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# LATTICE OF IDEMPOTENT SEPARATING CONGRUENCES IN A $\mathbb{P}$-REGULAR SEMIGROUP

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In this paper we introduce the concept of $\mathbb{P}$-regular, normal subset of a $\mathbb{P}$-regular semigroup, give an alternate characterisation of the maximum idempotent separating congruence in a $\mathbb{P}$-regular semigroup and finally describe the lattice of the idempotent separating congruences in a $\mathbb{P}$-regular semigroup.

## 1. INTRODUCTION

Yamada and Sen introduced the concept of $\mathbb{P}$-*regularity* [4] in a regular semigroup as a generalisation of both the concept of "orthodox" and the concept of "(special) involution" [3]. In [4] they characterised the maximum idempotent separating congruence $\tau$ on a $\mathbb{P}$-regular semigroup $S$. In this paper firstly we give an alternate characterisation of $\tau$ and secondly we give a description of the lattice of idempotent separating congruences on a $\mathbb{P}$-regular semigroup $S$, which generalises Feigenbaum's result [1] for orthodox semigroups. Unless otherwise defined, our notation will be that of [2].

## 2. PRELIMINARY RESULTS AND DEFINITIONS

Let $S$ be a regular semigroup and $E_S$ the set of idempotents of $S$. Let $P \subseteq E_S$. Then $(S, P)$ is called a $\mathbb{P}$-*regular* semigroup [5] if it satisfies the following:

(1) $P^2 \subseteq E_S$
(2) for each $q \in P$, $qPq \subseteq P$.
(3) for any $a \in S$, there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$. ($P^1 = P \cup \{1\}$ and $V(a) = \{a^+ \in S : aa^+a = a,\ a^+aa^+ = a^+\}$).

Hereafter $(S, P)$ will be denoted by $S(P)$. Let $S(P)$ be a $\mathbb{P}$-regular semigroup. For any $a \in S$, $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$ is called a $\mathbb{P}$-*inverse of* $a$ and $V_P(a)$ denotes the set of all $\mathbb{P}$-inverses of $a$. Now $V_P(a) \neq \emptyset$ for any $a \in S$. Also for any $p \in P$, $p \in V(P)$ and $pP^1p \subseteq P$. So $p \in V_P(p)$. The following lemma follows from Lemma 2.5 of [5].

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**Lemma 2.1.** *In a $\mathbb{P}$-regular semigroup $S(P)$, if $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, then $b^+a^+ \in V_P(ab)$.*

The following lemma, which will be used frequently in this paper, is taken from [5].

**Lemma 2.2.** *Let $a, b$ be two elements of a $\mathbb{P}$-regular semigroup $S(P)$. Then $a\mathcal{H}b$ (where $\mathcal{H}$ is Green's $\mathcal{H}$-relation) if and only if there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$.*

Note. Actually if $a\mathcal{H}b$ then for all $a^+ \in V_P(a)$ there exists $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$. Because for any $a^+ \in V_P(a)$ we have $a^+a\mathcal{L}a\mathcal{R}aa^+$. Consequently $a^+a\mathcal{L}b\mathcal{R}aa^+$. So by Lemma 2.7 of [5] there exists a unique $b^+ \in V_P(b)$ such that $bb^+ = aa^+$ and $b^+b = a^+a$. Moreover we notice that if $(a, p) \in \mathcal{H}$ for some $p \in P$ then as $p \in V_P(p)$ there exists a unique $a^+ \in V_P(a)$ such that $aa^+ = pp (= p) = a^+a$. But this implies $(a^+, p) \in \mathcal{H}$ by Lemma 2.2. Hence $a^+ \in V_P(a) \cap H_a$.

A subset $B$ of a $\mathbb{P}$-regular semigroup $S(P)$ will be called *$\mathbb{P}$-regular* if for every $x \in B$ there exists $x^+ \in V_P(x)$ such that $x^+ \in B$.

A subset $N$ of a $\mathbb{P}$-regular semigroup $S(P)$ will be called *normal* if for any $a, b \in S$, and $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, such that $aa^+ = bb^+$, $a^+a = b^+b$ then $ab^+$, $a^+b \in N$ imply $aNb^+ \subseteq N$ and $a^+Nb \subseteq N$.

**Lemma 2.3.** *In a $\mathbb{P}$-regular semigroup $S(P)$ the set $P$ is a $\mathbb{P}$-regular normal subset of $S$.*

Proof: For each $p \in P$, $p \in V(p)$ and $pP^1p \subseteq P$. So $p \in V_P(p) \cap P$. Thus $P$ is a $\mathbb{P}$-regular subset of $S$. Also we note that if for any $a, b \in S$ there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$ then $ab^+$, $a^+b \in P$ imply $aPb^+ = aa^+aPb^+bb^+ = bb^+(aPa^+)ab^+ = bb^+bb^+(aPa^+)ab^+ab^+ = b(a^+(ab^+(aPa^+)ab^+)a)b^+ \subseteq P$ and $a^+Pb = a^+aa^+Pbb^+b = a^+b(b^+Pb)a^+a = a^+ba^+b(b^+Pb)a^+aa^+a = a^+(b(a^+b(b^+Pb)a^+b)b^+)a \subseteq P$. Thus $P$ is a normal subset of $S$. ∎

## 3. The lattice of idempotent separating congruences

Yamada and Sen [4] characterised $\tau$, the maximum idempotent separating congruence on a $\mathbb{P}$-regular semigroup $S(P)$, as $\tau = \{(a, b) \in S \times S :$ there exist $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ such that $axa^+ = bxb^+$ and $a^+xa = b^+xb$ for all $x \in P\}$. Here we shall present an alternate characterisation of $\tau$. For this we define *centraliser of $P$* to be $C(P) = \{x \in S : (x, p) \in \tau$ for some $p \in P\}$.

**Theorem 3.1.** *Let $\delta = \{(a, b) \in S \times S :$ there exists $a^+ \in V_P(a), b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$, $ab^+$, $a^+b \in C(P)\}$. Then $\tau = \delta$.*

PROOF: Let $(a, b) \in \tau$. Then as $\tau$ is an idempotent separating congruence on a regular semigroup, $\tau \subseteq \mathcal{H}$. Hence by Lemma 2.2, there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$. Also $(ab^+, bb^+) \in \tau$, $(a^+a, a^+b) \in \tau$ where $bb^+$ and $a^+a \in P$. Therefore $ab^+$, $a^+b \in C(P)$ and so $(a, b) \in \delta$. Conversely let $(a, b) \in \delta$. Then there are inverses $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+$, $a^+b \in C(P)$. So $a\mathcal{H}b$ and $a^+\mathcal{H}b^+$ which gives $ab^+\mathcal{H}bb^+$. Since $ab^+ \in C(P)$, we have $(ab^+)\tau = (bb^+)\tau$. Therefore $a\tau = (a\tau)(a^+a)\tau = (a\tau)(b^+b)\tau = (ab^+)\tau b\tau = (bb^+)\tau b\tau = b\tau$. Thus $\delta = \tau$. ∎

**LEMMA 3.2.** *$C(P)$ is a $\mathbb{P}$-regular, normal subset of $S$.*

PROOF: Let $a \in C(P)$; then $(a, p) \in \tau$, for some $p \in P$. So $(a, p) \in \mathcal{H}$. Hence by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Also $(a^+a, a^+p) \in \tau$ implies that $(p, a^+) \in \tau$. Thus $a^+ \in C(P)$. Next let $a, b \in S$ be such that there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+$, $a^+b \in C(P)$. Then by Theorem 3.1 $(a, b) \in \tau$. So $(axb^+, bxb^+) \in \tau$ for all $x \in C(P)$. Now $x \in C(P)$ implies that $(x, p) \in \tau$ for some $p \in P$. So $(bxb^+, bpb^+) \in \tau$ where $bpb^+ \in P$. Therefore $axb^+ \in C(P)$. Similarly, $a^+xb \in C(P)$ for all $x \in C(P)$. So $C(P)$ is a $\mathbb{P}$-regular, normal subset of $S$. ∎

**LEMMA 3.3.** *Let $A = \{a \in S :$ there exists $a^+ \in V_P(a)$ for which $a^+pap = aa^+p$, and $papa^+ = pa^+a$ for each $p \in P\}$. Then $C(P) = A$.*

PROOF: Let $a \in C(P)$. Then $(a, q) \in \tau$ for some $q \in P$. Since $\tau \subseteq \mathcal{H}$ by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = q$. Now $(a, q) \in \tau$ implies that $(a^+a, a^+q) \in \tau$, that is $(a^+, q) \in \tau$. For each $p \in P$, we have $(apa^+, qpq) \in \tau$ and $(a^+pa, qpq) \in \tau$ which implies that $apa^+ = qpq = a^+pa$. So we have, $a^+pap = qpqp = qp = aa^+p$ and $papa^+ = pqpq = pq = pa^+a$. Conversely, if $a \in A$, then there exists $a^+ \in V_P(a)$ such that $a^+pap = aa^+p$ and $papa^+ = pa^+a$ for all $p \in P$. Since $aa^+ \in P$, $a^+a \in P$, we have $aa^+ = aa^+aa^+ = a^+(aa^+)a(aa^+) = a^+aaa^+ = (a^+a)a(a^+a)a^+ = (a^+a)a^+a = a^+a$. So for any $p \in P$, $apa^+ = a(a^+a)pa^+ = a(aa^+p)a^+ = a(a^+pap)a^+ = (aa^+)(papa^+) = aa^+p(a^+a) = aa^+paa^+$ and $a^+pa = a^+p(aa^+)a = a^+(pa^+a)a = a^+(papa^+)a = (a^+pap)a^+a = aa^+p(a^+a) = (aa^+)p(aa^+)$. Hence $(a, aa^+) \in \tau$. Therefore $a \in C(P)$. Thus $A = C(P)$. ∎

The following lemma will be used frequently in the proof of the next theorem.

**LEMMA 3.4.** *Let $K$ be a normal subset of a $\mathbb{P}$-regular semigroup $S(P)$ such that $P \subseteq K$. For $a, b \in S$ if there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+$, $a^+b \in K$ then $ba^+$, $b^+a \in K$ and $ab^*$, $a^*b \in K$ for all $a^* \in V_P(a)$, $b^* \in V_P(b)$.*

PROOF: First we note that in a normal subset $K(\supseteq P)$, $aKa^+ \subseteq K$ for all $a \in S$,

$a^+ \in V_P(a)$. Now

$$ba^+ = bb^+ba^+ = aa^+ba^+ \in aKa^+ \subseteq K,$$
$$b^+a = b^+aa^+a = b^+ab^+b \in b^+Kb \subseteq K,$$
$$ab^* = aa^+ab^* = bb^+ab^* \in bKb^* \subseteq K,$$

and

$$a^*b = a^*bb^+b = a^*ba^+a \in a^*Ka \subseteq K.$$

∎

The following theorem gives a description of the lattice of idempotent separating congruences on a $\mathbb{P}$-regular semigroup $S(P)$.

Let $N_r = \{K \subseteq S : P \subseteq K \subseteq C(P)$ where $K$ is normal and $\mathbb{P}$-regular$\}$. Clearly $P$ and $C(P)$ belong to $N_r$.

**THEOREM 3.5.** *The map $K \to (K) = \{(a, b) \in S \times S :$ there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ for which $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+$, $a^+b \in K\}$ is a one to one order preserving map of $N_r$ onto the set of all idempotent separating congruences on $S$.*

PROOF: First we shall show that if $K \in N_r$ then $(K)$ is an idempotent separating congruence on $S$. Since $P \subseteq K$, $(K)$ is a reflexive relation. Furthermore, $K \subseteq C(P)$ implies $(K) \subseteq \tau$ (by Theorem 3.1), so that $(K)$ is an idempotent separating relation. Also if $(a, b) \in (K)$ then by Lemma 3.4 $(b, a) \in (K)$. Hence $(K)$ is symmetric. To prove that $(K)$ is transitive, let $(a, b) \in (K)$, $(b, c) \in (K)$. Then there exist $a^+ \in V_P(a)$, $b^+, b^* \in V_P(b)$, $c^* \in V_P(c)$ such that $aa^+ = bb^+$, $a^+a = b^+b$, $bb^* = cc^*$, $b^*b = c^*c$. But $(K) \subseteq \tau \subseteq \mathcal{H}$ implies that $a\mathcal{H}b\mathcal{H}c$ so that by the Note after Lemma 2.2 there exists $c^+ \in V_P(c)$ such that $aa^+ = bb^+ = cc^+$ and $a^+a = b^+b = c^+c$. Also $ab^+$, $a^+b$, $bc^+$, $b^+c \in K$ by Lemma 3.4. Therefore $ac^+ = aa^+ac^+cc^+ = ab^+bc^+bb^+ \in K$ and $a^+c = a^+aa^+cc^+c = a^+bb^+cb^+b \in K$, as $K$ is normal. So $(a, c) \in (K)$. To see that $(K)$ is compatible, let $(a, b) \in (K)$, $(c, d) \in (K)$. Since $K \subseteq C(P)$ we have $(K) \subseteq \tau$. So $(a, b)$, $(c, d) \in \tau$. Hence there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $c^+ \in V_P(c)$, $d^+ \in V_P(d)$ such that $apa^+ = bpb^+$, $a^+pa = b^+pb$, $cpc^+ = dpd^+$, $c^+pc = d^+pd$ for all $p \in P$. Then it follows that $aa^+ = bb^+$, $a^+a = b^+b$, $cc^+ = dd^+$, $c^+c = d^+d$ and by Lemma 2.1 $c^+a^+ \in V_P(ac)$, $d^+b^+ \in V_P(bd)$. Now $(ac)(c^+a^+) = a(cc^+)a^+ = a(dd^+)a^+ = b(dd^+)b^+ = (bd)(d^+b^+)$, $(c^+a^+)(ac) = c^+(a^+a)c = c^+(b^+b)c = d^+(b^+b)d = (d^+b^+)(bd)$, $(ac)(d^+b^+) = a(cd^+)b^+ \in aKb^+ \subseteq K$ and $(c^+a^+)(bd) = c^+(a^+b)d \in c^+Kd \subseteq K$ as $cd^+$ and $a^+b \in K$ by Lemma 3.4. Thus $(ac, bd) \in (K)$. Therefore $(K)$ is an idempotent separating congruence. Now if $\mu$ is an idempotent separating congruence on $S$ then we define the $P$-kernel of $\mu$, written as $P$-ker $\mu$, by $P\text{-ker } \mu = \{a \in S : a\mu p$ for some $p \in P\}$. We shall show that $P$-ker $\mu \in N_r$ and $(P\text{-ker}\mu) = \mu$. To prove that $P$-ker $\mu$ is a $\mathbb{P}$-regular subset of $S$, let $a \in P$-ker

$\mu$. Then $(a, p) \in \mu \subseteq \mathcal{H}$ for some $p \in P$. So by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Now $(a^+a, a^+p) \in \mu$ or, $(p, a^+) \in \mu$. Therefore $a^+ \in P$-ker $\mu$. To prove that $P$-ker $\mu$ is normal, let $a, b \in S$, $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $a^+b$, $ab^+ \in P$-ker $\mu$. We shall show that $axb^+$ and $a^+xb$ belong to $P$-ker $\mu$ for all $x \in P$-ker $\mu$. From the given condition $a\mathcal{H}b$, $a^+\mathcal{H}b^+$. Therefore $ab^+\mathcal{H}bb^+$. As $ab^+ \in P$-ker $\mu$, we have $(ab^+)\mu = (bb^+)\mu$. Therefore $a\mu = a\mu(a^+a)\mu = a\mu(b^+b)\mu = (ab^+)\mu b\mu = (bb^+)\mu b\mu = b\mu$. Hence $(a, b) \in \mu$. So $(axb^+, bxb^+) \in \mu$ for all $x \in P$-ker $\mu$. Now $x \in P$-ker $\mu$ implies that $(x, p) \in \mu$ for some $p \in P$. Hence $(bxb^+, bpb^+) \in \mu$. Thus $(axb^+, bpb^+) \in \mu$. But $bpb^+ \in P$. So $axb^+ \in P$-ker $\mu$. Similarly $a^+xb \in P$-ker $\mu$ for all $x \in P$-ker $\mu$. Also $P \subseteq P$-ker $\mu \subseteq C(P)$. Therefore $P$-ker $\mu \in N_r$. Now we shall show that $(P\text{-ker } \mu) = \mu$. For this let $(a, b) \in (P\text{-ker } \mu)$; then there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+$, $a^+b \in P$-ker $\mu$. Therefore $a\mathcal{H}b$, $a^+\mathcal{H}b^+$ which implies that $ab^+\mathcal{H}bb^+$. As $ab^+ \in P$-ker $\mu$ we have $(ab^+)\mu = (bb^+)\mu$. Therefore $a\mu = a\mu(a^+a)\mu = a\mu(b^+b)\mu = (ab^+)\mu b\mu = (bb^+)\mu b\mu = b\mu$. Next let $(a, b) \in \mu$. Then $(a, b) \in \mathcal{H}$. So there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$. Also $(ab^+, bb^+) \in \mu$ and $(a^+a, a^+b) \in \mu$. Therefore $ab^+, a^+b \in P$-ker $\mu$. Hence $(a, b) \in (P\text{-ker } \mu)$. Thus the given map is onto. The given map is clearly order preserving. We shall now show that the given map is one to one. For this let $K, L \in N_r$ with $(K) = (L)$. Let $a \in K$, since $K \subseteq C(P)$, $a \in C(P)$. Therefore $(a, p) \in \tau$ for some $p \in P$. Since $\tau \subseteq \mathcal{H}$ by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Now $K$ is $\mathbb{P}$-regular so there exists $a^* \in V_P(a) \cap K$. Then $a^+ = a^+aa^+ = (a^+a)a^*(aa^+) = pa^*p \in K$. Thus $a\mathcal{H}a^+a$, $a(a^+a) = a \in K$ and $a^+(a^+a) = a^+(aa^+) = a^+ \in K$. Therefore $(a, a^+a) \in (K) = (L)$. So $a(a^+a)^* \in L$ for all $(a^+a)^* \in V_P(a^+a)$ by Lemma 3.4. So in particular $aa^+a \in L$ so that $a \in L$. Thus $K \subseteq L$. Similarly we can prove $L \subseteq K$. Thus $K = L$. ∎

NOTE. If we take $P = E$ then $S$ becomes an orthodox semigroup and $N_r$ becomes the collection of all self conjugate, regular subsemigroups $K$ such that $E \subseteq K \subseteq C(E)$ so we get the Theorem 3.3 of [1] due to Feigenbaum, because in that case $a, b \in K$ implies $a, b \in C(E)$. So $a\tau e$, $b\tau f$ for some $e, f \in E$ and there exist $a^+ \in V(a)$, $b^+ \in V(b)$ such that $aa^+ = a^+a = e$, $bb^+ = b^+b = f$ and $ae = a \in K$, $a^+e = a^+ \in K$ and $abef = ab = efab$. Since $K$ is normal we have $aKe \subseteq K$. Therefore $efabeef \in efKef$ or, $ab \in efKef \subseteq K$. If we take $P = \{e\}$ then $S$ becomes a group and $N_r$ becomes the collection of all normal subgroups of the group $S$.

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