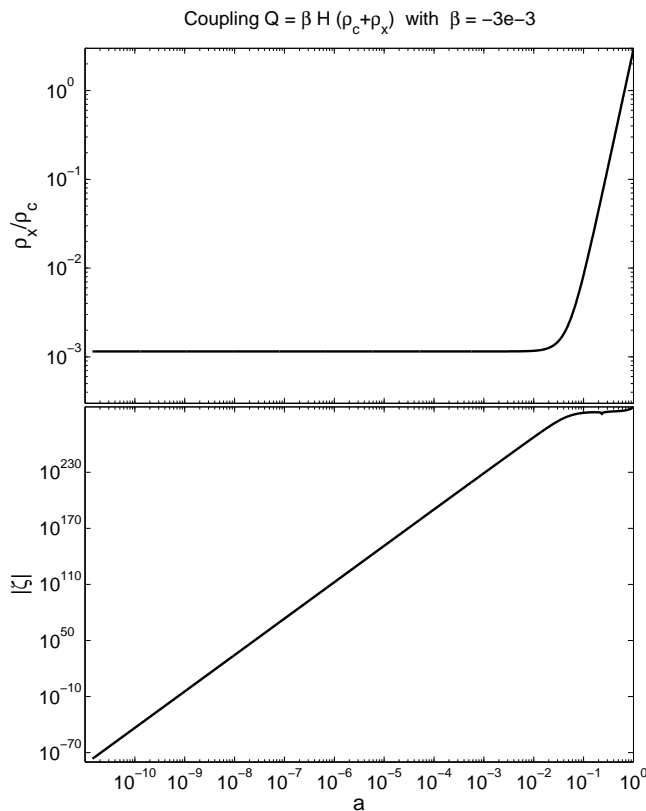


FIG. 3: The evolution of the background ρ_x/ρ_c (top panel), and gauge-invariant curvature perturbation ζ for a super-Hubble scale $k = 7 \times 10^{-5} \text{ Mpc}^{-1}$ (bottom panel) for the model with coupling given by Eq. (92). The figure shows a full numerical solution for $w_x = -0.87$ and $\beta = -0.003$, starting with the initial value $\zeta = 10^{-80}$ and ending up with oscillations of amplitude $|\zeta| > 10^{+300}$.

Hence our analytical solution again shows an instability, and this is confirmed by numerical solution, as illustrated in Fig. 3.

We have also checked numerically that the instability persists with negligible changes if we make alternative choices of $Q_c^\mu = -Q_x^\mu$. The two obvious choices are to align $Q_c^\mu = -Q_x^\mu$ along the dark energy four-velocity or along the “centre-of-mass” four-velocity in the dark sector:

$$aQ_c^\mu = -aQ_x^\mu = -\beta\mathcal{H}(\rho_c + \rho_x)u_x^\mu, \quad aQ_c^\mu = -aQ_x^\mu = -\beta\mathcal{H}(\rho_c + \rho_x)u_{cx}^\mu, \quad (100)$$

where

$$(\rho_c + \rho_x)u_{cx}^\mu = \rho_c u_c^\mu + \rho_x u_x^\mu. \quad (101)$$

V. CONCLUSIONS

We have given a detailed general analysis of the relativistic perturbations for a cosmology with coupled dark energy and dark matter fluids, paying particular attention to the non-adiabatic features in the dark energy sound speed and to the correct three-momentum transfer in the dark sector. We specialized to the case of a constant dark energy equation of state w_x , and with energy-momentum transfer four-vector of the form $Q_c^\mu = -Q_x^\mu = -\Gamma\rho_c u_c^\mu$, where Γ is the constant interaction rate. We were able to find the fastest growing early-radiation regular solution of the perturbation equations to leading order in $k\tau$, and this solution shows a strong blow-up of the gauge-invariant curvature perturbation on super-Hubble scales. Even if adiabatic initial conditions are chosen, the dominant mode quickly overwhelms the adiabatic mode.

The instability occurs no matter how weak the coupling is. Decreasing $|\Gamma|$ only shifts the blow-up to earlier times. This is in contrast with the strong-coupling instabilities discussed in [7, 8]. The instability is also non-adiabatic,

since growth of $|\zeta|$ is driven by the total non-adiabatic pressure. This is also in contrast to the adiabatic instabilities of [7, 8].

The origin of this large-scale non-adiabatic instability is not simply the fact that the dark energy fluid is non-adiabatic, i.e., $c_{ax}^2 \neq c_{sx}^2$. In the uncoupled case, the same non-adiabatic fluid behaviour is also present, but there is no instability [15]. The coupling plays an essential role in the large-scale non-adiabatic instability. It appears that the key driver for the instability is the coupling term that enters the non-adiabatic dark energy pressure perturbation δP_x . This leads to a runaway growth of the dark energy velocity, in order to maintain momentum balance in the presence of energy-momentum transfer between the perturbed dark fluids.

We also showed that the instability is not specific to the coupling model that we introduced. Similar coupling models show the same qualitative behaviour. We proposed and analyzed covariant forms of $Q_c^\mu = -Q_x^\mu$ that reduce in the background to two previously studied coupling models, $aQ = \alpha\mathcal{H}\rho_c$ and $aQ = \beta\mathcal{H}(\rho_c + \rho_x)$, and showed that in both cases the non-adiabatic large-scale instability is present. The perturbations in the second case were previously considered in [17, 18], but they omitted the coupling term in δP_x and this inadvertently removes the instability. We confirmed numerically that different choices of $Q_c^\mu = -Q_x^\mu$, i.e., aligning them along different four-velocities, has a negligible effect on the blow-up of the perturbations.

Uncoupled models with constant w_x and $c_{sx}^2 = 1$ are perfectly well behaved. But it appears that these models are unstable to the inclusion of coupling, at least for simple forms of coupling. What are the ways to avoid this instability? We can relax the assumption $w'_x = 0$, using a quintessence model instead of a fluid model for dark energy. Or more simply, we can use the parametrization $w_x = w_0 + (1 - a)w_a$. In this parametrization, w_x is effectively constant, $w_x = w_0 + w_a$, in the radiation and early matter eras. Therefore, our analysis applies unmodified also to this case. This may place a tight theoretical lower bound on w_a , since w_0 is the late-time value of w_x . Whether we can produce good fits to CMB and matter power data is under investigation [25].

The results of this paper, together with previous results on adiabatic strong-coupling instabilities, strongly constrain the model space for coupled dark energy. Further constraints on certain models arise from the background dynamics [3] and from “fifth-force”-type limits in cases where couplings extend to non-dark particles (e.g. in the case of supergravity-based quintessence [26]).

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