Latin Routers, Design and Implementation [1]

Richard A. Barry and Pierre A. Humblet

MIT, Laboratory for Information and Decision Systems

Cambridge, MA 02139

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Abstract

We present a class of designs for very large all optical wavelength routing networks. The designs use a relatively small number of components and can be implemented distributively.

1 Introduction

This paper presents designs of large wavelength routing networks built with small wavelength routing devices and minimal interconnections. The approach is analogous to designing large switching networks from 2×2 components. We now define the desired functionality of the networks.

Consider an optical system where the available bandwidth is divided into F frequency bands numbered from 0 to $F-1$. An $N \times N$ wavelength routing device, or an all optical network with passive wavelength routing, can be specified by an $N \times N$ matrix S, where the $(i,j)^{th}$ element, $S(i,j)$, is the set of wavelengths connecting input i to output j. We say that the device or network is *connected* if each input can reach every output on some wavelength, i.e. $|S(i,j)| \geq 1$ for all (i,j). A special case of a connected device is when the matrix S is a latin square. A latin square is an $N \times N$ matrix where each element (i,j) is one of N symbols such that no symbol appears in a row or column more than once. Two examples of a 4×4 Latin Router are shown in tables 1a and 1b. Notice that these squares are distinct, in the sense that there is no relabeling (of columns, rows, or elements) that will turn table 1b into table 1a. We call such a device a *Latin Router.*

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0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

Table 1a

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

Table 1b

Table 1: Two Examples of 4×4 Latin Routers/Squares

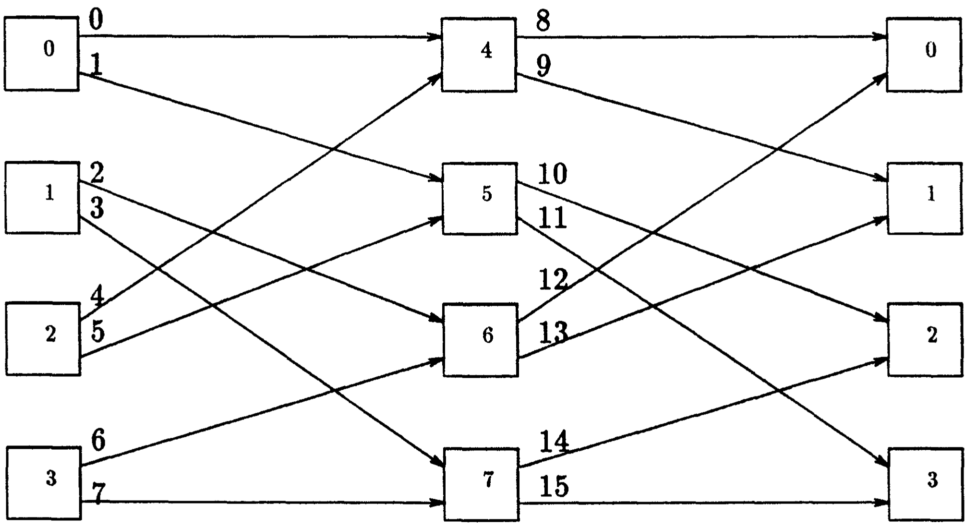

Figure 1: $(2,2)$ ShuffleNet with conventional wavelength assignment

In a Latin Router, all F wavelengths can be simultaneously applied to each input without any output contention. It has long been known that Latin Routers[2] can be used to provide full connectivity between F users with only F wavelengths. That is, a total of F^2 sessions can be accommodated simultaneously, [1, 2, 3, 4, 5]. Another use is as an $F \times F$ non-blocking switch with an appropriate frequency assignment protocol.

Latin Routers also have applications in multihop lightwave networks. In particular, consider the (p,k) ShuffleNet topology presented in [6] where p is the number of transceivers per user, k is the number of stages, and $N = kp^k$ is the number of users. A $(2,2)$ example is shown in Fig. 1. If the underlying physical topology is an $N \times N$ broadcast star, then pN wavelengths are required for full connectivity between connected transceivers without frequency sharing [6, 7].

[2]although never before given a name as far as the authors know

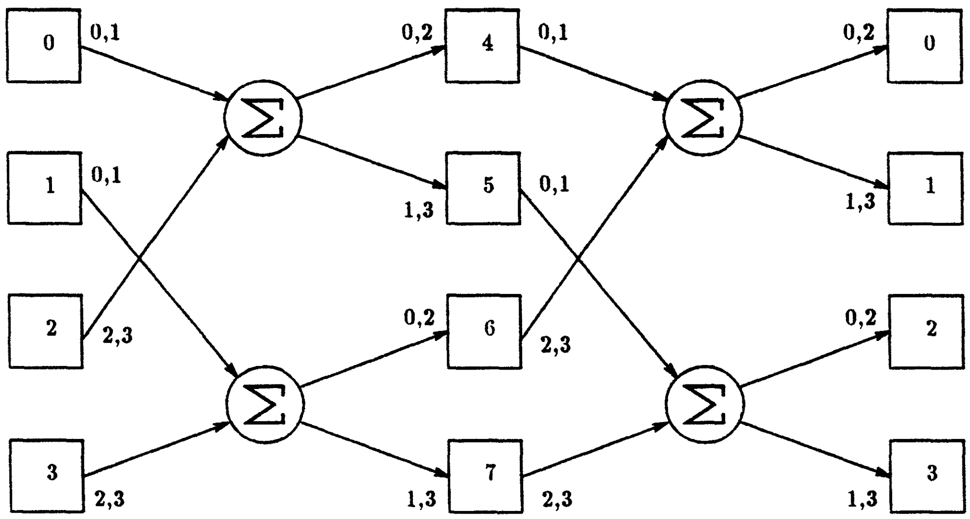

Figure 2: (2,2) ShuffleNet implementation with multiple stars and new wavelength assignment

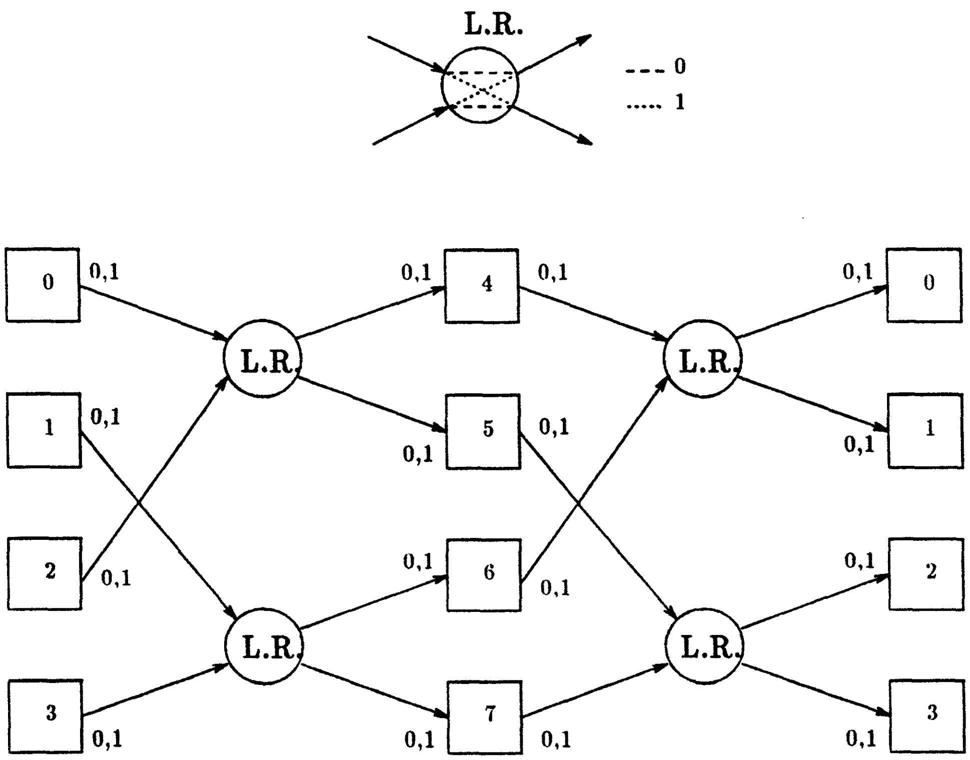

Figure 3: (2,2) ShuffleNet implementation with multiple Latin Routers (LR)

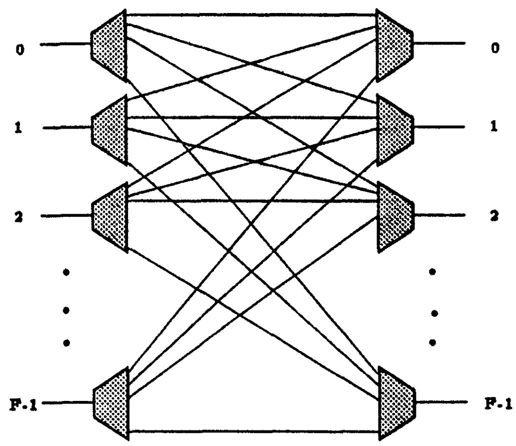

Figure 4: Standard implementation of a Latin Router

This wavelength assignment is also shown in Fig. 1. However if N/p $p \times p$ star couplers are used instead, the transceivers can be grouped such that only p^2 wavelengths are used. This is shown in Fig. 2 for the 8 node example of Fig. 1. Now, if the N/p star couplers are replaced by Latin Routers, only p wavelengths are needed, see Fig. 3. This last fact has been previously reported by [7].

A well known design for a Latin Router is the 2-stage network shown in Fig. 4. The first stage consists of F frequency demultiplexers of size $1 \times F$ each. The second stage consists of F multiplexers of size $F \times 1$ each. The demultiplexers separate the wavelengths $\{0, 1, ...F-1\}$ on each input fiber onto a unique output fiber. The interconnection between the stages consists of F^2 fibers connecting each of the F demultiplexers to each of the F multiplexers in the final stage. By properly choosing the output and input ports of the demultiplexers and multiplexers any Latin Router can be implemented. The design uses $2F$ devices and F^2 interconnections. In addition, as F grows the size of each device grows. For large F, this is not feasible.

For the rest of this paper, we use the terms *device* to mean a wavelength routing device and *network* to mean an interconnection of *devices*. Our goal is to design large Latin Routing networks from relatively few small devices. In addition, we will attempt to minimize the number of interconnections. The following section describes the devices we will use and generalizes the problem. Section 3 describes the network topologies, and then presents some necessary and sufficient conditions on the devices for the network to be a Latin Router. Two designs, the *Coarse/Fine* and *Vernier* are presented in section 4. An example is presented in section 5. In section 6, we extend our results to networks with devices which operate on time/frequency slots. After we have established the conditions under which the networks are Latin Routers, we analyze, in section 7, the number of components and interconnections required for our design and show that the number of components can be made $<< F$, with devices of size $<< F \times F$, and only F interconnections.

2 Wavelength Routing Devices

We consider optical systems where the bandwidth is divided into frequency bands numbered by non-negative integers. It is more convenient at this stage to assume an infinite number of wavelengths. There is certainly no loss of generality since the number of wavelengths can always be limited after the network has been designed.

Recall that an $N \times N$ wavelength routing device, or an all optical network with passive wavelength routing, can be specified by an $N \times N$ matrix S, where the $(i,j)^{th}$ element, $S(i,j)$, is the set of wavelengths connecting input i to output j. Let $J(i,f)$ be the set of output ports reachable from input port i on wavelength f. Similarly, let $I(j,f)$ be the set of input ports that can reach output j on f. Specifically,

$$J(i,f) = \{j : f \in S(i,j)\} \quad (1)$$

$$I(j,f) = \{i : f \in S(i,j)\} \quad (2)$$

We say that the device or network is *connected* if each input can reach every output on some wavelength, i.e. $|S(i,j)| \geq 1$ for all (i,j). When $|J(i,f)| = 0$, we say that f is blocked at i. If $|J(i,f)| > 1$ then input i can reach more than one output on a wavelength f. In this case we say that f is *split* at i. Similarly, if two inputs can reach the same output j on f, then we say that f is *combined* at j, i.e. $|I(j,f)| > 1$.

A Latin Router is a device for which each input is connected to each output on exactly 1 wavelength, $|S(i,j)| = 1$ for all (i,j), and there is no splitting, combining, or blocking of any wavelength $f \in [0,F)$. Such a situation provides full connectivity between the inputs and outputs with the maximum re-use of wavelengths. Since there is no splitting and since $|S(\cdot,\cdot)| = 1$, it follows that there are F inputs and outputs.

DEF: 1 Latin Router :

An $F \times F$ wavelength routing network, or device, satisfying

1) $|S(i,j)| = 1$, $\forall (i,j) \in [0,F)^2$

2) $|J(i,f)| = |I(i,f)| = 1$, $\forall i \in [0,F), f \in [0,F)$

Equivalently, the matrix S is an $F \times F$ latin square.

Consider a Latin Router built by interconnecting smaller wavelength routing devices. Splitting within a device of the network is undesirable because only one output of the network is connected to any input of the network on any wavelength in a Latin Router. Therefore any split signal would either have to be recombined or blocked from reaching all but one output.[3] Devices without splitting, combining, or blocking are called *pure* devices. In theory, pure devices can be lossless even if they are single mode.

DEF: 2 Pure Wavelength Routing Device:

A device, S, is a **pure** *wavelength routing device if*

$$|J(i,f)| = |I(j,f)| = 1 \text{ , for all } i,j, \text{ and } f \tag{3}$$

That is, wavelengths are never split, combined, or blocked.

A device is *periodic* if there is an integer P such that $f \in S(i,j)$ implies $f + nP \in S(i,j)$ for all integers n. The period of the device is defined to be the smallest integer P for which the device is periodic. By convention we consider the set of wavelengths $[nP,(n+1)P)$ to be the n^{th} period. We limit our discussion to a particular kind of periodic pure device. The devices are described below. Our motivation for using this type of device is two-fold: the devices are practical [8, 9] and are easily described.

DEF: 3 Periodic Latin Router, (N,C,L): *A periodic latin router is completely specified by the triplet (N,C,L) where N is the device size, C is a positive integer, and L is an $N \times N$ latin square. Also,*

$$S(i,j) = \left\{ f \mid \left\lfloor \frac{f}{C} \right\rfloor \equiv L(i,j) \pmod N \right\} \tag{4}$$

From eqn. (4), the device is periodic with period $P = NC$. Notice that if $C = 1$ and $N = F$, the device is a Latin Router (def. 1). C represents the size of the passband and is called the *coarseness* of the device. When $C > 1$, we say the device is *coarse.* If $C = 1$, we say the device is *fine.*

[3]The first property is also undesirable since it would lead to multi-path.

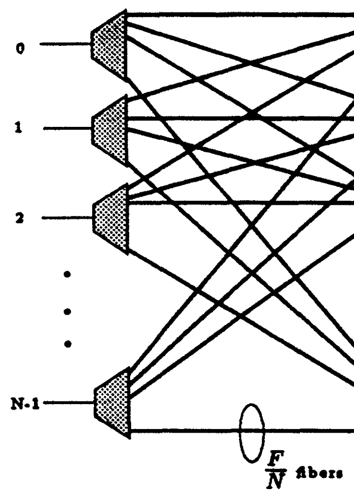

Figure 5: Periodic Latin Router Design

One possible implementation of a Periodic Latin Router is to use a design similar to Fig. 4. This design is shown in Fig. 5. Here, N periodic frequency demultiplexers and N periodic frequency multiplexers are used in the first and second stages, respectively. Each have a period of NC. The demultiplexers (multiplexers) are each of size $1 \times NC$ ($NC \times 1$). Each demultiplexer is connected to each multiplexer with C fibers instead of 1. The design requires $2N$ components and N^2C connections. We are not proposing this design, just pointing out that the device assumptions are both feasible and more practical than Fig. 4. The design can probably be made more compact.

A simple example of a fine router is a Mach Zehnder interferometer when the channels are spaced to fall in the peaks and nulls of the frequency response. The Mach Zehnder is a 2×2 device with power frequency response $H(0,0) = H(1,1) = \cos^2(\frac{\pi}{2}f)$ and $H(0,1) = H(1,0) = \sin^2(\frac{\pi}{2}f)$, where $H(i,j)$ is the power frequency response from input i to output j. Therefore, the Mach Zehnder is a Periodic Latin Router characterized by $N = 2$, $C = 1$, $L(1,1) = L(0,0) = 0$, and $L(0,1) = L(1,0) = 1$. The matrix S is given by

$$S(1,1) = \quad S(0,0) \quad = \{0,2,4,...\} \tag{5}$$

$$S(1,0) = \quad S(0,1) \quad = \{1,3,5,...\} \tag{6}$$

The Mach Zehnder can be generalized to create an $(N,1,L)$ Periodic Latin Router. Such devices have been demonstrated by [8, 9] for various values of N, $N \leq 20$. The devices are compact as they can be integrated onto silicon. Similar devices have also been demonstrated by [10] and

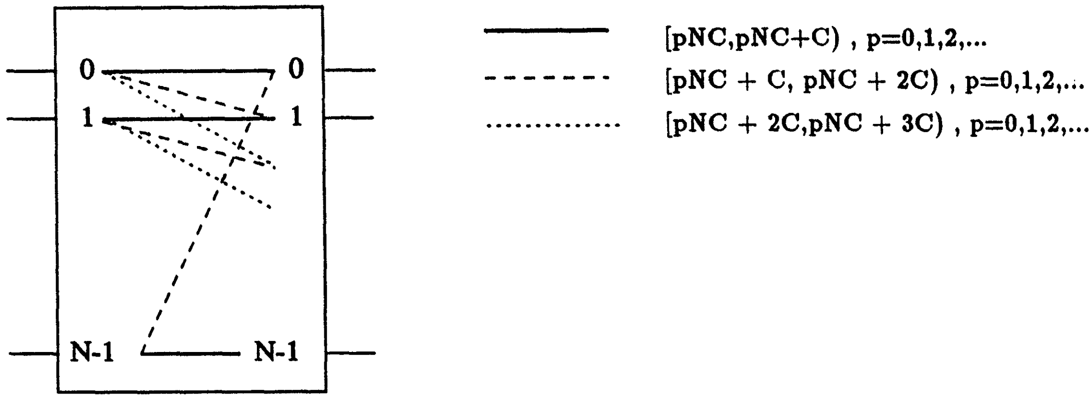

Figure 6: Periodic Diagonal Latin Router

[11] using a grating technique; however the grating frequency response limits the periodicity. Note that these devices cannot be used as an (N, C, L) device with channel spacing reduce by $1/C$ since the closer channels will not necessarily fall in the peaks and nulls of the frequency response of the $(N, 1, L)$ device.

If the latin square describing the Periodic Latin Router has diagonals with constant elements and the first row is $L(0, i) = i$ (see table 1a), then we say the device is *diagonal.* The Mach Zehnder and generalized Mach Zehnder are examples of diagonal Periodic Latin Routers. For a diagonal Periodic Latin Router,

$$L(i,j) = (j - i) \bmod N \quad , \forall(i,j) \in [0,N)^2 \tag{7}$$

$$J(i,f) = (i + \delta(f)) \bmod N \quad , \forall(i,j) \in [0,N)^2 \tag{8}$$

$$I(j,f) = (j - \delta(f)) \bmod N \quad , \forall(i,j) \in [0,N)^2 \tag{9}$$

where

$$\delta(f) = \left\lfloor \frac{f}{C} \right\rfloor \bmod N \tag{10}$$

This is shown pictorially in Fig. 6. We call $\delta(f)$ the *deflection* of wavelength f. Notice that wavelengths $0, 1, ...C - 1$ pass through the device with 0 deflection, $[C, 2C)$ pass through with 1 deflection, etc.

In the following sections we will be considering networks of Perodic Latin Routers. For such networks, it may not be possible to express S explicitly. However, we are interested in finding networks for which S is a Periodic Latin Router, i.e. there exists an (N, C, L) such that S is described by eqn. (4). The following theorem gives three conditions, which if all satisfied, insure that S is a Periodic Latin Router.

Theorem 1 *An $N \times N$ device, or network, S, is a Periodic Latin Router with coarseness C iff the following 3 conditions are satisfied*

1. *S is pure*
2. *S is periodic with period NC*
3. *$S(i,j) \bigcap [0, NC)$ has at least C contiguous elements. Specifically, for each $(i,j) \in [0,N)^2$*

$$\exists f(i,j)\,, \quad \textit{such that} \quad [f(i,j), f(i,j)+C) \subseteq S(i,j) \bigcap [0, NC) \tag{11}$$

If the conditions are satisfied, then S is a periodic Latin Router described by (N, C, L) where $L(i,j) = \frac{f(i,j)}{C}$.

Proof: If S is a Periodic Latin Router, (N, C, L), then by definition

$$\begin{aligned} S(i,j) &= \left\{ f \mid \left\lfloor \frac{f}{C} \right\rfloor \equiv L(i,j) \pmod N \right\} &(12)\\ &= \left\{ f \mid \left\lfloor \frac{f \bmod NC}{C} \right\rfloor = L(i,j) \right\} &(13)\\ &= \bigcup_{p \in Z} [pNC + L(i,j)C\,,\ pNC + L(i,j)C + C) &(14) \end{aligned}$$

It follows immediately that S must satisfy the three conditions.

S is an $N \times N$ Periodic Latin Router with coarseness C iff $\exists$ an $N \times N$ latin square L, such that S is described by eqn. (4). We show that 1.,2., and 3. imply that S is given by eqn. (4) with $L(i,j) = \frac{f(i,j)}{C}$ and that L is a latin square. Define $S_o(i,j) = S(i,j) \bigcap [0, NC)$ to be the restriction of S to the first period. From 3.,

$$[L(i,j)C + b(i,j), L(i,j)C + b(i,j) + C) \subseteq S_o(i,j) \quad , \forall (i,j) \tag{15}$$

where $b(i,j) = f(i,j) \bmod C$. First we show that $S_o(i,j)$ is given by the left hand side of eqn. (15). To do this we show that $|S_o(i,j)| = C$ for all (i,j). Clearly, $|S_o(i,j)| \geq C$. Suppose $|S_o(i,j)| > C$ for some (i,j). Since the device is pure, each wavelength between 0 and $NC - 1$ reaches a single output $J(i,f)$. There are N possible outputs and NC

wavelengths in the first period. The average number of wavelengths per output is C. Since $|S_o(i,j)| > C$, $|S(i,j')| < C$ for some j'. Since $|S_o(i,j)| \geq C$ for all (i,j) it follows that $|S_o(i,j)| = C$ for all (i,j). Therefore,

$$S_o(i,j) = [L(i,j)C + b(i,j), L(i,j)C + b(i,j) + C) \tag{16}$$

Therefore, all that remains to be shown is that $b(i,j) = 0$ for all (i,j) and that L is a latin square. First we show that $b(i,j) = 0$, i.e. that $f(i,j)$ is divisible by C for all (i,j). To do this, consider the wavelengths between 0 and $NC - 1$ not in $S_o(i,j)$, i.e.

$$[0, L(i,j)C + b(i,j)) \bigcup [(L(i,j)+1)C + b(i,j), NC) \tag{17}$$

These wavelengths, in groups of C, make up the $N-1$ sets $S_o(i,j')$, for $j' \neq j$. There are at most

$$\left\lfloor \frac{L(i,j)C + b(i,j)}{C} \right\rfloor = L(i,j) \tag{18}$$

groups of C in the interval $[0, L(i,j)C + b(i,j))$ and at most

$$\left\lfloor \frac{(N - L(i,j) - 1)C - b(i,j)}{C} \right\rfloor = \begin{cases} N - L(i,j) - 1 & \text{if } b(i,j) = 0 \\ N - L(i,j) - 2 & \text{else} \end{cases} \tag{19}$$

groups of C in the other interval. Since there must be a total of $N-1$ groups of C, $b(i,j) = 0$. So $S_o(i,j)$ is given by

$$\begin{aligned} S_o(i,j) &= [L(i,j)C, L(i,j)C + C) & (20) \\ &= \left\{ f \mid \left\lfloor \frac{f}{C} \right\rfloor = L(i,j) \right\} & (21) \end{aligned}$$

And since S is periodic with period NC,

$$\begin{aligned} S(i,j) &= \left\{ f \mid \left\lfloor \frac{f \pmod{NC}}{C} \right\rfloor = L(i,j) \right\} & (22) \\ &= \left\{ f \mid \left\lfloor \frac{f}{C} \right\rfloor \equiv L(i,j) \pmod{N} \right\} & (23) \end{aligned}$$

Now we show that L is a latin square. First notice that $f(i,j) \neq f(i,j')$ for $j \neq j'$ since the device is pure. Similarly $f(i,j) \neq f(i',j)$ for $j \neq j'$. Also, since $[f(i,j), f(i,j)+C) \subseteq S_o(i,j)$, it follows that $0 \leq f(i,j) \leq (N-1)C$. Therefore, if C divides $f(i,j)$ for all (i,j), L is a latin square. Since $b(i,j) = 0$ for all (i,j), C divides $f(i,j)$ and L is a latin square. □

This theorem greatly simplifies proving if a given network is a Periodic Latin Router since it is only necessary to show conditions 1.,2., and 3. In particular, we need not show that the C contiguous elements are the only elements of the first period in $S(i,j)$. Nor must we explicitly show that L is a latin square.

3 Networks of Wavelength Routing Devices

The topology of the network is specified by the directed graph $(\mathcal{N}, \mathcal{A})$, where the set of nodes, $\mathcal{N}$, are the devices, network inputs, and network outputs. The set of arcs, $\mathcal{A}$, are the interconnections between the devices, network inputs and network outputs. For simplicity of notation, we define our network such that all routing functions are performed within the devices. That is,

DEF: 4 Network: *An interconnection of wavelength routing devices such that*

1. *Each device output is either a network output, or is connected to exactly one device input.*
2. *Each device input is either a network input, or is connected to exactly one device output.*

These conditions do not restrict the networks we can consider because multiple or no connections can be included in the device descriptions. An intuitively obvious result is that a network of pure devices has no splitting, combining, or blocking of any wavelength. That is, a network of pure devices is a pure network. This is shown in Appendix A.

Since we are designing Periodic Latin Routers, and since Periodic Latin Routers are pure devices, it is natural to only consider networks of pure devices. In particular, we consider networks of Periodic Latin Routers. We are interested in finding the conditions for which the networks are Periodic Latin Routers. The topologies we consider are multi-stage interconnection networks (MIN), where stage k consists of M_k, $N_k \times N_k$, devices. The outputs of stage k are connected only to the inputs of stage $k+1$. For simplicity of notation, the inputs to the network are considered to be the outputs of stage 0 and the outputs to be the inputs of stage $n+1$, where

n is the number of device stages. From def. 4, each output of a stage k device is connected to a unique input of a stage $k+1$ device and vice-versa. It follows that there are exactly $M_k = N/N_k$ devices in stage k, where N is the size of the network (the number of inputs and outputs).

We restrict our attention to MINs with the unique path property (UPMIN). That is for each input $i \in [0, N)$ and each output $j \in [0, N)$ there is a unique path connecting i and j in the graph $(\mathcal{N}, \mathcal{A})$. For N a power of 2, a UPMIN of 2×2 Mach Zehnders can be used to make a Periodic Latin Router. This is an easy generalization of the system considered [12] for demultiplexing. Stage k of the design consists of $\frac{N}{2}$, 2×2, Mach Zehnders, each with a different period.

In our design, all devices in stage k are identical. In this case, a UPMIN built with Periodic Latin Routers is completely described by the topology $(\mathcal{N}, \mathcal{A})$, and (N_k, C_k, L_k) for $k = 1, 2, ..., n$ where n is the number of stages and (N_k, C_k, L_k) describe the routers in stage k.

Let p_{ij} be the unique path connecting input i to output j. Also, let $\text{in}_k(p_{ij})$ be the unique input port to the stage k device used in p_{ij}. Similarly let $\text{out}_k(p_{ij})$ be the unique output port to the stage k device used in p_{ij}. The set of wavelengths connecting i to j is given by

$$S(i,j) = \bigcap_{k=1}^{n} S_k(\text{in}_k(p_{ij}), \text{out}_k(p_{ij})) \tag{24}$$

where $S_k(m,l)$ is the set of wavelengths connecting input m to output l for a stage k device. Using eqn. (4), $S(i,j)$ is given by

$$S(i,j) = \left\{ f \mid \left\lfloor \frac{f}{C_k} \right\rfloor \equiv d_k(p_{ij}) \pmod{N_k} \quad , \ \forall k = 1, 2, ..., n \right\} \tag{25}$$

where $d_k(p_{ij}) = L_k(in_k(p_{ij}), out_k(p_{ij}))$. We are interested in the conditions on the topologies and devices for which eqn. (25) defines a Periodic Latin Router. Since Periodic Latin Routers are pure devices, any UPMIN of Periodic Latin Routers is a pure network by theorem 4 in Appendix A. So all that remains is to find the conditions on $(\mathcal{N}, \mathcal{A})$ and $\{(N_k, C_k, L_k) \mid k = 1, 2, ..., n\}$ such that conditions 2. and 3. of theorem 1 are satisfied. Using the structure of the UPMIN and the definition of Periodic Latin Routers, we get the following equivalent conditions.

Theorem 2 *An n-stage UPMIN of Periodic Latin Routers $\{(N_k, C_k, L_k) \mid k = 1, 2, ..., n\}$ with topology $(\mathcal{N}, \mathcal{A})$ is a Periodic Latin Router of size N and coarseness C iff all of the following are satisfied:*

1. $N = \prod_{k=1}^{n} N_k$

2. $C = \min\{C_1, C_2, ..., C_n\}$

3. $NC = \text{lcm}(N_1C_1, N_2C_2, ..., N_nC_n)$*, where* lcm *stands for least common multiple*

4. $\gcd(C_1, C_2, ..., C_n) = \min\{C_1, C_2, ..., C_n\}$*, where* gcd *stands for greatest common divisor*

5. *There exists an* $f(\mathbf{d})$ *for each* $\mathbf{d} \in [0, N_1) \times [0, N_2) \times ... \times [0, N_n)$ *such that*

$$\left\lfloor \frac{f(\mathbf{d})}{C_k} \right\rfloor = d_k \quad , k = 1, 2, ..., n \tag{26}$$

If 1.-5. are satisfied, then the network is an (N, C, L) *Periodic Latin Router with*

6. $L(i,j) = \left\lfloor \frac{f(\mathbf{d}_{ij})}{C} \right\rfloor$*, where* $\mathbf{d}_{ij} = (d_1(p_{ij})d_2(p_{ij})...d_n(p_{ij}))$*.*

Proof: First we show that the conditions are necessary.

To see 1.: Notice that the right hand side is the number of paths leading out from any user i and the left hand side is the number of outputs. The result follows from the unique path property.

To see 2.: Consider the wavelengths $[0, C)$ from any input. These wavelengths all reach the same output. By the unique path property, each wavelength must follow the same path. Therefore, $\min\{C_1, ..., C_n\} \geq C$. If $C < \min\{C_1, ..., C_n\}$, then the $C+1$ wavelengths $f = 0, 1, 2, ...C$ would have the same path for any input, so $C = \min\{C_1, C_2, ..., C_n\}$.

To see 3.: By definition, NC is the smallest period of the Periodic Latin Router. Therefore, each device period must divide NC. The smallest such number is $\text{lcm}(N_1C_1, ..., N_nC_n)$.

To see 4.: If C does not divide C_k for some k, then there exists an integer a such that the wavelengths $[aC, (a+1)C)$ are divided in some device. Specifically, let $a = \lfloor C_k/C \rfloor$, $f = aC$, and $f' = C_k$. Then $J_k(i, f) \neq J_k(i, f')$ for any i since $\lfloor f/C_k \rfloor = 0$ and $\lfloor f'/C_k \rfloor = 1$. Therefore f and f' take different paths through the network from any input. This is a contradiction.

To see 5.: From theorem 1 and eqn. (25), if S is a Periodic Latin Router then

$$\left\lfloor \frac{f}{C_k} \right\rfloor \equiv d_k(p_{ij}) \pmod{N_k} \tag{27}$$

has at least C contiguous solutions in $[0, NC)$. Define $\mathbf{d}_{ij} = (d_1(p_{ij}), d_2(p_{ij}), ..., d_n(p_{ij}))$. $\mathbf{d}_{ij} \neq \mathbf{d}_{ij'}$ since $p_{ij} \neq p_{ij'}$ and hence p_{ij} and $p_{ij'}$ split in some device. Therefore, for each

i, $\{\mathbf{d}_{ij} \mid j = 0, 1, ...N-1\} = \{(d_1, d_2, ..., d_n) \mid d_k \in [0, N_k)\}$. 5. is necessary since eqn. (26) must have at least C contiguous solutions for each $\mathbf{d}$, so trivially it must have at least 1 solution.

Now we show sufficiency.

From theorem 1, the network is a Periodic Latin Router if S is pure, periodic with period NC, and

$$\exists f(i,j) \text{ s.t. } [f(i,j), f(i,j)+C) \in S_o(i,j) \ \forall (i,j) \tag{28}$$

where $S_o(i,j) = S(i,j) \bigcap [0, NC)$. S is pure since it is a network of pure devices. It is periodic since all the devices are periodic. So S is a Periodic Latin Router iff the period is NC, i.e. iff $NC = \text{lcm}(N_1C_1, N_2C_2, ..., N_nC_n)$, and for all (i,j), there are at least C contiguous solutions in $[0, NC)$ to the system of congruences

$$\left\lfloor \frac{f}{C_k} \right\rfloor \equiv d_k(p_{ij}) \pmod{N_k} \quad , \quad k = 1, 2, ..., n \tag{29}$$

where $d_k(p_{ij}) = L_k(in_k(p_{ij}), out_k(p_{ij}))$ as before.

If $f(\mathbf{d})$ is a solution to eqn. (26), then so is $\lfloor f(\mathbf{d})/C \rfloor C$ since C_k is a multiple of C for each k. Also since C_k is a multiple of C,

$$\left\lfloor \frac{\left\lfloor \frac{f(\mathbf{d})}{C} \right\rfloor C + l}{C_k} \right\rfloor \equiv d_k \pmod{N_k} \tag{30}$$

for $l = 0, 1, ..., C-1$. So if eqn. (26) has one solution for each $\mathbf{d}$, it has C contiguous solutions for each $\mathbf{d}$. Since for any (i,j), $d_k(p_{ij}) \in [0, N_k)$, if eqn. (26) has at least C contiguous solutions in $[0, NC)$ for any $\mathbf{d}$, then eqn. (29) has at least C contiguous solutions in $[0, NC)$.

Now 6. follows from the last statement of theorem 1. □

Therefore, in order to determine if an n-stage UPMIN of Periodic Latin Routers is a Periodic Latin Router, only the device specifications are needed. That is, the important parameters are contained in the set $D = \{(N_k, C_k) \mid k = 1, 2, ...n\}$. The topology is irrelevant as long as it is a UPMIN and the matrices $L_1, L_2, ..., L_n$ can be arbitrary latin squares. We call D the design. The system of equations are independent of any ordering, so we will assume without loss of generality that $C_1 \geq C_2 \geq \ldots \geq C_n$. Let $\mathcal{D}$ be the family of designs that produce a Periodic Latin Router (i.e. those D which satisfy the first five conditions in theorem 2).

In Appendix B, we show that for a 2-stage design, conditions 1.-4. imply condition 5. This enables us to determine all 2-stage designs. For $n > 2$, conditions 1.-4. are not sufficient. Two counter examples are presented in Appendix B.

Now we are led to the conclusion that all $(N, 1, L)$ designs are (N, C, L) designs and vice-versa. In other words, if a UPMIN of Periodic Latin Routers is a Periodic Latin Router with coarseness C, then the network formed by dividing the coarseness of all devices by C is a Periodic Latin Router with coarseness 1. Similarly, if a UPMIN of Periodic Latin Routers is a Periodic Latin Router with coarseness 1, then the network formed by multiplying the coarseness of all devices by C is a Periodic Latin Router with coarseness C. This is shown in the following corollary

Corollary 3 $D = \{(N_k, C_k)\} \in \mathcal{D}$ *iff* $D' = \{(N_k, CC_k)\} \in \mathcal{D}$.

Proof: Notice that

$$\left\lfloor \frac{f}{C_k} \right\rfloor \equiv d_k \pmod{N_k} \Rightarrow \left\lfloor \frac{fC}{C_k C} \right\rfloor \equiv d_k \pmod{N_k} \tag{31}$$

Also that

$$\left\lfloor \frac{f}{CC_k} \right\rfloor \equiv d_k \pmod{N_k} \Rightarrow \left\lfloor \frac{\left\lfloor \frac{f}{C} \right\rfloor C}{C_k C} \right\rfloor \equiv d_k \pmod{N_k} \Rightarrow \left\lfloor \frac{\left\lfloor \frac{f}{C} \right\rfloor}{C_k} \right\rfloor \equiv d_k \pmod{N_k} \tag{32}$$

□

4 Designs

Two n-stage solutions are the Coarse/Fine solutions and the Vernier Solutions. These are described in the following two sections. All 2-stage designs are determined in Appendix B. In addition, Appendix B presents an efficient algorithm to determine if a given design is a Periodic Latin Router.

4.1 Coarse/Fine Periodic Latin Router

The n-stage Coarse/Fine design is valid for any N that can be factored into n inte an N and n, write N as $N_1 N_2 ... N_n$, where N_k is the device size in stage k. Let C_n C_k equal to the period of the next stage, specifically let $C_k = C_{k+1} N_{k+1}$ for $k =$ Since $N = N_1 N_2 ... N_n$, it follows that

$$C_k = \frac{P}{N_1 N_2 ... N_k} \quad , \text{ for } k = 1, 2, ..., n$$

where $P = NC$. Each $f \in [0, NC)$ has a unique expansion,

$$\begin{aligned} f &= a + \sum_{k=1}^{n} a_k C_k \\ &= a + \sum_{k=1}^{n} a_k C (N_{k+1} N_{k+2} ... N_n) \end{aligned}$$

where $a < C$. For instance, if $C = N_k = 2$ for all k then $a_n a_{n-1} ... a_1$ is the binary expansi f.

It is an easy matter to check that 1.-4. of theorem 2 are satisfied. All that remains to sl is that for any $\mathbf{d}$, a solution to the system of congruences is

$$f(\mathbf{d}) = \sum_{k=1}^{n} d_k C_k \quad , \text{ for } l = 0, 1, ... C - 1 \qquad (3$$

To see this notice that

$$\left\lfloor \frac{f(\mathbf{d})}{C_m} \right\rfloor = d_m + \sum_{i=1}^{m-1} d_i \frac{C_i}{C_m} + \left\lfloor \sum_{i=m+1}^{n} d_i \frac{C_i}{C_m} \right\rfloor \qquad (37)$$

$$= d_m + \sum_{i=1}^{m-1} d_i \frac{C_i}{C_m} \qquad (38)$$

since

$$\sum_{i=m+1}^{n} d_i \frac{C_i}{C_m} \leq \sum_{i=m+1}^{n} \frac{d_i}{N_i} \frac{1}{2^{i-m-2}} < 1 \qquad (39)$$

Now taking both sides of eqn. (38) modN_m, we get the desired result because

$$\frac{C_i}{C_m} = N_{i+1}N_{i+2}...N_m \quad \text{for } i < m. \tag{40}$$

4.2 Vernier Latin Router

The n-stage Vernier design is valid only for those N such that $N = N_1N_2...N_n$ and the N_k are relatively prime. For those N and n, pick $C_k = C$ for $k = 1, 2, ...n$ and let N_k be the device size in stage k. Again it is trivial to check that 1.-4. of theorem 2 are satisfied. Given $\mathbf{d}$, f must satisfy

$$\left\lfloor \frac{f}{C} \right\rfloor \equiv d_k \pmod{N_k} \quad , \text{for } k = 1, 2, ...n \tag{41}$$

Let $x = \left\lfloor \frac{f}{C} \right\rfloor$. By the Chinese Remainder Theorem [13], each n-tuple has a unique solution $x(\mathbf{d})$ in $[0, N)$. This solution can be found by solving for a in

$$\frac{N}{N_k}a \equiv d_k \pmod{N_k} \tag{42}$$

for each $k = 1, 2, ...n$. Let a_k be the solution to the k^{th} congruence. Then $x(\mathbf{d})$ is given by

$$x(\mathbf{d}) = \sum_{k=1}^{n} a_k \frac{N}{N_k} \pmod{N} \tag{43}$$

Thus $f(\mathbf{d}) = x(\mathbf{d})C$ is a solution to eqn. (41).

5 A Useful Example

Although the sufficiency conditions from theorem 2 are independent of the UPMIN topology, $(\mathcal{N}, \mathcal{A})$, this section will describe a particularly useful n-stage topology for which the mapping from (i, j) to the vector $\mathbf{d}$ is simple.

Since $N_1N_2...N_{k-1}N_{k+1}...N_n = N/N_k$, we can uniquely label the $M_k = N/N_k$ devices in stage k by the modified n-tuple $(i_1, i_2, ..., i_{k-1}, *, i_{k+1}, ..., i_n)$ for $i_l \in [0, N_l)$, $l = 1, 2, ...k-1, k+1, ...n$. Label the i^{th} input to device $(i_1, i_2, ..., i_{k-1}, *, i_{k+1}, ..., i_n)$ by replacing the $*$ with i. Since $i \in [0, N_k)$, there is exactly one n-tuple per input (output). An arbitrary device in stage k is shown in Fig. 7.

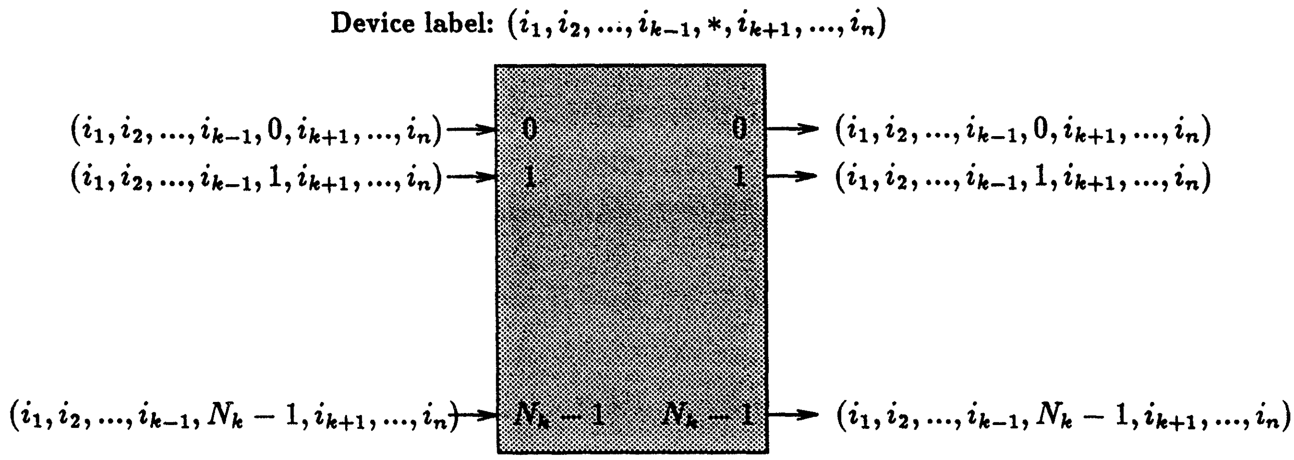

Figure 7: Single device of stage k

Connect each output of stage k to the unique input of stage $k+1$ with a matching label, for $k = 1, 2, ...n-1$. A 2-stage example is shown in Fig. 8 and redrawn in Fig. 9 with $N_1 = 5$ and $N_2 = 3$. This design seems very attractive for packaging relatively small size routers. A 3-stage example, with the device labels shown, for $N_1 = 2$, $N_2 = 3$, and $N_3 = 2$ is drawn in Fig. 10.

The inputs and outputs of each device can be thought of as lying on the integer points in n dimensional rectangle with sides $[0, N_k - 1]$ in the k^{th} dimension. A line of the rectangle in the k^{th} dimension is defined to be $(i_1, i_2, ..., i_{k-1}, i, i_{k+1}, ..., i_n)$ where i varies from 0 to $N_k - 1$. For instance, if $n = 2$, a line is a column or row. Since there are N points and N_k points per line in the k^{th} dimension, the solid is divided into a total of N/N_k lines. Let $i = (i_1, i_2, ..., i_n)$ and $j = (j_1, j_2, ..., j_n)$. Then the unique path from i to j is the Manhattan walk where the k^{th} stage changes only the k^{th} component. Therefore,

$$\mathrm{in}(p_{ij})_k = i_k \tag{44}$$

$$\mathrm{out}(p_{ij})_k = j_k \tag{45}$$

so that $d_k(p_{ij}) = L(i_k, j_k)$. Therefore, for the Coarse/Fine Latin Router, the C wavelengths connecting input i to output j in $[0, NC)$ are

$$f = \sum_{k=1}^{n} L(i_k, j_k)C_k + l \quad , \; l = 0, 1, ..., C-1 \tag{46}$$

For the Vernier, the C wavelengths are $x + l$, $l = 0, 1, ..., C-1$, where x is the unique solution

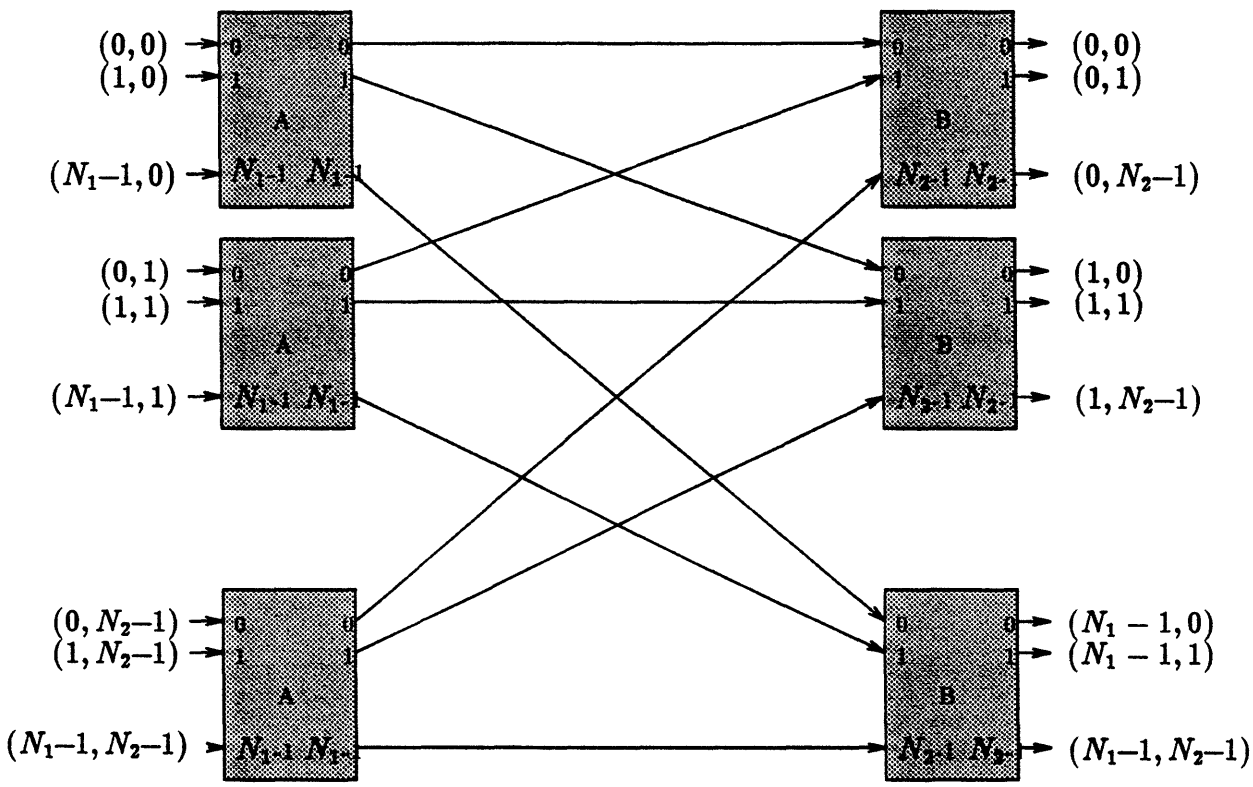

Figure 8: 2 Stage Design

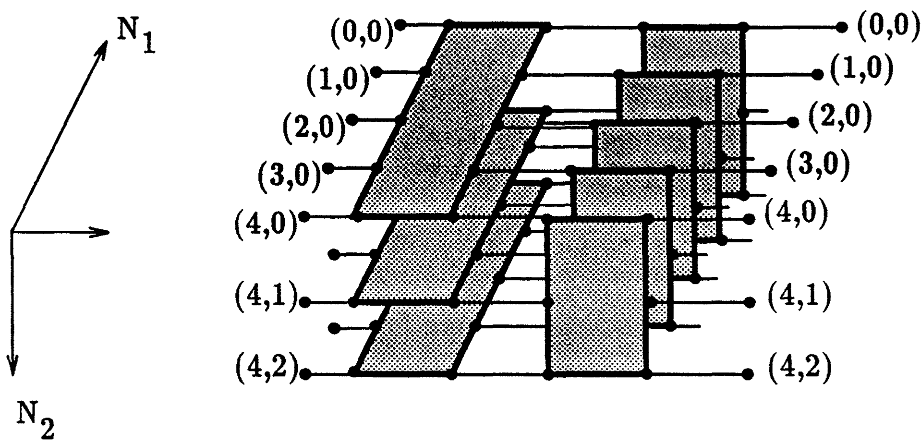

Figure 9: 2-stage design, $N_1 = 5, N_2 = 3$

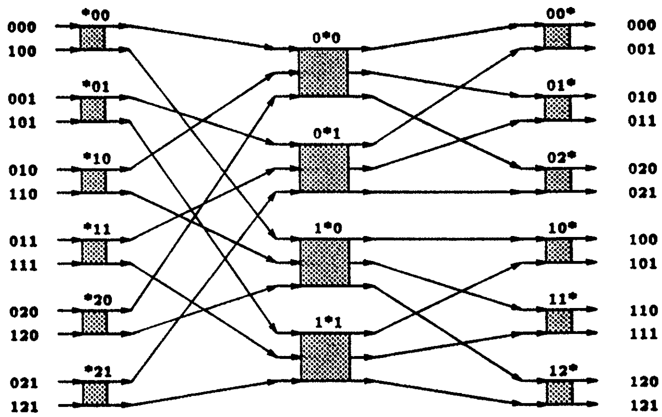

Figure 10: 3-stage design,$N_1 = 2, N_2 = 3, N_3 = 2$

(mod N) to

$$x \equiv L(i_k, j_k) \pmod{N_k} \quad , k = 1, 2, \ldots n \tag{47}$$

Recall that a Periodic Latin Router is diagonal if $L(i,j) = j - i \pmod{N}$. If all the devices in the network are diagonal, then eqns. (46) and (47) simplify to

$$f = \sum_{k=1}^{n} ((j_k - i_k) \bmod N_k) C_k + l \quad , l = 0, 1, \ldots, C-1 \tag{48}$$

$$x \equiv j_k - i_k \pmod{N_k} \quad , k = 1, 2, \ldots n \tag{49}$$

6 TDM Implementation

Let $0, 1, \ldots T-1$ represent periodic time slots. All the definitions made in this paper can be generalized to time slots, or for that matter time/frequency slots. Call each time/frequency slot a channel. A *channel routing device* is described by a matrix S, where $S(i,j)$ is the set of channels connecting input i to output j. The definitions of splitting, combining, Latin Router, and Periodic Latin Router immediately generalize. We have so far assumed a frequency implementation. However, it should be kept in mind that except for the specific devices used to implement the design, everything generalizes to channels. A particularly interesting design for a time/frequency slot Periodic Latin Router is the 2-stage design, see Figs. 8 and 9, where the first stage devices are wavelength Periodic Latin Routers and the second stage devices are time slot Periodic Latin Routers.

7 Number of Components and Connections

Recall that there are $M_k = N/N_k$ devices in stage k and there are exactly N interconnections between each stage. The maximum device size, total number of devices, and total number of fibers (interconnections) used are

$$\begin{aligned} \text{Size} &= \max_k N_k && (50) \\ \text{Devices} &= \sum_{k=1}^{n} \frac{N}{N_k} && (51) \\ \text{Fibers} &= nN && (52) \end{aligned}$$

A lower bound to the number of devices under the constraint $\prod_{k=1}^{n} N_k = N$ is obtained by minimizing eqn. (51) neglecting integer and relative primeness constraints. The minimum is achieved when $N_k = N^{1/n}$ for all k. In this case,

$$\begin{aligned} \text{Size} &= N^{1/n} && (53) \\ \text{Devices} &\geq nN^{1-\frac{1}{n}} = n\frac{N}{\text{Size}} && (54) \\ \text{Fibers} &= nN && (55) \end{aligned}$$

The number of devices grows with n. The minimum occurs with $n = 1$ in which only one $N \times N$ device is needed. For $n = 2$, only about $2\sqrt{N}$ devices are needed, each of size $\sqrt{N}$. This is a vast improvement over the standard design of Fig. 4. As n grows, the number of devices grows and the device size shrinks. Notice that for moderate n, the growth in the number of devices becomes relatively slow, however, the device size shrinks dramatically. Therefore, it may be appropriate to increase the number of devices for the benefit of decreasing device size.

Notice that there are N fibers used in each interconnection stage, or equivalently N number of connections between stages. This is the minimum possible. The reason is each fiber can support at most NC sessions per period, and the Latin Router can support N^2C sessions per period. Therefore, there must be at least N fibers connecting any two stages.

8 Conclusions

It is possible to build large Latin Routers in time and frequency from a relatively small number of practical devices. The UPMIN can be implemented distributively with the minimum growth in the number of interconnections, F per stage, possible. This is a dramatic improvement over the well known design which requires F^2 interconnections. In addition, the device size can be reduced to $F^{1/n} \times F^{1/n}$ where n is the number of stages. This is also a dramatic decrease over the well known 2-stage design (Fig. 4) which requires $1 \times F$ and $F \times 1$ devices. For instance, if $F = 1000$, only around 66, 33×33, devices are needed for a 2-stage UPMIN.

The Latin Routers are built from devices called Periodic Latin Routers. Periodic Latin Routers can, in turn, be build from smaller Periodic Latin Routers. A particularly convenient n-stage topology as well as two designs were presented.

In addition, a computationally efficient algorithm to determine if a given set of device parameters is a design for a Periodic Latin Router is presented in Appendix B.

9 Appendix A

In this section we show that under only the assumptions of def. 4, a network of pure devices is a pure network. The network need not be a UPMIN, and may include loops.

Theorem 4 *A network of pure devices is a pure network.*

> **Proof:** By contradiction. Suppose the network was not pure. Let S describe the network. Then at least one of the following is true,
>
> (1) $\exists$ a row of S that uses a wavelength twice
>
> (2) $\exists$ a column of S that uses a wavelength twice
>
> (3) $\exists$ a frequency f that is blocked from an input i or an output j
>
> **(1) leads to a contradiction:** Consider the row $S(i,*)$. Let f be used twice in the row, i.e. $\exists j, j'$, $j \neq j'$ s.t. $f \in S(i,j)$ and $f \in S(i,j')$. Let p be a path from network input i to network output j. Suppose that p contains $n(p)$ devices. Let $\text{in}_k(p)$ and $\text{out}_k(p)$ be the input and output ports of the k^{th} device in p. The set of wavelengths connecting input i

to output j on p is given by

$$S(p) = \bigcap_{k=1}^{n(p)} S_{p,k}(\text{in}_k(p), \text{out}_k(p)) \tag{56}$$

where $S_{p,k}(m,l)$ is the set of wavelengths connecting input m to output l for the k^{th} device in p. If $f \in S(p)$ we say that p *supports* f. The the set of wavelengths connecting i to j, $S(i,j)$, is the set of frequencies supported by some path between i and j. Therefore, if P_{ij} is the set of paths connecting i and j,

$$S(i,j) = \bigcup_{p \in P_{ij}} S(p) = \bigcup_{p \in P_{ij}} \bigcap_{k=1}^{n(p)} S_{p,k}(\text{in}_k(p), \text{out}_k(p)) \tag{57}$$

Since $f \in S(i,j)$, there must be some path $p \in P_{ij}$ which supports f. Similarly, there must be a path p' from i to j' that support f. Since $p \neq p'$, the two paths must split somewhere. Suppose they split in the k^{th} device common to p and p'. They at least have one common device since input i is connected to at most one device input. Let in_k be the common input port to the k device for p and p'. Also let out_k and out'_k be the distinct output ports of the two paths. Then

$$f \in S_{p,k}(\text{in}_k, \text{out}_k) \bigcap S_{p',k}(\text{in}_k, \text{out'}_k) \tag{58}$$

and since $S_{p,k} = S_{p',k}$, the device is not pure, which is a contradiction. **so (1) is not true**

(2) leads to a contradiction: The proof follows the same line as proof of (1). Consider the column $S(*,j)$. Then $f \in S(i,j)$ and $f \in S(i',j)$ for some $i \neq i'$. Then there exists two paths $p \in P_{ij}$ and $p' \in P_{i'j}$ which must join at some device. So that device is not pure which is a contradiction. **so (2) is not true**

(3) leads to a contradiction: Any blocking must occur within a device since each device output (input) is connected to a device input (output). But each device is pure. **so (3) is not true**

Hence, we have a contradiction. □

10 Appendix B

10.1 General Solutions

We show that for a 2-stage design, conditions 1.-4. imply condition 5. This enables us to determine all 2-stage designs. However, for $n > 2$, conditions 1.-4. are not sufficient. A counter example is $D = \{(2,12),(3,8),(4,1)\}$. In fact any D such that $N_iC_i = N_jC_j$ for $j \neq i$ violates condition 5. Another counter example is $D = \{(2,6),(3,8),(4,1)\}$. Notice that $N_iC_i \neq N_jC_j$ for $j \neq i$.

First we need a lemma which we reproduce here without proof from [13].

Lemma 5 *The congruence*

$$kx \equiv l \pmod{m} \tag{59}$$

has a unique solution $\pmod{m}$ *iff k and m are relatively prime.*

Then the following theorem holds.

Theorem 6 *Let $\mathcal{D}_2$ be the set of 2-stage designs in $\mathcal{D}$. Then $\mathcal{D}_2$ can be explicitly written as*

$$\mathcal{D}_2 = \{(N_1, bC),(ab, C) \mid \gcd(N_1, a) = 1 \ \textit{and} \ b \geq 1\} \tag{60}$$

Proof: Assume without loss of generality that $C_1 \geq C_2 = C$. Now if, $D \in \mathcal{D}_2$, then from 1.-3. of theorem 2

$$N_1N_2C = \mathrm{lcm}(N_1C_1, N_2C) \tag{61}$$

From 4. of theorem 2, $C_1 = bC$ for some $b \geq 1$. Plugging this into eqn. (61) and pulling out the C on both sides of the equation,

$$N_1N_2 = \mathrm{lcm}(N_1b, N_2) \tag{62}$$

so b divides N_2. Write $N_2 = ab$, for some $a \geq 1$. Plugging this into eqn. (62)

$$N_1ab = \mathrm{lcm}(N_1b, ab) \tag{63}$$

So a and N_1 are relatively prime, that is $\gcd(a, N_1) = 1$. We have shown that if $D \in \mathcal{D}_2$, then D is of the form

$$D = \{(N_1, bC), (ab, C) \mid \gcd(N_1, a) = 1\ ,\ b \geq 1\} \tag{64}$$

We now show that if D is of the form given in eqn. (64), then $D \in \mathcal{D}_2$. Or, in other words when $n = 2$, conditions 1.-4. imply condition 5. in theorem 2. Using corollary 3, it is sufficient only to consider the $C = 1$ case. Consider the equations,

$$\left\lfloor \frac{f}{b} \right\rfloor \equiv d_1 \pmod{N_1} \tag{65}$$

$$\lfloor f \rfloor \equiv d_2 \pmod{ab} \tag{66}$$

The second equation has solutions of the form $f = nab + d_2$. Plugging this into the first equation,

$$\left\lfloor \frac{nab + d_2}{b} \right\rfloor = na + \left\lfloor \frac{d_2}{b} \right\rfloor \equiv d_1 \pmod{N_1} \tag{67}$$

So

$$na \equiv d_1 - \left\lfloor \frac{d_2}{b} \right\rfloor \pmod{N_1} \tag{68}$$

which has a solution from the lemma since $\gcd(a, N_1) = 1$. □

Notice that the $a = 1$ designs are the Coarse/Fine designs and the $b = 1$ designs are the Vernier designs.

10.2 Algorithm

Testing whether or not a given D is in $\mathcal{D}$ can be done in $O(N)$ time for any C. The algorithm is trivial and makes use of corollary 3.

Instance: $D = \{(N_k, C_k) : k = 1, 2, ...n\}$

Question: Is $D \in \mathcal{D}$?

Check necessary conditions 1.-4. in theorem 2. If not satisfied, STOP. $D \notin \mathcal{D}$. If satisfied and $n = 2$, stop $D \in \mathcal{D}$. If satisfied and $n \geq 3$, initialize $g(\mathbf{d}) = FALSE$ for all F values of

$\mathbf{d} = (d_1, d_2, ..., d_n)$.

FOR $f = 0, 1, ...N - 1$, DO

Calculate $\mathbf{d}$, where $d_k = \left\lfloor \frac{fC}{C_k} \right\rfloor$

IF $g(\mathbf{d}) = TRUE$ then STOP. $D \notin \mathcal{D}$

ELSE set $g(\mathbf{d}) = TRUE$

NEXT f

The storage requirements of the algorithm are $O(N)$ since there are N possible values of $\mathbf{d}$.

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