Large time asymptotics of nonlinear drift-diffusion systems with Poisson coupling

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Abstract

We study the asymptotic behavior as $t \to +\infty$ of a system of densities of charged particles satisfying nonlinear drift-diffusion equations coupled by a damped Poisson equation for the drift-potential.

In plasma physics applications the damping is caused by a spatio-temporal rescaling of an “unconfined” problem, which introduces a harmonic external potential of confinement. We present formal calculations (valid for smooth solutions) which extend the results known in the linear diffusion case to nonlinear diffusion of *e.g.* Fermi-Dirac or fast diffusion/porous media type.

Key words and phrases: nonlinear drift-diffusion systems, asymptotic behavior of solutions, logarithmic Sobolev inequalities, fast diffusion, porous media

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1 Introduction

Consider the system

$$\begin{aligned} u_t &= \nabla\cdot(\nabla f(u) + u\nabla V + \beta(t)u\nabla\phi) \\ v_t &= \nabla\cdot(\nabla f(v) + v\nabla V - \beta(t)v\nabla\phi) \\ \Delta\phi &= v - u \end{aligned} \tag{1.1}$$

in $\mathbb{R}_t^+ \times \mathbb{R}_x^d$, $d \geq 3$, and assume that β is a nonnegative decreasing function of time t with $\lim_{t\to+\infty}\beta(t)=0$. V is the exterior potential with $V(x)\to+\infty$ as $|x|\to+\infty$. The initial data $u_0=u(t=0)$, $v_0=v(t=0)$ are assumed to be in $L^1_+(\mathbb{R}^d_x)$. The function f satisfies

$$f(0)=0\,, \quad f'(s)>0 \quad \forall\, s\in(0,\infty). \tag{1.2}$$

The system (1.1) can be regarded as a model for a bipolar plasma, where both types of particles are confined by a potential $V(x)$, and where the Poisson coupling (mean field) becomes asymptotically weaker as $t\to+\infty$. In the next section, we derive such a model by a spatio-temporal rescaling from a system without confinement and without damping of the mean field. The function f defines the density-pressure constitutive relation, which is taken equal for both particle species (*cf* [8]).

Note that the minimum principle implies $u(t)$, $v(t)\geq 0$ (since we assumed u_0, $v_0\geq 0$). We remark that for the following we always take the Newtonian potential ψ of g as solution of $-\Delta\psi=g$ in $\mathbb{R}^d$.

In this paper the nonlinearities we have in mind are either

$$f(s) = s^m\,, \quad s \geq 0 \tag{1.3}$$

where the cases $m<1$, $m=1$ and $m>1$ correspond to the fast diffusion equation, the heat equation (linear diffusion) and the porous media equation respectively, or the following diffusion equation corresponding to "physical" 3-dimensional flows in the Fermi-Dirac thermodynamical framework. Define, with $\epsilon>0$ a parameter, $F:\mathbb{R}\to(0,\infty)$ by

$$F(\sigma) := \int_{\mathbb{R}^3_v} \frac{dv}{\epsilon + \exp(|v|^2/2-\sigma)}\,. \tag{1.4}$$

Clearly, $F(-\infty)=0$, $F(\infty)=\infty$. The nonlinearity f in (1.1) then reads

$$f(s) = sF^{-1}(s) - \int_0^s F^{-1}(\tau)\,d\tau\,, \quad 0 \leq s < \infty \tag{1.5}$$

(where F^{-1} denotes the inverse function of F).

Note that stationary solutions of the equation

$$z_t = \nabla\cdot(\nabla f(z) + z\nabla V) = \nabla\cdot(z(\nabla h(z) + \nabla V))$$

where $h'(s) = f'(s)/s$, are of the form

$$z(x) = \Big(C - V(x)\Big)_+^{1/(m-1)} \quad \text{if} \quad m \neq 1\,,$$
$$z(x) = C\,e^{-V(x)} \quad \text{if} \quad m = 1$$

for (1.3), and

$$z(x) = \int_{\mathbb{R}^3_v} \frac{dv}{\epsilon + C\,\exp(V(x) + |v|^2/2)}$$

for (1.4)-(1.5).

At the end of this introduction, let us mention a (nonexhaustive) list of references related to this work. Concerning the Gross logarithmic Sobolev inequalities in a PDE framework, we refer to [2] and references therein. The extension to the porous media or fast diffusion cases have been studied in [5, 6, 9]. For systems with a Poisson coupling and a linear diffusion, let us quote [2, 1, 3]. References [4, 7, 8] are relevant for the modelization and the analysis in the plasma physics or semiconductor context.

Notation. In the sequel the $L^p(\mathbb{R}^d)$ norms shall be denoted by $|\,.\,|_p$.

2 Derivation from a drift-diffusion system without confinement

Systems of the form (1.1) can be obtained by a spatio-temporal rescaling from drift-diffusion systems without confinement, and with a nonlinear diffusion of power-law type.

Consider the system for the densities n and p of oppositely charged particles

$$\begin{aligned} n_t &= \nabla\cdot(\nabla f(n) + n\nabla\psi) \\ p_t &= \nabla\cdot(\nabla f(p) - p\nabla\psi) \\ \Delta\psi &= p - n \end{aligned} \tag{2.1}$$

where, with $m > 0$

$$f(s) = s^m \quad \text{for} \quad s \geq 0, \tag{2.2}$$

and define u and v by

$$\begin{aligned} n(t,x) &= \frac{1}{R^d(t)} u\left(\log R(t), \frac{x}{R(t)}\right), \\ p(t,x) &= \frac{1}{R^d(t)} v\left(\log R(t), \frac{x}{R(t)}\right), \end{aligned} \tag{2.3}$$

with an increasing function $R > 0$.

Lemma 2.1 *A solution $\langle n,p\rangle$ of (2.1) (with f given by (2.2)) corresponds by the change of variables (2.3) to a solution $\langle u,v\rangle$ of (1.1) if and only if*

$$\begin{aligned} \dot{R}R^{d(m-1)+1} &= 1, \\ V(x) &= \frac{1}{2}|x|^2, \\ \beta(t) &= R(t)^{2-d}. \end{aligned} \tag{2.4}$$

Moreover

$$\psi(t,x) = \frac{1}{R^{d-2}(t)} \phi\left(\log R(t), \frac{x}{R(t)}\right).$$

Note that $\langle n,p\rangle$ and $\langle u,v\rangle$ have the same initial data if $R(0)=1$. Contrarily to (1.1), the strength of the Poisson coupling in (2.1) is assumed to be constant in time: the damping in (1.1) appears as a consequence of the rescaling.

3 Asymptotic (uncoupled) problem

Consider now the system (1.1) with $\beta=0$. Both u and v then solve an equation of the form

$$z_t=\nabla\cdot(\nabla f(z)+z\nabla V)\,,\quad z(0)=z_0\geq 0\,. \tag{3.1}$$

Formally we have

$$\int z(x,t)\,dx=\int z_0(x)\,dx\quad \text{for all } t>0$$

(all the integrals are over $\mathbb{R}^d$, unless specified differently). Let

$$W[z]=\int[z(V+h(z))-f(z)]\,dx \tag{3.2}$$

with the enthalpy defined by

$$h(z)=\int_1^z\frac{f'(s)}{s}\,ds\,. \tag{3.3}$$

For a solution of (3.1), a standard computation (formally) gives

$$\begin{aligned}\frac{d}{dt}W[z](t)\;&=\int[V+(h(z)+zh'(z)-f'(z))]z_t\,dx\\ &=-\int z\,|\nabla(V+h(z))|^2\,dx\,.\end{aligned} \tag{3.4}$$

Consider then a steady state z_∞ such that, for a constant $C_z\in\mathbb{R}$ with

$$C_z\leq\inf_{\mathbb{R}^d}V+h(\infty) \tag{3.5}$$

we have

$$z_\infty(x)=\widetilde{h}^{-1}(C_z-V(x))\,. \tag{3.6}$$

Here $\widetilde{h}^{-1}$ is the extension of h^{-1} given by

$$\widetilde{h}^{-1}(\sigma)=\begin{cases} h^{-1}(\sigma) & \text{if}\quad \sigma\in(h(0^+),h(\infty))\,,\\ 0 & \text{if}\quad \sigma\leq h(0^+)\,.\end{cases}$$

Remark 3.1 *In the fast diffusion / porous media cases (1.3)* $h(s)=m(s^{m-1}-1)/(m-1)$ *is such that*

$$h(0+)=-\infty\,,\quad h(\infty)=\frac{m}{m-1}\quad \textit{if}\quad m<1\,,$$
$$h(0+)=-\frac{m}{m-1}\,,\quad h(\infty)=+\infty\quad \textit{if}\quad m>1\,,$$

while $h(0+)=-\infty$ *and* $h(\infty)=+\infty$ *if* $m=1$*. In the case (1.4)-(1.5) we have* $h(s)=F^{-1}(s)$*,* $h(0+)=-\infty$ *and* $h(\infty)=+\infty$*.*

Note that (3.6) implies

$$\begin{aligned} V(x)+h(z_\infty(x))=C_z &\quad \text{if}\ \ h(0^+)\leq C_z-V(x)\\ \text{and}\quad z_\infty(x)=0 &\quad \text{if}\ \ h(0^+)\geq C_z-V(x). \end{aligned} \tag{3.7}$$

Assume now that V is such that for all $C\in\inf_{\mathbb{R}^d}V+(h(0^+),h(\infty))$

$$\int \widetilde{h}^{-1}(C-V(x))\,dx<\infty. \tag{3.8}$$

Now let $M<\infty$ satisfy

$$0\leq M\leq \int \widetilde{h}^{-1}\left(\inf_{\mathbb{R}^d}V+h(\infty)-V(x)\right)\,dx \tag{3.9}$$

(the right hand side may very well be $+\infty$!). Then the steady state z_∞ is uniquely determined by the requirement

$$\int z_\infty(x)\,dx=M\,. \tag{3.10}$$

Note that this is the case for all $M\geq 0$ if $f(s)=s^m$ with $m>d/2-1$ and $V(x)=\frac{1}{2}|x|^2$ (*cf* [6]), or in the Fermi-Dirac case.

Assuming $W[z_0]<+\infty$, the entropy $W[z](t)$ decays monotonically with respect to t, and under additional regularity assumptions, it was shown in [2, 5, 6, 9] that

$$\lim_{t\to+\infty}W[z](t)=W[z_\infty] \tag{3.11}$$

if

$$\int z_0\,dx=\int z_\infty\,dx=M\,. \tag{3.12}$$

In the following, we define the relative entropy

$$W[z|z_\infty]=W[z]-W[z_\infty] \tag{3.13}$$

of the nonnegative states z, z_∞ with equal integrals.

Remark 3.2 *Set*

$$\widetilde{W}[z|z_\infty]=\int\Big(\int_{z_\infty(x)}^{z(x)}(h(s)-h(z_\infty(x))\,ds\Big)\,dx\geq 0\,. \tag{3.14}$$

Since, by the definition (3.3) of h

$$\int_{z_1}^{z_2}(h(s)-h(z_1))\,ds=z_2(h(z_2)-h(z_1))-f(z_2)+f(z_1)\,, \tag{3.15}$$

we conclude

$$\begin{aligned} W[z|z_\infty]-\widetilde{W}[z|z_\infty] &=\int(V(x)+h(z_\infty))(z-z_\infty)\,dx \\ &=\int_{\{h(0^+)\geq C_z-V(x)\}} z(V(x)+h(0^+)-C_z)\,dx\geq 0\,, \end{aligned}$$

where (3.7) and (3.12) were used for the last equality. Therefore $W[z|z_\infty]\geq 0$ *follows and* $W[z|z_\infty]=\widetilde{W}[z|z_\infty]$ *if* $h(0^+)=-\infty$.

Remark 3.3 *Let* $h(0^+)=-\infty$ *and take a function* $\Phi=\Phi(\gamma)$ *with* $\Phi(0)=0$ *and* $\Phi'(\gamma)>0$ *for* $\gamma\in\mathbb{R}$. *We define the functional*

$$\widetilde{W}_\Phi[z|z_\infty]=\int\Big(\int_{z_\infty(x)}^{z(x)}\Phi(h(s)-h(z_\infty(x))\,ds\Big)\,dx\geq 0\,. \tag{3.16}$$

and compute its time-derivative along the solution $z(t)$ *of (3.1):*

$$\frac{d}{dt}W_\Phi[z|z_\infty](t)=-\int z\,|\nabla(V+h(z))|^2\,\Phi'(h(z)-h(z_\infty))\,dx\,. \tag{3.17}$$

Thus, W_Φ *is another relative entropy for (3.1).*

4 A Lyapunov functional

Consider now a solution $\langle u,v\rangle$ of (1.1) such that

$$\int u_0\,dx = M_u \geq 0\,, \tag{4.1}$$

$$\int v_0\,dx = M_v \geq 0\,, \tag{4.2}$$

(with M_u, M_v satisfying (3.9) and $M_u+M_v>0$), and define the relative entropy

$$\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle] = W[u|u_\infty]+W[v|v_\infty]+\frac{\beta}{2}|\nabla\phi|_2^2\,. \tag{4.3}$$

Similarly to the case studied in [1], [3], we obtain

Lemma 4.1 *For $d\geq 3$, if u and v are smooth and decay sufficiently fast as $|x|\to+\infty$, and if f satisfies (1.2) we have*

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle](t)\Big) &= -2J-\beta^2\int (u+v)|\nabla\phi|^2\,dx\\ -2\beta\int\Big[f(u)-f(v)\Big](u-v)\,dx &+ 2\beta\int \Delta\phi\nabla\phi\cdot\nabla V\,dx+\frac{1}{2}\frac{d\beta}{dt}|\nabla\phi|_2^2\,,\end{aligned} \tag{4.4}$$

where

$$J=\frac{1}{2}\int u|\nabla h(u)+\nabla V|^2\,dx+\frac{1}{2}\int v|\nabla h(v)+\nabla V|^2\,dx\,. \tag{4.5}$$

Proof: Assuming a sufficient decay of ϕ in $x\in\mathbb{R}^d$ (with $d\geq 3$) as $|x|\to+\infty$, we obtain

$$\frac{d}{dt}|\nabla\phi|_2^2(t) = 2\int(-\Delta\phi)_t\phi\,dx = 2\int(u_t-v_t)\phi\,dx\,, \tag{4.6}$$

and thus

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle](t)\Big)-\frac{1}{2}\frac{d\beta}{dt}|\nabla\phi|_2^2(t)\\ &=\int(V+h(u)+\beta\phi)u_t\,dx+\int(V+h(v)-\beta\phi)v_t\,dx\,.\end{aligned} \tag{4.7}$$

Then, replacing u_t and v_t by their expressions in (1.1) and integrating by parts, we obtain

$$\frac{d}{dt}\ \Big(\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle](t)\Big)-\frac{1}{2}\frac{d\beta}{dt}|\nabla\phi|_2^2(t)$$

$$
\begin{aligned}
&= -\int \nabla(V+h(u)+\beta\phi)\cdot\Big[\nabla f(u)+u\nabla V+\beta u\nabla\phi\Big]\,dx\\
&\quad -\int \nabla(V+h(v)-\beta\phi)\cdot\Big[\nabla f(v)+v\nabla V-\beta v\nabla\phi\Big]\,dx\,.
\end{aligned}
$$

The evaluation of the cross-terms between u or v and ϕ goes as follows

$$
\begin{aligned}
&- \int \beta\nabla\phi\cdot\Big[\nabla f(u)+u\nabla V\Big]\,dx-\int \nabla(V+h(u))\cdot\beta u\nabla\phi\,dx\\
&+ \int \beta\nabla\phi\cdot\Big[\nabla f(v)+v\nabla V\Big]\,dx+\int \nabla(V+h(v))\cdot\beta v\nabla\phi\,dx\\
&= -2\beta\int \nabla\phi\cdot\Big[\nabla f(u)+u\nabla V\Big]\,dx\\
&\quad +2\beta\int \nabla\phi\cdot\Big[\nabla f(v)+v\nabla V\Big]\,dx
\end{aligned}
$$

using $z\nabla h(z)=\nabla f(z)$ since $sh'(s)=f'(s)$. Collecting the terms and using the Poisson equation, we first obtain

$$
\begin{aligned}
&-2\beta\int \nabla\phi\cdot\Big[\nabla f(u)-\nabla f(v)\Big]\,dx\\
&=2\beta\int \Delta\phi\Big[f(u)-f(v)\Big]\,dx\\
&=-2\beta\int (u-v)\Big[f(u)-f(v)\Big]\,dx\,,
\end{aligned}
$$

and then

$$
-2\beta\int \nabla\phi\cdot(u\nabla V-v\nabla V)\,dx=2\beta\int(\nabla V\cdot\nabla\phi)\Delta\phi\,dx\,, \tag{4.8}
$$

so (4.4) follows. □

Later on we shall use the identity

$$
\int \Delta\phi\nabla\phi\cdot\nabla V\,dx=\frac{1}{2}\int|\nabla\phi|^2\Delta V\,dx-\int\nabla\phi^\top(D^2V)\nabla\phi\,dx\,, \tag{4.9}
$$

where D^2V denotes the Hessian of V and "$^\top$" stands for transposition.

5 Another relative entropy

In this section (only) we shall assume $h(0^+)=-\infty$, $h(\infty)=\infty$, which hold in the Maxwell and Fermi-Dirac cases.

We define the "t-local Maxwellian" functions $\bar u=\bar u(t)$ and $\bar v=\bar v(t)$ respectively by

$$\begin{aligned}
&\bar u(x,t)=h^{-1}(C_u(t)-V(x)-\beta(t)\bar\phi(x,t))\,,\\
&\bar v(x,t)=h^{-1}(C_v(t)-V(x)+\beta(t)\bar\phi(x,t))\,,\\
&-\Delta\bar\phi=\bar u-\bar v\,,\\
&\int\bar u(x,t)\,dx=M_u\,,\\
&\int\bar v(x,t)\,dx=M_v\,.
\end{aligned}\tag{5.1}$$

Note that, due to the dependence of β on t, the normalization constants C_u and C_v depend on t.

The potential $\bar\phi$ then solves the nonlinear elliptic problem

$$\begin{aligned}
&-\Delta\bar\phi=h^{-1}(C_u-V-\beta\bar\phi)-h^{-1}(C_v-V+\beta\bar\phi),\\
&\int h^{-1}(C_u-V-\beta\bar\phi)\,dx=M_u\,,\\
&\int h^{-1}(C_v-V+\beta\bar\phi)\,dx=M_v\,.
\end{aligned}\tag{5.2}$$

Remark 5.1 *The problem (5.2) has the following variational formulation: $\bar\phi$ minimizes the functional $\mathcal{E}[\phi]$ on $\mathcal{D}^{1,2}(\mathbb{R}^d)=\{\phi\in L^{2d/(d-2)}(\mathbb{R}^d):\nabla\phi\in L^2(\mathbb{R}^d)\}$, where*

$$\begin{aligned}
\mathcal{E}[\phi]=\frac12\int|\nabla\phi|^2\,dx\ &+\frac1\beta\int G\Big(D_1[\phi]-V-\beta\phi\Big)\,dx-\frac{M_u}{\beta}D_1[\phi]\\
&+\frac1\beta\int G\Big(D_2[\phi]-V+\beta\phi\Big)\,dx-\frac{M_v}{\beta}D_2[\phi]\,.
\end{aligned}$$

Here G is a primitive of h^{-1}, i.e. $G'=h^{-1}$, and $D_1[\phi]$, $D_2[\phi]\in\mathbb{R}$ are determined from the normalizations

$$\int h^{-1}\Big(D_1[\phi]-V-\beta\phi\Big)\,dx=M_u\,,$$

$$\int h^{-1}\Big(D_2[\phi]-V+\beta\phi\Big)\,dx=M_v\,.$$

A simple computation gives (5.2) as the Euler–Lagrange equations of $\mathcal{E}$. In the linear diffusion case (i.e. $h(s)=\log s$, $h^{-1}(\sigma)=e^\sigma$), this variational *problem has been studied in [7], [4]), and has been shown to have "good" properties (boundedness from below, weak lower semicontinuity and strict convexity). The problem is under investigation in the nonlinear case.*

Consider now the functional

$$\Sigma[u,v] = W[u|\bar{u}] + W[v|\bar{v}] + \frac{\beta}{2}\left(|\nabla\phi|_2^2 - |\nabla\bar{\phi}|_2^2\right). \tag{5.3}$$

A simple computation shows that

$$\begin{aligned}\Sigma[u,v] &= \int [u(h(u)-h(\bar{u})) - f(u) + f(\bar{u})]\,dx \\ &+ \int [v(h(v)-h(\bar{v})) - f(v) + f(\bar{v})]\,dx \\ &+ \int (u-\bar{u})(V+h(\bar{u}))\,dx + \int (v-\bar{v})(V+h(\bar{v}))\,dx \\ &+ \frac{\beta}{2}(|\nabla\phi|_2^2 - |\nabla\bar{\phi}|_2^2).\end{aligned}$$

The integrands in the first two terms on the right hand side can be expressed using (3.15). On the other hand, by the definition of $\bar{u}$ and $\bar{v}$,

$$\begin{aligned}&\int (u-\bar{u})(V+h(\bar{u}))\,dx + \int (v-\bar{v})(V+h(\bar{v}))\,dx \\ &= \int (u-\bar{u})(C_u - \beta\bar{\phi})\,dx + \int (v-\bar{v})(C_v + \beta\bar{\phi})\,dx \\ &= \int (-\Delta(\phi-\bar{\phi}))(-\beta\bar{\phi})\,dx \\ &= -\beta\int \nabla\phi\cdot\nabla\bar{\phi}\,dx + \beta|\nabla\bar{\phi}|_2^2.\end{aligned}$$

Thus the representation

$$\Sigma[u,v] = \int\left(\int_{\bar{u}}^{u}(h(s)-h(\bar{u}))ds + \int_{\bar{v}}^{v}(h(s)-h(\bar{v}))ds\right)dx + \frac{\beta}{2}|\nabla\phi - \nabla\bar{\phi}|_2^2 \geq 0 \tag{5.4}$$

holds and the inequality is strict unless $\langle u,v\rangle = \langle\bar{u},\bar{v}\rangle$.

Remark 5.2 *Since for solutions of (5.2),*

$$\begin{aligned}
&\frac{d}{dt}\Big(\int (\bar u(V+h(\bar u))-f(\bar u))\,dx + \int (\bar v(V+h(\bar v))-f(\bar v))\,dx + \frac{\beta}{2}|\nabla\bar\phi|_2^2\Big)\\
&\quad = \int (h(\bar u)+V+\beta\bar\phi)\frac{\partial \bar u}{\partial t}\,dx + \int (h(\bar v)+V-\beta\bar\phi)\frac{\partial \bar v}{\partial t}\,dx + \frac{1}{2}\frac{d\beta}{dt}|\nabla\bar\phi|_2^2\\
&\quad = \frac{1}{2}\frac{d\beta}{dt}|\nabla\bar\phi|_2^2
\end{aligned} \tag{5.5}$$

by the definition of $\bar u$ and $\bar v$, for any solution of (1.1),

$$\frac{d}{dt}(\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle]-\Sigma[u,v]) = \frac{1}{2}\frac{d\beta}{dt}|\nabla\bar\phi|_2^2\,, \tag{5.6}$$

and we conclude

$$\lim_{t\to+\infty}(\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle](t)-\Sigma[u,v](t)) = 0\,. \tag{5.7}$$

Thus $\Sigma[u,v]$ is another relative entropy of (1.1).

6 Exponential decay in the bipolar case

The method used in [3] extends (formally) to the system (1.1) for which we assume from now on the existence of a smooth solution for $t\in[0,+\infty)$, which decays sufficiently fast for large $|x|$. We also assume in the following that V and f are chosen such that

$$W[z|z_\infty] \le \frac{K}{2}\int z\,|\nabla h(z)+\nabla V|^2\,dx \tag{6.1}$$

for all sufficiently regular nonnegative functions z on $\mathbb{R}^d$ with $\int z dx = \int z_\infty dx$, where $K>0$ is independent of z. In the following, we shall refer to this inequality as the *Generalized Sobolev inequality* (see [6], [5]). Note that the Gross logarithmic Sobolev inequality is an example of the Generalized Sobolev inequality for $f(s)=s$, *i.e.* for $h(s)=\log s$, and $V(x)=\frac{1}{2}|x|^2$.

Theorem 6.1 *Let $d\ge 3$ and consider f satisfying (1.2). Assume that f and V, $V(x)\to+\infty$ as $|x|\to+\infty$, are such that the Generalized Sobolev inequality (6.1) holds. Consider a sufficiently regular, global solution of (1.1) (which* *decays sufficiently fast for $|x|$ large) corresponding to initial data $u_0, v_0 \geq 0$ and assume that M_u, M_v (as defined in (4.1), (4.2)) satisfy (3.9). Moreover, assume that there are constants $c_1 \in \mathbb{R}$ and $\omega > 0$ such that*

(i) $2D^2V(x) - \mathrm{Tr}(D^2V(x))I \geq c_1 I \quad \text{for all } x \in \mathbb{R}^d$,

(ii) $\beta_t(t) \leq -2\omega\beta(t) \quad \text{for all } t \geq 0$.

Then there exists a constant $\widetilde{\lambda} > 0$, explicitly computable in terms of K, c_1, ω such that

$$\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle](t) \quad \leq \quad e^{-\widetilde{\lambda}t}\mathcal{W}[\langle u_0,v_0\rangle|\langle u_\infty,v_\infty\rangle] \tag{6.2}$$

for each solution $\langle u,v\rangle$ of (1.1) with initial data $\langle u_0,v_0\rangle$ and all $t \geq 0$.

Remark 6.2 *If we assume charge neutrality $M_u = M_v$, i.e. $\int(u_0 - v_0)\,dx = 0$, and $|\nabla\phi(0)|_2^2 < \infty$ (which follows from the finiteness of $\mathcal{W}[\langle u_0,v_0\rangle]|\langle u_\infty,v_\infty\rangle)$), then the results in Theorem 6.1 also hold true in the one- and two-dimensional cases $d=1$, $d=2$ (since $\lim_{|x|\to+\infty}|\nabla\phi(x,t)| = 0$ under the electroneutrality condition).*

Remark 6.3 *The potentials $V(x) = (1+|x|^2)^{\alpha/2}$ with $0 < \alpha \leq 2$ do satisfy assumptions (i) on confinement, while $V(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha > 2$ does not.*

Also note that Generalized Sobolev inequalities were so far proven only for uniformly convex potentials in the case of nonlinear functions $f = f(s)$. At least quadratic growth of $V(x)$ as $|x| \to \infty$ seems necessary for (6.1) to hold.

Remark 6.4 *If $\beta(t) = e^{-(d-2)t}$ as it is the case when β is obtained by the mean of time-dependent rescalings, we recover the results of [3] for $f(s) = s$. For the system (2.1)-(2.2), the (algebraic) rate of convergence of course depends on m because of the dependence of R on t.*

Remark 6.5 *As it is well known, cf [3, 5, 6], results on exponential decay of the relative entropy imply (via Csiszár–Kullback inequalities) the exponential convergence to steady states of (1.1) and convergence to self-similar solutions of (2.1) with an algebraic decay rate in the L^1-norms.*

Proof: For any positive λ

$$\begin{aligned}
&-\Big(\frac{d}{dt}\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle]+\lambda\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle]\Big)\\
&=\lambda\Big(KJ-W[u|u_\infty]-W[v|v_\infty]\Big)+(2-\lambda K)J+B+2E+F+C\,, \quad (6.3)
\end{aligned}$$

where

$$\begin{aligned}
B&=\beta^2\int(u+v)|\nabla\phi|^2\,dx\,,\\
E&=\beta\int\Big[f(u)-f(v)\Big](u-v)\,dx\,,\\
F&=-\frac{1}{2}(\beta_t+\lambda\beta)|\nabla\phi|_2^2\,,\\
C&=-2\beta\int\Delta\phi\nabla\phi\cdot\nabla V\,dx\,.
\end{aligned}$$

Observe that if we define

$$G_1=\beta\int u(\nabla h(u)+\nabla V)\cdot\nabla\phi\,dx\,,\quad G_2=\beta\int v(\nabla h(v)+\nabla V)\cdot\nabla\phi\,dx\,,$$

then

$$G_1-G_2=\beta\int\Big(u(\nabla h(u)+\nabla V)-v(\nabla h(v)+\nabla V)\Big)\cdot\nabla\phi\,dx=E+\frac{1}{2}C.$$

Now set

$$\begin{aligned}
f_1&=\sqrt{u}\,(\nabla h(u)+V)\,,\quad g_1=\beta\sqrt{u}\nabla\phi\,,\\
f_2&=\sqrt{v}\,(\nabla h(v)+V)\,,\quad g_2=\beta\sqrt{v}\nabla\phi\,,\\
a_1&=|f_1|_2\,,\quad b_1=|g_1|_2\,,\quad a_2=|f_2|_2\,,\quad b_2=|g_2|_2\,.
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$2|G_1-G_2|=2\Big|\int(f_1g_1-f_2g_2)\,dx\Big|\le 2(a_1b_1+a_2b_2)\,.$$

But

$$0\le(a_1b_2-a_2b_1)^2=(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2\,,$$

$$
\begin{aligned}
2\gamma(a_1b_1+a_2b_2) &\le 2\sqrt{\gamma^2(a_1^2+a_2^2)}\sqrt{b_1^2+b_2^2}\\
&\le \gamma^2(a_1^2+a_2^2)+(b_1^2+b_2^2)\\
&= 2\gamma^2 J+B\,,
\end{aligned}
$$

so that taking $\gamma=\sqrt{1-\lambda K/2}$ with $\lambda K<2$ we obtain

$$
\sqrt{1-\lambda K/2}\,|2E+C|\le(2-\lambda K)J+B.
$$

Using (6.3) we arrive at

$$
\begin{aligned}
&-\Big(\frac{d}{dt}\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle]+\lambda\mathcal{W}[\langle u,v\rangle|\langle u_\infty,v_\infty\rangle]\Big)\\
&\ge\sqrt{1-\lambda K/2}\,|2E+C|+2E+C+F \qquad (6.4)\\
&=(2E+C)\Big(1+\operatorname{sgn}(2E+C)\sqrt{1-\lambda K/2}\Big)+F\,.
\end{aligned}
$$

By the assumption (i) of Theorem 6.1 and (4.9), $C\ge c_1\beta|\nabla\phi|_2^2$. By assumptions (i) and (ii) of Theorem 6.1, $F\ge(\omega-\lambda/2)\beta|\nabla\phi|_2^2$.

Therefore if $2E+C\ge 0$, then $\widetilde{\lambda}=\min(2/K,2\omega)$ gives $-\Big(\frac{d}{dt}\mathcal{W}+\lambda\mathcal{W}\Big)\ge 0$ for $0<\lambda\le\widetilde{\lambda}$. Otherwise, since $E\ge 0$, $C\le 2E+C\le 0$ (then, in particular, $c_1<0$) and

$$
-\left(\frac{d}{dt}\mathcal{W}+\lambda\mathcal{W}\right)\ge\left[c_1\left(1-\sqrt{1-\lambda K/2}\right)+\omega-\lambda/2\right]\beta|\nabla\phi|_2^2.
$$

Hence, there exists a $\widetilde{\lambda}>0$ such that the expression in the brackets is positive for $0<\lambda<\widetilde{\lambda}$, which implies (6.2). $\square$

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