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Large-time Behavior of Solutions to the Equations of a Viscous Polytropic Ideal Gas (*).

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Abstract. – *First we prove for the equations of a viscous polytropic ideal gas in bounded annular domains in* $\mathbb{R}^n$ $(n = 2, 3)$ *that (generalized) spherically symmetric solutions decay to a constant state exponentially as time goes to infinity. Then we show that solutions of the Cauchy problem in* $\mathbb{R}$ *are asymptotically stable if the initial specific volume is close to a constant in* L^∞ *and weighted* L^2, *the initial velocity is small in weighted* $L^2 \cap L^4$, *and the initial temperature is close to a constant in weighted* L^2.

1. – Introduction.

In this paper we study the asymptotic behavior of solutions to the following equations in the domain G_n $(1 \leq n \leq 3)$:

$$\partial_t \varrho + \partial_r(\varrho v) + \frac{(n-1)}{r} \varrho v = 0 \,, \tag{1.1}$$

$$\varrho(\partial_t v + v \partial_r v) = (\lambda + 2\mu)\left(\partial_r^2 v + \frac{(n-1)}{r} \partial_r v - \frac{(n-1)}{r^2} v\right) - R\partial_r(\varrho\theta) \,, \tag{1.2}$$

$$\begin{aligned} c_V \varrho(\partial_t \theta + v \partial_r \theta) = \kappa \partial_r^2 \theta + \kappa \frac{(n-1)}{r} \partial_r \theta - R\varrho\theta\left(\partial_r v + \frac{(n-1)}{r} v\right) + \\ + \lambda\left(\partial_r v + \frac{(n-1)}{r} v\right)^2 + 2\mu(\partial_r v)^2 + 2\mu \frac{(n-1)}{r^2} v^2 \,, \\ r \in G_n,\ t > 0 \,, \end{aligned} \tag{1.3}$$

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where $G_n = \mathbb{R}$ for $n=1$ and $G_n = (a, b)$ $(a>0)$ for $n=2, 3$; R, c_V, κ, λ, μ are constants satisfying R, c_V, κ, $\mu > 0$, $\lambda + 2\mu/n \geqslant 0$. For (1.1)-(1.3) we will consider the Cauchy problem in the case of $n=1$ and the following initial boundary value problem in the case of $n=2, 3$:

$$\varrho(r, 0) = \varrho_0(r), \quad v(r, 0) = v_0(r), \quad \theta(r, 0) = \theta_0(r), \quad r \in G_n \quad \text{for } 1 \leqslant n \leqslant 3, \tag{1.4}$$

$$v(a, t) = v(b, t) = 0, \quad \theta_r(a, t) = \theta_r(b, t) = 0, \quad t \geqslant 0 \quad \text{for } n=2 \text{ or } 3. \tag{1.5}$$

The equations (1.1)-(1.3) describe the motion of a viscous polytropic ideal gas in $\mathbb{R}$ in the case of $n=1$, or the spherically symmetric motion of a viscous polytropic ideal gas in the annular domain $\{x \in \mathbb{R}^n \mid a < |x| < b\}$ in the case of $n=2, 3$ (cf. [1, 5, 10]), where ϱ, v, θ are the density, the velocity, and the absolute temperature, respectively; λ and μ are the constant viscosity coefficients, R, c_V, and κ are the gas constant, the specific heat capacity, and the thermal conductivity, respectively.

In two or three dimensions the global existence and large-time behavior of smooth solutions to the equations of a viscous polytropic ideal gas have been investigated for general domains only in the case of sufficiently small initial data (see e.g. [2, 3], [16]-[20], [27, 28], where more general constitutive equations were considered). For large initial data the global existence of (generalized) solutions was shown in [4, 5, 25] resp. in [10] for the spherically symmetric motion in a bounded annular domain resp. in an exterior domain. The asymptotic behavior of the (spherically symmetric) solutions in the bounded annular domain, however, is not discussed in [4, 5, 25] (in [10] some large-time behavior of ϱ, v was discussed only for the case $n=3$).

In one dimension it is well known that global solutions exist. Moreover, for initial boundary value problems in bounded domains a solution converges to a steady state (exponentially) as $t \to \infty$ (see [1, 8, 9], [21]-[24]). For the Cauchy problem the large-time behavior of solutions is investigated only for small initial data. In [12, 15] (also cf. [6]) decay rates of solutions were studied for the initial data sufficiently small at least in $H^3(\mathbb{R})$. Kanel, Kawashima, Nishida, and Okada [11, 13, 26] proved that if the $H^1(\mathbb{R})$-norm of the initial data is sufficiently small, then a (smooth) solution converges to a constant steady state as $t \to \infty$.

In the present paper first we prove the exponential decay of (generalized) solutions of (1.1)-(1.5) for $n=2$ or 3. Then we show that in the case of $n=1$ (generalized) solutions of the Cauchy problem (1.1)-(1.4) converge to a constant steady state as $t \to \infty$ provided that the initial specific volume is close to a constant in L^∞ and weighted L^2, the initial velocity is small in weighted $L^2 \cap L^4$, and the initial temperature is close to a constant in weighted L^2.

To show the time-asymptotic behavior it is convenient to transform the system (1.1)-(1.3) to that in Lagrangian coordinates. The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$r(\xi, t) = r_0(\xi) + \int_0^t \tilde{v}(\xi, \tau)\, d\tau, \tag{1.6}$$

where $\tilde{v}(\xi, t) := v(r(\xi, t), t)$, and

$$r_0(\xi) := \eta^{-1}(\xi), \qquad \eta(r) := \int_{d_n}^{r} s^{n-1} \varrho_0(s)\, ds, \qquad r \in G_n; \qquad d_n := \begin{cases} 0, & n = 1, \\ a, & n = 2, 3. \end{cases}$$

It should be noted that if $\inf\{\varrho_0(s);\, s \in \overline{G}_n\} > 0$ (which will be assumed later), then η as a function of $r \in \overline{G}_n$ is invertible. Denote $L := \int_a^b s^{n-1} \varrho_0(s)\, ds > 0$. Using the equation (1.1), (1.6), and (1.5), we obtain $\partial_t \int_{d_n}^{r(\xi, t)} s^{n-1} \varrho(s, t)\, ds = \delta_{n1} v(0, t)\, \varrho(0, t)$ with δ_{ij} being the Kronecker delta, which by integration turns into

$$\int_{d_n}^{r(\xi, t)} s^{n-1} \varrho(s, t)\, ds = \int_{d_n}^{r_0(\xi)} s^{n-1} \varrho_0(s)\, ds + \delta_{n1} \int_0^t (v\varrho)(0, \tau)\, d\tau = \xi + \delta_{n1} \int_0^t (v\varrho)(0, \tau)\, d\tau\,.$$

Thus, under the assumption $\inf\{\varrho(s, t);\, s \in \overline{G}_n,\, t \geq 0\} > 0$ (which is posteriori justified) we see that G_n is tranformed to Ω_n with $\Omega_n = \mathbb{R}$ if $n = 1$ and $\Omega_n = (0, L)$ if $n = 2, 3$. Moreover we have

$$\partial_\xi r(\xi, t) = [r(\xi, t)^{n-1} \varrho(r(\xi, t), t)]^{-1}. \tag{1.7}$$

For a function $\varphi(r, t)$ we write $\tilde{\varphi}(\xi, t) := \varphi(r(\xi, t), t)$. By virtue of (1.6) and (1.7),

$$\begin{cases} \partial_t \tilde{\varphi}(\xi, t) = \partial_t \varphi(r, t) + v \partial_r \varphi(r, t), \\ \partial_\xi \tilde{\varphi}(\xi, t) = \partial_r \varphi(r, t)\, \partial_\xi r(\xi, t) = \dfrac{1}{r^{n-1} \varrho(r, t)}\, \partial_r \varphi(r, t). \end{cases} \tag{1.8}$$

Without danger of confusion we denote $(\tilde{\varrho}, \tilde{v}, \tilde{\theta})$ still by (ϱ, v, θ) and (ξ, t) by (x, t). We use $u := 1/\varrho$ to denote the specific volume. Therefore, by virtue of (1.7)-(1.8), the equations (1.1)-(1.5) in the new variables (x, t) read:

$$u_t = (r^{n-1} v)_x, \tag{1.9}$$

$$v_t = r^{n-1} \left[\beta \frac{(r^{n-1} v)_x}{u} - R \frac{\theta}{u} \right]_x, \qquad x \in \Omega_n,\ t > 0, \tag{1.10}$$

$$c_V \theta_t = \kappa \left[\frac{r^{2n-2} \theta_x}{u} \right]_x + \frac{1}{u} [\beta (r^{n-1} v)_x - R\theta](r^{n-1} v)_x - 2\mu(n-1)(r^{n-2} v^2)_x \tag{1.11}$$

together with

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega_n, \quad 1 \leq n \leq 3, \tag{1.12}$$

$$v(0, t) = v(L, t) = 0, \quad \theta_x(0, t) = \theta_x(L, t) = 0, \quad t \geq 0, \quad n = 2, 3. \tag{1.13}$$

Here $\Omega_n = \mathbb{R}$ if $n = 1$ and $\Omega = (0, L)$ if $n = 2, 3$; $u_0 = 1/\varrho_0$, $\beta = \lambda + 2\mu$, and by virtue of (1.6), $r \equiv r(x, t)$ is determined by

$$\tag{1.14} \begin{cases} r(x, t) = r_0(x) + \int_0^t v(x, \tau)\, d\tau\,, & x \in [0, L], \quad t \geq 0\,, \\ r_0(x) := \left\{ (d_n)^n + n \int_0^x u_0(y)\, dy \right\}^{1/n}. \end{cases}$$

For the formulation of the main result we introduce the following notation: H^m and $\|\cdot\|_{H^m}$ ($m \geq 0$ integer) denote $H^m(\Omega_n)$ and its norm ($1 \leq n \leq 3$), respectively. $\|\cdot\|$ and $\|\cdot\|_{L^p}$ denote the norms in $L^2(\Omega_n)$ and $L^p(\Omega_n)$ ($1 \leq p \leq \infty$), respectively. Q_T stands for the domain $\Omega_n \times (0, T)$ ($1 \leq n \leq 3$). For a vector valued function $f = (f_1, \dots, f_m)$ we put $|||f||| := |||f_1||| + \dots + |||f_m|||$, where $|||\cdot|||$ denotes a norm.

As mentioned in the introduction the global existence of (generalized) solutions to (1.9)-(1.14) has been established. In the case of $n = 1$ Kazhikhov [1, 14] proved that if for some positive constants $\bar{u}$, $\bar{\theta}$, $u_0 - \bar{u}$, v_0, $\theta_0 - \bar{\theta} \in H^1$, and $u_0(x)$, $\theta_0(x) > 0$ on $\mathbb{R}$, then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ with positive u and θ to the Cauchy problem (1.1)-(1.12) on $\mathbb{R} \times [0, \infty)$ such that for every $T > 0$

$$\tag{1.15} u - \bar{u} \in L^\infty([0, T], H^1), \qquad v, \theta - \bar{\theta} \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2), \qquad u_t, v_t, \theta_t \in L^2(Q_T).$$

In the case of $n = 2, 3$ Nikolaev [25] (also cf. [4, 5]) showed that if u_0, v_0, $\theta_0 \in H^1$, $u_0(x)$, $\theta_0(x) > 0$ on $[0, L]$ and the initial data are compatible with the boundary conditions (1.13), then there exists a unique solution $\{u(x, t), v(x, t), \theta(x, t)\}$ with positive u and θ to (1.9)-(1.14) on $[0, L] \times [0, \infty)$ such that for every $T > 0$

$$\tag{1.16} u \in L^\infty([0, T], H^1), \qquad v, \theta \in L^\infty([0, T], H^1) \cap L^2([0, T], H^2), \qquad u_t, v_t, \theta_t \in L^2(Q_T).$$

Denote

$$\tag{1.17} \begin{cases} u^* := \dfrac{1}{L} \displaystyle\int_0^L u_0(x)\, dx\,, \\ \theta^* := \dfrac{1}{c_V L} \displaystyle\int_0^L \left\{ c_V \theta_0 + \dfrac{v_0^2}{2} \right\}(x)\, dx\,; \\ r^*(x) := (a^n + n u^* x)^{1/n}, \quad x \in [0, L]. \end{cases}$$

We assume for $n=2$ or 3 that λ and μ satisfy

$$n\lambda + 2\mu > 0\,. \tag{1.18}$$

Then the main result of the paper reads:

THEOREM 1.1. – (i) *Let* $n=2$ *or* 3. *Assume that* (1.18) *is satisfied. Let* $\{u(x,t), v(x,t), \theta(x,t)\}$ *be a solution of* (1.9)-(1.14) *in the function class indicated in* (1.16). *Then there are positive constants* α, T_0, C, *independent of* t, *such that*

$$\|(u-u^*, v, \theta-\theta^*)(t)\|_{H^1} + \|r(t)-r^*\|_{H^2} \leq Ce^{-\alpha t}, \quad \textit{for any } t \geq T_0\,. \tag{1.19}$$

(ii) *Let* $n=1$ *and* $\{u(x,t), v(x,t), \theta(x,t)\}$ *be a solution of* (1.9)-(1.12) *in the function class indicated in* (1.15). *Denote*

$$e_0^2 := \|u_0-\bar{u}\|_{L^\infty}^2 + \int_{\mathbf{R}} (1+x^2)^\gamma \{(u_0-\bar{u})^2 + v_0^2 + (\theta_0-\bar{\theta})^2 + v_0^4\}\, dx$$

where $\gamma > 1/2$ *is an arbitray but fixed constant. Then there is a constant* $\varepsilon \in (0,1]$, *independent of* u_0, v_0, θ_0, *such that if* $e_0 \leq \varepsilon$, *then*

$$\|(u-\bar{u}, v, \theta-\bar{\theta})(t)\|_{L^\infty} + \|(u_x, v_x, \theta_x)(t)\| \to 0\,, \quad \textit{as } t\to\infty\,. \tag{1.20}$$

REMARK 1.1. – The same techniques work and a result analogous to Theorem 1.1 (i) holds when (1.13) is replaced by $v|_{\partial\Omega_n}=0$, $\theta|_{\partial\Omega_n}=1$.

We will prove (i) and (ii) of Theorem 1.1 in Sections 2 and 3, respectively.

2. – Proof of Theorem 1.1-(i).

In this section the same letter C (sometimes used as C_1, C_2) denotes various positive constants which are in particular independent of t and x. The proof of Theorem 1.1 (i) is essentially based on a careful examination of a priori estimates which are shown to be independent of t. The difficulties arise from the dependence on the time and spatial variables of the coefficients in the equations (1.9)-(1.11), but can be overcome in our approach by modifying an idea of Kazhikhov [1, 14] for the one-dimensional case. The proof will be partitioned into several steps.

The first observation is that, by virtue of (1.7) and (1.14),

$$r_t(x,t)=v(x,t)\,, \quad r^{n-1}(x,t)\,r_x(x,t)=u(x,t)\,, \quad x\in[0,L],\ t\geq 0\,. \tag{2.1}$$

By (1.13)-(1.14) and (2.1) we obtain $r_x(0,t)=r^{1-n}(0,t)\,u(0,t)=a^{1-n}u(0,t)>0$ for all $t\geq 0$. Thus, if $r_x(x,t)>0$ is violated on $[0,L]\times[0,\infty)$, there are $y\in(0,L]$ and $\tau\in[0,\infty)$ such that $r_x(x,t)>0$ for $0\leq x<y$, $0\leq t\leq\tau$, but $r_x(y,\tau)=0$. So by continuity, $r_x(x,t)\geq 0$ for $x\in[0,y]$ and $t\in[0,\tau]$, and we have $r(y,\tau)\geq r(0,\tau)=a>0$. From (2.1) we get $0=r_x(y,\tau)=r^{1-n}(y,\tau)u(y,\tau)>0$ which is a contradiction. This shows

$r_x(x, t) > 0$ for $0 \leq x \leq L$, $t \geq 0$. Therefore,

$$a = r(0, t) \leq r(x, t) \leq r(L, t) = b \quad \text{for } x \in [0, L],\ t \geq 0. \tag{2.2}$$

The following estimate embodies the dissipative character of viscosity and thermal diffusion and is motivated by the second law of thermodynamics.

LEMMA 2.1. – *There is a positive constant c_0, independent of t, such that*

$$\int_0^L U(x, t)\, dx + \int_0^t \int_0^L \left(\frac{v_x^2}{u\theta} + \frac{\theta_x^2}{u\theta^2} \right) dx\, ds \leq c_0, \qquad \forall t \geq 0, \tag{2.3}$$

where

$$U(x, t) := \{v^2/2 + R(u - \log u - 1) + c_V(\theta - \log\theta - 1)\}(x, t). \tag{2.4}$$

PROOF. – Using (1.9)-(1.11), we obtain after a straightforward calculation that

$$U_t + \frac{\beta}{u\theta}(r^{n-1}v)_x^2 + \frac{\kappa}{u\theta^2}(r^{n-1}\theta_x)^2 = \left[r^{n-1} v \left(\frac{\beta}{u}(r^{n-1}v)_x - R\frac{\theta}{u} \right) \right]_x + \tag{2.5}$$
$$+ R(r^{n-1}v)_x + \kappa \left[\left(1 - \frac{1}{\theta}\right) \frac{r^{2n-2}\theta_x}{u} \right]_x - 2(n-1)\mu \left(1 - \frac{1}{\theta}\right)(r^{n-2}v^2)_x.$$

Recalling $2\mu + n\lambda$, $2\mu + (n-1)\lambda > 0$, we utilise (2.1) to arrive at

$$\frac{\beta}{u\theta}(r^{n-1}v)_x^2 - 2\mu(n-1)\frac{(r^{n-2}v^2)_x}{\theta} = \tag{2.6}$$
$$= \frac{1}{u\theta}\left\{ (n-1)(2\mu + (n-1)\lambda)\left(r^{-1}uv + \frac{\lambda r^{n-1}v_x}{2\mu + (n-1)\lambda} \right)^2 + \frac{2\mu(2\mu + n\lambda)}{2\mu + (n-1)\lambda} r^{2n-2}v_x^2 \right\} \geq$$
$$\geq \frac{2\mu(2\mu + n\lambda)}{(2\mu + (n-1)\lambda)} \frac{r^{2n-2}v_x^2}{u\theta}.$$

By virtue of Taylor's theorem, $\int_0^L U(x, 0)\, dx \leq C(1 + \|(u_0, v_0, \theta_0)\|^2)$. So If we integrate (2.5) over $[0, L] \times [0, t]$ $(t \geq 0)$, use (1.13) and (2.6), we obtain (2.3). ■

As a corollary of Lemma 2.1 we have

LEMMA 2.2. – *There are positive constants α_1, α_2, independent of t, such that*

$$\alpha_1 \leq \int_0^L \theta(x, t)\, dx \leq \alpha_2 \qquad \forall t \geq 0, \tag{2.7}$$

and for each $t \geq 0$ *there is an* $a(t) \in [0, L]$ *satisfying*

$$\alpha_1 \leq \theta(a(t), t) \leq \alpha_2 . \tag{2.8}$$

PROOF. – (2.3) implies

$$c_V \int_0^L (\theta(x, t) - \log \theta(x, t) - 1)\, dx \leq c_0 , \qquad t \geq 0 . \tag{2.9}$$

Therefore by virtue of the mean value theorem, for each $t \geq 0$ there is an $a(t) \in [0, L]$ such that $\theta(a(t), t) - \log \theta(a(t), t) - 1 \leq (c_V L)^{-1} c_0$, from which it follows that $\zeta_1 \leq \leq \theta(a(t), t) \leq \zeta_2$ with ζ_1, ζ_2 being two (positive) roots of the equation: $y - \log y - 1 = = (c_V L)^{-1} c_0$. If we use (2.9) and apply Jensen's inequality to the convex function $y - - \log y - 1$, we obtain:

$$\int_0^L \theta(x, t)\, dx - \log \int_0^L \theta(x, t)\, dx - 1 \leq c_V^{-1} c_0 , \qquad t \geq 0 .$$

Therefore $0 < \zeta_3 \leq \int_0^L \theta(x, t)\, dx \leq \zeta_4$ for $t \geq 0$, where ζ_3, ζ_4 are two (positive) roots of the equation: $y - \log y - 1 = c_V^{-1} c_0$. Taking $\alpha_1 := \min \{\zeta_1, \zeta_3\}$ and $\alpha_2 := \max \{\zeta_2, \zeta_4\}$, we obtain (2.7)-(2.8). ■

Next we adapt and modify an idea of Kazhikhov [14] (also cf. [1]) for the one-dimensional case to give a representation for u.

Let

$$\sigma(x, t) := \beta \frac{(r^{n-1} v)_x}{u} - R \frac{u}{\theta} , \tag{2.10}$$

$$\phi(x, t) := \int_0^t \sigma(x, s)\, ds + \int_0^x r_0^{-(n-1)}(y)\, v_0(y)\, dy + (n-1) \int_0^t \int_x^L r^{-n}(y, s)\, v^2(y, s)\, dy\, ds . \tag{2.11}$$

Then by (1.10), a partial integration in the varivable t, and (2.1),

$$\phi_x(x, t) = r^{-(n-1)}(x, t)\, v(x, t) . \tag{2.12}$$

Note that in view of (2.1) ϕ satisfies

$$\phi_t = \beta \frac{(r^{n-1} v)_x}{u} - R \frac{\theta}{u} + \frac{(n-1)}{n} \frac{(r^n)_x}{u} \int_x^L r^{-n} v^2\, dy . \tag{2.13}$$

Multiplying (2.13) by u, using (1.9) and (2.12), we arrive at

$$(2.14)\qquad (u\phi)_t - (r^{n-1}v\phi)_x = -v^2 - R\theta + \beta(r^{n-1}v)_x + \frac{(n-1)}{n}(r^n)_x \int_x^L r^{-n}v^2\,dy =$$

$$= -\frac{v^2}{n} - R\theta + \beta(r^{n-1}v)_x + \frac{(n-1)}{n}\left[r^n \int_x^L r^{-n}v^2\,dy\right]_x .$$

Keeping in mind that v vanishes on the boundary and $r(0, t) = a$, we integrate (2.14) over $[0, L] \times [0, t]$ to infer

$$(2.15)\qquad \int_0^L (u\phi)(x, t)\,dx = \int_0^L u_0(x)\,\phi_0(x)\,dx - \int_0^t\int_0^L \left(\frac{v^2}{n} + R\theta\right) dx\,ds - \frac{(n-1)}{n}a^n \int_0^t\int_0^L r^{-n}v^2\,dx\,ds ,$$

where $\phi_0(x) := \phi(x, 0)$. It follows from integration of (1.9) over $[0, L] \times [0, t]$ and use of (1.13) that

$$(2.16)\qquad \int_0^L u(x, t)\,dx = \int_0^L u_0(x)\,dx \equiv u^* \qquad \text{for } t \geqslant 0 .$$

Note that $u > 0$. If we apply the mean value theorem to (2.15) and use (2.16), we conclude that for each $t \geqslant 0$ there is an $x_0(t) \in [0, L]$ such that

$$(2.17)\qquad \phi(x_0(t), t) = \frac{1}{u^*}\int_0^L \phi(x, t)\,u(x, t)\,dx .$$

Therefore from (2.11), (2.15), and (2.17) we get

$$(2.18)\qquad \int_0^t \sigma(x_0(t), s)\,ds = \phi(x_0(t), t) - \int_0^{x_0(t)} r_0^{-(n-1)}v_0\,dy - (n-1)\int_0^t\int_{x_0(t)}^L r^{-n}v^2\,dy\,ds =$$

$$= -\frac{1}{u^*}\int_0^t\int_0^L\left(\frac{v^2}{n} + R\theta\right) dx\,ds - \frac{(n-1)\,a^n}{nu^*}\int_0^t\int_0^L r^{-n}v^2\,dx\,ds -$$

$$-(n-1)\int_0^t\int_{x_0(t)}^L r^{-n}v^2\,dx\,ds + \frac{1}{u^*}\int_0^L u_0\phi_0\,dx - \int_0^{x_0(t)} r_0^{-(n-1)}v_0\,dy$$

for any $t \geqslant 0$. Using (2.18), we can show

LEMMA 2.3. – *We have the following representation*

$$u(x,t)=\frac{D(x,t)}{B(x,t)}\left\{1+\frac{R}{\beta}\int_0^t\frac{\theta(x,s)\,B(x,s)}{D(x,s)}\,ds\right\},\qquad x\in[0,L],\ t\geq 0\,, \tag{2.19}$$

where

$$D(x,t):=u_0(x)\exp\left\{\frac{1}{\beta}\left[\frac{1}{u^*}\int_0^L u_0\phi_0\,dx-\int_0^x r_0^{-(n-1)}v_0\,dy+\int_{x_0(t)}^x r^{-(n-1)}v\,dy\right]\right\}, \tag{2.20}$$

$$B(x,t):=\exp\left\{\frac{1}{\beta}\left[\frac{1}{u^*}\int_0^t\int_0^L\left(\frac{v^2}{n}+R\theta\right)dx\,ds+\right.\right.$$
$$\left.\left.+\frac{(n-1)\,a^n}{nu^*}\int_0^t\int_0^L r^{-n}v^2\,dy\,dx\,ds+(n-1)\int_0^t\int_x^L r^{-n}v^2\,dy\,ds\right]\right\}, \tag{2.21}$$

and $x_0(t)\in[0,L]$ *is the same as in* (2.17).

PROOF. – Using (1.9) we may write (1.10) in the form

$$r^{-(n-1)}v_t=\beta[\log u]_{xt}-R\left[\frac{\theta}{u}\right]_x\qquad(\Leftrightarrow r^{-(n-1)}v_t=\sigma_x)\,. \tag{2.22}$$

Integrate (2.22) over $[0,t]$, then integrate over $[x_0(t),x]$ with respect to x. If we integrate by parts with respect to t, utilise (2.1) and (2.18), we infer

$$\beta\log u-R\int_0^t\frac{\theta}{u}\,ds=\beta\log u_0+\int_0^t\sigma(x_0(t),s)\,ds+\int_{x_0(t)}^x\int_0^t r^{-(n-1)}v_t\,ds\,dy=$$
$$=\beta\log u_0-\frac{1}{u^*}\int_0^t\int_0^L\left(\frac{v^2}{n}+R\theta\right)dx\,ds-\frac{(n-1)\,a^n}{nu^*}\int_0^t\int_0^L r^{-n}v^2\,dx\,ds-$$
$$-(n-1)\int_0^t\int_x^L r^{-n}v^2\,dy\,ds+\int_{x_0(t)}^x r^{-(n-1)}v\,dy+\frac{1}{u^*}\int_0^L u_0\phi_0\,dx-\int_0^x r_0^{-(n-1)}v_0\,dy\,,$$

which, when the exponentials are taken, turns into

$$\frac{B(x,t)}{D(x,t)}=\frac{1}{u(x,t)}\exp\left(\frac{R}{\beta}\int_0^t\frac{\theta(x,s)}{u(x,s)}\,ds\right). \tag{2.23}$$

Multiplying (2.23) by $R\theta/\beta$ and integrating over $[0, t]$, we arrive at

$$\exp\left(\frac{R}{\beta}\int_0^t \frac{\theta(x, s)}{u(x, s)}\,ds\right) = 1 + \frac{R}{\beta}\int_0^t \frac{\theta(x, s)\,B(x, s)}{D(x, s)}\,ds\,.$$

Substituting this into (2.23), we obtain the lemma. ■

Now we are able to derive bounds on $u(x, t)$ by using the representation (2.19).

LEMMA 2.4. – *There are positive constants $\underline{u}$ and $\overline{u}$, independent of t, such that*

$$\underline{u} \leqslant u(x, t) \leqslant \overline{u} \quad \textit{for any } x \in [0, L],\ t \geqslant 0\,. \tag{2.24}$$

PROOF. – Recalling the definition of $D(x, t)$, we have by (2.2), Cauchy-Schwarz's inequality, and Lemma 2.1 that

$$0 < C^{-1} \leqslant D(x, t) \leqslant C\,, \qquad \forall x \in [0, 1],\ t \geqslant 0\,. \tag{2.25}$$

Noting that $u > 0$, we get from (2.2) and (2.7) that

$$\frac{B(x, s)}{B(x, t)} \leqslant \exp\left\{-\frac{R}{\beta u^*}\int_s^t\int_0^L \theta(x, s)\,dx\,ds\right\} \leqslant \exp\left\{-\frac{Ra_1(t-s)}{\beta u^*}\right\}, \quad t \geqslant s \geqslant 0\,. \tag{2.26}$$

Similarly,

$$B(x, s)/B(x, t) \geqslant Ce^{-C_1(t-s)}, \quad t \geqslant s \geqslant 0\,; \qquad e^{Ct} \geqslant B(x, t) \geqslant 1\,, \quad t \geqslant 0 \tag{2.27}$$

with C_1 being independent of t, where we have used (2.2)-(2.3) and (2.7).
It is easy to see by (2.2) and (2.7) that

$$|\theta^{1/2}(x, t) - \theta^{1/2}(a(t), t)| \leqslant \int_0^L \theta^{-1/2}(x, t)\,|\theta_x(x, t)|\,dx \leqslant$$

$$\leqslant \left(\int_0^L \frac{\theta_x^2}{u\theta^2}\,dx\right)^{1/2}\left(\int_0^L \theta u\,dx\right)^{1/2} \leqslant C\left[\int_0^L \left(\frac{\theta_x^2}{u\theta^2}\right)(x, t)\,dx\right]^{1/2} \max_{[0, L]} u^{1/2}(\cdot, t)\,,$$

which together with (2.8) gives

$$\frac{\alpha_1}{2} - C \max_{[0,L]} u(\cdot, t) \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx \leq \theta(x, t) \leq \tag{2.28}$$

$$\leq 2\alpha_2 + C \max_{[0,L]} u(\cdot, t) \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx \qquad \forall x \in [0, L],\ t \geq 0 .$$

Hence it follows from (2.19) and (2.25)-(2.28) that

$$u(x, t) \leq C + C \int_0^t \left(1 + \max_{[0,L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx \right) e^{-(t-s)/C}\,ds \leq \tag{2.29}$$

$$\leq C + C \int_0^t \max_{[0,L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx\,ds .$$

Applying Gronwall's inequality to (2.29) and utilising (2.3), one gets $u(x, t) \leq \overline{u}$ $\forall x \in [0, L]$ $\forall t \geq 0$ for some positive constant $\overline{u}$ independent of t and x.

To complete the proof it remains to show the lower boundedness of u. To this end we make use of (2.3), (2.19), (2.25), (2.27), and (2.28) to infer

$$u(x, t) \geq \frac{RD(x, t)}{\beta} \int_0^t \frac{\theta(x, s)\,B(x, s)}{D(x, s)\,B(x, t)}\,ds \geq \tag{2.30}$$

$$\geq C_2 \int_0^t \left(\frac{\alpha_1}{2} - C \max_{[0,L]} u(\cdot, s) \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx \right) e^{-C_1(t-s)}\,ds \geq$$

$$\geq \frac{C_2\alpha_1}{2C_1}(1 - e^{-C_1 t}) - Ce^{-C_1 t/2} \int_0^{t/2}\int_0^L \frac{\theta_x^2}{u\theta^2}\,dx\,ds - C \int_{t/2}^t \int_0^L \frac{\theta_x^2}{u\theta^2}\,dx\,ds \geq \frac{C_2\alpha_1}{4C_1} > 0$$

for all $t \geq T_0$ and some (large) $T_0 > 0$, where C_2 is independent of t. Furthermore, from (2.19), (2.25), and (2.27) we get $u(x, t) \geq D(x, t)/B(x, t) \geq C^{-1}e^{-Ct}$ for all $x \in [0, L]$ and $t \geq 0$. This combined with (2.30) shows that u is bounded from below. The proof is complete. ■

In the sequel we derive Sobolev-norm estimates of derivatives for u, v, θ by applying the energy method.

Recalling (2.10), using (1.9)-(1.10), we may write the equation (1.11) as follows

$$\left[c_V\theta + \frac{v^2}{2} \right]_t = \left[\kappa \frac{r^{2n-2}\theta_x}{u} + \sigma r^{n-1} v - 2(n-1)\,\mu r^{n-2} v^2 \right]_x . \tag{2.31}$$

Multiply (2.31) by $c_V\theta + v^2/2$ and integrate. If we integrate by parts with respect to x, and make use of (2.1)-(2.2), Cauchy-Schwarz's inequality, and (2.24), we obtain that

$$(2.32)\qquad \frac{1}{2}\int_0^L\left[c_V\theta+\frac{v^2}{2}\right]^2(x,t)\,dx\leqslant$$

$$\leqslant C-\frac{c_V K}{2}\int_0^t\int_0^L\frac{r^{2n-2}\theta_x^2}{u}\,dx\,ds+C\int_0^t\int_0^L(r^{2n-2}v_x^2v^2+v^4+\theta^2v^2)\,dx\,ds\,.$$

To bound the term $\int_0^t\int_0^L r^{2n-2}v_x^2v^2\,dx\,ds$, we multiply (1.10) by v^3, integrate over $[0,L]\times[0,t]$, integrate by parts with respect to x, and utilise (2.1)-(2.2), Cauchy-Schwarz's inequality, and (2.24) to get

$$(2.33)\qquad \frac{1}{4}\int_0^L v^4(x,t)\,dx\leqslant C-\frac{\beta}{\bar u}\int_0^t\int_0^L r^{2n-2}v^2v_x^2\,dx\,ds+C\int_0^t\int_0^L(v^4+v^2\theta^2)\,dx\,ds\,.$$

We multiply (2.32) by $\beta/(2\,\bar u C)$ and add the resulting inequality to (2.33) to obtain, with the help of (2.2)-(2.3) and (2.24), the result

$$(2.34)\qquad \int_0^L(\theta^2+v^4)(x,t)\,dx+\int_0^t\int_0^L(v^2v_x^2+\theta_x^2)\,dx\,ds\leqslant$$

$$\leqslant C+C\int_0^t\max_{[0,L]}v^2(\cdot,s)\,ds+C\int_0^t\max_{[0,L]}v^2(\cdot,s)\int_0^L\theta^2(x,s)\,dx\,ds\,.$$

On the other hand, by (2.2)-(2.3), (2.7) and (2.24),

$$(2.35)\qquad \int_0^t\max_{[0,L]}v^2(\cdot,s)\,ds\leqslant\int_0^t\left(\int_0^L|v_x|\,dx\right)^2ds\leqslant\int_0^t\int_0^L\frac{v_x^2}{u\theta}\,dx\int_0^L u\theta\,dx\,ds\leqslant C\,,\qquad t\geqslant 0\,.$$

In view of (2.35), we apply Gronwall's inequality to (2.34) to obtain

LEMMA 2.5.

$$(2.36)\qquad \int_0^L(\theta^2+v^4)(x,t)\,dx+\int_0^t\int_0^L(v^2v_x^2+\theta_x^2)\,dx\,ds\leqslant C\,,\qquad t\geqslant 0\,.$$

LEMMA 2.6.

$$\int_0^L u_x^2(x,t)\,dx + \int_0^t\int_0^L (v_x^2 + u_x^2 + \theta u_x^2)\,dx\,ds \le C, \qquad t \ge 0. \tag{2.37}$$

PROOF. – By virtue of (2.1)-(2.2) and (2.24),

$$(r^{n-1}v)_x^2 = (r^{n-1}v_x + (n-1)r^{-1}uv)^2 \ge r^{2n-2}v_x^2/2 - Cv^2 \ge a^{2n-2}v_x^2/2 - Cv^2. \tag{2.38}$$

So multiplying (1.10) by v and integrating, we integrate by parts with respect to x, use Cauchy-Schwarz's inequality, (2.7), (2.24), and (2.35)-(2.36), to deduce

$$\frac{1}{2}\int_0^L v^2(x,t)\,dx + \frac{\beta a^{2n-2}}{2\bar{u}}\int_0^t\int_0^L v_x^2\,dx\,ds \le \tag{2.39}$$

$$\le C + C\delta^{-1}\int_0^t\int_0^L (\theta_x^2 + v^2 + \theta v^2)\,dx\,ds + \delta\int_0^t\int_0^L \theta u_x^2\,dx\,ds \le$$

$$\le C\delta^{-1} + \delta\int_0^t\int_0^L \theta u_x^2\,dx\,ds, \qquad (0<\delta<1 \text{ constant}).$$

With the help of (1.9), we may write (1.10) in the form $\beta[u_x/u]_t = r^{-(n-1)}v_t + R[\theta_x/u - \theta u_x/u^2]$. Multiply this by u_x/u and integrate. After utilising (2.3), (2.24), and (2.36), we infer

$$\frac{\beta}{2}\int_0^L\left[\frac{u_x}{u}\right]^2(x,t)\,dx + \frac{R}{2}\int_0^t\int_0^L \frac{\theta u_x^2}{u^3}\,dx\,ds \le \tag{2.40}$$

$$\le C + \int_0^t\int_0^L r^{-(n-1)}v_t\frac{u_x}{u}\,dx\,ds + C\int_0^t\int_0^L \frac{\theta_x^2}{u}\left(1+\frac{1}{\theta^2}\right)dx\,ds.$$

Noting that $[u_x/u]_t = [u_t/u]_x$, the second term on the right hand side of (2.40) can be estimated, with the help of integration by parts, and (1.9), (2.1)-(2.3), (2.24) and (2.35), as follows:

$$\int_0^t\int_0^L r^{-(n-1)}v_t\frac{u_x}{u}\,dx\,ds = \int_0^L r^{-(n-1)}v\frac{u_x}{u}\,dx\,\Big|_0^t + \tag{2.41}$$

$$+(n-1)\int_0^t\int_0^L r^{-n}v^2\frac{u_x}{u}\,dx\,ds - \int_0^t\int_0^L r^{-(n-1)}v\left[\frac{u_t}{u}\right]_x dx\,ds \le$$

$$\le C + \frac{\beta}{4}\int_0^L\left[\frac{u_x}{u}\right]^2(x,t)\,dx + C\int_0^t \max_{[0,L]} v^2\int_0^L u_x^2\,dx\,ds + \frac{2}{\underline{u}}\int_0^t\int_0^L v_x^2\,dx\,ds.$$

Substituting (2.41) into (2.40), taking (2.2)-(2.3), (2.24), and (2.36) into account, one gets

$$\frac{\beta}{4\overline{u}^2}\int_0^L u_x^2(x,t)\,dx+\frac{R}{2\overline{u}^3}\int_0^t\int_0^L \theta u_x^2\,dx\,ds\leqslant \tag{2.42}$$

$$\leqslant C+C\int_0^t \max_{[0,L]} v^2\int_0^L u_x^2\,dx\,ds+\frac{2}{\underline{u}}\int_0^t\int_0^L v_x^2\,dx\,ds\,.$$

Multiplying (2.42) by $\underline{u}\beta a^{2n-2}/(8\,\overline{u})$, and adding the resulting inequality to (2.39), we obtain for an appropriately small but fixed $\delta\in(0,1)$ that

$$\int_0^L u_x^2(x,t)\,dx+\int_0^t\int_0^L (v_x^2+\theta u_x^2)\,dx\,ds\leqslant C+C\int_0^t \max_{[0,L]} v^2\int_0^L u_x^2\,dx\,ds\,,\qquad t\geqslant 0\,.$$

In view of (2.35), we apply Gronwall's inequality to the above inequality to obtain

$$\int_0^L u_x^2(x,t)\,dx+\int_0^t\int_0^L (v_x^2+\theta u_x^2)\,dx\,ds\leqslant C\,,\qquad \forall t\geqslant 0\,. \tag{2.43}$$

Finally, it follows from (2.24), (2.28), (2.43), and (2.3) that

$$\frac{\alpha_1}{2}\int_0^t\int_0^L u_x^2\,dx\,ds\leqslant \int_0^t\int_0^L \theta u_x^2\,dx\,ds+C\int_0^t\int_0^L \frac{\theta_x^2}{u\theta^2}\,dx\int_0^L u_x^2\,dx\,ds\leqslant C\,,\qquad t\geqslant 0\,,$$

from which and (2.43), (2.37) follows. This completes the proof. ■

In the following lemma we bound v_t in the $L^2((0,L)\times(0,\infty))$-norm.

LEMMA 2.7.

$$\int_0^L v_x^2(x,t)\,dx+\int_0^t\int_0^L v_t^2\,dx\,ds\leqslant C\,,\qquad \forall t\geqslant 0\,, \tag{2.44}$$

$$|v(x,t)|\leqslant C\,,\qquad \forall x\in[0,L],\ t\geqslant 0\,. \tag{2.45}$$

PROOF. – We first note that by (2.8) and Cauchy-Schwarz's inequality,

$$\max_{[0,L]}\theta(\cdot,t)\leqslant C+C\max_{[0,L]}|\theta(\cdot,t)-\theta(a(t),t)|\leqslant \tag{2.46}$$

$$\leqslant C+C\int_0^L |\theta_x|\,dx\leqslant C+C\|\theta_x(t)\|\,,\qquad t\geqslant 0\,.$$

Multiply (1.10) by v_t and integrate over $[0,L]\times[0,t]$. Integrating by parts, using (2.1)-(2.3), (2.24), (2.35)-(2.37), and (2.46), taking into account that $(r^{n-1}v_t)_x=$

$= (r^{n-1}v)_{xt} - (n-1)(r^{n-2}v^2)_x$ and $|(r^{n-2}v^2)_x| \leq C\{v^2 + (r^{n-1}v)_x^2\}$, we obtain

$$\frac{1}{2}\int_0^t \|v_t\|^2\,ds + \frac{\beta}{2\overline{u}}\|(r^{n-1}v)_x(t)\|^2 \leq \tag{2.47}$$

$$\leq C + C\int_0^t\int_0^L \left\{\frac{|(r^{n-1}v)_x|}{u}(v^2 + (r^{n-1}v)_x^2) + \theta_x^2 + \theta^2 u_x^2\right\} dx\,ds \leq$$

$$\leq C + C\int_0^t \left\{\max_{[0,L]}\left|\frac{(r^{n-1}v)_x}{u}\right| + \max_{[0,L]} v^2\right\}\|(r^{n-1}v)_x\|^2\,ds\,.$$

Here $\max|(r^{n-1}v)_x/u|$ can be bounded as follows, using (2.10), Sobolev's imbedding theorem ($H^1 \hookrightarrow L^\infty$), and (2.24), (1.10), (2.1)-(2.2), and (2.46)

$$\beta\left|\frac{(r^{n-1}v)_x}{u}\right|(x,t) \leq |\sigma| + R\frac{\theta}{u} \leq C\left(1 + \|\sigma\| + \|\sigma_x\| + \max_{[0,L]}\theta\right) \leq \tag{2.48}$$

$$\leq C\left(1 + \max_{[0,L]} v^2 + \|v_x\|^2 + \|\theta_x\|^2 + \|v_t\|\right), \qquad \forall x \in [0,L],\ t \geq 0\,.$$

Inserting (2.48) into (2.47) and recalling that $|(r^{n-1}v)_x^2| \leq C(v^2 + v_x^2)$, we get from (2.24), (2.35), and (2.37) that

$$\frac{1}{4}\int_0^t \|v_t\|^2\,ds + \frac{\beta}{2\,\overline{u}}\|(r^{n-1}v)_x(t)\|^2 \leq \tag{2.49}$$

$$\leq C + C\int_0^t \left(\max_{[0,L]} v^2 + \|v_x\|^2 + \|\theta_x\|^2\right)\|(r^{n-1}v)_x\|^2\,ds\,.$$

Applying Gronwall's inequality to (2.49) and taking account of (2.35)-(2.37), we conclude that $\int_0^t \|v_t\|^2\,ds + \|(r^{n-1}v)_x(t)\|^2 \leq C$ for $t \geq 0$, which combined with (2.3) and (2.37)-(2.38) yields (2.44). Finally, (2.45) follows from Sobolev's inequality, (2.3), and (2.44). The proof is complete. ■

As a result of Lemma 2.7 we have

$$\int_0^t\int_0^L (u_{xt}^2 + v_{xx}^2)(x,s)\,dx\,ds + \int_0^t \max_{[0,L]} v_x^2(\cdot,s)\,ds \leq C\,, \qquad t \geq 0\,. \tag{2.50}$$

In fact, by virtue of Sobolev's imbedding theorem ($W^{1,1} \hookrightarrow L^\infty$), $\max v_x^2(\cdot,t) \leq$ $\leq C\varepsilon^{-1}\|v_x(t)\|^2 + \varepsilon\|v_{xx}(t)\|^2$ $(0 < \varepsilon < 1)$, we get from (2.1)-(2.2), (2.24), and (2.35)-(2.37), (1.10), (2.46), and (2.44) that

$$\int_0^t\int_0^L (u_{xt}^2+v_{xx}^2)\,dx\,ds \leqslant C\int_0^t\int_0^L [(r^{n-1}v)_{xx}^2+v_x^2+u_x^2v^2+v^2]\,dx\,ds \leqslant$$

$$\leqslant C + C\int_0^t\int_0^L \left[\frac{(r^{n-1}v)_x}{u}\right]_x^2 dx\,ds + C\int_0^t\int_0^L u_x^2 v_x^2\,dx\,ds \leqslant$$

$$\leqslant C + C\int_0^t\int_0^L v_t^2\,dx\,ds + C\int_0^t \max_{[0,L]}\theta^2\int_0^L u_x^2\,dx\,ds + C\int_0^t \max_{[0,L]} v_x^2\,ds \leqslant C + \frac{1}{2}\int_0^t \|v_{xx}\|^2\,ds\,,$$

which implies (2.50).

We multiply (1.11) by θ_t and integrate, we obtain by the same arguments as used in Lemma 2.7 and (2.50) that

$$\int_0^L \theta_x^2(x,t)\,dx + \max_{[0,L]}|\theta(\cdot,t)| + \int_0^t\int_0^L\left(\theta_t^2+\theta_{xx}^2+\max_{[0,L]}\theta_x^2\right)(s)\,ds \leqslant C\,, \qquad \forall t\geqslant 0\,. \tag{2.51}$$

Now we are able to prove Theorem 2.1. By (2.37), (2.44), (2.50)-(2.51), and the identities

$$\int_0^L v_x v_{xt}\,dx = -\int_0^L v_{xx}v_t\,dx\,, \qquad \int_0^L \theta_x\theta_{xt}\,dx = -\int_0^L \theta_{xx}\theta_t\,dx\,,$$

we see that

$$\int_0^\infty\left\{\left|\frac{d}{dt}\|u_x(t)\|^2\right| + \left|\frac{d}{dt}\|v_x(t)\|^2\right| + \left|\frac{d}{dt}\|\theta_x(t)\|^2\right|\right\}dt \leqslant C\,,$$

which together with (2.36)-(2.37) implies

$$\|u_x(t)\|^2+\|v_x(t)\|^2+\|\theta_x(t)\|^2 \to 0\,, \qquad \text{as } t\to\infty\,. \tag{2.52}$$

From (2.16) and Poincaré's inequality we get $\|u(t)-u^*\|_{H^1}+\|v(t)\|_{H^1}\to 0$ as $t\to\infty$. Recalling (1.13) and the definition of θ^*, we integrate (2.31) over $[0,L]\times[0,t]$ to infer

$$\int_0^L \{(c_V\theta+v^2/2)(x,t)-c_V\theta^*\}\,dx = 0\,, \qquad t\geqslant 0\,,$$

from which it follows with the help of Poincaré's inequality and (2.45) that as $t\to\infty$

$$\|\theta(t)-\theta^*\| \leqslant C\|c_V\theta(t)+v^2(t)/2-c_V\theta^*\| + C\|v^2(t)\| \leqslant C(\|\theta_x\|+\|v(t)\|_{H^1})\to 0\,.$$

To show $r(x, t) \to (a^n + nu^* x)^{1/n}$ as $t \to \infty$ we note that by (2.1) and (1.14),

$$r^n(x, t) = r^n(0, t) + n\int_0^x u(y, t)dy = [r^*(x)]^n + n\int_0^x (u(y, t) - u^*)\,dy\,, \tag{2.53}$$

where $r^*(x)$ is defined by (1.17). It follows from (2.2) and (2.53) that $\|r(t) - r^*\| \leq C\|u(t) - u^*\|$, $t \geq 0$. Therefore, differentiating (2.53) with respect to x and recalling (2.2), we find that $\|r(t) - r^*\|_{H^2} \leq C\|u(t) - u^*\|_{H^1} \to 0$ as $t \to \infty$. We have known that for large $t\{u(x, t) - u^*, v(x, t), \theta(x, t) - \theta^*\}$ and $r(x, t) - r^*(x)$ become small in the H^1- and H^2-norms respectively, thus we can apply arguments similar to those used in [26, Theorem 2.2] to obtain (1.19) in Theroem 1.1 (the exponential decay). This completes the proof of Theorem 1.1 (i).

3. – Proof of Theorem 1.1-(ii).

We use and modify an idea of Hoff [7] for barotropic fluids to prove Theorem 1.1 (ii) for the system (1.9)-(1.12) in the case of $n = 1$. Let $e_0 \leq 1$ be satisfied in this section. In what follows C or $\tilde{C}$ denotes a generic constant (≥ 1) which may depend at most on $\overline{u}$, $\overline{\theta}$, β, R, c_V, κ, and γ.

Define $\phi(t) := \min\{1, t\}$. We first assume that u, θ satisfy

$$|u(x, t) - \overline{u}|\,, \qquad \phi(t)\,|\theta(x, t) - \overline{\theta}| \leq \min\{\overline{u}, \overline{\theta}\}/2 \qquad \text{for all } x \in \mathbb{R},\ t \geq 0\,. \tag{3.1}$$

In the sequel we derive a priori estimates for u, v, θ under (3.1).

Following the same procedure as in the proof of Lemma 2.1 (recalling $n = 1$), applying (3.1) and the mean value theorem, we can show

$$\int_{\mathbb{R}} U(x, t)\,dx + \overline{\theta}\int_1^t\int_{\mathbb{R}}\left(\beta\,\frac{v_x^2}{u\theta} + \kappa\,\frac{\theta_x^2}{u\theta^2}\right)dx\,ds = \int_{\mathbb{R}} U(x, 1)\,dx \leq \tag{3.2}$$

$$\leq C\int_{\mathbb{R}}\{v^2 + (u - \overline{u})^2 + (\theta - \overline{\theta})^2\}(x, 1)\,dx\,, \qquad \forall t \geq 1\,,$$

where

$$U(x, t) := \left\{\frac{v^2}{2} + R\overline{\theta}\left(\frac{u}{\overline{u}} - \log\frac{u}{\overline{u}} - 1\right) + c_V\left(\theta - \overline{\theta}\log\frac{\theta}{\overline{\theta}} - \overline{\theta}\right)\right\}(x, t)\,. \tag{3.3}$$

Now we estimate $\{u - \overline{u}, v, \theta - \overline{\theta}\}$ in a weighted L^2-norm for $0 \leq t \leq 1$. For simplicity we denote $\psi(x) := (1 + x^2)^\gamma$ with γ being the same as in Theorem 1.1. Multiply (1.10) by $2\psi v$ (recalling $n = 1$) and integrate over $\mathbb{R} \times (0, t)$ ($t \in [0, 1]$). We integrate by parts to arrive at

$$(3.4)\qquad \int_{\mathbb{R}} \psi v^2(x,t)\,dx + \int_0^t \int_{\mathbb{R}} \psi v_x^2\,dx\,ds \leqslant$$

$$\leqslant Ce_0^2 + C\int_0^t \int_{\mathbb{R}} \psi((u-\overline{u})^2 + v^2 + (\theta-\overline{\theta})^2)\,dx\,ds\,, \qquad t\in[0,1].$$

Multiplying (1.9) by $2\psi(u-\overline{u})$ and integrating, we easily see that

$$\int_{\mathbb{R}} \psi(u(x,t)-\overline{u})^2\,dx \leqslant e_0^2 + C\int_0^t\int_{\mathbb{R}} \psi(u-\overline{u})^2\,dx\,ds + \frac{1}{2}\int_0^t\int_{\mathbb{R}} \psi v_x^2\,dx\,ds\,,$$

which together with (3.4) gives

$$(3.5)\qquad \int_{\mathbb{R}} \psi((u-\overline{u})^2 + v^2)(x,t)\,dx + \int_0^t\int_{\mathbb{R}} \psi v_x^2\,dx\,ds \leqslant$$

$$\leqslant Ce_0^2 + C\int_0^t\int_{\mathbb{R}} \psi((u-\overline{u})^2 + v^2 + (\theta-\overline{\theta})^2)\,dx\,ds$$

for all $t\in[0,1]$.

Let us denote $h(t) := \sup\limits_{0\leqslant s\leqslant t} \int_{\mathbb{R}} \psi\{v^2 + (\theta-\overline{\theta})^2\}(x,s)\,dx$. Utilising (3.1), we obtain by the same arguments as used for (2.31)-(2.23) that

$$(3.6)\qquad \int_{\mathbb{R}} \psi((\theta-\overline{\theta})^2 + v^4)(x,t)\,dx + \int_0^t\int_{\mathbb{R}} \psi(v^2v_x^2 + \theta_x^2)\,dx\,ds \leqslant$$

$$\leqslant Ce_0^2 + Ch(t)\int_0^t \max_{\mathbb{R}}(\theta-\overline{\theta})^2(\cdot,s)\,ds + \int_0^t\int_{\mathbb{R}} \psi((\theta-\overline{\theta})^2 + v^4 + v^2)\,dx\,ds \leqslant$$

$$\leqslant C(e_0^2 + h^3(t)) + C\int_0^t\int_{\mathbb{R}} \psi((\theta-\overline{\theta})^2 + v^4 + v^2)\,dx\,ds + \frac{1}{2}\int_0^t\int_{\mathbb{R}} \psi\theta_x^2\,dx\,ds\,, \qquad t\in[0,1],$$

where we have also used the inequality $\|\cdot\|_{L^\infty} \leqslant C\|\cdot\|\|\partial_x\cdot\|$ for $\max(\theta-\overline{\theta})^2(\cdot,s)$.

Applying the generalized Gronwall inequality to (3.5) and (3.6), we find that for all $t\in[0,1]$

$$(3.7)\qquad \int_{\mathbb{R}} \psi((u-\overline{u})^2 + v^2 + v^4 + (\theta-\overline{\theta})^2)(x,t)\,dx +$$

$$+ \int_0^t\int_{\mathbb{R}} \psi(v_x^2 + v^2v_x^2 + \theta_x^2)\,dx\,ds \leqslant C(e_0^2 + h^3(t))\,.$$

By the definition of $h(t)$ and (3.7) we have $h(t) \leqslant C(e_0^2 + h^3(t))$ for all $t \in [0, 1]$, which gives $h(t) \leqslant 2Ce_0^2 \leqslant e_0$ for all $t \in [0, 1]$ provided $e_0 \leqslant 1/(2C)$. Therefore, in view of (3.7) we conclude

$$(3.8) \qquad \int_{\mathbb{R}} \psi((u-\overline{u})^2 + v^2 + v^4 + (\theta - \overline{\theta})^2)(x, t)\, dx + \\ + \int_0^t \int_{\mathbb{R}} \psi(v_x^2 + v^2 v_x^2 + \theta_x^2)\, dx\, ds \leqslant Ce_0^2, \qquad t \in [0, 1]$$

provided $e_0 \leqslant 1/(2C)$. Using (3.1) and the mean value theorem, we get from (3.8) and (3.2) that

$$(3.9) \qquad \int_{\mathbb{R}} \{v^2 + (u-\overline{u})^2 + (\theta-\overline{\theta})^2\}(x, t)\, dx + \int_0^t \int_{\mathbb{R}} (v_x^2 + \theta_x^2)\, dx\, ds \leqslant Ce_0^2, \qquad \forall t \geqslant 0$$

provided $e_0 \leqslant 1/(2C)$.

Next we derive Sobolev-norm estimates for u, v, θ. We define

$$(3.10) \qquad A(t) := \sup_{0 \leqslant s \leqslant t} \{\|u-\overline{u}\|_{L^\infty}^2 + \phi^2 \|v_x\|^2 + \phi^4 \|\theta_x\|^2\}(s) + \\ + \int_0^t \{\phi^2 \|v_t\|^2 + \phi^4 \|\theta_t\|^2 + \|v_x\|^2\}(s)\, ds .$$

Multiply (1.10) by $\phi^2 v_t$ and integrate. We integrate by parts, utilise (3.1), (3.9), and Cauchy-Schwarz's inequality to infer

$$\phi^2 \int_{\mathbb{R}} v_x^2\, dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2\, dx\, ds \leqslant$$

$$\leqslant Ce_0^2 + C\int_0^t \int_{\mathbb{R}} \phi^2 |v_x|^3\, dx\, ds + C\left| \int_0^t \int_{\mathbb{R}} \phi^2 \left[\left(\frac{1}{u} - \frac{1}{\overline{u}}\right)\theta\right]_x v_t\, dx\, ds \right| \leqslant$$

$$\leqslant Ce_0^2 + C\int_0^t \int_{\mathbb{R}} \phi^4 v_x^4\, dx\, ds + C\int_{\mathbb{R}} |u-\overline{u}|\, \phi\, |v_x|\, dx +$$

$$+ C\int_0^1 \int_{\mathbb{R}} \phi |u-\overline{u}|\, \theta\, |v_x|\, dx\, ds + C\left| \int_0^t \int_{\mathbb{R}} \phi^2 \left[\left(\frac{1}{u} - \frac{1}{\overline{u}}\right)\theta\right]_t v_x\, dx\, ds \right| \leqslant$$

$$\leqslant Ce_0^2 + C\int_0^t \int_{\mathbb{R}} \phi^4 v_x^4\, dx\, ds + C\int_0^t \int_{\mathbb{R}} |u-\overline{u}|^2 \phi^4 \theta_t^2\, dx\, ds + \frac{1}{2}\phi^2 \int_{\mathbb{R}} v_x^2\, dx,$$

whence

$$\phi^2 \int_{\mathbb{R}} v_x^2\, dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2\, dx\, ds \leqslant C e_0^2 + C \int_0^t \int_{\mathbb{R}} \phi^4 v_x^4\, dx\, ds + A^2(t), \qquad t \geqslant 0, \tag{3.11}$$

where the second term on the right-hand side of (3.11) can be bounded as follows, using (3.1), (3.9), and $\|\cdot\|_{L^\infty}^2 \leqslant C\|\cdot\|_{H^1}^2$, (1.10) and (3.2)

$$\int_0^t \int_{\mathbb{R}} \phi^4 v_x^4\, dx\, ds \leqslant C \int_0^t \phi^4 \max_{\mathbb{R}} v_x^2 \int_{\mathbb{R}} v_x^2\, dx\, ds \leqslant \tag{3.12}$$

$$\leqslant C e_0^2 + C \int_0^t \phi^4 \max_{\mathbb{R}} \left(\beta \frac{v_x}{u} - R \frac{\theta}{u} + R \frac{\overline{\theta}}{\overline{u}} \right)^2 \int_{\mathbb{R}} v_x^2\, dx\, ds \leqslant$$

$$\leqslant C e_0^2 + C \int_0^t \phi^4 (\|v_x\|^2 + \|u - \overline{u}\|^2 + \|\theta - \overline{\theta}\|^2 + \|v_t\|^2)\, \|v_x\|^2\, ds \leqslant C(e_0^2 + A^2(t)).$$

Inserting (3.12) into (3.11), one obtains

$$\phi^2 \int_{\mathbb{R}} v_x^2\, dx + \int_0^t \int_{\mathbb{R}} \phi^2 v_t^2\, dx\, ds \leqslant C(e_0^2 + A^2(t)), \qquad \forall t \geqslant 0. \tag{3.13}$$

Multiplying (1.11) by $\phi^4 \theta_t$ and integrating, following the same arguments as used for (3.11)-(3.13), we deduce that

$$\phi^4(t) \|\theta_x(t)\|^2 + \int_0^t \|\theta_t\|^2 \phi^4\, ds \leqslant C e_0^2 + C \int_0^t \int_{\mathbb{R}} (\phi^4 v_x^4 + \phi^4 |v_x| \theta_x^2)\, dx\, ds \leqslant \tag{3.14}$$

$$\leqslant C(e_0^2 + A^2(t)) + C \int_0^t \phi^8 \max_{\mathbb{R}} v_x^2 \int_{\mathbb{R}} \theta_x^2\, dx\, ds \leqslant C(e_0^2 + A^2(t)), \qquad \forall t \geqslant 0.$$

We are now able to derive pointwise bounds for $u - \overline{u}$. We may write (1.10) in the form (recalling $n = 1$): $v_t = \beta[\log(u/\overline{u})]_{tx} - R[\theta/u - \overline{\theta}/\overline{u}]_x$. Integrating this over $(-\infty, x) \times (0, t)$ $(t \in [0, 1])$ and then taking the absolute value, making use of (3.1) and (3.8)-(3.9), we see that

$$|u - \overline{u}| \leqslant C \left| \log \frac{u}{\overline{u}} \right| \leqslant \tag{3.15}$$

$$\leqslant C|u_0 - \overline{u}| + C \int_{-\infty}^{x} (|v| + |v_0|)\, dy + C \int_0^t (|u - \overline{u}| + |q - \overline{\theta}|)\, ds \leqslant$$

$$\leqslant Ce_0 + \|\psi^{1/2} v\| \|\psi^{-1/2}\| + C\int_0^t |u - \overline{u}|\, ds + C\int_0^t \|\theta - \overline{\theta}\|_{H^1}\, ds \leqslant$$

$$\leqslant Ce_0 + C\int_0^t |u - \overline{u}|\, ds + C\left(\int_0^t \|\theta - \overline{\theta}\|_{H^1}^2\, ds\right)^{1/2} \leqslant Ce_0 + C\int_0^t |u - \overline{u}|\, ds\,, \qquad t \in [0, 1].$$

An application of Gronwall's inequality to (3.15) yields

$$(3.16) \qquad |u(x, t) - \overline{u}| \leqslant Ce_0\,, \qquad \forall x \in \mathbb{R},\ t \in [0, 1].$$

To estimate $u - \overline{u}$ for $t \geqslant 1$ we denote $F := \beta[v_x/u] - R[u/\theta] + R[\overline{u}/\overline{\theta}]$ and find that

$$\beta\left[\frac{1}{u} - \frac{1}{\overline{u}}\right]_t + \frac{R\theta}{u}\left[\frac{1}{u} - \frac{1}{\overline{u}}\right] = -\frac{F}{u} - \frac{R(\theta - \overline{\theta})}{u\overline{u}}\,.$$

Multiplying this by $1/u - 1/\overline{u}$, using (3.1), (3.10), and (3.9), we get

$$\left[\frac{1}{u} - \frac{1}{\overline{u}}\right]_t^2 + C^{-1}\left[\frac{1}{u} - \frac{1}{\overline{u}}\right]^2 \leqslant C(\|F\|_{L^\infty}^2 + \|\theta - \overline{\theta}\|_{L^\infty}^2) \leqslant$$

$$\leqslant C(\|F\|^2 + \|F_x\|^2 + \|\theta - \overline{\theta}\|_{H^1}^2) \leqslant Ce_0^2 + \|(v_x, v_t, \theta_x)\|^2, \qquad t \geqslant 1\,,$$

which together with (3.9) and (3.13) gives

$$(3.17) \qquad |u(x, t) - \overline{u}|^2 \leqslant Ce_0^2 + C|u(x, 1) - \overline{u}|^2 +$$

$$+ C\int_1^t \|(v_x, v_t, \theta_x\|^2\, ds \leqslant C(e_0^2 + A^2(t))\,, \qquad \forall x \in \mathbb{R},\ t \geqslant 1\,.$$

Combining (3.9), (3.13)-(3.14), and (3.16)-(3.17), we obtain $A(t) \leqslant \tilde{c}\{e_0^2 + A^2(t)\}$ for $t \geqslant 0$, where $\tilde{c} \geqslant 1$ depends at most on $\overline{u}$, $\overline{\theta}$, β, R, c_V, κ, and γ. Hence we have

$$(3.18) \qquad A(t) \leqslant 2\tilde{c}e_0^2\,, \qquad \text{for all } t \geqslant 0$$

provided $e_0 < \min\{1/(2\tilde{c}), 1/(2C)\}$.

From (3.18), (3.10), and (3.9) we conclude that

$$(3.19) \qquad |u(x, t) - \overline{u}| + \phi(t)\,|\theta(x, t) - \overline{\theta}| \leqslant A^{1/2}(t) + C\|\theta - \overline{\theta}\|^{1/2}\phi\|\theta_x\|^{1/2} \leqslant$$

$$\leqslant A^{1/2}(t) + Ce_0A^{1/4}(t) \leqslant \sqrt{2\tilde{c}}\, e_0(1 + \tilde{C}\sqrt{e_0}) < \min\{\overline{u}, \overline{\theta}\}/3\,, \qquad \forall x \in \mathbb{R},\ t \geqslant 0$$

provided $e_0 < \min\{\overline{u}/(6\sqrt{\tilde{c}}), \overline{\theta}/(6\sqrt{\tilde{c}}), 1/(3\tilde{C})^2, 1/(2\tilde{c}), 1/(2C)\} =: \varepsilon$.

For the initial data satisfying $e_0 < \varepsilon$ we thus have proved that under (3.1) the estimate (3.19) holds. Since (3.19) is valid for $t = 0$, by virtue of the continuity of u and θ, (3.19) remains valid for all $t \geqslant 0$. Hence (3.9), (3.18) hold for all $t \geqslant 0$. We now multiply (1.10) by u_x/u and integrate over $\mathbb{R} \times (1, t)$. We make use of (1.9), (3.19), (3.9), and

(3.18) to deduce (cf. (2.40))

$$\int_{\mathbb{R}} u_x^2(x,t)\,dx + \int_1^t \|u_x\|^2\,ds \leq C(1+\|u_x(1)\|^2) < \infty\,, \qquad t \geq 1\,. \tag{3.20}$$

Similarly, if we multiply (1.10) resp. (1.11) by v_{xx} resp. by θ_{xx} and integrate over $\mathbb{R} \times \times (1, t)$, utilise (3.20), we have

$$\int_1^t (\|v_{xx}\|^2 + \|\theta_{xx}\|^2)\,ds \leq C(1+\|u_x(1)\|^2) < \infty\,, \qquad t \geq 1\,. \tag{3.21}$$

From (3.18) and (3.20)-(3.21) we get by the same argument as used for (2.52) that $\|u_x(t)\| + \|v_x(t)\| + \|\theta_x(t)\| \to 0$ as $t \to \infty$. From this, (3.9), and $\|\cdot\|_{L^\infty}^2 \leq C\|\cdot\|\,\|\partial_x \cdot\|$, (1.20) follows immediately. The proof is complete.

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