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LATTICE INVARIANTS AND THE CENTER OF THE GENERIC DIVISION RING

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ABSTRACT. Let G be a finite group, let M be a ZG-lattice, and let F be a field of characteristic zero containing primitive p^{th} roots of 1. Let $F(M)$ be the quotient field of the group algebra of the abelian group M. It is well known that if M is quasi-permutation and G-faithful, then $F(M)^G$ is stably equivalent to $F(ZG)^G$. Let C_n be the center of the division ring of $n \times n$ generic matrices over F. Let S_n be the symmetric group on n symbols. Let p be a prime. We show that there exist a split group extension G' of S_p by a p-elementary group, a G'-faithful quasi-permutation ZG'-lattice M, and a one-cocycle α in $\operatorname{Ext}^1_{G'}(M, F^*)$ such that C_p is stably isomorphic to $F_\alpha(M)^{G'}$. This represents a reduction of the problem since we have a quasi-permutation action; however, the twist introduces a new level of complexity. The second result, which is a consequence of the first, is that, if F is algebraically closed, there is a group extension E of S_p by an abelian p-group such that C_p is stably equivalent to the invariants of the Noether setting $F(E)$.

INTRODUCTION

Let C_p denote the center of the division ring of $p \times p$ generic matrices over a field F. We study the question of whether C_p is stably rational over F, when p is a prime and F is a field of characteristic zero containing primitive p^{th} roots of 1.

The question of rationality of the center of the generic division ring has been studied extensively, in particular for its connection to important problems in other fields such as geometric invariant theory and Brauer groups.

For the span of more than a century, stable rationality of the center has been shown for the primes 2, 3, 5 and 7, and for 4 ([S], [F1], [F2] and [BL]), with rationality proven for 2, 3 and 4. It was also shown that C_n is retract rational over F for all n square-free, [SD1].

Let G be a finite group and let F be a field. Given a ZG-lattice M, we may form the group algebra $F[M]$ of the abelian group M. There is an action of G on its quotient field $F(M)$ via the G-action on M. If $M = ZG$, then $F(M)^G$ is denoted by $F(G)$, and referred to as the Noether setting of G. The main tool for determining whether $F(M)^G$ is stably rational over F is given by work of Endo, Miyata, Lenstra and Swan, and it is as follows. For any finite group G and for any G-faithful ZG-lattices M and M', the fields $F(M)$ and $F(M')$ are stably isomorphic as F-algebras and the isomorphism respects their G-actions if and only if M and

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M' are in the same flasque class. In particular, if $F(G)^G$ is stably rational over F, then so is $F(M)^G$ for any G-faithful quasi-permutation ZG-lattice M.

Let S_n be the symmetric group on n letters. It was shown in [F1] that C_n is stably isomorphic to the fixed field under the action of S_n of $F(G_n)$, where G_n is a specific ZS_n-lattice which we define below. To determine directly whether the invariant field of $F(G_n)$ is stably rational over F turned out to be quite an intractable problem for primes p greater than or equal to 5. One of the reasons for this intractability is that for $p \geq 5$, G_p is not quasi-permutation [BL]. The main results of this article are as follows. We show that there is a split group extension G' of S_p by a p-elementary group, a G'-faithful quasi-permutation ZG'-lattice M, and a one-cocycle $\alpha \in \mathrm{Ext}^1_{G'}(M, F^*)$ such that C_p is stably isomorphic to $F_\alpha(M)^{G'}$. Furthermore the Noether setting $F(G')$ has stably rational invariants over the base field. This represents a reduction of the problem since we have a quasi-permutation action; however, the twist introduces a new level of complexity. The second result, which is a consequence of the first, is that, if F is algebraically closed, there is a group extension E of S_p by an abelian p-group, such that C_p is stably equivalent to the invariants of the Noether setting $F(E)$. These results are described below.

Section 1 consists mostly of preliminary results and definitions. Let N be the normalizer of a p-Sylow subgroup of S_p. The starting point is our result in [B1], which says that G_p and $ZS_p \otimes_{ZN} G_p$ are in the same flasque class, implying that $F(G_p)$ and $F(ZS_p \otimes_{ZN} G_p)$ are S_p-stably isomorphic. If we let H be a p-Sylow subgroup of S_p, then N is the semi-direct product of H by a cyclic group C, of order $p-1$. We show

Theorem 1.6. *Let h generate H and let I_C denote the augmentation ideal of ZC. The flasque classes of G_p and $ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2$ are equal. Consequently, C_p and $F(ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2)^{S_p}$ are stably isomorphic over F.*

The theorem is a consequence of the decomposition of G_p into indecomposable $\widehat{Z}_pN$-modules, described in [B2], and it represents the first step in the proof of the main theorem.

The second reduction step is done in Theorem 2.2. Let U be the standard rank p permutation representation of S_p, and let A be the kernel of the augmentation map U. We show that there exists a transcendency basis for $F(A)$ on which N acts linearly. This basis plays a crucial role in the proof. It is exhibited in Theorem 2.1.[1] Let M be a ZG-lattice and let $\alpha \in \mathrm{Ext}^1_G(M, F^*)$ be a one-cocycle. We have a new G-action on $F(M)$ via α. Such an action is said to be α-twisted, and the corresponding field is denoted by $F_\alpha(M)$. The lattice $ZS_p \otimes_{ZN} I_C \otimes ZH(h-1)^2$ of Theorem 1.6 embeds into the quasi-permutation ZS_p-lattice $ZS_p \otimes_{ZH} A$, which allows us to express C_p in terms of $ZS_p \otimes_{ZH} A$.

Theorem 2.2. *There exists a finite ZS_p-module P of exponent p, and a γ-twisted action of $G' = P \rtimes S_p$ on $F(ZS_p \otimes_{ZH} A)$ such that the center, C_p, is stably isomorphic to $F_\gamma(ZS_p \otimes_{ZH} A)^{G'}$.*

The key point is that the transcendency basis for $F_\gamma(ZS_p \otimes_{ZH} A)$, obtained by lifting the basis of Theorem 2.1 to S_p, behaves well with respect to the twisting, as is shown in Proposition 2.5 and Theorem 2.6. This basis allows us to shift the twisting from P to S_p, so that the action of P on the field is purely monomial, and

[1]The idea for this basis was given to us by William Chin, to whom we are deeply indebted.

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the action of S_p on the F-span of the basis is F-linear. Furthermore, and this is essential to the argument, the monomial action of P is a quasi-permutation action. This is the third reduction step.

Theorem 2.6. *There exist a G'-faithful quasi-permutation ZG'-lattice M and a one-cocycle $\beta \in \mathrm{Ext}^1_{G'}(M, F^*)$ such that the center C_p is stably isomorphic to $F_\beta(M)^{G'}$.*

In Proposition 2.7, we show that M is in fact induced from a p-Sylow subgroup of G'. Most of the results in Section 2 require a fair amount of computations to specifically determine certain group actions. We have chosen to show explicit computations in the simplest setting, and then induce from there.

The fourth and final reduction is done in Theorem 3.3. We show that there is a group extension E of S_p by an abelian p-group such that the invariants of the Noether setting $F(E)$ are stably isomorphic to the center.

1.

Let G be a finite group. An equivalence relation is defined in the category $\mathcal{L}_G$ of ZG-lattices as follows. The ZG-lattices M and M' are said to be equivalent if there exist permutation modules P and P' such that $M \oplus P \cong M' \oplus P'$. The equivalence class of a ZG-lattice M will be denoted by $[M]$. The set of equivalence classes forms an abelian monoid under the direct sum. Lattices equivalent to 0 are said to be stably permutation. These are lattices M, for which there exist permutation modules P and R such that $M \oplus P \cong R$. Lattices whose equivalence class has an inverse are called invertible.

For any integer n, $H^n(G, M)$ denotes the n^{th} Tate cohomology group of G with coefficients in M. A ZG-lattice M is flasque if $H^{-1}(H, M) = 0$ for all subgroups H of G. A flasque resolution of a ZG-lattice M is a ZG-exact sequence

$$0 \to M \to P \to E \to 0$$

with P permutation and E flasque. It follows directly from [EM, Lemma 1.1] that any ZG-lattice M has a flasque resolution. The flasque class of M is $[E]$ and will be denoted by $\Phi(M)$. By [CTS, Lemma 5, Section 1], $\Phi(M)$ is independent of the flasque resolution of M. Lattices whose flasque class is 0 are said to be quasi-permutation. Thus a lattice M is quasi-permutation if there exists an exact sequence

$$0 \to M \to P \to S \to 0$$

with P and S permutation.

Flasque classes play a crucial role with respect to stable rationality, as will be illustrated by the following results, which are direct consequences of the work of Swan [SR], Lenstra [L] and Endo and Miyata [EM]. These results are included for the reader's convenience. A detailed study of flasque classes can be found in [CTS].

Definition. Let K and L be extension fields of F, on which a finite group G acts as a subgroup of their groups of F-automorphisms. We say that L and K are G-isomorphic (G-stably isomorphic) if they are isomorphic (stably isomorphic) as F-algebras, and the isomorphism respects their G-actions.

Theorem 1.1 ([B2, Theorem 1.1])**.** *Let G be a finite group and let M and M' be G-faithful ZG-lattices. Then M and M' are in the same flasque class if and only if $F(M)$ and $F(M')$ are G-stably isomorphic.*

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Theorem 1.2. *If G is a finite group and M is a quasi-permutation G-faithful ZG-lattice, then $F(M)$ is G-stably isomorphic to $F(ZG)$.*

Proof. The proof of [L, Proposition 1.5] holds; the proof also follows directly from Theorem 1.1. □

We will also need the following results.

Lemma 1.3. *Let G be a finite group, and let K be a field. Let $L = K(v_1, \dots, v_n)$ be a rational extension of K, and suppose that there is a K-linear action of G on the K-span of the v_i.*

a) *If G acts faithfully on K, then L and K are G-stably isomorphic.*

b) *If G acts faithfully on L, then L is G-stably isomorphic to $K(ZG)$.*

Proof. a) Speiser's Lemma, e.g., [W].

b) The proof is basically that of [SD2, Proposition 3.1]. We include it for the reader's convenience. We have

$$L(ZG) = K(v_1, \dots, v_n)(ZG) = K(ZG)(v_1, \dots, v_n).$$

By a) $L(ZG)$ is stably isomorphic to L, and $K(ZG)(v_1, \dots, v_n)$ is stably isomorphic to $K(ZG)$. □

We now define the ZS_n-lattice G_n mentioned in the introduction. Let U be the standard rank p permutation representation of S_p, and let A be the kernel of the augmentation map on U. More precisely, U is the ZS_n-lattice with Z-basis $\{u_i \colon 1 \le i \le n\}$ with S_n-action given by $gu_i = u_{g(i)}$ for all $g \in S_n$, and A is defined by the exact sequence

$$\begin{gathered} 0 \to A \to U \to Z \to 0, \\ u_i \to 1. \end{gathered}$$

Then $G_n = A \otimes_Z A$ has the property that $F(G_n)^{S_n}$ is stably isomorphic to C_n, the center of the division ring of $n \times n$ generic matrices over F [F1, Theorem 3]. Note that U and Z are ZS_p-permutation lattices, and hence A is quasi-permutation.

Lemma 1.4. *If M is a quasi-permutation S_n-faithful ZS_n-lattice, then $F(M)^{S_n}$ is stably rational over F.*

Proof. The proof follows directly from Theorem 1.2, and the fact that $F(U)^{S_n}$ is rational over F, generated by the elementary symmetric functions on the u_i. □

Given a finite group G, a ZG-lattice M, and a field L on which G acts as automorphisms, the field $L(M)$ has a G-action induced from the action of G on M and on L. The reader should note that the action on L may be trivial. However there exist other G-actions on $L(M)$. These actions were found by Saltman [SD3], and called α-twisted actions. They are defined as follows.

Let α be in $\mathrm{Ext}^1_G(M, L^*)$, where L^* is the multiplicative group of L. Let the equivalence class of

$$1 \to L^* \to M' \to M \to 1$$

in $\mathrm{Ext}_G(M, L^*)$ be α. Writing M and M' as multiplicative abelian groups, we have

$$M' = \{x \cdot m \colon x \in L^*, m \in M\}$$

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and the G-action on M' is given by $g * x \cdot m = g(x)d_g(gm) \cdot gm$, where $d\colon G \to \mathrm{Hom}_Z(M, L^*)$ is the derivation corresponding to α. In particular, for $x = 1$, we have

$$g * m = d_g(gm) \cdot gm.$$

Thus we obtain an α-twisted action on $L(M)$. We denote by $L_\alpha(M)$ the field $L(M)$ with the corresponding G-action.

Given a finite group G and a ZG-lattice M with Z-basis $S = \{m_1, \ldots, m_s\}$, the set S will also represent a transcendency basis for $F(M)$ over F, when no confusion can arise; in other words, the operation in M will be denoted either by addition or multiplication, according to whether we are viewing M as a ZG-lattice or as a subgroup of the multiplicative group of $F(M)$, but the basis elements will be denoted by the same symbols.

Now let H be a subgroup of G of index r, and let $\{g_i\colon i = 1, \ldots, r\}$ be a transversal for H in G. Then a Z-basis for $ZG \otimes_{ZH} M$ is $\{g_i \otimes m_j\colon i = 1, \ldots, r, j = 1, \ldots, s\}$. Let $\{f_j = f_j(m_1, \ldots, m_s)\colon j = 1, \ldots, s\}$ be a transcendency basis for $F(M)$ over F, and let $f_{ij} = f_j(g_i \otimes m_1, \ldots, g_i \otimes m_s)$; then $F(ZG \otimes_{ZH} M) = F(f_{ij}\colon i = 1, \ldots, r, j = 1, \ldots, s)$.

Notation. Henceforth we adopt the following notation unless otherwise specified.

- $G = S_p$, the symmetric group on p letters for a prime p.
- $H = p$-Sylow subgroup of G.
- $N =$ normalizer of H in G. Thus $N = H \rtimes C$ is the semi-direct product of H by a cyclic group C of order $p - 1$.
- $a =$ primitive $(p-1)^{\text{st}}$ root of $1 \bmod p$.
- We will let h and c generate H and C respectively, and $chc^{-1} = h^a$.
- For any finite group G and any ZG-lattice $M, \widehat{M}$ will denote the p-adic completion of M, and for any prime q, M_q will denote the localization of M at q.
- $F =$ field of characteristic zero containing primitive p^{th} roots of 1.

Remark 1.5. Since $ZN/H \cong ZC \cong Z[x]/(x^{p-1} - 1)$ as ZN-lattices, the decomposition of $\widehat{Z}N/H$ into indecomposables is given by

$$\widehat{Z}N/H \cong \bigoplus_{k=1}^{p-1} Z_k,$$

where Z_k is the $\widehat{Z}N$-module of $\widehat{Z}$-rank 1 on which H acts trivially, and such that $cl = \vartheta^k$, where ϑ is a primitive $(p-1)^{\text{st}}$ root of 1 in $\widehat{Z}$ which is congruent to a mod p. We also set $X_k = Z_k/pZ_k$. This notation will be used throughout the article.

The lattice A defined above is isomorphic to $ZH(h-1)$ as a ZN-lattice. To see this, it suffices to note that $U \cong ZG/S_{p-1}$, and therefore $\mathrm{Res}_N^G(U) = ZN/C \cong ZH$. The action of C on ZH is given by $ch^i = h^{ai}$. Note that A is ZC-free, since C permutes its Z-basis $\{h^i - 1\colon i = 1, \ldots, p-1\}$. We denote by A' the ZN-lattice $ZH(h-1)^2$; from the ZN-exact sequence

$$(*) \qquad 0 \to A' \to A \to X_1 \to 0$$

where the map $A \to X_1$ is given by $(h-1) \to 1$, we see that A' is ZC-projective since X_1 is of order p, and hence $A'_q \cong A_q$ for all primes q dividing the order of C.

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Also A' is isomorphic to A as a ZH-lattice with the isomorphism given by

$$h^i(h-1)^2 \to h^i(h-1).$$

These facts will be used later.

Definition. A ZG-module is said to be invertible or permutation projective, if it is a direct summand of a permutation ZG-module.

Theorem 1.6. *Let A' denote the ZN-lattice $ZH(h-1)^2$ and let I_C denote the augmentation ideal of ZC. Then the flasque classes $\Phi(G_p)$ and $\Phi(ZG \otimes_{ZN} I_C \otimes A')$ are equal, consequently C_p and $F(ZG \otimes_{ZN} I_C \otimes A')^G$ are stably isomorphic over F.*

Proof. By [B2, Theorem 2.2], $\Phi(ZG \otimes_{ZN} A') = [G_p]$. Since G_p is invertible by [BL, Proposition 3, Section 3.1], $\Phi(G_p) = -[G_p]$. We have a ZN-exact sequence

$$0 \to A' \to ZC \otimes A' \to I_C \otimes A' \to 0$$

where the map $ZC \otimes A' \to I_C \otimes A'$ is $(c-1) \otimes 1_{A'}$. This map splits at the prime p since C is of order $p-1$, and hence $(I_C)_p$ is Z_pN-projective. At any prime q, we have an injection

$$\mathrm{Ext}^1_N((I_C \otimes A')_q, A'_q) \to \mathrm{Ext}^1_{N_q}((I_C \otimes A')_q, A'_q)$$

by [BK, Corollary 10.2 and Theorem 10.3, Chapter III], where N_q is the q-Sylow subgroup of N. If $q \neq p$, we may assume that N_q is contained in C, and then $\mathrm{Ext}^1_{N_q}((I_C \otimes A')_q, A'_q) = 0$ since A' is ZC-projective. Thus the sequence splits and

$$A' \oplus I_C \otimes A' \cong ZC \otimes A'.$$

Therefore

$$\Phi(ZG \otimes_{ZN} A') + \Phi(ZG \otimes_{ZN} I_C \otimes A') = \Phi(ZG \otimes_{ZN} ZC \otimes A').$$

By Remark 1.5, $A' = ZH(h-1)^2$ is isomorphic to A as a ZH-lattice, and hence $\Phi(ZG \otimes_{ZN} ZC \otimes A') = 0$, since $ZG \otimes_{ZN} ZC \otimes A' \cong ZG \otimes_{ZH} A' \cong ZG \otimes_{ZH} A$ which is quasi-permutation. Thus

$$\Phi(G_p) = -[G_p] = -\Phi(ZG \otimes_{ZN} A') = \Phi(ZG \otimes_{ZN} I_C \otimes A').$$

By Theorem 1.1 and [F1, Theorem 3], C_p and $F(ZG \otimes_{ZN} I_C \otimes A')^G$ are stably isomorphic. □

Remark 1.7. The following technique is outlined in [SD2] and will be used throughout the article. Let G be a finite group and let

$$0 \to A \to B \to C \to 0$$

be a ZG-exact sequence where A and B are ZG-lattices and C is a finite ZG-module of exponent n. Let F be a field containing primitive n^{th} roots of 1. Then the field extension

$$F(A) \subset F(B)$$

is a Kummer extension with Galois group $C' = \mathrm{Hom}(C, F^*)$. There is a natural action of G on C' via its action on C. Since the G-action of $F(A)$ extends to the G-action of $F(B)$, we have Galois extensions

$$F(A)^G \subset F(A) \subset F(B)$$

and the Galois group of $F(B)$ over $F(A)^G$ is $C' \rtimes G$.

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2.

We adopt the following notation. If $x_1, \dots, x_r$ are commuting indeterminates over F, we will denote $F(x_1, \dots, x_r)$ by $F(x_i)$.

Theorem 2.1. *There exists a transcendency basis for $F(A)$ on which N acts linearly.*

Proof. Let $a_i = h^i - h^{i-1}$ for $i = 1, \dots, p-1$. Then $\{a_i \colon i = 1, \dots, p-1\}$ is a Z-basis for A. Let ε be a primitive p^{th} root of 1. For $k = 0, \dots, p-1$, let $m_k = 1 + \sum_{k=1}^{p-1} \varepsilon^{ki} a_1 a_2 \dots a_i$ and let $n_k = m_k/m_0$ for $k = 1, \dots, p-1$. Since the Van der Monde determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{p-1} \\ 1 & \varepsilon^2 & \dots & \dots & \varepsilon^{2(p-1)} \\ 1 & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{p-1} & \dots & \dots & \varepsilon \end{vmatrix}$$

is nonzero, we can express the m_k as F-linear combinations of the elements $1, a_1$, $a_1 a_2, \dots, a_1 a_2 \dots a_{p-1}$, and therefore $F(A) = F(m_k)$. Furthermore

$$\sum_{k=1}^{p-1} n_k = -1 + p/m_0,$$

hence $F(m_k) = F(n_k)$. Recall that in N, $chc^{-1} = h^a$, and note that $ha_i = a_{i+1}$ if $i = 1, \dots, p-2$ and $ha_{p-1} = a_1^{-1} a_2^{-1} \dots a_{p-1}^{-1}$ in multiplicative notation. For $k = 0, \dots, p-1$ we have

$$\begin{aligned} hm_k &= 1 + \varepsilon^k a_2 + \varepsilon^{2k} a_2 a_3 + \dots + \varepsilon^{(p-2)k} a_2 \cdots a_{p-1} + \varepsilon^{(p-1)k} a_2 \cdots a_{p-1} a_1^{-1} a_2^{-1} \cdots a_{p-1}^{-1} \\ &= 1 + \varepsilon^k a_2 + \varepsilon^{2k} a_2 a_3 + \dots + \varepsilon^{(p-2)k} a_2 \cdots a_{p-1} + \varepsilon^{(p-1)k} a_1^{-1} \\ &= \varepsilon^{-k} a_1^{-1} m_k. \end{aligned}$$

For $i = 1, \dots, p-1$, let $v_i = a_1 a_2 \cdots a_i$. Then in additive notation v_i is $h^i - 1$, and $c(h^i - 1) = h^{ai} - 1$. Hence $cv_i = v_{ia}$. Furthermore

$$m_k = 1 + \sum_{k=1}^{p-1} \varepsilon^{ki} v_i.$$

Thus

$$cm_k = m_{k/a}.$$

Consequently

$$hn_k = \varepsilon^{-k} n_k \quad \text{and} \quad cn_k = n_{k/a}$$

where the indices are computed mod p. ∎

Theorem 2.2. *Let k denote the field with p elements, let $P = kG \otimes_{kN} I_C \otimes X_{p-2}$, and let $G' = P \rtimes G$. Then there is a twisted action of P on $F(ZG \otimes_{ZN} ZC \otimes A)$ given by a one-cocycle $\gamma \in \operatorname{Ext}^1_G(ZG \otimes_{ZN} ZC \otimes A, F^*)$, such that the center C_p is stably isomorphic to $F_\gamma(ZG \otimes_{ZN} ZC \otimes A)^{G'}$.*

Proof. Consider the ZN-exact sequence $(*)$ of Remark 1.5, namely

$$0 \to A' \to A \to X_1 \to 0$$

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where $h-1 \to 1_{X_1}$. Tensoring by I_C over Z we get

$$0 \to I_C \otimes A' \to I_C \otimes A \to I_C \otimes X_1 \to 0. \tag{1}$$

Consider the ZN-exact sequence

$$0 \to A \to ZC \otimes A \to I_C \otimes A \to 0 \tag{2}$$

where the map $ZC \otimes A \to I_C \otimes A$ is $(1-c) \otimes 1_A$. As in the proof of Theorem 1.6, (2) splits at the prime p since $(I_C)_p$ is Z_pC-projective, and it splits at all other primes dividing the order of N since A is ZC-free. Thus the sequence splits, and

$$ZC \otimes A \cong I_C \otimes A \oplus A.$$

Adding A to the last two terms of (1) we get

$$0 \to I_C \otimes A' \oplus A \to ZC \otimes A \to I_C \otimes X_1 \to 0. \tag{3}$$

Tensoring this sequence by ZG over ZN, we obtain

(4)

$$0 \to ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A \to ZG \otimes_{ZN} ZC \otimes A \to ZG \otimes_{ZN} (I_C \otimes X_1) \to 0.$$

Clearly $\mathrm{Hom}(I_C \otimes X_1, F^*) \cong I_C \otimes X_{p-2}$. Thus $P \cong \mathrm{Hom}(ZG \otimes_{ZN} I_C \otimes X_1, F^*)$ and by Remark 1.7, $F_\gamma(ZG \otimes_{ZN} ZC \otimes A)^{G'} = F(ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A)^G$ for some $\gamma \in \mathrm{Ext}^1_{G'}(ZG \otimes_{ZN} ZC \otimes A, F^*)$. Now

$$\Phi(ZG \otimes_{ZN} I_C \otimes A' \oplus ZG \otimes_{ZN} A) = \Phi(ZG \otimes_{ZN} I_C \otimes A') + \Phi(ZG \otimes_{ZN} A).$$

But $\Phi(ZG \otimes_{ZN} A) = 0$ since the sequence

$$0 \to ZG \otimes_{ZN} A \to ZG \otimes_{ZN} U \to ZG/N \to 0$$

implies that $ZG \otimes_{ZN} A$ is quasi-permutation. $F_\gamma(ZG \otimes_{ZN} ZC \otimes A)^{G'}$ is stably isomorphic to $F(ZG \otimes_{ZN} I_C \otimes A')^G$ by Theorem 1.1, which in turn is stably isomorphic to C_p by Theorem 1.6. □

We will now go back through the process that led us to prove Theorem 2.2 in order to determine precisely the G'-action on $F_\gamma(ZG \otimes_{ZN} ZC \otimes A)$. Note that $ZG \otimes_{ZN} ZC \otimes A \cong ZG \otimes_{ZN} A$.

Remark 2.3. To simplify notation we let $X_{p-2} = Y = \mathrm{Hom}(X_1, F^*)$, and we let y generate Y. This notation will be used throughout the rest of the article. Recall that a was defined to be a primitive $(p-1)^{\text{st}}$ root of 1 mod p, and let $b = a^{p-2} = a^{-1} \mod p$. Then H acts trivially on Y and $cy = y^b$. Consider the ZN-sequence $(*)$ of Remark 1.5, namely

$$0 \to A' \to A \to X_1 \to 0.$$

By Remark 1.7, we have an action of $N' = Y \rtimes N$ on $F_\delta(A) = F(n_k)$ for some $\delta \in \mathrm{Ext}^1_{N'}(A, F^*)$, as follows. Let ε be a primitive p^{th} root of 1, which we fix once and for all; then

$$ya_k = \varepsilon a_k \quad \text{for all } k.$$

Thus

$$ym_k = y\left(1 + \sum_{k=1}^{p-1} \varepsilon^{ki} a_1 a_2 \cdots a_i\right) = 1 + \sum_{k=1}^{p-1} \varepsilon^{ki}\varepsilon^i a_1 a_2 \cdots a_i = m_{k+1}$$

for $k = 0, \ldots, p-1$, and the indices on the m_k are computed mod p. Therefore $yn_k = n_{k+1}n_1^{-1}$, for $k = 1, \ldots, p-2$, and $yn_{p-1} = n_1^{-1}$, hence Y acts on the Z-span of $\{n_k \colon k = 1, \ldots, p-1\}$ as on the Z-basis for I_Y, the augmentation ideal of the

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group ring ZY, where n_k corresponds to $y^k - 1$ in additive notation. The ring ZY has a ZN'-lattice structure via the action of N on Y. We thus have an isomorphism of N'-fields

$$F_\delta(A) = F(n_k) \to F_\gamma(I_Y)$$

for the appropriate $\gamma \in \operatorname{Ext}^1_{N'}(I_Y, F^*)$.

The proof the main theorem uses a generalization of this construction.

Definition. Let $K = F(x_1, \ldots, x_n)$ be a rational extension of F, and let G be a finite group acting on K. The G-action on K is said to be monomial if for all $i = 1, \ldots, n,\ gx_i = a_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $a_i \in F, \alpha_i \in Z$, and purely monomial if $a_i = 1$ for all i.

Lemma 2.4 and Proposition 2.5 are needed for the proof of Theorem 2.6 in the following sense. They illustrate the techniques used in Theorem 2.6 in a simpler setting. Their purpose is to allow us to write the proof of Theorem 2.6 in an abbreviated and much simpler form.

Lemma 2.4. *Let $Q = \operatorname{Hom}(I_C \otimes X_1, F^*) \cong I_C \otimes Y$ and let $N' = Q \rtimes N$. There exists an N'-action on $ZC \otimes ZY$, such that $ZC \otimes ZY$ is a ZN'-permutation module.*

Proof. Recall that Z_{p-2} was defined to be the $\widehat{Z}N$-module of $\widehat{Z}$-rank 1 with trivial H-action and such that $c \cdot 1_{\widehat{Z}} = \vartheta 1_{\widehat{Z}}$, where ϑ is a primitive $(p-1)^{\text{st}}$ root of 1 in $\widehat{Z}$ congruent to $b \mod p$. Also Y was defined to be Z_{p-2}/pZ_{p-2}. There is a natural action of N on $ZC \otimes ZY$ via the action of C on ZC and on Y, and with trivial H-action. Specifically

$$c(c^i \otimes y^j) = c^{i+1} \otimes y^{jb}.$$

Let k be the field of p elements, and let

$$R = kC \otimes Y \cong \frac{\widehat{Z}C \otimes Z_{p-2}}{p\widehat{Z}C \otimes Z_{p-2}}.$$

An additive $\widehat{Z}$-basis for $\widehat{Z}C \otimes Z_{p-2}$ is $\{c^i \otimes 1_{\widehat{Z}_{p-2}} \colon i = 1, \ldots, p-1\}$, and $c(c^i \otimes 1_{\widehat{Z}_{p-2}}) = c^{i+1} \otimes \vartheta 1_{\widehat{Z}_{p-2}}$. Let $\{t_i \colon i = 1, \ldots, p-1\}$ be a multiplicative generating set for R, where t_i corresponds to $c^i \otimes 1_{\widehat{Z}_{p-2}} \operatorname{mod} p\widehat{Z}C \otimes Z_{p-2}$. Letting t_i generate T_i, we may write $R \cong kC \otimes Y \cong T_1 \times \cdots \times T_{p-1}$ as an abelian group, with $ct_i = t_{i+1}^b$. We define an action of $N'' = R \rtimes N$ on $ZC \otimes ZY$ as follows. We set

$$\operatorname{Res}^{N''}_R ZC \otimes ZY \cong ZT_1 \oplus \cdots \oplus ZT_{p-1},$$
$$c^i \otimes y^j \to t_i^j.$$

The action of R is obtained via this isomorphism. More specifically, the set $\{c^i \otimes y^j \colon i = 1, \ldots, p-1, j = 1, \ldots, p\}$ is a Z-basis for $ZC \otimes ZY$, and we have

$$t_k(c^i \otimes y^j) = c^i \otimes y^j \quad \text{if } k \neq i,$$
$$t_i(c^i \otimes y^j) = c^i \otimes y^{j+1}.$$

The action of C is given by $c(c^i \otimes y^j) = c^{i+1} \otimes y^{jb}$. A computation shows that this is in fact a group action, and clearly N'' permutes the basis elements. Now

$$Q = I_C \otimes Y \cong I_C \otimes \frac{Z_{p-2}}{pZ_{p-2}}.$$

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Thus Q is the N-submodule of R generated by $y_i = t_i t_{p-1}^{-1}$, with

$$cy_i = y_{i+1}^b y_1^{-b} \quad \text{if } i \neq p-2 \qquad \text{and} \qquad cy_{p-2} = y_1^{-b}.$$

Letting Y_i be the subgroup of Q generated by y_i we have

$$Q = Y_1 \times \cdots \times Y_{p-2}$$

as an abelian group. The action of N' on $ZC \otimes ZY$ is just the action of the subgroup of N'' generated by the y_i and N. Specifically the action of N' on $ZC \otimes ZY$ is given by

$$\begin{aligned} y_k(c^i \otimes y^j) &= c^i \otimes y^j \quad \text{if } k \neq i, i \neq p-1, \\ y_i(c^i \otimes y^j) &= c^i \otimes y^{j+1}, \\ y_i(1 \otimes y^j) &= 1 \otimes y^{j-1}, \\ c(c^i \otimes y^j) &= c^{i+1} \otimes y^{jb}. \end{aligned}$$

Here $1 = c^{p-1}$ and the powers of y are computed mod p. □

Proposition 2.5. *Let $Q = \operatorname{Hom}(I_C \otimes X_1, F^*) \cong I_C \otimes Y$ and let $N' = Q \rtimes N$. There exist one-cocycles $\gamma \in \operatorname{Ext}^1_{N'}(ZC \otimes A, F^*)$ and $\beta \in \operatorname{Ext}^1_{N'}(ZC \otimes I_Y, F^*)$ such that $F_\gamma(ZC \otimes A)$ and $F_\beta(ZC \otimes I_Y)$ are isomorphic as N'-fields.*

Proof. In the proof of Theorem 2.2 we have seen that there is a twisted action of N' on $F(ZC \otimes A)$. We need to determine precisely what this action is. We keep the notation of Lemma 2.4, and use the following Z-basis for $I_C \otimes A$:

$$\{(c^i - c^{i+1}) \otimes a_j \colon i = 1, \ldots, p-2; j = 1, \ldots, p-1\}.$$

Consider sequence (1) of Theorem 2.2, namely

$$0 \to I_C \otimes A' \to I_C \otimes A \to I_C \otimes X_1 \to 0.$$

By Remark 1.7, we have an action of Q on $F(I_C \otimes A)$ given in multiplicative notation by

$$y_k(c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} = \begin{cases} \varepsilon(c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} & \text{if } k = i, \\ (c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1} & \text{otherwise,} \end{cases}$$

for $k = 1, \ldots, p-2$. The reader should note that $(c^i \otimes a_j)(c^{i+1} a_j)^{-1}$ represents an element of the transcendency basis of $F(I_C \otimes A)$ over F.

The action of N' on $ZC \otimes A$ is obtained from sequences (2) and (3) of Theorem 2.2. We have already seen that sequence (2) splits, and we let the splitting map for $A \to ZC \otimes A$ be denoted by δ. Then

$$\begin{aligned} f \colon ZC \otimes A &\to I_C \otimes A \oplus A \\ c^i \otimes a_j &\to c^i(1-c) \otimes a_j + \delta(c^i \otimes a_j) \end{aligned}$$

is an isomorphism. Note that $f(1 \otimes a_j) = -\sum_{i=1}^{p-1} c^i(1-c) \otimes a_j + \delta(\otimes a_j)$. In multiplicative notation

$$f(1 \otimes a_j) = \left(\prod_{i=1}^{p-1} \{(c^i \otimes a_j)(c^{i+1} \otimes a_j)^{-1}\}^{-1} \right) \delta(1 \otimes a_j).$$

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With this in mind, we see that the Q-action on $F(ZC \otimes A)$ given via f is:

$$y_k c^i \otimes a_j = \begin{cases} \varepsilon c^i \otimes a_j & \text{if } k = i \text{ and } i \neq p-1, \\ \varepsilon^{-1} c^i \otimes a_j & \text{if } i = p-1, \\ c^i \otimes a_j & \text{otherwise.} \end{cases}$$

We now change the transcendency basis for $F(ZC \otimes A)$. For $k = 1, \ldots, p-1$, let $n_k = n_k(a_1, \ldots, a_{p-1})$ be defined as in Theorem 2.1. Define

$$n_{ik} = n_k(c^i \otimes a_1, \ldots, c^i \otimes a_{p-1})$$

for $i, k = 1, \ldots, p-1$. Then by an argument similar to that of Remark 2.3, we have the following action of Q:

$$\begin{aligned} &y_i \text{ acts trivially on } \{n_{jk} \colon i \neq j \text{ and } j \neq p-1\}, \\ &y_k n_{ik} = n_{ik+1} n_{i1}^{-1} \quad \text{if } k \neq p-1, \\ &y_i n_{i\,p-1} = n_{i1}^{-1}, \\ &y_i n_{p-1\,k} = n_{p-1\,k-1} n_{p-1\,p-1}^{-1} \quad \text{if } k \neq 1, \\ &y_i n_{p-1\,1} = n_{p-1\,p-1}^{-1}. \end{aligned}$$

The action of N is given by $hn_{ik} = \varepsilon^{-k} n_{ik}$ and $cn_{ik} = n_{i+1\,kb}$. The indexes are computed mod $p-1$ but powers of b are computed mod p. Thus we have an isomorphism of N'-fields

$$F_\gamma(ZC \otimes A) \cong F(n_{ik})$$

for some one-cocycle $\gamma \in \mathrm{Ext}^1_{N'}(ZC \otimes A, F^*)$.

On the other hand $ZC \otimes ZY$ is a ZN'-lattice by Lemma 2.4. Now $ZC \otimes I_Y$ is the ZN'-submodule of $ZC \otimes ZY$ generated by $c^i \otimes (y^k - 1)$, thus the N'-action is given in additive notation:

$$\begin{aligned} &y_i \text{ acts trivially on } \{c^j \otimes (y^k - 1) \colon i \neq j \text{ and } j \neq p-1\}, \\ &y_i c^i \otimes (y^{k-1} - 1) = c^i \otimes (y^{k+1} - 1) - (c^i \otimes (y-1)) \quad \text{if } i \neq p-1, \\ &y_i c^i \otimes (y^{p-1} - 1) = -c^i \otimes (y-1), \\ &y_i 1 \otimes (y^k - 1) = 1 \otimes (y^{k-1} - 1) - (1 \otimes (y^{p-1} - 1)) \quad \text{if } k \neq 1, \\ &y_i 1 \otimes (y-1) = -1 \otimes (y^{p-1} - 1). \end{aligned}$$

Therefore as a Q-lattice the Z-span of the n_{ik} is isomorphic to $ZC \otimes I_Y$. There is an N'/H-isomorphism f, from the Z-span of the n_{ik} to $ZC \otimes I_Y$, sending n_{ik} to $c^i \otimes (y^k - 1)$ in additive notation. We define a new N'-action on $F(ZC \otimes I_Y)$ by having H act on the element of the transcendency basis corresponding to $c^i \otimes (y^k - 1)$ as on n_{ik}. The map f clearly induces an N'-isomorphism from $F(n_{ik})$ to $F_\beta(ZC \otimes I_Y)$ for the appropriate $\beta \in \mathrm{Ext}^1_{N'}(ZC \otimes I_Y, F^*)$. Thus $F_\gamma(ZC \otimes A)$ and $F_\beta(ZC \otimes I_Y)$ are isomorphic as N'-fields. $\square$

The crucial point here is that now the action of Q on $F(n_{ik})$ is purely monomial, and the action of N is linear. The proof of the next theorem uses this argument induced up to G'.

Theorem 2.6. *There exist a G'-faithful quasi-permutation ZG'-lattice M and a one-cocycle $\beta \in \mathrm{Ext}^1_G(M, F^*)$ such that the center C_p is stably isomorphic to $F_\beta(M)^{G'}$.*

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Proof. For $k = 1, \ldots, p-1$ let $n_k = n_k(a_1, \ldots, a_{p-1})$ be defined as in Theorem 2.1. Let S_{p-2} be the subgroup of G fixing $p-1$ and p, and let $\{g_i \colon g_i \in S_{p-2}\}$ be a transversal for N in G. This transversal will be fixed throughout the paper. As in Proposition 2.5, let $n_{ijk} = n_k(g_i \otimes c^j \otimes a_1, \ldots, g_i \otimes c^j \otimes a_{p-1})$ for $i = 1, \ldots, (p-2)!$, $j, k = 1, \ldots, p-1$. Then

$$F_\gamma(ZG \otimes_{ZN} \ ZC \otimes A) = F(n_{ijk})$$

for some $\gamma \in \operatorname{Ext}^1_{G'}(ZG \otimes_{ZN} \ ZC \otimes A, F^*)$, and the G-action is given by: Let $gg_i = g_s c^t h^l$, where s, t and l depend on g and g_i. Then

$$gn_{ijk} = \varepsilon^{-lk} n_{s, j+t, k\delta^t}.$$

Write $P = \sum_{ij} Y_{ij}$, where $Y_{ij} = g_i \otimes Y_j$, and let y_{ij} generate Y_{ij}. Then

$$\begin{aligned}
&y_{st} \text{ acts trivially on } n_{ijk} \text{ if } s \neq i \text{ or } (i = s, t \neq j \ \text{ and } k \neq p-1),\\
&y_{ij} n_{ijk} = n_{ijk+1} n_{ij1}^{-1} \quad \text{if } k \neq p-1,\\
&y_{it} n_{ijp-1} = n_{ij1}^{-1},\\
&y_{it} n_{ip-1k} = n_{ip-1k-1} n_{ip-1p-1}^{-1} \quad \text{if } k \neq 1,\\
&y_{it} n_{ip-11} = n_{ijp-1}^{-1}.
\end{aligned}$$

By an argument similar to that of Proposition 2.5, P acts on the Z-span of the n_{ijk} as on the ZG'-lattice

$$M = ZG \otimes_{ZN} \ ZC \otimes I_Y$$

with n_{ijk} corresponding to $g_i \otimes c^j \otimes (y^k - 1)$ in additive notation. This induces a G'-isomorphism from $F(n_{ijk})$ to $F_\beta(M)$ for the appropriate $\beta \in \operatorname{Ext}^1_{G'}(M, F^*)$. $\square$

Proposition 2.7. *Let PH denote the p-Sylow subgroup of G'. There exists a $Z[PH]$-lattice I such that the center C_p is stably isomorphic to $F_\beta(ZG' \otimes_{Z[PH]} I)$ for some $\beta \in \operatorname{Ext}^1_{G'}(ZG' \otimes_{Z[PH]} I, F^*)$.*

Proof. The ZG'-lattice $ZG \otimes_{ZN} \ ZC \otimes ZY$ is a permutation lattice with Z-basis

$$\{g_i \otimes c^j \otimes y^k \colon 1 \le i \le (p-2)!, \quad 1 \le j \le p-1, 1 \le k \le p\}.$$

If we let $g_1 = 1_G$, then, for any i, j, k, $y_i^k c^j g_i(1 \otimes 1 \otimes 1) = g_i \otimes c^j \otimes y^k$. Therefore it is a transitive permutation lattice. The stabilizer in G' of $1 \otimes 1 \otimes 1$ is the semi-direct product of the subgroup, K, of P generated by $\{y_{ij}, y_{1j} y_{1k}^{-1} \colon i \neq 1, j \neq k\}$, by H. We denote this stabilizer by KH. Thus $ZG \otimes_{ZN} \ ZC \otimes ZY \cong ZG'/KH$. Let I be the $Z[PH]$-lattice defined by the exact sequence

$$0 \to I \to Z[PH/KH] \to Z \to 0.$$

Then it is immediate that $M = ZG \otimes_{ZN} ZC \otimes I_Y$ is isomorphic to $ZG' \otimes_{Z[PH]} I$. $\square$

3.

In this section we assume the base field F to be algebraically closed of characteristic zero. We will keep all the notation of the previous sections unless otherwise specified. Recall that

$$\widehat{Z}N/H \cong \bigoplus_{k=1}^{p-1} Z_k,$$

where Z_k is the $\widehat{Z}N$-module of $\widehat{Z}$-rank 1 on which H acts trivially, and such that $c1 = \vartheta^k$, where ϑ is a primitive $(p-1)^{\text{st}}$ root of 1 in $\widehat{Z}$ which is congruent to a

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mod p. We also set $X_k = Z_k/pZ_k$. Recall also that we had set $X_{p-2} = Y$, and to simplify notation we set $X_{p-1} = X$, so X is the trivial ZN-module of p elements. Finally k was defined to be the field of p elements.

Notation 3.1. If G is a finite group, the Noether setting of G is $F(ZG)$ and it is denoted by $F(G)$. For any ZG-lattice S, $\widetilde{S}$ will denote $\mathrm{Hom}(S, F^*)$.

The ZG'-lattice $ZG' \otimes_{Z[PH]} I$ will be denoted by M, as above.

Theorem 3.2. *The center C_p of the division ring of $p \times p$ generic matrices over an algebraically closed field F of characteristic zero is stably isomorphic to the Noether setting of the group G'' defined by*

$$\gamma\colon 1 \to kG/H \to G'' \to G' \to 1$$

where $\gamma \in H^2(G', kG/H)$. Furthermore the sequence

$$0 \to kG/H \to \widetilde{Z}[G'/KH] \to \widetilde{M} \oplus \widetilde{Z}G'/PH \to 0$$

is exact and γ is the image of an element $\beta \in \mathrm{Ext}^1_{G'}(M, F^*)$ under its connecting homomorphism.

Proof. Recall that P was defined to be $kG \otimes_{KN} I_C \otimes Y$, and that $G' = P \rtimes G$. By Theorem 2.6, C_p is stably isomorphic to $F_\beta(M)^{G'}$. Furthermore, the action of P on $F_\beta(M)$ is not twisted. We have the ZG'-exact sequence

$$0 \to M \xrightarrow{f} ZG'/KH \to ZG'/PH \to 0$$

from which we obtain

$$0 \to M \oplus ZG'/PH \to ZG'/KH \to kG'/PH \to 0$$

by sending M to $f(M)$ and $\overline{g}$ to $\sum_{j=1}^{p} g \otimes y^j$ for $g \in G'/PH$, where y generates PH/KH.

Since $\mathrm{Hom}(kG'/PH, F^*) \cong kG/H$ as G'-modules, [BJ, Theorem I-2.1] says that there exists a group G'' given by the exact sequence

$$\gamma\colon 1 \to kG/H \to G'' \to G' \to 1$$

such that $F(ZG'/KH)^{G''} = F_\beta(M \oplus ZG'/PH)^{G'}$. The class of $\gamma \in H^2(G', kG/H)$ is the image of β by the connecting homomorphism δ of the exact sequence

$$0 \to kG/H \to \widetilde{Z}[G'/KH] \to \widetilde{M} \oplus \widetilde{Z}G'/PH \to 0.$$

Furthermore the action of G'' on $F(ZG'/KH)$ is linear. Also note that this action is G''-faithful since the action of G' on $F_\beta(M \oplus ZG'/PH)$ is G'-faithful. By Theorem 1.2, $F(ZG'/KH)^{G''}$ is stably isomorphic to $F(G'')$, and by Lemma 1.3 $F_\beta(M \oplus ZG'/PH)^{G'}$ is stably isomorphic to $F_\beta(M)^{G'}$. Thus $F(G'')^{G''}$ is stably isomorphic to the center C_p. □

Theorem 3.3. *There is a group extension E of G by an abelian p-group such that the center C_p of the division ring of $p \times p$ generic matrices over an algebraically closed field F of characteristic zero is stably isomorphic to the invariants of the Noether setting $F(E)$.*

Proof. Consider the group G'' of Theorem 3.2. It is defined by the exact sequence

$$\gamma\colon 1 \to kG/H \to G'' \to G' \to 1$$

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where $\gamma \in H^2(G', kG/H)$ is the image of β, the one-cocycle from G' to $\mathrm{Hom}(M, F^*)$ which gives the twisting of the action of G' on $F_\beta(M)$. Now $\mathrm{Res}_P^{G'}(\beta) = 0$ since the action of P on $F_\beta(M)$ is not twisted. By Theorem 3.2 $\gamma = \delta(\beta)$ where δ is the connecting homomorphism of the exact sequence

$$0 \to kG/H \to \widetilde{Z}[G'/KH] \to \widetilde{M} \oplus \widetilde{Z}G'/PH \to 0.$$

Thus $\mathrm{Res}\,\gamma = \mathrm{Res}(\delta(\beta)) = \delta(\mathrm{Res}(\beta)) = 0$, and hence there is a group monomorphism from P to G''. Let S be the subgroup of G'' generated by kG/H and the image of P. Then S is normal and we have a group extension

$$1 \to S \to G'' \to G \to 1.$$

The result now follows from Theorem 3.2. $\square$

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