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Large-Time Behavior of Solutions of Initial and Initial-Boundary Value Problems of a General System of Hyperbolic Conservation Laws*

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Abstract. We study the asymptotic behavior of the solution of the initial and initial-boundary value problem of hyperbolic conservation laws when the initial and boundary data have bounded total variation. It is shown that the solution converges to the linear superposition of traveling waves, shock waves and rarefaction waves. The strength and speed of these waves depend only on the values of the data at infinity.

§ 1. Introduction

We consider a system of conservation laws

$$U_t + F(U)_x = 0, \tag{1.1}$$

where $F(U)$ and U are n-vectors, $F = (F_1, \ldots, F_n)$, $U = (U_1, \ldots, U_n)$, $x \in R$ and $t \geqq 0$. We assume that the system is *strictly hyperbolic* and each characteristic field is either *genuinely-non-linear* or *linearly degenerate* in the sense of Lax [10]. We study the Cauchy problem (1.1) with initial data

$$U(x, 0) = U_0(x) \tag{1.2}$$

which is assumed to have *bounded total variation* so that the limiting values of U_0 at $x = \pm\infty$ exist:

$$U_l \equiv U_0(-\infty), \qquad U_r \equiv U_0(+\infty).$$

Our main purpose is to compare the solution $U(x, t)$ of (1.1), (1.2) with the solution $U_*(x, t)$ of the corresponding Riemann problem (1.1) with

$$U(x, 0) = \begin{cases} U_l & \text{for} \quad x < 0, \\ U_r & \text{for} \quad x > 0. \end{cases} \tag{1.3}$$

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We show that $U(x,t)$ converges to $U_*(x,t)$ as t tends to infinity in the following sense: In the primary i-th region, (5.1), all j-th waves, $i \neq j$, decay, (Theorem 5.2). If the i-th wave in $U_*(x,t)$ is a shock wave, then an i-th shock wave will appear in $U(x,t)$ as t becomes large, and will approach the corresponding i-th shock in $U_*(x,t)$ and dominates $U(x,t)$ in the primary i-th region, [Theorem 5.7, (iii)]. If the i-th wave in $U_*(x,t)$ is a rarefraction wave, then all i-th shocks in $U(x,t)$ decay to zero and $U(x,t)$ approaches $U_*(x,t)$ in the primary i-th region, [Theorem 5.7, (ii)]. When the i-th wave in $U_*(x,t)$ is a contact discontinuity, i.e. when the i-th characteristic field is linearly degenerate, then all i-th waves have speed approaching that of the contact discontinuity of $U_*(x,t)$, in other words, the i-th waves become increasingly linear.

In particular if $U_l = U_r$, then all i-th waves decay except those associated with linear degenerate characteristic fields and thus if η is a Riemann invariant for all linear degenerate fields, then η tends to a constant. If, moreover, the system is genuinely nonlinear in all characteristic fields, then the solution decays to the constant $U_l = U_r$.

Our main assumption is that the *total amount of interactions* is finite (cf. Section 3). We carry out our analysis with Glimm's difference scheme, [5]. The scheme has a stochastic feature. By a compactness argument based on Helly's theorem [5], it is shown that the approximate solution U_h converge to an exact solution U if

$$\text{total var}_x\ U_h(x,t) \leqq \text{const total var}_x\ U_h(x,0) \tag{1.4}$$

for some constant independent of t. When the initial data have small total variation, the estimate (1.4) was established in Glimm [5] by introducing a nonlinear functional defined on the approximate solutions. The functional consists of a linear and a quadratic term. The quadratic term measures the potential amount of interactions. It follows from the boundedness of the functional that the total amount of interactions is bounded. Thus our results apply for the Glimm solutions. Estimate (1.4) has also been established for certain systems where initial data need not be of small total variation, [1, 3, 14, 15, 17, 18, 19]. In particular, Nishida [17] solves the Cauchy problem for the model equations of gas dynamics

$$\begin{aligned} &u_t - p(v)_x = 0, \\ &\quad v_t - u_x = 0, \qquad p(v) = \text{const}\, v^{-1} \end{aligned} \tag{1.5}$$

when the initial data have arbitrary finite total variation. For isentropic equations of a polytropic gas, $p(v) = \text{const}\, v^{-\gamma}$, $\gamma > 1$, Nishida and Smoller [18] obtain the uniform bound (1.4) under the assumption that $(\gamma - 1)$ total $\text{var}_x\, U(x,0)$ is less than a constant independent of γ. Solutions for general equations of a polytropic gas have been constructed by Liu [14] under similar hypothesis. The functionals used in [14, 18] contain quadratic terms. Other aforementioned works do not use functionals containing quadratic terms, nevertheless we will show that for those solutions the total amount of interactions are finite and our results apply.

Our methods also apply to solutions of initial-boundary value problems in the quadrant $x \geqq 0$, $t \geqq 0$. It is shown that the asymptotic behavior of the solution is determined by the initial data at $x = +\infty$ and the boundary data at $t = +\infty$. We will illustrate this for general gas equations, Theorem 6.1. The initial-boundary value problems for general gas equations with pressures or velocity given at $x=0$ have been studied by Liu [15].

The theory of decay for genuinely nonlinear systems of conservation laws has been developed by Glimm and Lax [6]. The Glimm-Lax theory is developed for systems of two conservation laws when initial data have small oscillation. In the case when the initial data are constant outside a finite interval, the solution decay to zero at the rate $t^{-1/2}$. With periodic initial data, the total variation of the solution per period decays uniformly at the rate t^{-1}. When the initial data equal U_l for $x<-N$ and U_r for $x>N$ for some $N>0$, and the solution contains only weak shock waves, it was shown by Liu [16] that the solution of (1.1), (1.2) converges to the solution of (1.1), (1.3) at algebraic rates.

Our results on the asymptotic behavior of solutions of general systems of n-conservation laws, $n\geqq 2$, reduces to results on the decay of solutions when the initial data are constant outside a finite interval. Such decay results have been obtained by DiPerna [4] under an additional assumption that system (1.1) possesses a *convex extension* in the sense of Lax [11]. Since we do not assume the initial data to be constant for $|x|$ large, we do not expect the solution to converge at algebraic rates. When $U_0(x)$ equals U_l for $x<-N$, and U_r for $x>N$, $N>0$, we believe that the solution converges to that of the corresponding Riemann problem at algebraic rates. However, new techniques are required for the proof; we leave this for the future. For the asymptotic behavior of special solutions see [2, 7] and [8].

The primary reasons for the simple large-time behavior of the solution are the spreading of rarefaction waves which forces the cancellation of shock and rarefaction waves of the same genuinely nonlinear characteristic field. Waves of the same linearly degenerate family do not cancel and behave like linear waves. For general systems, the interaction of waves may change the speeds and magnitudes of waves and may produce new waves. Since we assume the total amount of interaction to be finite, the amount of interaction after large time is small. It follows that the solution is almost *uncoupled*, (Lemma 5.1). We then use the asymptotic results for scalar equations, Liu [16], to show that the solution approaches that of the corresponding Riemann problem. Our main tool is the theory of generalized characteristics developed by Glimm and Lax [6].

It is essential for our methods that the Riemann problem has a unique solution. For general systems of conservation laws, Lax [11] uses the implicit function theorem to show that the Riemann problem has a unique self-similar solution in a small neighborhood of a constant. For a wide class of two-conservation laws, Smoller [21, 22] solved the Riemann problem when the initial states may not be close. The Riemann problem for general gas equations has been solved by Liu [13] and Smith [20] for arbitrary initial states. In proving that the system is increasingly uncoupled, we assume that the characteristic speeds are strictly separated for any approximate solutions under consideration, [(5.1)]. This assumption is satisfied for nearconstant solutions of general systems and also solutions of general gas equations in the Lagrangian coordinates which are bounded away from the vacuum. This assumption can be relaxed, however.

The space of functions of bounded variation is a natural space for the solution operator of a system of conservation laws. Even if the initial data are analytic, in general the solution is not smooth due to the nonlinearity of the system. On the other hand, results on decay and asymptotic behavior of solutions show that the nonlinearity of the system has certain smoothing effects, and these results may be viewed as results on the regularity of the solutions.

In the next section we will describe briefly the Riemann problem, the Glimm difference scheme, Glimm and Lax's notions of approximate conservation laws. In Section 3 we investigate the assumption on the boundedness of total amount of interactions for existing existence theorems. Section 4 studies the spreading of rarefaction waves. The main results on the asymptotic behavior of solutions of initial value problems are proved in Section 5. The initial-boundary value problems for gas equations are studied in Section 6.

§ 2. Preliminary

We assume that system (1.1) is strictly hyperbolic, i.e. $\partial F(U)/\partial U$ has real and distinct eigenvalues $\lambda_1(U)<\lambda_2(U)<\ldots<\lambda_n(U)$. Assume that each characteristic field is either genuinely nonlinear or linearly degenerate, i.e. for any U under consideration,

$$r_{\alpha_i}(U)\cdot \nabla_U \lambda_{\alpha_i}(U) \neq 0, \qquad i=1,2,\ldots,p, \tag{2.1}$$

$$r_{\beta_j}(U)\cdot \nabla_U \lambda_{\beta_j}(U) \equiv 0, \qquad j=1,2,\ldots,n-p, \tag{2.2}$$

where $r_i(U)$, $i=1,2,\ldots,n$, is an i-th right eigenvector of $\partial F(U)/\partial U$ and $\{\alpha_1,\alpha_2,\ldots,\alpha_p,\beta_1,\beta_2,\ldots,\beta_{n-p}\}=\{1,2,\ldots,n\}$. The *rarefaction curve* $R_i(U_0)$, $i=1,2,\ldots,n$ is the integral curve of r_i through the point U_0; and the *shock curve* $S_i(U_0)$, $i=1,2,\ldots,n$, is a curve tangent to $R_i(U_0)$ at U_0 and for all $U\in U_0$, (U_0,U) satisfies the following *Rankine-Hugoniot condition*

$$\sigma(U,U_0)(U-U_0)=F(U)-F(U_0) \tag{R-H}$$

for some scalar $\sigma=\sigma(U_0,U)$, the *shock speed* for (U_0,U). When $i\in\{\beta_1,\ldots,\beta_{n-p}\}$, $R_i(U_0)=S_i(U_0)$, (Lax [11]), and for any $U\in R_i(U_0)$, $\sigma(U_0,U)=\lambda_i(U_0)=\lambda_i(U)$, U_0 is connected to U by an i-th *contact discontinuity*. For $i\in\{\alpha_1,\ldots,\alpha_p\}$ and $U\in R_i^+(U_0)\equiv\{U\in R_i(U_0)|\lambda_i(U)>\lambda_i(U_0)\}$, then U_0 can be connected to U on the right by an i-th *rarefaction wave*; if $U\in S_i^-(U_0)\equiv\{U\in S_i(U_0)|\lambda_i(U)<\lambda_i(U_0)\}$, then U_0 can be connected to U on the right by an i-th *shock wave* satisfying the shock inequality of Lax [11]:

$$\lambda_i(U)<\sigma(U_0,U)<\lambda_i(U_0). \tag{L}$$

We set

$$T_i(U_0)=\begin{cases} R_i(U_0)=S_i(U_0), & \text{for } i\in\{\beta_1,\ldots,\beta_{n-p}\},\\ S_i^-(U_0)UR_i^+(U_0), & \text{for } i\in\{\alpha_1,\ldots,\alpha_p\},\end{cases} \tag{2.3}$$

so that U_0 can be connected to any U on $T_i(U_0)$ on the right by an i-th wave.

The Riemann problem (1.1), (1.3) is solved by finding U_i, $i=0,1,2,\ldots,n$, $U_0=U_l$, $U_n=U_r$, $U_i\in T_i(U_{i-1})$ so that U_{i-1} is connected to U_i on the right by a centered i-th wave, denoted as (U_{i-1},U_i). Solutions of Riemann problems are the building blocks for the Glimm's difference scheme, [5]. Let r,s be mesh lengths so chosen that $r/s\geqq\max\{\lambda_i\}$. The *Glimm's approximate solution* $U_s(x,t)$ is exact in the strip $ns\leqq t\leqq(n+1)s$ and consists of elementary waves generated at $t=ns$, $x=mr$, $m+n=$even.

At time $(n+1)s$, the value of $U_s(x,t)$ in the interval $(m-1)r<x<(m+1)r$ is set to be the value of the exact solution constructed in the strip $t=(n+1)s$ and $x=(m+\alpha_{n+1})r$. Here $\{\alpha_n\}$ is a *randomly* chosen sequence, equidistributed in $(-1,1)$. An *I-curve* is a space-like curve consisting of a segment joining neighboring mesh points $((m+\alpha_n)r, ns)$, $m+n=$ even. The upper half plane $t\geqq 0$ is covered by diamonds $\Delta_{n,m}$ with vertices $((m+\alpha_{n-1})r,\ (n-1)s)$, $((m+\alpha_{n+1})r,\ (n+1)s)$, $((m-1-\alpha_n)r, ns)$, $((m+1+\alpha_n)r, ns)$.

The *strength* of the i-th wave (U_{i-1}, U_i) in the solution of the Riemann problem (U_l, U_r) is defined as

$$(U_l, U_r)_i \equiv w_i(U_{i-1}) - w_i(U_i), \qquad i=1,2,\dots,n, \tag{2.4}$$

where $w_i=\lambda_i$ if $i=\alpha_1,\dots,\alpha_p$ and w_i is any increasing function along T_i for $i=\beta_1,\dots,\beta_{n-p}$. The first step in establishing estimates such as (1.4) is to investigate the *interaction* of waves of solutions of two Riemann problems. Suppose that the Riemann problems (U_l, U_m), (U_m, U_r) and (U_l, U_r) can be solved. Then in rather general circumstances, there exists a quantity $Q(U_l, U_m, U_r)$, the potential amount of interactions, so that for some constant $0(1)$ depending only on the system (1.1),

$$(U_l, U_r)_i = (U_l, U_m)_i + (U_m, U_r)_i + 0(1)Q(U_l, U_m, U_r). \tag{2.5$_i$}$$

Given any diamond Δ in the Glimm scheme, if the waves entering Δ from the right and left as solutions of the Riemann problems (U_m, U_r) and (U_l, U_m) respectively, then we set $Q(\Delta)=Q(U_l, U_m, U_r)$. We also set the amount of *cancellation* in Δ as

$$C_i(\Delta)=\tfrac{1}{2}[|(U_l, U_m)|_i + |(U_m, U_r)_i| - |(U_l, U_m)_i + (U_m, U_r)_i|]. \tag{2.6$_i$}$$

Let Λ be a collection of diamonds. We denote by $E_i^+(\Lambda)$ and $E_i^-(\Lambda)$ the total amount of i-th rarefaction and shock waves, respectively, entering Λ. The amount of waves leaving Λ is denoted by $L_i^\pm(\Lambda)$. Summing up (2.5) for all diamonds in Λ we obtain the following *approximate conservation laws*

$$L_i^\pm(\Lambda) = E_i^\pm(\Lambda) \mp C_i(\Lambda) + 0(1)Q(\Lambda), \tag{2.7$_i$}$$

where $C_i(\Lambda)=\sum_{\Delta\in\Lambda} C_i(\Delta)$, $Q(\Lambda)=\sum_{\Delta\in\Lambda} Q(\Delta)$. In the next section we will investigate the amount of interactions Q.

§ 3. The Amount of Interactions

For a general system of n-conservation laws, Glimm [5] obtains the following estimate for any nearby states U_l, U_m, U_r:

$$(U_l, U_r)_i = (U_l, U_m)_i + (U_m, U_r)_i + 0(1)D(U_l, U_m, U_r) \tag{3.1}$$

where $D(U_l, U_m, U_r)$ is the sum of products of the strength of *approaching waves.* An i-th wave α approaches a j-th wave β if either $i>j$ and α lies toward the left of β, or $i=j$ and at least one of α and β is a shock wave. Given any I-curve J, we define $D(J)$ as the sum of products of strength of approaching waves which cross J. If J_2 is an immediate successor of J_1, i.e. J_1 and J_2 sandwich a diamond Δ and J_2 lies toward larger time than J_1, then it follows directly that

$$D(J_2)-D(J_1) \leqq -D(\Delta) + 0(1)L(J_1)D(\Delta)$$

where $L(J_1)$ is the total amount of waves crossing J_1. Thus for $L(J_1)$ small enough $D(J_2)-D(J_1)\leqq -\frac{1}{2}D(\Delta)$. If we sum up this inequality over all diamonds in a region Λ, one gets

$$D(\Lambda)\leqq 2D(J) \tag{3.2}$$

for any I-curve J containing the domain of dependence of Λ. This shows in particular if $L(J)$ is small, then $D(\Lambda)$ is bounded for all Λ. Thus the total mount of interaction $D(\Lambda)$ is finite if the initial data have small total variation.

For isentropic gas equations (1.5) with $p(v)=\text{const}\, v^{-\gamma}$, $\gamma>1$, Nishida and Smoller [8] show that global solution exists if $(\gamma-1)$ times the total variation of the initial data is sufficiently small. Under this assumption it is not hard to see from their estimates that

$$Q(\Lambda)\leqq \text{const}\, F(0)\,(\gamma-1)^{-1}, \tag{3.3}$$

where $F(J)=L(J)+(\gamma-1)D(J)$ and $L(J)$ is the total mount of shock waves crossing J, $D(J)$ is a quadratic term measuring the potential amount of interaction. The inequality (3.3) is obtained from

$$F(J_2)-F(J_1)\leqq -\frac{(\gamma-1)}{2}D(\Delta).$$

Estimate (3.3) is crude when γ is close to 1. Suppose that the strength of waves is measured by a linear combination of Riemann invariants (cf. Liu [15]) and we set

$$\tilde{F}(J)=\tilde{L}(J)+\tilde{F}(J),$$

$$\tilde{L}(J)=\text{total mount of waves crossing } J,$$

$$\tilde{D}(J)=K\sum_J\{|\alpha\beta|\,|\alpha \text{ and } \beta \text{ are strengths of approaching shock waves}\}$$
$$+H\sum_J\{|\alpha\beta|\,|\alpha \text{ and } \beta \text{ are strengths of approaching waves and not both are shock waves}\},$$

where K and H are constants independent of γ. Then if we choose K and H sufficiently small, H small compared to K, it follows that

$$\tilde{F}(J_2)-\tilde{F}(J_1)\leqq -\tfrac{1}{2}\tilde{D}(\Delta) \tag{3.4}$$

whence we obtain an estimate stronger than (3.3):

$$Q(\Lambda)\leqq \text{const}\, F(0). \tag{3.5}$$

The inequality (3.4) is proved by detailed analysis of wave interactions, [8]. Analogous estimate also holds for general gas equation, [15]. We omit the details.

The result of Nishida was generalized by Bakharov [1] where the existence theorem was proved under the assumption that shock strength does not increase after interaction. If one measures the strength of the wave not by Riemann invariants but instead by linear combination of Riemann invariants, [15], then after detailed analysis of wave interactions, one sees that

$$F^*(J_2)-F^*(J_1)\leqq -D^*(\Delta)/L(0) \tag{3.6}$$

where $D^*(\Delta)$ is given as in (3.1), $F^*=L^*+D^*/L(0)$ and $L^*(J)$ measures only the amount of shock wave crossing J. Thus one concludes by summing up (3.6) that

$$Q(\Lambda)\leqq F(0)L(0)\leqq 2(L(0))^2 \tag{3.7}$$

where $L(0)$ is the total amount of shock waves for I-curve connecting points on $t=0$ and $t=s$ in Glimm's scheme. We thus obtain the estimate (3.7) which is as strong as (3.2) which holds only for solutions near the constant.

§ 4. Expansion of Rarefaction Waves

Generalized characteristics are Lipschitz continuous curves in the xt-space which propagate with either shock or characteristic speeds. Such curves can be constructed by the recipe of Glimm-Lax's [6]. The two-sided limits of the solution exist along a generalized characteristic except for a countable value of t. Given any k-characteristics χ_k^1 and χ_k^2 issued from time t_0, $k\in\{1,2,\dots,n\}$, χ_k^1 lies to the left of χ_k^2, we set

$D_k(t)=$ distance between χ_k^1 and χ_k^2 at time t, $t\geqq t_0$,

$X_k^{\pm}(t)=$ amount of k-rarefaction and k-shock waves, respectively, between (but not on) χ_k^1 and χ_k^2 at time t,

$\tilde{X}_k(t)=$ total amount of j-th waves, $j\neq k$, between χ_k^1 and χ_k^2 at time t,

$U_k^{\pm i}(t)=$ the one-sided limit from right and left, respectively, of $U(x,t)$ at the point (x,t) on χ_k^i, $i=1, 2$. (4.1)

We now assume that the k-characteristic family is *genuinely nonlinear* so that we have

$$\lambda_k^{+i}(t)\equiv\lambda_k(U_k^{+i}(t))\leqq\sigma_k(U_k^{+i}(t),U_k^{-i}(t))\leqq\lambda_k(U_k^{-i}(t))\equiv\lambda_k^{-i}(t).$$

It follows easily from the Rankine-Hugoniot condition and the mean-value theorem that for some $\theta(t)$, $0<\theta(t)<1$,

$$\begin{aligned}\dot{D}_k(t)&=\sigma_k(U_k^{+2}(t),U_k^{-2}(t))-\sigma_k(U_k^{+1}(t),U_k^{-2}(t))\\&=\theta(t)\,[\lambda_k^{-2}(t)-\lambda_k^{+1}(t)]+(1-\theta(t))\,[\lambda_k^{+2}(t)-\lambda_k^{-1}(t)].\end{aligned} \tag{4.2}$$

We note that $\theta(t)=\theta(\lambda_k^{\pm 1}(t),\lambda_k^{\pm 2}(t))$ and $\lambda_k^{\pm 1}(t)$, $\lambda_k^{\pm 2}(t)$ range over a compact set in U-space. Thus there exists a constant θ, $0<\theta<1$, independent of t such that

$$\dot{D}_k(t)\leqq\theta[\lambda_k^{-2}(t)-\lambda_k^{+1}(t)]+(1-\theta)\,[\lambda_k^{+2}(t)-\lambda_k^{-1}(t)]. \tag{4.3}$$

Since $\lambda_k^{-2}(t)-\lambda_k^{+1}(t)=X_k^+(t)+X_k^-(t)+0(1)\tilde{X}_k(t)$ as is easily seen, it follows from (4.2) that

$$\begin{aligned}\dot{D}_k(t)=&X_k^+(t)+X_k^-(t)+0(1)\tilde{X}_k(t)\\&+(1-\theta(t))\,[\operatorname{str}\chi_1(t)+\operatorname{str}\chi_2(t)].\end{aligned} \tag{4.4}$$

Since the characteristic speeds are assumed to be strictly separated, there exists a finite $t_1>t_0$ such that all i-th generalized characteristics χ_i^1 and χ_i^2, $i\neq k$, meet χ_k^1 or χ_k^2 before time t_1. Similarly, for any $t>t_1$, there exists $s<t$ such that the $(k-1)$-th

[$(k+1)$-th] generalized characteristics through a point on $\chi_k^2(\chi_k^1)$ at time s meets $\chi_k^1(\chi_k^2)$ before time t, and for some $0(1)$ independent of t,

$$t-s=0(1)D_k(t). \tag{4.5}$$

We denote by $h_k(t_0,t)$ the amount of i-th waves, $i\neq k$, crossing X_k^1 or X_k^2 between time t_0 and t. Let $Q(t_0,t)$ be the amount of interactions in the region between χ_k^1 and χ_k^2 between t_0 and t. We have from (2.7),

$$\tilde{X}_k(t)=0(1)\int_s^t d(Q(t_0,\tau)+h_k(t_0,\tau)). \tag{4.6}$$

Integrating (4.4) from t_1 to t, $t>t_1$, and using (4.5), (4.6) we obtain

$$\begin{aligned}D_k(t)\leqq D_k(t_1)&+\int_{t_1}^t [X_k^+(\tau)+X_k^-(\tau)\\&+(1-\theta(t))(\operatorname{str}\chi_1(\tau)+\operatorname{str}\chi_2(\tau))]d\tau\\&+0(1)\int_{t_1}^t D_k(\tau)d(Q(t_0,\tau)+h_k(t_0,\tau)).\end{aligned} \tag{4.7}$$

Since k-th waves may cross χ_k^1 or χ_k^2 only due to interactions, we have from (2.7)

$$X_k^+(s)\geqq X_k^+(t)-0(1)Q(s,t), \qquad t_0\leqq s\leqq t,$$
$$X_k^-(s)\geqq X_k^-(t_0)-0(1)Q(t_0,s),$$

and thus we may solve the linear integral inequality (4.6) to obtain

$$X_k^+(t)\leqq\frac{D_k(t)}{t-t_1}+0(1)[Q(t_0,t)+h_k(t_0,t)-X_k^-(t)-\max\operatorname{str}]$$

where max str is the maximum strength of χ_k^1 and χ_k^2 between t_0 and t. We may apply this inequality to subregions which contain predominantly k-th rarefraction waves and the boundary of these subregions may be so chosen that it consists of k-characteristics with small strengths ([6], pp. 88—92). Thus the above inequality holds without the last two terms on the right. We list this as a theorem.

Theorem 4.1. *Let χ_k^1 and χ_k^2, $k=1,2,\ldots,n$, be generalized k-characteristics issued from two points on $t=t_0$, χ_k^1 lying to the left of χ_k^2. Let t_1, $t_1>t_0$, be any time after which χ_i^1 and χ_i^2 do not intersect χ_j^1 and χ_j^2 for $i\neq j$. We denote by $D_k(t)$ the distance between χ_k^1 and χ_k^2 at time t, $X_k^\pm(t)$ the amount of k-rarefaction and k-shock waves respectively between χ_k^1 and χ_k^2 at time t. Then*

$$X_k^+(t)\leqq\frac{D_k(t)}{t-t_1}+0(1)[Q_k(t_0,t)+h_k(t_0,t)] \tag{4.8}$$

where $Q_k(t_0,t)$ is the amount of interaction between t_0 and t and $h_k(t_0,t)$ is the amount of i-th waves crossing $\chi_k^1(\chi_k^2)$ for all $i>k(i<k)$ between t_0 and t.

§ 5. Initial Value Problems

The main purpose of this section is to investigate the asymptotic behavior of the solution $U(x,t)$ of the initial value problem (1.1), (1.2). We assume that the characteristic speeds $\lambda_i(U(x,t))$ are strictly separated, i.e. there exist μ_i, $i=0,1,2,\ldots,n$, and a positive constant δ such that

$$\mu_0 < \min_{(x,t)} \lambda_1(U(x,t)) - \delta,$$

$$\max_{(x,t)} \lambda_i(U(x,t)) + \delta < \mu_i < \min_{(x,t)} \lambda_{i+1}(U(x,t)) - \delta, \quad i=1,2,\ldots,n-1,$$

$$\max_{(x,t)} \lambda_n(U(x,t)) + \delta < \mu_n. \tag{5.1$_1$}$$

We will investigate the asymptotic shape of $U(x,t)$ in each *i-th primary region* Ω_i defined as

$$\Omega_0 = \left\{(x,t) \mid \frac{x}{t} < \mu_0\right\},$$

$$\Omega_i = \left\{(x,t) \mid \mu_{i-1} < \frac{x}{t} < \mu_i\right\}, \quad i=1,2,\ldots,n,$$

$$\Omega_{n+1} = \left\{(x,t) \mid \mu_n < \frac{x}{t}\right\}. \tag{5.1$_2$}$$

We set

$W_{i,j}^+(t)$ = total amount of i-rarefaction waves contained in Ω_j at time t,

$W_{i,j}^-(t)$ = total amount of i-shock waves contained in Ω_j at time t, $i=1,2,\ldots,n$, $j=0,1,\ldots,n+1$.

Since the total amount of interactions is finite and $U(\cdot,t)$ has uniformly bounded total variation for each t, for any given $\varepsilon>0$, there exist $t_0 = t_0(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that for any $t \geqq t_0$,

$$Q(t_0, t) = Q\{(x,\tau) \mid t_0 \leqq \tau \leqq t\} < \varepsilon,$$

$$\text{total var}_x \{U(x,t_0) \mid |x| \geqq M\} < \varepsilon. \tag{5.2}$$

We denote by χ_k^1 and χ_k^2, $k=1,2,\ldots,n$, the k-th generalized characteristics issued from $(-M, t_0)$ and (M, t_0) respectively. The quantities $D_k(t), X_k^{\pm}(t)$ and t, are defined as in Section 4 for each given χ_k^1 and χ_k^2 and t_0. In what follows, $0(1)$ are bounded functions independent of t and ε.

Lemma 5.1. *Let Γ_i, $i=1,2,\ldots,n$, be the region between χ_i^1 and χ_i^2 and Λ_0 the region left of χ_1^1, Λ_i, $i=1,2,\ldots,n-1$, the region between χ_i^2 and χ_{i+1}^1, and Λ_n the region right of χ_n^2. Then for any $t \geqq t_1$, $j=1, 2,\ldots,n$, $i=0,1,2,\ldots,n$.*

(i) *The amount of j-waves outside Γ_j at time t is* $0(1)\varepsilon$.

(ii) *The total variation of U in regions Λ_i at time t is* $0(1)\varepsilon$.

(iii) *For any (x_1, t_1) and (x_2, t_2) in Λ_i,* $|U(x_1, t_1) - U(x_2, t_2)| = 0(1)\varepsilon$.

(iv) $X_j^+(t) \leqq \dfrac{D_j(t)}{t-t_1} + 0(1)\varepsilon$.

Proof. We apply the conservation law (3.1) to the region right of χ_k^2 to obtain that the amount of i-waves, $i<k$, which cross χ_k^2 is less than the total variation of $U(x, t_0)$ for $x>M$, plus the amount of interactions in the region. Similar estimates hold for the amount of i-waves, $i>k$, which cross χ_k^1. Thus (i) follows from (5.2); and as a direct consequence of (i), we have (ii) and (iii). Finally, (iv) follows from (i) and estimate (4.7) Q.E.D.

Theorem 5.2. *The amount of i-waves, $i=1,2,\ldots,n$, in the region Ω_j, $j=0,1,2,\ldots,n+1$, $i\neq j$, at time t approaches zero as $t\to+\infty$.*

Proof. According to Lemma 5.1, since ε is arbitrary, we need only to show that Γ_j is contained in Ω_j for large t, but this is obvious from the definitions of Ω_j and Γ_j. Q.E.D.

Lemma 5.3. *Suppose that $i=\beta_1,\ldots,\beta_{n-p}$, i.e. $r_i\cdot\nabla\lambda_i\equiv 0$. Then for any (x_k,t_k) in Λ_k, $k=i,\ i-1$,*

(i) $\lambda_o(U(x_i,t_i))=\lambda_i(U(x_{i-1},t_{i-1}))+0(1)\varepsilon$,

(ii) $U(x_i,t_i)\in T_i(U(x_{i-1},t_{i-1}))+0(\varepsilon)$.

Proof. Since λ_i changes value only across j-waves, $j\neq i$, (i) is a consequence of Lemma 5.1, (i). We note that T_i are integral waves of the vector field $r_i\cdot\nabla$, thus (ii) follows also from Lemma 5.1, (i). Q.E.D.

Lemma 5.4. *Suppose that $i=\alpha_1,\ldots,\alpha_p$, i.e. $r_i\cdot\nabla\lambda_i\neq 0$, and $\lambda_i(U(x_i,t_i))\leqq\lambda_i(U(x_{i-1},t_{i-1}))-k\varepsilon$ for some $(x_i,t_i)\in\Lambda_i$ and $(x_{i-1},t_{i-1})\in\Lambda_{i-1}$ and $k>0$. Then there exists a constant k_0 independent of t and ε such that for t sufficiently large and $k>k_0$,*

(i) $X_i^+(t)=0(1)\varepsilon$,

(ii) *χ_i^1 and χ_i^2 coalesce to form an i-shock with strength* $\lambda_i(U(x_i,t_i))-\lambda_i(U(x_{i-1},t_{i-1}))+0(1)\varepsilon$.

Proof. We will use the notation in Section 4 [cf. (4.1), (4.2)]. Since $\tilde{X}_i(t)=0(1)\varepsilon$ from Lemma 5.1, (i) and $\lambda_i^{+2}(t)=\lambda_i(U(x_i,t_i))+0(1)\varepsilon$, $\lambda_i^{-1}(t)=\lambda_i(U(x_{i-1},t_{i-1})+0(1)\varepsilon$ from Lemma 5.1, (iii), we have from (4.2) that for some $\theta\in(0,1)$,

$$\dot{D}_i(t)\leqq\theta[X_i^+(t)+X_i^-(t)]+(1-\theta)[\lambda_i(U(x_i,t_i)) - \lambda_i(U(x_{i-1},t_{i-1}))]+0(1)\varepsilon,\quad t\geqq t_1.$$

Thus it follows from Lemma 5.1, (iv) that

$$\dot{D}_i(t)\leqq\theta\frac{D_i(t)}{t-t_1}+(1-\theta)[\lambda_i(U(x_i,t_i)) - \lambda_i(U(x_{i-1},t_{i-1}))]+0(1)\varepsilon,\quad t\geqq t_1. \tag{5.3}$$

If we set

$$H_i(t)=D_i(t)-[\lambda_i(U(x_i,t_i))-\lambda_i(U(x_{i-1},t_{i-1}))](t-t_1),$$

then (5.3) yields

$$\dot{H}_i(t)\leqq\theta\frac{H_i(t)}{t-t_1}+0(1)\varepsilon,\quad t\geqq t_1.$$

This is a differential inequality which can be easily solved to yield

$$D_i(t) \leqq \text{const}(t-t_i)^\theta + [\lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1}))] \cdot (t-t_1) + 0(1)\varepsilon(t-t_1), \qquad t \geqq t_1. \tag{5.4}$$

Since $\theta \in (0,1)$, it follows from (5.4) that if k_0 is so chosen that $k_0 > 0(1)$ on RHS of (5.4), then $D_i(t)=0$ for t large. Thus the lemma follows from Lemma 5.1. Q.E.D.

Lemma 5.5. *Suppose that* $i=\alpha_1, \dots, \alpha_p$, *i.e.* $r_i \cdot \nabla\lambda_i \neq 0$, *and* $\lambda_i(U(x_i, t_i)) > \lambda_i(U(x_{i-1}, t_{i-1})) - 0(1)\varepsilon$ *for some* $0(1)>0$. *Then for t sufficiently large,*

(i) $|X_i^-(t)| = 0(1)\varepsilon$,

(ii) $U(x_i, t_i) \in R^+(U(x_{i-1}, t_{i-1})) + 0(1)\varepsilon$.

Proof. It follows from Lemma 5.1, (iv) and estimate (5.4) that

$$X_i^+(t) \leqq \text{const}(t-t_1)^{\theta-1} + \lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1})) + 0(1)\varepsilon, \qquad t \geqq t_1,$$

and so for t large,

$$X_i^+(t) \leqq \lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1})) + 0(1)\varepsilon. \tag{5.5}$$

Since

$$X_i^+(t) + X_i^-(t) + 0(1)\varepsilon = \lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1}))$$

as is easily seen from Lemma 5.1, it follows from (5.5) that

$$X_i^+(t) = \lambda_i(U(x_i, t_i)) - \lambda_i(U(x_{i-1}, t_{i-1})) + 0(1)\varepsilon,$$
$$X_i^-(t) = 0(1)\varepsilon. \tag{5.6}$$

The lemma follows from Lemma 5.1. and (5.6). Q.E.D.

Lemma 5.6. *Suppose that the Riemann problem* (1.1), (1.3) *is solved by centered i-waves* (U_{i-1}, U_i), $i=1, \dots, n$. *Then for any* $(x_i, t_i) \in \Lambda_i$, *we have*

$$|U(x_i, t_i) - U_i| = 0(1)\varepsilon.$$

Proof. It follows from Lemma 5.3, (ii), Lemma 5.4, (ii) and Lemma 5.5, (ii) that there exist $\tilde U_i$, $i=0, 1, \dots, n$, such that

$$|\tilde U_i - U(x_i, t_i)| = 0(1)\varepsilon,$$
$$\tilde U_i \in T(\tilde U_{i-1}).$$

Thus $(\tilde U_{i-1}, \tilde U_i)$, $i=1,2,\dots,n$, solves the Riemann problem with data $(\tilde U_0, \tilde U_n)$. But the above inequality and Lemma 5.1, (iii) imply that $|\tilde U_0 - U(-\infty)| + |\tilde U_n - U(+\infty)| = 0(1)\varepsilon$. Because the solution of the Riemann problem depends differentiably on its data, we have proved the lemma. Q.E.D.

Theorem 5.7. *Suppose that the Riemann problem* (1.1), (1.3) *is solved by i-th centered waves* (U_{i-1}, U_i), $i=1, 2, \dots, n$. *Then*

(i) $U(x,t) \to U_i$ *as* $t \to +\infty$ *for* $\frac{x}{t} = \mu_i$.

(ii) *If* $r_i \cdot \nabla\lambda_i \neq 0$, *i.e.* $i=\alpha_1, \alpha_2, \ldots, \alpha_p$, *and* (U_{i-1}, U_i) *is a centered rarefaction wave, i.e.* $\lambda_i(U_{i-1}) \leqq \lambda_i(U_i)$, *then the amount of i-shock waves in* Ω_i *approaches zero as* $t \to +\infty$ *and* $U(x,t)$ *approaches the centered rarefaction wave* (U_{i-1}, U_i) *pointwise in* Ω_i *as* $t \to +\infty$.

(iii) *If* $r_i \cdot \nabla\lambda_i \neq 0$, *i.e.* $i=\alpha_1, \alpha_2, \ldots, \alpha_p$ *and* (U_{i-1}, U_i) *is centered shock wave, i.e.* $\lambda_i(U_{i-1}) > \lambda_i(U_i)$, *then there exists an i-shock wave in* Ω_i *which approchaes* (U_{i-1}, U_i) *both in strength and speed and, moreover, the total variation of the solution in* Ω_i *outside of this shock wave approaches zero as* $t \to +\infty$.

(iv) *If* $r_i \cdot \nabla\lambda_i \equiv 0$, *i.e.* $i=\beta_1, \beta_2, \ldots, \beta_{n-p}$, *then in* Ω_i, $\lambda_i(U(x,t)) \to \lambda_i(U_{i-1}) = \lambda_i(U_i)$ *as* $t \to +\infty$ *and the distance between* $U(x,t)$, $(x,t) \in \Omega_i$, *and* $T(U_{i-1}) = T(U_i)$ *approaches zero uniformly as* $t \to +\infty$.

Proof. It is easy to see that given any $\varepsilon > 0$ and associated Γ_i, $i=0,1,2,\ldots,n$, the point (x,t) with $\frac{x}{t} = \mu_i$ belongs to Γ_i if t is sufficiently large. Thus (i) of the theorem follows from Lemma 5.6 and the arbitrariness of ε. Similarly, the first half of (ii) follows from Lemma 5.5, (i) and Lemma 5.6. We now prove that $U(x,t)$ approaches the centered rarefaction wave (U_{i-1}, U_i) in Ω_i.

Given any $\varepsilon > 0$, we construct χ_i^1 and χ_i^2 as above. By Lemma 5.5, (i) there exists $t_2 \geqq t_1$ such that $|X_i^-(t)| \leqq 0(1)\varepsilon$ for $t \geqq t_2$. Thus it follows from Lemma 5.6 that the speeds of χ_k^1 and χ_k^2 for $t \geqq t_2$ are $\lambda_i(U_{i-1}) + 0(1)\varepsilon$ and $\lambda_i(U_i) + 0(1)\varepsilon$, respectively. Thus for $t \geqq t_2 + D_i(t_2)$,

$$\text{distance}\{\chi_i^1, l_i^1\} + \text{distance}\{\chi_i^2, l_i^2\} = 0(1)\varepsilon(t-t_2), \tag{5.7}$$

where $l_i^j = \{(x,t)\} \Big| \frac{x}{t} = \lambda_i(U_{i-2+j})$, $j=1,2$, are the edges of the centered wave (U_{i-1}, U_i). For $(x,t) \in \Lambda_{i-1}$, it follows from (5.7), Lemma 5.6, and the structure of centered rarefaction waves that

$$\begin{aligned} |U^*(x,t) - U(x,t)| &= |U^*(x,t) - U_{i-1}| + |U_{i-1} - U(x,t)| \\ &= 0(1)\varepsilon \frac{t-t_2}{t} + 0(1)\varepsilon = 0(1)\varepsilon \end{aligned} \tag{5.8}$$

where $U^*(x,t)$ denotes the centered wave (U_{i-1}, U_i). Similarly, for $(x,t) \in \Lambda_i$,

$$|U^*(x,t) - U(x,t)| = 0(1)\varepsilon. \tag{5.9}$$

For any $(x,t) \in \Gamma_i$, $t \geqq t_2 + D_i(t_2)$, we can choose (x^*, t) between l_i^1 and l_i^2 such that $|U^*(x^*,t) - U(x,t)| = 0(1)\varepsilon$ as is easily seen from Lemmas 5.1 and 5.6. Through (x,t) we draw an i-th generalized characteristic χ backward in time. If χ meets χ_i^1 and χ_i^2 we continue χ with χ_i^1 or χ_i^2. Since χ may change speed only due to shock waves entering χ or j-waves, $i \neq j$, crossing χ, one sees that for $t \geqq t_2$, χ has speed $\lambda_i(U(x,t)) + 0(1)\varepsilon$. As a result we see that

$$|x^* - x| = 0(1)\varepsilon(t-t_2). \tag{5.10}$$

Similarly we may draw an i-th generalized characteristic χ^* through (x^*, t) and apply estimate (4.7) for χ and χ^* [cf. Lemma 5.1, (iv)] together with (5.10) to yield that the total amount of i-rarefaction waves between χ and χ^* is $0(1)\varepsilon$. This along with Lemma 5.5, (i), yields that for $(x,t)\in\Gamma_i$, t large

$$\operatorname{tot\,var}\{U(\cdot,t) \text{ between } x \text{ and } x^*\}=0(1)\varepsilon$$

and so

$$|U^*(x,t)-U(x,t)|=0(1)\varepsilon. \tag{5.11}$$

Thus it follows from (5.8), (5.9), (5.11) and the arbitrariness of ε that $U(x,t)\to U^*(x,t)$ is uniformly in Ω_i as $t\to+\infty$.

Statements (iii) and (iv) of the theorem follow from Lemmas 5.3 and 5.4 by analogous arguments. We omit the details. Q.E.D.

The following corollary is a direct consequence of the above theorem. We omit the proof.

Corollary. *Suppose that $U(-\infty)=U(+\infty)$ and let ψ be any i-Riemann invariant, $i=\beta_1,\beta_2,\dots,\beta_{n-p}$, i.e. ψ is constant along all T_i curves for all $i\in\{\beta_1,\beta_2,\dots,\beta_{n-p}\}$*

$$\operatorname{tot\,var}\{\psi(x,t)|-\infty<x<\infty\}\to 0 \quad as \quad t\to+\infty.$$

If $p=n$, i.e., all characteristic fields are genuinely nonlinear, then

$$\operatorname{tot\,var}\{U(x,t)|-\infty<x<\infty\}\to 0 \quad as \quad t\to+\infty.$$

§6. Initial-Boundary Value Problems

In this section we investigate the large-time behavior of solutions of (1.1) defined in the quadrant $x\geqq 0$, $t\geqq 0$. We will illustrate our basic ideas for gas equations:

$$\left.\begin{aligned} &u_t+p_x=0,\\ &v_t-u_x=0,\\ &E_t+(pu)_x=0, \qquad p=p(s,e)=p(s,v), \qquad E=e+\tfrac{1}{2}u^2,\\ &\qquad\qquad\qquad\quad p_v(s,v)<0, \qquad p_{vv}(s,v)>0, \end{aligned}\right\} \tag{6.1}$$

$$(u,v,E)(x,0)=(u_0,v_0,E_0)(x), \qquad x\geqq 0, \tag{6.2}$$

$$u(0,t)=u_b(t), \qquad t\geqq 0, \tag*{(6.3)$_1$}$$

or,

$$p(0,t)=p_b(t), \qquad t\geqq 0. \tag*{(6.3)$_2$}$$

It follows from the estimates in Liu [15] that there exists a finite amount of interactions Q_0 in the interior $t\geqq 0$, $x>0$ and Q_1 on the boundary $x=0$ provided that either the data (6.2), (6.3) have small total variation or the gas is polytropic, i.e. $p(s,v)=\text{const}\exp\left(\frac{(\gamma-1)s}{R}\right)v^{-\gamma}$, $1<\gamma\leqq 5/3$, and $(\gamma-1)$ times the total variation of the data (6.2), (6.3) is small, (see also Section 3). Given any region Λ in the quadrant which intersects the boundary $x=0$ from $t=a$ to $t=b$, the following approximate conservation laws hold [cf. $(2.7)_i$] when $(6.3)_1$ is the boundary data

$$L_1^\pm(\Lambda)=E_1^\pm(\Lambda)-R_1^\pm(a,b)\mp C_1(\Lambda)+0(1)\left[Q_0(\Lambda)+Q_1(a,b)\right], \tag*{(6.4)$_1$}$$

$$L_3^\pm(\Lambda)=E_3^\pm(\Lambda)+R_1^\pm(a,b)\mp C_3(\Lambda)+0(1)\left[Q_0(\Lambda)+Q_1(a,b)+B_3^\pm(a,b)\right] \tag*{(6.4)$_3$}$$

and when $(6.3)_2$ is the boundary data then the first equation in (6.4) is replaced by

$$L_3^{\pm}(\Lambda) = E_3^{\pm}(\Lambda) - R_1^{\pm}(a,b) \mp C_3(\Lambda) + 0(1)\,[Q_0(\Lambda) + Q_1(a,b) + B_3^{\pm}(a,b)].$$

Here $R_1^{\pm}(a,b)$ denotes the amount of 1-waves hitting the line $x=0$ between time a and b, and $B_3^{\pm}(a,b)$ the amount of 3-waves issuing from $x=0$ due to the boundary data (6.3).

Given any $\varepsilon > 0$, we choose t_0, M so large that $Q_2(\Lambda) + Q_1(t_0, \infty) \leqq \varepsilon$ for $\Lambda = \{(x,t) | t \geqq t_0\}$, and $|B_3^{\pm}(t_0, \infty)| < \varepsilon$, tot var $\{U(x,t), x \geqq M\} < \varepsilon$. Through (M, t_0) we construct a 1-characteristic χ which intersects $x=0$ at time t_1. Applying (6.4), to the region right of χ, we find that for any $t \geqq t_1$

$$|X_1^{\pm}(t)| = 0(1)\varepsilon, \quad |R_1^{\pm}(t, \infty)| = 0(1)\varepsilon, \tag{6.5}$$

and so, $(6.4)_3$ becomes

$$L_3^{\pm}(\Lambda) = E_3^{\pm}(\Lambda) \mp C_3(\Lambda) + 0(1)\varepsilon$$

for all Λ in the region $\{(x,t) | t \geqq t_1\}$. That is, for t sufficiently large, the amount of 1-waves is small and thus we may use the techniques used in the last section to prove the following theorem whose proof is omitted:

Theorem 6.1. *Suppose that either TV is sufficiently small or the gas is polytropic with exponent γ, $1 < \gamma < 5/3$ and $(\gamma - 1)TV$ is sufficiently small. Here TV is the total variation of the data* (6.2), (6.3). *Then the initial-boundary value problems have a global solution $(u, v, E)(x,t)$ which approaches the solution $(u^*, v^*, E^*)(x,t)$ of* (6.1) *such that*

$$(u, v, E)(x, 0) = (u_0, v_0, E_0)(+\infty), \quad x \geqq 0, \tag{6.2*}$$

$$u(0,t) = u_b(+\infty), \quad t \geqq 0, \tag{6.3_1^*}$$

or,

$$p(0,t) = p_b(+\infty), \quad t \geqq 0. \tag{6.3_2^*}$$

More precisely, if $(u^*, v^*, E^*)(x,t)$ is a rarefaction wave or a constant, then shock waves decay and $(p, u)(x,t)$ approaches $(p^*, u^*)(x,t)$ pointwise as $t \to +\infty$, and if $(u^*, v^*, E^*)(x,t)$ is a shock wave, then a shock wave emerges from $(u, v, E)(x,t)$ for t large such that the shock wave approaches the shock wave $(u^*, v^*, E^*)(x,t)$ both in speed and strength and outside the shock wave $(u, p)(\cdot, t)$ has total variation approaching zero as $t \to +\infty$.

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