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LATTICE ISOMORPHISMS OF INVERSE SEMIGROUPS

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A largely untouched problem in the theory of inverse semigroups has been that of finding to what extent an inverse semigroup is determined by its lattice of inverse subsemigroups. In this paper we discover various properties preserved by lattice isomorphisms, and use these results to show that a free inverse semigroup $\mathscr{F}\mathscr{I}_X$ is determined by its lattice of inverse subsemigroups, in the strong sense that every lattice isomorphism of $\mathscr{F}\mathscr{I}_X$ upon an inverse semigroup T is induced by a *unique* isomorphism of $\mathscr{F}\mathscr{I}_X$ upon T. (A similar result for free groups was proved by Sadovski (**12**) in 1941. An account of this may be found in Suzuki's monograph on the subject of subgroup lattices (**14**)).

If S is an inverse semigroup, denote by $\langle X\rangle$ the inverse subsemigroup of S generated by X, and write $\langle X\rangle \leqslant S$. We denote by $\mathscr{L}(S)$ the lattice of inverse subsemigroups of S, with meet and join defined in the usual way, and with the empty inverse semigroup $\Box$ as zero element. To cover the group case consistently, we will treat $\Box$ as a subgroup of every group.

By an *L-isomorphism* (also called structural isomorphism, or projectivity) of an inverse semigroup S upon another T we mean a lattice isomorphism Φ of $\mathscr{L}(S)$ upon $\mathscr{L}(T)$. Thus Φ is a bijection such that $U \leqslant V$ if and only if $U\Phi \leqslant V\Phi$, for all $U, V \in \mathscr{L}(S)$. Equivalently, Φ and Φ^{-1} preserve finite meets and joins.

If ϕ is an isomorphism of an inverse semigroup S upon another T, the *induced L-isomorphism* of S upon T is that defined by $U\Phi = U\phi$, for $U \leqslant S$, $U \neq \Box$, and $\Box\Phi = \Box$.

1. L-isomorphisms

Throughout this section S and T are inverse semigroups, with semilattices of idempotents, $E(S)$ and $E(T)$ respectively, and Φ is an L-isomorphism of S upon T. For the basic properties of inverse semigroups the reader is referred to (**1**).

If $(P, \leqslant)$ is any poset, and $p, q \in P$, we write $p \| q$ if p and q are incomparable, that is neither $p \leqslant q$ nor $q \leqslant p$. Otherwise we write $p \nparallel q$ and say p and q are comparable. Also if q covers p (that is $p < q$ and no $r \in P$ satisfies $p < r < q$) we write $p \prec q$.

Lemma 1.1. *Φ induces a bijection ϕ of $E(S)$ upon $E(T)$ such that $\{e\}\Phi = \{e\phi\}$, for all $e \in E(S)$, and satisfying*

(i) *$e \nparallel f$ if and only if $e\phi \nparallel f\phi$,*

(ii) *$e \| f$ implies $(ef)\phi = (e\phi)(f\phi)$.*

Thus $E(S)\Phi = E(T)$, and $\mathscr{L}(E(S)) \cong \mathscr{L}(E(T))$.

Proof. For any inverse semigroup U, the sets $\{e\}$, $e \in E(U)$, are clearly the atoms of the lattice $\mathcal{L}(U)$. Thus for each $e \in E(S)$, there is an idempotent f of T such that $\{e\}\Phi = \{f\}$. Put $f = e\phi$. Then ϕ is bijective since Φ is.

To obtain (i), note that $e \nparallel f$ if and only if $\langle e, f\rangle > \langle e\rangle$. To prove (ii) suppose $e \parallel f$. Then $\langle e, f\rangle\Phi = \{e\phi\} \vee \{f\phi\} = \{e\phi, f\phi, (e\phi)(f\phi)\}$. But $\{(ef)\phi\} = \{ef\}\Phi \subset \langle e, f\rangle\Phi$, and hence $(ef)\phi = (e\phi)(f\phi)$.

Finally, if $f \in E(T)$, then $\{f\} = \langle f\phi^{-1}\rangle\Phi \leqslant E(S)\Phi$, so that $E(T) \leqslant E(S)\Phi$. Similarly $E(S) \leqslant E(T)\Phi^{-1}$. Therefore $E(S)\Phi = E(T)$.

(In the special case of L-isomorphisms of *semilattices*, Ševrin showed **(13)** that, conversely, each such mapping ("weak isomorphism" in his terminology) between semilattices E and F induces an L-isomorphism. In Section 3 it will be seen that ϕ is not, in general, an isomorphism).

Corollary 1.2. *The L-isomorphisms Φ and Φ^{-1} preserve [maximal] subgroups; that is, G is a [maximal] subgroup of S if and only if $G\Phi$ is a [maximal] subgroup of T.*

Proof. For any inverse semigroup U, G is a subgroup of U if and only if in $\mathcal{L}(U)$, G lies above exactly one atom.

Corollary 1.3. *If S is $\mathcal{H}$-trivial (or, commonly, combinatorial), so is T.*

An inverse semigroup S is said to be *completely semisimple* if no two $\mathcal{D}$-related idempotents of S are comparable under the natural partial order of $E(S)$:

$$e \leqslant f \text{ iff } ef = e, \quad e, f \in E(S).$$

An equivalent property is that each principal factor of S is either completely 0-simple or completely simple. By Theorem 2.54 of **(1)**, S is then completely semisimple if and only if it contains no inverse subsemigroup isomorphic with the bicyclic semigroup. (Thus every finite inverse semigroup is completely semisimple). In any such semigroup $\mathcal{D} = \mathcal{J}$.

Proposition 1.4. *If S is completely semisimple, so is T.*

Proof. Suppose T contains an inverse subsemigroup U isomorphic with the bicyclic semigroup: thus U is $\mathcal{H}$-trivial and $E(U)$ is a chain. By Corollary 1.3, $V = U\Phi^{-1}$ *is* $\mathcal{H}$-trivial, and by Lemma 1.1 (i), $E(V)$ is a chain. However, if V contains a non-idempotent x, then $xx^{-1} \nparallel x^{-1}x$ and $xx^{-1}\mathcal{D}x^{-1}x$, which is impossible in S. So V, and therefore U, is a semilattice, a contradiction.

The next sequence of results shows that if S is both completely semisimple and $\mathcal{H}$-trivial, Φ preserves much of the structure of S.

Lemma 1.5. *If S is completely semisimple, suppose the element x of S is not in a subgroup of S (or, equivalently, $xx^{-1} \neq x^{-1}x$). Then $\langle x\rangle\Phi = \langle y\rangle$ for some y in T.*

Proof. Since S is completely semisimple, $\langle x\rangle \cap J_x = \{x, x^{-1}, xx^{-1}, x^{-1}x\}$. Further $\langle x\rangle \backslash \{x, x^{-1}, xx^{-1}, x^{-1}x\}$ is contained in the ideal $I(x)$ of S. Then $M = \langle x\rangle\backslash\{x, x^{-1}\}$ is the *maximum* proper inverse subsemigroup of $\langle x\rangle$. Since Φ maps $\mathscr{L}(\langle x\rangle)$ isomorphically upon $\mathscr{L}(\langle x\rangle\Phi)$, $M\Phi$ is the maximum proper inverse subsemigroup of $\langle x\rangle\Phi$. Let $y \in \langle x\rangle\Phi | M\Phi$; then $\langle x\rangle\Phi = M\Phi \vee \langle y\rangle$. So $\langle x\rangle = \langle y\rangle\Phi^{-1} \vee M$. But then $x \in \langle y\rangle\Phi^{-1}$, that is $\langle x\rangle\Phi = \langle y\rangle$.

We now wish to show that in this situation, y can be chosen (in fact uniquely) so that $(xx^{-1})\phi = yy^{-1}$ and $(x^{-1}x)\phi = y^{-1}y$. To do this, we must examine the structure of the semilattice of idempotents of a completely semisimple *elementary* inverse semigroup (that is, one generated by a single element).

Suppose x generates $\langle x\rangle$ freely, that is $\langle x\rangle \cong \mathscr{F}\mathscr{I}_1$, the free inverse semigroup on one generator. Then (Gluskin, (3)), every element of $\langle x\rangle$ has a *unique* representation in the form $(x^{-p}x^p)x^q(x^rx^{-r})$ for integers p, q, r, with $p, r, p+q, q+r \geq 0$, $p+q+r \geq 1$. (If p, q or r is zero, the corresponding term is deleted). The idempotents are then the elements $(x^{-p}x^p)(x^rx^{-r})$, $p, r \geq 0$, $p+r \geq 1$, with $(x^{-p}x^p)(x^rx^{-r}) \leq (x^{-k}x^k)(x^mx^{-m})$ iff $p \geq k$ and $r \geq m$. (Note that $x^{-p}x^p \| x^rx^{-r}$ for $p, r \geq 1$, and an idempotent $(x^{-p}x^p)(x^rx^{-r})$ is a product of 2 incomparable idempotents if and only if $r \neq 0$ and $s \neq 0$). Further $(x^{-p}x^p)x^q(x^rx^{-r})\mathscr{D}(x^{-k}x^k)x^l(x^mx^{-m})$ iff $p+q+r = k+l+m$. As a consequence $\langle x\rangle$ is completely semisimple, $\mathscr{D} = \mathscr{J}$, and the ideals are all principal of the form $\langle x\rangle x^n\langle x\rangle = I_n$, $n \geq 1$. We will denote also by I_n the Rees congruence $(I_n \times I_n) \cup \iota$.

Recall (11) that a congruence on $E = E(\langle x\rangle)$ is said to be *normal* if it is the restriction to $E \times E$ of some congruence on $\langle x\rangle$. Denote by I'_n the restriction of the Rees congruence I_n to $E \times E$. Eberhart and Selden showed (2; Theorem 3.4) that each normal congruence on E is of the form I'_n, $I'_n \cap A$ or $I'_n \cap B$, for some $n \geq 1$, where $(x^{-p}x^p)(x^rx^{-r})A(x^{-k}x^k)(x^mx^{-m})$ iff $r = m$, and B is defined dually. Let ρ be any congruence on $\langle x\rangle$ such that $\rho \cap (E \times E) = I'_n \cap A$. In $\langle x\rangle/\rho$, then, we have $(x^{-n}x^n)\rho\mathscr{D}(x^nx^{-n})\rho$ and $(x^{-n}x^n)\rho > (x^nx^{-n})\rho$ (since $((x^{-n}x^n)(x^nx^{-n}))A = (x^nx^{-n})A$ but $(x^{-n}x^n, x^nx^{-n}) \notin A$). So $\langle x\rangle/\rho$ is not completely semisimple.

Since every elementary inverse semigroup $\langle a\rangle$ is a factor of $\mathscr{F}\mathscr{I}_1$ by some congruence, if it is completely semisimple and not free, $E(\langle a\rangle) \cong E(\langle x\rangle/I_n)$ for some $n \geq 1$. Put $\mathscr{E}^n = \{\langle a\rangle | E(\langle a\rangle) \cong E(\langle x\rangle/I_n)\}$, $n \geq 1$. (For example, $\mathscr{E}^1$ consists of the cyclic groups). If $\langle a\rangle \in \mathscr{E}^n$, $n \geq 2$, write 0_a for its minimum idempotent; then any non-zero idempotent of $\langle a\rangle$ can be uniquely expressed in the form $(x^{-p}x^p)(x^rx^{-r})$, $0 \leq p, r \leq n-1$, $1 \leq p+r \leq n-1$. Thus $|E(\langle a\rangle)| = (2+3+\cdots+n)+1 = n(n+1)/2$.

Lemma 1.6. *Suppose $\langle x\rangle$ is any completely semisimple elementary inverse semigroup, not a group, and Φ is an L-isomorphism of $\langle x\rangle$ upon $\langle y\rangle$. Then, by replacing y by y^{-1}, if necessary, $(xx^{-1})\phi = yy^{-1}$ and $(x^{-1}x)\phi = y^{-1}y$.*

Proof. From the above discussion, we have two cases to consider (note that from Lemma 1.1, $|E(\langle y\rangle)| = |E(\langle x\rangle)|$).

(i) If $\langle x\rangle$ is free, then $\langle y\rangle$ must also be. Since the maximal elements of $E(\langle y\rangle)$ are yy^{-1} and $y^{-1}y$, we may, by replacing y by y^{-1} if necessary, assume that $(xx^{-1})\phi \leq yy^{-1}$; thus $(xx^{-1})\phi \neq y^{-p}y^p$, $p \geq 1$. Now if $(xx^{-1})\phi = (y^{-p}y^p)(y^ry^{-r})$ for $p, r \geq 1$, then $(xx^{-1})\phi$ is a product of incomparable idempotents, and xx^{-1} is also (by Lemma 1.1 (ii)), a contradiction. So $(xx^{-1})\phi = y^r y^{-r}$, for some $r \geqslant 1$. In view of Lemma 1.1 (i) a similar argument shows that if $s > 1$, $(x^s x^{-s})\phi = y^q y^{-q}$ for some $q > 1$ (depending on s), that $(x^{-1}x)\phi = y^{-p}y^p$ for some $p \geqslant 1$ and that $(y^{-1}y)\phi = x^{-k}x^k$ for some $k > 1$. Consider the idempotent $g = (x^{-k}x^k)(xx^{-1}) = (x^{-k}x^k)(x^{-1}x)(xx^{-1})$ of $\langle x \rangle$. Now $g\phi = (x^{-k}x^k)\phi(xx^{-1})\phi = (y^{-1}y)(y^r y^{-r})$, since $xx^{-1} \| x^{-k}x^k$. But similarly if $k \neq 1$, $g\phi = (x^{-k}x^k)\phi((x^{-1}x)(xx^{-1}))\phi = (x^{-k}x^k)\phi(x^{-1}x)\phi(xx^{-1})\phi = (y^{-1}y)(y^{-p}y^p)(y^r y^{-r}) = (y^{-p}y^p)(y^r y^{-r})$. Thus $p = 1$, that is $(x^{-1}x)\phi = y^{-1}y$, contradicting $k \neq 1$. Thus $k = 1$ and $(x^{-1}x)\phi = y^{-1}y$. Similarly $(xx^{-1})\phi = yy^{-1}$.

(ii) If $\langle x \rangle \in \mathscr{E}^n$, $n \geqslant 2$, then since $|E(\langle y \rangle)| = |E(\langle x \rangle)|$, $y \in \mathscr{E}^n$ also. Again we may assume $(xx^{-1})\phi \leqslant yy^{-1}$. Note that $0_x\phi = 0_y$ (since for any $\langle a \rangle \in \mathscr{E}^n$, $n \geqslant 2$, 0_a is the only idempotent comparable with every other). By similar arguments to those used in (i), $(xx^{-1})\phi = y^r y^{-r}$ for some r, $1 \leqslant r \leqslant n-1$. Now $(y^{-p}y^p)(y^r y^{-r}) = 0_y$ for each p such that $n - r \leqslant p \leqslant n-1$, that is $(y^{-p}y^p)\phi^{-1} \cdot (xx^{-1}) = 0_x$. But $(y^{-p}y^p)\phi^{-1} = x^{-k}x^k$, for some $k \geqslant 1$, as above; therefore $(x^{-k}x^k)(xx^{-1}) = 0_x$, possible only for $k = n-1$. Since k has only one value and since ϕ^{-1} is injective, p can have only one value; that is, $r = 1$. Hence $(xx^{-1})\phi = yy^{-1}$. Similarly $(x^{-1}x)\phi = y^{-1}y$.

Combining this lemma with the previous one, we obtain the following.

Proposition 1.7. *Suppose S is completely semisimple, and the element x of S is not in a subgroup. Then there exists a unique element y of T such that $\langle x \rangle \Phi = \langle y \rangle$, $(xx^{-1})\phi = yy^{-1}$ and $(x^{-1}x)\phi = y^{-1}y$.*

Proof. Only the uniqueness remains to be shown, and this follows immediately from the complete semisimplicity of $\langle y \rangle$, for only $\{y\}$ and $\{y^{-1}\}$ can generate $\langle y \rangle$ irredundantly.

Recall that the set $\mathcal{J}(U)$ of $\mathcal{J}$-classes of an inverse semigroup U forms a poset under the ordering $J_x \leqslant J_y$ if and only if $x \in UyU$ $(x, y \in U)$. Note that $J_x \leqslant J_y$ if and only if there are idempotents $e \in J_x$, $f \in J_y$ such that $e \leqslant f$, (by, for example (**1**; §8.4 Exercise 3)).

Proposition 1.8. *If S is completely semisimple, and $e, f \in E(S)$, then $e\phi \mathscr{D} f\phi$ if and only if $e \mathscr{D} f$. Thus Φ induces a bijection ϕ of $\mathcal{J}(S)$ upon $\mathcal{J}(T)$ such that*

(i) *$J_1 \nparallel J_2$ if and only if $J_1\phi \nparallel J_2\phi$, $J_1, J_2 \in \mathcal{J}(S)$, and*

(ii) *$|E(T) \cap J\phi| = |E(S) \cap J|$ for all $J \in \mathcal{J}(S)$.*

Proof. If $e, f \in E(S)$, $e \mathscr{D} f$ and $e \neq f$, then $e = xx^{-1}$, $f = x^{-1}x$ for some $x \in S$, not in a subgroup. Thus $\langle x \rangle \Phi = \langle y \rangle$, for some $y \in T$ such that $e\phi = yy^{-1}$ and $f\phi = y^{-1}y$. Thus $e\phi \mathscr{D} f\phi$. The converse follows similarly.

For any $\mathcal{J}$-class J of S, define $J\phi = J_{e\phi} \in \mathcal{J}(T)$ where $e \in E(S) \cap J$. Since $\mathscr{D} = \mathcal{J}$, ϕ is well-defined, and is a bijection of $\mathcal{J}(S)$ upon $\mathcal{J}(T)$ since $\phi\colon E(S) \to E(T)$ is a bijection. The statement (i) follows from Lemma 1.1 (i), and (ii) is immediate.

If we now require in addition that S be $\mathscr{H}$-trivial, no non-idempotent of S lies in a subgroup, and so we may extend the bijection ϕ of $E(S)$ upon $E(T)$ to a bijection ϕ of S upon T, by defining $x\phi$, for each non-idempotent x of S, to be the (unique) element y of T obtained in Proposition 1.7.

Theorem 1.9. *Let S be completely semisimple and $\mathcal{H}$-trivial. Then Φ induces a unique bijection ϕ of S upon T satisfying* (i) $\langle x\rangle\Phi = \langle x\phi\rangle$, *and* (ii) $(x\phi)^{-1} = x^{-1}\phi$, *and* $(xx^{-1})\phi = (x\phi)(x\phi)^{-1}$, *for all $x \in S$. Thus if Φ is induced by an isomorphism of S upon T, it is induced by a unique such isomorphism, namely, ϕ.*

Proof. With ϕ defined as above, properties (i) and (ii) are immediate from Proposition 1.7. The uniqueness of ϕ follows from the uniqueness of the element y chosen in the lemma. Finally, if θ: $S \to T$ is an isomorphism inducing Φ, properties (i) and (ii) are clearly satisfied by θ, so $\theta = \phi$.

As an immediate consequence of this theorem, Aut(S), the group of automorphisms of S (completely semisimple and $\mathcal{H}$-trivial) is isomorphic with a subgroup of Aut($\mathcal{L}(S)$), the group of automorphisms of $\mathcal{L}(S)$.

Corollary 1.10. *If S is finite $\mathcal{H}$-trivial, so is T, and $|S| = |T|$.*

Proof. As noted earlier, every finite inverse semigroup is completely semisimple.

Corollary 1.11. *If S is completely semisimple and $\mathcal{H}$-trivial, and $x \in S$, then x has finite order iff $x\phi$ has.*

Proof. From Gluskin's representation of $\mathcal{FI}_1$ given earlier, every element of $\langle x\rangle$ can be expressed (not necessarily uniquely) in the form $(x^{-p}x^{p})x^{q}(x^{r}x^{-r})$ for some integers p, q, r. Thus $\langle x\rangle$ is finite iff x has finite order.

2. Free Inverse Semigroups

Denote by $\mathcal{FI}_X$ the free inverse semigroup on the set X. If X is finite, $|X| = n$, we may instead write $\mathcal{FI}_n$.

Theorem 2.1. *Every L-isomorphism of $\mathcal{FI}_X$ upon an inverse semigroup T is induced by a unique isomorphism of $\mathcal{FI}_X$ upon T.*

To prove this result we need firstly some properties of $\mathcal{FI}_X$. We use Munn's representation of $\mathcal{FI}_X$ by birooted trees, and refer the reader to (**8**) for all definitions and elementary results. The following theorem summarises the various properties which we will need.

Theorem 2.2. (Munn (**8**), Reilly (**10**), McAlister and McFadden (**5**)). *For any set X,*

(i) *$\mathcal{FI}_X$ is $\mathcal{H}$-trivial and completely semisimple;*

(ii) *$\mathcal{FI}_X$ is E-unitary (alternatively "reduced" or "proper"), that is, $x, e \in \mathcal{FI}_X$, $e^2 = e$ and $xe = e$ imply $x^2 = x$;*

(iii) *there is a one-to-one correspondence $J \mapsto T(J)$ between the $\mathcal{J}$-classes of $\mathcal{FI}_X$ and the (isomorphism classes of) word-trees on X, such that $J_1 < J_2$ if and only if $T(J_1) \supset T(J_2)$, and such that $J = \{(T(J), \alpha, \beta) | \alpha, \beta \in V(T(J))\}$;*

(iv) *the idempotents of $\mathcal{FI}_X$ are the birooted trees (T, α, α), $\alpha \in V(T)$, and $(T_1, \alpha, \alpha) \leq (T_2, \beta, \beta)$ if and only if $T_1 \supseteq T_2$ and $\alpha = \beta$. Thus each $\mathcal{J}$-class is finite, having $|T(J)|$ idempotents.*

If, then, $J \in \mathscr{J}(\mathscr{F}\mathscr{I}_X)$, define $d(J)=|T(J)|-1$, and for each idempotent e of $\mathscr{F}\mathscr{I}_X$, let $d(e)=d(J_e)$ be its *depth* (following Munn (**8**)). From Theorem 3.5 of (**8**), every chain of idempotents between two comparable idempotents of $\mathscr{F}\mathscr{I}_X$ is finite, and $E(\mathscr{F}\mathscr{I}_X)$ (and thus $\mathscr{J}(\mathscr{F}\mathscr{I}_X)$) has the ascending chain condition.

By (i) of Theorem 2.2, we can apply Theorem 1.9. For the remainder of this section, we assume that Φ is an L-isomorphism of $\mathscr{F}\mathscr{I}_X$ upon an inverse semigroup T.

Proposition 2.3. *The induced map ϕ of $\mathscr{J}(\mathscr{F}\mathscr{I}_X)$ upon $\mathscr{J}(T)$ is an order-isomorphism.*

Proof. Since $\mathscr{J}(\mathscr{F}\mathscr{I}_X)$ has the ascending chain condition, to show ϕ is isotone, it is clearly sufficient to show that $J_1\phi < J_2\phi$ for any $J_1, J_2 \in \mathscr{J}(\mathscr{F}\mathscr{I}_X)$ such that $J_1 < J_2$. Put $T_1 = T(J_1)$, and $T_2 = T(J_2)$. Then $T_1 \supset T_2$ and T_1 is obtained from T_2 by the addition of exactly one extra vertex β to T_2 and a corresponding edge $\alpha\beta$, for some vertex α of T_2.

Let $e=(T_1, \alpha, \alpha)$ and $f=(T_2, \alpha, \alpha)$. Thus $e<f$ and $e \in J_1$, $f \in J_2$. Since T_2 has at least two vertices, it has an extremity $\gamma \neq \alpha$ and a corresponding edge $\delta\gamma$, for some vertex δ of T_2. Let T_3 be the tree obtained from T_1 by removing this vertex γ and edge $\delta\gamma$, and put $g=(T_3, \alpha, \alpha)$. Then $e=fg$ and $f\|g$, so by Lemma 1.1, $e\phi=(f\phi)(g\phi)$. Therefore $e\phi < f\phi$, and so $J_1\phi < J_2\phi$.

On the other hand, suppose $J_1\|J_2$. Then $e\|f$ for all e in $E(S)\cap J_1$ and for all f in $E(S)\cap J_2$, that is $e\phi\|f\phi$, by Lemma 1.1(i). By Proposition 1.8, $J_1\phi\|J_2\phi$.

Corollary 2.4. *The induced map ϕ of $E(\mathscr{F}\mathscr{I}_X)$ upon $E(T)$ is an isomorphism.*

Proof. If $e, f \in E(\mathscr{F}\mathscr{I}_X)$ with $e<f$, then $J_e < J_f$, so $J_{e\phi} = J_e\phi < J_f\phi = J_{f\phi}$. Since $e\phi \not\| f\phi$, $e\phi < f\phi$. If $e\|f$, then $e\phi\|f\phi$ by Lemma 1.1(i).

Corollary 2.5. *T is E-unitary.*

Proof. Suppose T contains a non-idempotent y and an idempotent f such that $yf=f$. By Theorem 1.9, $y=x\phi$, $f=e\phi$, for some non-idempotent x and idempotent e of $\mathscr{F}\mathscr{I}_X$. Now by a result of Reilly (**10**), $\langle x\rangle \cong \mathscr{F}\mathscr{I}_1$, so it has infinite order. By Corollary 1.11, then, so does y. Since T is completely semisimple and $\mathscr{H}$-trivial (by Proposition 1.4), $x^{k+1} \in I(x^k)$ for each $k \geq 1$, and so the idempotents $y^{-k}y^k$, $k \geq 1$ are distinct. But $yf=f$ implies $y^kf=f$, $k\geq 1$, and so $y^{-k}y^k \geq f$, $k \geq 1$. Thus $E(T)$ has an infinite chain of idempotents $y^{-1}y > y^{-2}y^2 > \cdots > y^{-k}y^k > \cdots > f$. From the previous result, however, $E(T) \cong E(\mathscr{F}\mathscr{I}_X)$, contradicting the remarks following Theorem 2.2. So T is E-unitary.

$\mathscr{F}\mathscr{I}_X$ is generated by a set which we may clearly identify with X. Further, if $y \in T$, then $y\phi^{-1} = x_1 \ldots x_n$, for some $x_i \in X \cup X^{-1}$, $1 \leq i \leq n$. (For any subset A of an inverse semigroup, by A^{-1} we mean $\{a^{-1} | a \in A\}$). Thus $y\phi^{-1} \in \langle x_1\rangle \vee \cdots \vee \langle x_n\rangle$, so $y \in \langle x_1\phi\rangle \vee \cdots \vee \langle x_n\phi\rangle$. Hence $T = \langle X\phi\rangle$. The freeness of $\mathscr{F}\mathscr{I}_X$ thus asserts the existence of a unique homomorphism θ of $\mathscr{F}\mathscr{I}_X$ upon T such that $x\theta = x\phi$, $x \in X$. In fact $x\theta = x\phi$ and $(xx^{-1})\theta = (xx^{-1})\phi$ for all $x \in X \cup X^{-1}$ (since ϕ satisfies (ii) of Theorem 1.9).

Theorem 2.1 will be proved by showing that θ and ϕ are identical. Firstly, however, we need some further properties of the idempotents of $\mathcal{FI}_X$.

An element a of $\mathcal{FI}_X$ is said to be a *reduced word* (in $X \cup X^{-1}$) if $a = x_1 \ldots x_n$ for some $x_1, \ldots, x_n \in X \cup X^{-1}$ such that $x_i \neq x_{i+1}^{-1}$, $1 \leq i \leq n-1$. In this case, the description is unique and the number n is called the *length* of a. If $e = aa^{-1}$, then $d(e) = n$. It is an easy consequence of Theorem 2.2 that any idempotent of $\mathcal{FI}_X$ can be expressed in the form $\Pi_{i=1}^{n} r_i r_i^{-1}$, for some reduced words r_i in $X \cup X^{-1}$, $1 \leq i \leq n$. Since θ and ϕ are morphisms on $E(\mathcal{FI}_X)$, to show $\theta = \phi$ on $E(\mathcal{FI}_X)$ it is therefore sufficient to prove the following result.

Lemma 2.6. *Let a be a reduced word in $\mathcal{FI}_X$. Then $(aa^{-1})\theta = (aa^{-1})\phi$.*

Proof. We proceed by induction on the length $l(a)$ of a. From the comments above, the result is true for words of length 1.

Assume, then, that the result is true for all reduced words of length less than n. Since ϕ is injective, this implies in particular that $b\theta$ is non-idempotent for all reduced words b, $l(b) < n$.

Let $a = x_1 \ldots x_n$ be a reduced word of length n, $n \geq 2$, $x_i \in X \cup X^{-1}$. Denote by b_i the subword $x_1 \ldots x_i$ of length i, $1 \leq i \leq n$, by c_j the subword $x_{j+1} \ldots x_n$ of length $n-j$, $0 \leq j \leq n-1$, and by d_i the idempotent $(b_i^{-1}b_i)(c_ic_i^{-1})$, $1 \leq i \leq n-1$. By Lemma 1.1(ii) and the induction hypothesis, $d_i\theta = d_i\phi$, $1 \leq i \leq n-1$.

Now $E(J_a) = \{aa^{-1}, a^{-1}a\} \cup \{d_i \mid 1 \leq i \leq n-1\}$. From Proposition 1.8, $E(J_a\phi) = \{(aa^{-1})\phi, (a^{-1}a)\phi\} \cup \{d_i\phi \mid 1 \leq i \leq n-1\}$. Since $n \geq 2$, $aa^{-1}\mathcal{D}d_1$, so $(aa^{-1})\theta \in E(J_a\phi)$. Now $d_i\mathcal{L}b_i(c_ic_i^{-1}) = ac_i^{-1}\mathcal{R}aa^{-1}$, so that if $(aa^{-1})\theta = d_i\phi (= d_i\theta)$ for some i, $1 \leq i \leq n-1$, then $d_i\theta\mathcal{H}(b_i\theta)(c_ic_i^{-1})\theta$. But T is $\mathcal{H}$-trivial and E-unitary, so $b_i\theta$ is idempotent, a contradiction. Thus $(aa^{-1})\theta$ is either $(aa^{-1})\phi$ or $(a^{-1}a)\phi$. If $(aa^{-1})\theta = (a^{-1}a)\phi$, we note that $aa^{-1} < b_{n-1}b_{n-1}^{-1}$, so $(aa^{-1})\theta \leq (b_{n-1}b_{n-1}^{-1})\theta = (b_{n-1}b_{n-1}^{-1})\phi$. But ϕ is an isomorphism on $E(\mathcal{FI}_X)$, so $a^{-1}a \leq b_{n-1}b_{n-1}^{-1}$. By considering lengths of words we deduce that $a^{-1}a < b_{n-1}b_{n-1}^{-1}$. By representing a by its birooted tree (Theorem 2.2) it follows that $x_k^{-1} = x_{n-k+1}$, $1 \leq k \leq n$. If n is even, $n = 2m$, say, then $x_m^{-1} = x_{m+1}$, a contradiction, and if n is odd, $n = 2m+1$, say, then $x_{m+1}^{-1} = x_{m+1}$, again a contradiction. Therefore $(aa^{-1})\theta = (aa^{-1})\phi$, and the result follows by induction.

Theorem 2.1 now follows immediately, for by the comments immediately preceding Lemma 2.6, $\theta = \phi$ on idempotents. Therefore for any $a \in \mathcal{FI}_X$, $(a\theta)(a\theta)^{-1} = (aa^{-1})\phi = (a\phi)(a\phi)^{-1}$ and $(a\theta)^{-1}(a\theta) = (a^{-1}a)\phi = (a\phi)^{-1}(a\phi)$, so $a\theta = a\phi$ (from triviality of $\mathcal{H}$ on T). The uniqueness of θ is a consequence of the uniqueness of ϕ, in Theorem 1.9.

An immediate consequence of Theorem 2.1 is that $\mathrm{Aut}(\mathcal{L}(\mathcal{FI}_X))$ is isomorphic with $\mathrm{Aut}(\mathcal{FI}_X)$. O'Carroll (**9**) has shown that $\mathrm{Aut}(\mathcal{FI}_n) \cong \{\psi \in S_B \mid (-x)\psi = -(x\psi)$, for all $x \in B\}$, where $B = \{k \in Z \mid -n \leq k \leq n\}$, and S_B is the symmetric group on B. Thus $\mathrm{Aut}(\mathcal{FI}_n)$ has $2^n n!$ elements.

3. Finite Inverse Semigroups

Since every finite inverse semigroup semigroup is completely semisimple, the results of Section 1 apply to finite $\mathcal{H}$-trivial inverse semigroups. In particular we see from Lemma 1.1 and Theorem 1.9 that if S and T are finite $\mathscr{H}$-trivial and are L-isomorphic, $|E(S)| = |E(T)|$ and $|S| = |T|$. In addition, such inverse semigroups have the *basis property* (4), whereby any two minimal generating sets (*bases*) for an inverse subsemigroup have the same number of elements, this number being the *rank* of the inverse subsemigroup. From Theorem 1.9, it is clear that rank S = rank T. (See the discussion following Corollary 2.5).

However, the following non-isomorphic semilattices can be easily shown to have isomorphic lattices of inverse subsemigroups (that is, subsemilattices).

E_1 E_2

Fig. 1

For any semilattice E, define T_E as the inverse subsemigroup of $\mathscr{I}_E$ (the symmetric inverse semigroup on E) consisting of the isomorphisms between principal ideals of E. Munn introduced T_E and showed (6, 7) that $E(T_E) \cong E$, that T_E is fundamental (that is, T_E has no non-trivial congruences contained in $\mathscr{H}$), and further that any fundamental inverse semigroup with semilattice isomorphic with E can be embedded in T_E. Also, T_E is $\mathscr{H}$-trivial if and only if each principal ideal has trivial group of automorphisms.

Theorem 3.1. *Let E be a finite semilattice such that*

(i) *each non-maximal idempotent of E is covered by at least two idempotents of E.*

Then every L-isomorphism of E upon an inverse semigroup F is induced by a (unique) isomorphism of E upon F.

If further, E satisfies

(ii) *each principal ideal has trivial group of automorphisms, then any inverse semigroup L-isomorphic with T_E is isomorphic with T_E.*

Proof. Suppose E satisfies (i), and Φ is an L-isomorphism of E upon F, inducing the bijection ϕ of Lemma 1.1. Now if $e, f \in E$ and $e < f$, then e is not maximal, and so $e = fg$ for some $g \in E$, $g \| f$. Thus $e\phi = (f\phi)(g\phi)$, so $e\phi < f\phi$. Since E is finite, for any $e, f \in E$, with $e < f$, there is a chain $e = e_0 < e_1 < e_2 < \cdots < e_n = f$ of elements of E. Thus $e\phi < f\phi$, and so ϕ is an isomorphism, (as in the proof of Corollary 2.4).

Now suppose E satisfies (ii) in addition, and Φ is an L-isomorphism of T_E upon T. Then $\mathscr{L}(E) \cong \mathscr{L}(E(T))$, and so, from (i), $E \cong E(T)$. From (ii), T_E, and so T, is $\mathscr{H}$-trivial. Then T is isomorphic with an inverse subsemigroup of T_E. But T_E is finite, and $|T| = |T_E|$. So $T \cong T_E$.

The free semilattice $\mathscr{SL}_n$ on a set of n elements satisfies (i)—thus we have an analogue of Theorem 2.1 for free semilattices. There are numerous examples of semilattices satisfying both (i) and (ii) (since, for example, any finite semilattice whose principal ideals are chains satisfies (ii)).

In general, however, T_E is not determined by $\mathscr{L}(T_E)$, even in the $\mathscr{H}$-trivial case. For example the semilattices E_1 and E_2 cited earlier satisfy (ii) of Theorem 3.1 (but not (i)), but direct calculation shows $\mathscr{L}(T_{E_1}) \cong \mathscr{L}(T_{E_2})$.

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