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Large Sharpening Intertemporal Prospect Theory

Pushpa Rathie

Department of Statistics, University of Brasilia
70910-900 Brasilia DF, Brazil
pushpanrathie@yahoo.com

Carlos Radavelli

Department of Economics, Federal University of Santa Catarina
88049-970 Florianopolis SC, Brazil
carlosradavelli@hotmail.com

Sergio Da Silva[1]

Department of Economics, Federal University of Santa Catarina
88049-970 Florianopolis SC, Brazil
professorsergiodasilva@gmail.com

Abstract

Prospect theory [4] of risky choices has been extended to encompass intertemporal choices [6]. Presentation of intertemporal prospect theory suffers from minor mistakes, however [2]. To clarify the theory we restate it and show further mistakes in current presentations ([6], [2]) of value and discount functions.

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1 Introduction

Choice under risk trades off utility between alternative situations whereas choice over time trades off utility between alternative periods.

[1]Corresponding author

Microeconomics tackles risk by simply extending fundamental choice theory; this is expected utility theory [11]. Intertemporal choices are explained by discount utility theory [8].

As it happens, several anomalies have been well documented through experiments. Prospect theory has stepped in to take these into account and suggest an alternative paradigm to explain risky choices [4]. Both the anomalies and prospect theory are likely to be brain wired ([3], [10]).

Prospect theory of risky choices itself has been extended to encompass intertemporal choices [6]. Yet it has been claimed that such an "intertemporal" prospect theory suffers from minor mathematical errors [2].

Our task in this note is to reexamine Loewenstein and Prelec's [6] theory by taking into account Al-Nowaihi and Dhami's [2] claims. The assumptions underlying value and discount functions are restated more neatly. Incidentally we point out further mistakes in both papers. Our aim is to help perfect presentation of intertemporal prospect theory.

Section 2 restates the theory. Section 3 evaluates the previous corrections to the theory, and Section 4 concludes.

2 Restating the assumptions

To find value and discount functions, it is useful restating Loewenstein and Prelec (LP)'s assumptions as follows.

A1 (impatience). Discount function $\phi : [0, \infty) \longrightarrow (0, \infty)$ is strictly decreasing in an arbitrarily small interval. (A1 is not made explicit in LP, as Al-Nowaihi and Dhami (AD) observe.)

A2 (gain-loss asymmetry). If $0 < x < y$ and $v(x) = v(y)\phi(t)$, then $v(-x) > v(-y)\phi(t)$. ($v(\cdot)$ is the value function for consumption plan outcomes x and y.)

A3 (absolute magnitude effect). If $0 < x < y$, $v(x) = v(y)\phi(t)$ and $a > 1$, then $v(ax) < v(ay)\phi(t)$.

A4 (common difference effect). If $0 < x < y$, $v(x) = v(y)\phi(t)$ and $s > 0$, then $v(x)\phi(s) < v(y)\phi(s + t)$. (Common difference effect is one behavioral finding contradicting discount utility theory; it captures the fact that preferences between two delayed outcomes often switch when both delays are incremented by a given constant amount.)

The assumptions above come from (experimentally) well documented anomalies in discount utility theory. (LP (p. 578) refer to delay-speedup asymmetry as another key anomaly, though they do not consider it in their theory.) In particular, assumptions A1 and A4 suffice to derive a generalized hyperbolic discount function ϕ that encompasses the exponential function as a particular case.

Rather than A4, AD employ

A5 (common difference effect with quadratic delay). If $0 < x < y$, $v(x) = v(y)\phi(t)$, and $s > 0$, then $v(x)\phi(s) = v(y)\phi(s + t + \alpha st)$, $\alpha > 0$.

Obviously A4 follows from A1 and A5. Then AD use A1 and A5 to uniquely derive discount function

$$\phi(t) = (1 + \alpha t)^c, \;\; c < 0 \tag{1}$$

as a solution to functional equation

$$\phi(s + t + \alpha st) = \phi(s)\phi(t). \tag{2}$$

(There are a number of possibilities to get equations of type

$$\phi(s)\phi(t) = g(s, t)\phi(s + t)$$

where $g(s, t)$ can be defined depending on the type of solution needed, including the generalized hyperbolic.) We note that this result in well known in the literature of functional equations and was first proved by Thielman [9]. (See also [1] p. 81.) Yet A5 is not needed, and AD's presumption of (2) is unnecessary.

Indeed observe that A4 (including also the equality) gives

$$\phi(t)\phi(s) \le \phi(s + t). \tag{3}$$

Here Petrovic [7] proves that continuous and convex functions ϕ do satisfy (3). (See also [5] pp. 197, 205.) It is straightforward to verify that exponential function

$$\phi(t) = e^{-bt} \tag{4}$$

and generalized hyperbolic function

$$\phi(t) = (1 + at)^{-b/a} \tag{5}$$

are only two from a number of continuous convex functions satisfying (3). (The a tracks how much (5) departs from constant discounting; as a approaches zero, (4) obtains.) Thus it follows that there is no need to artificially assume (2) (as done by AD) to get the exponential and generalized hyperbolic discount functions.

To justify (2), AD borrow from LP (p. 580)'s derivation and incorrectly get $k = 1+\alpha t$. This is incorrect because k does not depend on t. LP explicitly say (p. 579) that k depends on x and y, but not on t; and LP's solution (at p. 580) is only possible assuming k to be constant, i.e. independent of t. So the solution of LP does not satisfy their functional equation

$$\phi(s)\phi(t) = \phi(t + ks). \tag{6}$$

In short, LP's solution is one of the solutions to (3). Yet their proof to derive the generalized hyperbolic solution is flawed. What is more, AD's remark based on LP's incorrect proof that $k = 1 + \alpha s$ is flawed too.

3 Evaluating the previous corrections

AD claim four major mistakes made by LP. Now we evaluate these claims. The corrections are as follows.

1. When defining the elasticity of value function, LP (p. 584) assert that the value function is more elastic for outcomes that are larger in absolute magnitude, i.e.

$$\varepsilon_\upsilon(x) < \varepsilon_\upsilon(y), \ \ 0 < x < y \quad \text{or} \quad y < x < 0$$

where $\varepsilon_\upsilon(x) = x\frac{\upsilon'(x)}{\upsilon(x)}$.

AD claim the correct proposition to be the following. The value function is more elastic for outcomes that are *smaller* in absolute magnitude, i.e.

$$\varepsilon_\upsilon(x) > \varepsilon_\upsilon(y), \ \ 0 < x < y \quad \text{or} \quad y < x < 0.$$

This correction is appropriate, and the proof given by AD is quite straightforward.

2. As seen, when considering the common difference effect, LP (p. 579) impose a linear delay (A4). So if $\upsilon(x) = \upsilon(y)\phi(s)$ then $\upsilon(x)\phi(t) = \upsilon(y)\phi(kt + s)$, and constant k depends only on x and y. AD warn that the hyperbolic discount with $k = 1 + \alpha s$ depends only on s, not on x and y. This claim is misleading, for the reasons discussed in previous section. Over there we fix LP's solution by stating that it is one of the solutions to (3). Their proof to derive the generalized hyperbolic solution is not correct. And AD's remark based on LP's incorrect proof that $k = 1 + \alpha s$ is not correct either.

3. LP employ their model to predict the shape of the optimal intertemporal allocation of benefits under a constant market present value

constraint. Their equation for optimal consumption plan in the gain domain is (p. 592)

$$c'^{*} = r - \left(-\frac{\phi'(t)}{\phi(t)}\right)\left(-\frac{\upsilon'}{\upsilon''}\right)$$

where $r > 0$ is a constant real interest rate.

AD suggest that this needs correction. The equation to replace the above one is

$$c'^{*} = \left[r - \left(-\frac{\phi'(t)}{\phi(t)}\right)\right]\left(-\frac{\upsilon'}{\upsilon''}\right).$$

We have evaluated this claim and found AD to have a point.

Actually an alternative proof to that of AD can be obtained by going back to Yaari's [12] benchmark paper on consumption allocation over time when preferences depend not only on consumption but also on final wealth (or bequest). Yaari's optimal consumption plan (obtained after maximizing a utility function subject to wealth constraint) yields

$$e^{r(t-T)}\phi(t)\upsilon'(c^{*}(t)) = \kappa,\ \kappa > 0$$

as a necessary and sufficient condition (Yaari's equation (12) at p. 307). Obviously a value function υ is absent from Yaari's paper. Rather than υ, he takes a utility associated with the rate of consumption at every moment of time. He assumes the utility to be twice differentiable and strictly concave. Yaari concedes this assumption to be strong (p. 305) though without it no specific implications can be obtained. Taking the value function instead renders the equation above more robust theoretically.

Taking the derivative of the equation produces

$$e^{r(t-T)}r\phi\upsilon'(c^{*}) + e^{r(t-T)}\phi'\upsilon'(c^{*}) + e^{r(t-T)}\phi\upsilon''(c^{*})c'^{*} = 0$$

$$e^{r(t-T)}[r\phi\upsilon'(c^{*}) + \phi'\upsilon'(c^{*}) + \phi\upsilon''(c^{*})c'^{*}] = 0$$

$$-(r\phi + \phi')\upsilon'(c^{*}) = \phi\upsilon''(c^{*})c'^{*}$$

$$c'^{*} = \left[r - \left(-\frac{\phi'(t)}{\phi(t)}\right)\right]\left(-\frac{\upsilon'}{\upsilon''}\right).$$

And this confirms the correction.

4. In the loss domain, LP (p. 592) state that

$$r < -\frac{\frac{\phi'(t)}{\phi(t)}}{\varepsilon_v(Ie^{rt})}$$

where I is initial wealth, and $\varepsilon_v(Ie^{rt})$ is increasing.

AD again suggest that this expression is wrong. The correct expression is

$$r > -\frac{\frac{\phi'(t)}{\phi(t)}}{(\frac{Ie^{rt}}{Ie^{rt}-\bar{c}})\varepsilon_v(Ie^{rt}-\bar{c})}.$$

By checking this claim we have found AD to be right again.

4 Conclusion

We restate the assumptions in intertemporal prospect theory [6] that underlie its value and (generalized hyperbolic) discount functions. When reexamining the theory we have taken the recent Al-Nowaihi and Dhami's [2] corrections into account. By doing so, we have found remaining errors in both LP and AD papers. In particular, we find LP's proof to derive the generalized hyperbolic solution to be flawed. AD's remark based on LP's incorrect proof ends up flawed, as a result. These mathematical errors are minor and do not affect the theory's critical results. However, by pointing out these mistakes we hope to contribute to sharpen presentation of intertemporal prospect theory.

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