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LARGE TIME BEHAVIOR AND GLOBAL EXISTENCE OF SOLUTION TO THE BIPOLAR DEFOCUSING NONLINEAR SCHRÖDINGER-POISSON SYSTEM

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Abstract. In this paper, we study the large time behavior and the existence of globally defined smooth solutions to the Cauchy problem for the bipolar defocusing nonlinear Schrödinger-Poisson system in the space $\mathbb{R}^3$.

1. Introduction. In the present paper, we study the global existence and large time behavior for the bipolar defocusing nonlinear Schrödinger-Poisson (BDNLSP) system

$$i\varepsilon\dot{\psi}_j = -\frac{\varepsilon^2}{2}\Delta\psi_j + (q_j V + h_j(|\psi_j|^2))\psi_j, \quad j = 1, 2, \tag{1.1}$$

$$-\lambda^2\Delta V = |\psi_1|^2 - |\psi_2|^2, \tag{1.2}$$

with the initial data

$$\psi_j(0, \cdot) = \varphi_j, \quad j = 1, 2, \tag{1.3}$$

where the wave function $\psi_j = \psi_j(t, x) : \mathbb{R}^{1+d} \to \mathbb{C}$, $j = 1, 2$, $\dot{\psi}_j = \partial\psi_j/\partial t$, Δ is the Laplace operator on $\mathbb{R}^d$, and the electrostatic potential $V = V(t, x)$. The nonlinear self-interacting potential $h_j(s)$ is assumed to be given by

$$h_j(s) = a_j^2 s^{\gamma_j}, \quad \text{for } s \geqslant 0 \text{ and some } a_j > 0, \ \frac{2}{d} < \gamma_j < \alpha(d),$$

where $\alpha(d) = \frac{2}{d-2}$ if $d \geqslant 3$ and $\alpha(d) = \infty$ if $d = 1, 2$. The charges of the particles described by the wave functions ψ_j are defined by $q_1 = 1$, $q_2 = -1$, respectively. ε is the scaled Planck constant and λ is the scaled Debye length.

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We assume that the initial value

$$\varphi_j(x) \in \Sigma(\mathbb{R}^d) := \{u \in H^1(\mathbb{R}^d) : |x|u \in L^2(\mathbb{R}^d)\}, \ j = 1, 2, \tag{1.4}$$

with the norm

$$\|\psi_j\|_\Sigma = \|\psi_j\|_{H^1} + \||x|\psi_j\|_{L^2}.$$

This system appears in quantum mechanics as well as semi-conductor and plasma physics. A large amount of interesting work has been devoted to the study of the Schrödinger-Poisson systems (see [2], [3], [4], [6], [7] and references therein). In [4], by applying the estimates of a modulated energy functional and the Wigner measure method, Jüngel and Wang discussed the combined semi-classical and quasineutral limit of the (BDNLSP) system with the initial data (1.3) in the whole space where $a_1 = a_2$ and $\gamma_1 = \gamma_2$, provided the solution of (1.1)–(1.3) exists. But they only declared the existence and uniqueness of global small smooth solution under the assumption that the initial data were sufficiently small in H^s where $s > d/2 + 2$. And in [3], Castella proved the global existence and the asymptotic behavior of solutions in the function space L^2 for the mixed-state unipolar Schrödinger-Poisson systems without the defocusing nonlinearity. In [6], with the help of madelung transform and WKB expansion, Li and Lin discussed the following unipolar nonlinear Schrödinger-Poisson system:

$$\begin{aligned} &i\varepsilon\psi_t^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - (V^\varepsilon(x,t) + f'(|\psi^\varepsilon|^2))\psi^\varepsilon - (arg\psi^\varepsilon)\psi^\varepsilon = 0, \\ &-\Delta V^\varepsilon = |\psi^\varepsilon|^2 - \mathcal{C}(x), \quad V \to 0 \text{ as } |x| \to \infty, \end{aligned}$$

subject to the rapidly oscillating (WKB) initial condition

$$\psi^\varepsilon(x, 0) = \psi_0^\varepsilon = A_0^\varepsilon(x)e^{\frac{i}{\varepsilon}S_0(x)}, \tag{1.5}$$

where $f \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R})$, $S_0 \in H^s(\mathbb{R}^d)$, $d \geqslant 1$, for $s \geqslant d/2 + 2$, A_0^ε was a function, polynomial in ε, with coefficients of Sobolev regularity in x, and the function $C(x) > 0$ denoted the background ions. They obtained the existence of smooth solution where the wave function was of the form $\psi^\varepsilon(x,t) = A^\varepsilon(x,t)e^{\frac{i}{\varepsilon}S^\varepsilon(x,t)}$, with A^ε and ∇S^ε bounded in $L^\infty([0,T]; H^s(\mathbb{R}^d))$ and the initial data being sufficiently small in $H^s(\mathbb{R}^d)$. However, to our knowledge, there is no previous result on the global existence and the asymptotic behavior of solutions for the (BDNLSP) system with arbitrary initial data in $\Sigma(\mathbb{R}^3)$. In this paper, by using the pseudo-conformal conservation law of the (BDNLSP) system and applying the time-space $L^p - L^{p'}$ estimate method, we shall establish the global existence and uniqueness of the solution to the (BDNLSP) system with initial data in $\Sigma(\mathbb{R}^3)$. As a byproduct, the large time behavior to the solution is also obtained. Although the above results are established for the single bipolar defocusing nonlinear Schrödinger-Poisson system, the results can be extended to the mixed-state bipolar defocusing nonlinear Schrödinger-Poisson system within the same framework.

For convenience, we first introduce some notation. For any $p \in [2, \infty)$, we denote $\frac{1}{\gamma(p)} = \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$. $S(t)$ denotes the unitary group generated by $\frac{\varepsilon}{2}i\Delta$ in $L^2(\mathbb{R}^3)$. For $p \in [1, \infty]$, we denote by p' the conjugate exponent of p, defined by $1/p + 1/p' = 1$. $\bar{z}$ denotes the conjugate of the complex number z.

Now we state the main result of this paper.

THEOREM 1.1 (Existence and uniqueness). Let $\varphi_j \in \Sigma(\mathbb{R}^3)$. Assume that $\rho \in [2,6)$. Then, there exists a unique solution

$$\psi_j \in \mathcal{C}(\mathbb{R};\, \Sigma(\mathbb{R}^3)) \cap L^\infty(\mathbb{R};\, H^1(\mathbb{R}^3)) \cap L^{\gamma(\rho)}_{loc}(\mathbb{R};\, H^1_\rho(\mathbb{R}^3)), \text{ for } j = 1,2$$

to the (BDNLSP) system with the initial data (1.3).

Moreover, the solution (ψ_1, ψ_2, V) satisfies the L^2-norm, the energy, and the pseudo-conformal conservation laws (for details, one can see Proposition 2.1 in the case $d = 3$).

THEOREM 1.2 (Large time behavior). Let (ψ_1, ψ_2, V) and ρ be as in Theorem 1.1. Then, there exist constants C depending only on $\|\varphi_j\|_{H^1}$ and $\||x|\varphi_j\|_2$ such that

$$\|\psi_j\|_\rho \leqslant C|t|^{-\frac{1}{\gamma(\rho)}}, \quad \forall \rho \in [2,6),\ \forall |t| \geqslant 1, \tag{1.6}$$

$$\|\nabla V(t)\|_\rho \leqslant C|t|^{-(1-\frac{3}{2\rho})}, \quad \forall \rho \in (\frac{3}{2}, \infty),\ \forall |t| \geqslant 1, \tag{1.7}$$

$$\|V(t)\|_\rho \leqslant C|t|^{-\frac{1}{2}(1-\frac{3}{\rho})}, \quad \forall \rho \in (3,\infty),\ \forall |t| \geqslant 1. \tag{1.8}$$

In the next section we shall derive three conservation identities including the L^2-norm, the energy, and the pseudo-conformal conservation laws in the whole space $\mathbb{R}^d$ for any $d \in \mathbb{N}$. In Sec. 3, we will give some basic estimates used in our proofs for $d = 3$. Section 4 is devoted to the proof of the existence and uniqueness of the solutions for the initial data in $\Sigma(\mathbb{R}^3)$. Finally, the large time behavior of the solution is obtained in Sec. 5.

2. Derivation of the conservation laws.

PROPOSITION 2.1. Let $d \in \mathbb{N}$, $\{\psi_j\}$ be a solution of the (BDNLSP) system with the initial value $\varphi_j(x) \in \Sigma(\mathbb{R}^d)$. Then, we have the following conservation laws for all $t \in \mathbb{R}$:

(i) L^2-norm law:

$$\|\psi_j(t)\|_2 = \|\varphi_j\|_2 \quad \text{for } j = 1,2; \tag{2.1}$$

(ii) Energy conservation law:

$$\varepsilon^2 \sum_{j=1}^{2} \|\nabla\psi_j(t)\|_2^2 + \lambda^2\|\nabla V\|_2^2 + 2\sum_{j=1}^{2} \frac{a_j^2}{\gamma_j + 1}\|\psi_j(t)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} = \text{const}; \tag{2.2}$$

(iii) Pseudo-conformal conservation law (cf. [9]):

$$\begin{aligned} \sum_{j=1}^{2} \|x\psi_j + i\varepsilon t\nabla\psi_j\|_2^2 + \lambda^2 t^2\|\nabla V\|_2^2 + 2t^2 \sum_{j=1}^{2} \frac{a_j^2}{\gamma_j + 1}\|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} \\ + 2\sum_{j=1}^{2} \frac{a_j^2(d\gamma_j - 2)}{\gamma_j + 1} \int_0^t \tau\|\psi_j(\tau)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}\, \mathrm{d}\tau \\ = \sum_{j=1}^{2} \||x|\varphi_j\|_2^2 + (4-d)\lambda^2 \int_0^t \tau\|\nabla V(\tau)\|_2^2\, \mathrm{d}\tau. \end{aligned} \tag{2.3}$$

Proof. Denote $eq(\psi_j) =: i\varepsilon\dot{\psi}_j + \frac{\varepsilon^2}{2}\Delta\psi_j - (q_jV + h_j(|\psi_j|^2))\psi_j$.

(i) It is well known that (2.1) holds for $j = 1, 2$. We omit its proof.

(ii) We consider

$$\Re(eq(\psi_j), \dot{\psi}_j) = 0$$

where $(\cdot, \cdot)$ denotes the L^2-inner product. From the above, we can get

$$\int_{\mathbb{R}^d} \Big\{\frac{\varepsilon}{4}\partial_t|\nabla\psi_j|^2 + \frac{1}{2}(q_jV + h_j(|\psi_j|^2))\partial_t|\psi_j|^2\Big\}\,\mathrm{d}x = 0.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^d} \Big\{\frac{\varepsilon}{4}\partial_t(|\nabla\psi_1|^2 + |\nabla\psi_2|^2) + \frac{1}{2}V\partial_t(|\psi_1|^2 - |\psi_2|^2)\\ + \frac{1}{2}h_1(|\psi_1|^2)\partial_t|\psi_1|^2 + \frac{1}{2}h_2(|\psi_2|^2)\partial_t|\psi_2|^2\Big\}\,\mathrm{d}x = 0.\end{aligned} \tag{2.4}$$

Integrating (2.4) over $[0, t]$, we obtain the desired identity.

(iii) Considering

$$\Re(eq(\psi_j)\bar{\psi}_j) = 0,$$

we have

$$-\varepsilon\Im\partial_t\psi_j\bar{\psi}_j + \frac{\varepsilon^2}{2}\Re\nabla\cdot(\nabla\psi_j\bar{\psi}_j) - \frac{\varepsilon^2}{2}|\nabla\psi_j|^2 - (q_jV + h_j(|\psi_j|^2))|\psi_j|^2 = 0. \tag{2.5}$$

Noticing

$$\Re(eq(\psi_j)\bar{\psi}_{jr}r) = 0 \quad \text{with } r := |x|,$$

we obtain

$$\begin{aligned}-\varepsilon\Im\partial_t\psi_j\bar{\psi}_{jr}r + \frac{\varepsilon^2}{2}\Re\nabla\cdot(\nabla\psi_j\bar{\psi}_{jr}r - \frac{x}{2}|\nabla\psi_j|^2) + \frac{\varepsilon^2}{4}(d-2)|\nabla\psi_j|^2\\ - \frac{1}{2}(q_jV + h_j(|\psi_j|^2))r\partial_r|\psi_j|^2 = 0.\end{aligned} \tag{2.6}$$

Due to

$$\partial_t(\psi_j\bar{\psi}_{jr}r) + \nabla\cdot(x\bar{\psi}_j\partial_t\psi_j) = 2\partial_t\psi_j\bar{\psi}_{jr}r + 2\Re\psi_j x\cdot\nabla\partial_t\bar{\psi}_j + d\partial_t\psi_j\bar{\psi}_j,$$

we have, by taking the imaginary part, that

$$\Im\partial_t(\psi_j\bar{\psi}_{jr}r) + \Im\nabla\cdot(x\bar{\psi}_j\partial_t\psi_j) = d\Im\partial_t\psi_j\bar{\psi}_j + 2\Im(\partial_t\psi_j\bar{\psi}_{jr}r).$$

From (2.5) and (2.6), we have

$$\begin{aligned}-\varepsilon\Im\partial_t(\psi_j\bar{\psi}_{jr}r) - \Im\nabla\cdot(x\bar{\psi}_j\partial_t\psi_j) - \frac{\varepsilon^2}{2}d\Re\nabla\cdot(\nabla\psi_j\bar{\psi}_j) - d(q_jV + h_j(|\psi_j|^2))|\psi_j|^2\\ + \varepsilon^2\Re\nabla\cdot(\nabla\psi_j\bar{\psi}_{jr}r - \frac{x}{2}|\nabla\psi_j|^2) - (q_jV + h_j(|\psi_j|^2))r\partial_r|\psi_j|^2 = 0.\end{aligned}$$

We also have

$$\partial_t|x\psi_j + i\varepsilon t\nabla\psi_j|^2 = \partial_t(|\psi_j r|^2 + \varepsilon^2t^2|\nabla\psi_j|^2 + 2\varepsilon t\Im\psi_j x\cdot\nabla\bar{\psi}_j).$$

Since

$$\begin{aligned}\frac{\varepsilon}{2}\partial_t|\psi_j r|^2 &= \varepsilon\Re(\partial_t\psi_j\bar{\psi}_j)r^2 = \Im i\varepsilon\partial_t\psi_j\bar{\psi}_j r^2\\ &= \Im[-\frac{\varepsilon^2}{2}\Delta\psi_j + (q_jV + h_j(|\psi_j|^2))\psi_j]\bar{\psi}_j r^2\\ &= -\frac{\varepsilon^2}{2}\Im\Delta\psi_j\bar{\psi}_j r^2,\\ \Im\nabla\cdot(\nabla\psi_j\bar{\psi}_j r^2) &= \Im\Delta\psi_j\bar{\psi}_j r^2 - 2\Im(\nabla\bar{\psi}_j\cdot x)\psi_j,\end{aligned}$$

we have

$$\begin{aligned}\partial_t|x\psi_j + i\varepsilon t\nabla\psi_j|^2 = 2\varepsilon^2 t|\nabla\psi_j|^2 + \varepsilon^2t^2\partial_t|\nabla\psi_j|^2 - \varepsilon\Im\nabla\cdot(\nabla\psi_j\bar{\psi}_j r^2)\\ + 2\varepsilon t\Im\partial_t(\psi_j x\cdot\nabla\bar{\psi}_j).\end{aligned} \tag{2.7}$$

Integrating (2.7) over $\mathbb{R}^d$, we obtain that for $j = 1, 2$

$$\begin{aligned}\partial_t\|x\psi_j + i\varepsilon t\nabla\psi_j\|_2^2 = \varepsilon^2t^2\partial_t\|\nabla\psi_j\|_2^2 - 2t\int_{\mathbb{R}^d}\big\{d(q_jV + h_j(|\psi_j|^2))|\psi_j|^2\\ + (q_jV + h_j(|\psi_j|^2))r\partial_r|\psi_j|^2\big\}\,\mathrm{d}x.\end{aligned} \tag{2.8}$$

From the above, we obtain

$$\begin{aligned}\partial_t\sum_{j=1}^{2}\|x\psi_j + i\varepsilon t\nabla\psi_j\|_2^2 + 4t^2\partial_t\big(\frac{\lambda^2}{4}\|\nabla V\|_2^2 + \sum_{j=1}^{2}\frac{a_j^2}{2(\gamma_j+1)}\|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}\big)\\ + 2t\int_{\mathbb{R}^d}\big\{dV(-\lambda^2\Delta V) + V\partial_r(-\lambda^2\Delta V)r + d\sum_{j=1}^{2}a_j^2|\psi_j|^{2(\gamma_j+1)}\\ + \sum_{j=1}^{2}h_j(|\psi_j|^2)r\partial_r|\psi_j|^2\big\}\,\mathrm{d}x = 0.\end{aligned} \tag{2.9}$$

Noticing that

$$h_j(s)r\partial_r s = a_j^2 s^{\gamma_j}r\partial_r s = \frac{a_j^2}{\gamma_j+1}[\nabla\cdot(xs^{\gamma_j+1}) - ds^{\gamma_j+1}] \tag{2.10}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d}Vr\partial_r(-\Delta V)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\nabla V_r\cdot\nabla(Vr)\,\mathrm{d}x = \int_{\mathbb{R}^d}(\nabla V_r\cdot\nabla V)r + (\nabla V_r\cdot\nabla r)V\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\frac{1}{2}r\partial_r|\nabla V|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}V\Delta V\,\mathrm{d}x \qquad (2.11)\\ &= \int_{\mathbb{R}^d}\frac{1}{2}[\nabla\cdot(x|\nabla V|^2) - d|\nabla V|^2]\,\mathrm{d}x - \int_{\mathbb{R}^d}\nabla V\cdot\nabla V\,\mathrm{d}x\\ &= -(\frac{d}{2}+1)\|\nabla V\|_2^2,\end{aligned}$$

we can obtain, in view of (2.4) and (2.8)–(2.11), that

$$\partial_t\Big[\sum_{j=1}^{2}\|x\psi_j+i\varepsilon t\nabla\psi_j\|_2^2+\lambda^2t^2\|\nabla V\|_2^2+\sum_{j=1}^{2}\frac{2a_j^2}{\gamma_j+1}t^2\|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}\Big]$$
$$+2\sum_{j=1}^{2}\frac{a_j^2(d\gamma_j-2)}{\gamma_j+1}t\|\psi_j(t)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}-(4-d)\lambda^2t\|\nabla V(t)\|_2^2=0,$$

which yields the pseudo-conformal conservation law. □

3. Basic estimates.

LEMMA 3.1 (Estimate I). Let $\frac{1}{\rho'}=\frac{2}{a}+\frac{1}{\rho}+\frac{1}{3}-1$, $u,v\in L^{\gamma(a)}(0,T;\,H_a^1)$ and $w\in L^{\gamma(\rho)}(0,T;\,H_\rho^1)$. Then we have the estimate

$$\|(\frac{1}{r}*uv)w\|_{L^{\gamma(\rho)'}(0,T;\,H^1_{\rho'})}\leqslant CT^{1/2}\|u\|_{L^{\gamma(a)}(0,T;\,H^1_a)}\cdot\|v\|_{L^{\gamma(a)}(0,T;\,H^1_a)}\|w\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}. \tag{3.1}$$

Proof. By the known estimate as in [8], we have the following:

$$\|(\frac{1}{r}*uv)w\|_{H^1_{\rho'}}\leqslant C\|\frac{1}{r}*uv\|_{L^p}\|w\|_{H^1_q}+C\|\frac{1}{r}*uv\|_{H^1_q}\|w\|_{L^p}$$
$$\leqslant C\|u\|_{H^1_{2m}}\|v\|_{H^1_{2m}}\|w\|_{H^1_q}$$

where $\frac{1}{\rho'}=\frac{1}{p}+\frac{1}{q}$, $\frac{1}{p}=\frac{1}{m}+\frac{1}{3}-1$. Let $2m=a$, $q=\rho$, i.e., $\frac{1}{\rho'}=\frac{2}{a}+\frac{1}{\rho}+\frac{1}{3}-1$. By the Sobolev embedding theorem (cf. [1]), we obtain

$$\|(\frac{1}{r}*uv)w\|_{H^1_{\rho'}}\leqslant C\|u\|_{H^1_a}\|v\|_{H^1_a}\|w\|_{H^1_\rho}.$$

Since

$$\frac{1}{\gamma(\rho)'}=1-\frac{3}{2}(\frac{1}{\rho'}-\frac{1}{2})=\frac{1}{2}+\frac{2}{\gamma(a)}+\frac{1}{\gamma(\rho)},$$

we have the desired result. □

LEMMA 3.2 (Estimate II). Let $\rho\in[2,6)$; we have

$$\||u|^pu\|_{L^{\gamma(\rho)'}(0,T;\,H^1_{\rho'})}\leqslant CT^{1-\frac{2}{\gamma(\rho)}}\|u\|^p_{L^\infty(0,T;\,H^1)}\|u\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}. \tag{3.2}$$

Proof. From the identity

$$\nabla(|u|^pu)=\nabla(|u|^p)u+|u|^p\nabla u=\frac{p}{2}|u|^{p-2}(\nabla u\bar u+u\nabla\bar u)u+|u|^p\nabla u$$
$$=(\frac{p}{2}+1)|u|^p\nabla u+\frac{p}{2}|u|^{p-2}u^2\nabla\bar u,$$

we have, in view of $\frac{1}{\rho'}=\frac{p}{q}+\frac{1}{\rho}$, that

$$\|\nabla(|u|^pu)\|_{L^{\rho'}}\leqslant C\||u|^p\nabla u\|_{L^{\rho'}}+C\||u|^{p-2}u^2\nabla\bar u\|_{L^{\rho'}}$$
$$\leqslant C\|u\|^p_{L^q}\|\nabla u\|_{L^\rho}.$$

By the Sobolev embedding theorem, it yields

$$\||u|^pu\|_{H^1_{\rho'}}\leqslant C\|u\|^p_{H^1}\|u\|_{H^1_\rho}.$$

which implies the desired result in view of the Hölder inequality with respect to the time variable. □

Now we introduce the Galilei-type operator

$$J(t) = x + i\varepsilon t\nabla. \tag{3.3}$$

Let $\mathcal{M}(t) = e^{\frac{i|x|^2}{2\varepsilon t}}$ and $w_j = \mathcal{M}(-t)\psi_j$; we easily see that

$$J(t) = S(t)xS(-t) = \mathcal{M}(t)(i\varepsilon t\nabla)\mathcal{M}(-t), \tag{3.4}$$

$$|w_j(t)| = |\psi_j(t)|, \quad |J(t)\psi_j(t)| = \varepsilon|t||\nabla w_j(t)|. \tag{3.5}$$

LEMMA 3.3 (Estimate III). It holds

$$\|J(|\psi_j|^p\psi_j)\|_{L^{\gamma(\rho)'}(0,T;\,L^{\rho'})} \leqslant CT^{1-\frac{2}{\gamma(\rho)}}\|\psi_j\|^p_{L^\infty(0,T;\,H^1)}\|J\psi_j\|_{L^{\gamma(\rho)}(0,T;\,L^\rho)}. \tag{3.6}$$

Proof. We have, in view of (3.5), that

$$\|J(|\psi_j|^p\psi_j)\|_{L^{\rho'}} = \varepsilon|t|\|\nabla(|w_j|^p w_j)\|_{L^{\rho'}}. \tag{3.7}$$

Since

$$\nabla(|w_j|^p w_j) = \nabla(|w_j|^p)w_j + |w_j|^p\nabla w_j,$$

we obtain from Hölder's inequality

$$\|\nabla(|w_j|^p w_j)\|_{L^{\rho'}} \leqslant C\|w_j\|^p_{H^1}\|\nabla w_j\|_{L^\rho}. \tag{3.8}$$

Then, by (3.5), (3.7), and (3.8), we see that

$$\|J(|\psi_j|^p\psi_j)\|_{L^{\rho'}} \leqslant C\|\psi_j\|^p_{H^1}\|J\psi_j\|_{L^\rho},$$

which implies the desired result (3.6) in view of the Hölder inequality with respect to the time variable t. □

LEMMA 3.4 (Estimate IV). We have the estimate

$$\|J(V\psi_j)\|_{L^{\gamma(\rho)'}(0,T;\,L^{\rho'})} \leqslant CT^{1/2}\|(\psi_1,\psi_2)\|_{L^{\gamma(a)}(0,T;\,L^a)} \cdot\|(J\psi_1,J\psi_2)\|_{L^{\gamma(\rho)}(0,T;\,L^\rho)} \tag{3.9}$$

where $\|(u,v)\|_X := \|u\|_X + \|v\|_X$.

Proof. Noticing that

$$\begin{aligned}\nabla(Vw_j) &= \nabla V w_j + V\nabla w_j,\\ \nabla V &= C\nabla\Big(\frac{1}{r} * (|w_1|^2 - |w_2|^2)\Big) = \frac{C}{r} * \nabla(|w_1|^2 - |w_2|^2)\\ &= \frac{C}{r} * (w_1\nabla\bar{w}_1 + \nabla w_1\bar{w}_1 - w_2\nabla\bar{w}_2 - \nabla w_2\bar{w}_2),\end{aligned}$$

we have for $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{2}{a} + \frac{1}{3} - 1$

$$\|\nabla(Vw_j)\|_{L^{\rho'}} \leqslant C(\|w_1\|^2_{L^a} + \|w_2\|^2_{L^a})(\|\nabla w_1\|_{L^\rho} + \|\nabla w_2\|_{L^\rho}).$$

Thus, we can get

$$\begin{aligned}\|J(V\psi_j)\|_{L^{\rho'}} &= \varepsilon|t|\|\nabla(Vw_j)\|_{L^{\rho'}} \leqslant C\varepsilon|t|(\|w_1\|^2_{L^a} + \|w_2\|^2_{L^a})(\|\nabla w_1\|_{L^\rho} + \|\nabla w_2\|_{L^\rho})\\ &\leqslant C(\|\psi_1\|^2_{L^a} + \|\psi_2\|^2_{L^a})(\|J\psi_1\|_{L^\rho} + \|J\psi_2\|_{L^\rho}),\end{aligned}$$

which implies the desired result. □

LEMMA 3.5 (Estimate V). We have the following estimate

$$\begin{aligned}&\|(V_k\psi_{1k}-V_l\psi_{1l},V_k\psi_{2k}-V_l\psi_{2l})\|_{L^{\gamma(\rho)'}(0,T;H^1_{\rho'})}\\ \leqslant\,& CT^{1/2}\|(\psi_{1k},\psi_{2k},\psi_{1l},\psi_{2l})\|^2_{L^{\gamma(a)}(0,T;H^1_a)}\\ &\cdot\|(\psi_{1k}-\psi_{1l},\psi_{2k}-\psi_{2l})\|_{L^{\gamma(\rho)}(0,T;H^1_\rho)}.\end{aligned}\tag{3.10}$$

Proof. Since

$$\begin{aligned}&C(V(\psi_{1k},\psi_{2k})\psi_{jk}-V(\psi_{1l},\psi_{2l})\psi_{jl})\\ =&(\frac{1}{r}*(|\psi_{1k}|^2-|\psi_{2k}|^2))\psi_{jk}-(\frac{1}{r}*(|\psi_{1l}|^2-|\psi_{2l}|^2))\psi_{jl}\\ =&(\frac{1}{r}*(|\psi_{1k}|^2-|\psi_{2k}|^2))(\psi_{jk}-\psi_{jl})+\psi_{jl}[\frac{1}{r}*((|\psi_{1k}|^2-|\psi_{1l}|^2)-(|\psi_{2k}|^2-|\psi_{2l}|^2))]\\ =&(\frac{1}{r}*(|\psi_{1k}|^2-|\psi_{2k}|^2))(\psi_{jk}-\psi_{jl})+[\frac{1}{r}*((\bar\psi_{1k}-\bar\psi_{1l})\psi_{1k})+\frac{1}{r}*((\psi_{1k}-\psi_{1l})\bar\psi_{1l})\\ &+\frac{1}{r}*((\bar\psi_{2k}-\bar\psi_{2l})\psi_{2k})+\frac{1}{r}*((\psi_{2k}-\psi_{2l})\bar\psi_{2l})]\psi_{jl},\end{aligned}$$

we have the desired result by the Hölder inequality. □

4. The proof of the existence. In this section, we will prove the local existence of the Cauchy problem (BDNLSP) with the initial data (1.3) first. Let $S(t):=e^{\frac{1}{2}i\varepsilon\Delta t}$ and consider the integral equation

$$\psi_j(t)=S(t)\varphi_j-\frac{1}{\varepsilon}i\int_0^t S(t-\tau)(q_jV(\tau)+h_j(|\psi_j(\tau)|^2))\psi_j(\tau)\,\mathrm{d}\tau.\tag{4.1}$$

Define the workspace $(\mathcal{D},d)$ as

$$\mathcal{D}:=\big\{(\psi_j)_{j=1,2}:\ \|\psi_j\|_{L^\infty(0,T;H^1)\cap L^{\gamma(a)}(0,T;H^1_a)}\leqslant M,\ \text{for any } a\in[2,6)\big\},\tag{4.2}$$

with the distance

$$d((\psi_{1k},\psi_{2k}),(\psi_{1l},\psi_{2l}))=\|(\psi_{1k}-\psi_{1l},\psi_{2k}-\psi_{2l})\|_{L^{\gamma(\rho)}(0,T;H^1_\rho)}\tag{4.3}$$

where $M\geqslant 2\max_{j=1,2}\|\varphi_j\|_{H^1}$ and $\rho\in[2,6)$. It is clear that $(\mathcal{D},d)$ is a Banach space. Let us consider the mapping $\mathcal{T}=\mathcal{T}_1\otimes\mathcal{T}_2:(\mathcal{D},d)\to(\mathcal{D},d)$ defined by

$$\mathcal{T}_j:\psi_j(t)\mapsto S(t)\varphi_j-\frac{1}{\varepsilon}i\int_0^t S(t-\tau)(q_jV(\tau)+h_j(|\psi_j(\tau)|^2))\psi_j(\tau)\,\mathrm{d}\tau,\ j=1,2.\tag{4.4}$$

By Lemmas 3.1–3.5 and the Strichartz estimates (cf. [5]), we have

$$\begin{aligned}\|\mathcal{T}_j\psi_j\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)} &\leqslant \|\varphi_j\|_{H^1} + \|V\psi_j\|_{L^{\gamma(\rho)'}(0,T;\,H^1_{\rho'})} + \|h_j(|\psi_j|^2)\psi_j\|_{L^{\gamma(\rho)'}(0,T;\,H^1_{\rho'})}\\ &\leqslant \|\varphi_j\|_{H^1} + CT^{1/2}\Big(\sum_{j=1}^{2}\|\psi_j\|^2_{L^{\gamma(a)}(0,T;\,H^1_a)}\Big)\|\psi_j\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}\\ &\qquad + CT^{1-2/\gamma(\rho)}\|\psi_j\|^{2\gamma_j}_{L^\infty(0,T;\,H^1)}\|\psi_j\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}\\ &\leqslant M/2 + (CT^{1/2}M^2 + CT^{1-2/\gamma(\rho)}M^{2\gamma_j})M\\ &\leqslant M,\end{aligned} \tag{4.5}$$

where we have taken T so small that $CT^{1/2}M^2 + CT^{1-2/\gamma(\rho)}M^{2\gamma_j} < \frac{1}{2}$. Similar to the above, a straightforward computation shows that it holds

$$\begin{aligned}&\|\mathcal{T}(\psi_{1k},\psi_{2k}) - \mathcal{T}(\psi_{1l},\psi_{2l})\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}\\ \leqslant&\frac{1}{2}\|(\psi_{1k}-\psi_{1l},\psi_{2k}-\psi_{2l})\|_{L^{\gamma(\rho)}(0,T;\,H^1_\rho)}.\end{aligned} \tag{4.6}$$

Hence, $\mathcal{T}$ is a contracted mapping from the Banach space $(\mathcal{D}, d)$ to itself. By the Banach contraction mapping principle, we know that there exists a unique solution $(\psi_1, \psi_2) \in L^{\gamma(\rho)}(0,T;\,H^1_\rho) \times L^{\gamma(\rho)}(0,T;\,H^1_\rho)$ to the (BDNLSP) system with the initial data (1.3). From (4.1), (3.4) and (3.5), we may easily obtain $J\psi_1, J\psi_2 \in L^{\gamma(\rho)}(0,T;\,L^\rho)$ with the help of Lemmas 3.3–3.4. Thus, we can use the standard argument (cf. [3]) to extend it to a global one satisfying for any $T > 0$

$$\psi_1(t,x),\ \psi_2(t,x) \in \mathcal{C}(\mathbb{R};\,\Sigma(\mathbb{R}^3)) \cap L^\infty(\mathbb{R};\,H^1(\mathbb{R}^3)) \cap L^{\gamma(\rho)}(-T,T;\,H^1_\rho(\mathbb{R}^3)),$$

and prove the uniqueness of the global solution. We omit the details.

5. Large time behavior of the solution. By the pseudo-conformal conservation law, we get for $d = 3$

$$\lambda^2 t^2\|\nabla V\|_2^2 \leqslant I + \lambda^2\int_1^t \tau\|\nabla V(\tau)\|_2^2\,\mathrm{d}\tau, \tag{5.1}$$

where $I := \sum_{j=1}^2 \||x|\psi_j\|_2^2 + \lambda^2\int_0^1 \tau\|\nabla V(\tau)\|_2^2\,\mathrm{d}\tau$.

From the Gronwall inequality, we have

$$\|\nabla V\|_2 \leqslant \frac{I^{1/2}}{\lambda}|t|^{-1/2}. \tag{5.2}$$

By the energy conservation law and the Sobolev embedding theorem, we obtain

$$\begin{aligned}\lambda^2\|\nabla V\|_2^2 &\leqslant \varepsilon^2\sum_{j=1}^{2}\|\nabla\varphi_j\|_2^2+\|\nabla V(0)\|_2^2+\sum_{j=1}^{2}\frac{2a_j^2}{\gamma_j+1}\|\varphi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)}\\ &\leqslant C(\varepsilon,\gamma_j,\|\varphi_j\|_{H^1})+\|\nabla V(0)\|_2^2.\end{aligned}\tag{5.3}$$

$$\begin{aligned}\|\nabla V(0)\|_2 &= C\|\nabla(\frac{1}{|x|}*(|\varphi_1|^2-|\varphi_2|^2))\|_2 \leqslant C\|\frac{1}{|x|^2}*(|\varphi_1|^2-|\varphi_2|^2)\|_2\\ &\leqslant C\sum_{j=1}^{2}\|\varphi_j\|_{L^{12/5}}^2 \leqslant C\sum_{j=1}^{2}\|\varphi_j\|_{H^1}^2\\ &\leqslant C(\varepsilon,\gamma_j,\|\varphi_j\|_{H^1}).\end{aligned}\tag{5.4}$$

Therefore, we have the estimate

$$\|\nabla V\|_2 \leqslant \frac{C}{\lambda}|t|^{-\frac{1}{2}}.\tag{5.5}$$

By the Sobolev embedding theorem and the pseudo-conformal conservation law, we have

$$\begin{aligned}\|\psi_j\|_\rho = \|\mathcal{M}(-t)\psi_j\|_\rho &\leqslant C\|\nabla\mathcal{M}(-t)\psi_j\|_2^{2/\gamma(\rho)}\|\mathcal{M}(t)\psi_j\|_2^{1-\gamma(\rho)}\\ &\leqslant C\|\mathcal{M}(-t)(\frac{x}{i\varepsilon t}\psi_j+\nabla\psi_j)\|_2^{2/\gamma(\rho)}\|\psi_j\|_2^{1-\gamma(\rho)}\\ &\leqslant C|t|^{-1/\gamma(\rho)}.\end{aligned}\tag{5.6}$$

From the above and the Hardy-Littlewood-Sobolev inequality, we obtain

$$\|\nabla V(t)\|_\rho \leqslant C|t|^{-(1-\frac{3}{2\rho})},\quad \forall\rho\in(\frac{3}{2},\infty),\ \forall|t|\geqslant 1,\tag{5.7}$$

$$\|V(t)\|_\rho \leqslant C|t|^{-\frac{1}{2}(1-\frac{3}{\rho})},\quad \forall\rho\in(3,\infty),\ \forall|t|\geqslant 1.\tag{5.8}$$

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