Late-time acceleration in $f(Q)$ gravity: Analysis and constraints in an anisotropic background

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This paper is devoted to investigate the anisotropic locally rotationally symmetric (LRS) Bianchi type-I space-time in the context of the recently proposed $f(Q)$ gravity in which Q is the non-metricity scalar. For this purpose, we consider a linear form of $f(Q)$ gravity model, specifically, $f(Q) = \alpha Q + \beta$, where α and β are free parameters and we analyzed the exact solutions of LRS Bianchi type-I space-time. The modified Friedmann equations are solved by presuming an expansion scalar $\theta(t)$ is proportional to the shear scalar $\sigma(t)$ which leads to the relation between the metric potentials as $A = B^n$ where n is an arbitrary constant. Then we constrain our model parameters with the observational Hubble datasets of 57 data points. Moreover, we discuss the physical behavior of cosmological parameters such as energy density, pressure, EoS parameter, and deceleration parameter. The behavior of the deceleration parameter predicts a transition from deceleration to accelerated phases in an expanding Universe. Finally, the EoS parameter indicates that the anisotropic fluid behaves like the standard ΛCDM model.

I. INTRODUCTION

Observations of high redshift supernovae and cosmic microwave background fluctuations (CMBR) [1–4] indicated that the present acceleration epoch of the Universe is accelerated. This late-time acceleration is due to an unidentified fluid called dark energy (DE). Many suggestions have been considered as a candidate to explain the true nature of DE. The first is the cosmological constant that encounters problems such as the incredibly small value required by general relativity (GR) theory. On other hand, the cosmological constant provided by particle physics predictions is generally more than 50 orders of magnitude than the actual value assumed by GR [5]. This mysterious DE that is responsible for the cosmological acceleration and is estimated as 68% of the total energy density of the Universe may require us to reconsider the theory of gravity on cosmological scales. The DE can be tested using an effective tool namely the equation of state (EoS) parameter of the form $\omega = \frac{p}{\rho}$,

which is the ratio of the cosmic pressure p to the cosmic energy density ρ. Each DE model has a different EoS parameter value, for example, in the case of the cosmological constant mentioned above $\omega = -1$, also for the quintessence model ω is bounded as $-1 < \omega < -0.33$, and finally $\omega < -1$ for the phantom DE model.

Modified gravity theories (MGT) provide intriguing theoretical concepts for addressing the cosmological constant problem and explaining the late-time acceleration of the Universe. Several DE models started from the simplest modified gravity $1/R$ theory [6, 7]. In general, MGT appears to be quite appealing since it provides subjective solutions to a number of key problems concerning DE. An alternative theory to GR is teleparallel gravity by which gravitational interaction is described by the torsion scalar T [8–10] in a space-time with zero curvature. This theory is named *teleparallel equivalent to general relativity* (TEGR) and formulated by tetrad fields on the tangent space in the Weitzenbock connection which is different from the Levi–Civita connection in GR. The advantage of working with $f(T)$ models is the order of the field equations, this allows simplifying the dynamics and finding easily exact solutions. Symmetric teleparallel $f(Q)$ gravity is also an alternative theory in which the covariant derivative of the metric tensor does not vanish, i.e. $Q_{\gamma\mu\nu} = \nabla_\gamma g_{\mu\nu}$. This theory is called *symmetric teleparallel equivalent to general rel-*

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ativity (STEGR) [11, 12]. This new modified $f(Q)$ gravity where Q is the non-metricity scalar attracted interest of many researchers [13–18]. Moreover, this theory is based on the generalization of Riemannian geometry described by Weyl geometry [19]. Generally, the gravitational interaction is classified through three types of geometries: the curvature of space-time, torsion, and non-metricity. For this reason, in recent decades researchers have been attracted to MGT because they reflect the current phenomena of the Universe. Therefore, gravitational interactions have been calculated using several forms of geometrics [20–22].

It has been stated by observations that the Universe is homogeneous and isotropic when the inflationary phase was successfully produced [23]. However, anomalies in the CMBR lead to conclude that an anisotropic phase in the early Universe which make it not exactly uniform [24]. Thus, constructing cosmological models that describe the anisotropic and inhomogeneous properties of the Universe must be taken into consideration. Toward this goal, Bianchi-type models provide a good description of the anisotropic background and investigate the cosmic evolution in the early Universe. In fact, there exist nine types of Bianchi models in the literature. Here, we consider the anisotropic locally rotationally symmetric (LRS) Bianchi type-I model which is assumed to be a more general cosmological metric than Friedmann-Lemaitre-Robertson-Walker (FLRW) metrics [25]. The Bianchi type-I model is used to test the possible effects of anisotropy in the early Universe [26]. Recently, cosmological models have been constructed using anisotropic fluid in Bianchi type-I space-time. Moreover, some exact Bianchi type-I solutions have also been investigated in $f(Q)$ modified gravity [27, 28]. The Bianchi type I model usually presents good consistency with the most simple mathematical form, considering the nature of this model. Bianchi type I theory was studied in the context of a viscous fluid to discuss the behavior of the early Universe near the singularity [29].

The current article is organized as follows: In Sec. II we discuss the theoretical basis for $f(Q)$ gravity. In Sec. III, we derive the field equations in the LRS Bianchi type-I model. In Sec. IV, the cosmological solutions of the field equations are calculated with anisotropic relation. In Sec. V we analyze the physical and geometrical parameters of the cosmological model. Further, we constrain our model parameters with the observational Hubble datasets of 57 data points. Finally, the conclusion of the results is given in Sec VI.

II. $f(Q)$ GRAVITY FORMALISM

In differential geometry, the symmetric metric tensor $g_{\mu\nu}$ is used based on the definition of the length of a vector, and an asymmetric connection $\Sigma^\gamma{}_{\mu\nu}$ is used to define the covariant derivatives and parallel transport. Hence, the general affine connection can be decayed into three components: the Christoffel symbol $\Gamma^\gamma{}_{\mu\nu}$, the contortion tensor $C^\gamma{}_{\mu\nu}$, and the disformation tensor $L^\gamma{}_{\mu\nu}$, respectively, which is given by [19]

$$\Sigma^\gamma{}_{\mu\nu} = \Gamma^\gamma{}_{\mu\nu} + C^\gamma{}_{\mu\nu} + L^\gamma{}_{\mu\nu}, \tag{1}$$

where the Levi-Civita connection $\Gamma^\gamma{}_{\mu\nu}$ of the metric $g_{\mu\nu}$ has the form

$$\Gamma^\gamma{}_{\mu\nu} \equiv \frac{1}{2} g^{\gamma\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right), \tag{2}$$

the contorsion tensor $C^\gamma{}_{\mu\nu}$ can be written as

$$C^\gamma{}_{\mu\nu} \equiv \frac{1}{2} T^\gamma{}_{\mu\nu} + T_{(\mu}{}^\gamma{}_{\nu)}, \tag{3}$$

where $T^\gamma{}_{\mu\nu} \equiv 2\Sigma^\gamma{}_{[\mu\nu]}$ in Eq. (3) is the torsion tensor. Finally, the disformation tensor $L^\gamma{}_{\mu\nu}$ is derived from the non-metricity tensor $Q_{\gamma\mu\nu}$ as

$$L^\gamma{}_{\mu\nu} \equiv \frac{1}{2} g^{\gamma\sigma} \left(Q_{\nu\mu\sigma} + Q_{\mu\nu\sigma} - Q_{\gamma\mu\nu} \right). \tag{4}$$

In the above equation, the non-metricity tensor $Q_{\gamma\mu\nu}$ is specific as the (minus) covariant derivative of the metric tensor with regard to the Weyl-Cartan connection $\Sigma^\gamma{}_{\mu\nu}$, i.e. $Q_{\gamma\mu\nu} = \nabla_\gamma g_{\mu\nu}$, and it can be obtained

$$Q_{\gamma\mu\nu} = -\partial_\gamma g_{\mu\nu} + g_{\nu\sigma}\Sigma^\sigma{}_{\mu\gamma} + g_{\sigma\mu}\Sigma^\sigma{}_{\nu\gamma}. \tag{5}$$

The connection is presumed to be torsionless and curvatureless within the current background. It corresponds to the pure coordinate transformation from the trivial connection mentioned in [11]. Thus, for a flat and torsion-free connection, the connection (1) can be parameterized as

$$\Sigma^\gamma{}_{\mu\beta} = \frac{\partial x^\gamma}{\partial \xi^\rho} \partial_\mu \partial_\beta \xi^\rho. \tag{6}$$

Now, $\xi^\gamma = \xi^\gamma(x^\mu)$ is an invertible relation. It is always possible to get a coordinate system so that the connection $\Sigma^\gamma{}_{\mu\nu}$ vanish. This condition is called coincident

gauge and has been used in many studies of STEGR [19] and in this condition the covariant derivative ∇_γ reduces to the partial derivative ∂_γ. Thus, in the coincident gauge coordinate, we get

$$Q_{\gamma\mu\nu} = -\partial_\gamma g_{\mu\nu}. \tag{7}$$

The symmetric teleparallel gravity is a geometric description of gravity equivalent to GR (STEGR) within coincident gauge coordinates in which $\Sigma^\gamma{}_{\mu\nu} = 0$ and $C^\gamma{}_{\mu\nu} = 0$, and consequently from Eq. (1) we can conclude that

$$\Gamma^\gamma{}_{\mu\nu} = -L^\gamma{}_{\mu\nu}. \tag{8}$$

The modified Einstein-Hilbert action in symmetric teleparallel gravity can be considered as

$$S = \int \left[\frac{1}{2\kappa} f(Q) + \mathcal{L}_m \right] d^4x \sqrt{-g}, \tag{9}$$

where $\kappa = 8\pi G = 1$, $f(Q)$ can be expressed as the arbitrary function of non-metricity scalar Q, g is the determinant of the metric tensor $g_{\mu\nu}$, and $\mathcal{L}_m$ is the matter Lagrangian density. Now, the non-metricity tensor $Q_{\gamma\mu\nu}$ and its traces can be written as

$$Q_{\gamma\mu\nu} = \nabla_\gamma g_{\mu\nu}, \tag{10}$$

$$Q_\gamma = Q_\gamma{}^\mu{}_\mu, \qquad \widetilde{Q}_\gamma = Q^\mu{}_{\gamma\mu}. \tag{11}$$

In addition, the superpotential tensor (non-metricity conjugate) can be expressed as

$$4P^\gamma{}_{\mu\nu} = -Q^\gamma{}_{\mu\nu} + 2Q_{(\mu}{}^\gamma{}_{\nu)} - Q^\gamma g_{\mu\nu} - \widetilde{Q}^\gamma g_{\mu\nu} - \delta^\gamma_{(\gamma} Q_{\nu)}, \tag{12}$$

where the trace of the non-metricity tensor can be obtained as

$$Q = -Q_{\gamma\mu\nu} P^{\gamma\mu\nu}. \tag{13}$$

Now, the matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{14}$$

By varying the modified Einstein-Hilbert action (9) with respect to the metric tensor $g_{\mu\nu}$, the gravitational field equations obtained as

$$\frac{2}{\sqrt{-g}} \nabla_\gamma(\sqrt{-g} f_Q P^\gamma{}_{\mu\nu}) - \frac{1}{2} f g_{\mu\nu} + f_Q(P_{\nu\rho\sigma} Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu} Q^{\rho\sigma}{}_\nu) = \kappa T_{\mu\nu}. \tag{15}$$

where $f_Q = \frac{df}{dQ}$.

III. BIANCHI TYPE-I SPACE-TIME WITH FIELD EQUATIONS

As mentioned in the Introduction, the standard FLRW space-time is isotropic and homogeneous. Hence, to address the anisotropic nature of the Universe in $f(Q)$ gravity, which manifests as anomalies found in the CMB, the LRS Bianchi type-I space-time is indeed important because it represents a spatially homogeneous, but not isotropic. Thus, we consider a Bianchi-type I space-time in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2), \tag{16}$$

where metric potentials $A(t)$ and $B(t)$ depend only on cosmic time t. Here, to complete the choice of the anisotropic type space-time, the equation of state (EoS) parameter of the gravitational fluid must also be generalized, and from another point of view, to give a more reasonable model, an anisotropic nature must be presented as described in [30].

Thus, the energy-momentum tensor for the anisotropic fluid can be expressed as

$$\begin{aligned} T_\nu^\mu &= \mathrm{diag}(-\rho, p_x, p_y, p_z), \\ &= \mathrm{diag}(-1, \omega_x, \omega_y, \omega_z)\rho, \\ &= \mathrm{diag}(-1, \omega, (\omega+\delta), (\omega+\delta))\rho, \end{aligned} \tag{17}$$

where ρ is the energy density of the anisotropic fluid, p_x, p_y, p_z are the pressures and $\omega_x, \omega_y, \omega_z$ are the directional EoS parameters along x, y and z coordinates respectively. The deviation from isotropy is parametrized by setting $\omega_x = \omega$ and then introducing the deviations along y and z axes by the skewness parameter δ, where

ω and δ are functions of cosmic time t [27].

The non-metricity scalar of the anisotropic fluid leads to

$$Q = -2\left(\frac{\dot{B}}{B}\right)^2 - 4\frac{\dot{A}}{A}\frac{\dot{B}}{B}. \tag{18}$$

From the gravitational field equations (15), the corresponding modified Friedmann equations of LRS Bianchi type-I space-time (16) for the anisotropic fluid of energy-momentum tensor (17) can be written as [27]

$$\frac{f}{2} + f_Q\left[4\frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\left(\frac{\dot{B}}{B}\right)^2\right] = \rho, \tag{19}$$

$$\frac{f}{2} - f_Q\left[-2\frac{\dot{A}}{A}\frac{\dot{B}}{B} - 2\frac{\ddot{B}}{B} - 2\left(\frac{\dot{B}}{B}\right)^2\right] + 2\frac{\dot{B}}{B}\dot{Q}f_{QQ} = -\omega\rho, \tag{20}$$

$$\frac{f}{2} - f_Q\left[-3\frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2\right] + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{Q}f_{QQ} = -(\omega + \delta)\rho. \tag{21}$$

where the dot (·) denote derivative with respect to cosmic time t.

The directional Hubble parameters in the direction of the x, y, and z-axis, respectively are given by

$$H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B}. \tag{22}$$

The average Hubble parameter, which expresses the volumetric expansion rate of the Universe is given by

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right], \tag{23}$$

where the average scale factor and spatial volume as

$$V = a^3 = AB^2. \tag{24}$$

The mean anisotropy parameter is given by

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2}{9H^2}\left(H_x - H_y\right)^2. \tag{25}$$

The expansion scalar $\theta(t)$ and the shear scalar $\sigma(t)$ of the fluid are defined as follows

$$\theta(t) = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}, \quad \sigma(t) = \frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right). \tag{26}$$

In order to simplify the form of the field equations (19)-(21) and write them in terms of the non-metricity scalar Q, the directional Hubble parameters H_x, H_y and average Hubble parameter H, we use the following relations: $\frac{\partial}{\partial t}\left(\frac{\dot{A}}{A}\right) = \frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2$ and $Q = -2H_y^2 - 4H_xH_y$. The field equations (19)-(21) becomes

$$\frac{f}{2} - Qf_Q = \rho, \tag{27}$$

$$\frac{f}{2} + 2\frac{\partial}{\partial t}\left[H_yf_Q\right] + 6Hf_QH_y = -\omega\rho, \tag{28}$$

$$\frac{f}{2} + \frac{\partial}{\partial t}\left[f_Q(H_x + H_y)\right] + 3Hf_Q\left(H_x + H_y\right) = -(\omega + \delta)\rho. \tag{29}$$

Lastly, here we have three differential equations with six unknowns namely, f, H_x, H_y, ρ, ω, and δ. The exact solutions of these equations are examined in the next section.

IV. COSMOLOGICAL SOLUTIONS OF FIELD EQUATIONS

In order to completely solve the field equations, some other constraints must be added. Although the problems discussed in the introduction above, the cosmological constant Λ in GR is by far the most successful model

among all the proposed alternatives, and thus this motivates us to examine following linear form of $f(Q)$ gravity model [31],

$$f(Q) = \alpha Q + \beta, \tag{30}$$

where α and β are free model parameters.

Now, using (30) and subtracting (28) from (29), we get

$$\frac{d}{dt}\left(H_x - H_y\right) + \left(H_x - H_y\right)\frac{\dot{V}}{V} = -\frac{\delta\rho}{\alpha}. \tag{31}$$

This on integrating gives

$$\left(H_x - H_y\right) = \frac{c}{\alpha V}e^{\int \frac{\delta\rho}{\alpha\left(H_y - H_x\right)}dt}, \tag{32}$$

where c is constant of integration.

In order to find the exact solutions to the above equation, we will follow the work of Adhav [32] and Sahni [33], and uses the condition that

$$\delta = \frac{\alpha}{\rho}\left(H_y - H_x\right). \tag{33}$$

Using Eq. (33) in Eq. (32), we obtain this expression

$$\left(H_x - H_y\right) = \frac{c}{\alpha V}e^t. \tag{34}$$

The above equation can be written in terms of the metric potentials $A(t)$ and $B(t)$ as $\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = \frac{c}{\alpha AB^2}e^t$. By looking at this last equation, we are left with one differential equation and two unknowns, namely $A(t)$ and $B(t)$. Hence, we need a supplementary constraint to finally solve Eq. (34). In this work, we use the anisotropic relation i.e. the physical condition that the expansion scalar $\theta(t)$ is proportional to the shear scalar $\sigma(t)$ ($\theta^2 \propto \sigma^2$), which leads to the relation between the metric potentials as

$$A = B^n, \tag{35}$$

where n is an arbitrary real number and we think $n \neq 0$, and 1 for non-trivial solutions. According to Thorne [34] this physical law is justified on the basis of the observations of the velocity redshift relation for extragalactic sources which suggest that the Hubble expansion of the Universe is isotropic at present time within 30% [35]. More exactly, the redshift studies place the limit $\frac{\sigma}{\theta} \leq 0.3$, the ratio of the shear to the expansion scalar in the vicinity of our galaxy at present time. Collins et al. [36] pointed out that the normal congruence to the homogeneous expansion for spatially homogeneous metric satisfies the condition $\frac{\sigma}{\theta} =$ constant. Bunn et al. [37] conducted statistical analysis on 4-yr data from CMB and set a limit for primordial anisotropy to be less than 10^{-3} in Planck epoch. Many researchers have used this condition to find exact solutions of field equations in many backgrounds [38, 39].

Hence, using the above considerations and condition (35), Eq. (34) takes the form

$$\frac{\dot{B}}{B} - \frac{c}{\alpha(n-1)B^{n+2}}e^t = 0, \tag{36}$$

which yields a solution

$$A(t) = c_1^n\left[\frac{(n+2)e^t}{\alpha(n-1)} + c_2\right]^{\frac{n}{n+2}}, \tag{37}$$

$$B(t) = c_1\left[\frac{(n+2)e^t}{\alpha(n-1)} + c_2\right]^{\frac{1}{n+2}}, \tag{38}$$

where $c_1 = c^{\frac{1}{n+2}}$ and c_2 both are the constants of integration. Thus, using Eqs. (37) and (38), the metric in Eq. (16) takes the form

$$ds^2 = -dt^2 + c_1^{2n}\left[\frac{(n+2)e^t}{\alpha(n-1)} + c_2\right]^{\frac{2n}{n+2}}dx^2 + c_1^2\left[\frac{(n+2)e^t}{\alpha(n-1)} + c_2\right]^{\frac{2}{n+2}}(dy^2 + dz^2). \tag{39}$$

V. EVOLUTION OF COSMOLOGICAL PARAMETERS

In this section, we will discuss some basic physical and geometrical parameters to validate the cosmologi-

cal model, such as the spatial volume, expansion scalar,

shear scalar, average Hubble parameter, anisotropic parameter, energy density, pressure, EoS parameter, skewness parameter, and deceleration parameter.

Firstly, from Eq. (18), the non-metricity scalar becomes

$$Q = -\frac{2(2n+1)e^{2t}}{\left[c_2(n-1)\alpha + (n+2)e^t\right]^2} \tag{40}$$

$$\theta(t) = \frac{(n+2)e^t}{c_2(n-1)\alpha + (n+2)e^t}, \quad \sigma(t) = \frac{(n-1)e^t}{\sqrt{3}\left[c_2(n-1)\alpha + (n+2)e^t\right]}. \tag{42}$$

The average Hubble parameter is obtained as

$$H = \frac{(n+2)e^t}{3\left[c_2(n-1)\alpha + (n+2)e^t\right]} \tag{43}$$

Using Eqs. (37), (38), and (43) in (25), we obtain the anisotropy parameter as follows

$$\Delta = \frac{2(n-1)^2}{(n+2)^2}. \tag{44}$$

From Eqs. (40) and (41), we observed that the non-metricity scalar is time-dependent, and the spatial vol-

The spatial volume of the Universe becomes

$$V = B^{n+2} = c_1^{n+2}\left[\frac{(n+2)e^t}{\alpha(n-1)} + c_2\right]. \tag{41}$$

The expansion scalar and the shear scalar becomes

ume of the Universe is zero in the initial time $t = 0$ and increasing function of cosmic time. Thus, it can be said that in our model the evolution of the Universe begins with the Big Bang scenario. Also, from Eqs. (42)-(43) we can see that the expansion scalar, shear scalar and average Hubble parameter diverge at $t = 0$ and have a finite value at $t \to \infty$. It is also possible to look at the isotropic condition $\frac{\sigma^2}{\theta^2}$, as it takes a constant value from the early time to the late time. Therefore, our model appears that, it does not come close to the isotropy throughout the evolution of the Universe, and this is confirmed by Eq. (44) where we see that the anisotropic parameter is constant for our model.

Using Eqs. (30) and (40) in Eq. (27), we obtain the energy density of the Universe as

$$\rho = \frac{\alpha^2\beta c_2^2(n-1)^2 + 2\alpha\beta c_2\left(n^2+n-2\right)e^t + e^{2t}\left[\alpha(4n+2) + \beta(n+2)^2\right]}{2\left[\alpha c_2(n-1) + (n+2)e^t\right]^2} \tag{45}$$

Similarly, using Eqs. (30), (37), (38), (40), and (43) in Eq. (28), we obtain the pressure of the Universe as

$$p = -\frac{\alpha^2\beta c_2^2(n-1)^2 + 2\alpha c_2(n-1)e^t\left[2\alpha + \beta(n+2)\right] + e^{2t}\left[6\alpha + \beta(n+2)^2\right]}{2\left[\alpha c_2(n-1) + (n+2)e^t\right]^2} \tag{46}$$

Thus, the EoS parameter of the Universe is obtained as

$$\omega = -\frac{\alpha^2\beta c_2^2(n-1)^2 + 2\alpha c_2(n-1)e^t\left[2\alpha + \beta(n+2)\right] + e^{2t}\left[6\alpha + \beta(n+2)^2\right]}{\alpha^2\beta c_2^2(n-1)^2 + 2\alpha\beta c_2\left(n^2+n-2\right)e^t + e^{2t}\left[\alpha(4n+2) + \beta(n+2)^2\right]} \tag{47}$$

From Eqs. (33), (37), and (38), the skewness parameter is obtained as

$$\delta = -\frac{2\alpha(n-1)e^t\left[\alpha c_2(n-1) + (n+2)e^t\right]}{\alpha^2\beta c_2^2(n-1)^2 + 2\alpha\beta c_2\left(n^2+n-2\right)e^t + e^{2t}\left[\alpha(4n+2) + \beta(n+2)^2\right]} \tag{48}$$

A. Observational constraints

In the above discussions, we have described the $f(Q)$ gravity and solved the field equation. The expressions

of the Hubble parameter $H(t)$ in (43) can be expressed
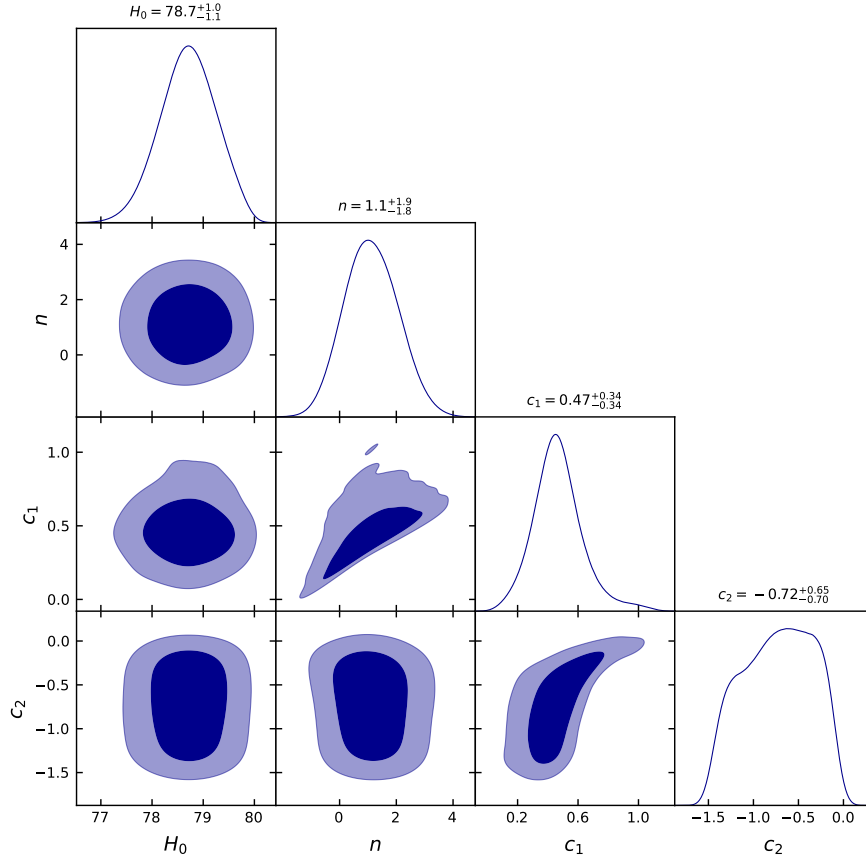
in terms of redshift z as,

FIG. 1. The $1 - \sigma$ and $2 - \sigma$ likelihood contours for the model parameters using Hubble datasets.

$$H(z) = H_0 \frac{-1 + c_1^{n+2} c_2 (1+z)^3}{-1 + c_1^{n+2} c_2}, \qquad (49)$$

where $a(t) = (1+z)^{-1}$ with $a(t_0) = a_0 = 1$, suffix 0 representing the value of parameter at $t = t_0$ and t_0 is the present time. The functional form of $H(z)$ contains three model parameters c_1, c_2 and n together with H_0. In order to describe the evolotion of some cosmological parameters in our obtained model, we need to choose some appropriate values of these model parameters. So, we consider here the Observational Hubble Datasets (OHD) to get some best fit values of these model parameters. We have used recently compiled 57

data points from OHD as in the reference [40], which is used in several papers. Scipy optimization technique from Python library is used here together with the consideration of a Gaussian prior with a fixed $\sigma = 1.0$ as the dispersion using Python's emcee package. The results are shown in the contour plots (two-dimensional) with $1 - \sigma$ and $2 - \sigma$ errors. The Chi-square function for our analysis is given by,

$$\chi_H^2(c_1, c_2, n) = \sum_{i=1}^{57} \frac{[H_{th}(z_i, c_1, c_2, n) - H_{obs}(z_i)]^2}{\sigma_{H(z_i)}^2}, \quad (50)$$

where H_{obs} is the observed value of the Hubble parameter and H_{th} is its theorised value and the symbol $\sigma_{H(z_i)}$
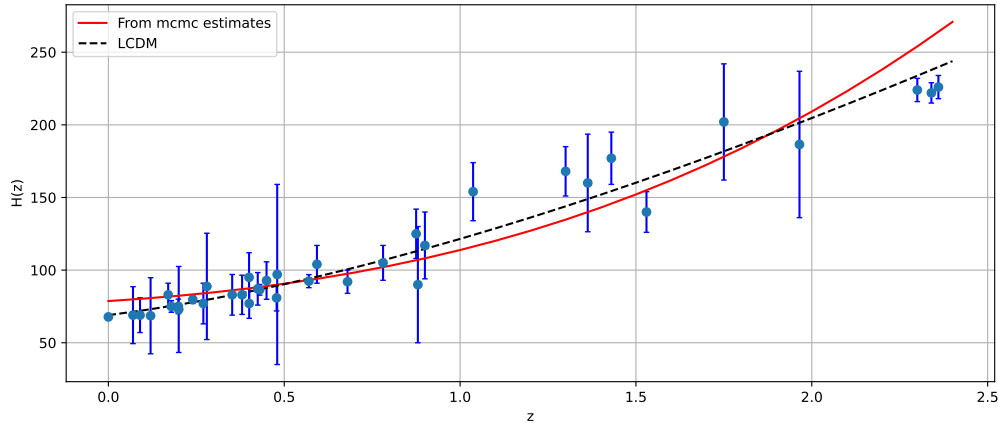
FIG. 2. The figure shows the error bar plot of the considered 57 points of Hubble datasets together with the fitting of Hubble function $H(z)$ vs. redshift z for our obtained model (red line) compared with that of standard ΛCDM model (black dashed line).

is the standard error in the observed value of the $H(z)$. With the above set up, we have found the best fit values of the model parameters for the observational Hubble dataets as $c_1 = 0.191^{+0.093}_{-0.093}$, $c_2 = 1.21^{+0.13}_{-0.13}$, $n = 1.21^{+0.13}_{-0.13}$. The result is shown in Fig. (1) as a two dimensional contour plots with $1-\sigma$ and $2-\sigma$ errors.

Additionally, we observed our derived model has nice fit to the aforementioned Hubble datasets. The error bars for the considered datasets and the ΛCDM model (with $\Omega_{\Lambda 0} = 0.7$ and $\Omega_{m0} = 0.3$) are also plotted along with our model for comparision. This is displayed in Fig. (2),

B. Physical interpretation of some cosmological parameters of the model

In cosmology, the deceleration parameter q is a measure of the variation in the expansion of the Universe, if $q < 0$ the Universe is in a phase of accelerated expansion and if $q > 0$ the Universe is in a phase of decelerated expansion and is defined as $q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right)$. For the model under discussion, the deceleration parameter is obtained as,

$$q = \frac{e^{-t}\left[(n+2)\left(-e^t\right) - 3c_2(n-1)\alpha\right]}{n+2},$$ (51)

which can be written in terms of redshift z as,

$$q(z) = -1 + \frac{3c_1^{n+2}c_2\left(1+z\right)^3}{-1 + c_1^{n+2}c_2\left(1+z\right)^3}.$$ (52)

The plot for the deceleration parameter in Fig. (3) exhibits a phase transition from early deceleration to the current acceleration of the Universe with current value corresponding to the observational Hubble datasets $q_0 \sim -0.7804$.
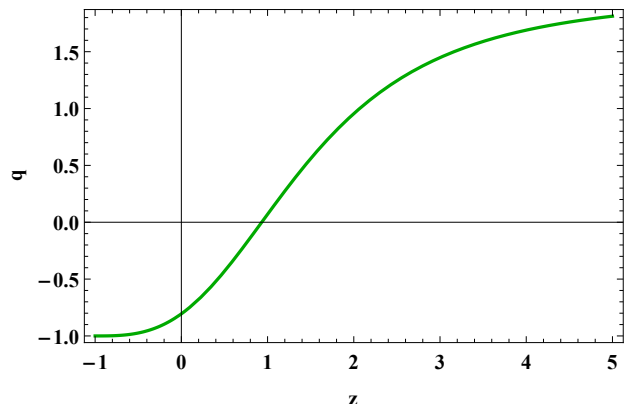

FIG. 3. q versus redshift z corresponding to the values of the parameters constrained by the Hubble datasets.

From Fig. (4), it is clear that the energy density of the Universe is an increasing function of cosmic redshift z and also it remains positive for all redshift values. It starts with large positive values and approaches zero in the future i.e. $z \to -1$. Fig. (5) indicates that the pressure of the Universe is also an increasing function of cosmic redshift z, which starts with large positive values and then becomes negative in the present period ($z = 0$) and in the future, the negative pressure pushes the Universe to the phase of acceleration, as indicated by astronomical observations. The EoS parameter is a relationship between energy density and pressure, and helps us

determine the phases the Universe has gone through. The matter phase at $\omega = 0$. Next, $\omega = \frac{1}{3}$ exhibit the radiation-dominated phase, while $\omega = -1$ corresponds to the ΛCDM model. In addition, the acceleration phase of the Universe is described at $\omega < -\frac{1}{3}$ which includes the quintessence $(-1 < \omega \leq -\frac{1}{3})$ and phantom model $(\omega < -1)$. Moreover, the EoS parameter presented in Fig. (6) indicates that the anisotropic fluid behaves like the ΛCDM model [41]. The current value of EoS parameter corresponding to the observational Hubble datasets is $\omega_0 \sim -1$. In Fig. (7) we see that the skewness parameter evolves in the range of negative values and tends towards values close to zero in the future, which confirms the previous discussion that our model remains under anisotropic behavior throughout the expansion of the Universe. Thus, this anisotropic cosmological $f(Q)$ model simulates the standard ΛCDM model.
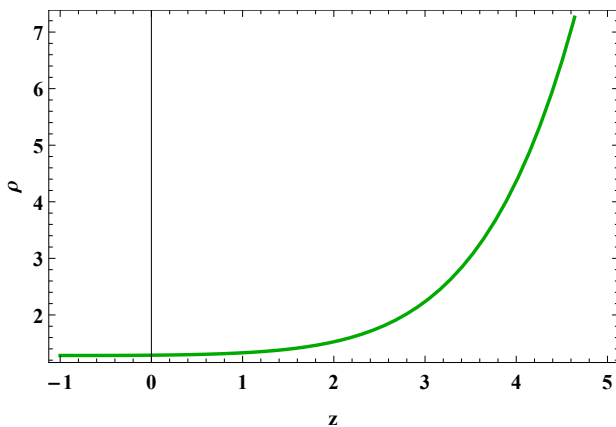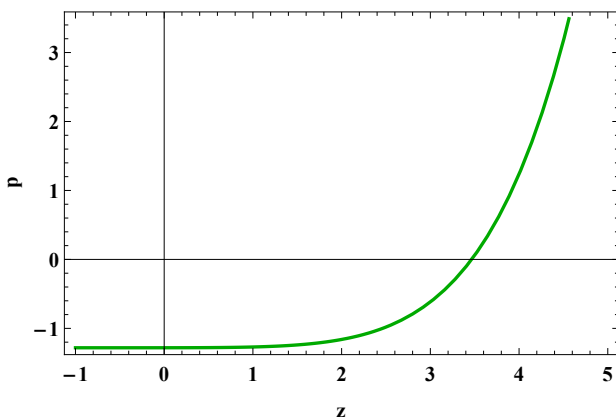

FIG. 4. ρ versus redshift z corresponding to the values of the parameters constrained by the Hubble datasets.

FIG. 5. p versus redshift z corresponding to the values of the parameters constrained by the Hubble datasets.
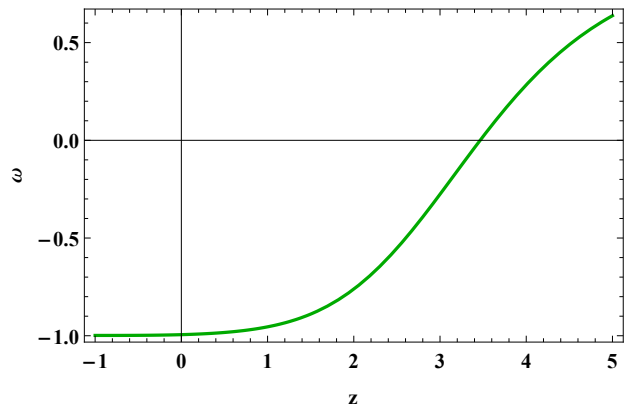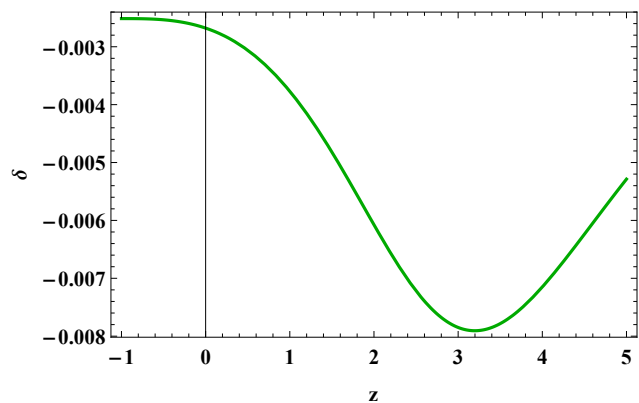

FIG. 6. ω versus redshift z corresponding to the values of the parameters constrained by the Hubble datasets.

FIG. 7. δ versus redshift z corresponding to the values of the parameters constrained by the Hubble datasets.

VI. CONCLUDING REMARKS

In this paper, we investigated the homogeneous and anisotropic LRS Bianchi-I space-time in the framework of $f(Q)$ modified gravity, where the non-metricity Q is the basis of gravitational interactions with zero curvature and torsion. The physical motivation for exploring the anisotropic Universe is the small deviations from the isotropy observed by the nine-year Wilkinson Microwave Anisotropy probe (WMAP) [42], which could yield more realistic results, especially with the $f(Q)$ modified theory of gravity. First, we briefly presented the mathematical formalism of the theory, then we derived the field equations for the LRS Bianchi-I space-time for the content of the Universe in the form of a perfect anisotropic fluid as in references [27, 28]. To get the exact solutions and study dark energy in $f(Q)$ gravity, motivated by the cosmological constant (Λ), we have considered the following linear model $f(Q) = \alpha Q + \beta$, where α and β are free model parameters. Further, to

complete the solutions we used the assumption that the scalar expansion $\theta(t)$ is proportional to the shear scalar $\sigma(t)$, which leads to the relation between the metric potentials in the form $A = B^n$, where n is an arbitrary constant. We obtained the best fit values of the model parameters by using the observational Hubble datasets of 57 data points. The obtained best fit values are $c_1 = 0.191^{+0.093}_{-0.093}$, $c_2 = 1.21^{+0.13}_{-0.13}$, $n = 1.21^{+0.13}_{-0.13}$.

Under these considerations, we found the complete solutions to the field equations and we have investigated the behavior of some cosmological parameters such as: The spatial volume of the Universe is zero in the initial time $t = 0$, which suggests that the evolution of the Universe begins with the Big Bang scenario and thus the model has a point type singularity [43]. The expansion scalar, shear scalar, and average Hubble parameter diverges at $t = 0$ and become a finite value at $t \to \infty$. To test the anisotropy of the model, we have studied the behavior of the anisotropic parameter and found that it takes a constant value throughout the expansion of the Universe. Further, for physical properties, we have discussed the behavior of energy density ρ, pressure p, equation of state (EoS) parameter ω, and skewness parameter δ with the help of Figs. (4)-(7). We have found the positive energy density and negative pressure, which results in the EoS parameter behaving like the standard ΛCDM model and its current value corresponding to the observational Hubble datasets is $\omega_0 \sim -1$. Thus, this value is consistent with the observational constraints on the dark energy EoS ω such as

$\omega_0 = -1.03 \pm 0.03$ [24], which suggests its value should be highly close to -1. In our model, we have discussed the behavior of skewness parameter which is an effective tool for checking whether the model is anisotropic or not, because in the case of an isotropic Universe, $\delta = 0$, and we have found that it changes in the range of negative values and tends towards values close to zero in the future, which confirms that our model remains under anisotropic behavior throughout the expansion of the Universe. In addition, we observed that for all values of n the deceleration parameter exhibits a phase transition from early deceleration to the current acceleration of the Universe with current value corresponding to the observational Hubble datasets $q_0 \sim -0.7804$. Finally, it can be said that this type of result agrees well with the accelerating scenario of the Universe.

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