Large Stable Pulse Solutions in Reaction-diffusion Equations

ARJEN DOELMAN, ROBERT A. GARDNER & TASSO J. KAPER

ABSTRACT. In this paper we study the existence and stability of asymptotically large stationary multi-pulse solutions in a family of singularly perturbed reaction-diffusion equations. This family includes the generalized Gierer-Meinhardt equation. The existence of N-pulse homoclinic orbits ($N \geq 1$) is established by the methods of geometric singular perturbation theory. A theory, called the NLEP (=NonLocal Eigenvalue Problem) approach, is developed, by which the stability of these patterns can be studied explicitly. This theory is based on the ideas developed in our earlier work on the Gray-Scott model. It is known that the Evans function of the linear eigenvalue problem associated to the stability of the pattern can be decomposed into the product of a slow and a fast transmission function. The NLEP approach determines explicit leading order approximations of these transmission functions. It is shown that the zero/pole cancellation in the decomposition of the Evans function, called the NLEP paradox, is a phenomenon that occurs naturally in singularly perturbed eigenvalue problems. It follows that the zeroes of the Evans function, and thus the spectrum of the stability problem, can be studied by the slow transmission function. The key ingredient of the analysis of this expression is a transformation of the associated nonlocal eigenvalue problem into an inhomogeneous hypergeometric differential equation. By this transformation it is possible to determine both the number and the position of all elements in the discrete spectrum of the linear eigenvalue problem. The method is applied to a special case that corresponds to the classical model proposed by Gierer and Meinhardt. It is shown that the one-pulse pattern can gain (or lose) stability through a Hopf bifurcation at a certain value μ_{Hopf} of the main parameter μ. The NLEP approach not only yields a leading order approximation of μ_{Hopf}, but it also shows that there is another bifurcation value, μ_{edge}, at which a new (stable) eigenvalue bifurcates from the edge of the essential spectrum. Finally, it is shown that the N-pulse patterns are always unstable when $N \geq 2$.

1. INTRODUCTION

In this paper we study the existence and stability of stationary pulse or multi-pulse solutions $(U_h(x), V_h(x))$ to a singularly perturbed two-dimensional system of reaction-diffusion equations on the unbounded, one-dimensional domain. The unboundedness of the domain reflects our choice to study a spatially extended system: the spatial scale of the patterns is much smaller than the length scale of the domain. This implies that the structure and the stability of the patterns does not depend on the domain and/or boundary conditions.

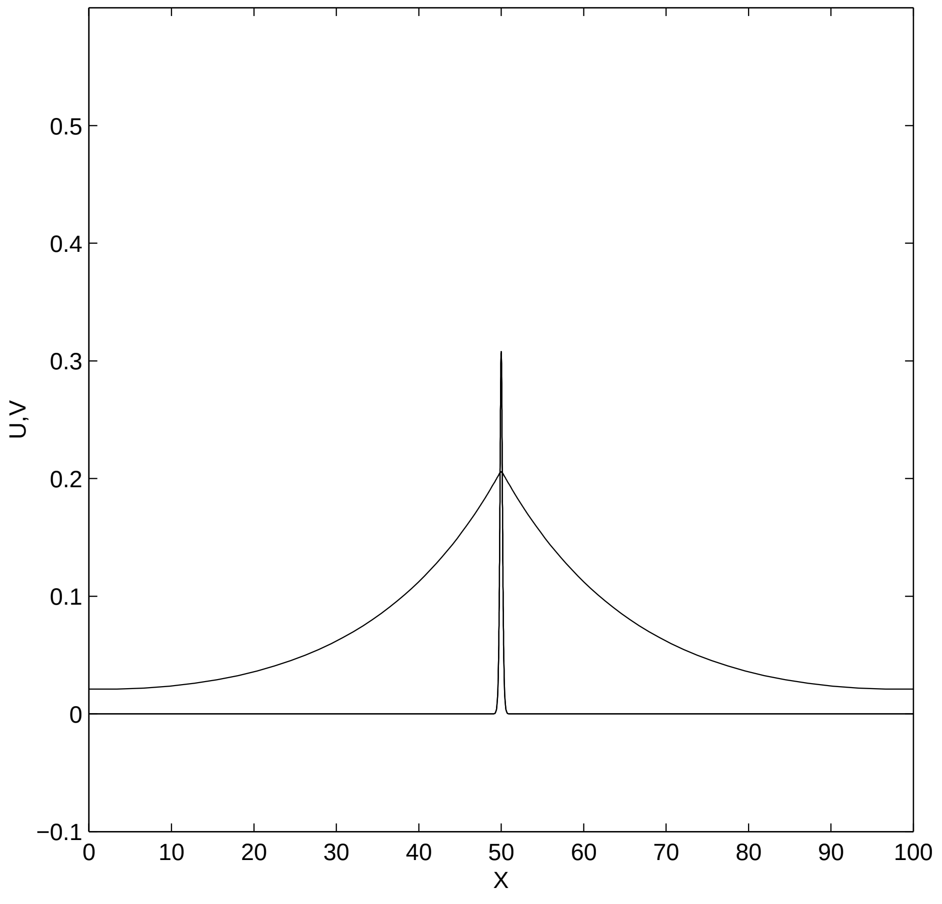

FIGURE 1.1. The stable 1-pulse homoclinic solution of the scaled equation (1.7) in the classical Gierer-Meinhardt case (1.9) with $\mu = 0.38$ and $\varepsilon = 0.1$. Note that one has to multiply by $1/\varepsilon$ (1.6) to obtain the amplitudes of the corresponding solution to the unscaled equation (1.2).

The pulse solutions are assumed to be of a homoclinic nature:

$$\lim_{|x|\to\infty} (U_h(x), V_h(x)) = (U_0, V_0),$$

where (U_0, V_0) is an asymptotically stable trivial state that can be set to be $(0,0)$ by a translation, see Figure 1.1 on the facing page. In the most general setting, the pulse pattern $(U_h(x), V_h(x))$ is assumed to be a solution of the system

$$\begin{cases} U_t = d_U U_{xx} + a_{11}U + a_{12}V + H_1(U,V), \\ V_t = d_V V_{xx} + a_{21}U + a_{22}V + H_2(U,V), \end{cases} \tag{1.1}$$

with $x \in \mathbb{R}$ and $d_V \ll d_U$, so that we can define the small parameter ε by $\varepsilon^2 = d_V/d_U \ll 1$. The nonlinear terms $H_i(U,V)$, $i = 1, 2$, are assumed to be smooth enough for $(U,V) \neq (0,0)$; the coefficients a_{ij} of the linear terms are chosen such that the trivial pattern $(U_0, V_0) \equiv (0,0)$ is asymptotically stable as a solution to (1.1) with $H_i(U,V) \equiv 0$; see Remarks 1.2 and 1.3.

The (generalized) Gierer-Meinhardt system in morphogenesis ([15], [28], and the references there) and the Gray-Scott model for autocatalytic reactions ([17], [9], and the references there) are among the most well-known examples of systems of the type (1.1) that exhibit (stable) singular pulse patterns of the type studied in this paper. We refer to [27], [24] for many other explicit examples of systems of the type (1.1) originating from applications in biology, chemistry, and physics.

The methods to be developed in this paper can be applied to the existence and stability analysis of singular pulse solutions $(U_h(x), V_h(x))$ of (1.1) that can either be large or small (with respect to ε), and positive or negative (see Remark 1.1). In this paper, we will focus on pulses that are *large* and *positive*, i.e., the components of the solutions are positive, $U_h(x) > 0$ and $V_h(x) > 0$, and their amplitudes scale with negative powers of ε, see Figure 1.1 on the preceding page. The pulse solutions $(U_h(x), V_h(x))$ are also singular in the sense that the V-component evolves on a spatial scale that is shorter than the U-component (since $d_V \ll d_U$ in (1.1)): $V_h(x)$ is exponentially small except on x-intervals that are so short that $U_h(x)$ can be assumed to be constant (to leading order) on such an interval, see Figure 1.1.

The leading order behavior for U, $V \gg 1$ of the nonlinear terms $H_i(U,V)$ in (1.1) can be represented by a small number of parameters. Nevertheless, we consider a less general version of (1.1), in order to simplify the analysis:

$$\begin{cases} U_t = U_{xx} \quad -\mu U \; + F_1(U)G_1(V), \\ V_t = \varepsilon^2 V_{xx} - V \quad + F_2(U)G_2(V), \end{cases} \tag{1.2}$$

with $\mu > 0$ and $\mu = \mathcal{O}(1)$. The decomposition of the nonlinear terms $H_i(U,V)$ into the products $F_i(U)G_i(V)$ is motivated by the fact that it is a straightforward procedure to describe the leading order behavior of $F_i(U)G_i(V)$ for U and V

large and positive, while this leading order behavior will depend on the relative magnitudes of U and V for general nonlinearities $H_i(U,V)$, see Appendix A on page 501. Thus, the restriction of (1.1) to (1.2) is merely a technical one; it has no influence on the essence of the method. The same is true for the removal of the linear coupling terms (i.e., $a_{12} = a_{21} = 0$) (Appendix A).

The 'critical' or 'significant' magnitudes of the homoclinic patterns $(U_h(x), V_h(x))$ as functions of ε are determined by a straightforward scaling analysis. We introduce

$$\tag{1.3} \begin{aligned} \tilde{U}(x,t) &= \varepsilon^r U(x,t), \quad r > 0, \\ \tilde{V}(x,t) &= \varepsilon^s V(x,t), \quad s > 0, \end{aligned}$$

so that the leading order behavior of $F_i(U)$ and $G_i(V)$ can be described by the constants α_i, β_i and f_i, g_i ($\neq 0$):

$$\tag{1.4} \begin{aligned} F_i\left(\frac{\tilde{U}}{\varepsilon^r}\right) &= \left(\frac{\tilde{U}}{\varepsilon^r}\right)^{\alpha_i} (f_i + \varepsilon^r \tilde{F}_i(\tilde{U};\varepsilon)), \\ G_i\left(\frac{\tilde{V}}{\varepsilon^s}\right) &= \left(\frac{\tilde{V}}{\varepsilon^s}\right)^{\beta_i} (g_i + \varepsilon^s \tilde{G}_i(\tilde{V};\varepsilon)), \end{aligned} \qquad i = 1,2.$$

Thus, we have implicitly assumed that $F_i(U)$ and $G_i(V)$ are smooth functions for U, $V > 0$ that have a leading order behavior that is *algebraic* in U, respectively V, when U, V become large. Both in the existence and in the stability analysis of this paper we will have to assume that $\beta_i > 1$ and that $\lim_{V\to 0} G_i(V) = 0$ ($i = 1, 2$), which implies that the functions $G_i(V)$ are at least C^1 functions, for all V. However, the α_i's might become negative (in fact, we need to impose that $\alpha_2 < 0$, see Theorem 2.1), so that the functions $F_i(U)$ are allowed to have algebraic singularities as $U \to 0$ (see Remark 1.3). The leading order behavior of the nonlinear terms in (1.2) is thus described by r, s, α_i, β_i, h_i, $\tilde{H}_i$, and $\hat{\varepsilon}$, where the latter three expressions are defined by

$$\tag{1.5} \begin{aligned} (f_i + \varepsilon^r \tilde{F}_i(\tilde{U},\varepsilon))(g_i + \varepsilon^s \tilde{G}_i(\tilde{U},\varepsilon)) &= h_i + \hat{\varepsilon}\tilde{H}_i(\tilde{U},\tilde{V};\varepsilon) \\ &: h_i = f_i g_i,\ \hat{\varepsilon} = \varepsilon^{\min(r,s)}. \end{aligned}$$

The details of the scaling procedure are given in Appendix A on page 501. There, it is shown that, with $x = \sqrt{\varepsilon}\tilde{x}$, $\tilde{\varepsilon} = \sqrt{\varepsilon}$, and

$$\tag{1.6} r = \frac{\beta_2 - 1}{D} > 0, \quad s = -\frac{\alpha_2}{D} > 0, \quad D = (\alpha_1 - 1)(\beta_2 - 1) - \alpha_2\beta_1 \neq 0,$$

equation (1.2) can be written in the 'normal form'

$$(1.7)\qquad \begin{cases} ^2U_t = U_{xx} \quad - \varepsilon^2\mu U + U^{\alpha_1}V^{\beta_1}(h_1 + \hat{\varepsilon}H_1(U,V;\varepsilon)), \\ \ V_t = \varepsilon^2 V_{xx} - V \qquad + U^{\alpha_2}V^{\beta_2}(h_2 + \hat{\varepsilon}H_2(U,V;\varepsilon)), \end{cases}$$

where we have dropped the tildes. This equation is the subject of the analysis in this paper.

An unusual aspect of the pulse solutions under consideration is that they are homoclinic solutions to a point $(U,V) = (0,0)$ which is a singularity of the nonlinearities in the equations. While it is possible to control the singularities in the ODE's arising in both the steady state existence analysis and the linearized stability analysis, standard parabolic theory cannot immediately be applied to such basic questions as the well-posedness of the partial differential equations (1.7) for solutions in a neighborhood of the steady state solutions studied in this paper. The abstract semigroup techniques that are usually employed when deriving nonlinear stability from linearized stability are similarly unavailable. Standard treatments usually require some amount of smoothness [18]. These are interesting and important issues, but they are not directly related to the present analysis. They are the subject of work in progress, and will be addressed in future publications; see also Remark 1.3.

The singularly perturbed ordinary differential equation for the *stationary* solutions to (1.7) can be studied by the methods of geometric singular perturbation theory [11], [21] in combination with the topological approach developed in [8] (for the $N > 1$-pulses). It is shown in Section 5.3 that for any parameter combination in an open, unbounded part of the $(\alpha_1, \alpha_2, \beta_1, \beta_2, h_1, h_2)$-parameter space (2.5), there exists a homoclinic N-pulse solution ($N = 1, 2, \ldots, N = \mathcal{O}(1)$) *for any* $\mu > 0$ (there is one additional condition on $G_i(V)$, $i = 1, 2$, for V small, (2.6), see Theorem 2.1). Here, the N describes the number of (fast) circuits the V-component makes through the 4-dimensional phase space; the N-pulse solution is homoclinic to the critical point $(0,0,0,0)$ that corresponds to the trivial state $(U_0, V_0) \equiv (0,0)$. The function $V_h(x)$ has, as a function of x, N distinct narrow pulses at an $\mathcal{O}(|\log\varepsilon|)$ distance apart. The function $U_h(x)$ is to leading order constant in the region where $V_h(x)$ has its peaks, $U_h(x)$ decreases to 0 on a much slower spatial scale (Figure 1.1 on page 444). As was the case for a similar analysis in the Gray-Scott model [9], the existence of these solutions depends crucially on the reversibility symmetry (with respect to x) in (1.7) and the fact that the limit problem for the slow system is *'super slow'*, see Section 2. The geometric analysis also implies that (stationary) N-pulse homoclinic patterns of the type described by Theorem 2.1 can only exist in the non-scaled system (1.2) when U and V are scaled according to (1.3) with r and s as in (1.6).

After the existence is established, a theory is developed by which the stability of the $N = 1$-pulse patterns can be determined *explicitly* as function of the parameters in the problem (in Sections 3, 4, 5). We will show that the method, called *the NLEP approach* [6], [7], enables us to determine the number and position (to leading order) of *all* elements in the discrete spectrum of the linear eigenvalue problem associated to the stability of the pattern. Here NLEP stands for *NonLocal Eigenvalue Problem*. The NLEP approach can also be applied to the stability analysis of the N-pulse patterns ($N \geq 2$), as we shall show in Section 6.

The first step of the NLEP approach (Section 3) follows the ideas developed by Alexander, Gardner, and Jones ([1], [13]): it is shown that the 4-dimensional linear eigenvalue problem associated to the stability problem for the 1-pulse patterns can be studied through the Evans function $\mathcal{D}(\lambda, \varepsilon)$. $\mathcal{D}(\lambda, \varepsilon)$ is in essence the determinant of the 4×4-matrix formed by 4 independent functions that span the solution space of the linear eigenvalue problem. By the general theory developed in [1], one can choose these functions such that $\mathcal{D}(\lambda, \varepsilon) = 0$ if and only if λ is an eigenvalue (counting multiplicities). It follows from the singular character of the equations that $\mathcal{D}(\lambda, \varepsilon)$ can be decomposed into a product of a fast and a slow 'transmission' component [13]: $\mathcal{D}(\lambda, \varepsilon) = f t_1(\lambda, \varepsilon) t_2(\lambda, \varepsilon)$, where $f = f(\lambda, \mu, \varepsilon) > 0$ is an explicitly known non-zero factor, see (3.21).

For $N = 1$, the fast component, $t_1(\lambda, \varepsilon)$, corresponds to the Evans function associated to the stability problem for the stationary homoclinic orbit $V_h^{\mathrm{red}}(\xi)$ of the scalar fast reduced limit problem

$$V_t = V_{\xi\xi} - V + h_2 U_0^{\alpha_2} V^{\beta_2}, \tag{1.8}$$

with $h_2 > 0$, $\beta_2 > 1$, and $U_0 > 0$ an explicitly known constant (see (3.13)); $\xi = x/\varepsilon$ is the fast spatial scale. Note that this eigenvalue problem can be written as $(\mathcal{L}_f(\xi) - \lambda)v = 0$, where $v(\xi)$ is defined by $V(\xi, t) = V_h^{\mathrm{red}}(\xi) + e^{\lambda t} v(\xi)$. It is shown, by a topological winding number argument (as in [1], [13]), that there corresponds to any eigenvalue λ_f^j of this fast reduced stability problem a zero $\lambda^j(\varepsilon)$ of $t_1(\lambda, \varepsilon)$ such that $\lim_{\varepsilon \to 0} \lambda^j(\varepsilon) = \lambda_f^j$ (Section 4). Since it is well-known that the stationary homoclinic solution to (1.8) is unstable with one eigenvalue $\lambda_f^* = \lambda_f^0 > 0$ (see also Proposition 5.6), we encounter the same *'NLEP paradox'* for this general problem as was studied in [6], [7] for the Gray-Scott model: although $t_1(\lambda^*(\varepsilon), \varepsilon) = 0$, we will find that

$$\mathcal{D}(\lambda^*(\varepsilon), \varepsilon) = f t_1(\lambda^*(\varepsilon), \varepsilon) t_2(\lambda^*(\varepsilon), \varepsilon) \neq 0.$$

The resolution to this paradox lies in a detailed analysis of the slow component $t_2(\lambda, \varepsilon)$ of $\mathcal{D}(\lambda, \varepsilon)$. Using the matched asymptotics approach originally developed in [6], we determine an explicit expression for $t_2(\lambda, \varepsilon)$ in terms of an integral involving the (uniquely determined) solution to an *inhomogeneous* version of the reduced fast limit problem: $(\mathcal{L}_f(\xi) - \lambda)v = g(\xi)$, where $g(\xi)$ is an explicitly known function (see (4.7) and (4.11)). Since the operator $(\mathcal{L}_f(\xi) - \lambda)$ is not invertible at the eigenvalue λ_f^*, we deduce that $t_2(\lambda, \varepsilon)$ has *a pole of order* 1 at $\lambda = \lambda^*(\varepsilon)$. Hence, $\mathcal{D}(\lambda^*(\varepsilon), \varepsilon) \neq 0$.

Due to the reversibility symmetry $x \to -x$ in (1.2) and (1.7), we know that all eigenfunctions $v_f^j(\xi)$ of the fast reduced stability problem are either even (for j even) or odd (for j odd) as functions of ξ. We show that the zero of $t_1(\lambda, \varepsilon)$ at $\lambda^j(\varepsilon)$ is cancelled by a pole of $t_2(\lambda, \varepsilon)$ for all even j; however, the inhomogeneous term $g(\xi)$ is such that there is no pole at $\lambda^j(\varepsilon)$ when j is odd (Corollary 4.4). This implies that every eigenvalue λ_f^j of the fast reduced limit problem with j odd persists, for $\varepsilon \neq 0$, as an eigenvalue for the full problem. However, neither of these eigenvalues is positive when $N = 1$ (see Proposition 5.6), so that we can conclude that all 'relevant' eigenvalues of the full instability problem are determined by the zeroes of $t_2(\lambda, \varepsilon)$.

Using the leading order approximation of $t_2(\lambda, \varepsilon)$ we prove, by topological winding number arguments, a number of general results on the instability of the 1-pulse as function of the parameters μ, α_1, α_2, β_1, and β_2. The most important of these results, Theorem 5.1, states that there exists a $\mu^U > 0$ such that the homoclinic pattern $(U_h(x), V_h(x))$ is unstable for all $\mu < \mu^U$. This for instance implies that the pulse pattern cannot lose its stability by an 'essential instability', i.e., an instability caused by the essential spectrum (see [37], [38]).

Next, it is shown that the inhomogeneous problem $(\mathcal{L}_f(\xi) - \lambda)v = g(\xi)$ can be solved explicitly in terms of integrals involving hypergeometric functions. Such a reduction is known in the literature on (linear) Schrödinger problems in the case that the potential, in our case $V_h^{\rm red}(\xi)$, is explicitly known (see for instance [26]). This is only the case in (1.2) or (1.8), when $\beta_2 = 2$ or $\beta_2 = 3$ (see [6], which corresponds to $\beta_2 = 2$; here $V_h^{\rm red}(\xi) = c_1/(\cosh c_2\xi)^2$ for certain c_1, c_2). In this paper, we present a general transformation, that does not depend on an explicit expression for $V_h^{\rm red}(\xi)$, by which the inhomogeneous problem can be written as an inhomogeneous hypergeometric differential equation. This equation can be solved by the classical Green's function method (see Appendix B on page 502). This solution is substituted into the expression for $t_2(\lambda, \varepsilon)$.

Thus, we have obtained a completely explicit leading order approximation of the slow transmission function $t_2(\lambda, \varepsilon)$ that determines the stability of the pattern $(U_h(x), V_h(x))$, (5.21). Since we also can determine all zeroes of the fast transmission coefficient $t_1(\lambda, \varepsilon)$, it follows that one can obtain *all relevant information on the spectrum associated to the stability of the* 1*-pulse patterns* by the NLEP approach.

It should be noted that the transformation into a hypergeometric form has also been used to get an *exact* description of the spectrum of the reduced linear operator $\mathcal{L}_f(\xi) - \lambda$. It is for instance shown that $\lambda_f^* = \frac{1}{4}(\beta_2 + 1)^2 - 1$ and that the number of (discrete) eigenvalues equals $J + 1$, where $J = J(\beta_2)$ satisfies $J < (\beta_2 + 1)/(\beta_2 - 1) \leq J + 1$. Hence, $J(\beta_2) \to \infty$ as $\beta_2 \downarrow 1$ and $J \equiv 1$ for all $\beta_2 \geq 3$. See Proposition 5.6.

The general theory can be applied to the generalized Gierer-Meinhardt model [20], [28], [29], [30]. This model corresponds to $h_1 = h_2 = 1$ and $H_1(U, V; \varepsilon) = H_2(U, V; \varepsilon) \equiv 0$ in (1.7). Note that the elimination of the higher order nonlinear terms in (1.7) is a rather strong restriction, since the magnitude of these terms can be $\gg \mathcal{O}(\varepsilon^2)$ (1.5). The special case

$$(1.9)\qquad \begin{gathered} \alpha_1 = 0, \quad \alpha_2 = -1, \quad \beta_1 = \beta_2 = 2, \quad h_1 = h_2 = 1, \\ H_1(U,V;\varepsilon) = H_2(U,V;\varepsilon) \equiv 0, \end{gathered}$$

in (1.7) corresponds to the original biological values of the parameters by Gierer and Meinhardt [15], see also [20], [28], [29], [30]. It is shown in Section 5.3 that there are two unstable (real) solutions of the equation $t_2(\lambda,\varepsilon) = 0$, i.e., eigenvalues, when μ is 'too small'. As μ increases, these solutions merge, at $\mu = \mu_{\text{complex}} = 0.053\ldots(+\mathcal{O}(\varepsilon))$ and become a pair of complex conjugate eigenvalues. This pair crosses the imaginary axis and enters the stable part of the complex plane at $\mu = \mu_{\text{Hopf}} = 0.36\ldots$ (to leading order in ε). Hence, we conclude that the homoclinic 1-pulse solution of the 'classical' Gierer-Meinhardt problem (1.2) with (1.9) is spectrally stable for $\mu > \mu_{\text{Hopf}}$ (Theorem 5.11). Moreover, the method also shows that there is a so-called 'edge bifurcation' at $\mu = \mu_{\text{edge}} = 0.77\ldots(+\mathcal{O}(\varepsilon))$: a new, fourth, eigenvalue appears from the edge of the essential spectrum as μ increases through μ_{edge}; this eigenvalue remains negative for all $\mu > \mu_{\text{edge}}$. The leading order approximations of μ_{complex}, μ_{Hopf}, and μ_{edge} have been computed with the aid of Mathematica; the stability result has been confirmed by a direct numerical simulation of (1.7), see Figure 1.1 on page 444.

It is stressed that this result is only an application of the 'NLEP approach' to a special case: the method can be applied to the stability problem for any (singular) homoclinic pattern $(U_h(x), V_h(x))$ to (1.2) (and in principle to (1.1), see Appendix A).

The stability problem for the N-pulse patterns with $N \geq 2$ can now be studied along the lines of the machinery developed for the 1-pulse patterns. We can define the Evans functions $\mathcal{D}^N(\lambda,\varepsilon)$ and determine its decomposition into the transmission functions $t_1^N(\lambda,\varepsilon)$ and $t_2^N(\lambda,\varepsilon)$. The most important 'new' insight for this problem is that the linear problem associated to the fast reduced limit problem will have more than one unstable eigenvalue. Intuitively this is clear, since instead of studying the linearization of (1.8) around $V_h^{\text{red}}(x)$, one now has to linearize around, roughly speaking, N copies of the $V_h^{\text{red}}(x)$-pulse (at $\mathcal{O}(|\log\varepsilon|)$ distances apart). Therefore, one expects N unstable eigenvalues $\lambda_f^{N,j}$, $j = 1, 2, \ldots, N$ that all merge with λ_f^* in the limit $\varepsilon \to 0$ [35]. It is shown in Section 6 that, for $N \geq 2$, there is at least one such an eigenvalue, $\lambda_f^{N,1}$, that is associated to an *odd* eigenfunction. By the same mechanism as for the case $N = 1$, we can conclude that there thus is a $\lambda^{N,1}(\varepsilon)$ such that $t_1^N(\lambda^{N,1}(\varepsilon),\varepsilon) = 0$, while $t_2^N(\lambda^{N,1}(\varepsilon),\varepsilon)$ is well-defined (i.e., $t_2^N(\lambda,\varepsilon)$ has no pole at $\lambda^{N,1}(\varepsilon)$). Hence, we establish that all N-pulse patterns are unstable for $N \geq 2$ (Theorem 6.4).

Several aspects of the contents of this paper are related to existing literature, such as: the existence and stability of singular 'localized' patterns in the Gray-Scott model [9], [6], [7], [41]; the shadow system approach [20], [28], [29], [30], [2];

the stability of multi-pulse solutions [35], [36], [2], [3]; and the SLEP method [32], [33], [31], [19]. A section in which these relations are discussed concludes the paper.

Remark 1.1. We focus on the existence and stability of *large* and *positive* solutions to (1.1) and (1.2) in this paper. However, the methods we develop here can also be used when one is interested in small solutions, or in negative solutions or any 'mixture' (for instance $0 \le U \ll 1$ and $V \ll -1$). See also Remarks A.1, 2.9, and 3.1.

Remark 1.2. If we assume that $a_{ij} = \mathcal{O}(1)$, then the solution $(U, V) \equiv (0, 0)$ is asymptotically stable as solution to (1.1), with $H_i(U, V) \equiv 0$ when $a_{22} < 0$, $a_{11} + a_{22} < 0$, and $a_{11}a_{22} - a_{12}a_{21} > 0$.

Remark 1.3. We do allow for singular behavior in the nonlinearities in equations (1.1), (1.2), (1.7) as $U \to 0$, as is the case in the (generalized) Gierer-Meinhardt model. We shall see in Section 2 that what makes the singularity at the origin manageable in the existence (ODE) analysis is that the U and V components of the wave decay respectively at slow and fast exponential rates as $|x| \to \infty$. In work in progress, we use semigroup theory in a suitable class of exponentially decaying perturbations to study both well-posedness and nonlinear stability in a neigborhood of the various waves analyzed in the present paper. This technique was first introduced in [39] for wave solutions of parabolic systems, but a fundamental difference here is that the two components of the wave require different exponential decay rates in order to control the singularities in the nonlinear terms. It appears that this leads to a coherent theory of local existence and uniqueness for the partial differential equations (1.7).

The issue of nonlinear stability is more delicate than well-posedness. A new complication arises in the equations of perturbations for the (exponentially) weighted variables, due to the exponential weights that are required to control the singularities at the origin. It turns out that the linearized operator about the wave (in the new, exponentially weighted variables) necessarily has essential spectrum that is tangent to the imaginary axis at the origin, and that this portion of the spectrum cannot be removed by the approach of [39] involving the introduction of additional exponential weights. Hence the general results in [39], [18] still cannot be applied. A similar problem with the essential spectrum occurs in the nonlinear stability analysis of traveling wave solutions of a class of Ginzburg-Landau equations. In [22], a nonlinear stability theory for these solutions for classes of perturbations that decay *algebraically* at infinity is presented. It appears that the technique developed in [22] is also relevant for the nonlinear stability of the (linearly stable) waves studied in this paper.

2. THE EXISTENCE OF LARGE-AMPLITUDE MULTI-CIRCUIT HOMOCLINIC SOLUTIONS

In this section, we will study the existence of *stationary* multi-circuit homoclinic solutions to (1.7). The associated ODE can be written in two ways: as the *slow* system

$$
\begin{cases} u' = p \\ p' = -h_1 u^{\alpha_1} v^{\beta_1} - \hat{\varepsilon} u^{\alpha_1} v^{\beta_1} H_1(u, v; \varepsilon) + \varepsilon^2 \mu u \\ \varepsilon v' = q \\ \varepsilon q' = v - h_2 u^{\alpha_2} v^{\beta_2} - \hat{\varepsilon} u^{\alpha_2} v^{\beta_2} H_2(u, v; \varepsilon), \end{cases} \tag{2.1}
$$

where $'$ denotes the derivative with respect to the slow spatial variable x, and as the *fast* system

$$
\begin{cases} \dot{u} = \varepsilon p \\ \dot{p} = \varepsilon[-h_1 u^{\alpha_1} v^{\beta_1} - \hat{\varepsilon} u^{\alpha_1} v^{\beta_1} H_1(u, v; \varepsilon)] + \varepsilon^3 \mu u \\ \dot{v} = q \\ \dot{q} = v - h_2 u^{\alpha_2} v^{\beta_2} - \hat{\varepsilon} u^{\alpha_2} v^{\beta_2} H_2(u, v; \varepsilon), \end{cases} \tag{2.2}
$$

where $\dot{}$ denotes the derivative with respect to the *fast* spatial variable defined by $\xi = x/\varepsilon$. Note that both equations possess the reversibility symmetry

$$
x,\ \xi \to -x,\ -\xi, \quad p \to -p, \quad q \to -q. \tag{2.3}
$$

This symmetry will play a crucial role in the forthcoming analysis.

The central feature of interest in (2.2) is the semi-infinite, two-dimensional plane

$$
\mathcal{M} = \{(u, p, v, q) : v = q = 0,\ u > 0\}. \tag{2.4}
$$

Note that the vector field defined by (2.2) might be singular in the limit $u \to 0$ (for $\alpha_i < 0$, see Remark 1.3). As may be seen from a direct inspection of (2.2), when $\varepsilon = 0$, this manifold $\mathcal{M}$ is invariant under (2.2). In addition, it is also invariant for all $\varepsilon \in \mathbb{R}$, due to the assumptions (1.4) on $G_i(V)$, and the conditions $\beta_1, \beta_2 > 0$ (which we will explicitly assume, see (2.5) in the hypotheses of Theorem 2.1). Finally, there is one fixed point, S, on the boundary of $\mathcal{M}$, precisely at $(0, 0, 0, 0)$. We can now state the main result of this section.

Theorem 2.1. *Let* $F_i(U)$ *and* $G_i(V)$ *in* (1.2) *be such that* (1.4) *holds with*

$$
\begin{gathered} f_1 g_1 = h_1 > 0, \quad f_2 g_2 = h_2 > 0, \\ \alpha_1 > 1 + \frac{\alpha_2 \beta_1}{\beta_2 - 1}, \quad \alpha_2 < 0, \quad \beta_1 > 1, \quad \beta_2 > 1, \end{gathered} \tag{2.5}
$$

and let $G_i(V)$ satisfy

$$\lim_{V \to 0} \frac{G_i(V)}{V} = 0, \quad i = 1, 2. \tag{2.6}$$

Then, for any $N \geq 1$ with $N = \mathcal{O}(1)$, (2.2) *possesses an N-loop orbit $\gamma_h^N(\xi)$ homoclinic to $S = (0,0,0,0)$; the u, v coordinates of $\gamma_h^N(\xi)$ are non-negative; and $\gamma_h^N(\xi)$ is exponentially close to $\mathcal{M}$, except for N circuits through the fast field during which $\gamma_h^N(\xi)$ remains at least $\mathcal{O}(\sqrt{\varepsilon})$ away from $\mathcal{M}$. Moreover, each γ_h^N lies in the transverse intersection of $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$.*

This theorem will be proven in Sections 2.1-2.4. The fact that the fixed point S lies on the boundary of $\mathcal{M}$, where $u = 0$, introduces a technical difficulty for the application of the Fenichel geometric singular perturbation theory, since the theory applies for $u > 0$. Moreover, since $\alpha_2 < 0$, there is a singularity in the vector field at $u = 0$ that, a priori, could prevent the existence of orbits homoclinic to S. Both of these issues are treated and resolved in the conclusion of the proof of the theorem in Section 2.4.

Remark 2.2. The same geometric method by which the existence of the (multi-pulse) homoclinic solutions is established can also be used to construct several families of singular stationary *spatially periodic* patterns. This has been done in [9], [25] in the case of Gray-Scott model and the ideas developed there can be applied in a straightforward fashion to this more general case. Moreover, unlike the Gray-Scott case, it is also possible to construct singular stationary *aperiodic* patterns in (2.2). The periodic, respectively the aperiodic, orbits consist of a periodic, resp. arbitrary, arrangement of various kinds of fast N-loop excursions interspersed with long 'periods' close to $\mathcal{M}$. Both types of orbits are exponentially close to certain N-loop homoclinic orbits to $\mathcal{M}$ that are not homoclinic to S. Furthermore, it has been shown in [6] that the NLEP approach can also be used, at least on formal grounds, to study the stability of non-homoclinic solutions (see also [12]). We do not go any further into this subject of future research in this paper.

2.1. The geometry of the slow manifolds $\mathcal{M}$ and the dynamics of (2.2) **on and normal to them.** The fast reduced limit $\varepsilon \downarrow 0$ of (2.2) is:

$$\ddot{v} = v - h_2 u^{\alpha_2} v^{\beta_2}, \tag{2.7}$$

where $u > 0$ is a constant. System (2.7) is integrable and has a saddle fixed point at $(v = 0, \dot{v} = 0)$ that has an orbit $(v_h(\xi), q_h(\xi) = \dot{v}_h(\xi))$ homoclinic to it when

$$h_2 > 0, \tag{2.8}$$

and

$$\beta_2 > 1, \tag{2.9}$$

which are two conditions explicitly contained in the hypotheses (2.5) of Theorem 2.1. Note by (1.6) that the latter condition implies that

$$D = (\alpha_1 - 1)(\beta_2 - 1) - \alpha_2\beta_1 > 0, \tag{2.10}$$

and that $\alpha_2 < 0$. We have assumed in the scaling that $v = \mathcal{O}(1)$. We now see that even when $0 < v \ll 1$, i.e., when $0 < V \ll \mathcal{O}(1/\varepsilon^s)$ (and one must analyze the stationary version of problem (1.2) with $U = U_0$ constant, instead of the reduced system associated to (1.7)), the point $(V = 0, dV/d\xi = 0)$ is a saddle equilibrium as long as the condition (2.6) formulated in Theorem 2.1 holds. Also, we note that in this case (when $0 < V \ll \mathcal{O}(1/\varepsilon^s)$), there are only two fixed points, i.e., two roots of

$$F_2(U_0)G_2(V) = V, \tag{2.11}$$

and the homoclinic orbit connecting $(V = 0, dV/d\xi = 0)$ is the same as that of (2.7). These statements follow since $U_0 \gg 1$ (we study large amplitude solutions) implies by (1.4), (1.6), and (2.10) that $|F_2(U)| \ll 1$, and hence, by (2.6), (2.11) can only have solutions for $|V| \gg 1$. We can then apply the scaling analysis of Appendix A on page 501 to draw the desired conclusion.

The manifold $\mathcal{M}$, defined in (2.4), which is simply the union of the saddle points $(0,0)$ over all $u > 0$ and all $p \in \mathbb{R}$, is normally hyperbolic relative to (2.2) when $\varepsilon = 0$ for all v. Specifically, $\mathcal{M}$ has three-dimensional stable and unstable manifolds which are the unions of the two-parameter (u, p) families of one-dimensional stable and unstable manifolds, respectively, of the saddle points $(v, q) = (0, 0)$.

The Fenichel persistence theory (see [10], [11], and [21]) implies that system (2.2) with $0 < \varepsilon \ll 1$ has a locally invariant, slow manifold, under the condition that the vector field is at least C^1. Hence, we have to impose (2.6) (Remark 2.4). Here, we know even more already, since the manifold $\mathcal{M}$ is also invariant in the full system (2.2) with $\varepsilon \neq 0$, as noted above, so that it is a locally invariant, persistent slow manifold. In addition, the Fenichel theory states that, in the system (2.2) with $0 < \varepsilon \ll 1$, $\mathcal{M}$ has three-dimensional local stable and local unstable manifolds, which we denote $W^S_{\text{loc}}(\mathcal{M})$ and $W^U_{\text{loc}}(\mathcal{M})$ in the full system, and that these manifolds are $\mathcal{O}(\varepsilon, \hat{\varepsilon})$ close to their $\varepsilon = 0$ counterparts.

The flow on $\mathcal{M}$ is obtained by setting $v, q = 0$ in (2.1):

$$u'' = \varepsilon^2 \mu u. \tag{2.12}$$

Hence, it is linear and slow. Moreover, it is actually *super slow*, since $d/dx = \mathcal{O}(\varepsilon)$ and x is already the slow variable. We will find that this 'super slowness' is crucial both for the existence of the singular pulse solutions and for their stability. Finally, on $\mathcal{M}$, there are one-dimensional stable and unstable manifolds (restricted to $\mathcal{M}$) that are asymptotic to the saddle $S = (0, 0, 0, 0)$ on the boundary of $\mathcal{M}$:

$$\ell^{U,S} : p = \pm\varepsilon\sqrt{\mu}u. \tag{2.13}$$

Remark 2.3. The conditions (2.8) and (2.9) arise naturally in the search for homoclinic orbits. First, the condition $h_2 > 0$ arises since we look for positive solutions u, and so the two terms on the right hand side of (2.7) are of opposite signs, as is necessary. Second, if instead of (2.9) one has $0 < \beta_2 < 1$, then $\mathcal{M}$ is still invariant but no longer normally hyperbolic, as may be verified directly on (2.7). Therefore, if the original nonlinearities $H_2(U, V)$ of (1.1) or $F_2(U)$ and $G_2(V)$ of (1.2) are such that $\beta_2 < 1$ and/or $h_2 < 0$, then there cannot be a mechanism that enables (positive) solutions to (2.1), (2.2) to be biasymptotic to $\mathcal{M}$.

Remark 2.4. Since one has to use the original scalings of (1.2) when v becomes small, it is not necessary to impose the condition $\beta_1 > 1$ to apply the persistence results of [10], [11]. Nevertheless, we will see in Section 4.1 that we need to assume that $\beta_1 > 1$ in order to develop the 'NLEP approach' (see, however, Remark 3.2). Therefore, we have added this condition to (2.5). Note that the conditions on the $G_i(V)$'s and the β_i's in Theorem 2.1 can be replaced by the slightly stronger but simpler assumptions that, for $i = 1, 2$, $G_i(V) = V\widetilde{G}_i(V)$ with $\widetilde{G}_i(0) = 0$ and $\widetilde{G}_i(V) \to \infty$ algebraically as $V \to \infty$.

Remark 2.5. When $G_2(V) \sim V$ in the limit $V \to 0$, i.e., when (2.6) no longer holds, the character of the critical points depends on $F_2(U)$. We do not consider this case in the paper. Also, the above arguments about $G_2(V)$ and the number of fixed points need not hold for general (nonseparable) reaction terms, $H_2(U, V) \neq F_2(U)G_2(V)$. In the more general case, one needs to impose another (nontrivial) non-degeneracy condition to avoid additional complications.

2.2. One-circuit orbits homoclinic to $\mathcal{M}$. One-circuit orbits homoclinic to $\mathcal{M}$ in the full system (2.2) with $0 < \varepsilon \ll 1$ will lie in the transverse intersection of the extensions $W^U(\mathcal{M})$ and $W^S(\mathcal{M})$ of the local manifolds $W^U_{\text{loc}}(\mathcal{M})$ and $W^S_{\text{loc}}(\mathcal{M})$, and their excursions in the fast field will lie close to a homoclinic orbit of (2.7) for some particular value of u.

In order to detect these solutions, we use a Melnikov method for slowly varying systems. Here, we refine the approach of [34] a little bit so that we have better control over the $\mathcal{O}(\hat{\varepsilon})$ terms. In particular, it is helpful to make use of the integrable planar system

$$\ddot{v} = v - h_2 u^{\alpha_2} v^{\beta_2} - \hat{\varepsilon} u^{\alpha_2} v^{\beta_2} H_2(u, v; \varepsilon), \tag{2.14}$$

with $q = \dot{v}$, and $u > 0$ fixed, rather than the reduced system (2.7). System (2.14) has a conserved quantity, or energy, given by:

$$K(\xi) = \frac{1}{2}q^2 - \frac{1}{2}v^2 + \frac{h_2}{\beta_2 + 1}u^{\alpha_2}v^{\beta_2+1} + \hat{\varepsilon}u^{\alpha_2}\int_0^v \tilde{v}^{\beta_2}H_2(u, \tilde{v}; \varepsilon)\, d\tilde{v}. \tag{2.15}$$

By construction, $K|_{\mathcal{M}} = 0$, and $K < 0$ for orbits inside the homoclinic orbit $(\tilde{v}_h(\xi; u_0), \tilde{q}_h(\xi; u_0))$ of (2.14), while $K > 0$ for orbits outside. In addition, a direct calculation yields:

$$\begin{aligned}(2.16) \quad \dot{K} = \varepsilon\alpha_2 u^{\alpha_2-1} p\Big[\frac{h_2}{\beta_2+1}v^{\beta_2+1} \\ + \hat{\varepsilon}\int_0^v \tilde{v}^{\beta_2}\Big(H_2(u,\tilde{v};\varepsilon) + \frac{1}{\alpha_2}u\frac{\partial H_2}{\partial u}(u,\tilde{v};\varepsilon)\Big)\,d\tilde{v}\Big].\end{aligned}$$

Hence, $\dot{K} = \mathcal{O}(\varepsilon)$, also by construction.

Now, since $K|_{\mathcal{M}} \equiv 0$, any orbit $\gamma(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi))$ of (2.2) that is homoclinic to $\mathcal{M}$ must satisfy the condition

$$(2.17) \qquad \Delta K(u_0, p_0) = \int_{-\infty}^{\infty} \dot{K}(\gamma(\xi))\,d\xi = 0.$$

Here, without loss of generality, we assume that the orbits $\gamma(\xi)$ homoclinic to $\mathcal{M}$, if they exist, are parameterized such that $\gamma(0) = (u_0, p_0, v_0, 0)$. Therefore, (2.16) implies:

$$\begin{aligned}(2.18) \quad \varepsilon\alpha_2\int_{-\infty}^{\infty} u^{\alpha_2-1}p\Big[\frac{h_2}{\beta_2+1}v^{\beta_2+1} + \hat{\varepsilon}\int_0^v \tilde{v}^{\beta_2}\Big(H_2(u,\tilde{v};\varepsilon) \\ + \frac{1}{\alpha_2}u\frac{\partial H_2}{\partial u}(u,\tilde{v};\varepsilon)\Big)\,d\tilde{v}\Big]\,d\xi = 0.\end{aligned}$$

The condition (2.18) is *exact* in the sense that we did not introduce any approximations so far. Moreover, as we now show, if the zero of ΔK is a simple one, then the homoclinic orbit $\gamma(\xi)$ lies in the transverse intersection of $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$.

Now, $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$ are three-dimensional manifolds. Thus, in the four-dimensional phase space of (2.2), one expects that $W^S(\mathcal{M}) \cap W^U(\mathcal{M})$ is a two-dimensional manifold, or, equivalently, that there is a one-parameter family of orbits γ that are homoclinic to $\mathcal{M}$. The analysis carried out in the remainder of this subsection reveals that this is indeed the case (if (2.5) holds).

Since $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$ are $\mathcal{O}(\varepsilon, \hat{\varepsilon})$ close to the (u_0, p_0)-family of homoclinic orbits to (2.7), as stated above, both $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$ intersect the three-dimensional hyperplane $\{q = 0\}$ transversely in two-dimensional manifolds, defined as $\mathcal{I}^{-1}(\mathcal{M})$ and $\mathcal{I}^{+1}(\mathcal{M})$, respectively. These manifolds can be parameterized by (u_0, p_0):

$$(2.19) \qquad \mathcal{I}^{\pm 1}(\mathcal{M}) = \{(u_0, p_0, v_0^{\pm 1}(u_0, p_0), 0),\ u_0 > 0\} \subset \{q = 0\}.$$

Thus, for every $u_0 > 0$ and $p_0 \in \mathbb{R}$, there exists a v_0^{-1} such that the solution $\gamma(\xi)$ of (2.2) with $\gamma(0) = (u_0, p_0, v_0^{-1}, 0)$ satisfies $\lim_{\xi\to\infty}\gamma(\xi) \in \mathcal{M}$. Similarly, there exists a v_0^{+1} such that the solution $\gamma(\xi)$ of (2.2) with $\gamma(0) = (u_0, p_0, v_0^{+1}, 0)$ satisfies $\lim_{\xi\to-\infty}\gamma(\xi) \in \mathcal{M}$ (where the superscripts are indices, not powers). Note that using the above limits is, in fact, a slight abuse of notation. The slow manifold $\mathcal{M}$ has a boundary $\partial\mathcal{M} = \{(u,p,v,q) : u = v = q = 0\}$ (2.4) and the vector field (2.2) can be singular when $u \to 0$. However, in this paper we are only interested in orbits that are homoclinic to $S \in \partial\mathcal{M}$ and we will show in Section 2.4 how to extend the geometric analysis to the boundary of $\mathcal{M}$.

Having established that the sets $\mathcal{I}^{-1}(\mathcal{M})$ and $\mathcal{I}^{+1}(\mathcal{M})$ are nonempty, we now show they intersect in the hyperplane $\{p = 0\}$. We remark that we have a choice in how to show this. We can use either the homoclinic orbit $(v_h(\xi), q_h(\xi))$ of (2.7) or the homoclinic orbit $(\tilde{v}_h(\xi), \tilde{q}_h(\xi))$ of (2.14) to approximate the solutions on $W^U(\mathcal{M})$ and $W^S(\mathcal{M})$, and we choose the latter, as is consistent with our choice of K. In particular, using $\tilde{v}_h$ to approximate the fast-field behavior of the solutions to (2.2) in $W^U(\mathcal{M})$ and $W^S(\mathcal{M})$, the exact condition (2.18) implies that, to leading order, one-circuit homoclinic solutions must satisfy:

$$\begin{aligned}\varepsilon\alpha_2 u_0^{\alpha_2-1} p_0 \int_{-\infty}^{\infty}\Bigg[\frac{h_2}{\beta_2+1}\tilde{v}_h^{\beta_2+1} + \hat{\varepsilon}\int_0^{\tilde{v}_h}\tilde{v}^{\beta_2}\Big(H_2(u_0,\tilde{v};\varepsilon)\\ + \frac{1}{\alpha_2}u_0\frac{\partial H_2}{\partial u}(u_0,\tilde{v};\varepsilon)\Big)\,d\tilde{v}\Bigg]\,d\xi + \mathcal{O}(\varepsilon^2) = 0.\end{aligned} \tag{2.20}$$

The improper integral exists because $\tilde{v}_h$ converges exponentially to zero as $\xi \to \pm\infty$, and hence so does the entire integrand. Then, since the integrand in (2.20) is positive and since u_0 is assumed to be positive, we see that it is only possible to satisfy (2.20) if p_0 is $\mathcal{O}(\varepsilon)$. In addition, we conclude that

$$\Delta K(u_0,p_0) = \mathcal{O}(\varepsilon^2) \quad \text{for } p_0 = \mathcal{O}(\varepsilon). \tag{2.21}$$

Finally, for the one circuit homoclinic orbits we are after here, we now show that not only is it necessary that $p_0 = \mathcal{O}(\varepsilon)$, but it is in fact the case that $p_0 \equiv 0$. We go back to the exact condition (2.18). For any solution

$$\gamma(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi)) \in W^U(\mathcal{M})$$

with $\gamma(0) = (u_0, 0, v_0^{+1}(u_0,0), 0) \in \mathcal{I}^{+1}(\mathcal{M})$, the reversibility symmetry (2.3) implies that

$$\begin{aligned} u(-\xi) &= u(\xi), & p(-\xi) &= -p(\xi),\\ v(-\xi) &= v(\xi), & q(-\xi) &= -q(\xi),\end{aligned} \tag{2.22}$$

and, hence also, $v_0^{+1} = v_0^{-1}$. Therefore, along $\gamma(\xi)$, the integrand in (2.18) is an odd function of ξ, and the integral vanishes identically. This, in turn, implies that $W^U(\mathcal{M}) \cap W^S(\mathcal{M})$ precisely in the orbit $\gamma(\xi)$ and that the set $\mathcal{I}^{+1}(\mathcal{M}) \cap \mathcal{I}^{-1}(\mathcal{M}) \subset \{p = 0\}$.

2.3. Multi-circuit orbits homoclinic to $\mathcal{M}$. In this subsection, we extend the results for one-circuit homoclinic orbits to $\mathcal{M}$ from the previous subsection to multi-circuit orbits homoclinic to $\mathcal{M}$.

As we shall show below, the global stable and unstable manifolds of $\mathcal{M}$ intersect the hyperplane $\{q = 0\}$ many times. The sets $\mathcal{I}^{\pm 1}(\mathcal{M})$ defined above can be seen to be the *first* intersections of $W^S(\mathcal{M})$ and $W^U(\mathcal{M})$ with $\{q = 0\}$: an orbit $\gamma(\xi)$ with initial condition in $\mathcal{I}^{-1}(\mathcal{M})$ only follows the reduced fast flow for half a circuit and 'gets caught' by $\mathcal{M}$, i.e., it does not leave an exponentially small neighborhood of $\mathcal{M}$ anymore. Orbits that have their initial conditions in the *second* intersections of $W^{U,S}(\mathcal{M})$ with $\{q = 0\}$, whose existence we shall show shortly, follow the fast flow for two half circuits through the fast field before settling down on $\mathcal{M}$. Hence, they each make one full circuit. We label these sets of initial conditions $(u_0, p_0, v^{\pm 2}(u_0, p_0), 0)$ by $\mathcal{I}^{\pm 2}(\mathcal{M})$. For initial conditions in them, $v_0^{\pm 2}(u_0, p_0)$ are strictly $\mathcal{O}(\sqrt{\varepsilon})$.

Similar definitions can be given for the n-th intersection sets $\mathcal{I}^{\pm n}(\mathcal{M})$. These sets are also two-dimensional manifolds. For n even, $v_0^{\pm n}$ is strictly $\mathcal{O}(\sqrt{\varepsilon})$, since these solutions make $n/2$ full circuits in the fast field; while, for n odd, $v_0^{\pm n}$ is strictly $\mathcal{O}(1)$ (and $\mathcal{O}(\varepsilon, \hat{\varepsilon})$ close to the intersection of the corresponding unperturbed homoclinic orbit of (2.7) with $\{q = 0\}$), since these solutions make a half-integer number of circuits in the fast field. Below, we will show that all $\mathcal{I}^{\pm n}(\mathcal{M})$ exist. Finally, we will show that $\mathcal{I}^{-m}(\mathcal{M}) \cap \mathcal{I}^{n}(\mathcal{M})$ for all $m + n$ even, and it is precisely in these intersections in which the orbits homoclinic to $\mathcal{M}$, that make $(m + n)/2$ full circuits through the fast field of (2.2), lie.

Remark 2.6. Intersections with $m + n$ odd are ruled out due to the locations of v_0^{-m} and v_0^{n}, since one of these is strictly $\mathcal{O}(\sqrt{\varepsilon})$, while the other is strictly $\mathcal{O}(1)$.

We first establish that the curves $\mathcal{I}^{\pm n}(\mathcal{M})$ exist for all $n > 1$ (if (2.5) holds), focusing on the case of $\mathcal{I}^{+n}(\mathcal{M})$, since the case of $\mathcal{I}^{-n}(\mathcal{M})$ may be done similarly. The plane $\{p_0 = 0\}$ separates $\mathcal{I}^{+1}(\mathcal{M})$ into two parts. Orbits with initial conditions in the 'wrong' part of $\mathcal{I}^{+1}(\mathcal{M})$ are 'outside' the three-dimensional manifold $W^S(\mathcal{M})$ and follow the unbounded part of the integrable flow (2.14) in forward 'time' ξ. Hence, they do not return to $\{q = 0\}$. On the other hand, orbits with initial conditions in the 'right' part of $\mathcal{I}^{+1}(\mathcal{M})$ are 'inside' the three-dimensional manifold $W^S(\mathcal{M})$ and follow the bounded part of the integrable flow (2.14) in forward 'time' ξ. Hence, there is the possibility that they can return to $\{q = 0\}$.

In order to deduce which part of $\mathcal{I}^{+1}(\mathcal{M})$ does return to $\{q = 0\}$, i.e., which part of $\mathcal{I}^{+1}(\mathcal{M})$ is the 'right' part, we consider an orbit

$$\gamma^{+1}(\xi) = (u^{+1}(\xi), p^{+1}(\xi), v^{+1}(\xi), q^{+1}(\xi))$$

with $\gamma^{+1}(0) = (u_0^{+1}, p_0^{+1}, v_0^{+1}, 0) \in I^{+1}(\mathcal{M})$. We assume that p_0^{+1} is strictly $\mathcal{O}(\varepsilon)$, i.e., $\gamma^{+1}(0)$ is not too close to $\mathcal{I}^{-1}(\mathcal{M})$. Thus, γ^{+1} is at its minimal distance

$(\mathcal{O}(\sqrt{\varepsilon}))$ from $\mathcal{M}$ when $\xi = \Xi = \mathcal{O}(|\log \varepsilon|)$. Then, since $\gamma^{+1}(\xi) \to \mathcal{M}$ as $\xi \to -\infty$ and since $K \equiv 0$ on $\mathcal{M}$, we see that

$$
\begin{aligned}
(2.23)\qquad K(\gamma^{+1}(\Xi)) &= \int_{-\infty}^{\Xi} \dot{K}(\gamma(\xi))\, d\xi \\
&= \varepsilon \frac{\alpha_2 h_2}{\beta_2 + 1} (u_0^{+1})^{\alpha_2 - 1} p_0^{+1} \int_{-\infty}^{\infty} \tilde{v}_h^{\beta_2+1}\, d\xi + \mathcal{O}(\varepsilon^{1+\sigma})
\end{aligned}
$$

for some $\sigma > 0$ (since $\tilde{v}_h(\xi)$ approaches 0 exponentially fast), where we have made the same approximation as in (2.20). Thus, since $h_2 > 0$ and $\alpha_2 < 0$ (1.6), (2.10), we have

$$
(2.24)\qquad K(\gamma^{+1}(\Xi)) < 0 \iff p_0^{+1} > 0.
$$

Finally, since $K < 0$, we know from the definition (2.15) of K that $\gamma^{+1}(\Xi)$ is 'inside' $W^S(\mathcal{M})$; and, also that $\gamma^{+1}(\Xi)$ intersects $\{q = 0\}$ again, i.e., $\mathcal{I}^{+2}(\mathcal{M})$ is nonempty. Correspondingly, the above argument shows that if $p_0^{+1} < 0$, then $K > 0$ and the orbit $\gamma^{+1}(\Xi)$ is 'outside' $W^S(\mathcal{M})$. Hence, it cannot intersect the hyperplane $\{q = 0\}$ again. The same argument in backwards 'time' yields that orbits $\gamma^{-1}(\xi)$ with $\gamma^{-1}(0) = (u_0^{-1}, p_0^{-1}, v_0^{-1}, 0) \in I^{-1}(\mathcal{M})$ intersect $\{q = 0\}$ again when $p_0^{-1} < 0$ (but not when $p_0^{-1} > 0$). Thus, both $\mathcal{I}^{\pm 2}(\mathcal{M})$ exist. Furthermore, the above argument may be extended to show that all $\mathcal{I}^{\pm n}(\mathcal{M})$ exist, and we denote the points in these sets by $(u_0^{\pm n}, 0, v_0^{\pm n}(u_0^{\pm n}, 0), 0)$, respectively.

Next, we show that the intersections $\mathcal{I}^{+2}(\mathcal{M}) \cap \mathcal{I}^{-2}(\mathcal{M})$, and their higher order equivalents, exist. Orbits with initial conditions in $\mathcal{I}^{\pm 2}(\mathcal{M})$ can also be approximated by $\tilde{v}_h(\xi)$ (2.14) to leading order, since both circuits must be $\mathcal{O}(\varepsilon)$ close to $\tilde{v}_h(\xi)$. Thus, to leading order,

$$
(2.25)\qquad \Delta K = 2\varepsilon \frac{\alpha_2 h_2}{\beta_2 + 1} (u_0^{+1})^{\alpha_2 - 1} p_0^{+1} \int_{-\infty}^{\infty} \tilde{v}_h^{\beta_2+1}\, d\xi,
$$

and we find that the p-coordinate p_0 of the initial condition must also be 0, *to leading order*, for a two-circuit homoclinic orbit with initial conditions in $\mathcal{I}^{+2}(\mathcal{M}) \cap \mathcal{I}^{-2}(\mathcal{M})$. Moreover, not only is $p_0 = 0$ to leading order, but $p_0 \equiv 0$ *exactly*, since the reversibility symmetry (2.3) implies that the homoclinic orbits with initial conditions $p_0 = 0$ in $\mathcal{I}^{+2}(\mathcal{M}) \cap \mathcal{I}^{-2}(\mathcal{M})$ are also symmetric, just as we saw for the one-circuit orbits. That is, we have shown

$$
(2.26)\qquad \mathcal{I}^{+2}(\mathcal{M}) \cap \mathcal{I}^{-2}(\mathcal{M}) \subset \{p = 0\} \textit{ exactly}.
$$

Finally, the same argument may be repeated inductively to show that

$$
(2.27)\qquad \mathcal{I}^{+n}(\mathcal{M}) \cap \mathcal{I}^{-n}(\mathcal{M}) \subset \{p = 0\} \quad \text{for all } n = \mathcal{O}(1) \textit{ exactly}.
$$

It remains to determine, for which pairs $m \neq n$ and $m+n$ even, whether or not

$$\mathcal{I}^{+n}(\mathcal{M}) \cap \{p=0\} \neq \emptyset \quad \text{and} \quad \mathcal{I}^{-m}(\mathcal{M}) \cap \{p=0\} \neq \emptyset, \tag{2.28}$$

so that we also have nonsymmetric multi-circuit homoclinic orbits to $\mathcal{M}$. As shown above, only those orbits with initial conditions in $\mathcal{I}^{+1}(\mathcal{M}) \cup \{p>0\}$ can 'build' $\mathcal{I}^{+2}(\mathcal{M})$. The main question then is: which of those orbits satisfy $p(\xi)=0$ at their second (or higher-order) intersection with $\{q=0\}$?

We begin by calculating the change in p during the half-circuit from $\{q=0\}$ back to itself. Consider the orbit $\gamma^{+1}(\xi) = (u^{+1}(\xi), p^{+1}(\xi), v^{+1}(\xi), q^{+1}(\xi))$ with $\gamma^{+1}(0) = (u_0^{+1}, p_0^{+1}, v_0^{+1}, 0) \in I^{+1}(\mathcal{M})$, where now $p_0^{+1} = \varepsilon\tilde{p}_0 > 0$ with $\tilde{p}_0$ strictly $\mathcal{O}(1)$. Let Ξ $(= \mathcal{O}(|\log\varepsilon|))$ be such that $\gamma^{+1}(\Xi) \in \mathcal{I}^{+2}(\mathcal{M}) \subset \{q=0\}$. We define Δp by $p^{+1}(\Xi) \stackrel{\text{def}}{=} \varepsilon\tilde{p}_0 + \Delta p$. Hence, by the second component of (2.2),

$$\Delta p^{+1}(u_0, \varepsilon\tilde{p}_0) = -\varepsilon\int_0^{\Xi} [h_1 u^{\alpha_1} v^{\beta_1} + \hat{\varepsilon} u^{\alpha_1} v^{\beta_1} H_1(u,v;\varepsilon))]\, d\xi + \mathcal{O}(\varepsilon^3|\log\varepsilon|), \tag{2.29}$$

and we see that $\Delta p^{+1}(u_0, \varepsilon\tilde{p}_0)$ is finite, since β_1, $\beta_2 > 0$. Using the approximation $\tilde{v}_h(\xi)$ defined by (2.14), we find

$$\Delta p^{+1}(u_0, \varepsilon\tilde{p}_0) = -\varepsilon h_1 u_0^{\alpha_1} \int_0^{\infty} (\tilde{v}_h(\xi))^{\beta_1}\, d\xi + \mathcal{O}(\varepsilon^{1+\sigma}), \tag{2.30}$$

for some $\sigma > 0$. Note that we can replace $\tilde{v}_h(\xi)$ by $v_h(\xi) \geq 0$, the corresponding homoclinic solution to (2.7). The p-coordinate of $\gamma^{+1}(\Xi) \in \mathcal{I}^{+2}(\mathcal{M})$ is given to leading order by

$$\varepsilon\left[\tilde{p}_0 - h_1 u_0^{\alpha_1} \int_0^{\infty} (v_h(\xi))^{\beta_1}\, d\xi\right], \quad \text{with } \tilde{p}_0 > 0. \tag{2.31}$$

Since $u_0 > 0$ and $v_h(\xi) > 0$, this expression can only change sign when

$$h_1 > 0, \tag{2.32}$$

as assumed in (2.5). We see again therefore that, for $h_1 > 0$, all intersections $\mathcal{I}^{+n}(\mathcal{M}) \cap \mathcal{I}^{-n}(\mathcal{M})$ exist and satisfy (2.27). Moreover, by following the fast flow for j half circuits, we see that all $\mathcal{I}^{n+j}(\mathcal{M}) \cap \mathcal{I}^{-n+j}(\mathcal{M})$ exist, although these sets are not subsets of $\{p=0\}$, since $\Delta p \neq 0$, by (2.30). Summarizing,

$$\mathcal{I}^{+n}(\mathcal{M}) \cap \mathcal{I}^{-m}(\mathcal{M}) \neq \emptyset, \quad \text{for all } n+m \text{ even, and } m,n = \mathcal{O}(1). \tag{2.33}$$

Remark 2.7. The condition $\beta_1 > 0$ in the hypotheses (2.5) of Theorem 2.1, which was imposed to help establish the existence of $\mathcal{M}$, has been crucial here for a different reason. The quantity Δp^{+1} becomes unbounded as $\tilde{p}_0 \downarrow 0$ when $\beta_1 < 0$ (recall that the v-coordinate of $\gamma^{+1}(\Xi)$ is at most $\mathcal{O}(\sqrt{\varepsilon})$). In addition, the case $\beta_1 = 0$ is special, and we do not consider the details of this degenerate case here.

Remark 2.8. We have also just seen the reason for imposing the requirement (2.32) in Theorem 2.1 (recall (2.5)). In the opposite case when $h_1 < 0$, the p-coordinate of $\gamma^{+1}(\Xi)$ is strictly positive. Thus, $\mathcal{I}^{+2}(\mathcal{M}) \subset \{p > \varepsilon p_*^{+2}\}$ for some $0 < p_*^{+2} = \mathcal{O}(1)$. Analogously, it follows that $\mathcal{I}^{-2}(\mathcal{M}) \subset \{p < -\varepsilon p_*^{-2}\}$ for some $0 < p_*^{-2} = \mathcal{O}(1)$, and hence that $\mathcal{I}^{+2}(\mathcal{M}) \cap \mathcal{I}^{-2}(\mathcal{M}) = \emptyset$. Then, by repeating this argument, one readily sees also that

$$\mathcal{I}^{+n}(\mathcal{M}) \cap \mathcal{I}^{-n}(\mathcal{M}) = \emptyset, \quad \text{for } n > 1, \text{ when } h_1 < 0. \tag{2.34}$$

In other words, if there exist homoclinic orbits to $\mathcal{M}$ when $h_1 < 0$ (with non-negative u and v coordinates), then they can only be of the type that make at most one circuit through the fast field.

2.4. Take off and touch down curves and the proof of Theorem 2.1. So far, we have focused on the dynamics in the fast field of (2.2), and for every $\mathcal{O}(1)$ $N > 0$, we have constructed a one-parameter family of multi-circuit orbits homoclinic to $\mathcal{M}$, under the conditions in the hypotheses (2.5) of Theorem 2.1. We now turn our attention for each $N \geq 1$ to locating special pairs of curves on $\mathcal{M}$ that are essential for determining the slow segments of these same multi-circuit orbits. In particular, we will need the ideas developed in [11].

Let $\gamma^N(\xi)$ be an N-circuit orbit homoclinic to $\mathcal{M}$, of the type whose existence has been shown in the previous subsection, with $\gamma^N(0) \in \mathcal{I}^N(\mathcal{M}) \cap \mathcal{I}^{-N}(\mathcal{M}) \subset \{p = 0\}$. By geometrical singular perturbation theory (see [11] and [21]), there are two orbits $\gamma_{\mathcal{M}}^{+N} = \gamma_{\mathcal{M}}^{+N}(\xi; (u_0^{+N}, p_0^{+N})) \subset \mathcal{M}$ and $\gamma_{\mathcal{M}}^{-N}(\xi; (u_0^{-N}, p_0^{-N})) \subset \mathcal{M}$, respectively (where $\gamma_{\mathcal{M}}^{\pm N}(0; (u_0^{\pm N}, p_0^{\pm N})) = (u_0^{\pm N}, p_0^{\pm N}) \in \mathcal{M}$), such that $\|\gamma^N(\xi) - \gamma_{\mathcal{M}}^{+N}(\xi; (u_0^{+N}, p_0^{+N}))\|$ is exponentially small for $\xi > 0$ with $\xi \geq \mathcal{O}(1/\varepsilon)$ and $\|\gamma^N(\xi) - \gamma_{\mathcal{M}}^{-N}(\xi; (u_0^{-N}, p_0^{-N}))\|$ is exponentially small for $\xi < 0$ with $-\xi \geq \mathcal{O}(1/\varepsilon)$. As a consequence,

$$d(\gamma^N(\xi), \mathcal{M}) = \mathcal{O}(e^{-k/\varepsilon}) \quad \text{for } |\xi| \geq \mathcal{O}(1/\varepsilon) \text{ or larger}, \tag{2.35}$$

for some $k > 0$. The orbits $\gamma_{\mathcal{M}}^{\pm N}(\xi; (u_0^{\pm N}, p_0^{\pm N}))$ determine the behavior of $\gamma^N(\xi)$ near $\mathcal{M}$. Moreover, $\gamma^N(\xi)$ satisfies the reversibility symmetry (2.3) by the choice of initial conditions, and thus

$$\gamma_{\mathcal{M}}^{-N}(\xi; (u_0^{-N}, p_0^{-N})) = \gamma_{\mathcal{M}}^{+N}(-\xi; (u_0^{+N}, -p_0^{+N})). \tag{2.36}$$

We now define the curves $T_d^N \subset \mathcal{M}$ ('touch down') and $T_o^N \subset \mathcal{M}$ ('take off') as

$$(2.37)\qquad \begin{aligned} T_d^N &= \bigcup_{\gamma^N(0)} \{(u_0^{+N}, p_0^{+N}) = \gamma_{\mathcal{M}}^N(0; (u_0^{+N}, p_0^{+N}))\}, \\ T_o^N &= \bigcup_{\gamma^N(0)} \{(u_0^{+N}, -p_0^{+N})\}, \end{aligned}$$

where the unions are over all $\gamma^N(0) \in \mathcal{I}^N(\mathcal{M}) \cap \mathcal{I}^{-N}(\mathcal{M}) \subset \{p = 0\} \cap \{q = 0\}$. For each $N = 1, 2, \ldots$, the take off set T_o^N (respectively, the touch down set T_d^N) is the collection of base points of all of the Fenichel fibers in $W^U(\mathcal{M})$ (respectively, $W^S(\mathcal{M})$) that have points in the transverse intersection of $W^U(\mathcal{M})$ and $W^S(\mathcal{M})$.

Detailed asymptotic information about the locations of T_o^N and T_d^N can be obtained explicitly by determining the relations between $\gamma^N(0) = (u_0, 0, v_0, 0)$ and $(u_0^{+N}, p_0^{+N}, 0, 0)$. First, we observe that $\dot{p} = \mathcal{O}(\varepsilon^3)$ on $\mathcal{M}$ (2.2), thus, the p-coordinate of $\gamma_{\mathcal{M}}^{+N}$ remains constant to leading order during the fast excursions of $\gamma^N(\xi)$. Therefore, p_0^{+N} is completely determined (to leading order) by the accumulated change in p of $\gamma^N(\xi)$ during its 'time' $\xi > 0$ in the fast field. These changes have already been calculated for $N = 1$ in (2.29) and (2.30). For $N > 1$, the calculation is exactly the same, except for the fact that $\gamma^N(\xi)$ now makes N half circuits before 'touching down' on $\mathcal{M}$:

$$(2.38)\qquad p_0^{+N} = -\varepsilon h_1 N u_0^{\alpha_1} \int_0^\infty (v_h(\xi))^{\beta_1}\, d\xi + \mathcal{O}(\varepsilon^{1+\sigma}),$$

for some $\sigma > 0$. From the first component of (2.2) and the fact that $p = \mathcal{O}(\varepsilon)$, we also conclude that $u_0^{+N} = u_0$ to leading order. Thus, we find

$$(2.39)\qquad \begin{aligned} &T_d^N : \left\{p \stackrel{\text{def}}{=} p_d^N(u) = -\varepsilon h_1 N u^{\alpha_1} \int_0^\infty (v_h(\xi))^{\beta_1}\, d\xi + \mathcal{O}(\varepsilon^{1+\sigma})\right\}, \\ &T_o^N : \{p = -p_d^N(u)\} \end{aligned}$$

($\sigma > 0$), where $v_h(\xi) = v_h(\xi; u)$, the homoclinic solution of (2.7).

To more fully determine the behavior of p_d^N as a function of u, we introduce $w_h = w_h(\xi; \beta_2) \geq 0$, which is the (positive) homoclinic solution of a rescaled version of (2.7):

$$(2.40)\qquad \ddot{w} = w - w^{\beta_2}.$$

Without loss of generality, we take the solutions to be parameterized such that $w_h(\xi)$ is symmetric with respect to $\xi = 0$. Thus,

$$(2.41)\qquad v_h(\xi; u) = v_h(\xi; u, h_2, \alpha_2, \beta_2) = (h_2 u^{\alpha_2})^{1/(1-\beta_2)} w_h(\xi; \beta_2),$$

which yields

$$\int_{-\infty}^{\infty}(v_h(\xi))^{\beta_1}\,d\xi = 2\int_0^{\infty}(v_h(\xi))^{\beta_1}\,d\xi = h_2^{-\beta_1/(\beta_2-1)}u^{-\alpha_2\beta_1/(\beta_2-1)}W(\beta_1,\beta_2), \tag{2.42}$$

where

$$W(\beta_1,\beta_2) = \int_{-\infty}^{\infty}(w_h(\xi;\beta_2))^{\beta_1}\,d\xi. \tag{2.43}$$

We can now rewrite (2.39) to leading order as

$$T_{d,o}^N : \{p = \pm p_d^N(u)\}, \quad \text{with } p_d^N(u) = -\frac{N}{2}\varepsilon h_1 h_2^{-\beta_1/(\beta_2-1)}u^{1+D/(\beta_2-1)}W(\beta_1,\beta_2), \tag{2.44}$$

with $D > 0$ (2.10). Note that the higher order corrections, which are not needed here, can be obtained by a straightforward asymptotic approximation scheme, see [9].

In Figure 2.1 on the following page, we have plotted the curves $T_{d,o}^N$ for a few values of N superimposed onto the linear flow on $\mathcal{M}$ given by (2.12). Since $D > 0$ and $\beta_2 > 1$, the curve T_o^N given by (2.44) to leading order is tangent to the u-axis for each N; and, thus, by (2.13), $T_o^N \cap \ell^U$ exists for all N. In fact, the unique intersection is given to leading order by

$$u_{h,N} = \left[\frac{2h_2^{\beta_1/(\beta_2-1)}\sqrt{\mu}}{h_1 N W(\beta_1,\beta_2)}\right]^{(\beta_2-1)/D}. \tag{2.45}$$

Then, the reversibility symmetry (2.3) implies that $T_d^N \cap \ell^S$ also, with precisely the same value of u given by (2.45). Therefore, for each $N > 0$, there is a unique homoclinic orbit to the saddle point $S = (0,0,0,0)$ in (2.2) that (i) flows outward from S staying exponentially close to ℓ^U (and hence also to $\mathcal{M}$) until the u coordinate reaches a neighborhood of $u_{h,N}$, (ii) makes N full circuits through the fast field near the homoclinic orbit $(v_h(\xi), q_h(\xi))$ of (2.7) while u is constant $(= u_{h,N})$, and (iii) returns to an exponentially small neighborhood of $\mathcal{M}$, flowing in toward S along ℓ^S.

We have shown in Sections 2.2 ($N = 1$) and 2.3 ($N > 1$) that this homoclinic orbit lies in the transverse intersection of the manifolds $W^U(\mathcal{M})$ and $W^S(\mathcal{M})$. Therefore, we can conclude that there exists an orbit $\gamma_h^N(\xi) = (u_h^N(\xi), p_h^N(\xi), v_h^N(\xi), q_h^N(\xi)) \in W^U(\mathcal{M}) \cap W^S(\mathcal{M})$ that is asymptotically close to an orbit on $\ell^U \subset \mathcal{M}$ for $\xi \ll -1/\varepsilon$ and asymptotically close to an orbit on $\ell^S \subset \mathcal{M}$ for

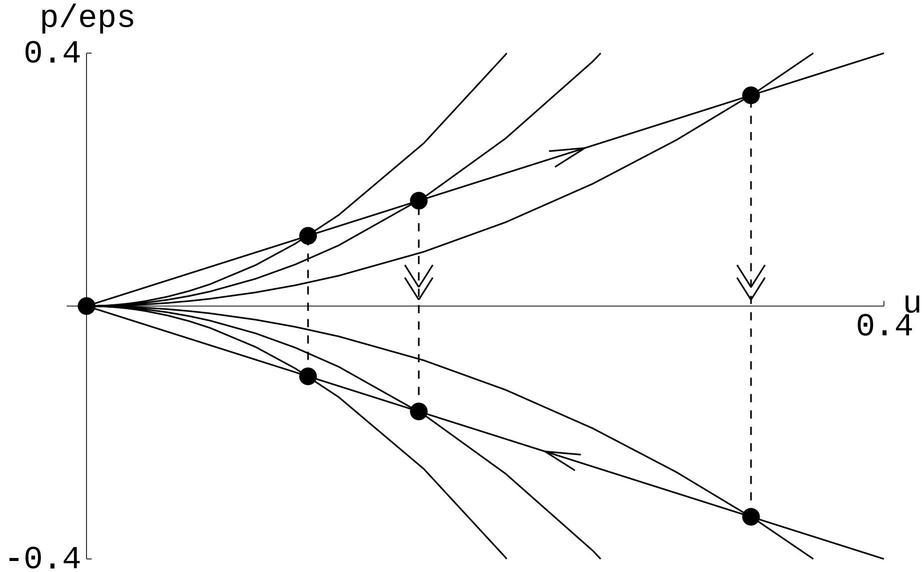

FIGURE 2.1. The stable and unstable manifolds ℓ^U and ℓ^S in $\mathcal{M}$ and the take off and touch down curves T_o^N ($p > 0$) and T_d^N ($p < 0$), $N = 1, 2, 3$ (with parameters as in (1.9) and $\mu = 1$). The dashed lines indicate the projections of the leading order approximations of the fast N-circuit 'jumps' from ℓ^U to ℓ^S through the 4-dimensional phase space ($N = 1$ (right), 2 (middle), 3 (left)).

$\xi \gg 1/\varepsilon$. However, it is not yet clear whether $\gamma_h^N(\xi)$ approaches the critical point $S = (0,0,0,0)$ as $|\xi| \to \infty$, since S is on the boundary of $\mathcal{M}$. Recall that the line $\{u = 0\}$ had to be excluded from the definition of $\mathcal{M}$ (2.4), since the vector field is not smooth as $u \to 0$ (by (2.5): $\alpha_2 < 0$, and $\alpha_1 < 0$ is also possible; see also Remark 1.3).

Nevertheless, the methods of geometric singular perturbation theory can be and have been applied in the half space $\{u \geq \delta\}$ for any $0 < \delta \ll 1$. The above analysis shows that $\gamma_h^N(\xi)$ intersects the hyperplane $\{u = \delta\}$ twice for any $0 < \delta \ll 1$. At the intersection points, $v_h^N(\xi)$ and $|q_h^N(\xi)|$ are at least $\mathcal{O}(\exp(-1/\varepsilon))$ small, since $|\xi| \gg 1/\varepsilon$, due to the super slow flow on $\mathcal{M}$, and they decrease with the rate $\exp(-|\xi|)$. Moreover, $u_h^N(\xi)$ and $p_h^N(\xi)$ are $\mathcal{O}(\exp(-1/\varepsilon))$ close to orbits on ℓ^U or ℓ^S that decrease as $\exp(-\varepsilon^2|\xi|)$. Thus, since $\beta_i > 1$ (2.5), the 'dangerous' terms in the vector field (2.2), $u^{\alpha_i}v^{\beta_i}$, $i = 1, 2$, are also at least $\mathcal{O}(\exp(-1/\varepsilon))$ small. A local analysis of (2.2) near S shows that the unstable and stable manifolds of S, $W^U_{\text{loc}}(S)$ and $W^S_{\text{loc}}(S)$, do exist in a cusp shaped region $\Omega_{\text{cusp}} \subset \{u > 0\}$. By considering (for instance) Ω_{cusp} as the region bounded by the manifolds $\{v = 0\}$ and $\{v^2 + q^2 = u^{-2/\varepsilon}\}$ with $0 < u < 2\delta$ and $-\delta < p < \delta$, we establish an overlap region between the local and the global analysis. Therefore, $W^U_{\text{loc}}(S) \subset W^U(\mathcal{M})$ and $W^S_{\text{loc}}(S) \subset W^S(\mathcal{M})$ can both be extended as globally existing manifolds $W^U(S)$ and $W^S(S)$. The above global analysis (in the space $\{u \geq 0\}$) has shown that $\gamma^N_h(\xi) \subset W^U(S) \cap W^S(S) \neq \emptyset$. Hence, $\gamma^N_h(\xi) \to S$ as $|\xi| \to \infty$. $\square$

Remark 2.9. The choice to consider only *large, positive* solutions does not have an essential influence on the analysis in this section. One can, for instance, immediately formulate an existence theorem for 'small' N-loop homoclinic solutions to (2.2) – both $T^N_{d,o}$ can now become singular at $p \downarrow 0$, but $T^N_o \cap \ell^U$ always defines a unique point. The only difference is that for small loops one has $r, s < 0$ (1.6) and thus, by (2.9), the conditions on α_i flip. In fact, only the conditions on β_1 and β_2 are crucial for the existence of any type of singular homoclinic solutions. We have seen in the above analysis that violating the assumptions that $\beta_1 > 0$ and $\beta_2 > 1$ truly obstructs this construction (see also Remarks 2.4 and 3.2). The conditions on the signs of h_1 and h_2 are caused by our choice for positive (U, V)-solutions of (1.7) and (1.2), (1.1). The conditions on α_1 and α_2 are clearly determined by our 'preferences' for the signs of r and s.

Remark 2.10. For $h_1 < 0$, we found in Remark 2.8 that $T^N_{o,d}$ exists only for $N = 1$. However, it turns out that $T^N_o \cap \ell^U = \emptyset$, due to the sign change. Hence, there is no 1-loop homoclinic orbit at all. The intersection $T^N_o \cap \ell^S$ corresponds to an unbounded orbit that 'jumps' from ℓ^S to ℓ^U without getting close to S.

3. LINEAR STABILITY ANALYSIS

In Section 3.1 below we will derive the linearized stability problem associated to a general N-loop homoclinic pattern (see Theorem 2.1). However, in Sections 3.2, 4, and 5 we will focus on the stability analysis of the homoclinic 1-pulse patterns. The more general case of the homoclinic N-pulse solutions is considered in Section 6 on page 492.

Remark 3.1. The issue of nonlinear stability is a separate issue with its own subtleties due to the singularities in the nonlinear terms of (1.1), (1.2), (1.7) as $U \to 0$, as was explained in Remark 1.3. These singularities occur due to the conditions on α_1 and α_2 in (2.5) which arise from our choice to consider large solutions (see Remark 2.9). The linear stability theory to be developed in this paper can also be applied to the small solutions (or the solutions of 'mixed type', Remark 1.1), where the α_1 and α_2 are not necessarily negative. Hence, in such cases, the nonlinear terms will not be singular and one can apply the nonlinear stability results of (for instance) [18].

3.1. The linearized equations. Let $(U_0(\xi), V_0(\xi))$ be a homoclinic N-pulse associated to $\gamma^N_h(\xi)$ described in Theorem 2.1. We introduce $u(\xi)$, $v(\xi)$, and λ

by

$$U(\xi,t) = U_0(\xi) + u(\xi)e^{\lambda t}, \quad V(\xi,t) = V_0(\xi) + v(\xi)e^{\lambda t},$$

where $(U(\xi,t), V(\xi,t))$ is a solution of (1.7) in the fast spatial scale ξ. The linearized equations for (u,v) read

$$\begin{aligned} u_{\xi\xi} &= -\varepsilon^2[(\alpha_1 h_1 U_0^{\alpha_1-1}V_0^{\beta_1} + \hat{\varepsilon}R_1(U_0,V_0))u \\ &\qquad + (\beta_1 h_1 U_0^{\alpha_1}V_0^{\beta_1-1} + \hat{\varepsilon}S_1(U_0,V_0))v] + \varepsilon^4(\mu+\lambda)u, \\ v_{\xi\xi} &+ [\beta_2 h_2 U_0^{\alpha_2}V_0^{\beta_2-1} + \hat{\varepsilon}S_2(U_0,V_0) - (1+\lambda)]v \\ &\qquad = -[\alpha_2 h_2 U_0^{\alpha_2-1}V_0^{\beta_2} + \hat{\varepsilon}R_2(U_0,V_0)]u, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} R_i(U_0,V_0) &= \alpha_i H_i(U_0,V_0)U_0^{\alpha_i-1}V_0^{\beta_i} + h_i U_0^{\alpha_i}V_0^{\beta_i}\frac{\partial H_i}{\partial u}(U_0,V_0), \\ S_i(U_0,V_0) &= \beta_i H_i(U_0,V_0)U_0^{\alpha_i}V_0^{\beta_i-1} + h_i U_0^{\alpha_i}V_0^{\beta_i}\frac{\partial H_i}{\partial v}(U_0,V_0), \end{aligned} \tag{3.2}$$

for $i = 1, 2$. Note that $u(\xi)$ remains constant to leading order on ξ-intervals of length $\ll \mathcal{O}(1/\varepsilon)$. This system can be written in vector form

$$\dot{\varphi} = A(\xi;\lambda,\varepsilon)\varphi, \tag{3.3}$$

where, with abuse of notation (see Section 2), $\varphi(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi))^t$ and

$$A(\xi;\lambda,\varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ -\varepsilon[\alpha_1 h_1 U_0^{\alpha_1-1}V_0^{\beta_1}+\hat{\varepsilon}R_1]+\varepsilon^3[\mu+\lambda] & 0 & -\varepsilon[\beta_1 h_1 U_0^{\alpha_1}V_0^{\beta_1-1}+\hat{\varepsilon}S_1] & 0 \\ 0 & 0 & 0 & 1 \\ -[\alpha_2 h_2 U_0^{\alpha_2-1}V_0^{\beta_2}+\hat{\varepsilon}R_2] & 0 & -[\beta_2 h_2 U_0^{\alpha_2}V_0^{\beta_2-1}+\hat{\varepsilon}S_2]+(1+\lambda) & 0 \end{pmatrix}. \tag{3.4}$$

Taking into account that the fast pulse $V_0(\xi)$ decays much faster than $U_0(\xi)$, we can take the limits $|\xi| \to \infty$ in $A(\xi)$ (using (2.6)). The result is the constant coefficient matrix

$$(3.5) \qquad A_\infty(\lambda, \varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon^3[\mu + \lambda] & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (1+\lambda) & 0 \end{pmatrix}.$$

Using the structure of $(U_0(\xi), V_0(\xi))$ it can be concluded by (2.6) that there exist positive, $\mathcal{O}(1)$, constants C_1 and C_2 such that

$$(3.6) \qquad \|A(\xi; \lambda, \varepsilon) - A_\infty(\lambda, \varepsilon)\| \leq C_1 e^{-C_2|\xi|} \quad \text{for } |\xi| > 1/\varepsilon^\sigma, \ \sigma > 0,$$

see Remark 3.2. Thus, $A(\xi)$ is exponentially close to A_∞ as soon as $|\xi|$ is algebraically large in $1/\varepsilon$.

The eigenvalues of $A_\infty(\lambda, \varepsilon)$ are given by

$$(3.7) \qquad \Lambda_{1,4}(\lambda) = \pm\sqrt{1+\lambda}, \quad \Lambda_{2,3}(\lambda, \varepsilon) = \pm\varepsilon^2\sqrt{\mu + \lambda},$$

so that $\mathrm{Re}(\Lambda_1(\lambda)) > \mathrm{Re}(\Lambda_2(\lambda, \varepsilon)) > \mathrm{Re}(\Lambda_3(\lambda, \varepsilon)) > \mathrm{Re}(\Lambda_4(\lambda))$. The associated eigenvectors are given by

$$(3.8) \qquad E_{1,4}(\lambda) = (0, 0, 1, \pm\sqrt{1+\lambda})^t, \quad E_{2,3}(\lambda, \varepsilon) = (1, \pm\varepsilon\sqrt{\mu+\lambda}, 0, 0)^t.$$

As a consequence, we find for the essential spectrum of the linear eigenvalue problem (3.1), that

$$(3.9) \qquad \sigma_{\mathrm{ess}} \subset \{\lambda \in \mathbb{R} : \lambda \leq \max(-\mu, -1)\}.$$

Since $\mu > 0$, we conclude that the stability of the pattern (U_0, V_0) is determined by the discrete spectrum of (3.1). We denote the complement of the essential spectrum by

$$(3.10) \qquad C_e = \mathbb{C} \setminus \{\lambda \in \mathbb{R} : \lambda \leq \max(-\mu, -1)\}.$$

Remark 3.2. Note that it again (Remark 2.4) has not been necessary to impose $\beta_1 > 1$ in order to obtain (3.6). For $0 < \beta_1 < 1$ the (2, 3)-coefficient of $A(\xi)$ will become algebraically large for $|\xi|$ logarithmically large (i.e., ξ is smaller than $1/\varepsilon^\sigma$ for all $\sigma > 0$). As soon as $|\xi|$ is algebraically large, V_0 is exponentially small, so that the original *unscaled* V in (1.2) is also (exponentially) small. Conditions (2.6) are thus enough to guarantee that (3.6) holds (in other words:

in the case $0 < \beta_1 < 1$ there is a singularity in S_1 (3.2) that cancels the growth of the $U_0^{\alpha_1}V_0^{\beta_1-1}$-term for algebraically large ξ). Nevertheless, with $\beta_1 > 1$ it is straightforward to obtain uniform bounds on the coefficients of $A(\xi;\lambda,\varepsilon)$ as $\varepsilon \to 0$, which will be necessary in the proof of Lemma 4.2. We do not go further here into the possibility of relaxing the condition on β_1.

Remark 3.3. In this paper we do not consider the case that $0 < \mu \to 0$ as $\varepsilon \to 0$ in (1.2), or, more generally, a_{ij} with $a_{22} < 0$, $a_{11} + a_{22} < 0$, and $0 < a_{11}a_{22} - a_{12}a_{21} \to 0$ as $\varepsilon \to 0$ in (1.1) – see Remark 1.2. In these cases, the essential spectrum approaches 0 (from below) as $\varepsilon \to 0$. Hence, one has to take the essential spectrum into account in the stability analysis (by extending the Evans function into the essential spectrum, see [14], [23], and [7] for more details).

3.2. The Evans function and its decomposition. An eigenfunction associated to a point in the discrete spectrum of (3.3) decays exponentially fast as $|\xi| \to \infty$. Since $A(\xi)$ is exponentially close to A_∞ for algebraically large $|\xi|$, we can give a more precise description of the behavior of bounded solutions to (3.3) for large $|\xi|$. We first consider the case $\xi \ll -1$.

Lemma 3.4. *For all $\lambda \in C_e$ there is a two-dimensional family of solutions $\Phi_-(\xi;\lambda,\varepsilon)$ to* (3.3) *such that* $\lim_{\xi\to-\infty} \varphi_-(\xi;\lambda,\varepsilon) = (0,0,0,0)^t$ *for all* $\varphi_-(\xi;\lambda,\varepsilon) \in \Phi_-(\xi;\lambda,\varepsilon)$. $\Phi_-(\xi;\lambda,\varepsilon)$ *depends analytically on* λ.

Proof. This result is very natural: $A(\xi) \to A_\infty$ and limit system $\dot{\tilde{\varphi}} = A_\infty\tilde{\varphi}$ has a two-dimensional unstable subspace (spanned by $E_i e^{\Lambda_i\xi}$, $i = 1, 2$). See [1, Lemma 3.3] for the details. ∎

Of course, a similar result can be formulated for $\xi > 1/\varepsilon^\sigma$ and functions $\varphi_+(\xi;\lambda,\varepsilon) \in \Phi_+(\xi;\lambda,\varepsilon)$. An eigenfunction $\varphi_e(\xi)$ is an element of $\Phi_-(\xi;\lambda,\varepsilon) \cap \Phi_+(\xi;\lambda,\varepsilon)$. This observation is the starting point for the definition of the Evans function, $\mathcal{D}(\lambda,\varepsilon)$, associated to (3.3) (see [1], [13], [7] for more details). The Evans function $\mathcal{D}(\lambda,\varepsilon)$ is defined by

$$\mathcal{D}(\lambda,\varepsilon) = \det[\varphi_1(\xi;\lambda,\varepsilon), \varphi_2(\xi;\lambda,\varepsilon), \varphi_3(\xi;\lambda,\varepsilon), \varphi_4(\xi;\lambda,\varepsilon)], \tag{3.11}$$

where $\{\varphi_1,\varphi_2\}$ (respectively $\{\varphi_3,\varphi_4\}$) span the space Φ_- (resp. Φ_+). It is shown in [1] that the Evans function is analytic in λ for λ outside of the essential spectrum (i.e., $\lambda \in C_e$) and that the zeroes of $\mathcal{D}(\lambda,\varepsilon)$ correspond to eigenvalues of (3.3) – counting multiplicities. Note that $\mathcal{D}(\lambda,\varepsilon)$ does not depend on ξ since $\mathrm{Tr}(A) \equiv 0$. In general, there is a certain freedom of choice in $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\varphi_3(\xi)$, $\varphi_4(\xi)$. However, here we will determine them uniquely.

Lemma 3.5. *For all $\lambda \in C_e$ there is a unique solution $\varphi_1(\xi;\lambda,\varepsilon) \in \Phi_-(\xi;\lambda,\varepsilon)$ of* (3.3) *such that*

$$\lim_{\xi\to-\infty} \varphi_1(\xi;\lambda,\varepsilon)e^{-\Lambda_1(\lambda)\xi} = E_1(\lambda). \tag{3.12}$$

Proof. As in the case of Lemma 3.4, see [1, Lemma 3.3] for the details. □

These two results hold for any N-homoclinic pattern $(U_0(\xi), V_0(\xi))$. However, we need to distinguish between the cases $N = 1$ and $N \geq 2$ to define $\varphi_2(\xi)$. This is because we initially have to exclude certain regions from the domain C_e (3.10) of potential eigenvalues. In order to do so, we first need to consider the fast reduced scalar equation associated to (1.7):

$$V_t = V_{\xi\xi} - V + h_2 u_{h,N}^{\alpha_2} V^{\beta_2}, \tag{3.13}$$

where $u_{h,N}$ (2.45) is the (leading order) constant value of $U_0(\xi)$ during the fast excursion(s) of $V_0(\xi)$. When $N = 1$ we can approximate $V_0(\xi)$ by the stationary homoclinic solution $V(\xi) = v_h(\xi; u_{h,1})$ (2.41) of this equation (that exists under the conditions of Theorem 2.1). This is not the case for $N \geq 2$: $V_0(\xi)$ can be seen, to leading order, to be N copies of the homoclinic solution $v_h(\xi; u_{h,N})$ of (3.13). This situation is technically a bit more involved, therefore we focus on the case $N = 1$ in this (sub)section, and in Sections 4 and 5. The case $N \geq 2$ is considered in Section 6 on page 492.

The linearized stability problem associated to $v(\xi) = V(\xi) - v_h(\xi; u_{h,1})$ (2.41), (2.45) can be written as

$$\dot{\psi} = A_f(\xi; \lambda)\psi, \quad \text{where } \psi(\xi) = (v(\xi), \dot{v}(\xi)). \tag{3.14}$$

Here, $A_f(\xi; \lambda)$ = the lower diagonal 2×2 block of $A(\xi; \lambda, \varepsilon)$ (3.4) in the limit $\varepsilon \to 0$, with $U_0(\xi)$ replaced by $u_{h,1}$ and $V_0(\xi)$ by $v_h(\xi)$. The Evans function associated to this problem can be written as

$$\mathcal{D}_f(\lambda) = \det[\psi_1(\xi, \lambda), \psi_4(\xi, \lambda)], \tag{3.15}$$

where

$$\lim_{\xi \to -\infty} \psi_1(\xi) e^{-\Lambda_1 \xi} = (1, \sqrt{1+\lambda})^t \quad \text{and} \quad \lim_{\xi \to \infty} \psi_4(\xi) e^{-\Lambda_4 \xi} = (1, -\sqrt{1+\lambda})^t$$

(compare to $E_{1,4}$ (3.8)). This (stationary) solution is unstable, there is *one* real eigenvalue $\lambda_f^* > 0$. We denote the set of all eigenvalues of (3.14) by λ_f^j, $j = 0, 1, \ldots, J$, where $\lambda_f^0 = \lambda_f^*$, $\lambda_f^1 = 0$, and $-1 < \lambda_f^j < \lambda_f^{j-1} \leq 0$ for $J \geq j \geq 2$ (both $J = J(\beta_2)$ and λ_f^j will be explicitly determined in Proposition 5.6). Note that $\mathcal{D}_f(\lambda_f^j) = 0$ for $j = 0, \ldots, J$.

We introduce a second asymptotically small parameter, $0 < \delta \ll 1$ that is independent of ε, and define $B_{\lambda_f^j, \delta}$ as the disk around the eigenvalue λ_f^j of (3.14) with radius δ ($j = 0, \dots, J$). Moreover, we introduce the complement of a δ-neighborhood of the essential spectrum C_e:

$$C_\delta = \mathbb{C} \setminus \{\lambda \in \mathbb{C} : (\mathrm{Re}(\lambda) < \max(-\mu, -1),\ |\mathrm{Im}(\lambda)| < \delta) \text{ or } \|\lambda - \max(-\mu, -1)\| < \delta\}. \tag{3.16}$$

Clearly, $C_\delta \subset C_e$.

Lemma 3.6. *Let $\lambda \in C_\delta$, then*

$$\lim_{\xi \to \infty} \varphi_1(\xi; \lambda, \varepsilon) e^{-\Lambda_1(\lambda)\xi} = t_1(\lambda, \varepsilon) E_1(\lambda), \tag{3.17}$$

where $t_1(\lambda, \varepsilon)$ is an analytic (transmission) function of $\lambda \in C_\delta$. Moreover, $t_1(\lambda, \varepsilon) \neq 0$ for $\lambda \in C_\delta \setminus \{\bigcup_{j=0}^{J} B_{\lambda_f^j, \delta}\}$.

Proof. This result is again quite natural: in general a solution to (3.3) will grow as $e^{\Lambda_1 \xi}$ as $\xi \to \infty$. This observation can be made precise by the 'elephant trunk' procedure [13, 7] for $\varphi_1(\xi)$. By construction, in the limit $\varepsilon \to 0$, the (v, q)-part of $\varphi_1(\xi)$ approaches the solution $\psi_1(\xi)$ of the fast reduced eigenvalue problem (3.14) defined in (3.15). Since $\psi_1(\xi) = \psi_4(\xi)$ at $\lambda = \lambda_f^j$, an eigenvalue of (3.14), $\psi_1(\xi)$ will not grow like $e^{\Lambda_1 \xi}$ at $\lambda = \lambda_f^j$: the elephant trunk procedure cannot be applied for λ close to λ_f^j, in the sense that $\varphi_1(\xi)$ might also not grow like $e^{\Lambda_1 \xi}$ for $\xi \to \infty$ (here 'close' means in a neighborhood of λ_f^j that shrinks to 0 in the limit $\varepsilon \to 0$). Note, however, that this implies that $t_1(\lambda, \varepsilon) \neq 0$ for $\lambda \in C_\delta$ and not in a $B_{\lambda_f^j, \delta}$ disk, while $t_1(\lambda, \varepsilon)$ can be zero for λ close to λ_f^j (see also Section 4.1). Since the technical details are very similar to the corresponding results in [13, 7], we omit the details here. $\square$

Note that Lemmas 3.5 and 3.6 have natural counterparts that describe the existence of the unique solution $\varphi_4(\xi; \lambda, \varepsilon)$ and its behavior for $\xi \to \infty$ and $\xi \to -\infty$.

Lemma 3.7. *Let $\lambda \in C_\delta$ be such that $t_1(\lambda, \varepsilon) \neq 0$. There is a uniquely determined solution $\varphi_2(\xi; \lambda, \varepsilon) \in \Phi_-(\xi; \lambda, \varepsilon)$ such that*

$$\Phi_-(\xi; \lambda, \varepsilon) = \mathrm{span}\{\varphi_1(\xi; \lambda, \varepsilon), \varphi_2(\xi; \lambda, \varepsilon)\},$$

$$\lim_{\xi \to -\infty} \varphi_2(\xi; \lambda, \varepsilon) e^{-\Lambda_2(\lambda, \varepsilon)\xi} = E_2(\lambda, \varepsilon), \tag{3.18}$$

and

$$\lim_{\xi \to \infty} \varphi_2(\xi; \lambda, \varepsilon) e^{-\Lambda_1(\lambda)\xi} = (0, 0, 0, 0)^t. \tag{3.19}$$

Proof. Define the three-dimensional space $\Phi_{s,+}(\xi)$ of solutions $\varphi(\xi)$ to (3.3) by

$$\lim_{\xi\to\infty} \varphi(\xi)e^{-\Lambda_1(\lambda)\xi} = (0,0,0,0)^t$$

($\Phi_{s,+}(\xi)$ is three-dimensional by [1, Lemma 3.3], as in Lemmas 3.4, 3.5). Thus, by a dimension count, the intersection $\Phi_-(\xi;\lambda,\varepsilon) \cap \Phi_{s,+}(\xi)$ exists and is at least one-dimensional. Lemma 3.5 establishes the existence of the solution $\varphi_1(\xi;\lambda,\varepsilon) \in \Phi_-(\xi;\lambda,\varepsilon)$ that cannot be an element of $\Phi_{s,+}(\xi)$ by Lemma 3.6, since $t_1(\lambda,\varepsilon) \neq 0$. Hence, $\Phi_-(\xi;\lambda,\varepsilon)$ intersects $\Phi_{s,+}(\xi)$ transversally and the intersection is one-dimensional. The solution $\varphi_2(\xi;\lambda,\varepsilon)$ is now uniquely determined by the 'boundary conditions' (3.18), (3.19). ◻

The definition (3.11) of the Evans function $\mathcal{D}(\lambda,\varepsilon)$ is based on the functions $\varphi_1(\xi;\lambda,\varepsilon), \varphi_2(\xi;\lambda,\varepsilon)$ constructed above, and their natural counterparts $\varphi_3(\xi;\lambda,\varepsilon), \varphi_4(\xi;\lambda,\varepsilon)$. The above results also yield a natural fast-slow decomposition of $\mathcal{D}(\lambda,\varepsilon)$. This decomposition can be obtained by noticing that, by (3.19) and the limit behavior of $A(\xi)$ (3.6), there must be a transmission function $t_2(\lambda,\varepsilon)$ such that

$$\lim_{\xi\to\infty} \varphi_2(\xi;\lambda,\varepsilon)e^{-\Lambda_2(\lambda,\varepsilon)\xi} = t_2(\lambda,\varepsilon)E_2(\lambda,\varepsilon). \tag{3.20}$$

for all $\lambda \in C_\delta$ such that $t_1(\lambda,\varepsilon) \neq 0$. Thus, since $\mathcal{D}$ does not depend on ξ and $\sum_{i=1}^4 \Lambda_i(\lambda) \equiv 0$, we have:

$$\begin{aligned} \mathcal{D}(\lambda,\varepsilon) &= \lim_{\xi\to\infty} \det[\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi),\varphi_4(\xi)] \\ &= \lim_{\xi\to\infty} \det[\varphi_1(\xi)e^{-\Lambda_1\xi},\varphi_2(\xi)e^{-\Lambda_2\xi},\varphi_3(\xi)e^{-\Lambda_3\xi},\varphi_4(\xi)e^{-\Lambda_4\xi}] \\ &= \det[t_1E_1, t_2E_2, E_3, E_4] \\ &= 4\varepsilon t_1(\lambda,\varepsilon)t_2(\lambda,\varepsilon)\sqrt{(\mu+\lambda)(1+\lambda)}, \end{aligned} \tag{3.21}$$

where $t_1(\lambda,\varepsilon)$, $t_2(\lambda,\varepsilon)$ have been defined in (3.17), (3.20).

Corollary 3.8. *The eigenvalues of* (3.3) *in the region* C_δ *coincide with the roots of* $\mathcal{D}(\lambda,\varepsilon)$*, counting multiplicities. Furthermore, the set of eigenvalues in this region lies in the union of the roots of the coefficients* $t_1(\lambda,\varepsilon)$ *and* $t_2(\lambda,\varepsilon)$ *defined in* (3.17) *and* (3.20)*, respectively.*

The first statement is a consequence of a general theorem in [1]. The second statement is obvious from (3.21). However, we will see below that a zero of $t_1(\lambda,\varepsilon)$ does not necessarily imply that $\mathcal{D}(\lambda,\varepsilon) = 0$. A similar 'NLEP paradox' has been studied in [7] in the analysis of pulses in the Gray-Scott model.

The fast reduced Evans function $\mathcal{D}_f(\lambda)$ (3.15) can also be transformed as $\mathcal{D}(\lambda, \varepsilon)$ in (3.21), using the obvious fact that there must exist an analytic (transmission) function $t_f(\lambda)$ such that $\psi_1(\xi)e^{-\Lambda_1(\lambda)} \to t_f(\lambda)(1, \sqrt{1+\lambda})^t$ as $\xi \to \infty$:

$$\begin{aligned} \mathcal{D}_f(\lambda) &= \lim_{\xi\to\infty} \det[\psi_1(\xi), \psi_4(\xi)] \\ &= \det[t_f(\lambda)(1, \sqrt{1+\lambda})^t, (1, -\sqrt{1+\lambda})^t] \\ &= -2t_f(\lambda)\sqrt{1+\lambda}. \end{aligned} \tag{3.22}$$

4. THE NLEP APPROACH

4.1. The NLEP paradox and its resolution. The fast-slow decomposition of $\mathcal{D}(\lambda, \varepsilon)$ obtained in the previous section has become quite standard in singularly perturbed systems (see [1], [13], [19], [7] and the references cited there). However, here this decomposition of $\mathcal{D}(\lambda, \varepsilon)$ exhibits the singular behavior identified above as 'the NLEP paradox' (see also [7]). This behavior is closely related to the fact that the 'elephant trunk' procedure might fail near eigenvalues of the eigenvalue problem (3.14) associated to the homoclinic solution $v_h(\xi)$ of the fast reduced limit (3.13) – see the proof of Lemma 3.6. As was already noted there: (3.17) holds for any $\lambda \in C_\delta$, and $t_1(\lambda) = 0$ when λ is such that $\varphi_1(\xi)$ does not grow as $e^{\Lambda_1\xi}$ for $\xi \to \infty$. This is exactly what happens near λ_f^* (the unstable eigenvalue of the fast reduced limit).

Lemma 4.1. *There is a unique* $\lambda^*(\varepsilon) \in \mathbb{R}$ *such that* $\lim_{\varepsilon\to 0} \lambda^*(\varepsilon) = \lambda_f^* > 0$ *and* $t_1(\lambda^*(\varepsilon), \varepsilon) = 0$ (*with multiplicity* 1).

Proof. Consider a contour $K_\delta \in C_\delta$ around λ_f^* such that K_δ is independent of ε, has the disk $B_{\lambda_f^*,\delta}$ in its interior, and encloses no other eigenvalues of (3.14). Since $t_1(\lambda, \varepsilon)$ is analytic in $\lambda \in K_\delta$ (Lemma 3.6), we can define the winding number $W(t_1; K_\delta)$ of $t_1(\lambda)$ with respect to K_δ. Likewise, we define $W(\mathcal{D}_f; K_\delta)$, the winding number over K_δ of the Evans function $\mathcal{D}_f(\lambda)$ (3.15) associated to the fast reduced problem (3.14). Since $\mathcal{D}_f(\lambda_f^*) = 0$ with multiplicity 1 (see also Proposition 5.6), we know that $W(\mathcal{D}_f; K_\delta) = 1$. By construction $2t_1(\lambda, \varepsilon)\sqrt{1+\lambda}$ approaches $2t_f(\lambda)\sqrt{1+\lambda} = -\mathcal{D}_f(\lambda)$ (3.22) in the limit $\varepsilon \to 0$ (see [1], [13], [7] for the technical details). Thus, since the winding number is a topological invariant, $W(t_1(\lambda, \varepsilon); K_\delta) = W(\mathcal{D}_f(\lambda); K_\delta) = 1$ for all possible K_δ and ε small enough. The zero $\lambda^*(\varepsilon)$ must be real since $t_1(\lambda, \varepsilon)$ is defined in terms of real expressions (and a pair of complex conjugate zeroes would give $W(t_1(\lambda, \varepsilon); K_\delta) = 2$). □

The NLEP paradox, where NLEP stands for *NonLocal Eigenvalue Problem* (see below), can now be formulated as follows.

> *A homoclinic solution* $(U_0(\xi), V_0(\xi))$ *to* (1.7) *that corresponds to an unstable solution with an* $\mathcal{O}(1)$ *positive eigenvalue* λ_f^* *of the fast reduced scalar equation* (3.13) *can be stable* (*with* $\mathcal{O}(1)$ *eigenvalues* λ *with* $\mathrm{Re}(\lambda) < 0$).

Or, in more technical terms:

> *Although* $\mathcal{D}(\lambda, \varepsilon) = 4\varepsilon t_1(\lambda, \varepsilon) t_2(\lambda, \varepsilon)\sqrt{(\mu + \lambda)(1 + \lambda)}$ *and there exists a* $\lambda^*(\varepsilon)$ *such that* $t_1(\lambda^*(\varepsilon), \varepsilon) = 0$, $\mathcal{D}(\lambda^*(\varepsilon), \varepsilon) \neq 0$.

The resolution of this 'paradox' lies in the definition of $\varphi_2(\xi; \lambda, \varepsilon)$. It follows from Lemma 3.7 that $t_2(\lambda, \varepsilon)$ can be defined for all $\lambda \in C_\delta$ such that $t_1(\lambda, \varepsilon) \neq 0$, but it is a priori not clear what happens at (or near) $\lambda^*(\varepsilon)$: we need to determine an expression for $t_2(\lambda, \varepsilon)$ so that we can study the behavior of $t_2(\lambda, \varepsilon)$ near $\lambda^*(\varepsilon)$. This is done by the so-called NLEP approach, as was originally developed in [6] for the Gray-Scott model. Like in [7] (for the Gray-Scott model), we will find that $t_2(\lambda, \varepsilon)$ has a pole of order 1 at $\lambda = \lambda^*(\varepsilon)$ which yields the resolution to the NLEP paradox. Note that since both $\mathcal{D}(\lambda, \varepsilon)$ and $t_1(\lambda, \varepsilon)$ are analytic in C_δ ([1] and Lemma 3.6), it follows by (3.21) that $t_2(\lambda, \varepsilon)$ is at least meromorphic as function of λ in C_δ.

Apart from the necessity to understand the behavior of $\mathcal{D}(\lambda, \varepsilon)$ near $\lambda^*(\varepsilon)$, there is another reason to search for a more explicit expression of $t_2(\lambda, \varepsilon)$. It follows from Lemma 3.6 that $t_1(\lambda)$ can only be zero near an eigenvalue of the fast reduced limit system (3.14). Therefore, all possible eigenvalues of (3.3) that are not close to a reduced fast eigenvalue must be zeroes of $t_2(\lambda, \varepsilon)$. We will even find that all 'dangerous' eigenvalues of (3.3) correspond to zeroes of t_2.

We first need to derive an approximation of $\varphi_2(\xi)$. The leading order behavior of $\varphi_2(\xi)$ for algebraically large $|\xi|$ is described by the following lemma.

Lemma 4.2. *Suppose that* (2.5) *and* (2.6) *hold. Fix* $\lambda \in C_\delta$ (3.16) *such that* $t_1(\lambda, \varepsilon) \neq 0$ *and let* $\varphi_2(\xi) = (u_2(\xi), p_2(\xi), v_2(\xi), q_2(\xi))$ *be defined by Lemma 3.7. Define the interval* I_t *by*

$$I_t = \left[-\frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}\right]. \tag{4.1}$$

(a) *There exists an* $\mathcal{O}(1)$ *constant* $C_- > 0$ *such that*

$$\varphi_2(\xi; \lambda, \varepsilon) = E_2(\lambda, \varepsilon) e^{\Lambda_2(\lambda, \varepsilon)\xi} + \mathcal{O}(e^{C_- \xi}) \quad \text{for } \xi < -1/\sqrt{\varepsilon}. \tag{4.2}$$

(b) *There exists an* $\mathcal{O}(1)$ *constant* $C_t > 0$ *such that* $\|\varphi_2(\xi)\| \leq C_t$ *for* $\xi \in I_t$ *and*

$$u_2(\xi) = 1 + \mathcal{O}(\hat{\varepsilon}, \sqrt{\varepsilon}) \quad \text{for } \xi \in I_t. \tag{4.3}$$

(c) *Let* $t_2(\lambda, \varepsilon)$ *be defined as in* (3.20). *There exists a meromorphic function* $t_3(\lambda, \varepsilon)$ *and an* $\mathcal{O}(1)$ *constant* $C_+ > 0$, *such that*

$$t_2(\lambda, \varepsilon) + t_3(\lambda, \varepsilon) = 1 + \mathcal{O}(\hat{\varepsilon}, \sqrt{\varepsilon}) \quad \text{for } \xi \in I_t, \tag{4.4}$$

and

$$(4.5)\quad \varphi_2(\xi;\lambda,\varepsilon) = t_2(\lambda,\varepsilon)E_2(\lambda,\varepsilon)e^{\Lambda_2(\lambda,\varepsilon)\xi} + t_3(\lambda,\varepsilon)E_3(\lambda,\varepsilon)e^{\Lambda_3(\lambda,\varepsilon)\xi} + \mathcal{O}(e^{-C_+\xi}) \quad \text{for } \xi > 1/\sqrt{\varepsilon}.$$

The constants $C_\pm$ and C_t depend only on the distance of λ to the roots of the analytic function $t_1(\lambda,\varepsilon)$. In fact, these constants might become unbounded as λ approaches a root of $t_1(\lambda,\varepsilon)$. However, for small but fixed ε, they are uniformly bounded in any compact subdomain of C_δ that is disjoint from the set of roots of $t_1(\lambda,\varepsilon)$. This uniform boundedness is the essential property for the results here and below.

Proof. The conditions on $G_i(V)$ and β_i ($i = 1, 2$) are required in order to ensure that the coefficients of $A(\xi)$ are uniformly bounded as $\varepsilon \to 0$ (see also Remark 3.2). The proof of Lemma 4.2 is lengthy. The main points are to first track the behavior of $\varphi_2(\xi)$ over both the left and the right slow fields, $|\xi| \geq 1/\sqrt{\varepsilon}$, and then, to use this information to determine its behavior over the fast field I_t, to leading order (note that the boundary between the fast and the slow fields is not uniquely determined, the length of I_t could be $\mathcal{O}(\varepsilon^{-\sigma})$ for some $\sigma > 0$). The details are similar to and simpler than those presented in in [7, Lemmas 4.4-4.7] for the Gray-Scott model, and for brevity's sake they will be omitted here. There is however one important difference between the approach used here and that used in [7]. For those readers interested in referring to [7], we briefly outline the differences. It is then an easy matter to see how the estimates in [7] apply to the present situation.

Here, the transmission coefficients t_1, t_2 are defined in terms of the limiting behavior of $\varphi_{1,2}$ at $\xi = +\infty$. This is no longer possible inside the essential spectrum of the wave, and in the Gray-Scott model the essential spectrum is asymptotically close to the origin (see also Remark 3.3). In order to work in a fixed region including $\{\lambda = 0\}$ in its interior, it was therefore necessary to obtain estimates inside the essential spectrum and to use the Gap Lemma to analytically continue the Evans function into this region [14], [23]. The slow transmission coefficient t_2 must then be defined in a different manner. In [7], this coefficient is determined by the φ_2 solution immediately after it emerges from the fast field at $\xi = +1/\xi^\sigma$ (for some $\sigma > 0$), not at $\xi = +\infty$. Indeed, the latter definition would give the wrong result inside the essential spectrum. Nevertheless, outside the essential spectrum the difference between the transmission coefficients obtained from these two distinct definitions is exponentially small in ε and can therefore be neglected. □

The next task is to determine the leading order behavior of t_2. From the above, it suffices to determine the leading order behavior of the components of φ_2 as the solution emerges from the fast field at $1/\sqrt{\varepsilon}$. To this end, we consider the linearized problem (3.3) once again as the coupled system of two second order equations (3.1). We know by Lemma 4.2 that $u_2(\xi) = 1$ to leading order

(4.3). This means that the v-equation in (3.1) decouples from the u-equation (to leading order):

$$v_{\xi\xi} + [\beta_2 h_2 U_0^{\alpha_2} V_0^{\beta_2 - 1} - (1+\lambda)]v = -\alpha_2 h_2 U_0^{\alpha_2 - 1} V_0^{\beta_2} \quad \text{for } \xi \in I_t, \tag{4.6}$$

where the right hand side is now an explicitly known inhomogeneous term. Furthermore, $U_0(\xi)$ can be approximated by the constant $u_{h,1}$ (2.45) in I_t, V_0 by the homoclinic solution $v_h(\xi; u_{h,1})$ (2.41) of the fast reduced limit (3.13) and, I_t by $\mathbb{R}$ due to the exponential decay. These approximations can also be inserted in (4.6). We thus recover on the left hand side the linear operator associated to the fast reduced system (3.14). Note that the leading order corrections are $\mathcal{O}(\sqrt{\varepsilon}, \hat{\varepsilon})$.

The $\mathcal{O}(1)$ problem is of Sturm-Liouville type:

$$\begin{aligned}(\mathcal{L}_f(\xi) - \lambda)v &= v_{\xi\xi} + [\beta_2 h_2 (u_{h,1})^{\alpha_2} (v_h(\xi))^{\beta_2 - 1} - (1+\lambda)]v \\ &= -\alpha_2 h_2 (u_{h,1})^{\alpha_2 - 1} (v_h(\xi))^{\beta_2}.\end{aligned} \tag{4.7}$$

Therefore, there exists a unique bounded solution $v_{in}(\xi;\lambda)$ for $\lambda \in C_\delta$ (3.16) and $\lambda \neq \lambda_f^j$, an eigenvalue of (3.14), see [40], and also Section 5.2, Appendix B on page 502) that is the leading order approximation of $v_2(\xi, \lambda)$ (by construction, see also [7]). The leading order (slow) behavior of $u_2(\xi)$ through I_t is thus governed by

$$\begin{aligned}u_{\xi\xi} = -\varepsilon^2 [&\alpha_1 h_1 (u_{h,1})^{\alpha_1 - 1} (v_h(\xi))^{\beta_1} \\ &+ \beta_1 h_1 (u_{h,1})^{\alpha_1} (v_h(\xi))^{\beta_1 - 1} v_{in}(\xi)]\end{aligned} \tag{4.8}$$

(recall that $u_2 = 1$ to leading order (4.3)). We can thus obtain an approximation for the total change in u_ξ through I_t:

$$\begin{aligned}\Delta_{\text{fast}} u_\xi = -\varepsilon^2 \int_{-\infty}^{\infty} &[\alpha_1 h_1 (u_{h,1})^{\alpha_1 - 1} (v_h(\xi))^{\beta_1} \\ &+ \beta_1 h_1 (u_{h,1})^{\alpha_1} (v_h(\xi))^{\beta_1 - 1} v_{in}(\xi)]\, d\xi.\end{aligned} \tag{4.9}$$

Note that the integration should be over $I_t \subset (-\infty, \infty)$, however this does not have a leading order effect. This expression should be equal to the change (or 'jump') in u_ξ over I_t as is described by the slow evolution outside I_t. Lemma 4.2 yields, to leading order,

$$\begin{aligned}\Delta_{\text{slow}} u_\xi &= u_\xi\left(\frac{1}{\sqrt{\varepsilon}}\right) - u_\xi\left(-\frac{1}{\sqrt{\varepsilon}}\right) \\ &= [\Lambda_2(\lambda,\varepsilon) t_2(\lambda,\varepsilon) + \Lambda_3(\lambda,\varepsilon) t_3(\lambda,\varepsilon)] - \Lambda_2(\lambda,\varepsilon).\end{aligned} \tag{4.10}$$

By the combination of the condition $\Delta_{\text{fast}} u_\xi = \Delta_{\text{slow}} u_\xi$ with (4.4) we obtain an explicit *nonlocal* (leading order) expression for the transmission function $t_2(\lambda, \varepsilon)$ in terms of the solution v_{in} of the singular Sturm-Liouville problem (4.7):

$$
\begin{aligned}
(4.11) \quad t_2(\lambda, 0) = 1 - \frac{1}{2\sqrt{\mu + \lambda}} \int_{-\infty}^{\infty} \big[&\alpha_1 h_1 (u_{h,1})^{\alpha_1 - 1} (v_h(\xi))^{\beta_1} \\
&+ \beta_1 h_1 (u_{h,1})^{\alpha_1} (v_h(\xi))^{\beta_1 - 1} v_{in}(\xi)\big]\, d\xi.
\end{aligned}
$$

Note that the first order approximation term to (4.11) is $\mathcal{O}(\hat{\varepsilon}, \sqrt{\varepsilon})$.

Since $t_1(\lambda, \varepsilon)$ is an analytic function of λ for each fixed but small ε in the region C_δ and has isolated roots there, it follows that the function $t_2(\lambda, \varepsilon)$ is a meromorphic function of λ in this region for each *fixed* but small ε. This is because the constants $C_\pm$ and C_t in Lemma 4.2 depend only on the distance between λ and a root of $t_1(\lambda, \varepsilon)$, and this distance is uniform in ε for all sufficiently small ε, as stated above.

The calculation of $t_2(\lambda)$ is valid away from the eigenvalues λ_f^j of the fast reduced problem (3.14) inside C_δ. In order to complete the description of $t_2(\lambda)$, the behavior of $t_2(\lambda)$ near these eigenvalues must be determined. The inhomogeneous problem (4.7) can, in general, not be solved at an eigenvalue, since then the operator $\mathcal{L}_f(\xi) - \lambda$ is not invertible. The eigenfunction $v_f^*(\xi)$ of (3.14) at $\lambda = \lambda_f^*$ is positive and even ([40, Appendix B]), like $v_h(\xi)$. Thus, the solvability condition for the inhomogeneous problem at λ_f^*,

$$\int_{-\infty}^{\infty} (v_h(\xi))^{\beta_2} v_f^*(\xi)\, d\xi = 0,$$

cannot be satisfied, which yields that v_{in} has a simple pole as function of λ at λ_f^* (the Wronskian in the denominator of the Green's function associated to the operator $\mathcal{L}_f - \lambda$ has a simple zero at λ_f^*, see (B.4) for an explicit expression). Therefore, $t_2(\lambda, \varepsilon)$ must have a simple pole at $\lambda_p^*(\varepsilon)$, with $\lim_{\varepsilon \to 0} \lambda_p^*(\varepsilon) = \lambda_f^*$. Since $\mathcal{D}(\lambda, \varepsilon)$ is analytic in C_δ, it follows by (3.21) that $\lambda_p^*(\varepsilon) = \lambda^*(\varepsilon)$.

The transmission function $t_2(\lambda, \varepsilon)$ does not have a pole at the eigenvalue $\lambda_f^1 = 0$ of (3.14) in C_δ, since the fast reduced eigenfunction at $\lambda = 0$, $\dot{v}_{h,1}$, is odd so that

$$\int_{-\infty}^{\infty} (v_h(\xi))^{\beta_2} \dot{v}_{h,1}\, d\xi = 0,$$

i.e., v_{in} exists at $\lambda = 0$ but is not uniquely determined. Nevertheless, $t_2(\lambda, \varepsilon)$ is well-defined and analytic at $\lambda = 0$, see the proof of the more general result, Corollary 4.4.

We summarize the above results in the following lemma.

Lemma 4.3. *The transmission function* $t_2(\lambda, \varepsilon)$ *is meromorphic as function of* λ *for* $\lambda \in C_\delta$*. It has a pole of order* 1 *at the point* $\lambda^*(\varepsilon)$*, and is analytic for all* λ *with* $\text{Re}(\lambda) > -\delta$*,* $\lambda \neq \lambda^*(\varepsilon)$*. The leading order behavior of* $t_2(\lambda)$ *is given by the expression in* (4.11)*, where* $v_{in}(\xi)$ *is the unique bounded solution of* (4.7)*. This approximation is accurate to* $\mathcal{O}(\hat{\varepsilon}, \sqrt{\varepsilon})$*.*

This yields the resolution to the NLEP paradox: $\mathcal{D}(\lambda, \varepsilon) = Ct_1(\lambda, \varepsilon)t_2(\lambda, \varepsilon) \neq 0$ at $\lambda^*(\varepsilon)$ for a certain positive constant C. The solution $(U_0(\xi), V_0(\xi))$ is not necessarily unstable. Moreover, $\mathcal{D}(0, \varepsilon) \equiv 0$, as it should be. We will see in Section 4.2 on the next page that $t_2(0, \varepsilon) \neq 0$: $\lambda = 0$ is always a simple eigenvalue of (3.3).

Transmission function $t_1(\lambda, \varepsilon)$ has been defined for all $\lambda \in C_\delta$ (3.17), transmission function $t_2(\lambda, \varepsilon)$ has been defined for all $\lambda \in C_\delta$ such that $t_1(\lambda, \varepsilon) \neq 0$ (3.20). The above observed behavior of $t_2(\lambda; \varepsilon)$ near the eigenvalues λ_f^* and 0 occurs at every eigenvalue of the fast reduced eigenvalue equation (3.14). Recall that λ_f^j, $j = 0, 1, \ldots, J$, with $\lambda_f^0 = \lambda_f^*$, $\lambda_f^1 = 0$ and $-1 < \lambda_f^j < \lambda_f^{j-1} \leq 0$ for $J \geq j \geq 2$, denote the set of all eigenvalues of (3.14) (see Proposition 5.6). The associated eigenfunctions $v_f^j(\xi)$ are even functions of ξ when j is even and odd when j is odd ((3.14) is symmetric with respect to $\xi \to -\xi$, see Remark B.4 in Appendix B). Thus, the above observations can be generalized by the following corollary.

Corollary 4.4. *Let* $\lambda_f^j > \max(-1, -\mu)$ *be an eigenvalue of* (3.14)*. There exists a* $\lambda^j(\varepsilon)$ *such that* $t_1(\lambda^j(\varepsilon), \varepsilon) = 0$ *and* $\lim_{\varepsilon \to 0} \lambda^j(\varepsilon) = \lambda_f^j$*. If* j *is even, then* $t_2(\lambda, \varepsilon)$ *has a pole of order* 1 *at* $\lambda^j(\varepsilon)$*, so that* $\mathcal{D}(\lambda^j(\varepsilon), \varepsilon) \neq 0$*. If* j *is odd, then* $t_2(\lambda, \varepsilon)$ *is analytic at* $\lambda^j(\varepsilon)$*, and* $\lambda^j(\varepsilon)$ *is an eigenvalue of* (3.3)*. The transmission function* $t_2(\lambda, \varepsilon)$ *is analytic as function of* λ *for all* $\lambda \in C_\delta \setminus \{\bigcup_{j=\text{even}} \lambda^j(\varepsilon)\}$*.*

Note that the condition $\lambda_f^j > \max(-1, -\mu)$ is crucial: if not, λ is in the essential spectrum of (3.3), (3.9).

Proof. It follows by the winding number arguments of the proof of Lemma 4.1 that there exists $\lambda^j(\varepsilon)$ such that $t_1(\lambda^j(\varepsilon), \varepsilon) = 0$ and $\lambda^j(\varepsilon) \to \lambda_f^j$ as $\varepsilon \to 0$. The proof of the singular behavior of $t_2(\lambda, \varepsilon)$ near λ_f^j for j even is identical to that of Lemma 4.3. When j is odd, (4.7) can be solved, but the solution is not uniquely determined: $v_{in}(\xi) = v_{in}^p(\xi) + Cv_f^j(\xi)$ for any $C \in \mathbb{R}$. Yet, $t_2(\lambda_f^j, 0)$, and thus $t_2(\lambda^j(\varepsilon), \varepsilon)$ is well-defined since $v_f^j(\xi)$ is odd: the term $Cv_f^j(\xi)$ does not give a contribution to (4.11). Since both $\mathcal{D}(\lambda, \varepsilon)$ and $t_1(\lambda, \varepsilon)$ are analytic for $\lambda \in C_\delta$, we conclude by (3.21) and Lemma 4.2 that $t_2(\lambda, \varepsilon)$ is analytic as function of λ (for ε fixed, and small enough) for all $\lambda \in C_\delta$ such that $\lambda \neq \lambda^j(\varepsilon)$ with j even. $\square$

Since $\lambda_f^0 = \lambda_f^*$ is the only non-zero eigenvalue of (3.14) with real part $\geq -\delta$ (Proposition 5.6), we can conclude that all possible unstable eigenvalues of (3.3) correspond to zeroes of $t_2(\lambda, \varepsilon)$. Thus, the stability of $(U_0(\xi), V_0(\xi))$ can be established by (4.7) and (4.11).

4.2. The NLEP equation. The function $v_h = v_h(\xi; u_{h,1})$ is given by (2.41) and (2.45). Inserting (2.41) in the equation for $v_{in}(\xi)$ (4.7) yields

$$v_{\xi\xi} + [\beta_2(w_h(\xi))^{\beta_2-1} - (1+\lambda)]v = -\alpha_2 h_2^{-1/(\beta_2-1)}(u_{h,1})^{(1-\alpha_2-\beta_2)/(\beta_2-1)}(w_h(\xi))^{\beta_2}, \tag{4.12}$$

where $w_h(\xi; \beta_2)$ is the positive homoclinic solution of (2.40). Likewise, (4.11) becomes

$$\begin{aligned} t_2(\lambda, 0) &= 1 - \frac{1}{2\sqrt{\mu+\lambda}}\Big[\alpha_1 h_1 h_2^{-\beta_1/(\beta_2-1)}(u_{h,1})^{[(\alpha_1-1)(\beta_2-1)-\alpha_2\beta_1]/(\beta_2-1)} W(\beta_1, \beta_2) \\ &\quad + \beta_1 h_1 h_2^{-(\beta_1-1)/(\beta_2-1)}(u_{h,1})^{[\alpha_1(\beta_2-1)-\alpha_2(\beta_1-1)]/(\beta_2-1)} \\ &\quad \cdot \int_{-\infty}^{\infty} v_{in}(\xi)(w_h(\xi))^{\beta_1-1}\, d\xi\Big], \end{aligned} \tag{4.13}$$

where $W(\beta_1, \beta_2)$ was defined in (2.43). We now define $w_{in}(\xi; \lambda, \beta_2)$ as the unique bounded solution of

$$w_{\xi\xi} + [\beta_2(w_h(\xi))^{\beta_2-1} - (1+\lambda)]w = (w_h(\xi))^{\beta_2} \tag{4.14}$$

(for $\lambda \in C_\delta$, $\lambda \neq \lambda_f^j$, $j = 0, \ldots, J$) so that by (4.12)

$$v_{in}(\xi) = -\alpha_2 h_2^{-1/(\beta_2-1)}(u_{h,1})^{(1-\alpha_2-\beta_2)/(\beta_2-1)} w_{in}(\xi; \lambda, \beta_2).$$

Next, we use (2.10) and (2.45) to derive a considerable simplification of (4.13):

$$t_2(\lambda, 0) = 1 - \frac{\sqrt{\mu}}{\sqrt{\mu+\lambda}}\left[\alpha_1 - \frac{\alpha_2\beta_1}{W(\beta_1, \beta_2)}\int_{-\infty}^{\infty} w_{in}(\xi)(w_h(\xi))^{\beta_1-1}\, d\xi\right]. \tag{4.15}$$

Since the leading order approximation of an eigenvalue λ of (3.3) (with $\mathrm{Re}(\lambda) > -\delta$ for some $\varepsilon \ll \delta \ll 1$) must correspond to a zero of $t_2(\lambda, 0)$, we can now write down the *NonLocal Eigenvalue Problem*:

$$\begin{cases} w_{\xi\xi} + [\beta_2(w_h(\xi))^{\beta_2-1} - (1+\lambda)]w = (w_h(\xi))^{\beta_2} \\ \alpha_1 - \dfrac{\alpha_2\beta_1}{W(\beta_1, \beta_2)}\displaystyle\int_{-\infty}^{\infty} w(\xi)(w_h(\xi))^{\beta_1-1}\, d\xi = \sqrt{1+\dfrac{\lambda}{\mu}} \end{cases} \tag{4.16}$$

for $\mu > 0$ and $G_i(V)$, h_i, α_i, β_i ($i = 1, 2$) as in (2.5), (2.6). This equation determines $w(\xi;\lambda)$ uniquely, which is not usual in eigenvalue problems. A more standard equation can be obtained by introducing $\widetilde{w}(\xi;\lambda) = Cw(\xi;\lambda)$ for some $C \neq 0$ into (4.16) and eliminating C. This gives the following, more compact version of the NLEP problem,

$$(4.17)\quad w_{\xi\xi} + [\beta_2(w_h(\xi))^{\beta_2-1} - (1+\lambda)]w = \frac{\alpha_2\beta_1}{(\alpha_1 - \sqrt{1+\lambda/\mu})W(\beta_1,\beta_2)}(w_h(\xi))^{\beta_2}\int_{-\infty}^{\infty} w(\xi)(w_h(\xi))^{\beta_1-1}\,d\xi,$$

where we have dropped the tilde on w. The NLEP problem was originally introduced in [6] (for the Gray-Scott model) in this form. Note that neither (4.16) nor (4.17) depends explicitly on h_1 and h_2 (but $h_{1,2}$ must satisfy $h_{1,2} > 0$ by Theorem 2.1).

The explicit expression (4.11) can also be used to compute the leading order expression for $t_2(\lambda,\varepsilon)$ at $\lambda = 0$. It is a straightforward calculation to check that the solution $w_{in}(\xi)$ of (4.14) at $\lambda = 0$ is given by

$$(4.18)\qquad w_{in}(\xi;0,\beta_2) = \frac{1}{\beta_2 - 1}w_h(\xi) + C\dot{w}_h(\xi),$$

where C may be any real number, since $w_{in}(\xi)$ is not uniquely determined in $\lambda = 0$. The extra contribution of the kernel to w_{in} disappears from (4.15) since $\dot{w}_h(\xi)$ is odd:

$$(4.19)\qquad \int_{-\infty}^{\infty} w_{in}(\xi;0)(w_h(\xi))^{\beta_1-1}\,d\xi = \frac{W(\beta_1,\beta_2)}{\beta_2 - 1}$$

(see (2.43)). Therefore, by (2.10),

$$(4.20)\qquad t_2(0,0) = 1 - \frac{\sqrt{\mu}}{\sqrt{\mu}}\left[\alpha_1 - \frac{\alpha_2\beta_1}{W(\beta_1,\beta_2)}\frac{W(\beta_1,\beta_2)}{\beta_2 - 1}\right] = -\frac{D}{\beta_2 - 1} < 0.$$

Thus, $t_1(0) = 0$ (with multiplicity 1) and $t_2(0) < 0$ so that the eigenvalue $\lambda = 0$ is *always* simple with the obvious eigenfunction $(\dot{V}_0(\xi), \dot{U}_0(\xi))$, the derivative of the 'wave' $(V_0(\xi), U_0(\xi))$.

Corollary 4.5. *The homoclinic pattern* $(V_0(\xi), U_0(\xi))$ *can only lose or gain stability when a pair of complex conjugate eigenvalues* (*with non-zero imaginary parts*) *crosses the imaginary axis: the associated bifurcation is of Hopf type.*

5. THE STABILITY OF THE 1-CIRCUIT HOMOCLINIC PATTERNS

In this section we first obtain a number of general instability results using topological winding number arguments (Section 5.1 below). It is shown that all 1-circuit homoclinic patterns must be unstable for $\mu > 0$ small enough (Theorem 5.1). The 1-pulse patterns can gain stability by increasing μ, as we shall show in Section 5.3.

5.1. Asymptotic results. The combination of the explicit leading order approximation (4.15) for $t_2(\lambda, \varepsilon)$ and the fact that $t_2(\lambda, \varepsilon)$ has a pole near λ_f^* (Lemma 4.3) yields a number of results on the *instability* of the homoclinic 1-pulse pattern $(U_0(\xi), V_0(\xi))$.

Theorem 5.1. *Let* $G_i(V)$, h_i, α_i, β_i $(i = 1, 2)$ *satisfy* (2.5), (2.6) *and let* $(U_0(\xi), V_0(\xi))$ *correspond to the homoclinic orbit* $\gamma_h^1(\xi)$ *defined in Theorem 2.1. Then there exists a* $\mu^U = \mu^U(\alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ *such that* (3.3) *has an unstable eigenvalue* $0 < \lambda_{s-f}^0(\varepsilon) \in \mathbb{R}$ *for all* $0 < \mu < \mu^U$.

This 'new' eigenvalue $\lambda_{s-f}^0(\varepsilon)$ that exists due to the slow-fast interaction should not be confused with the pole $\lambda^*(\varepsilon) = \lambda^0(\varepsilon)$ (Lemma 4.3 and Corollary 4.4).

Proof. Consider δ with $0 < \varepsilon \ll \delta \ll 1$ and the contour $K_{\lambda_f^*,\delta}$, a circle around λ_f^* with radius δ. Thus, the pole $\lambda^*(\varepsilon)$ of $t_2(\lambda, \varepsilon)$ is in the interior of $K_{\lambda_f^*,\delta}$ for all ε small enough. By the analyticity of $t_2(\lambda, \varepsilon)$ on $K_{\lambda_f^*,\delta}$ there must be constants $C_{1,2}^*(\delta)$ such that

$$\begin{aligned} \left| \int_{-\infty}^{\infty} w_{in}(\xi)(w_h(\xi))^{\beta_1 - 1}\, d\xi \right| &< C_1^*(\delta), \\ \left| \frac{d}{d\lambda} \int_{-\infty}^{\infty} w_{in}(\xi)(w_h(\xi))^{\beta_1 - 1}\, d\xi \right| &< C_2^*(\delta) \end{aligned} \tag{5.1}$$

for all $\lambda \in K_{\lambda_f^*,\delta}$. Since $|\sqrt{\mu + \lambda}| > \frac{1}{2}\sqrt{\lambda_f^*}$ for $\lambda \in K_{\lambda_f^*,\delta}$ and δ small enough, we have, for $0 < \mu$ small enough

$$\begin{aligned} |t_2(\lambda)| &\geq 1 - 2\sqrt{\frac{\mu}{\lambda_f^*}} \left(|\alpha_1| + \frac{|\alpha_2|\beta_1}{W(\beta_1, \beta_2)} C_1^*(\delta) \right) \\ &> \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} \left| \frac{d}{d\lambda} t_2(\lambda) \right| &\leq 2\sqrt{\frac{\mu}{\lambda_f^*}} \left[\frac{2}{\lambda_f^*} \left(|\alpha_1| + \frac{|\alpha_2|\beta_1}{W(\beta_1, \beta_2)} C_1^*(\delta) \right) + \frac{|\alpha_2|\beta_1}{W(\beta_1, \beta_2)} C_2^*(\delta) \right] \\ &< \frac{1}{3\delta}, \end{aligned}$$

for all $\lambda \in K_{\lambda_f^*,\delta}$. Thus, we obtain for the winding number $W(t_2; K_{\lambda_f^*,\delta})$ of $t_2(\lambda)$ over $K_{\lambda_f^*,\delta}$:

$$|W(t_2; K_{\lambda_f^*,\delta})| = \left| \frac{1}{2\pi i} \oint_K \frac{(d/d\lambda) t_2(\lambda)}{t_2(\lambda)} d\lambda \right| \leq \frac{1}{2\pi} \left(\frac{2}{3\delta} \right) (2\pi\delta) = \frac{2}{3},$$

(with $K = K_{\lambda_f^*,\delta}$) for μ small enough. Then, since the winding number $W(t_2; K_{\lambda_f^*,\delta})$ must be an integer, we see that

$$W(t_2; K_{\lambda_f^*,\delta}) = 0.$$

Hence, because the simple pole of $t_2(\lambda, \varepsilon)$ at $\lambda^*(\varepsilon)$ gives a contribution of -1 to $W(t_2; K_{\lambda_f^*,\delta})$, $t_2(\lambda; \varepsilon)$ must have one simple zero $\lambda_{s-f}^0(\varepsilon)$ inside $K_{\lambda_f^*,\delta}$. Finally, since complex eigenvalues can only occur in pairs ((3.3) is real), we have $\lambda_{s-f}^0(\varepsilon) \in \mathbb{R}$. □

This method cannot give information on the ordering of λ_f^* and λ_{s-f}^0. We will see in Corollary 5.10 that stability can only be recovered by increasing μ when $\lambda_{s-f}^0 < \lambda_f^*$ for all μ small enough.

Although μ can be considered to be the main parameter in (1.2) and/or (1.7), we can also formulate a similar result in terms of the parameters α_1 and α_2.

Theorem 5.2. *Let* $\mu > 0$ *and* $G_i(V)$, h_i, α_i, β_i $(i = 1, 2)$ *as in* (2.5), (2.6) *and let* $(U_0(\xi), V_0(\xi))$ *correspond to the homoclinic orbit* $\gamma_h^1(\xi)$ *defined in Theorem* 2.1. *Then there exists a* $\alpha_1^U = \alpha_1^U(\mu, \alpha_2, \beta_1, \beta_2) > 0$ *such that* (3.3) *has an unstable eigenvalue* $0 < \lambda_1(\varepsilon) \in \mathbb{R}$ *for all* $\alpha_1 > \alpha_1^U$. *Analogously, there exists a* $\alpha_2^U = \alpha_2^U(\mu, \alpha_2, \beta_1, \beta_2) < 0$ *such that* (3.3) *has an unstable eigenvalue* $0 < \lambda_1(\varepsilon) \in \mathbb{R}$ *for all* $\alpha_2^U < \alpha_2 < 0$.

Proof. The proof is again based on a winding number calculation over $K_{\lambda_f^*,\delta}$ using the uniform estimates (5.1). It follows from (4.15) that $|[dt_2(\lambda)/d\lambda]/t_2(\lambda)|$ can again be made as small as necessary (either for α_1 large enough or $\alpha_2 < 0$ close enough to zero), so that $W(t_2; K_{\lambda_f^*,\delta}) = 0$ again. □

These arguments can also be applied near each of the other possible poles of $t_2(\lambda, \varepsilon)$. Recall from Corollary 4.4 that λ_f^j is associated to a pole $\lambda^j(\varepsilon)$ of $t_2(\lambda, \varepsilon)$ when $j \geq 2$ even:

Corollary 5.3. *Let* $\mu > 0$ *and* $G_i(V)$, h_i, α_i, β_i $(i = 1, 2)$ *as in* (2.5), (2.6). *Let* λ_f^j *be an eigenvalue of* (3.14) *such that* $-\mu < \lambda_f^j < 0$, *with* $j \geq 2$ *and* j *even. Then there exists a* $\alpha_1^{U,j} = \alpha_1^{U,j}(\mu, \alpha_2, \beta_1, \beta_2) > 0$ *such that* (3.3) *has a stable eigenvalue* $0 > \lambda_{s-f}^j(\varepsilon) \in \mathbb{R}$ *for all* $\alpha_1 > \alpha_1^{U,j}$.

A similar result can once again be formulated in terms of $\alpha_2 < 0$, close enough to 0. However, the equivalent to Theorem 5.1 will in general not be true: the condition $-\mu < \lambda_f^j < 0$ prohibits us from taking μ 'small enough'. Nevertheless, an estimate like (5.1) can also be used to establish the existence of another *positive* eigenvalue $\lambda_{s-f}^1(\varepsilon)$ near $\lambda_f^1 = 0$ for μ small enough:

Lemma 5.4. *There exists a value* $\mu^{U,1} = \mu^{U,1}(\alpha_1, \alpha_2, \beta_1, \beta_2) > 0$ *such that* (3.3) *has an uniquely determined second positive eigenvalue* $0 < \lambda^1_{s-f}(\varepsilon) < \lambda^0_{s-f}(\varepsilon)$ *that satisfies* $\lim_{\mu\to 0} \lambda^1_{s-f}(\varepsilon) = 0$ *for all* $\mu < \mu^{U,1}$ *(with* $\mu^{U,1}$*).*

Proof. It follows from the smoothness of t_2 as function of λ and (4.19) that for μ small enough

$$\int_{-\infty}^{\infty} w_{in}(\xi;\lambda)(w_h(\xi))^{\beta_1-1}\, d\xi < 2\frac{W(\beta_1,\beta_2)}{\beta_2-1} \quad \text{for all } \lambda \in (0,\sqrt{\mu}).$$

Since $\sqrt{\mu}/\sqrt{\mu+\sqrt{\mu}}$ can be made as small as necessary by decreasing μ, it follows from (4.15) that $t_2(\sqrt{\mu},0) > 0$ (recall that $\alpha_2 < 0$). Then, since $t_2(0,0) < 0$ by (4.20), we see that $t_2(\lambda)$ must have a zero in $(0,\sqrt{\mu})$. An estimate (for small λ) like in (5.1) on the derivative of the integral shows that λ^1_{s-f} is uniquely determined. ◻

For asymptotically small μ, it is even possible to derive a leading order approximation of $\lambda^1_{s-f}(\varepsilon)$:

$$\lambda^1_{s-f}(0) = \frac{D^2 + 2(\beta_2-1)D}{(\beta_2-1)^2}\mu + \mathcal{O}(\mu^2), \tag{5.2}$$

since (4.19) only has an $\mathcal{O}(\mu)$ correction for $\lambda = \mathcal{O}(\mu)$ (and μ asymptotically small). Finally we note that $t_2(\lambda,\varepsilon) = 1$ to leading order for μ asymptotically small and λ not too close to either 0 or the pole $\lambda^*(\varepsilon)$. Therefore, (3.3) will only have 3 eigenvalues for μ small enough: $\lambda^1_f = 0 < \lambda^1_{s-f} < \lambda^0_{s-f}$ (to conclude this one needs an estimate of the type (5.1) that is uniform in $\lambda \in \mathbb{C}\backslash$(the union of $\mathcal{O}(\delta)$ neighborhoods of λ^*, 0 and the negative $\mathrm{Re}(\lambda)$-axis), which is a mostly technical affair). However, when μ increases the essential spectrum moves away from $\lambda = 0$ and poles and zeroes of $t_2(\lambda,\varepsilon)$ might bifurcate out of the 'edge' of the essential spectrum at $\lambda = -\mu$ (Corollaries 4.4 and 5.3, and Section 5.2: this will only happen for $\beta_2 < 3$). We will see in Section 5.3 that such an edge bifurcation exists in the classical Gierer-Meinhardt problem.

Remark 5.5. The situation described here (and in Lemma 5.4, Theorem 5.1) is in essence the same as in the Gray-Scott model in the 'strongly unstable regime' [6, 7]. In the Gray-Scott problem there are two unstable eigenvalues, one near 0 and one near $\lambda^*_f = \frac{5}{4}$ (Proposition 5.6 with $\beta_2 = 2$), in the strongly unstable regime. The pattern $(U_0(\xi), V_0(\xi))$ becomes stable in the Gray-Scott model when these two eigenvalues merge and become a pair of complex conjugate eigenvalues that can cross through the $\mathrm{Re}(\lambda) = 0$ axis. We will see in Section 5.3 that the same happens when μ is increased in the classical Gierer-Meinhardt problem (note that this agrees with Corollary 4.5).

5.2. A reduction to hypergeometric functions. An inhomogeneous second order ODE can of course be solved as soon as one knows two independent solutions of the homogeneous problem. Therefore, we first study the homogeneous problem associated to (4.14):

$$w_{\xi\xi} + [\beta_2 (w_h(\xi))^{\beta_2 - 1} - (1+\lambda)]w = 0. \tag{5.3}$$

It can be checked by (2.41) and (2.45) that this is equal to the fast reduced linear stability problem (3.14). In the Gray-Scott model $\beta_2 = 2$, so that this equation can be transformed to the hypergeometric differential equation using the explicit expression for the homoclinic solution of (2.40), $w_h(\xi) = 3/(2\cosh^2(\xi/2))$ [6]. Such an expression does not exist for general $\beta_2 > 1$, however, the classical procedure to transform (5.3) to the hypergeometric differential equation for $\beta_2 = 2$ [26] can be modified so that it can be applied for all $\beta_2 > 1$, without using an explicit formula for $w_h(\xi)$. We first introduce P by

$$P = +\sqrt{1+\lambda} \in \mathbb{C}, \tag{5.4}$$

i.e., $\mathrm{Re}(P) > 0$ for $\lambda \in \{\mathrm{Re}(\lambda) > -1\} \subset C_e$. Since $w(\xi)$ must remain bounded as $\xi \to -\infty$, or $\xi \to \infty$, it will decay as $e^{-P|\xi|}$, therefore we introduce $F = F(\xi; P, \beta_2)$ by

$$w(\xi) = F(\xi)(w_h(\xi))^P, \tag{5.5}$$

so that $F(\xi)$ approaches a constant when $w(\xi) \to 0$ for $\xi \to \pm\infty$. Using (2.40) and the integral relation

$$\frac{1}{2}\dot{w}_h^2 = \frac{1}{2}w_h^2 - \frac{1}{\beta_2+1}w_h^{\beta_2+1}, \tag{5.6}$$

it follows that $F(\xi)$ satisfies

$$\ddot{F} + 2P\frac{\dot{w}_h}{w_h}\dot{F} + \frac{(\beta_2 - P)(\beta_2+1) - 2P(P-1)}{\beta_2+1}w_h^{\beta_2-1}F = 0, \tag{5.7}$$

where the 'dots' denote differentiation with respect to ξ. Next, we introduce the new independent variable z by

$$z = \frac{1}{2}\left(1 - \frac{\dot{w}_h(\xi)}{w_h(\xi)}\right). \tag{5.8}$$

Note that $z = \frac{1}{2}$ corresponds to $\xi = 0$: the axis of symmetry for $w_h = w_h(\xi)$. The relation (5.6) implies that $\lim_{\xi\to\pm\infty} \dot{w}_h(\xi)/w_h(\xi) = \mp 1$. Thus, (5.8) defines a one-to-one map from $\xi \in (-\infty, \infty)$ onto $z \in (0,1)$. Moreover, (2.40) and (5.6) imply that

$$\frac{\dot{w}_h}{w_h} = 1 - 2z, \quad w_h^{\beta_2 - 1} = 2(\beta_2 + 1)z(1-z), \qquad \frac{d}{d\xi} = (\beta_2 - 1)z(1-z)\frac{d}{dz}. \tag{5.9}$$

This yields the following equation for F as function of z:

$$z(1-z)F'' + (1-2z)\frac{\beta_2 + 2P - 1}{\beta_2 - 1}F' + 2\frac{(\beta_2 - P)(\beta_2 + 1) - 2P(P-1)}{(\beta_2 - 1)^2}F = 0. \tag{5.10}$$

Equation (5.10) is a special form of the hypergeometric differential equation

$$z(1-z)F'' + [c - (a+b+1)z]F' - abF = 0,$$

with

$$a = \frac{2P + 2\beta_2}{\beta_2 - 1}, \quad b = \frac{2P - \beta_2 - 1}{\beta_2 - 1}, \quad c = \frac{2P + \beta_2 - 1}{\beta_2 - 1}. \tag{5.11}$$

Thus, the following two hypergeometric functions span the solution space of (5.10)

$$F(a\,, b \mid c \mid z) \quad \text{and} \quad z^{1-c}F(a-c+1\,, b-c+1 \mid 2-c \mid z), \tag{5.12}$$

with a, b, c as in (5.11) [26]. However, since (5.7) is symmetric with respect to $z \to 1 - z$, we will use a different set of independent solutions that exploit this symmetry (see Appendix B on page 502). Note that the symmetry around $z = \frac{1}{2}$ is a transformation of the reversibility symmetry $\xi \to -\xi$ in (5.3) and in the underlying PDE.

The reduction of the eigenvalue problem to a hypergeometric differential equation can be used to determine the eigenvalues λ_f^j and the associated eigenfunctions of the fast reduced system. This result will be necessary for the analysis of the inhomogeneous problem.

Proposition 5.6. *Let* $J = J(\beta_2) \in \mathbb{N}$ *be such that* $J < (\beta_2 + 1)/(\beta_2 - 1) \leq J + 1$. *The eigenvalue problem* (5.3) *has* $J + 1$ *eigenvalues given by*

$$\lambda_f^j = \frac{1}{4}[(\beta_2 + 1) - j(\beta_2 - 1)]^2 - 1, \quad \text{for } j = 0, 1, \ldots, J. \tag{5.13}$$

The eigenfunctions $w_f^j(\xi)$ *are polynomial solutions of degree* j (*in* z) *of the associated problem* (5.7) *and can be explicitly expressed in terms of* $w_h(\xi)$ *and* $\dot{w}_h(\xi)$ (*through* (5.5) *and* (5.8)).

Proof. A solution of (5.3) that decays for $\xi \to \pm\infty$ corresponds to a solution $F(z)$ of (5.10) that is regular at both $z = 0$ and $z = 1$ (5.5). Since (5.3) is self-adjoint, we know that the eigenvalues λ must be real; moreover, $\lambda > -1$, thus $P > 0$. It follows from $1 - c = -2P/(\beta_2 - 1) < 0$ (5.4) that $z^{1-c}F(a-c+1\ ,\ b-c+1 \mid 2-c \mid z)$ is singular at $z = 0$ whereas $F(a\ ,\ b \mid c \mid z)$ is by construction regular at $z = 0$, but has a singularity in $z = 1$ in general [26]:

$$\lim_{z\to 1}(1-z)^{-(c-a-b)}F(a\ ,\ b \mid c \mid z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{5.14}$$

since $c - a - b = 1 - c < 0$ by (5.11); $\Gamma(z)$ is the Gamma function. Hence, a solution of (5.10) that is regular at both $z = 0, 1$ can only exist at the poles of $\Gamma(a)$ or $\Gamma(b)$: $a, b = 0, -1, -2, \ldots$. It follows by (5.11) and the fact that $P > 0$, $\beta_2 > 1$ that $a > 0$, hence only the $\Gamma(b)$ term can have a pole:

$$P^j = \frac{1}{2}(\beta_2 + 1) - \frac{1}{2}(\beta_2 - 1)j, \quad \text{for } j = 0, 1, 2, \ldots, \tag{5.15}$$

which transforms into (5.13) by (5.4). Note that the superscripts are indices, not powers. Since $b = -j$ at $P = P^j$, it follows that the power series expansion of $F(a\ ,\ b \mid c \mid z)$ has finite length: it is a polynomial of degree j [26]. □

Remark 5.7. Note that $\lambda_f^j \in (-1, 0)$ for $j \geq 2$ and that $J(\beta_2) \equiv 1$ for $\beta_2 \geq 3$ (i.e., there are no 'stable' eigenvalues for $\beta_2 \geq 3$). On the other hand, $\lim_{\beta_2 \downarrow 1} J(\beta_2) = \infty$: the number of stable eigenvalues becomes unbounded as β_2 decreases to 1. The new eigenvalues that appear as β_2 decreases from $\beta_2 = 3$ are generated as so-called 'edge bifurcations' at the endpoint -1 of the essential spectrum associated to (5.3). Furthermore, we observe that

$$\lambda_f^* = \lambda_f^0 = \frac{1}{4}(\beta_2 + 1)^2 - 1 > 0, \quad \text{with } w_f^*(\xi) = w_f^0(\xi) = (w_h(\xi))^{1/2(\beta_2+1)}, \tag{5.16}$$

since $F(z) \equiv 1$ is the zeroth order polynomial solution of (5.7) at $P = P^0$. Moreover, we confirm that $\lambda_f^1 \equiv 0$ with $w_f^1(\xi) = \dot{w}_h(\xi)$. In Section 5.3 we will study the special case corresponding to the classical Gierer-Meinhardt problem, where $\beta_2 = 2$ (1.9). In this case there are 3 eigenvalues ($J = 2$): $\lambda_f^* = \lambda_f^0 = \frac{5}{4}$, $\lambda_f^1 = 0$, and $\lambda_f^2 = -\frac{3}{4}$.

The inhomogeneous problem (4.14) can be transformed in precisely the same fashion as (5.3). Introducing $G(z)$ by

$$F(z;P,\beta_2) = \frac{[2(\beta_2+1)]^{(\beta_2-P)/(\beta_2-1)}}{(\beta_2-1)^2} G(z;P,\beta_2), \tag{5.17}$$

we obtain the inhomogeneous hypergeometric equation

$$\begin{aligned} z(1-z)G'' &+ (1-2z)\frac{\beta_2+2P-1}{\beta_2-1}G' \\ &+ 2\frac{(\beta_2-P)(\beta_2+1)-2P(P-1)}{(\beta_2-1)^2}G \\ &= [z(1-z)]^{(1-P)/(\beta_2-1)}. \end{aligned} \tag{5.18}$$

Using the independent solutions of the homogeneous problem (5.10), we can determine the unique solution $G(z;P,\beta_2)$ of this equation by a classical variation of constants (or Green's function) approach. The details of this analysis are given in Appendix B on page 502. In Section 5.3, we will apply the general procedure to a special case, the classical Gierer-Meinhardt equation (see also [6]). However, the procedure works in general, for all possible parameter combinations.

Remark 5.8. The right hand side of (5.18) is a polynomial in z when $(1-P)/(\beta_2-1) = k$ or $P = 1-k(\beta_2-1)$, $k = 0, 1, 2, \dots$. This corresponds to $j = 2k+1$ in (5.15). Therefore, we can determine the solution $G(z;P,\beta_2)$ as a polynomial in z when $P = P^j$ with j odd. By Corollary 4.4 we know that there is an eigenvalue of the full problem (3.3) near λ_f^j for odd values of j (for all μ when $j = 1$, for $\mu > -\lambda_f^j$ when $j \geq 3$). For such values we can thus determine $G(z;P^j,\beta_2)$ explicitly (but not uniquely) and calculate $t_2(\lambda_f^j, 0)$. We have already done this in Section 4.2 for $k = 0$ (i.e., $j = 1$, (4.18)) without using the transformation to a hypergeometric differential equation. It also follows from the analysis in Section 4 that $G(z;P,\beta_2)$ does not exist for $P = P^j$ with j even ($P = P^j$ corresponds to the pole of $t_2(\lambda,0)$ at $\lambda = \lambda_f^j$ in this case, see Corollary 4.4).

The leading order expressions for the eigenvalues of (3.3) are now determined by imposing $t_2(\lambda,0) = 0$, where we recall that $t_2(\lambda,0)$ is given by (4.15). We introduce

$$\mathcal{R}(P;\beta_1,\beta_2) = \int_0^1 G(z;P,\beta_2)[z(1-z)]^{(P+\beta_1-\beta_2)/(\beta_2-1)}\,dz, \tag{5.19}$$

and

$$\mathcal{B}(\beta_1,\beta_2) = \frac{(\beta_2-1)^2}{2\beta_1(\beta_2+1)} \int_0^1 [z(1-z)]^{(\beta_1-\beta_2+1)/(\beta_2-1)}\,dz. \tag{5.20}$$

The integral in the definition of $\mathcal{B}(\beta_1, \beta_2)$ comes from the term $W(\beta_1, \beta_2)$ (2.43). It is a beta-function and can thus be written as an expression in Gamma functions [26]. It follows from the expression for $t_2(\lambda, 0)$ (4.15) and (5.4), (5.5), (5.8), (5.17), (5.19), (5.20) that

$$t_2(P, 0) = 1 - \left[\alpha_1 - \frac{\alpha_2}{\mathcal{B}(\beta_1, \beta_2)}\mathcal{R}(P; \beta_1, \beta_2)\right]\sqrt{\frac{\mu}{\mu + P^2 - 1}}. \tag{5.21}$$

Although the expression $\mathcal{R}(P; \beta_1, \beta_2)$ is explicitly known (B.6), it is almost impossible to handle $\mathcal{R}(P; \beta_1, \beta_2)$ (5.19) by hand: here a program like Mathematica is indispensable, especially when P is complex valued. With the use of such a program $\mathcal{R}(P; \beta_1, \beta_2)$, and thus $t_2(P, 0)$, can be considered as known functions: an eigenvalue $\lambda = \lambda(\mu, \alpha_1, \alpha_2, \beta_1, \beta_2)$ of (3.3) corresponds to a solution $P(\mu, \alpha_1, \alpha_2, \beta_1, \beta_2)$ of the equation $t_2(P, 0) = 0$,

$$\sqrt{\mu + P^2 - 1} = \left[\alpha_1 - \frac{\alpha_2}{\mathcal{B}(\beta_1, \beta_2)}\mathcal{R}(P; \beta_1, \beta_2)\right]\sqrt{\mu}. \tag{5.22}$$

If one is interested in real eigenvalues of (3.3) (for a given value of μ), one can determine the (non-)existence 'graphically' by solving $t_2(P, 0) = 0$ for $\mu = \mu_{\text{real}}(P)$:

$$\mu_{\text{real}}(P; \alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{P^2 - 1}{\left[\alpha_1 - \frac{\alpha_2}{\mathcal{B}(\beta_1, \beta_2)}\mathcal{R}(P; \beta_1, \beta_2)\right]^2 - 1}, \quad \text{if } \alpha_1 - \frac{\alpha_2}{\mathcal{B}(\beta_1, \beta_2)}\mathcal{R}(P; \beta_1, \beta_2) \geq 0, \tag{5.23}$$

by (5.22); $\mu_{\text{real}}(P)$ is not defined when the right hand side of (5.22) is negative. Thus, if $\mu > 0$ is given and there is a $P \in (P^1, \infty) = (1, \infty)$ such that $\mu_{\text{real}}(P) = \mu$, then the 1-pulse solution of (1.7) is unstable, with (at least one) real positive eigenvalue. Note that there can still be complex valued unstable eigenvalues if this is not the case (see Section 5.3 on the next page).

The asymptotic results of the previous sections give some insight in the graphs of $\mu_{\text{real}}(P; \alpha_1, \alpha_2, \beta_1, \beta_2)$ and $\mathcal{R}(P; \beta_1, \beta_2)$. Such insight can serve as an extremely useful check of the calculations in Appendix B on page 502 and their implementation in a Mathematica code (see Figures 5.1 on page 489 and 5.2 on page 490).

Corollary 5.9. *Let $P > 0$ and k be such that $2k = j = 0, 2, \cdots \leq J$ (defined in Proposition 5.6):*

(i) $\mathcal{R}(P; \beta_1, \beta_2) \to R^k/(P - P^{2k})$ *as* $P \to P^{2k}$ *for some* $R^k = R^k(\beta_1, \beta_2) \neq 0$*;* $\mathcal{R}(P; \beta_1, \beta_2)$ *is well-defined for all* $P \neq P^{2k}$*;*

(ii) $\mathcal{R}(1; \beta_1, \beta_2) = \beta_1 \mathcal{B}(\beta_1, \beta_2)/(\beta_2 - 1) > 0$*;* $\alpha_1 - (\alpha_2/\mathcal{B}(\beta_1, \beta_2))\mathcal{R}(1; \beta_1, \beta_2) > 1$*;*

(iii) $\mu_{\rm real}(P^{2k}) = 0$ *and* $\mu_{\rm real}(1) = 0$*; the graph of* $\mu_{\rm real}(P)$ *is tangent to the* P*-axis at* $P = P^{2k}$*;* $\mu_{\rm real}(P) \neq 0$ *for* $P \neq P^{2k}$, 1*;*

(iv) *Let* $P \in V^{2k}$*, a (small enough) neighborhood of* P^{2k}*:* $\mu_{\rm real}$ *is only defined either for* $P < P^{2k}$ *or for* $P > P^{2k}$*;*

(v) $\mu_{\rm real}(P) > 0$ *for* $P \in (1, \hat{P}^1)$ *and for* $P \in (\hat{P}^0_\ell, \hat{P}^0_r)$*, for some* $\hat{P} > 0$ *and* $\hat{P}^0_\ell < \hat{P}^0_r$ *with either* $\hat{P}^0_\ell = P^0$ *or* $\hat{P}^0_r = P^0$.

Proof. Corollary 4.4 yields that $G(P;\beta_2)$ only has poles of order one at $P = P^j$ for j even; it follows from the behavior of $G(z)$ near $z = 0$, 1 and (5.19) that the same is true for $\mathcal{R}(P;\beta_1,\beta_2)$, see Remark B.1 in Appendix B. This proves (i) and, by (5.23) it also establishes parts (iii) (since the denominator vanishes) and (iv) (since μ must be positive). Then, part (ii) is a consequence of (5.21) and (4.20), or equivalently (5.2), and the fact that $\beta_1/(\beta_2 - 1) > (\alpha_1 - 1)/\alpha_2$, i.e., $D > 0$ (2.10). Finally, it follows from Theorem 5.1 and Lemma 5.4 that the graph of $\mu_{\rm real}(P)$ intersects the line $\mu = \delta$ for some δ small enough at two points: P^1_{s-f} near (and larger than) P^1 (that corresponds to λ^1_{s-f}) and P^0_{s-f} near P^0 (that corresponds to λ^0_{s-f}); this implies (v). □

Note that the position of the interval $(\hat{P}^0_\ell, \hat{P}^0_r)$ with respect to P^0 is determined by the sign of R^0 (see (i), (v)) and that R^0 thus also determines the relative position of λ^0_{s-f} with respect to λ^0; R^k determines the domain of definition of $\mu_{\rm real}$ in (iv).

Finally, we observe by (5.23) that $\mu_{\rm real}(P)$ can have singularities if $\mathcal{R}(P;\beta_1,\beta_2) = (\alpha_1 - 1)\mathcal{B}(\beta_1,\beta_2)/\alpha_2$ (see Figure 5.2 on page 490 in the next section). If there are singularities in the interval $(1, P^0)$, then it follows from Corollary 5.9 that $(0,\infty) \subset \{\mu = \mu_{\rm real}(P) : P \geq 1\}$: there is at least one unstable (real) eigenvalue for any $\mu > 0$. This observation immediately yields an instability result.

Corollary 5.10. *If* $\lambda^0_{s-f} > \lambda^*_f$ *for* $\mu < \delta$ *(for some* δ *small enough), then there exists at least one unstable eigenvalue to* (3.3) *for any* $\mu > 0$.

Proof. It follows from Corollary 5.9 that $\mu_{\rm real}(P)$ must have a singularity in $(1, P^0)$ (note that, in this case, $\mu_{\rm real}(P)$ is not defined for $P < P^0$ and close to P^0). □

We will see in the next section that $\mathcal{R}(P;\beta_1,\beta_2)$ is monotonically increasing as function of P for $P \in (1, P^0)$ in the special case of the classical Gierer-Meinhardt problem (1.9). This implies by Corollary 5.9 that $\lambda_{s-f} < \lambda^*_f$ and that $\mu_{\rm real}(P)$ cannot have singularities in $(1, P^0)$: $\mu_{\rm real}(P)$ must have a maximum $\mu_{\rm complex}$ inside $(1, P^0)$. Thus, there are no unstable real eigenvalues for $\mu > \mu_{\rm complex}$ in this case. See Figures 5.1 on the facing page and 5.2 on page 490.

5.3. An application: the classical Gierer-Meinhardt equation. The choice (1.9) of $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ corresponds to the original biological values of the exponents of $F_1(U)$, $F_2(U)$, $G_1(V)$, $G_2(V)$ in (1.2) by Gierer and Meinhardt [20],

[28], [29], [30]. Recall, however, that the theory developed in this paper is valid for a much wider class than the functions considered in [20], [28], [29]: there $F_1(U) = U^{\alpha_1}$, $F_2(U) = U^{\alpha_2}$, $G_1(V) = V^{\beta_1}$, $G_2(V) = V^{\beta_2}$. In this paper the exponents $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are only determined by the leading order behavior of $F_1(U)$, $F_2(U)$, $G_1(V)$, and $G_2(V)$ for large U, V – see (1.4). Note that this case is not covered by the stability results in [30] (see Section 7 on page 498).

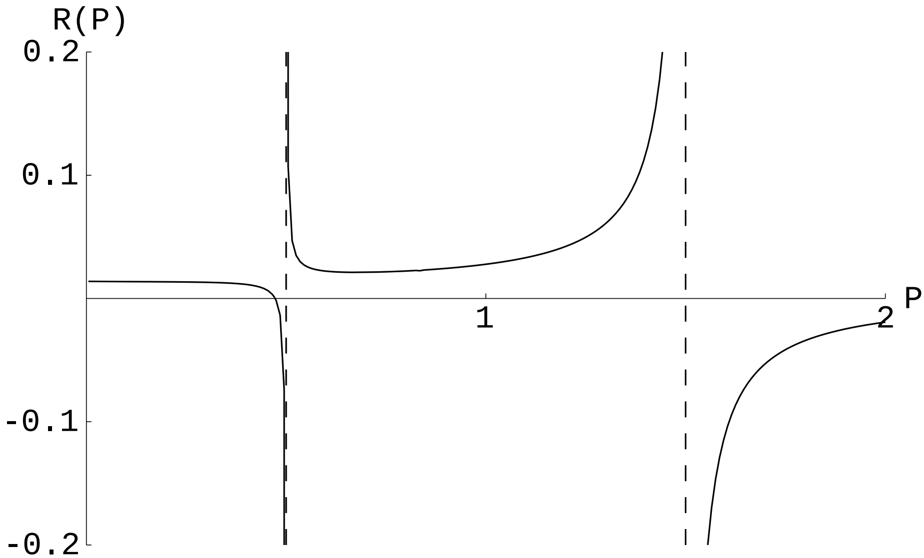

FIGURE 5.1. The function $\mathcal{R}(P;2,2)$ for $P \in (0,2)$. The (simple) poles are located at $P^0 = \frac{3}{2}$ and $P^2 = \frac{1}{2}$; $\mathcal{R}(P;2,2) > 0$ for $P \in (0, P_{\mathrm{edge}}) \cup (\frac{1}{2}, \frac{3}{2})$ where $P_{\mathrm{edge}} = 0.47\ldots + \mathcal{O}(\varepsilon)$.

In Figure 5.1, $\mathcal{R}(P;2,2)$ is plotted for $P \in (0,2)$, i.e., $\lambda \in (-1,3)$. We see that $\mathcal{R}(1;2,2) = \frac{1}{36}$ ((5.20) and Corollary 5.9). The behavior of $\mu_{\mathrm{real}}(P)$ as function of $P > 0$ is shown in Figure 5.2 on the following page. It follows from Figure 5.1 that $\mu_{\mathrm{real}}(P)$ can only be used to find real eigenvalues for $P \in (\frac{1}{2}, \frac{3}{2})$ and $P < P_{\mathrm{edge}} = 0.47\ldots$ (to leading order), because $\mathcal{R}(P;2,2) \leq 0$ for all other values of P (see (5.23), and recall that $\alpha_1 = 0$, $\alpha_2 = -1$). At the critical value $P = P_{\mathrm{edge}}$, $\mathcal{R}(P;2,2) = 0$ and the graph of $\mu_{\mathrm{real}}(P)$ is tangent to the 'edge' of the essential spectrum given by $P = \sqrt{1-\mu}$ (for $0 < \mu \leq 1$, (5.4), (3.9)). Thus, it follows that there is an *edge bifurcation* at $\mu = \mu_{\mathrm{edge}} = \mu_{\mathrm{real}}(P_{\mathrm{edge}}) = 0.77\ldots(+\mathcal{O}(\varepsilon))$. A fourth (stable) eigenvalue appears from the edge of the essential spectrum at this value of μ. This eigenvalue remains real and negative for all $\mu > \mu_{\mathrm{edge}}$. Inside the interval $(\frac{1}{2}, \frac{3}{3})$, $\mu_{\mathrm{real}}(P)$ has a maximum at $P = P_{\mathrm{complex}} = 1.17\ldots + \mathcal{O}(\varepsilon)$ (corresponding to $\lambda_{\mathrm{complex}} = 0.38\ldots + \mathcal{O}(\varepsilon)$). Hence, $\mu = \mu_{\mathrm{complex}} = \mu_{\mathrm{real}}(P_{\mathrm{complex}}) =$

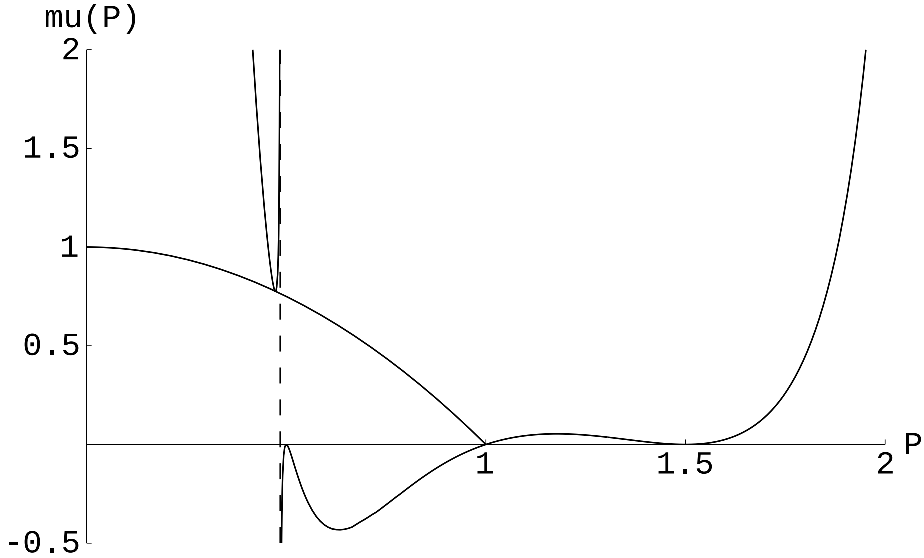

FIGURE 5.2. The function $\mu_{\text{real}}(P;0,-1,2,2)$ for $P \in (0,2)$. This graph only has meaning when $\mathcal{R}(P;2,2) > 0$ (5.23), i.e., for $P \in (0, P_{\text{edge}}) \cup (\frac{1}{2}, \frac{3}{2})$. The maximum of $\mu_{\text{real}}(P;0,-1,2,2)$ inside $(\frac{1}{2}, \frac{3}{2})$ defines the value $\mu_{\text{complex}} = 0.053\ldots + \mathcal{O}(\varepsilon)$: there are no real unstable eigenvalues for $\mu > \mu_{\text{complex}}$. The parabola connecting $(0,1)$ to $(1,0)$ represents the edge of the essential spectrum (3.9), (5.4); $\mu_{\text{real}}(P;0,-1,2,2)$ is tangent to this parabola at $P_{\text{edge}} = 0.47\ldots + \mathcal{O}(\varepsilon)$, the zero of $\mathcal{R}(P;2,2)$. A new eigenvalue bifurcates from the edge of the essential spectrum at $\mu = \mu_{\text{edge}} = \mu_{\text{real}}(P_{\text{edge}}) = 0.77\ldots + \mathcal{O}(\varepsilon)$. Note that $\mu_{\text{real}}(P;0,-1,2,2)$ is tangent to the P-axis at $P = P^0 = \frac{3}{2}$, $P = P^2 = \frac{1}{2}$ (Corollary 5.9, (iii)).

$0.053\ldots + \mathcal{O}(\varepsilon)$. Thus, there are two unstable real eigenvalues, $\lambda^1_{s-f}/P^1_{s-f}$ and $\lambda^0_{s-f}/P^0_{s-f}$, for all $\mu \in (0, \mu_{\text{complex}})$, as was already established qualitatively by Theorem 5.1 and Lemma 5.4 (see also Corollary 5.9). These eigenvalues merge at $\mu = \mu_{\text{complex}}$.

In Figure 5.3 on the facing page the 'orbit' (as function of μ) of the pair of complex conjugate eigenvalues that is created at μ_{complex} is shown in the complex λ-plane. This orbit is determined by solving (5.22) for complex values of P. The complex eigenvalues cross the imaginary axis at $\mu = \mu_{\text{Hopf}} = 0.36\ldots + \mathcal{O}(\varepsilon)$. Note that this agrees with Corollary 4.5. Finally, the pair of complex eigenvalues does

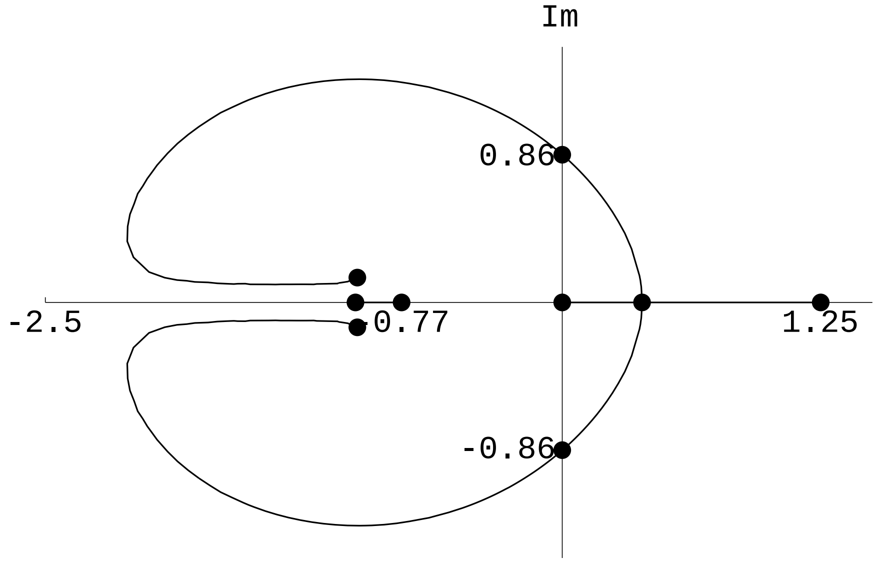

FIGURE 5.3. The 'orbits' of the eigenvalues through the complex plane as functions of $\mu > 0$ (with parameters as in (1.9)). The eigenvalues λ^1_{s-f} and λ^0_{s-f} approach 0 and $\frac{5}{4}$ as $\mu \downarrow 0$ and merge at $\mu = \mu_{\text{complex}} = 0.053\ldots + \mathcal{O}(\varepsilon)$, where $\lambda^1_{s-f} = \lambda^0_{s-f} = 0.38\ldots + \mathcal{O}(\varepsilon)$; λ^1_{s-f} and λ^0_{s-f} are complex for all $\mu > \mu_{\text{complex}}$ and cross the imaginary axis at $\mu = \mu_{\text{Hopf}} = 0.36\ldots + \mathcal{O}(\varepsilon)$ where $\lambda^1_{s-f} = -\lambda^0_{s-f} = 0.86\ldots i + \mathcal{O}(\varepsilon)$; they limit at $-0.99\ldots \pm 0.14\ldots i + \mathcal{O}(\varepsilon)$ as $\mu \to \infty$. The eigenvalue that bifurcates from the edge of the essential spectrum at $\mu = \mu_{\text{edge}} = 0.77\ldots + \mathcal{O}(\varepsilon)$ decreases monotonically as function of μ to $\lambda = -1$, the edge of the essential spectrum for $\mu > 1$ (3.9).

not return to the unstable half plane ($\lambda^1_{s-f}, \lambda^0_{s-f} \to -0.99\ldots \pm 0.14\ldots i + \mathcal{O}(\varepsilon)$ as $\mu \to \infty$). Hence, we have shown:

Theorem 5.11. *Let equation* (1.7) *be given with the parameters fixed by* (1.9). *Let* $(U_0(\xi), V_0(\xi))$ *be the homoclinic* 1*-pulse solution given by Theorem 2.1. Then,* $(U_0(\xi), V_0(\xi))$ *is an unstable solution of* (1.7) *if* $0 < \mu < \mu_{\text{Hopf}} = 0.36\ldots + \mathcal{O}(\varepsilon)$, *and* $(U_0(\xi), V_0(\xi))$ *is spectrally stable as solution to* (1.7) *for all* $\mu > \mu_{\text{Hopf}}$.

It has been checked by direct numerical integration of (1.7) that the Hopf bifurcation occurs at $\mu \approx 0.37$ (see Figure 1.1 on page 444: $(U_0(\xi), V_0(\xi))$ can be observed as an asymptotically stable pattern at $\mu = 0.38$, and for $\mu > 0.38$). This agrees completely with the leading order result given by the NLEP approach.

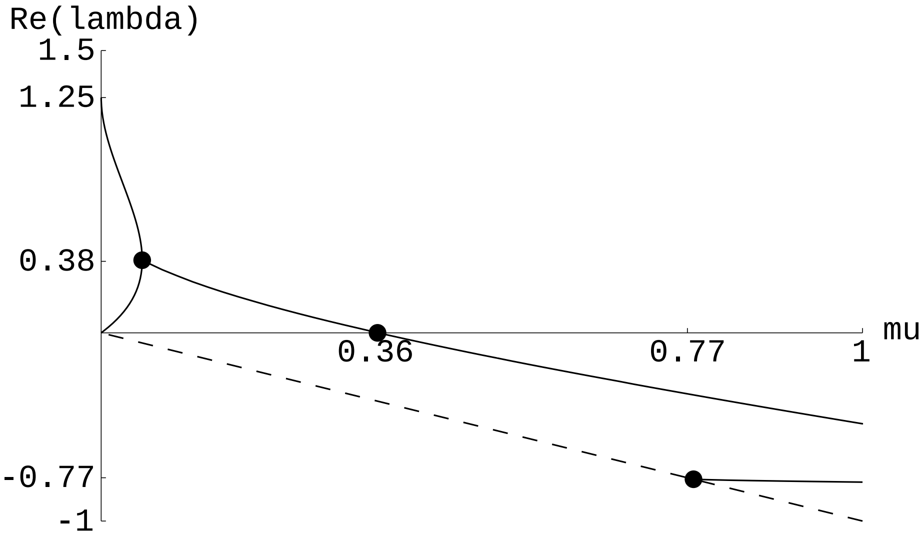

FIGURE 5.4. The real parts of the three non-trivial eigenvalues as functions of μ (with parameters as in (1.9)). The dashed line represents the edge of the essential spectrum (3.9).

In Figure 5.4 the graphs of the real parts of the 3 non-trivial eigenvalues of the linear problem (3.1) associated to the stability of $(U_0(\xi), V_0(\xi))$ have been plotted as functions of μ.

Remark 5.12. For the special case of (1.9), the inhomogeneous problem (5.18) is identical to the corresponding NLEP problem for the Gray-Scott model [6]. Moreover, by coincidence, the choice $\beta_1 = 2$ leads to the same hypergeometric NLEP analysis as in the Gray-Scott model. The exponents in the integrand of $\mathcal{R}(P; 2, 2)$ (5.19) are identical to those in the corresponding term in [6]! (Note that $P^2_{\mathrm{G-S}} = 4(1 + \lambda)$ in [6]: $P_{\mathrm{G-S}} = 2P$.)

6. THE INSTABILITY OF THE N-CIRCUIT HOMOCLINIC PATTERNS ($N \geq 2$)

The formulation of the linearized stability problem (3.1), or equivalently (3.3), is valid for any homoclinic N-pulse solution $(U_0(\xi), V_0(\xi))$ associated to an orbit $\gamma_h^N(\xi)$ (Theorem 2.1). As we noted in Section 3.2: one has to distinguish between the case $N = 1$ and the case $N \geq 2$ as soon as the structure of the stability problem in the fast reduced limit becomes important (i.e., as soon as one wants to define the component $\varphi_2(\xi)$ of the Evans function). Equation (3.13) is still the fast reduced scalar equation when $N \geq 2$. However, the equivalent of the fast reduced linearized problem (3.14) is now determined by linearizing around the *full* multi-pulse solution $V_0(\xi)$ instead of around the leading order *approximation* of $V_0(\xi)$, which we recall is the homoclinic solution $v_h(\xi)$ of (2.7) (with $u = u_{h,1}$ (2.45)). It is necessary to work with $V_0 = V_0(\xi;\varepsilon)$, since its leading order approximation is not an exact solution of (3.13). Moreover, one cannot take the limit $\varepsilon \to 0$ in $V_0(\xi)$ (for $N \geq 2$), since the distance between its N pulses becomes unbounded in this limit (see Lemma 6.1 below). Therefore, we need a precise description, and thus approximation, of $V_0(\xi)$ in the forthcoming analysis.

Lemma 6.1. *Let $V_0(\xi)$ be the v-component of the homoclinic orbit $\gamma_h^N(\xi)$ described in Theorem 2.1. There exist $-\infty = \xi_0 < \xi_1 < \cdots < \xi_{2N-1} < \xi_{2N} = +\infty$ such that:*

(i) $\partial V_0(\xi)/\partial\xi = 0$ *if and only if* $\xi = \xi_j$, $j = 0, 1, \ldots, 2N$.
(ii) $V_0(\xi_j) = \mathcal{O}(1)$ *for* $1 \leq j \leq 2N-1$ *odd, and* $V_0(\xi_j) = \mathcal{O}(\sqrt{\varepsilon})$ *for* $0 \leq j \leq 2N$ *even.*
(iii) $|\xi_j - \xi_k| = \mathcal{O}(|\log\varepsilon|)$ *for* $j, k = 1, 2, \ldots, 2N-1$, *and* $j \neq k$.
(iv) $\xi_j = -\xi_{2N-j}$ *for* $j = 0, 1, \ldots, 2N$ (*in particular:* $\xi_N = 0$).

Moreover, there exists a $\rho > 0$ *such that the following estimate holds:*

$$(6.1)\qquad \sup_{\xi\in(\xi_{2k-2},\xi_{2k})} |V_0(\xi) - v_h(\xi - \xi_{2k-1}; u_{h,N})| = \mathcal{O}(\varepsilon^\rho) \quad \text{for } k = 1, 2, \ldots, N,$$

where $v_h(\xi; u_{h,N})$ is defined by (2.40), (2.41), (2.45).

Proof. This Lemma is just a technical formulation of results obtained in Section 2 on page 451; (i), (ii), (iii), (iv), and the estimate follow immediately from the definition of the ξ_j's by $\gamma_h^N(\xi_j) \in \mathcal{I}^j(\mathcal{M}) \cap \mathcal{I}^{-2N+j}(\mathcal{M})$ and the leading order perturbation results. Note that ρ will in general depend on α_1, α_2, β_1, β_2 through $\hat{\varepsilon}$ (1.5). ◻

The homogeneous linearized stability eigenvalue problem associated to the fast reduced limit reads

$$(6.2)\quad (\mathcal{L}_f^N(\xi;\varepsilon) - \lambda)v = v_{\xi\xi} + [\beta_2 h_2 (u_{h,N})^{\alpha_2}(V_0(\xi;\varepsilon))^{\beta_2-1} - (1+\lambda)]v = 0.$$

Note that this operator can be identified with the leading order approximation of the lower diagonal 2×2 block of $A(\xi;\lambda,\varepsilon)$ (3.4), as is the case with A_f in (3.14) for $N = 1$ (see Section 3.2). Thus, replacing the linear operator A_f in (3.14) by $\mathcal{L}_f^N - \lambda$ (in matrix notation), we can proceed in the same manner as we did for the case $N = 1$ in Sections 3 and 4: we can define the functions $\varphi_j^N(\xi;\lambda,\varepsilon)$ ($j = 1, 2, 3, 4$), the Evans function $\mathcal{D}^N(\lambda,\varepsilon)$, and the decomposition of $\mathcal{D}^N(\lambda,\varepsilon)$ into a product of transmission functions $t_1^N(\lambda,\varepsilon)$ and $t_2^N(\lambda,\varepsilon)$ (see (3.21)). By construction, we again have that the transmission function $t_f^N(\lambda,\varepsilon)$ associated to the Evans function for the fast reduced limit problem (see (3.22)) is a leading order approximation of $t_1^N(\lambda,\varepsilon)$. Therefore, as in Lemma 4.1 and Corollary 4.4, we find that the zeroes of $t_1^N(\lambda,\varepsilon)$ are to leading order given by the eigenvalues of the fast reduced limit problem (6.2). However, it should be noted that the proof of this result is based on a topological winding number argument over a contour K_δ that must be independent of ε: the argument in the proof of Lemma 4.1 implies that the number of zeroes of $t_1^N(\lambda, \varepsilon)$ inside K_δ is equal to the number of eigenvalues of (6.2) inside K_δ. We shall see that this number is greater than 1 for $N > 1$.

Eigenvalue problem (6.2) is of singular Sturm-Liouville type [40]. The essential spectrum is identical to that of the corresponding problem for $N = 1$, (5.3), since $V_0(\xi)$ decays exponentially to 0 as $|\xi| \to \infty$. In general one expects that every eigenvalue of the associated 1-pulse potential Schrödinger problem will split into N nearby eigenvalues in the N-pulse potential Schrödinger problem (6.2), see also [35], [36]. We do not need a result as general as this here (see Remark 6.5). Note, however, that we know by the general theory on Sturm-Liouville problems [40] that the eigenvalues of (6.2) can be ordered

$$\lambda_f^{N,0} > \lambda_f^{N,1} > \cdots > \lambda_f^{N,J}$$

(for some finite number $J = J^N$) and that the eigenfunction associated to $\lambda_f^{N,j}$ must have exactly j zeroes. The application of this result to (6.2) immediately implies that $J^N \geq 2N - 1$ (independent of β_2). This can be seen as follows. The derivative of the wave $\partial V_0/\partial\xi$ is an eigenfunction of the full problem (3.1) at $\lambda = 0$. There must be an eigenvalue and an eigenfunction of reduced problem (6.2) associated to it ($\partial V_0/\partial\xi$ is only an approximation of the eigenfunction of (6.2) since $V_0(\xi)$ is not an exact solution of (3.13)). Just like the function $\partial V_0/\partial\xi$, this eigenfunction will have $2N - 1$ zeroes (Lemma 6.1(i)), thus, there must be $2N - 1$ eigenvalues $\lambda_f^{N,j} > \lambda_f^{N,2N-1} = 0+\text{h.o.t.}$ of (6.2). This yields that the number of eigenvalues is at least N times that of the 1-pulse pattern for $\beta_2 \geq 3$ (Remark 5.7). It is natural to expect that

$$J^N = (J + 1)N - 1,$$

with $J = J(\beta_2)$ as defined in Proposition 5.6 (see Remark 6.5).

The same classical result on the eigenvalues and their eigenfunctions of Sturm-Liouville problems also implies that all eigenfunctions of (6.2) must be either even or odd as functions of ξ, as in the case $N = 1$. $V_0(\xi)$ is even, thus, an eigenfunction that is neither even nor odd transforms by $\xi \to -\xi$ into another, independent, eigenfunction with the same (finite) number of zeroes, which is not possible. Thus, we conclude that the eigenfunction $v_f^{N,j}(\xi)$ of (6.2) that corresponds to the eigenvalue $\lambda_f^{N,j}$ is even for even j and odd for odd j. This enables us to formulate explicit expressions for the first two eigenvalues

$$\lambda_f^{N,0} \quad \text{and} \quad \lambda_f^{N,1}.$$

Let $L^2(\mathbb{R};\mathrm{e})$, respectively $L^2(\mathbb{R};\mathrm{o})$, denote the space of all even (resp. odd) L^2 functions on $\mathbb{R}$: $\lambda_f^{N,0}$ is the critical eigenvalue of (6.2) for v restricted to $L^2(\mathbb{R};\mathrm{e})$ and $\lambda_f^{N,1}$ is the critical eigenvalue of (6.2) for v restricted to $L^2(\mathbb{R};\mathrm{o})$. Hence,

$$\begin{aligned}\lambda_f^{N,0} &= \sup_{\substack{v\in L^2(\mathbb{R};\mathrm{e})\\ v\neq 0}} \frac{(\mathcal{L}_f^N(\xi;\varepsilon)v,v)}{\|v\|^2},\\ \lambda_f^{N,1} &= \sup_{\substack{v\in L^2(\mathbb{R};\mathrm{o})\\ v\neq 0}} \frac{(\mathcal{L}_f^N(\xi;\varepsilon)v,v)}{\|v\|^2},\end{aligned} \tag{6.3}$$

with the standard L^2 inner product and norm, see for instance [16]. Using Lemma 6.1 we obtain the following result.

Lemma 6.2. *There exists a* $\rho > 0$ *such that*

$$\lambda_f^* + \mathcal{O}(\varepsilon^\rho) = \lambda_f^{N,0} > \lambda_f^{N,1} = \lambda_f^* + \mathcal{O}(\varepsilon^\rho),$$

where $\lambda_f^* > 0$ *is the critical eigenvalue of the* 1*-pulse eigenvalue problem* (5.3).

Proof. Define $v_\ell(\xi) \in L^2(\mathbb{R})$ by

$$v_\ell(\xi) = \ell_k v_f^*(\xi - \xi_{2k-1}; u_{h,N}) \quad \text{for } \xi \in (\xi_{2k-2},\ \xi_{2k}),\ k = 1,2,\ldots,N, \tag{6.4}$$

with $\ell_k \in \mathbb{R}$ and ξ_j as defined in Lemma 6.1; $v_f^*(\xi; u_{h,N})$ is the eigenfunction corresponding to the critical eigenvalue λ_f^* of the 1-pulse eigenvalue problem

$$\begin{aligned}(\mathcal{L}_f(\xi;u_{h,N}) - \lambda)v &= v_{\xi\xi}\\ &\quad + [\beta_2 h_2(u_{h,N})^{\alpha_2}(v_h(\xi;u_{h,N}))^{\beta_2-1} - (1+\lambda)]v = 0.\end{aligned} \tag{6.5}$$

Note that for $N = 1$, $\mathcal{L}_f(\xi;u_{h,1}) = \mathcal{L}_f(\xi)$ as defined in (4.7). The operator $\mathcal{L}_f^N(\xi;\varepsilon)$, defined by (6.2), can be approximated by N translated copies of $\mathcal{L}_f$, $\mathcal{L}_f(\xi - \xi_{2k-1}; u_{h,N})$ ($k = 1,\ldots,N$), by Lemma 6.1:

$$\begin{aligned}(\mathcal{L}_f^N v_\ell, v_\ell) &= \sum_{k=1}^N \int_{\xi_{2k-2}}^{\xi_{2k}} \ell_k^2 \mathcal{L}_f^N(\xi)[v_f^*(\xi - \xi_{2k-1}; u_{h,N})] v_f^*(\xi - \xi_{2k-1}; u_{h,N})\, d\xi\\ &= \sum_{k=1}^N \ell_k^2 \int_{\xi_{2k-2}}^{\xi_{2k}} \mathcal{L}_f(\xi - \xi_{2k-1}; u_{h,N})[v_f^*(\xi - \xi_{2k-1}; u_{h,N})]\\ &\qquad\qquad \cdot v_f^*(\xi - \xi_{2k-1}; u_{h,N})\, d\xi + R_1^N \quad =\end{aligned}$$

$$= \lambda_f^* \sum_{k=1}^{N} \ell_k^2 \int_{\xi_{2k-2}}^{\xi_{2k}} (v_f^*(\xi - \xi_{2k-1}; u_{h,N}))^2 \, d\xi + R_1^N$$
$$= \lambda_f^* \|v_\ell\|^2 + R_1^N,$$

where, by (6.2) and (6.5),

$$|R_1^N| \le \beta_2 h_2(u_{h,N})^{\alpha_2} \sum_{k=1}^{N} \ell_k^2 \int_{\xi_{2k-2}}^{\xi_{2k}} |(V_0(\xi))^{\beta_2 - 1} - (v_h(\xi - \xi_{2k-1}))^{\beta_2-1}| \cdot (v_f^*(\xi - \xi_{2k-1}))^2 \, d\xi$$
$$= \mathcal{O}(\varepsilon^\rho),$$

by estimate (6.1). Thus, we have that $(\mathcal{L}_f^N v_\ell, v_\ell)/\|v_\ell\|^2 = \lambda_f^* + \mathcal{O}(\varepsilon^\rho)$ for any $(\ell_1, \dots, \ell_N) \in \mathbb{R}^N$. Since, $v_\ell(\xi) \in L^2(\mathbb{R}; \mathrm{e})$, respectively $v_\ell(\xi) \in L^2(\mathbb{R}; \mathrm{o})$, for $\ell_k = \ell_{N-k}$, resp. $\ell_k = -\ell_{N-k}$ (Lemma 6.1 (iv)) we know by (6.3) that $\lambda_f^{N,0} > \lambda_f^{N,1} \ge \lambda_f^* + \mathcal{O}(\varepsilon^\rho)$.

A similar decomposition of $(\mathcal{L}_f^N v, v)$ for a general $v \in L^2(\mathbb{R})$ shows that the critical eigenvalue $\lambda_f^{N,0}$ of (6.2) must approach λ_f^* as ε^ρ for some $\rho > 0$ in the limit $\varepsilon \to 0$ ($\lambda_f^{N,0}$ is the supremum of $(\mathcal{L}_f^N v, v)/\|v\|^2$ over all $v \in L^2(\mathbb{R})$ ($v \neq 0$); it follows from the decomposition of $\mathbb{R}$ into the (ξ_{2k-2}, ξ_{2k})-intervals ($k = 1, \dots, N$) that $(\mathcal{L}_f^N v, v)/\|v\|^2 \le (\mathcal{L}_f v_f^*, v_f^*)/\|v_f^*\|^2 + \mathcal{O}(\varepsilon^\rho) = \lambda_f^* + \mathcal{O}(\varepsilon^\rho)$ for all $v \in L^2(\mathbb{R})$). Hence, the statement of the Lemma follows. □

As we already observed above, this result on the eigenvalues of the fast reduced limit problem implies that the component $t_1^N(\lambda, \varepsilon)$ of the Evans function $\mathcal{D}^N(\lambda, \varepsilon)$ has at least two zeroes, $\lambda^{N,0}(\varepsilon)$ and $\lambda^{N,1}(\varepsilon)$, that both approach λ_f^* as $\varepsilon \to 0$. A zero of $t_1^N(\lambda, \varepsilon)$ corresponds to a zero of $\mathcal{D}^N(\lambda, \varepsilon)$ when $t_2^N(\lambda, \varepsilon)$ is non-singular at this value of λ. We can now, once again, follow the arguments developed for the case $N = 1$ in Section 4 in order to derive a leading order approximation of $t_2^N(\lambda, \varepsilon)$.

By Lemma 6.1 we know that we can approximate $\varphi_2^N(\xi)$ as in Lemma 4.2. Thus, since the u-component of $\varphi_2^N(\xi)$ is equal to a constant ($= 1$) to leading order on I_t (4.1), the leading order equation for the v-component of $\varphi_2^N(\xi)$ reads

$$(\mathcal{L}_f^N(\xi; \varepsilon) - \lambda)v = -\alpha_2 h_2(u_{h,N})^{\alpha_2 - 1} V_0^{\beta_2} \tag{6.6}$$

(see also (4.6)), where we can extend the ξ-interval I_t to $\mathbb{R}$ without altering the leading order effect. Instead of proceeding along the lines of Section 4.1 and deriving an explicit expression for $t_2^N(\lambda, \varepsilon)$, we immediately establish the equivalent of Corollary 4.4.

Lemma 6.3. *Let $\lambda_f^{N,j}$ be an eigenvalue of* (6.2) *and let $\lambda^{N,j}(\varepsilon)$ denote the corresponding zero of $t_1^N(\lambda,\varepsilon)$: $t_2^N(\lambda,\varepsilon)$ has a pole* (*of order one*) *at $\lambda^{N,j}(\varepsilon)$ when j is even, $t_2^N(\lambda,\varepsilon)$ is regular at $\lambda^{N,j}(\varepsilon)$ when j is odd.*

Proof. The operator $\mathcal{L}_f^N - \lambda$ is not invertible at an eigenvalue $\lambda = \lambda_f^{N,j}$. The solvability condition for the inhomogeneous equation (6.6) at $\lambda = \lambda_f^{N,j}$ reads

$$\int_{-\infty}^{\infty} (V_0(\xi))^{\beta_2} v_f^{N,j}(\xi)\, d\xi = 0,$$

where $v_f^{N,j}(\xi)$ is the eigenfunction of (6.2) associated to $\lambda_f^{N,j}$. As was already observed: it follows from classical results [40] that $v_f^{N,j}(\xi)$ is even, respectively odd, as function of ξ, for j even, resp. odd. Since $V_0(\xi)$ is even, we find that (6.6) has no solutions for j even, and a 1-parameter family of solutions for j odd. Hence, $t_2^N(\lambda,\varepsilon)$ has a simple pole at $\lambda = \lambda^{N,j}(\varepsilon)$ for j even (by the argument in the proof of Lemma 4.3), and $t_2^N(\lambda,\varepsilon)$ is well-defined at $\lambda = \lambda^{N,j}(\varepsilon)$ for j odd (by the argument in the proof of Corollary 4.4). $\square$

The combination of Lemmas 6.2 and 6.3 yields the following result.

Theorem 6.4. *Let $\mu > 0$ and $G_i(V)$, h_i, α_i, β_i ($i = 1, 2$) as in* (2.5), (2.6). *For $N \geq 2$, the eigenvalue problem* (3.1) *has at least one eigenvalue $\lambda = \lambda^{N,1}(\varepsilon) > 0$ that approaches the critical eigenvalue $\lambda_f^* > 0$ of* (5.3) *in the limit $\varepsilon \to 0$.*

Thus, the N-pulse solutions $(U_0(\xi), V_0(\xi))$ described by Theorem 2.1 are *unstable* for $N \geq 2$. The NLEP procedure might be able to eliminate the critical eigenvalue $\lambda_f^{N,0}$ of (6.2) by a zero-pole cancellation, but it does not have a leading order influence on the eigenvalues associated to odd eigenfunctions. This is no problem for $N = 1$, since there are no positive eigenvalues of the reduced linearized stability problem associated to an odd eigenfunction, but, Lemma 6.2 shows that this feature of the NLEP procedure yields the above instability result for all $N \geq 2$.

Remark 6.5. It is possible to obtain explicit approximations of all N eigenvalues of (6.2) in an $\mathcal{O}(\varepsilon^\rho)$ neighborhood of the eigenvalues λ_f^j of the 1-pulse problem (5.3) for $j = 1, 2, \ldots, J$ (Proposition 5.6). One can do this using the equivalent of the N-dimensional linear space spanned by the functions v_ℓ, as was defined in the proof of Lemma 6.2 for λ near λ_f^0 (6.4). For general λ_f^j, the functions v_ℓ^j consist of N independent copies of the eigenfunction $\ell_k v_f^j$ of (6.5) on the intervals (ξ_{2k-2}, ξ_{2k}), $k = 1, 2, \ldots, N$. The space $\{v_\ell^j\}_{\ell\in\mathbb{R}^N}$ is the leading order approximation of the N-dimensional space spanned by the N eigenfunctions of (6.2) associated to the N eigenvalues near λ_f^j. It is necessary to compute the next order approximations of $V_0(\xi)$ (Lemma 6.1 only gives the leading order approximation) and use these results to obtain the next order corrections to $\{v_\ell^j\}$; this is a computationally cumbersome, but straightforward procedure. The N eigenvalues and eigenfunctions can then be obtained by maximizing variational expressions as (6.3) with respect to $(\ell_1, \dots, \ell_N)$. Note, however, that one has to be extra careful as β_2 decreases through a value at which a new eigenvalue λ_f^J of (5.3) appears from the essential spectrum (Proposition 5.6, Remark 5.7): it is a priori not clear how the N eigenvalues of (6.2) that will accompany λ_f^J are created.

Remark 6.6. In principle it is possible to keep track of all eigenvalues of (3.1) for $N \geq 2$, just as was done for $N = 1$ in the previous sections. One can compute leading order expressions for $t_2^N(\lambda, \varepsilon)$ comparable to (4.11). Here, one has to take for instance into account that $\gamma_h^N(\xi)$ spirals N-times through phase space when one computes $\Delta_{\text{fast}} u_\xi$ (see (4.9)). It is also possible to derive asymptotic results on $t_2^N(\lambda, \varepsilon)$ and explicit formulae for the eigenvalues in terms of hypergeometric functions, as was done in Section 5 on page 479 for $N = 1$.

7. DISCUSSION

In this section we discuss the relation between this paper and (part of) the literature on related subjects.

The Gray-Scott model. The analysis in this paper can be seen as an extension and a generalization of the results in [9] (existence of homoclinic patterns) and [6], [7] (stability) on pattern formation in the Gray-Scott model. We have shown here that the 'NLEP paradox', i.e., the zero/pole cancellation in the decomposition of the Evans function encountered in the analysis of the Gray-Scott model, is not a special feature of this model. This zero/pole cancellation occurs in a large family (1.1) of singularly perturbed reaction-diffusion equations. Moreover, we have extended the NLEP approach so that it also is possible to study homoclinic N-pulse patterns. Without going into the details, we can state here that the methods developed in Section 6 on page 492 immediately yield that all N-pulse patterns in the Gray-Scott model [9] are unstable for $N \geq 2$.

Another relevant extension of the ideas developed in [6] is the transformation of the inhomogeneous problem that is essential to the NLEP approach into an inhomogeneous hypergeometric differential equation by a method that does not depend on an explicit formula for the 'potential' $w_h(\xi)$ of the Schrödinger problem (5.3). This transformation can also be applied to other nonlocal eigenvalue problems that appear in the literature, including those in [41] where a generalized Gray-Scott model has been studied by the shadow system approach (see below).

The shadow system approach. The shadow system approach can be applied to systems of the type (1.1) with $0 < d_v \ll 1 \ll d_u$ on a *bounded* domain $\Omega \subset \mathbb{R}^n$. As a consequence of these assumptions, one finds that $U(x,t) \equiv$ a constant, to leading order. This constant can be expressed in terms of an integral over V by solving the U equation. Hence, one obtains a scalar equation for V with a *nonlocal* term as leading order approximation, and this equation is known as the shadow system (see also [2] and [3] for more examples of reaction-diffusion equations with nonlocal terms). The shadow system can then be studied by variational techniques, and the results can be extended to the full system. This has been done in [28], [29], [30] for the generalized Gierer-Meinhardt problem, and in [41] for a generalized Gray-Scott model.

The main difference with the analysis in this paper is that here we cannot assume (or derive) that $U(x,t)$ is constant (to leading order): $U(x,t)$ varies slowly as a function of x, but it certainly has an $\mathcal{O}(1)$ range (Figure 1.1 on page 444). This has a significant effect on both the existence and the stability problem for the homoclinic pulse patterns: the results obtained in [41] for the Gray-Scott model on the bounded domain cannot be extended to results on an unbounded domain (see [6], [7], [4], [5]).

Nevertheless, both in the NLEP approach and in the shadow system approach one obtains an eigenvalue equation with a nonlocal term that is associated to the stability of the 1-pulse pattern (only 1-pulse patterns are considered in [28], [29], [30], [41]). These equations can all be studied by the transformation into hypergeometric form presented in this paper. Thus, for instance, the (in)stability results on pulses in the generalized Gierer-Meinhardt problem in [28], [29], [30] can be extended to cases where the parameter α does not satisfy the condition '*α is small enough*'. Here, α corresponds to $D/(\beta_2 - 1)$ in our notation. Note that this implies that the original Gierer-Meinhardt problem considered in Section 5.3 has not been covered by [28], [29], [30] (since $\alpha = 1$ here (1.9)); and, the application of the hypergeometric functions analysis to the shadow system approach of [28], [29], [30] will yield a stability result similar to that of Section 5.3. A similar observation can be made about [41], since the stability analysis presented there is also only valid in special (small) regions of the $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-parameter space.

However, it should also be noted that many of the results obtained in [28], [29], [30], [41] are valid in spatial domains of dimension greater than 1. Such results cannot be obtained by the more geometrical/dynamical systems methods of this paper. Furthermore, many of the more general results obtained by the shadow system approach that establish stability properties in the spirit of Corollary 4.5 and Theorems 5.1, 5.2 cover regions of the parameter space that are not considered explicitly here; these results are often also more specific than Corollary 4.5 and Theorems 5.1, 5.2.

Thus, one can conclude that the NLEP approach and the shadow system approach are truly complementary. It will be the subject of future research to find out the details of the relation between the two methods, with special focus on the relation between the different nonlocal eigenvalue problems.

The stability of homoclinic multi-pulse patterns. There is an essential difference between the homoclinic multi-pulses studied in this paper and those in [35], [36]. The N-circuit orbits that are homoclinic to a hyperbolic point $\tilde{S}$ studied in [35], [36], and related papers (see the references in [35]) can be 'built' from N-copies of a basic 1-circuit homoclinic orbit. As a consequence, the N-circuit orbits return $N - 1$ times to a small neighborhood of $\widetilde{S}$. This is not the case for the N-circuit homoclinic orbits constructed in this paper: these orbits approach the slow manifold $\mathcal{M}$ $N - 1$ times, but remain an $\mathcal{O}(1)$ distance away from the saddle S during these approaches. Moreover, the N-circuit orbits ($N \geq 2$) are not at all close to the 1-circuit orbit in the 4-dimensional phase space (see Figure 2.1 on page 464).

As a consequence, the general theory developed in [35] cannot be applied to the N-circuit orbits in this paper. In contrast to the results in [35], we find in this paper that the discrete part of the spectrum of the linear problem associated to the stability of the N-pulse pattern is not at all close to that of the 1-pulse pattern. The multi-pulse solutions in [35], [36] can be stable, while we have found that the multi-pulse solutions are always unstable. In a sense one can say that the multi-pulse patterns studied in this paper are 'strongly interacting', since the U-components of the patterns do not approach $U \equiv 0$ in between the fast excursions. The multi-pulse patterns considered in [35] are in this sense 'weakly interacting' and can thus be treated, to leading order, as N copies of the original 1-pulse pattern (see also [4] for a similar distinction between strongly and weakly interacting pulses).

The stability properties of the N-pulse patterns studied in this paper ($N \geq 2$) are similar to those of the N-pulse patterns in *scalar* equations with a *nonlocal term* studied in [2], [3]. The spectrum of the problem associated to the stability of the N-pulse patterns ($N \geq 2$) in [2], [3] has a number of features in common with the spectrum for the N-pulse patterns in this paper. However, in the problems studied in [2], [3], the eigenvalues cannot have non-zero imaginary parts.

The SLEP method. The NLEP approach developed in this paper is in spirit (and in name) very similar to the SLEP method developed in [32], [33], [31]. Both methods exploit the slow-fast structure of the linear problem associated to the stability of a 'localized' pattern. However, while the SLEP method has been developed to study the stability of *heteroclinic* patterns (traveling waves) in bi-stable reaction-diffusion equations, the NLEP approach has been developed to study *homoclinic* (stationary) patterns that are biasymptotic to a single stable homogeneous state. Moreover, while the patterns studied here by the NLEP approach possess unstable $\mathcal{O}(1)$ eigenvalues in the fast reduced limit, leading to the 'NLEP paradox', the traveling waves studied by the SLEP method do not have such unstable $\mathcal{O}(1)$ eigenvalues in the fast reduced limit. Thus, there is no 'NLEP paradox' for these waves. Moreover, in contrast to the analysis in this paper, the 'full' (non-reduced) eigenvalue problem for the traveling waves in a bi-stable system studied by the SLEP method has no $\mathcal{O}(1)$ unstable eigenvalues.

Both methods reduce the 4-dimensional linear eigenvalue problem to a 2-dimensional limit problem. In the SLEP method this is done, in essence, by solving the fast (v) equation and substituting this solution into the slow (u) equation. By taking the limit $\varepsilon \to 0$, a 2-dimensional eigenvalue problem is obtained in which the fast components are represented by a Dirac delta-function. By contrast, we have seen that in the NLEP approach, the 4-dimensional linear eigenvalue problem is reduced to the 2-dimensional problem associated to the fast (v) components in which the slow components are represented by the nonlocal term. In this sense, the SLEP method and the NLEP approach can be interpreted as the two opposite limits of a singularly perturbed eigenvalue problem. A priori, it seems that the nature of this eigenvalue problem determines which one of the two limits (SLEP or NLEP) is the most appropriate. At this point, the relation between both methods has not been investigated in detail. This is an important issue for the understanding of singularly perturbed eigenvalue problems and thus for the theory of pattern formation in (singular) reaction-diffusion equations. Therefore, this will be the topic of future research. Finally, we note that the SLEP method has also been linked to the Evans function [19].

APPENDIX A. THE DERIVATION OF THE SCALED SYSTEM

The substitution of (1.3) into (1.2) yields, by (1.4) and (1.5):

$$\text{(A.1)} \qquad \begin{cases} \tilde{U}_t = \tilde{U}_{xx} - \mu\tilde{U} + \varepsilon^{-(\alpha_1-1)r-\beta_1 s}\tilde{U}^{\alpha_1}\tilde{V}^{\beta_1}(h_1 + \hat{\varepsilon}\tilde{H}_1(\tilde{U},\tilde{V};\varepsilon)) \\ \tilde{V}_t = \varepsilon^2\tilde{V}_{xx} - \tilde{V} + \varepsilon^{-\alpha_2 r-(\beta_2-1)s}\tilde{U}^{\alpha_2}\tilde{V}^{\beta_2}(h_2 + \hat{\varepsilon}\tilde{H}_2(\tilde{U},\tilde{V};\varepsilon)). \end{cases}$$

Homoclinic solutions in the fast variable $\tilde{V}$ can be constructed if the linear and nonlinear terms in the $\tilde{V}$-equation are of the same magnitude (with respect to ε), therefore we impose $\alpha_2 r + (\beta_2 - 1)s = 0$ as a first condition on r and s. By introducing the new spatial scale $\tilde{x} = \varepsilon^{-(1/2)((\alpha_1-1)r+\beta_1 s)}x$ we can rewrite (A.1) as

$$\text{(A.2)} \qquad \begin{cases} \varepsilon^{(\alpha_1-1)r+\beta_1 s}\tilde{U}_t = \tilde{U}_{\tilde{x}\tilde{x}} - \varepsilon^{(\alpha_1-1)r+\beta_1 s}\mu\tilde{U} \\ \qquad\qquad + \tilde{U}^{\alpha_1}\tilde{V}^{\beta_1}(h_1 + \hat{\varepsilon}\tilde{H}_1(\tilde{U},\tilde{V})) \\ \tilde{V}_t = \varepsilon^{2-((\alpha_1-1)r+\beta_1 s)}\tilde{V}_{\tilde{x}\tilde{x}} - \tilde{V} \\ \qquad\qquad + \tilde{U}^{\alpha_2}\tilde{V}^{\beta_2}(h_2 + \hat{\varepsilon}\tilde{H}_2(\tilde{U},\tilde{V})). \end{cases}$$

The parameters r and s are now determined by the second condition $(\alpha_1 - 1)r + \beta_1 s = 1$ so that (1.6) follows (see also [20]). Introducing $\tilde{\varepsilon} = \sqrt{\varepsilon}$ in (A.2) yields (1.7) (after dropping the tildes). A related rescaling for a generalized Gierer-Meinhardt model has been performed in [20], [28], [29], [30].

Note that the second condition on r and s is motivated by the existence analysis in Section 2 on page 451: the balance in the magnitudes of the linear term in the $\tilde{U}$ or U equation and the diffusivity in the $\tilde{V}$ or V equation is necessary for the construction of the homoclinic solutions described in Theorem 2.1. Moreover, since the coefficient of the U_t term is of exactly the same order in ε, the homoclinic solutions can be stabilized by the NLEP mechanism.

Generically, it is also possible to reduce a more general model as (1.1) to 'normal form' (1.7). This can be done by including the linear coupling terms $a_{12}V$ and $a_{21}U$ in the definitions of respectively $H_1(U,V)$ and $H_2(U,V)$ in (1.1), and determining the leading order behavior of the 'new' $H_i(\tilde{U}/\varepsilon^r, \tilde{V}/\varepsilon^s)$ *as functions of* s *and* r. Once again, it has to be assumed that both functions $H_i(\tilde{U}/\varepsilon^r, \tilde{V}/\varepsilon^s)$ ($i = 1, 2$) have a leading order behavior that is *algebraic* in $\tilde{U}$ and $\tilde{V}$. Unlike the separable case $H_i(\tilde{U}/\varepsilon^r, \tilde{V}/\varepsilon^s) = F_i(\tilde{U}/\varepsilon^r)G_i(\tilde{V}/\varepsilon^s)$, the leading order exponents α_i and β_i will in general depend on r and s (consider for example $H_1(U,V) = a_{12}V + b_1U^2 + b_2UV^3$). Thus, determining (1.7) will now require a bit more 'book keeping'. Moreover, one has to impose a non-degeneracy condition now, since it cannot be excluded in general that the critical values of r and s (determined by (1.6)) are on the set of co-dimension 1, where the leading order behavior of $H_i(\tilde{U}/\varepsilon^r, \tilde{V}/\varepsilon^s)$ is given by $\varepsilon^\gamma(h_i^1\tilde{U}^{\alpha_i^1}\tilde{V}^{\beta_i^1} + h_i^2\tilde{U}^{\alpha_i^2}\tilde{V}^{\beta_i^2})$, for some γ, $(\alpha_i^1, \beta_i^1) \neq (\alpha_i^2, \beta_i^2)$, and $h_i^{1,2} \neq 0$. Since we do not want to go into the details of formulating an exact non-degeneracy condition, and since it is rather straightforward to apply the above procedure to a given pair of non-separable nonlinearities $H_i(U,V)$, we decided to focus on model problem (1.2) in this paper.

Remark A.1. The above procedure can be applied without any modifications when one is interested in *small* solutions, i.e., when r, $s < 0$ in (1.3). For 'mixed' solutions ($rs \leq 0$) one encounters the same problems in (1.2) as for large solutions in (1.1): the leading orders α_i and β_i will depend on r and s. Again, this is no more than a small technical complication.

Remark A.2. Note that one has to be extra careful in reducing the general problem (1.1) to a normal form of the type (1.7) when $a_{11} > 0$ (which is possible, see Remark 1.2). In this case one cannot automatically replace '$-\mu U$' by '$+a_{11}U$' in (1.7) since this will imply that the basic pattern $(0,0)$ is no longer asymptotically stable. Thus, one cannot scale the linear coupling terms '$a_{12}V$' and '$a_{21}U$' into higher order effects. As a consequence, the scaling analysis will produce a 'normal form' that differs slightly from (1.7). Nevertheless, the analysis can be developed along the same lines as for (1.7). We do not go further into the details here.

APPENDIX B. THE INHOMOGENEOUS HYPERGEOMETRIC DIFFERENTIAL EQUATION

Since (5.7) is symmetric with respect to $z \to 1-z$ we know by (5.12) that the pair

$$F(a\,,b \mid c \mid 1-z) \quad \text{and} \quad (1-z)^{1-c}F(a-c+1\,,b-c+1 \mid 2-c \mid 1-z)$$

also spans the solution space of (5.7). We exploit the symmetrical nature of the problem by introducing

$$\text{(B.1)} \qquad \begin{aligned} H_1(z;P,\beta_2) &= F(a\,,\,b\mid c\mid z),\\ H_2(z;P,\beta_2) &= H_1(1-z;P,\beta_2) = F(a\,,\,b\mid c\mid 1{-}z), \end{aligned}$$

with a, b, c as in (5.11). By (5.14), we observe that

$$\text{(B.2)} \quad \begin{aligned} F(a\,,\,b\mid c\mid 1{-}z) = {}& L(P;\beta_2)F(a\,,\,b\mid c\mid z)\\ &+ M(P;\beta_2)z^{1-c}F(a{-}c{+}1\,,\,b{-}c{+}1\mid 2{-}c\mid z), \end{aligned}$$

with

$$\text{(B.3)} \qquad \begin{aligned} L(P;\beta_2) &= -\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \quad \text{and}\\ M(P;\beta_2) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \end{aligned}$$

with a, b, c as in (5.11). Note that $M(P;\beta_2)$ already appeared in the right hand side of (5.14). Thus, the pair $\{H_1(z;P,\beta_2),\ H_2(z;P,\beta_2)\}$ spans the solutions space of (5.7) when $M(P;\beta_2) \neq 0$, or equivalently, when $P \neq P^j$ (see the proof of Proposition 5.6). The linear operator of (5.7), or equivalently of (5.3), is non-invertible at these (eigen)values of P, therefore, (5.18) has to be studied in an independent way at $P = P^j$, see Remark 5.8 and Remark B.4.

We define the Wronskian $W(z) = H_1(z)H_2'(z) - H_1'(z)H_2(z)$ so that, by (5.7), $W(z) = C_W(z(1-z))^{-c}$ for some constant

$$C_W = \lim_{z\to 0}(z(1-z))^c W(z) = \lim_{z\to 0} z^c(H_1(z)H_2'(z) - H_1'(z)H_2(z)).$$

It follows from (B.1) and (5.14) that

$$\lim_{z\to 0} z^c H_2'(z) = (1-c)M(P;\beta_2),$$

hence, by (5.11),

$$\text{(B.4)} \qquad W(z;P,\beta_2) = -\frac{2P}{\beta_2-1}M(P;\beta_2)(z(1-z))^{2P/(\beta_2-1)},$$

where $M(P;\beta_2)$ is given by (B.3). To solve the inhomogeneous problem (5.18), we introduce the functions $g_1(z)$ and $g_2(z)$ by:

$$\text{(B.5)} \qquad G(z;P,\beta_2) = g_1(z)H_1(z) + g_2(z)H_2(z).$$

Since (5.18) is also symmetrical with respect to $z \to 1 - z$ and since $H_2(z) = H_1(1-z)$ (B.1), we can set $g_2(z) = g_1(1-z)$. By the classical variation of constants method we obtain (using (B.4)) that

$$g_1(z) = g_2(1-z) = \frac{\beta_2 - 1}{2PM(P;\beta_2)}\left[\int_0^z [\zeta(1-\zeta)]^{(1+P)/(\beta_2-1)} H_1(1-\zeta)\,d\zeta + g_0\right],$$

where g_0 is a constant that has to be determined by the conditions on the behavior of $G(z)$ near $z = 0$ and $z = 1$. We observe by (5.14) and (B.5) that

$$\lim_{z\to 1}(1-z)^{c-1}G(z) = \frac{\beta_2 - 1}{2P}\left[\int_0^1 [\zeta(1-\zeta)]^{(1+P)/(\beta_2-1)} H_1(1-\zeta)\,d\zeta + g_0\right],$$

i.e., near $z = 1$, $G(z)$ has for general g_0 the same singular behavior as $H_1(z)$; $G(z)$ behaves similarly near $z = 0$, due to the symmetry. Therefore, we set

$$g_0 = -\int_0^1 [\zeta(1-\zeta)]^{(1+P)/(\beta_2-1)} H_1(1-\zeta)\,d\zeta,$$

which determines $G(z;P,\beta_2)$ uniquely. By (5.19) we thus finally arrive at

$$\text{(B.6)}\quad \begin{aligned} \mathcal{R}(P;\beta_1,\beta_2) = {} & -\frac{\beta_2 - 1}{PM(P;\beta_2)} \\ & \cdot \int_0^1 \int_z^1 [\zeta(1-\zeta)]^{(1+P)/(\beta_2-1)} \\ & \cdot [z(1-z)]^{(P+\beta_1-\beta_2)/(\beta_2-1)} H_1(1-\zeta)H_1(z)\,d\zeta\,dz. \end{aligned}$$

Remark B.1. The function $G(z;P,\beta_2)$ is the uniquely determined *bounded* solution of (5.18) as long as the inhomogeneous term in (5.18) is bounded. However, this term is singular at $z = 0$ and $z = 1$ when $\mathrm{Re}(P) > 1$. For these values of P, $G(z;P,\beta_2)$ is still uniquely determined by the above procedure, but not necessarily bounded. The leading order behavior of $G(z;P,\beta_2)$ can be obtained from the above explicit expressions: $G(z;P,\beta_2) \sim z^{(\beta_2-P)/(\beta_2-1)}$ as $z \to 0$ (and, by the symmetry, $G(z;P,\beta_2) \sim (1-z)^{(\beta_2-P)/(\beta_2-1)}$ as $z \to 1$). This singular behavior does not cause problems in the evaluation of $\mathcal{R}(P;\beta_1,\beta_2)$ (5.19):

$$G(z;P,\beta_2)[z(1-z)]^{(P+\beta_1-\beta_2)/(\beta_2-1)} \sim z^{\beta_1/(\beta_2-1)} \quad \text{as } z \to 0.$$

Thus, $\mathcal{R}(P;\beta_1,\beta_2)$ is a non-singular integral for all $P \neq P^j$ with j even (see Remark 5.8).

Remark B.2. Note that, by (B.3) and (5.11),

$$\text{(B.7)}\qquad M(P;\beta_2) = \frac{2P}{\beta_2 - 1}\frac{\left(\Gamma\left(\dfrac{2P}{\beta_2-1}\right)\right)^2}{\Gamma\left(\dfrac{2P}{\beta_2-1} + \dfrac{2\beta_2}{\beta_2-1}\right)\Gamma\left(\dfrac{2P}{\beta_2-1} - \dfrac{\beta_2+1}{\beta_2-1}\right)}.$$

Thus, the sign of $M(P;\beta_2)$ is determined by the sign of $\Gamma(Q(P))$ with $Q(P) = 2P/(\beta_2-1)-(\beta_2+1)/(\beta_2-1)$ (for $P>0$). For instance, $Q\in(-1,0)$ when $P\in(P^1,P^0)$ (5.15), hence $M(P;\beta_2)<0$ for $P\in(P^1,P^0)$. Combined with the minus sign in front of the right hand side of (B.6), it follows that $\mathcal{R}(P)$ can certainly be positive in this interval, see Corollary 5.9 and Figure 5.1 on page 489.

Remark B.3. Note by (5.11) and (B.3) that $L(P;\beta_2)\equiv 0$ when $(\beta_2+1)/(\beta_2-1)=k=2,3,\ldots$, i.e., $H_2(z)=Mz^{1-c}F(a-c+1\ ,\ b-c+1\mid 2-c\mid z)$ (B.1) for these values of β_2. Moreover, $F(a-c+1\ ,\ b-c+1\mid 2-c\mid z)$ is a k-th order polynomial in z in these cases (since $b-c+1=-k$). Thus, in the special case $\beta_2=2$, i.e., $k=3$, of the classical Gierer-Meinhardt problem (Section 5.3) and the Gray-Scott model [6], we have that $H_1(z;P,2)=M(P;2)(1-z)^{1-c}\times$ (a cubic polynomial in z), see [6]. Furthermore, by (B.7), $M(P;2)=((2P-3)(2P-2)(2P-1))/((2P+3)(2P+2)(2P+1))$: a quotient of cubic polynomials in P. This expression was called $L(P)$ in [6] (recall that $P_{\text{G-S}}=2P$, see Remark 5.12). This yields by (B.6) that $\mathcal{R}(P,\beta_1,\beta_2)$ can be expressed without the use of hypergeometric or Gamma functions for $\beta_2=2$ [6].

Remark B.4. It follows from (B.3), (5.11), and (5.15) that

$$L(P^j;\beta_2)=-\frac{\Gamma\left((1+j)-\dfrac{\beta_2+1}{\beta_2-1}\right)\Gamma\left(\dfrac{\beta_2+1}{\beta_2-1}-j\right)}{\Gamma\left(\dfrac{\beta_2+1}{\beta_2-1}+1\right)\Gamma\left(-\dfrac{\beta_2+1}{\beta_2-1}\right)}=(-1)^j,$$

hence, since $M(P^j;\beta_2)=0$,

$$H_2(z;P^j,\beta_2)=H_1(1-z;P^j,\beta_2)=(-1)^jH_1(z;P^j,\beta_2)\qquad\text{(B.1).}$$

Thus, $H_1(z;P^j,\beta_2)$ is either symmetric or anti-symmetric under the transformation $z\to 1-z$. Since $H_1(z;P^j,\beta_2)$ is regular at both $z=0$ and $z=1$ we observe that $\{H_1(z;P^j,\beta_2),\ H_2(z;P^j,\beta_2)\}$ spans the one-dimensional eigenspace associated to the eigenvalue $P=P^j$ of (5.7). We thus confirm by (5.8) that the eigenfunctions of the homogeneous eigenvalue problem (5.3) are even as functions of ξ for j even and odd for j odd.

Acknowledgments. We thank an anonymous referee for carefully reading the entire manuscript and for suggestions that helped to improve the presentation. A.D. thanks Yasumasa Nishiura for his hospitality in the winter of 1998 and for the stimulating discussions. T.K. gratefully acknowledges support from the National Science Foundation through CAREER grant DMS-9624471 and from the Alfred P. Sloan Foundation in the form of a Sloan Research Fellowship.

REFERENCES

[1] J. ALEXANDER, R.A. GARDNER & C.K.R.T. JONES, *A topological invariant arising in the stability of traveling waves*, J. reine angew. Math. **410** (1990), 167-212.

[2] A. BOSE, *A geometric approach to singularly perturbed non-local reaction diffusion equations*, SIAM J. Math. Anal. **31** (2000), 431-454.
[3] A. BOSE & G.A. KRIEGSMANN, *Stability of localized structures in non-local reaction-diffusion equations*, Meth. Appl. Anal. **5** (1998), 351-366.
[4] A. DOELMAN, W. ECKHAUS & T.J. KAPER, *Slowly-modulated two-pulse solutions in the Gray-Scott model I: asymptotic construction and stability*, SIAM J. Appl. Math. **61** (2000), 1080-1102.
[5] ———, *Slowly-modulated two-pulse solutions in the Gray-Scott model II: geometric theory, bifurcations and splitting dynamics*, SIAM J. Appl. Math. **61** (2001), 2036–2062.
[6] A. DOELMAN, R.A. GARDNER & T.J. KAPER, *Stability analysis of singular patterns in the* 1*-D Gray-Scott model: A matched asymptotics approach*, Physica D **122** (1998), 1-36.
[7] ———, *A stability index analysis of* 1*-D patterns of the Gray-Scott model*, to appear in Memoirs AMS.
[8] A. DOELMAN & P. HOLMES, *Homoclinic explosions and implosions*, Phil. Trans. R. Soc. Lond. Ser. A **354** (1996), 845-893.
[9] A. DOELMAN, T.J. KAPER & P. ZEGELING, *Pattern formation in the one-dimensional Gray-Scott model*, Nonlinearity **10** (1997), 523-563.
[10] N. FENICHEL, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. J. **21** (1971), 193-226.
[11] ———, *Geometrical singular perturbation theory for ordinary differential equations*, J. Diff. Eq. **31**(1979), 53-98.
[12] R.A. GARDNER, *Spectral analysis of long wavelength periodic waves and applications*, J. reine angew. Math. **491** (1997), 149-181.
[13] R.A. GARDNER & C.K.R.T. JONES, *Stability of the travelling wave solutions of diffusive predator-prey systems*, Trans. AMS **327** (1991), 465-524.
[14] R.A. GARDNER & K. ZUMBRUN, *The Gap Lemma and a geometric condition for the instability of viscous shock profiles*, Comm. Pure Appl. Math. **51** (1998), 797-855.
[15] A. GIERER & H. MEINHARDT, *A theory of biological pattern formation*, Kybernetik **12** (1972), 30-39.
[16] D. GILBARG & N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.
[17] P. GRAY & S.K. SCOTT, *Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability*, Chem. Eng. Sci. **38** (1983), 29-43.
[18] D. HENRY, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics Volume 840, Springer-Verlag, New York, 1981.
[19] H. IKEDA, Y. NISHIURA & H. SUZUKI, *Stability of traveling waves and a relation between the Evans function and the SLEP equation*, J. reine angew. Math. **475** (1996), 1-37.
[20] D. IRON & M.J. WARD, *A metastable spike solution for a non-local reaction-diffusion model*, SIAM J. Appl. Math. **60** (2000), 778-802.
[21] C.K.R.T. JONES, (1995), *Geometric singular perturbation theory*, In: Dynamical Systems, Montecatibi Terme, Lecture Notes in Mathematics Volume 1609 (R. Johnson eds.), Springer-Verlag, 1994.
[22] T. KAPITULA, *On the stability of travelling waves in weighted L^∞ spaces*, J. Diff. Eq. **112** (1994), 179-215.
[23] T. KAPITULA & B. SANDSTEDE, *Stability of bright and dark solitary wave solutions to perturbed nonlinear Schrödinger equations*, Physica D **124** (1998), 58-103.
[24] B.S. KERNER & V.V. OSIPOV, *Autosolitons: a New Approach to Problems of Self-Organization and Turbulence*, Kluwer, Dordrecht, the Netherlands, 1994.
[25] D.S. MORGAN, A. DOELMAN & T.J. KAPER, *Stationary periodic orbits in the* 1*-D Gray-Scott model*, Meth. Appl. Anal. **7** (2000), 105–150.
[26] P.M. MORSE & H. FESHBACH, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.
[27] J.D. MURRAY, *Mathematical Biology*, Biomathematics Texts Volume 19, Springer-Verlag, 1989.

[28] W.-M. NI, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices AMS **45** (1998), 9-18.
[29] W.-M. NI & I. TAKAGI, *Point condensation generated by a reaction-diffusion system in axially symmetric domains*, Japan J. Industr. Appl. Math. **12** (1995), 327-365.
[30] W.-M. NI, I. TAKAGI & E. YANAGIDA, *Stability analysis of point-condensation solutions to a reaction-diffusion system proposed by Gierer and Meinhardt*, preprint, 1998.
[31] Y. NISHIURA, *Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit*, Dynamics Reported **3** (1994), 25-103.
[32] Y. NISHIURA & H. FUJII, *Stability of singularly perturbed solutions to systems of reaction-diffusion equations*, SIAM J. Math. Anal. **18** (1987), 1726-1770.
[33] Y. NISHIURA, M. MIMURA, H. IKEDA & H. FUJII, *Singular limit analysis of stability of traveling wave solutions in bistable reaction-diffusion systems*, SIAM J. Math. Anal. **21** (1990), 85-122.
[34] C. ROBINSON, *Sustained resonance for a nonlinear system with slowly-varying coefficients*, SIAM J. Math. Anal. **14** (1983), 847-860.
[35] B. SANDSTEDE, *Stability of multi-pulse solutions*, Trans. AMS **350** (1998), 429-472.
[36] ———, *Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equation*, SIAM J. Math. Anal. **29** (1998), 183-207.
[37] B. SANDSTEDE & A. SCHEEL, *Essential instability of pulses, and bifurcations to modulated travelling waves*, Proc. Roy. Soc. Edinb. Ser. A **129** (1999), 1263-1290.
[38] ———, *Spectral stability of modulated travelling waves bifurcating near essential instabilities*, Proc. Roy. Soc. Edinb. Ser. A. **130** (2000), 419-448.
[39] D.H. SATTINGER, *On the stability of waves of nonlinear parabolic systems*, Adv. in Math. **22** (1976), 312-355.
[40] E.C. TITCHMARSH, *Eigenfunction Expansions Associated with Second-order Differential Equations* (2nd ed.), Oxford Univ. Press, 1962.
[41] J.C. WEI, *Existence, stability and metastability of point condensation patterns generated by the Gray-Scott system*, Nonlinearity **12** (1999), 593-616.

ARJEN DOELMAN:
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam, the Netherlands

ROBERT A. GARDNER:
Department of Mathematics
University of Massachusetts
Amherst, MA 01003, U. S. A.

TASSO J. KAPER:
Department of Mathematics
Boston University
111 Cummington Street
Boston, MA 02215, U. S. A.

1991 MATHEMATICS SUBJECT CLASSIFICATION, AMS (MOS): 35K57, 35B25, 35B32, 35B35, 34C37, 92C15.
Received: *September 16th, 1999; revised: April 11th, 2000.*