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Lateral Vibration of Axially Moving Wire or Belt Form Materials*

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Theoretical analysis was performed for the lateral vibration of an axially moving linear material which is supported by pulleys. The oscillation of this kind has some peculiar characters, by the effect of centrifugal force the oscillation degenerates to unstable character, the modes of natural oscillation take a form of wave motion, the frequencies of free oscillation diminish with increasing velocity and there occur self excited vibrations over the critical speeds.

The boundary conditions in the analysis are assumed to be constant curvature at two points of support ends. The method of solution is to simplify the characteristic equation and to unify the unknown variables. The numerical calculation by the iterative method leads to the solutions of frequency curves. Another method of analysis by the complex Fourier series gives an approximate but clear perspective of the solution.

Introduction

The problem of oscillation of axially moving material was partly treated by Watanabe⁽¹⁾ and Shimoyama⁽²⁾ for the case of flexible materials. They found some important characters of the oscillation. But the oscillation contains more interesting problems, for example, the modes of oscillation are not independent of each other, namely the energy of oscillation of each modes has correlation and the energy flows in or out from the supporting points. The rate of energy flow at one side may be expressed as follows for the case of flexible materials.

$$\text{rate of energy} = \frac{1}{2} \times \text{tension} \times [\text{slope}]^2 \times \text{moving velocity}$$

Furthermore, the energy of each modes are not constant, it fluctuates at a rate of 2 cycles per 1 cycle of oscillation, the phenomena originates from the wave motion of normal modes.

The present paper offers the solutions of oscillation for the material with flexural rigidity under the conditions of pulley support and slope free. Another solution for the boundary conditions of slope fixed was shown in detail by the author.⁽³⁾⁽⁴⁾

Fundamental Equation

Let the moving velocity v , and taking the coordinate ξ which is fixed to the space, the fundamental equation of lateral vibration may be written as

$$\rho A \left(\frac{\partial}{\partial t} - v \frac{\partial}{\partial \xi} \right)^2 w - P \frac{\partial^2 w}{\partial \xi^2} + EI \frac{\partial^4 w}{\partial \xi^4} = 0 \quad \dots (1)$$

where, ρ , E , A , I & P are linear density, Young's modulus, cross-sectional area, moment of inertia and tension of the moving material, respectively.

The boundary conditions at two ends of free range may be expressed by the following equations, if the material adhere closely to the pulleys and the slopes are continuous at these points,

$$\left. \begin{aligned} \xi=0 : w=0, \quad \partial^2 w / \partial \xi^2 = -1/R \\ \xi=l : w=0, \quad \partial^2 w / \partial \xi^2 = \mp 1/R \end{aligned} \right\} \quad \dots (2)$$

where, R is radius of pulleys, l is the free length of material. The expression of equation (2)

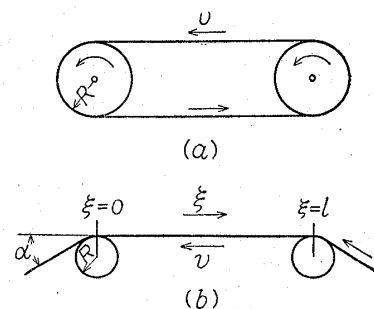


Fig. 1

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implies some approximation, but the error may be considered negligible in the limit of small oscillations.

We exchange the conditions (2) to the homogeneous form,

$$\xi=0 \text{ \& \; } \xi=l: w=0, \quad \partial^2 w / \partial \xi^2 = 0 \quad \dots\dots\dots (3)$$

then by the principle of superposition, the complete solution of equation (1) which satisfies the conditions of equation (2) will be given by the sum of general solution of equation (1) which satisfies the condition (3) and an arbitrary particular solution which satisfies (2).

First, we ask for the steady and time independent solution of equation (1). Eliminating the time derivatives in equation (1) we have

$$(v^2 - p^2) \frac{\partial^2 w}{\partial \xi^2} + q^2 \frac{\partial^4 w}{\partial \xi^4} = 0 \quad \dots\dots\dots (4)$$

$$\text{where,} \quad p = \sqrt{p/\rho A}, \quad q = \sqrt{EI/\rho A} \quad \dots\dots\dots (5)$$

The equation (4) is same as that of lateral buckling of a long beam under axial load, then we can assume a virtual or equivalent axial load,

$$P_0 = \rho A v^2 - P \quad \dots\dots\dots (6)$$

Corresponding to the critical load of buckling, there are critical speeds of moving materials. Solutions of equation (4) are

$$w = C \sin\left(\frac{n\pi\xi}{l}\right), \quad (n=1, 2, 3, \dots\dots) \quad \dots\dots (7)$$

and the critical speeds may be given by the relation

$$\beta = \sqrt{r^2 - k^2} = n\pi, \quad (n=1, 2, \dots\dots) \quad \dots\dots (8)$$

$$\text{where,} \quad r = vl/q, \quad k = pl/q \quad \dots\dots\dots (9)$$

The two constants r and k are velocity coefficient and tension coefficient respectively. Generally the tension of moving material increases with the speed and the critical speeds take higher value than those deduced from the tension at rest, because of the centrifugal force acting on the material, and the tension P increases by the amount

$$P = P' + \rho A v^2 \quad \dots\dots\dots (10)$$

where P' is the tensile component of the force transmitted from the pullies to the material.

To represent these relations with a simplified expression, we put

$$P = P_0 + \theta \rho A v^2 \quad \dots\dots\dots (11)$$

where P_0 is the tension at rest and θ is the coefficient of tension increase. Generally the value of θ is limited as $1 > \theta > 0$.

Using equation (11), we write

$$p^2 = p_0^2 + \theta v^2, \quad k^2 = k_0^2 + \theta r^2 \quad \dots\dots\dots (12)$$

then the velocity coefficient corresponding to the critical speeds may be given by the equation (8), that is

$$r_n = \sqrt{\frac{n^2 \pi^2 + k_0^2}{1 - \theta}} \quad \dots\dots\dots (13)$$

The values of θ are nearly equal to unity, and the critical speeds vanish to infinity, so long as the tension of pulley is controlled constant. Otherwise, there is a tendency that the value of θ increases near the first critical speed.

Characteristic Equation and Frequency Curves

Denoting by the equation (5), the equation of motion may be written,

$$\left(\frac{\partial}{\partial t} - v \frac{\partial}{\partial \xi}\right)^2 w - p^2 \frac{\partial^2 w}{\partial \xi^2} + q^2 \frac{\partial^4 w}{\partial \xi^4} = 0 \quad \dots\dots (14)$$

we put

$$w(\xi, t) = e^{i\lambda\xi} e^{i\mu t} \quad \dots\dots\dots (15)$$

where, λ and μ are unknown constants. Substituting equation (15) into equation (14) we have the following relation between λ and μ .

$$(\mu - v\lambda)^2 = p^2 \lambda^2 + q^2 \lambda^4$$

Rewriting this equation

$$\left. \begin{aligned} (\lambda l)^4 + (k^2 - r^2)(\lambda l)^2 + 2rh(\lambda l) - h^2 &= 0 \\ h &= \mu l^2 / q \end{aligned} \right\} \quad \dots\dots (16)$$

where

The parameter h represents the nondimensional frequency of free oscillation and when h is a complex number, its real part corresponds to frequency and the imaginary part corresponds to the coefficient of divergence or convergence. The other parameter λl represents wave form and it will be determined by the equation (16) and the boundary conditions.

The first of equation (16) is a biquadratic equation with respect to (λl) and there are four values of (λl) corresponding to one value of h in a free mode of oscillation, and the four roots of equation (16) may be expressed in terms of the three unknown parameters s_0, s_1 & s_2 , because it lacks in the term of the third order.

$$\left. \begin{aligned} \lambda_1 l &= (s_0 - s_1)/2, \quad \lambda_2 l = (s_0 + s_1)/2 \\ \lambda_3 l &= (-s_0 - s_2)/2, \quad \lambda_4 l = (-s_0 + s_2)/2 \end{aligned} \right\} \quad \dots\dots (17)$$

Comparing this with the equation (16), we get the following relations among parameters being derived from the relation of roots and coefficients of the biquadratic equation.

$$\left. \begin{aligned} 2s_0^2 + s_1^2 + s_2^2 &= 4(r^2 - k^2) \\ s_0(s_2^2 - s_1^2) &= 8rh \\ (s_0^2 - s_1^2)(s_0^2 - s_2^2) &= -16h^2 \end{aligned} \right\} \quad \dots\dots\dots (18)$$

On the other hand, the general solution of equation (1) may be written in the form of following equation.

$$w(\xi, t) = [C_1 e^{i\lambda_1 \xi} + C_2 e^{i\lambda_2 \xi} + C_3 e^{i\lambda_3 \xi} + C_4 e^{i\lambda_4 \xi}] e^{i\mu t} \quad \dots\dots\dots (19)$$

where, $C_1, C_2, \dots\dots$ are constants.

The right of equation (19) is a complex function and the conjugate function also exists as a solution, so we can take the real or imaginary

part of equation (19) as a solution in practice.

Substituting equation (19) into the homogeneous boundary conditions of equation (3) and eliminating C_1, C_2, \dots in a form of determinant, we get the following equation with respect to $\lambda_1, \lambda_2, \dots$.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ e^{i\lambda_1 l} & e^{i\lambda_2 l} & e^{i\lambda_3 l} & e^{i\lambda_4 l} \\ \lambda_1^2 e^{i\lambda_1 l} & \lambda_2^2 e^{i\lambda_2 l} & \lambda_3^2 e^{i\lambda_3 l} & \lambda_4^2 e^{i\lambda_4 l} \end{vmatrix} \dots\dots (20)$$

expanding,

$$\begin{aligned} & (\lambda_1^2 - \lambda_2^2)(\lambda_4^2 - \lambda_3^2)\{e^{i(\lambda_1 + \lambda_2)l} + e^{i(\lambda_4 + \lambda_3)l}\} \\ & + (\lambda_2^2 - \lambda_3^2)(\lambda_4^2 - \lambda_1^2)\{e^{i(\lambda_2 + \lambda_3)l} + e^{i(\lambda_4 + \lambda_1)l}\} \\ & + (\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)\{e^{i(\lambda_1 + \lambda_3)l} + e^{i(\lambda_2 + \lambda_4)l}\} = 0 \end{aligned} \dots\dots (21)$$

using equation (17) and operating calculation,

$$\begin{aligned} & \left(\frac{s_1 - s_2}{2}\right)^2 \left\{s_0^2 - \left(\frac{s_1 + s_2}{2}\right)^2\right\} \cos\left(\frac{s_1 - s_2}{2}\right) \\ & + \left(\frac{s_1 + s_2}{2}\right)^2 \left\{s_0^2 - \left(\frac{s_1 - s_2}{2}\right)^2\right\} \cos\left(\frac{s_1 + s_2}{2}\right) \\ & + s_0^2 s_1 s_2 \cos s_0 = 0 \end{aligned} \dots\dots (22)$$

This is the characteristic equation for the vibration of moving materials, under the condition of constant curvature at two ends. In equation (22) the parameters s_0 and s_2 take real value for the steady free oscillations, but s_1 takes real value only for the high speed region near the critical speed and takes imaginary value for the low speed region. Putting $s_1 = \pm i s_1'$, and denoting equation (22) for the low speed region,

$$\begin{aligned} & \left\{s_0^2(s_1'^2 - s_2^2) + \left(\frac{s_1'^2 + s_2^2}{2}\right)^2\right\} \sinh \frac{s_1'}{2} \sin \frac{s_2}{2} \\ & + 2s_0^2 s_1' s_2 \left(\cosh \frac{s_1'}{2} \cos \frac{s_2}{2} - \cos s_0\right) = 0 \end{aligned} \dots\dots (23)$$

Next, in equation (18), taking the square of the second, and adding the product of the first and the third,

$$(2s_0^2 - s_1'^2 - s_2^2)(s_0^4 - s_1'^2 s_2^2) = 64h^2 k^2$$

Combining this with the third of equations (18) and eliminating h^2 , we have

$$\frac{(2s_0^2 - s_1'^2 - s_2^2)(s_0^4 - s_1'^2 s_2^2)}{(s_0^2 - s_1'^2)(s_2^2 - s_0^2)} = 4k^2 \dots\dots (24)$$

To unify the parameters, we exchange the variables taking the relation of the 1st of equations (18) into consideration and using the nondimensional parameters α & δ .

$$\left. \begin{aligned} s_1'^2 &= s_1'^2 = \beta^2 - \delta - \alpha, \quad s_2^2 = \beta^2 - \delta + \alpha \\ s_0^2 &= \beta^2 + \delta, \quad \beta^2 = r^2 - k^2 \end{aligned} \right\} \dots\dots (25)$$

Substituting this into (24), we have

$$\frac{4\beta^2 \delta^2 - \alpha^2 \delta}{4\delta^2 - \alpha^2} = k^2$$

from which we get the relation between α and δ , namely

$$\alpha = 2r\delta / \sqrt{k^2 - \delta} \dots\dots (26)$$

Using the third of equations (18) and (25) we have

$$h = \sqrt{\alpha^2 - 4\delta^2} / 4 \dots\dots (27)$$

Furthermore, we transform the characteristic equation of (23) to make it easier for the iterative calculation, because the original form is not convenient for the purpose of numerical analysis.

$$\left. \begin{aligned} \tan \frac{s_2}{2} &= \frac{-2s_0^2 s_1' s_2 F}{\alpha^2 + 2\delta^2 - 2\beta^4} \\ F &= \left[\coth \frac{s_1'}{2} - \cos s_0 / \sinh \frac{s_1'}{2} \cos \frac{s_2}{2} \right] \end{aligned} \right\} \dots\dots (28)$$

The function F in these equations generally takes a value nearly equal to unity for the low speed region. And for higher speeds, we rewrite this using s_1 instead of s_1' , we have

$$\left. \begin{aligned} \tan \frac{s_2}{2} &= \frac{-2s_0^2 s_1 s_2 F}{\alpha^2 + 2\delta^2 - 2\beta^4} \\ F &= \left[\cot \frac{s_1}{2} - \cos s_0 / \tan \frac{s_1}{2} \cos \frac{s_2}{2} \right] \end{aligned} \right\} \dots\dots (29)$$

The boundary of application of equations (28) and (29) may be given by putting $s_1 = 0$ in equations (18) & (24), the results are

$$\left. \begin{aligned} s_0^2 &= (r^2 - 4k^2 + r\sqrt{8k^2 + r^2})/2 \\ s_2^2 &= s_0^2(4k^2 + 2s_0^2)/4(k^2 + s_0^2) \\ h &= s_0 \sqrt{s_2^2 + s_0^2} / 4 \end{aligned} \right\} \dots\dots (30)$$

By the use of equations (25) through (29) and by the iterative method, we get numerical solution of free oscillation. The process of iteration is as follows: First, we assume the value of δ for the given values of r and k and calculate α by the equation (26). Calculating s_1 or s_1' , s_0 and s_2 from the equations (25) and substituting them into the right side of equations (28) or (29), we have $\tan(s_2/2)$ and then s_2 . Comparing the value of s_2 with the foregoing value and correcting the assumption of the value of δ by the interpolation of s_2 , we repeat the iteration.

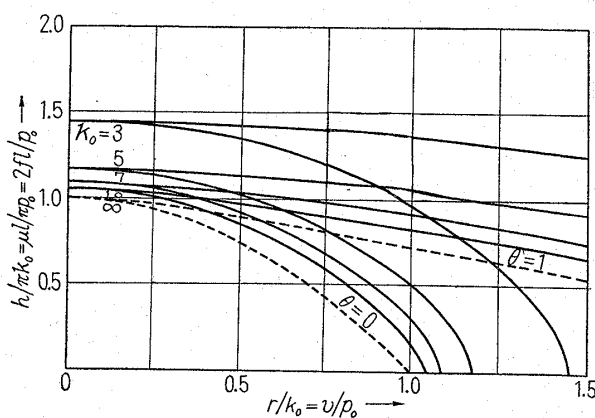
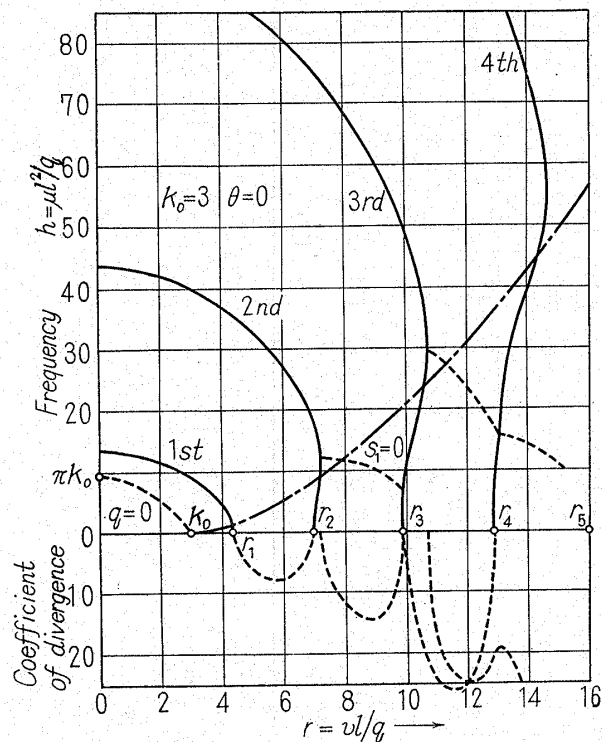
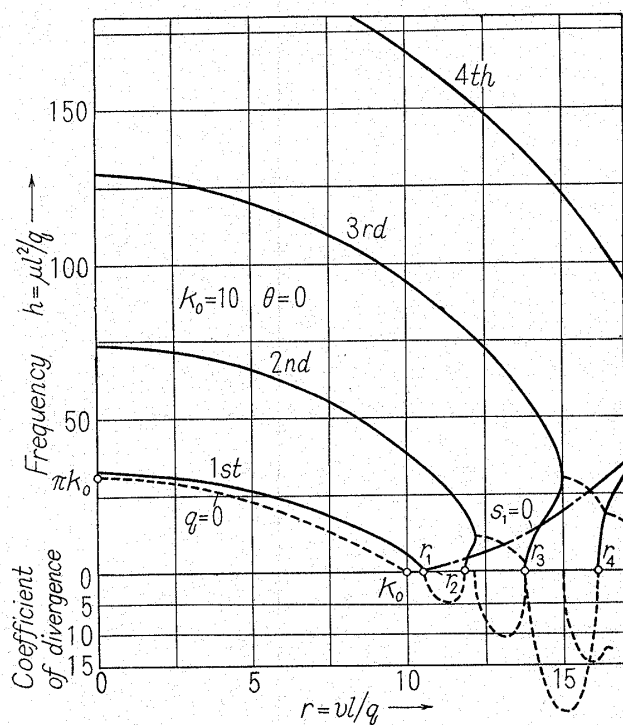


Fig. 2 Fundamental frequencies

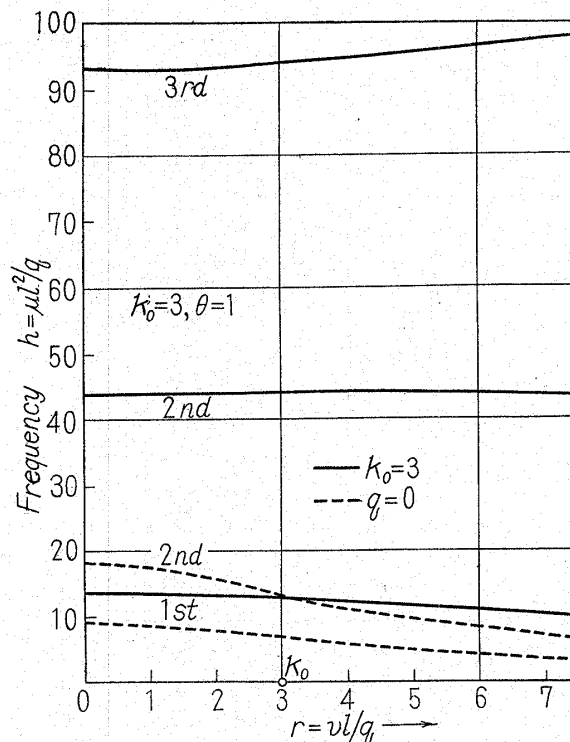
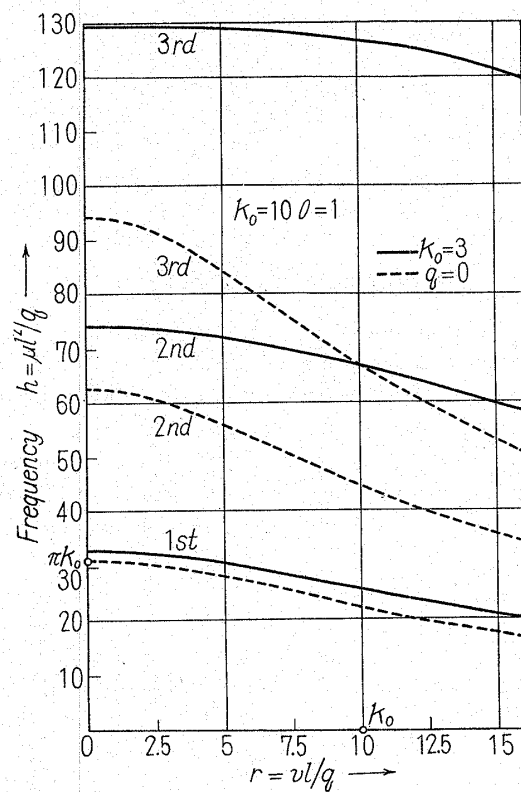
The method of calculation shows comparatively rapid convergence except in the high speed region. Fig. 2 shows the frequencies of the fundamental

mode of oscillation for the cases of $\theta=0$ and $\theta=1$ and for many values of the initial tension coefficient k_0 . The broken lines of $k_0=\infty$ in the figure represents the frequencies for the case of perfectly

Fig. 3 Frequency curves ($k_0=3, \theta=0$)Fig. 4 Frequency curves ($k_0=10, \theta=0$)

flexible material.

The results of calculation for the combination $k_0=3, 10$ and $\theta=0, 1$ are shown in Fig. 3, 4, 5 and 6. In these figures the broken lines of $q=0$

Fig. 5 Frequency curves ($k_0=3, \theta=1$)Fig. 6 Frequency curves ($k_0=10, \theta=1$)

correspond to the fundamental frequencies for the case of no flexural rigidity and may be expressed by the following equation.

$$h = \pi \frac{k_0^2 - (1 - \theta)r^2}{\sqrt{k_0^2 + \theta r^2}} \quad (31)$$

The curves in the upper half of the figures represent the frequencies in nondimensional expression and the lower half represent the coefficients of divergence and/or convergence. The full lines represent the frequencies of steady oscillations and the broken lines in the region of $r > r_1$ for the case of $\theta = 0$ correspond to the frequencies and the coefficient of divergence and/or convergence of periodic or aperiodic oscillations. The chain line represents the boundary of $s_1^2 \leq 0$, the character of frequency curves is bordered by the line and the most of unstable oscillations exist in the lower part from this line.

By the movement of material all the frequencies of free oscillations decrease with the increasing velocity in a similar manner like the beam oscillation under the axial thrust. The values of h corresponding to the unsteady motion are pure imaginary or complex conjugates and the divergent and convergent oscillations with same values of coefficients coexist on the same line. The values of coefficients of divergence become considerable order and vehement vibrations are expected in these regions. The aperiodic divergent or convergent motions exist in the region from an odd number of critical speed to the next even. Over the first critical speed the steady oscillations vanish one by one, degenerate to unstable oscillations and the unstable modes increase rapidly with increasing velocity.

Method of Solution by the Complex Fourier Series

The aforesaid method of solution by the use of characteristic equation is exact and we can have voluntary precision values of higher modes, but on the other hand, we may have no direct inspection by this method. The following method is an approximation, but offers comparatively precise solution, and is suitable for the practical purpose.

In equation (14) we put

$$w(\xi, t) = w(\xi) e^{i\mu t} \quad (32)$$

rewriting, using the notation of equations (7) & (16), the fundamental equation may be written.

$$\left(h' - r l \frac{d}{d\xi}\right)^2 w - k^2 l^2 \frac{d^2 w}{d\xi^2} + l^4 \frac{d^4 w}{d\xi^4} = 0 \quad (33)$$

where

$$h' = \pm i h \quad (34)$$

We expand $w(\xi)$ into Fourier series of ξ with

half wave length l , taking into account the boundary conditions (3).

$$w(\xi) = \sum_m C_m \sin \frac{m\pi\xi}{l} \quad (35)$$

Substituting this into the equation (33)

$$\sum_m C_m [h'^2 - (r^2 - k^2)m^2\pi^2 + m^4\pi^4] \sin \frac{m\pi\xi}{l} - \sum_m C_m 2h' r m \pi \cos \frac{m\pi\xi}{l} = 0$$

In the interior of the boundaries we use the following relations.

$$\cos \frac{m\pi\xi}{l} = \sum_n \frac{4n}{\pi(n^2 - m^2)} \sin \frac{m\pi\xi}{l}, \quad (m \pm n : \text{odd})$$

By this relation

$$\begin{aligned} \therefore \sum_m C_m [h'^2 - (r^2 - k^2)m^2\pi^2 + m^4\pi^4] \sin \frac{m\pi\xi}{l} \\ + \sum_m \sum_n C_n \frac{4mn}{(m^2 - n^2)} 2h' r \sin \frac{m\pi\xi}{l} = 0 \end{aligned} \quad (m \pm n : \text{odd})$$

Dividing by $8h'r$ and separating each term of m by the Fourier Analysis

$$C_m \phi_m + \sum_n C_n \frac{mn}{m^2 - n^2} = 0, \quad (m \pm n : \text{odd}) \quad (36)$$

where

$$\phi_m = [h'^2 - (r^2 - k^2)m^2\pi^2 + m^4\pi^4] / 8h'r \quad (37)$$

We eliminate C_m from these equations, then we have the following equation in a form of determinant

$$\begin{vmatrix} \phi_1 & \frac{2 \cdot 1}{2^2 - 1^2} & 0 & \frac{4 \cdot 1}{4^2 - 1^2} & 0 & \dots \\ \frac{1 \cdot 2}{1^2 - 2^2} & \phi_2 & \frac{3 \cdot 2}{3^2 - 2^2} & 0 & \frac{5 \cdot 2}{5^2 - 2^2} & \dots \\ 0 & \frac{2 \cdot 3}{2^2 - 3^2} & \phi_3 & \frac{4 \cdot 3}{4^2 - 3^2} & 0 & \dots \\ \frac{1 \cdot 4}{1^2 - 4^2} & 0 & \frac{3 \cdot 4}{3^2 - 4^2} & \phi_4 & \frac{5 \cdot 4}{5^2 - 4^2} & \dots \end{vmatrix} = 0 \quad (38)$$

The values of ϕ_m on the diagonal are proportional to m^4 and take large value compared with the other terms, and as the determinant converges rapidly, an approximate calculation with respect to a few columns and rows of this determinant gives comparatively precise results for lower modes. The results of numerical calculations of high order approximation almost perfectly coincide with those of the exact solution.

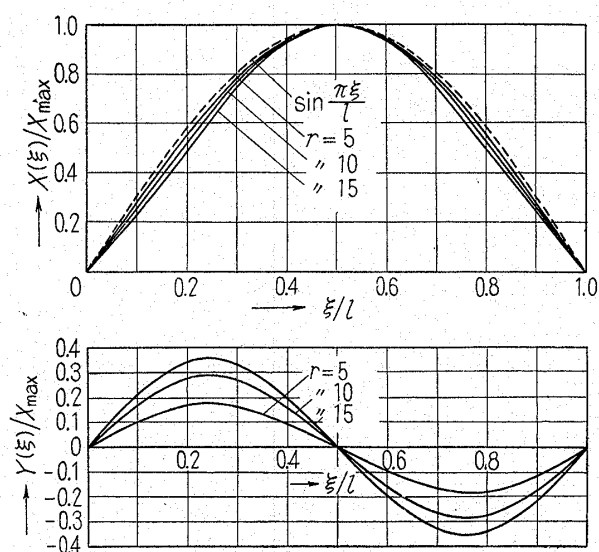
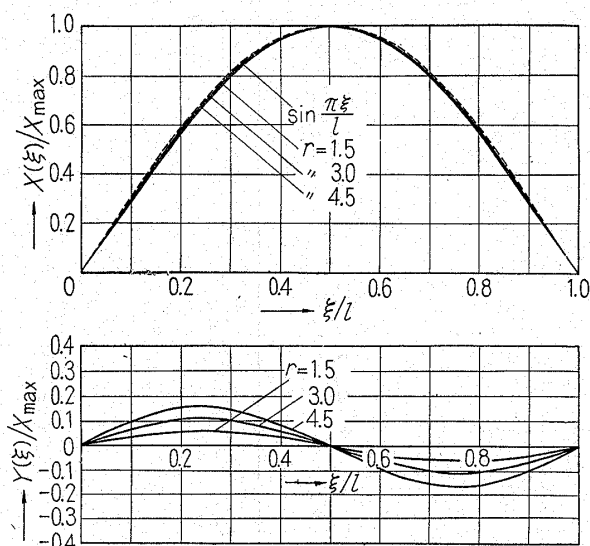
Next, we express the wave form by this way, we can write

$$w(\xi, t) = \Re C \left[a_1 \sin \frac{\pi\xi}{l} + a_2 \sin \frac{2\pi\xi}{l} + \dots \right] e^{i\mu t}$$

where \Re represents the real part of the function. Rewriting

$$\begin{aligned} w(\xi, t) &= \Re C [X(\xi) + iY(\xi)] e^{i\mu t} \\ &= A [X(\xi) \sin(\mu t + \delta) + Y(\xi) \cos(\mu t + \delta)] \end{aligned} \quad (39)$$

where, C , A , δ are arbitrary constants. The real

Fig. 7 Space wave functions ($k_0=3$, $\theta=1$)Fig. 8 Space wave functions ($k_0=10$, $\theta=1$)

and imaginary parts of the space wave function may be calculated from the equation (36) and take the following form.

$$\left. \begin{aligned} X(\xi) &= \left[a_1 \sin \frac{\pi \xi}{l} + a_3 \sin \frac{3\pi \xi}{l} + \dots \right] \\ iY(\xi) &= \left[a_2 \sin \frac{2\pi \xi}{l} + a_4 \sin \frac{4\pi \xi}{l} + \dots \right] \end{aligned} \right\} \dots (40)$$

Figs. 7 and 8 show the wave function. As is seen in the figures, the real and imaginary parts are of similar form to the 1st and 2nd of sine curve, and the ratio of the latter to the former gains with the increasing velocity and the distinct mode of wave motion comes to appear.

Conclusion

It may be mentioned as a conclusion that this oscillation has a similar character like a long beam under axial compression. Also the oscillation of a pipe line containing flowing liquid has quite similar character, and there is a study of Long⁽⁵⁾ for the case of two ends simply supported. His analysis shows a similar results for the fundamental mode.

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