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LARGE TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS*

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Abstract. We study the large time asymptotic behavior of solutions of the doubly degenerate parabolic equation $u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q$ with an initial condition $u(x,0) = u_0(x)$. Here the exponents m, p and q satisfy $m+p \geqslant 3$, $p > 1$ and $q > m+p-2$.

Keywords: degenerate parabolic equation, large time asymptotic behavior

MSC 2010: 35K55, 35K65, 35B40

1. Introduction

The objective of this article is to study the large time asymptotic behavior of weak solutions of nonlinear parabolic equations of the type

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - u^q \quad \text{in } S = \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N. \tag{1.2}$$

Here $p > 1$, $m(p-1) > 1$, $q > 1$, $N \geqslant 1$ and $u_0(x) \in L^1(\mathbb{R}^N)$ is a nonnegative function. Equation (1.1) has been suggested as a mathematical model for a variety of problems in mechanics, physics and biology, one can see [3], [5], [1] etc. The existence of a nonnegative solution of (1.1)–(1.2), defined in some weak sense, is well established (see [12] and [8]). In this paper we are interested in the behavior of solutions as $t \to \infty$. The elliptic method was used in several papers (see e.g. [4], [9]) to study the asymptotic behavior of the solutions of the porous media and the p-Laplacian equations. Also by the elliptic method, J. Manfredi and V. Vespri studied the large

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time behavior of the solution of the initial boundary problem without absorption $-u^q$ in [7]. In details the large time behavior of the solution of the problem

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } \Omega\times(0,\infty), \tag{1.3}$$

$$u(x,t) = 0 \quad \text{in } \partial\Omega\times(0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^N \tag{1.5}$$

was considered in [7].

In our paper we will study problem (1.1)–(1.2) in a way different from the elliptic method which is used in [7], namely, we will compare the large time behavior of the general solution of (1.1)–(1.2) to the Barenblatt-type solution of (1.1)–(1.2).

We begin with some preliminaries.

It is not difficult to verify that

$$E_c = t^{-l/\mu}\Big\{\Big[b - \frac{m(p-1)-1}{mp}(N\mu)^{-1/(p-1)}(|x|t^{-l/\mu})^{p/(p-1)}\Big]_+\Big\}^{(p-1)/(m(p-1)-1)}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) \quad \text{in } S = \mathbb{R}^N\times(0,\infty), \tag{1.6}$$

$$u(x,0) = c\delta(x) \quad \text{on } \mathbb{R}^N \tag{1.7}$$

where $l = (1+(m-1)/(p-1))^{1-p}$, $\mu = m+p-3+p/N$, $c = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$, b is a constant such that $b = \int_{\mathbb{R}^N} E_c(x,t)\,\mathrm{d}x$, and δ denotes the Dirac mass centered at the origin.

Let

$$B_R(x_0) = \{x\in\mathbb{R}^N\colon\ |x-x_0|<R\},\quad B_R = \{x\in\mathbb{R}^N\colon\ |x|<R\}.$$

Definition 1.1. A nonnegative function $u(x,t)$ is called a solution of (1.1)–(1.2) if u satisfies

$$u\in C(0,T;L^1(\mathbb{R}^N))\cap L^\infty(\mathbb{R}^N\times(\tau,T),\ u^{(m-1)/(p-1)}Du\in L^p_{\mathrm{loc}}(\mathbb{R}^N\times(0,T)),$$
$$u_t\in L^1(\mathbb{R}^N\times(\tau,T)),\quad \forall\,\tau>0; \tag{1.8}$$

$$\int_S [u(x,t)\varphi_t(x,t) - u^{m-1}|Du|^{p-2}Du\cdot D\varphi - u^q\varphi]\,\mathrm{d}x\,\mathrm{d}t = 0,\quad \forall\,\varphi\in C_0^1(S); \tag{1.9}$$

$$\lim_{t\to 0}|u(x,t)-u_0(x)|\,\mathrm{d}x = 0. \tag{1.10}$$

Definition 1.2. A nonnegative function $U\in C(\overline{S}\setminus(0))$, $U\neq 0$ is called a very singular solution of (1.1), if U satisfies (1.1) in the sense of distributions in S and

$$\lim_{t\to 0}\int_{B^R} U(x,t)\,\mathrm{d}x = 0,\quad \forall\,R>0.$$

Let $U(x,t) = t^{1/(q-1)} f(|x|t^{-1/\beta})$. Suppose f is the solution of the ordinary equation

$$(f^{m-1}|f'|^{p-2}f')' + \frac{1}{\eta} f^{m-1}|f'|^{p-2}f' + \frac{1}{\beta}\eta f' + \frac{1}{q} f - f^q = 0,$$
$$f(\eta) \geqslant 0, \quad f'(0) = 0, \quad \lim_{\eta\to\infty} \eta^{p/(q-(m+p-2))} f(\eta) = 0.$$

Then we can prove that $U(x,t)$ is a very singular solution of (1.1); we will publish this result in another paper.

Theorem 1.3. *Let $m(p-1) > 1$, $q > m+p-2$. If E_c is a unique solution of* (1.6)–(1.7), *then the solution u of* (1.1)–(1.2) *satisfies*

$$t^{l/\mu}|u(x,t) - E_c(x,t)| \to 0 \quad \text{as } t \to \infty \tag{1.11}$$

uniformly on the sets $\{x \in \mathbb{R}^N\colon\ |x| < at^{-l/\mu N},\ a > 0\}$, where

$$c = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x - \int_0^\infty\int_{\mathbb{R}^N} u^q(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 1.4. *Suppose $m(p-1) > 1$, $q > m+p-2$ and*

$$|x|^\alpha u_0(x) \leqslant B, \quad \lim_{|x|\to\infty} |x|^\alpha u_0(x) = C,$$

where α, B and C are constants with $\alpha \in (0, p/(q-(m+p-2)))$. Then the solution of (1.1)–(1.2) *satisfies*

$$t^{1/(q-1)} u(x,t) \to C^* \quad \text{as } t \to \infty$$

uniformly on the sets

$$\{x \in \mathbb{R}^N\colon\ |x| \leqslant at^{1/\beta},\ a > 0\},$$

where $C^* = (1/(q-1))^{1/(q-1)}$ and $\beta = (q-1)/(q-(m+p-2))$.

Theorem 1.5. *Suppose $1 < m(p-1)$, $m+p-2 < q < m+p-2+p/N$ and*

$$|x|^\alpha u_0(x) \leqslant B, \quad a > \frac{p}{q-(m+p-2)}, \quad \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x > 0.$$

Assume that (1.1) *has a unique very singular solution. Then the solution of* (1.1)–(1.2) *satisfies*

$$t^{1/(q-1)}|u(x,t) - U(x,t)| \to 0 \quad \text{as } t \to \infty$$

uniformly on the sets

$$\{x \in \mathbb{R}^N \colon |x| \leqslant a t^{1/\beta}\},$$

where $\beta = (q-1)/(q-(m+p-2))$.

R e m a r k 1.6. For $m = 1$, the uniqueness of solutions of (1.6)–(1.7) is known (see [2]). For $m = 1, p = 2$, the uniqueness of the very singular solution of (1.1) is known, too (see [11]).

2. Proof of Theorem 1.3

Let u be a solution of (1.1). We define the family of functions

$$u_k = k^N u(kx, k^{N\mu} t), \quad k > 0.$$

It is easy to see that they are solutions of the problems

$$u_t = \operatorname{div}(u^{m-1}|Du|^{p-2}Du) - k^{-v}u^q \quad \text{in } S = \mathbb{R}^N \times (0,\infty), \tag{2.1}$$

$$u(x,0) = u_{0k}(x) \quad \text{on } \mathbb{R}^N, \tag{2.2}$$

where $\mu = m+p-3+p/N$ as before and $v = q-m-p+2-p/N$, $u_{0k}(x) = k^N u_0(x)$.

Lemma 2.1. *For any* $s \in (0, m+p-2)$, u_k *satisfies*

$$\int_0^T \int_{B_R} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c(s, R, |u_0|_{L^1}), \tag{2.3}$$

$$\int_0^T \int_{B_R} u_k^{m+p-2+p/N-s} \,\mathrm{d}x\,\mathrm{d}t \leqslant c(s, R, |u_0|_{L^1}). \tag{2.4}$$

P r o o f. From Definition 1.1, we are able to deduce (see [10]): $\forall \varphi \in C^1(\overline{S})$, $\varphi = 0$ when $|x|$ is large enough,

$$\begin{aligned} \int_{\mathbb{R}^N} u_k(x,t)\varphi \,\mathrm{d}x - \int_0^T \int_{\mathbb{R}^N} (u_k\varphi_t - u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\varphi) \,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \int_{\mathbb{R}^N} u_{0k}(x)\varphi(x,0)\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

Let

$$\psi_R \in C_0^\infty(B_{2R}), \quad 0 \leqslant \psi_R \leqslant 1, \quad \psi_R = 1 \text{ on } B_R, \quad |D\psi_R| \leqslant cR^{-1}. \tag{2.6}$$

By an approximate procedure we can choose $\varphi = (u_k^s/(1+u_k^s))\psi_R^p$ in (2.5); then

$$
\begin{aligned}
(2.7)\qquad & \int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x \\
&+ s\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant -p\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s}|Du_k|^{p-2}\psi_R^{p-1}(x)Du_k\cdot D\psi_R\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_{\mathbb{R}^N}\int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x,
\end{aligned}
$$

where $0 < h < t$. Notice that

$$
\begin{aligned}
(2.8)\qquad & \left|\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-1}}{1+u_k^s}|Du_k|^{p-2}\psi_R^{p-1}(x)Du_k\cdot D\psi_R\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\leqslant \int_h^t\int_{\mathbb{R}^N}\Bigg[\varepsilon\Bigg(\frac{u_k^{(s+m-2)\cdot(p-1)/p}}{(1+u_k^s)^{2(p-1)/p}}|Du_k|^{p-1}\psi_R^{p-1}\Bigg)^{p/(p-1)} \\
&\quad + c(\varepsilon)\Bigg(\frac{u_k^{(s+m-1-(s+m-2))\cdot(p-1)/p}}{(1+u_k^s)^{1-2(p-1)/p}}|D\psi_R|\Bigg)^p\Bigg]\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_h^t\int_{\mathbb{R}^N}\Bigg[\varepsilon\Bigg(\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p + c(\varepsilon)\frac{u_k^{p+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\Bigg]\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

$$
(2.9)\qquad \int_{\mathbb{R}^N}\int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\,\mathrm{d}z\psi_R^p(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} u(x,k^{N\mu}h)\,\mathrm{d}x,
$$

hence by (2.7)–(2.9) we obtain

$$
\begin{aligned}
(2.10)\qquad & \sup_{0<t<T}\int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\,\mathrm{d}x + \int_h^t\int_{\mathbb{R}^N} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\psi_R^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant c\int_{\mathbb{R}^N} u(x,k^{N\mu}h)\,\mathrm{d}x + c\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Because $u_k \in L^\infty(\mathbb{R}^N\times(h,T))\cap L^1(S_T)$, $p+m-2>0$, we have

$$
(2.11)\qquad \lim_{R\to\infty}\int_h^t\int_{\mathbb{R}^N} \frac{u_k^{p+s+m-2}}{(1+u_k^s)^{2-p}}|D\psi_R|^p\,\mathrm{d}x\,\mathrm{d}\tau = 0.
$$

Let $R\to\infty$, $h\to 0$ in (2.10). Then

$$
\begin{aligned}
(2.12)\qquad & \sup_{0<t<T}\int_{\mathbb{R}^N}\int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\,\mathrm{d}z\,\mathrm{d}x + \iint_{S_t} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant c\int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x.
\end{aligned}
$$

Thus

$$\sup_{0<t<T} \int_{B_{2R}} u_k(x,t)\,\mathrm{d}x + \int_0^T\!\!\int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2} |Du_k|^p \,\mathrm{d}x\,\mathrm{d}\tau \leqslant c(R). \tag{2.13}$$

Let

$$u_1 = \max\{u_k(x,t), 1\}, \quad w = u_1^{(m+p-2-s)/p}.$$

By Sobolev's imbedding inequality (see [6]), for $\xi \in C_0^1(B_{2R})$, $\xi \geqslant 0$, we have

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} \xi^p w^r \,\mathrm{d}x\Big)^{1/r} \\ \leqslant c\Big(\int_{\mathbb{R}^N} |D(\xi w)|^p\Big)^{s/p} \Big(\int_{B_{2R}} w^{p/(m+p-2-s)}\,\mathrm{d}x\Big)^{((1-\theta)(m+p-2)-s)/p},\end{aligned}$$

where

$$\begin{aligned}\theta &= \Big(\frac{m+p-2-s}{p} - \frac{1}{r}\Big)\Big(\frac{1}{N} - \frac{1}{p} + \frac{m+p-2-s}{p}\Big)^{-1},\\ r &= \frac{p(m+p-2+p/N-s)}{m+p-2-s}.\end{aligned}$$

It follows that

$$\begin{aligned}\iint_{S_T} \xi^p w^r \,\mathrm{d}x\,\mathrm{d}t \\ \leqslant c\iint_{S_T} |D(\xi w)|^p \,\mathrm{d}x\,\mathrm{d}t \\ \times \sup_{t\in(0,T)} \Big(\int_{B_{2R}} w^{p/(m+p-2-s)}\,\mathrm{d}x\Big)^{(r-p)(m+p-2-s)/p}.\end{aligned} \tag{2.14}$$

Since

$$|Dw|^p \leqslant c\,\frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p \text{ a.e. on } \{u_k \geqslant 1\} \quad\text{and}\quad |Dw| = 0 \text{ on } \{u_k \leqslant 1\},$$

we have

$$\begin{aligned}\iint_{S_T} |D(\xi w)|^p \,\mathrm{d}x\,\mathrm{d}t &\leqslant c\iint_{S_T} (\xi^p|Dw|^p + w^p|D\xi|^p)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant c\Big[\iint_{S_T} |D\xi|^p u_1^{p+m-2-s}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\!\!\int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t\Big].\end{aligned} \tag{2.15}$$

Hence, by (2.14), (2.15) and (2.13), we get

$$\iint_{S_T} \xi^p u_1^{m+p-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t \leqslant c(s,R,|u_0|_{L^1})\Big(1+\iint_{S_T} |D\xi|^p u_1^{p+m-2-s}\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Let $\xi = \psi_R^b$, where ψ_R is the function satisfying (2.6) and $b = N(m+p-2-s)/p$. Then

$$\iint_{S_T} \psi_R^{pb} u_1^{m+p-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c(s,R,|u_0|_{L^1})\Big(1+\iint_{S_T} \psi_R^{pb} u_1^{p+m-2+p/N-s}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(m+p-2-s)/(m+p-2+p/N-s)},$$

which implies (2.4) is true. □

Let $Q_\varrho = B_\varrho(x_0)\times(t_0-\varrho^p,t_0)$ with $t_0 > (2\varrho)^p$ and $u_{k1} = \max\{u_k,1\}$.

Lemma 2.2. *Each u_k satisfies*

$$\sup_{Q_\varrho} u_k \leqslant c(\varrho,s_1)\Big(\iint_{Q_{2\varrho}} u_{k1}^{p+m-3+s_1}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/s_1}, \tag{2.16}$$

where $c(\varrho,s_1)$ depends on ϱ and s_1, and s_1 can be any number satisfying $0 < s_1 < 1+p/N$.

Lemma 2.3. *Each u_k satisfies*

$$\int_\tau^T\!\!\int_{B_R} u_k^{m-1}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant c(\tau,R),\qquad \int_\tau^T\!\!\int_{B_R} |u_{kt}|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant c(\tau,R). \tag{2.17}$$

Proof. By Lemma 2.1 and 2.2, u_k are uniformly bounded on every compact set $K\subset S_T$. Let ψ_R be a function satisfying (2.6) and let $\xi\in C_0^1(0,T+1)$ with $0\leqslant\xi\leqslant 1$, $\xi = 1$ if $t\in(\tau,T)$. We choose $\eta = \psi_R^p\xi u_k$ in (2.5) to obtain

$$\begin{aligned}\frac12\int_{\mathbb{R}^N} u_k^2(x,T)\psi_R^p\,\mathrm{d}x &+ \iint_{S_T} u_k^{m-1}|Du_k|^p\psi_R^p\xi\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \frac12\iint_{S_T} u_k^2\xi'\psi_R^p\,\mathrm{d}x\,\mathrm{d}t - p\iint_{S_T} u_k^m|Du_k|^{p-2}Du_k\cdot D\psi_R\psi_R^{p-1}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.18}$$

Notice that

$$\begin{aligned}&\iint_{S_T} u_k^m|Du_k|^{p-1}|D\psi_R|\psi_R^{p-1}\xi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqslant \varepsilon\iint_{S_T} u_k^{m-1}|Du_k|^p\psi_R^p\xi\,\mathrm{d}x\,\mathrm{d}t + c(\varepsilon)\iint_{S_T} u_k^{p+m-1}|D\psi_R|^p\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.19}$$

By (2.18), (2.19), one knows that the first inequality of (2.17) is true.

Now we will prove the second inequality of (2.17). Let

$$v(x,t) = u_{kr}(x,t) = r u_k(x, r^{m+p-3}t), \quad r \in (0,1).$$

Then

$$v_t(x,t) = \operatorname{div}(v^{m-1}|Dv|^{p-2}Dv) - r^{m+p-2-q}k^{-v}v^q, \tag{2.20}$$

$$v(x,0) = r u_k(x,0). \tag{2.21}$$

Notice that $r^{m+p-2-q}k^{-v} > k^{-v}$ using the argument similar to that in the proof of Theorem 1 of [12], one can prove

$$u_k \geqslant u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m+p-3}t) - u_k(x,t)}{(r^{m+p-3}-1)t} \geqslant \frac{r-1}{(1-r^{m+p-3})t} u_k(x, r^{m+p-3}t).$$

Letting $r \to 1$, we get

$$u_{kt} \geqslant -\frac{u_k}{(m+p-3)t}. \tag{2.22}$$

Denote $w = t^\beta u_k(x,t)$, $\beta = 1/(m+p-3)$. By (2.22), $w_t \geqslant 0$. By (2.1),

$$\begin{aligned}
&\int_\tau^T \int_{B_{2R}} t^\beta w_t \psi_R \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_\tau^T \int_{B_{2R}} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\psi_R \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_\tau^T \int_{B_{2R}} k^{-v}u_k^q \psi_R \,\mathrm{d}x\,\mathrm{d}t + \beta \int_\tau^T \int_{B_{2R}} t^{-1}u_k(x)\psi_R \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \frac{\beta}{\tau} \int_\tau^T \int_{B_{2R}} u_k \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \left(\int_\tau^T \int_{B_{2R}} u_k^{m-1}|Du_k|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{(p-1)/p} \left(\int_\tau^T \int_{B_{2R}} |D\psi_R|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p}.
\end{aligned} \tag{2.23}$$

From (2.13), (2.16) and (2.23) we obtain (2.17). □

P r o o f of Theorem 1.3.

By Lemmas 2.1–2.3 and [2], there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function v such that on every compact set $K \subset S$

$$u_{k_j} \to v \text{ in } C(K), \quad Du_{k_j}^m \rightharpoonup Dv^m \text{ in } L^p_{\mathrm{loc}}(S_T), \quad |u_{kt}|_{L^1_{\mathrm{loc}}(S_T)} \leqslant c.$$

Similar to what was done in the proof of Theorem 2 in [12], we can prove that v satisfies (1.1) in the sense of distributions.

We now prove $v(x,0) = c\delta(x)$. Let $\chi \in C_0^1(B_R)$. Then we have

$$\begin{aligned}\text{(2.24)}\qquad &\int_{\mathbb{R}^N} u_k(x,t)\chi \,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi_k \chi \,\mathrm{d}x \\ &= -\int_0^t \int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\chi \,\mathrm{d}x\,\mathrm{d}s - k^{-v}\int_0^t\int_{\mathbb{R}^N} u_k^q \chi \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

To estimate $\int_0^t \int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k \cdot D\chi \,\mathrm{d}x\,\mathrm{d}s$, without losing generality, one can assume that $u_k > 0$. By Hölder inequality and Lemma 2.1,

$$\begin{aligned}\text{(2.25)}\qquad &\left|\int_0^t\int_{\mathbb{R}^N} u_k^{m-1}|Du_k|^{p-2}Du_k\cdot D\chi\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\leqslant c\Big(\int_0^T\int_{B_{2R}} \frac{u_k^{s+m-2}}{(1+u_k^s)^2}|Du_k|^p\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p} \\ &\quad\times\Big(\int_0^T\int_{B_{2R}} (1+u_k^s)^{2(p-1)}u_k^{(p-1)(2-s-m)}\,\mathrm{d}x\,\mathrm{d}\tau\Big)^{1/p} \\ &\leqslant c\Big(\int_0^t\int_{B_{2R}} \big(u_{k1}^{(p-1)(2-s-m)} + u_{k1}^{(p-1)(2+s-m)}\,\mathrm{d}x\,\mathrm{d}\tau\big)^{1/p} \\ &\leqslant c\Big(\int_0^t\int_{B_{2R}} \big(u_{k1}^{(p-1)(2-s-m)}\big)^{\frac{m+p-2+p/N-s}{(p-1)(s+2-m)}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{(p-1)(s-2-m)}{m+p-2+p/N-s}\frac{1}{p}} t^d,\end{aligned}$$

where $s \in (0, 1/N)$, $d = ((m-s-1)Np + (s-2)N + p - s + 2)/((m+p-2)N + p - s) < 1$ because $p > (N+3)/(2N+1)$, $u_{k1} = \max(u_k, 1)$.

Hence from (2.24) we get

$$\begin{aligned}\text{(2.26)}\qquad &\left|\int_{\mathbb{R}^N} u_k(x,t)\chi\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi_k\chi\,\mathrm{d}x + k^{-v}\int_0^t\int_{\mathbb{R}^N} u_k^q\chi\,\mathrm{d}x\,\mathrm{d}s\right| \\ &= \left|\int_{\mathbb{R}^N} u_k(x,t)\chi\,\mathrm{d}x - \int_{\mathbb{R}^N}\varphi_k\chi(k^{-1}x)\,\mathrm{d}x + \int_0^{N\mu t}\int_{\mathbb{R}^N} u_k^q\chi(k^{-1}x)\,\mathrm{d}x\,\mathrm{d}\tau\right| \leqslant ct^d.\end{aligned}$$

Letting now $k \to \infty$, $t \to 0$, we obtain

$$\lim_{t\to 0}\int_{\mathbb{R}^N} v(x,t)\chi\,\mathrm{d}x = \chi(0)\Big(\int_{\mathbb{R}^N}\varphi(x)\,\mathrm{d}x - \int_0^\infty\int_{\mathbb{R}^N} u^q\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Thus

$$v(x,0) = c\delta(x), \quad c = \int_{\mathbb{R}^N} \varphi(x)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}^N} u^q\,\mathrm{d}x\,\mathrm{d}t,$$

$v(x,t)$ is a solution of (1.3)–(1.4). By the assumption on uniqueness of solution, we have $v(x,t) = E_c(x,t)$ and the whole sequence $\{u_k\}$ converges to E_c as $k \to \infty$. Set $t = 1$. Then

$$u_k(x,1) = k^N u(kx, k^{N\mu}) \to E_c(x,1)$$

uniformly on every compact subset of $\mathbb{R}^N$. Thus writing $kx = k'$, $k^{N\mu} = t'$, and dropping the prime again, we see that

$$t^{1/\mu} u(x,t) \to E_c(xt^{1/(N\mu)}, 1) = t^{1/\mu} E_c(x,t)$$

uniformly on the sets $\{x \in \mathbb{R}^N : |x| \leqslant at^{1/(N\mu)}\}$, $a > 0$. Thus Theorem 1.3 is true.

□

3. Proofs of Theorem 1.4 and 1.5

Let u be a solution of (1.1)–(1.2) and let $u_k(x,t) = k^\delta u(kx, k^\beta t)$, $k > 0$. If $\delta = 1/(q-(m+p-2))$, $\beta = (q-1)/(q-(m+p-2))$, then

$$u_{kt} = \operatorname{div}(u_k^{m-1}|Du_k|^{p-2}Du_k) - u_k^q, \tag{3.1}$$

$$u_k(x,0) = \varphi_k(x) = k^\delta \varphi(kx). \tag{3.2}$$

Lemma 3.1. *The solution u_k of* (3.1)–(3.2) *satisfies*

$$u_k(x,t) \leqslant C^* t^{-1/(q-1)}, \quad C^* = \Bigl(\frac{1}{q-1}\Bigr)^{1/(q-1)}. \tag{3.3}$$

Proof. We consider the regularized problem of (3.1), namely,

$$u_{kt} = \operatorname{div}((u_k^{m-1} + \varepsilon)(|Du_k|^2 + \varepsilon)^{(p-2)/2} Du_k) - u_k^q. \tag{3.4}$$

By the uniqueness of the solution of (3.1)–(3.2), we can prove that

$$u_{k\varepsilon} \to u_k \quad \text{as} \quad \varepsilon \to 0 \ \text{in} \ C(K)$$

on every compact set $K \subset S$, where $u_{k\varepsilon}$ are the solutions of (3.4), (3.2). By computation, it is easy to show that $C^*(t-t_0)^{-1/(q-1)}$ is a solution of (3.4) in $\mathbb{R}^N \times (t_0, \infty)$, $t_0 > 0$. For any $\delta_1 > 0$, we choose $\delta_0 \in (0, \delta_1)$ such that

$$|u_{k\varepsilon}(x,\delta_1)|_{L^\infty(\mathbb{R}^N)} \leqslant C^*(\delta_1 - \delta_0)^{-1/(q-1)}.$$

Hence by the comparison principle, we have

$$u_{k\varepsilon}(x,t) \leqslant C^*(t-t_0)^{-1/(q-1)}, \quad t > \delta_1.$$

The proof of Lemma 3.1 is completed by letting $\delta_1 \to 0$ and $\varepsilon \to 0$. □

Lemma 3.2. *Each u_k satisfies*

$$\int_\tau^T \int_{B_R} |Du_k|^p \leqslant c(\tau, R), \quad \int_\tau^T \int_{B_R} |u_{kt}| \, \mathrm{d}x \, \mathrm{d}t \leqslant c(\tau, R), \tag{3.5}$$

where $\tau \in (0, T)$.

Proof. The proof of Lemma 3.2 is similar to that of Lemma 2.3. □

Proof of Theorem 1.4. By Lemma 3.1, $\{u_k\}$ are uniformly bounded on every compact set of S. Hence by [2], there exists a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U \quad \text{in } C(K)$$

and

$$U(x,t) \leqslant C^* t^{-1/(q-1)}.$$

We now prove that $U(x,t) = C^* t^{-1/(q-1)}$. Let us introduce the function

$$\varphi_k^A = \min\{\varphi_k, A\} \tag{3.6}$$

and denote by $V_{K\varepsilon}^A$ the solution of (3.4) with initial value (3.6). By the comparison principle,

$$V_{K\varepsilon}^A \leqslant u_{k\varepsilon}, \tag{3.7}$$

where $u_{k\varepsilon}$ is the solution of (3.4), (3.2).

Define

$$V_A = C^* \Big(t + \frac{A^{1-q}}{q-1} \Big)^{-1/(q-1)},$$

which is the solution of (3.4) with initial value

$$V_A(x,0) = A. \tag{3.8}$$

Notice that

$$\lim_{k\to\infty} \varphi_k^A(x) = \lim_{k\to\infty} \min\Big\{ A, \frac{\varphi(kx)|kx|^\alpha k^{\delta-\alpha}}{|x|^\alpha} \Big\} = A.$$

Using the uniqueness of solution of (3.4), (3.8), we can prove (see [6])

$$V_{k\varepsilon}^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K),$$

where K is a compact set in S. Moreover, by [2] and [12]

$$V_{k\varepsilon}^A \to V_k^A u_{k\varepsilon} \to u_k \quad \text{as } k \to \infty \text{ in } C(K)$$

uniformly in K, where V_k^A is the solution of (1.1) with initial value (3.6). It follows that

$$V_k^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K).$$

Letting $\varepsilon \to 0$ and $k \to \infty$ in turn in (3.7), we get

$$V_A(x,t) \leqslant V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Since the lower bound holds for every $A > 0$, we conclude that

$$U(x,t) = V_\infty(x,t) = C^* t^{-1/(q-1)} \quad \text{in } S.$$

Thus

$$k^{p/(q-(m+p-2))} u(kx, k^\beta t) \to C^* t^{-1/(q-1)} \quad \text{as } k \to \infty.$$

Set $t = 1$. Then

$$k^{p/(q-(m+p-2))} u(kx, k^\beta) \to C^* \quad \text{as } k \to \infty$$

uniformly on every compact subset of $\mathbb{R}^N$. Therefore if we set $kx = x'$, $k^\beta = t'$, and omit the primes, we obtain

$$t^{1/(q-1)} u(x,t) \to C^* \quad \text{as } t \to \infty$$

uniformly on sets $\{x \in \mathbb{R}^N\colon\ |x| \leqslant \alpha t^{1/\beta}\}$ with $\alpha > 0$ for $t > 0$ and so Theorem 1.4 is proved. □

P r o o f of Theorem 1.5. By Lemma 3.1 and [2], there exist a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U \quad \text{in } C(K). \tag{3.9}$$

By Lemma 3.2, we can prove that U satisfies (1.1) in the sense of distributions in a manner similar to Theorem 2 of [12]. □

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