Lattice Points, Dedekind Sums, and Ehrhart Polynomials of Lattice Polyhedra*

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Dedicated to the Memory of Gian-Carlo Rota

Abstract. Let σ be a simplex of $\mathbf{R}^N$ with vertices in the integral lattice $\mathbf{Z}^N$. The number of lattice points of $m\sigma$ ($= \{m\alpha: \alpha \in \sigma\}$) is a polynomial function $L(\sigma, m)$ of $m \geq 0$. In this paper we present: (i) a formula for the coefficients of the polynomial $L(\sigma, t)$ in terms of the elementary symmetric functions; (ii) a hyperbolic cotangent expression for the generating functions of the sequence $L(\sigma, m), m \geq 0$; (iii) an explicit formula for the coefficients of the polynomial $L(\sigma, t)$ in terms of torsion. As an application of (i), the coefficient for the lattice n-simplex of $\mathbf{R}^n$ with the vertices $(0, \ldots, 0, a_j, 0, \ldots, 0)$ $(1 \leq j \leq n)$ plus the origin is explicitly expressed in terms of Dedekind sums; and when $n = 2$, it reduces to the reciprocity law about Dedekind sums. The whole exposition is elementary and self-contained.

1. Introduction

Let P be a lattice polytope of $\mathbf{R}^N$, i.e., the vertices of P are points of the integral lattice $\mathbf{Z}^N$. The dilation of P by a positive integer m is the polytope $mP = \{m\alpha: \alpha \in P\}$; the relative interior of P is the set $P^0 = P - \partial P$. Let $L(P, m)$ and $L(P^0, m)$ denote the number of lattice points of mP and mP^0, respectively. It is well known that $L(P, m)$ and $L(P^0, m)$ are polynomial functions of m, called the *Ehrhart polynomial*, and satisfy the functional equation

$$L(P, -t) = (-1)^{\dim P} L(P^0, t), \tag{1}$$

called the *reciprocity law* by Ehrhart [9]. Let $n = \dim P$ and write

$$L(P, t) = \sum_{j=0}^{n} c_j(P) t^j. \tag{2}$$

It is known that $c_n(P)$ is the volume vol P, $c_{n-1}(P)$ is the boundary surface area vol ∂P, and $c_0(P)$ is the (combinatorial) Euler characteristic $\chi(P)$, where vol is normalized by the determinants of the sublattices induced by the corresponding faces of P; see [20]. The Ehrhart polynomial $L(P, m)$ can be defined for any bounded lattice polyhedron P (a union of finitely many relatively open lattice polytopes). If P is not compact, the combinatorial Euler characteristic is usually different from that defined by the alternating sum of Betti numbers; while if P is compact, they are the same; see [4] and [12] for example. The other coefficients of Ehrhart polynomials are still mysterious, even for a general lattice 3-simplex, until the recent work of Morelli [16] in $\mathbf{R}^n$, Pommersheim [18] in $\mathbf{R}^3$, Kantor and Khovanskii [11] in $\mathbf{R}^4$, Cappell and Shaneson [3] in $\mathbf{R}^n$, Brion and Vergne [2] in $\mathbf{R}^n$, and Diaz and Robins [8] in $\mathbf{R}^n$. For instance, the coefficients for a lattice tetrahedron of $\mathbf{R}^3$ with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ are completely determined; Kanor and Khovanskii [11] gave a complete description of the codimension 2 coefficients of the Ehrhart polynomials. However, the work of [16] used the Bott theorem of differential geometry to express these coefficients in terms of rational functions on Grassmanians; [3] and [11] used the Riemann–Roch theorem of algebraic geometry to relate the Todd class and the Chern class of toric variety to express these coefficients in terms of Dedekind sums and cotangent functions; and the work of [2] used the technique of deformation for simple-polytopes and a combinatorial version of the Riemann–Roch theorem. The work of [8] is to express the generating functions of the sequences $L(\sigma, m)$ and $L(\sigma^0, m)$ in terms of hyperbolic cotangent functions for a lattice simplex σ, whose vertices are the column vectors of a lower triangular matrix plus the origin. These generating functions are important because the coefficients of Ehrhart polynomials can be easily computed from their explicit expressions. The method of [8] is analytic, using the Poisson summation and the Fourier analysis, and involves some lengthy estimations. It seems that it is still an open problem to find a general formula with geometric interpretation for the coefficients of Ehrhart polynomials.

This paper is inspired by the work of Brion and Vergne [2], Diaz and Robins [8], and the original work of Ehrhart [9]. We present in a simple direct way: (i) a formula for the coefficients of Ehrhart polynomials in terms of the elementary symmetric functions; (ii) a hyperbolic cotangent expression for the generating functions of the sequences $L(\sigma, m)$ and $L(\sigma^0, m)$ of any lattice simplex σ; (iii) an explicit formula for the coefficients of Ehrhart polynomials in terms of torsion. The whole exposition is elementary and self-contained.

To state our results, we begin with some notations. Let σ be an n-simplex of $\mathbf{R}^N$ with vertices $\alpha_0, \alpha_1, \ldots, \alpha_n$ in $\mathbf{Z}^N$; its relative interior is denoted by $\sigma^0 = \sigma - \partial \sigma$. We define the *determined set of σ at the vertex α_0* by

$$D(\sigma, \alpha_0) = \left\{ \sum_{j=1}^{n} u_j(\alpha_j - \alpha_0) \in \mathbf{Z}^N \colon 0 \leq u_j < 1 \right\}. \tag{3}$$

For each vector $\gamma = \sum_{j=1}^{n} u_j(\alpha_j - \alpha_0) \in D(\sigma, \alpha_0)$, we write $|\gamma| = \sum_{j=1}^{n} u_j$. Our first main result is on the jth coefficients of $L(\sigma^0, t)$ and $L(\sigma, t)$:

$$c_j(\sigma^0) = \frac{1}{n!} \sum_{\gamma \in D(\sigma, \alpha_0)} s_{n-j}(\lceil |\gamma| \rceil - 1, \ldots, \lceil |\gamma| \rceil - n), \tag{4}$$

$$c_j(\sigma) = \frac{1}{n!} \sum_{\gamma \in D(\sigma, \alpha_0)} s_{n-j}(1 - \lceil |\gamma| \rceil, \ldots, n - \lceil |\gamma| \rceil), \tag{5}$$

where $s_{n-j}(x_1, \ldots, x_n)$ is the $(n-j)$th elementary symmetric function of n variables; $\lceil |\gamma| \rceil$ is the smallest integer greater than or equal to $|\gamma|$. Now the reciprocity law is read to

$$c_j(\sigma) = (-1)^{n-j} c_j(\sigma^0).$$

The coefficients c_j are obviously well-defined for any bounded lattice polyhedra; thus c_j are unimodular invariant measures on bounded lattice polyhedra of $\mathbf{R}^N$. Coincidently, the measures c_j are discrete analogs of the intrinsic volumes V_j, which are unique rigid motion invariant measures on bounded polyhedra of $\mathbf{R}^N$, such that

$$V_j(P) = s_j(x_1, \ldots, x_n)$$

for rectangular parallelotopes P of side length $x_1, \ldots, x_n$; see [12].

Let A be the $N \times n$ matrix whose columns are the vectors $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$. The matrix A can be viewed as an Abelian group homomorphism from $\mathbf{Z}^n$ to $\mathbf{Z}^N$; the *torsion* of A is defined to be the order of the torsion subgroup of $\mathbf{Z}^N / \operatorname{Im} A$, denoted tor A. Set $p = \operatorname{tor} A$; our second main result is

$$c_j(\sigma^0) = \frac{1}{n!} \sum_{\substack{0 \le h_1, \ldots, h_n \le p-1 \\ h = (h_1, \ldots, h_n) \in \mathbf{Z}^n \\ p \mid Ah}} s_{n-j} \left(\left\lceil \frac{h_1 + \cdots + h_n}{p} \right\rceil - 1, \right.$$

$$\left. \ldots, \left\lceil \frac{h_1 + \cdots + h_n}{p} \right\rceil - n \right), \tag{6}$$

where $p \mid Ah$ means that p divides each entry of the vector Ah.

The matrix A can be transformed into an upper triangular form (7), having $a_{jj} > 0$, by the *modified Gaussian elimination* (adding an integral multiple of one row to another row; switching two rows):

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{7}$$

When $N = n$, this upper triangular matrix is known as the *Hermite normal form*; see [17]. For $(j_1, \ldots, j_n) \in \prod_{i=1}^{n}[0, a_{ii}) \cap \mathbf{Z}$, we define $u(j_1, \ldots, j_n) = (u_1, \ldots, u_n)$ inductively by

$$
\begin{cases}
u_n &= \dfrac{j_n}{a_{nn}}, \\[2mm]
u_i &= \dfrac{\lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil}{a_{ii}} \\[2mm]
&\quad - \dfrac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} + \dfrac{j_i}{a_{ii}}, \qquad 1 \le i \le n-1;
\end{cases}
\tag{8}
$$

and $|u(j_1, \ldots, j_n)| = u_1 + \cdots + u_n$. Then (4) and (5) for $c_j(\sigma^0)$ and $c_j(\sigma)$ can be further written as

$$
c_j(\sigma^0) = \frac{1}{n!} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} s_{n-j}(\lceil |u(j_1, \ldots, j_n)|\rceil - 1, \ldots, \lceil |u(j_1, \ldots, j_n)|\rceil - n),
\tag{9}
$$

$$
c_j(\sigma) = \frac{1}{n!} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} s_{n-j}(1 - \lceil |u(j_1, \ldots, j_n)|\rceil, \ldots, n - \lceil |u(j_1, \ldots, j_n)|\rceil).
\tag{10}
$$

Consequently, the generating functions of the sequences $L(\sigma, m)$ and $L(\sigma^0, m)$ are given by

$$
\sum_{m=0}^{\infty} L(\sigma, m)t^m = \frac{1}{(1-t)^{n+1}} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} t^{\lceil |u(j_1, \ldots, j_n)|\rceil};
\tag{11}
$$

$$
\sum_{m=1}^{\infty} L(\sigma^0, m)t^m = \frac{t^{n+1}}{(1-t)^{n+1}} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} t^{-\lceil |u(j_1, \ldots, j_n)|\rceil}.
\tag{12}
$$

If the matrix A is already in upper triangular form (7) without applying the modified Gaussian elimination, then the map $\tilde{u}\colon (j_1, \ldots, j_n) \mapsto \sum_{i=1}^{n} u_i(\alpha_i - \alpha_0)$ by (8) is a one-to-one correspondence between $\prod_{i=1}^{n}[0, a_{ii}) \cap \mathbf{Z}$ and the determined set $D(\sigma, \alpha_0)$, which gives an efficient algorithm to list all elements of $D(\sigma, \alpha_0)$.

Let B be the $(N + 1) \times (n + 1)$ matrix whose columns are the vectors $\beta_0 = (1, \alpha_0)$, $\beta_1 = (1, \alpha_1), \ldots, \beta_n = (1, \alpha_n)$. Let $\mathrm{adj}(B^{\mathsf{T}}B)$ denote the adjoint matrix of $B^{\mathsf{T}}B$. For $0 \le j \le n$ and $0 \le k \le N$, define positive integers:

$$
a_j = \text{gcd of the } j\text{th row of } \mathrm{adj}(B^{\mathsf{T}}B)B^{\mathsf{T}},
$$

$$
b_k = \text{gcd of the } k\text{th row of } B \operatorname{diag}[a_0, a_1, \ldots, a_n],
$$

$$
p_j = \frac{|\det B^{\mathsf{T}}B|}{a_j},
$$

$$
q_k = \frac{|\det B^{\mathsf{T}}B|}{\gcd(b_k, \det B^{\mathsf{T}}B)}.
$$

Set $G = \prod_{k=0}^{N} \mathbf{Z}/q_k\mathbf{Z}$; our third main result is on the generating functions of the sequences $L(\sigma, m)$ and $L(\sigma^0, m)$:

$$\sum_{m=0}^{\infty} L(\sigma, m)e^{-2\pi ms} = \frac{1}{2^{n+1}|G|} \sum_{g \in G} \prod_{j=0}^{n} \left(1 + \coth\frac{\pi}{p_j}(s + \iota\langle\beta_j, g\rangle)\right); \quad (13)$$

$$\sum_{m=1}^{\infty} L(\sigma^0, m)e^{-2\pi ms} = \frac{(-1)^{n+1}}{2^{n+1}|G|} \sum_{g \in G} \prod_{j=0}^{n} \left(1 - \coth\frac{\pi}{p_j}(s + \iota\langle\beta_j, g\rangle)\right). \quad (14)$$

Set $p = \operatorname{tor} A$; we further have

$$\sum_{m=0}^{\infty} L(\sigma, m)e^{-2\pi ms}$$

$$= \frac{1}{2^{n+1}p^{N+1}} \sum_{\substack{h_0,h_1,\dots,h_N=0 \\ h=(h_0,h_1,\dots,h_N)}}^{p-1} \prod_{j=0}^{n} \left(1 + \coth\frac{\pi}{p}(s + \iota\langle\beta_j, h\rangle)\right); \quad (15)$$

$$\sum_{m=1}^{\infty} L(\sigma^0, m)e^{-2\pi ms}$$

$$= \frac{(-1)^{(n+1)}}{2^{n+1}p^{N+1}} \sum_{\substack{h_0,h_1,\dots,h_N=0 \\ h=(h_0,h_1,\dots,h_N)}}^{p-1} \prod_{j=0}^{n} \left(1 - \coth\frac{\pi}{p}(s + \iota\langle\beta_j, h\rangle)\right). \quad (16)$$

In particular, when $N = n$, one can choose

$$a_j = \text{gcd of the } j\text{th row of adj } B,$$

$$b_k = \text{gcd of the } k\text{th row of } B \operatorname{diag}[a_0, a_1, \dots, a_n],$$

$$p_j = \frac{|\det B|}{a_j},$$

$$q_k = \frac{|\det B|}{\gcd(b_k, \det B)}$$

for (13) and (14). Moreover, in this special case $N = n$, if the $n \times n$ matrix A is in upper triangular form (7), one can simply choose $p_j = q_j = a_{jj}\cdots a_{nn}$, $0 \leq j \leq n$, where $a_{00} = 1$; $G = \prod_{j=0}^{n} \mathbf{Z}/p_j\mathbf{Z}$; and recover the main result of [8].

The hyperbolic cotangent functions can be also written as cotangent functions. A notable feature of our formulas is that the group G can be selected as long as certain conditions are satisfied. Since every bounded lattice polyhedron P can be decomposed into a disjoint union of relatively open lattice simplices, the coefficients of the Ehrhart polynomial $L(P, t)$ can be computed by the coefficient formulas (4)–(6), (9), (10) and the generating functions (11)–(16).

As an application of the coefficient formula (4), we consider the lattice n-simplex $\sigma(a_1, \dots, a_n)$ with the vertices $(0, \dots, 0, a_j, 0, \dots, 0)$ $(1 \leq j \leq n)$ plus the origin; and express the coefficient c_{n-2} explicitly in terms of Dedekind sums. When $n = 2$,

it reduces to the reciprocity law about Dedekind sums. The other coefficients can be similarly expressed in terms of Zagier's higher-dimensional Dedekind sums. While for an arbitrary lattice simplex, it is not clear whether the higher Dedekind sums are enough to express those coefficients. For this reason, we introduce a matrix Dedekind sum, which seems to be enough to do the job; and it should be of interest to study itself.

2.   Deriving Ehrhart Polynomials

To derive our first result and to make our exposition transparent and self-contained, we essentially follow the idea of Ehrhart [9] to derive some known (but not well-known) results, with more sophisticated notations and arguments.

Let Λ be an N-dimensional lattice of $\mathbf{R}^N$. Let σ be an n-simplex with vertices $\alpha_0, \alpha_1, \ldots, \alpha_n$. We assume that there is a positive real number d such that all $d\alpha_j$ are points of Λ. Choose a lattice basis of Λ; the vertices of σ may have irrational coordinates with respect to the lattice basis, and in this case d must be an irrational number. If the vertices of σ have rational coordinates with respect to the lattice basis, then d can be the lcm (least common multiple) of the denominators of the rational vertices. We want to count the number of points of Λ in the dilation $\lambda\sigma$ by a positive real number λ. Let $E(\sigma, \Lambda, \lambda)$ denote the number of points of $\lambda\sigma \cap \Lambda$, and $E(\sigma^0, \Lambda, \lambda)$ the number of points of $\lambda\sigma^0 \cap \Lambda$. Write each $\alpha \in \lambda\sigma$ and $\beta \in \lambda\sigma^0$ as the unique affine linear combinations:

$$\alpha = \sum_{j=0}^{n} a_j \lambda \alpha_j, \qquad \sum_{j=0}^{n} a_j = 1, \quad a_j \geq 0;$$

$$\beta = \sum_{j=0}^{n} b_j \lambda \alpha_j, \qquad \sum_{j=0}^{n} b_j = 1, \quad b_j > 0.$$

By the division algorithm, α and β can be further uniquely written as

$$\alpha = \sum_{j=0}^{n} u_j \alpha_j + \sum_{j=0}^{n} k_j \, d\alpha_j,$$

$$\beta = \sum_{j=0}^{n} v_j \alpha_j + \sum_{j=0}^{n} l_j \, d\alpha_j,$$

with real numbers u_j and v_j, non-negative integers k_j and l_j, such that

$$\sum_{j=0}^{n} u_j + d \sum_{j=0}^{n} k_j = \lambda, \qquad 0 \leq u_j < d, \quad k_j = \left\lfloor \frac{a_j \lambda}{d} \right\rfloor,$$

$$\sum_{j=0}^{n} v_j + d \sum_{j=0}^{n} l_j = \lambda, \qquad 0 < v_j \leq d, \quad l_j = \left\lceil \frac{b_j \lambda}{d} \right\rceil - 1.$$

Clearly, α is a point of Λ if and only if $\gamma = \sum_{j=0}^{n} u_j \alpha_j$ is a point of Λ; β is a point of Λ if and only if $\sum_{j=0}^{n} v_j \alpha_j$ is a point of Λ. Write $|\gamma| = \sum_{j=0}^{n} u_j$ and $k = \sum_{j=0}^{n} k_j$. For

a non-negative real number $\ell \leq (n+1)d$, define the *determined sets of σ at level ℓ*:

$$D(\sigma, \Lambda, \ell) = \left\{ \gamma = \sum_{j=0}^{n} u_j \alpha_j \in \Lambda \colon 0 \leq u_j < d, |\gamma| = \ell \right\}; \tag{17}$$

$$\bar{D}(\sigma, \Lambda, \ell) = \left\{ \gamma = \sum_{j=0}^{n} u_j \alpha_j \in \Lambda \colon 0 < u_j \leq d, |\gamma| = \ell \right\}. \tag{18}$$

For each $\gamma \in D(\sigma, \Lambda, \ell)$, since $0 \leq |\gamma| < (n+1)d$, the integer $k = (\lambda - |\gamma|)/d$ belongs to the half-open and half-closed interval $(\lambda/d - n - 1, \lambda/d]$; while for each $\gamma \in \bar{D}(\sigma, \Lambda, \ell)$, since $0 < |\gamma| \leq (n+1)d$, the integer $k = (\lambda - |\gamma|)/d$ belongs to the interval $[\lambda/d - n - 1, \lambda/d)$. Notice that when $|\gamma| = \lambda - dk$ is fixed, the number of tuples $(k_0, k_1, \ldots, k_n)$, with non-negative integer entries such that $\sum_{j=0}^{n} k_j = k$, is the binomial coefficient $\binom{k+n}{n}$; this is true even when k is a negative integer. We thus have the following proposition.

Proposition 2.1. *Let σ be an n-simplex whose vertices can be dilated into points of Λ by a single positive real number d. Then for any positive real number λ,*

$$E(\sigma, \Lambda, \lambda) = \sum_{k \in (\lambda/d - n - 1, \lambda/d] \cap \mathbf{Z}} |D(\sigma, \Lambda, \lambda - dk)| \binom{k+n}{n}; \tag{19}$$

$$E(\sigma^0, \Lambda, \lambda) = \sum_{k \in [\lambda/d - n - 1, \lambda/d) \cap \mathbf{Z}} |\bar{D}(\sigma, \Lambda, \lambda - dk)| \binom{k+n}{n}. \tag{20}$$

Notice that the right side of (19) and (20) are well-defined for non-positive real numbers λ. From now on we define $E(\sigma, \Lambda, \lambda)$ and $E(\sigma^0, \Lambda, \lambda)$ for all real numbers λ by (19) and (20), respectively.

Let r and s be non-negative integers such that $r + s = (n+1)d$. There is a one-to-one correspondence between $D(\sigma, \Lambda, r)$ and $\bar{D}(\sigma, \Lambda, s)$, sending the affine coordinates $(u_0, u_1, \ldots, u_n)$ to $(d - u_0, d - u_1, \ldots, d - u_n)$. This is well-defined because $\sum_{j=0}^{n} u_j \alpha_j$ is a point of Λ if and only if $\sum_{j=0}^{n} (d - u_j) \alpha_j$ is a point of Λ. Write $s = \lambda - dk$ in (20), then $\bar{D}(\sigma, \Lambda, \lambda - dk)$ can be replaced by $D(\sigma, \Lambda, r)$ with $r = (n+1)d - s = (n+1)d - \lambda + dk$. Thus

$$E(\sigma^0, \Lambda, \lambda) = \sum_{k \in [\lambda/d - n - 1, \lambda/d) \cap \mathbf{Z}} |D(\sigma, \Lambda, (n+1)d - \lambda + dk)| \binom{k+n}{n}.$$

Set $(n+1)d - \lambda + dk = -\lambda - dl$, we have $k = -l - n - 1$; l is an integer if and only if k is an integer; $-\lambda/d - n - 1 < l \leq -\lambda/d$. Therefore

$$E(\sigma^0, \Lambda, \lambda) = \sum_{l \in (-\lambda/d - n - 1, -\lambda/d] \cap \mathbf{Z}} |D(\sigma, \Lambda, -\lambda - dl)| \binom{-l-1}{n}$$

$$= \sum_{l \in (-\lambda/d - n - 1, -\lambda/d] \cap \mathbf{Z}} |D(\sigma, \Lambda, -\lambda - dl)| (-1)^n \binom{l+n}{n}.$$

In the last step we have made use of the reciprocity property of binomial coefficients:

$$\binom{n+l}{n} = (-1)^n \binom{-l-1}{n}.$$

We state this result in the following as a reciprocity law.

Proposition 2.2. *Let σ be an n-simplex whose vertices can be transformed into points of Λ multiplied by a single positive real number d. Then for any real number λ,*

$$E(\sigma^0, \Lambda, -\lambda) = (-1)^n E(\sigma, \Lambda, \lambda). \tag{21}$$

The reciprocity law (21) has been generalized to arbitrary rational polyhedra and rational polyhedral functions in [5] by using weight and boundary weight functions. The idea there can be further applied to get reciprocity laws for irrational polyhedra and polyhedral weight functions.

Consider the special case where σ is a Λ-lattice simplex; $d = 1$. If we restrict λ to integers m, then $E(\sigma, \Lambda, m)$ and $E(\sigma^0, \Lambda, m)$ are polynomial functions of integral variable m, called *Ehrhart polynomials*. We denote these polynomials by $L(\sigma, \Lambda, m)$ and $L(\sigma^0, \Lambda, m)$, respectively. Notice that when m is replaced by non-integral real numbers λ in $L(\sigma, \Lambda, m)$, the real-valued function $L(\sigma, \Lambda, \lambda)$ is *different* from $E(\sigma, \Lambda, \lambda)$. The function $E(\sigma, \Lambda, \lambda)$ is always integer-valued, while $L(\sigma, \Lambda, \lambda)$ is obviously not. We write $L(\sigma, \Lambda, m)$ and $L(\sigma^0, \Lambda, m)$ more explicitly in the following proposition by changing the summation indices of (19) and (20).

Proposition 2.3. *Let σ be an n-simplex with vertices in Λ. Then the polynomial functions $L(\sigma, \Lambda, m)$ and $L(\sigma^0, \Lambda, m)$ are given by*

$$L(\sigma, \Lambda, m) = \sum_{k=0}^{n} |D(\sigma, \Lambda, k)| \binom{m+n-k}{n}; \tag{22}$$

$$L(\sigma^0, \Lambda, m) = \sum_{k=0}^{n} |D(\sigma, \Lambda, k)| \binom{m+k-1}{n}. \tag{23}$$

Let s_j denote the jth elementary symmetric function of n variables, i.e.,

$$s_j(x_1, x_2, \ldots, x_n) = \sum_{k_1 < k_2 < \cdots < k_j} x_{k_1} x_{k_2} \cdots x_{k_j}.$$

With convention $s_0 \equiv 1$, the binomial coefficient in (23) can be written as

$$\binom{m+k-1}{n} = \frac{1}{n!}(m+k-1)(n+k-2)\cdots(m+k-n)$$

$$= \frac{1}{n!} \sum_{j=0}^{n} m^j s_{n-j}(k-1, k-2, \ldots, k-n).$$

This immediately gives rise to the following simple formulas on the coefficients of Ehrhart polynomials. Formulas (22) and (23) were first obtained by Ehrhart [9], then re-proved by Macdonald [13], [14], and appeared in Danilov's treatise [7] on toric varieties. It seems that the following coefficient formulas in elementary symmetric functions were overlooked, until it was recently observed in [6].

Theorem 2.4. *Let σ be an n-simplex with vertices in Λ. Then the jth coefficients of $L(\sigma, \Lambda, t)$ and $L(\sigma^0, \Lambda, t)$ are given by*

$$c_j(\sigma^0, \Lambda) \; = \; \frac{1}{n!} \sum_{k=0}^{n} |D(\sigma, \Lambda, k)| s_{n-j}(k-1, \ldots, k-n); \tag{24}$$

$$c_j(\sigma, \Lambda) \; = \; \frac{1}{n!} \sum_{k=0}^{n} |D(\sigma, \Lambda, k)| s_{n-j}(1-k, \ldots, n-k). \tag{25}$$

Since counting lattice points is invariant under integral translations, the number of lattice points in $m\sigma$ is the same as the number of lattice points in the dilation $m\sigma'$, where σ' is the simplex with the vertices $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$ and the origin. Then every $\alpha \in m\sigma'$ and $\beta \in m\sigma'^0$ can be uniquely written as the linear combinations:

$$\alpha \; = \; \sum_{j=1}^{n} a_j(\alpha_j - \alpha_0), \qquad \sum_{j=1}^{n} a_j \leq m, \quad a_j \geq 0;$$

$$\beta \; = \; \sum_{j=1}^{n} b_j(\alpha_j - \alpha_0), \qquad \sum_{j=1}^{n} b_j < m, \quad b_j > 0.$$

Similarly, α and β can be further uniquely written as

$$\alpha \; = \; \sum_{j=1}^{n} u_j(\alpha_j - \alpha_0) + \sum_{j=1}^{n} k_j(\alpha_j - \alpha_0),$$

$$\beta \; = \; \sum_{j=1}^{n} v_j(\alpha_j - \alpha_0) + \sum_{j=1}^{n} l_j(\alpha_j - \alpha_0),$$

with real numbers u_j and v_j, non-negative integers k_j and l_j, such that

$$\sum_{j=1}^{n} u_j + \sum_{i=1}^{n} k_j \leq m, \qquad 0 \leq u_j < 1;$$

$$\sum_{j=1}^{n} v_j + \sum_{j=1}^{n} l_j < m, \qquad 0 < v_j \leq 1.$$

Moreover, α and β are points of Λ if and only if $\sum_{j=1}^{n} u_j(\alpha_j - \alpha_0)$ and $\sum_{j=1}^{n} v_j(\alpha_j - \alpha_0)$ are points of Λ, respectively. Define the *determined sets*:

$$D(\sigma, \alpha_0; \Lambda) \; = \; \left\{ \sum_{j=1}^{n} u_j(\alpha_j - \alpha_0) \in \Lambda \colon 0 \leq u_j < 1 \right\};$$

$$\bar{D}(\sigma, \alpha_0; \Lambda) \; = \; \left\{ \sum_{j=1}^{n} v_j(\alpha_j - \alpha_0) \in \Lambda \colon 0 < v_j \leq 1 \right\}.$$

For each $\gamma = \sum_{j=1}^{n} u_j(\alpha_j - \alpha_0)$ of $D(\sigma, \Lambda) \cup \bar{D}(\sigma, \Lambda)$, we write $|\gamma| = \sum_{j=1}^{n} u_j$. It is easy to see

$$\sum_{j=1}^{n} k_j \leq m - |\gamma| \quad \Leftrightarrow \quad \sum_{j=1}^{n} k_j \leq m - \lceil |\gamma| \rceil;$$

$$\sum_{j=1}^{n} l_j < m - |\gamma| \quad \Leftrightarrow \quad \sum_{j=1}^{n} l_j \leq m - \lfloor |\gamma| \rfloor - 1.$$

Again, the numbers of non-negative integer solutions of the above inequalities are the binomial coefficients $\binom{m - \lceil |\gamma| \rceil}{n}$ and $\binom{m - \lfloor |\gamma| \rfloor - 1}{n}$, respectively; $D(\sigma, \Lambda)$ and $\bar{D}(\sigma, \Lambda)$ are in one-to-one correspondence, sending $u = (u_1, \ldots, u_n)$ to $(1 - u_1, \ldots, 1 - u_n)$. We thus have the following formulas on counting lattice points.

Theorem 2.5 [9]. *Let σ be an n-simplex whose vertices are points of Λ. Then*

$$L(\sigma, \Lambda, m) = \sum_{\gamma \in D(\sigma, \alpha_0; \Lambda)} \binom{m + n - \lceil |\gamma| \rceil}{n}; \tag{26}$$

$$L(\sigma^0, \Lambda, m) = \sum_{\gamma \in \bar{D}(\sigma, \alpha_0; \Lambda)} \binom{m + n - 1 - \lfloor |\gamma| \rfloor}{n} \tag{27}$$

$$= \sum_{\gamma \in D(\sigma, \alpha_0; \Lambda)} \binom{m + \lceil |\gamma| \rceil - 1}{n}. \tag{28}$$

Moreover, the jth coefficients of $L(\sigma^0, m)$ and $L(\sigma, m)$ are respectively given by

$$c_j(\sigma^0, \Lambda) = \frac{1}{n!} \sum_{\gamma \in D(\sigma, \alpha_0; \Lambda)} s_{n-j}(\lceil |\gamma| \rceil - 1, \lceil |\gamma| \rceil - 2, \ldots, \lceil |\gamma| \rceil - n); \tag{29}$$

$$c_j(\sigma, \Lambda) = \frac{1}{n!} \sum_{\gamma \in D(\sigma, \alpha_0; \Lambda)} s_{n-j}(1 - \lceil |\gamma| \rceil, 2 - \lceil |\gamma| \rceil, \ldots, n - \lceil |\gamma| \rceil). \tag{30}$$

Counting lattice points is also invariant under unimodular transformations. Let A be an $N \times n$ matrix whose columns are the vectors $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$. The matrix A can be transformed into an upper triangular matrix by multiplying an $N \times N$ unimodular matrix to the left. This can be done by using repeatedly the modified Gaussian elimination and the Euclidean algorithm. Interchanging columns of A is just to interchange the order of vertices of the simplex σ with α_0 fixed; we can also switch the columns of A and have the number of lattice points inside σ unchanged. Reversing the columns and the first n rows of the upper triangular matrix (7), one can obtain a lower triangular form which is considered in [8]; we do not choose this lower triangular form in our treatment.

Theorem 2.6. *Let the $N \times n$ matrix A be in the upper triangular form* (7) *and let $u = (u_1, \ldots, u_n) \in [0, 1)^n$. Then $Au \in \mathbf{Z}^N$ if and only if there exists $(j_1, \ldots, j_n) \in$*

$\prod_{i=1}^{n} [0, a_{ii}) \cap \mathbf{Z}$ *such that*

$$
\begin{cases}
u_n = \dfrac{j_n}{u_{nn}}, \\[2ex]
u_i = \dfrac{\lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil}{a_{ii}} \\[2ex]
\qquad - \dfrac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} + \dfrac{j_i}{a_{ii}}, \qquad 1 \le i \le n-1.
\end{cases}
\tag{31}
$$

In particular, $Au \in \mathbf{Z}^N$ implies that u_i must be certain rational numbers of the form $j/(a_{ii} \cdots a_{nn})$. Moreover, the map $\tilde{u}: (j_1, \ldots, j_n) \mapsto \sum_{i=1}^{n} u_i(\alpha_i - \alpha_0)$, where u_i are given by (31), is a one-to-one correspondence between $\prod_{i=1}^{n}[0, a_{ii}) \cap \mathbf{Z}$ and $D(\sigma, \alpha_0)$.

Proof. Obviously, u_n can have values j_n/a_{nn}, $0 \le j_n \le a_{nn} - 1$. Assume $u_{i+1}, \ldots, u_n$ are given values, we determine the possible values for u_i. Let J_i be the smallest integer such that

$$
a_{ii}u_i' + a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n = J_i
$$

for some real number $u_i' \in [0, 1)$. Then

$$
\frac{J_i}{a_{ii}} - \frac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} \ge 0
$$

and

$$
\frac{J_i - 1}{a_{ii}} - \frac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} < 0.
$$

This implies that $J_i = \lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil$. Thus u_i can take the values

$$
\frac{\lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil}{a_{ii}} - \frac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} + \frac{j_i}{a_{ii}},
$$

where $0 \le j_i \le a_{ii} - 1$, because when $j_i = a_{ii} - 1$,

$$
\frac{\lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil}{a_{ii}} - \frac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} + \frac{a_{ii} - 1}{a_{ii}} < 1,
$$

and when $j_i = a_{ii}$,

$$
\frac{\lceil a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n \rceil}{a_{ii}} - \frac{a_{i(i+1)}u_{i+1} + \cdots + a_{in}u_n}{a_{ii}} + \frac{a_{ii}}{a_{ii}} \ge 1.
$$

By definition of $D(\sigma, \alpha_0)$, the map $\tilde{u}$ is a one-to-one correspondence. $\qquad\square$

Now in Theorem 2.6, if the matrix A is viewed as a group homomorphism from $\mathbf{Z}^n$ to $\mathbf{Z}^N$, the product $a_{11} \cdots a_{nn}$ of non-zero diagonal entries is the order of the torsion subgroup of $\mathbf{Z}^N/\operatorname{Im} A$. The transpose A^{T} is a group homomorphism from $\mathbf{Z}^N$ to $\mathbf{Z}^n$; and $|\mathbf{Z}^n/\operatorname{Im} A^{\mathrm{T}}| = a_{11} \cdots a_{nn}$. We define torsion for any non-zero integral matrix in the following.

Definition 2.7.   Let φ be a group homomorphism from $\mathbf{Z}^n$ to $\mathbf{Z}^m$. The *torsion* of φ is the order of the torsion subgroup of the quotient Abelian group $\mathbf{Z}^m / \operatorname{Im} \varphi$. If A is the matrix of φ under the standard bases, the *torsion* of A is defined to be the torsion of φ, denoted $\operatorname{tor} A = \operatorname{tor} \varphi$.

The torsion of a group homomorphism $\varphi \colon \mathbf{Z}^n \to \mathbf{Z}^m$ can be computed from its matrix under the standard bases. Notice that changing the bases of $\mathbf{Z}^n$ and $\mathbf{Z}^m$ corresponds to the *modified Gaussian elimination*: (a) adding an integral multiple of one row (column) to another row (column); (b) switching two rows (columns). By repeated use of the Euclidean algorithm and the modified Gaussian elimination, any integral matrix can be reduced to a *Hermite normal form* whose non-zero entries are all on the diagonal line, see [17]; the product of these non-zero entries is just the torsion of φ. Theorem 2.6 immediately gives a rough upper bound on the number of lattice points of a lattice simplex σ.

Corollary 2.8.   *Let σ be a lattice n-simplex of $\mathbf{R}^N$ with vertices $\alpha_0, \alpha_1, \ldots, \alpha_n$; and let A be the $N \times n$ matrix whose columns are vectors $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$. Then $|D(\sigma, \alpha_0)| = \operatorname{tor} A$ and $|\sigma \cap \mathbf{Z}^N| \le n + \operatorname{tor} A$.*

3.   The Generating Functions of $L(\sigma, m)$ and $L(\sigma^0, m)$

Let Λ be a full-dimensional lattice of $\mathbf{R}^N$ with a lattice basis $\{\xi_1, \ldots, \xi_N\}$. Let $x = (x_1, \ldots, x_N)$ be an undetermined multi-variable. For any $\alpha = a_1 \xi_1 + \cdots + a_N \xi_N \in \Lambda$, define the Laurent monomial $x^\alpha = x_1^{a_1} \cdots x_N^{a_N}$; for a polyhedron P with vertices in Λ, define the formal power series

$$F(P, \Lambda, x) = \sum_{\alpha \in P \cap \Lambda} x^\alpha,$$

which is an element of $\mathbf{C}[[x_1, \ldots, x_N, x_1^{-1}, \ldots, x_N^{-1}]]$. As usual we identify rational functions of x as elements of $\mathbf{C}[[x_1, \ldots, x_N, x_1^{-1}, \ldots, x_N^{-1}]]$.

Let K be a convex cone generated by vectors $\alpha_1, \ldots, \alpha_n$ of Λ, i.e., $K = \bigoplus_{j=1}^n \mathbf{R}_{\ge 0} \alpha_j$. Let $K^0 = K - \partial K$ be the relative interior of K. Define the *determined sets* of K by

$$D(K, \Lambda) = \left\{ \sum_{j=1}^n u_j \alpha_j \in \Lambda \colon 0 \le u_j < 1 \right\};$$

$$\bar{D}(K^0, \Lambda) = \left\{ \sum_{j=1}^n v_j \alpha_j \in \Lambda \colon 0 < v_j \le 1 \right\}.$$

It is easy to see that a point α of K (or K^0) belongs to Λ if and only if there exist a unique vector γ of $D(K, \Lambda)$ (or $\bar{D}(K, \Lambda)$) and non-negative integers k_j such that $\alpha = \gamma + \sum_{j=1}^n k_j \alpha_j$. Then

$$F(K, \Lambda, x) = \sum_{\gamma \in D(K, \Lambda)} \sum_{k_1 \ge 0, \ldots, k_n \ge 0} x^{\gamma + k_1 \alpha_1 + \cdots + k_n \alpha_n}$$

$$= \left(\sum_{\gamma \in D(K, \Lambda)} x^\gamma \right) \prod_{j=1}^n \frac{1}{1 - x^{\alpha_j}}; \tag{32}$$

$$L(K^0, \Lambda, x) = \left(\sum_{\gamma \in \bar{D}(K, \Lambda)} x^\gamma \right) \prod_{j=1}^{n} \frac{1}{1 - x^{\alpha_j}} \qquad (33)$$

$$= \left(\sum_{\gamma \in D(K, \Lambda)} x^{-\gamma} \right) x^{\alpha_1 + \cdots + \alpha_n} \prod_{j=1}^{n} \frac{1}{1 - x^{\alpha_j}}. \qquad (34)$$

Throughout this section we denote by σ an n-simplex of $\mathbf{R}^N$ with the vertex set $\{\alpha_j\}_{j=0}^{n} \subset \Lambda$. We put $\{\alpha_j\}_{j=0}^{n}$ in $\mathbf{R}^{N+1}$ in the hyperplane $x_0 = 1$ by lifting upward one unit to have independent vectors $\{\beta_j = (1, \alpha_j)\}_{j=0}^{n}$. Let σ_1 denote the n-simplex with the vertex set $\{\beta_j\}_{j=0}^{n}$. Let Λ_1 be the lattice generated by Λ and the vector $\xi_0 = (1, 0, \ldots, 0) \in \mathbf{R}^{N+1}$; then $\{\xi_0, \xi_1, \ldots, \xi_N\}$ is a lattice basis of Λ_1; and the number of points in $m\sigma \cap \Lambda$ is the same as the number of points in $m\sigma_1 \cap \Lambda_1$. Set $V = \bigoplus_{j=0}^{n} \mathbf{R}\beta_j$, $M = V \cap \Lambda_1$, $C = \bigoplus_{j=0}^{n} \mathbf{R}_{\geq 0}\beta_j$; the lattice cone $C \cap \Lambda_1$ is a disjoint union of $m\sigma_1 \cap \Lambda_1$, $m \geq 0$. Set $x_0 = t$ and $x_1 = \cdots = x_N = 1$ in (32) and (34); we have the following generating functions for the sequences $L(\sigma, \Lambda, m)$ and $L(\sigma^0, \Lambda, m)$.

Proposition 3.1. *Let $D(\sigma_1, \Lambda_1, k)$ be the determined set of the simplex σ_1 at level k as defined in* (17). *Then*

$$\sum_{m=0}^{\infty} L(\sigma, \Lambda, m) t^m = \frac{1}{(1-t)^{n+1}} \sum_{\gamma \in D(C, \Lambda_1)} t^{|\gamma|}$$

$$= \frac{1}{(1-t)^{n+1}} \sum_{k=0}^{n} |D(\sigma_1, \Lambda_1, k)| t^k; \qquad (35)$$

$$\sum_{m=1}^{\infty} L(\sigma^0, \Lambda, m) t^m = \frac{1}{(1-t)^{n+1}} \sum_{\gamma \in \bar{D}(C, \Lambda_1)} t^{|\gamma|}$$

$$= \frac{t^{n+1}}{(1-t)^{n+1}} \sum_{k=0}^{n} |D(\sigma_1, \Lambda_1, k)| t^{-k}. \qquad (36)$$

Formulas (32)–(36) can be found in [20]. Let A be the $N \times n$ matrix whose columns are the vectors $\{\alpha_j - \alpha_0\}_{j=0}^{n}$; and let B be the $(N + 1) \times (n + 1)$ matrix whose columns are the vectors $\{\beta_j\}_{j=0}^{n}$. If A is in upper triangular form (7), then B is also in upper triangular form. The lattice points of the determined set $D(C, \Lambda_1)$ can be listed by a one-to-one correspondence $\tilde{u}: \prod_{i=0}^{n}[0, a_{ii}) \cap \mathbf{Z} \to D(C, \Lambda_1)$ similar to $\tilde{u}(j_1, \ldots, j_n)$. In fact, for any $(u_0, u_1, \ldots, u_n) \in [0, 1)^{n+1}$, $\sum_{i=0}^{n} u_i \beta_i \in \mathbf{Z}^{N+1}$ if and only if both $\sum_{i=0}^{n} u_i \in \mathbf{Z}$ and $\sum_{i=1}^{n} u_i \alpha_i \in \mathbf{Z}^N$. Since $0 \leq u_0 < 1$, we have $u_0 = \lceil \sum_{i=1}^{n} u_i \rceil - \sum_{i=1}^{n} u_i$. Thus $\tilde{u}(j_0, j_1, \ldots, j_n) = \sum_{i=0}^{n} u_i (\alpha_i - \alpha_0)$, where $(u_i)_{i=1}^{n}$ are given by (8) or (31), $u_0 = \lceil \sum_{i=1}^{n} u_i \rceil - \sum_{i=1}^{n} u_i$, $j_0 \equiv 1$. We have the following corollary.

Corollary 3.2. *Let A be the matrix whose columns are the vectors $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$. If A is in upper triangular form (7), then*

$$\sum_{m=0}^{\infty} L(\sigma, \Lambda, m)t^m = \frac{1}{(1-t)^{n+1}} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} t^{\lceil |u(j_1,\ldots,j_n)| \rceil}, \qquad (37)$$

$$\sum_{m=1}^{\infty} L(\sigma, \Lambda, m)t^m = \frac{t^{n+1}}{(1-t)^{n+1}} \sum_{j_1=0}^{a_{11}-1} \cdots \sum_{j_n=0}^{a_{nn}-1} t^{-\lceil |u(j_1,\ldots,j_n)| \rceil}, \qquad (38)$$

where $u(j_1, \ldots, j_n) = (u_1, \ldots, u_n)$ is defined by (8).

If we do not use the one-to-one correspondence $\tilde{u}$ by (8) to list all lattice points of the determined set $D(C, \Lambda_1)$, one may refine $D(C, \Lambda_1)$ so that all lattice points inside can be listed canonically at the cost of using characters. To this end, we need the following lemma.

Lemma 3.3. *Let L_1 and M_1 be lattices of finite rank of a vector space V_1; let L_2 and M_2 be lattices of the same rank of a vector space V_2 such that $L_2 \subset M_2$; $G_2 = M_2/L_2$. Let $\langle , \rangle$ be a bilinear pairing $V_1 \times V_2 \to \mathbf{R}$ such that $M_1 \times M_2 \to \mathbf{Z}$.*

  (i) *If $L_1 \times L_2 \to \mathbf{Z}$, then the character map $\chi \colon L_1 \times G_2 \to \mathbf{C}^*$ by $\chi_g(\alpha) = e^{2\pi \iota \langle \alpha, g \rangle}$ is a well-defined group bi-homomorphism.*
  (ii) *For $\alpha \in L_1 - M_1$, if there exists $g_\alpha \in M_2$ such that $\langle \alpha, g_\alpha \rangle \notin \mathbf{Z}$, then*

$$\sum_{g \in G} \chi_g(\alpha) = 0.$$

  (iii) *Let $\{\gamma_j\}_{j=0}^n$ be a lattice basis of L_1; $C = \bigoplus_{j=0}^n \mathbf{R}_{\geq 0}\gamma_j$, $C^0 = \bigoplus_{j=0}^n \mathbf{R}_{>0}\gamma_j$. If $L_1 \times L_2 \to \mathbf{Z}$, $C \cap M_1 \subset C \cap L_1$, and for each $\alpha \in C \cap (L_1 - M_1)$ there exists $g_\alpha \in M_2$ such that $\langle \alpha, g_\alpha \rangle \notin \mathbf{Z}$, then*

$$F(C, M_1, x) = \sum_{\alpha \in C \cap M_1} x^\alpha = \frac{1}{|G_2|} \sum_{g \in G_2} \prod_{j=0}^n \frac{1}{1 - \chi_g(\gamma_j)x^{\gamma_j}}; \qquad (39)$$

$$F(C^0, M_1, x) = \sum_{\alpha \in C^0 \cap M_1} x^\alpha = \frac{1}{|G_2|} \sum_{g \in G_2} \prod_{j=0}^n \frac{\chi_g(\gamma_j)x^{\gamma_j}}{1 - \chi_g(\gamma_j)x^{\gamma_j}}. \qquad (40)$$

Proof. (i) When we write $\langle \alpha, g \rangle$ for $\alpha \in L_1$ and $g \in G_2$, we understand that g is taken to be a representative of its coset $g + L_2$. Now it is trivial that $\chi_g(\alpha)$ is a bi-homomorphism.

(ii) Let α be a vector of $L_1 - M_1$ such that $\langle \alpha, g \rangle \notin \mathbf{Z}$ for some $g \in M_2$. Denote by $d(g)$ the order of g in G_2. Since $\chi_g(\alpha) = e^{2\pi \iota \langle \alpha, g \rangle}$ is a root of unity, then $\sum_{j=1}^{d(g)} \chi_{jg}(\alpha) = 0$. Fix one such group element g' and let $\langle g' \rangle$ denote the cyclic group generated by g'; write G_2 as a disjoint union $G_2 = \bigcup_k (g_k + \langle g' \rangle)$ of cosets for some representatives g_k. Obviously,

$$\sum_{g \in G_2} \chi_g(\alpha) = \sum_k \chi_{g_k}(\alpha) \sum_{g \in \langle g' \rangle} \chi_g(\alpha) = 0.$$

(iii) Let $\alpha \in C \cap L_1$. If $\alpha \in M_1$, then $\chi_g(\alpha) = 1$ for all $g \in G_2$. It follows from (ii) that

$$\frac{1}{|G_2|} \sum_{g \in G_2} \chi_g(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha \in M_1 \\ 0 & \text{for} \quad \alpha \notin M_1. \end{cases}$$

Write α uniquely as $\alpha = \sum_{j=0}^n k_j \gamma_j$ for some $k_j \geq 0$; we have

$$
\begin{aligned}
F(C, M_1, x) &= \sum_{\alpha \in C \cap L_1} x^\alpha \frac{1}{|G_2|} \sum_{g \in G_2} \chi_g(\alpha) \\
&= \frac{1}{|G_2|} \sum_{g \in G_2} \sum_{\alpha \in C \cap L_1} \chi_g(\alpha) x^\alpha \\
&= \frac{1}{|G_2|} \sum_{g \in G_2} \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty \chi_g \left(\sum_{j=0}^n k_j \gamma_j \right) x^{\sum_{j=0}^n k_j \gamma_j} \\
&= \frac{1}{|G_2|} \sum_{g \in G_2} \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty \prod_{j=0}^n \chi_g^{k_j}(\gamma_j)(x^{\gamma_j})^{k_j} \\
&= \frac{1}{|G_2|} \sum_{g \in G_2} \prod_{j=0}^n \frac{1}{1 - \chi_g(\gamma_j) x^{\gamma_j}}.
\end{aligned}
$$

The situation for $F(C^0, M_1, x)$ is similar, just pay attention to $k_0 \geq 1, k_1 \geq 1, \ldots, k_{n+1} \geq 1$ in the above summation. $\qquad\square$

Theorem 3.4. *Let $V = \bigoplus_{j=0}^n \mathbf{R}\beta_j$, $M = V \cap \Lambda_1$, $C = \bigoplus_{j=0}^n \mathbf{R}_{\geq 0}\beta_j$; and let p_j be positive integers such that $C \cap M \subset C \cap L$, where $L = \bigoplus_{j=0}^n \mathbf{Z}\beta_j/p_j$. Let L' and M' be lattices of the same rank of a vector space V' such that $L' \subset M'$; $G = M'/L'$. Let $\langle , \rangle$ be a bilinear pairing $V \times V' \to \mathbf{R}$ such that $M \times M' \to \mathbf{Z}$ and $L \times L' \to \mathbf{Z}$. If for any $\alpha \in C \cap (L - M)$ there exists $g_\alpha \in M'$ such that $\langle \alpha, g_\alpha \rangle \notin \mathbf{Z}$, then*

$$\sum_{m=0}^\infty L(\sigma, \Lambda, m) e^{2\pi ms} = \frac{1}{2^{n+1}|G|} \sum_{g \in G} \prod_{j=0}^n \left(1 + \coth \frac{\pi}{p_j}(s + \iota \langle \beta_j, g \rangle) \right); \quad (41)$$

$$\sum_{m=1}^\infty L(\sigma^0, \Lambda, m) e^{2\pi ms} = \frac{(-1)^{n+1}}{2^{n+1}|G|} \sum_{g \in G} \prod_{j=0}^n \left(1 - \coth \frac{\pi}{p_j}(s + \iota \langle \beta_j, g \rangle) \right). \quad (42)$$

Proof. This is an immediate consequence of Lemma 3.3. Set $V_1 = V$, $M_1 = M$, $L_1 = L$, $V_2 = V'$, $M_2 = M'$, $L_2 = L'$, $x = (e^{-2\pi s}, 1, \ldots, 1)$, $\gamma_j = \beta_j/p_j$; apply (iii) of Lemma 3.3, and keep in mind

$$\frac{1}{1 - e^{-2z}} = \tfrac{1}{2}(1 + \coth z), \qquad \frac{e^{-2z}}{1 - e^{-2z}} = -\tfrac{1}{2}(1 - \coth z).$$

We then have

$$
\frac{1}{1 - \chi_g(\gamma_j)x^{\gamma_j}} = \frac{1}{1 - \exp(-(2\pi/p_j)(s + \iota\langle\beta_j, g\rangle))}
$$

$$
= \frac{1}{2}\left(1 + \coth\frac{\pi}{p_j}(s + \iota\langle\beta_j, g\rangle)\right);
$$

$$
\frac{\chi_g(\gamma_j)x^{\gamma_j}}{1 - \chi_g(\gamma_j)x^{\gamma_j}} = \frac{\exp(-(2\pi/p_j)(s + \iota\langle\beta_j, g\rangle))}{1 - \exp(-(2\pi/p_j)(s + \iota\langle\beta_j, g\rangle))}
$$

$$
= -\frac{1}{2}\left(1 - \coth\frac{\pi}{p_j}(s + \iota\langle\beta_j, g\rangle)\right). \qquad \square
$$

Formulas (41) and (42) are naively simple, but they are very useful. A notable feature is that the group G can be selected as long as the lattices L, L', M, M' satisfy the required conditions. One can fix a pairing $V \times V' \to \mathbf{R}$ satisfying $M \times M' \to \mathbf{Z}$, then construct a lattice L such that $C \cap M \subset C \cap L$. Since L may contain some points not in M, usually we do not have $L \times M' \to \mathbf{Z}$. However, there always exists a sublattice $L' \subset M'$ of the same rank such that $L \times L' \to \mathbf{Z}$; and any sublattice $L'' \subset L'$ of the same rank does the same job. We give several examples for Theorem 3.4 in the following.

Example 3.5 [8].   Let σ be a lattice n-simplex of $\mathbf{R}^n$ with the vertex set $\{\alpha_j\}_{j=0}^n \subset \mathbf{Z}^n$. Let A be the matrix whose columns are the vectors $\{\alpha_j - \alpha_0\}_{j=0}^n$; and let B be the matrix whose columns are the vectors $\{\beta_j = (1, \alpha_j)\}_{j=0}^n$. Let A be in upper triangular form (7); set $p_j = a_{jj}\cdots a_{nn}, 0 \le j \le n$, where $a_{00} = 1$; $G = \prod_{j=0}^n \mathbf{Z}/p_j\mathbf{Z}$. Then we have (41) and (42).

Proof.   Set $N = n, V = V' = \mathbf{R}^{n+1}, M = M' = \Lambda_1 = \mathbf{Z}^{n+1}$. The pairing $V \times V' \to \mathbf{R}$ is given by the inner product $\langle, \rangle$ of $\mathbf{R}^{n+1}$, which is defined by $\langle\alpha, \beta\rangle = \sum_{j=0}^n u_j v_j$ for $\alpha = \sum_{j=0}^n u_j e_j$ and $\beta = \sum_{j=0}^n v_j e_j$, where $\{e_j\}_{j=0}^n$ is the standard lattice basis of $\mathbf{Z}^{n+1}$. Obviously, $M \times M' \to \mathbf{Z}$. Set $\gamma_j = \beta_j/p_j, \gamma_j' = p_j e_j, L = \sum_{j=0}^n \mathbf{Z}\gamma_j, L' = \sum_{k=0}^n \mathbf{Z}\gamma_k'$. By Theorem 2.6, $C \cap M \subset C \cap L$. Since

$$
\langle\gamma_j, \gamma_k'\rangle = \begin{cases} \dfrac{a_{kj}a_{kk}\cdots a_{nn}}{a_{jj}\cdots a_{nn}} & \text{for } j \ge k, \\ 0 & \text{for } j < k \end{cases}
$$

are integers, we have $L \times L' \to \mathbf{Z}$. For $\alpha \in C \cap (L - M)$, write $\alpha = \sum_{j=0}^n u_j e_j$; since $\alpha \notin M$, some u_j must be non-integral, say $u_{j'}$; we have $e_{j'} \in M'$ and $\langle\alpha, e_{j'}\rangle = u_{j'} \notin \mathbf{Z}$. Finally, $G = M'/L' = \mathbf{Z}^{n+1}/\prod_{j=0}^n p_j\mathbf{Z} = \prod_{j=0}^n \mathbf{Z}/p_j\mathbf{Z}$. Now all the conditions of Theorem 3.4 are satisfied. We thus recover the main result of [8], which was first obtained by the Poisson summation and Fourier analysis. $\square$

Sometimes it may not be easy to check the conditions of Theorem 3.4. In particular, it is important to find a canonical way to check the conditions: $C \cap M \subset C \cap L$; the existence of $g_\alpha \in M'$ such that $\langle\alpha, g_\alpha\rangle \notin \mathbf{Z}$ for each $\alpha \in C \cap (L - M)$; and $L_1 \times L_2 \to \mathbf{Z}$. The following lemma may help to do the job.

Lemma 3.6. *Let M_1 be a lattice with a lattice basis $\{\eta_k\}_{k=0}^l$ of a vector space V_1; and let M_2 be a lattice with a lattice basis $\{\eta_k'\}_{k=0}^l$ of a vector space V_2. Let $\langle,\rangle$ be a bilinear pairing $V_1 \times V_2 \to \mathbf{R}$ such that $\langle \eta_j, \eta_k' \rangle = \delta_{jk}$. Given independent vectors $\{\beta_j\}_{j=0}^n$ of M_1 and independent vectors $\{\beta_j'\}_{j=0}^n$ of M_2 such that $\langle \beta_j, \beta_k' \rangle = p_j \delta_{jk}$, where p_j are positive integers. Set $C = \bigoplus_{j=0}^n \mathbf{R}_{\geq 0}\beta_j$, $\gamma_j = \beta_j/p_j$, $L_1 = \bigoplus_{j=0}^n \mathbf{Z}\gamma_j$. We have:*

- (i) *$C \cap M_1 \subset C \cap L_1$.*
- (ii) *For any $\alpha \in C \cap (L_1 - M_1)$, there exists $\varphi \in M_2$ such that $\langle \alpha, \varphi \rangle \notin \mathbf{Z}$.*
- (iii) *Set $p = \operatorname{lcm}\{p_j\}_{j=0}^n$, $b_k = \gcd\{\langle \beta_j, \eta_k' \rangle\}_{j=0}^n$, $q_k = p/\gcd(p, b_k)$, $\gamma_k' = q_k\eta_k'$, $L_2 = \bigoplus_{k=0}^l \mathbf{Z}\gamma_k'$. Then $L_1 \times L_2 \to \mathbf{Z}$.*

Proof. (i) For any $\beta \in C \cap M_1$, write $\beta = \sum_{j=0}^n v_j\gamma_j$. Since $\beta \in M_1$ and $\beta_k' \in M_2$, $\langle \beta, \beta_k' \rangle$ is an integer; but $\langle \beta, \beta_k' \rangle = \langle \sum_{j=0}^n v_j\gamma_j, \beta_k' \rangle = v_k$. This shows that $\beta \in L_1$, so $\beta \in C \cap L_1$.

(ii) Let $\alpha \in C \cap (L_1 - M_1)$. Since α is a linear combination of $\{\beta_j\}_{j=0}^n$, it is a linear combination of $\{\eta_k\}_{k=0}^l$. Write $\alpha = \sum_{k=0}^l u_k\eta_k$; since $\alpha \notin M_1$, some u_k must be non-integral, say $u_{k'}$. We thus have $\eta_{k'}' \in M_2$ and $\langle \alpha, \eta_{k'}' \rangle = \langle \sum_{k=0}^l u_k\eta_k, \eta_{k'}' \rangle = u_{k'} \notin \mathbf{Z}$.

(iii) The result follows from

$$\langle \gamma_j, \gamma_k' \rangle = \frac{\operatorname{lcm}\{p_i\}_{i=0}^n}{p_j} \cdot \frac{\langle \beta_j, \eta_k' \rangle}{\gcd(p, b_k)} \in \mathbf{Z}, \qquad 0 \leq j \leq n, \quad 0 \leq k \leq l. \qquad \square$$

Example 3.7. Let $V = \bigoplus_{j=0}^n \mathbf{R}\beta_j$, $M = V \cap \Lambda_1$, $C = \bigoplus_{j=0}^n \mathbf{R}_{\geq 0}\beta_j$. Let V^* be the dual vector space of V and let M^* be the dual lattice of M, i.e., V^* is the vector space of all linear functionals on V and

$$M^* = \{\varphi \in V^*\colon \langle \alpha, \varphi \rangle = \varphi(\alpha) \in \mathbf{Z} \text{ for all } \alpha \in M\}.$$

Let $\beta_j' \in M^*$ be a normal vector of the facet of C without β_j such that $\langle \beta_j, \beta_j' \rangle = p_j$ is positive, i.e., $\langle \beta_j, \beta_k' \rangle = p_j\delta_{jk}$, $p_j > 0$. Set $L^* = \bigoplus_{j=0}^n \mathbf{Z}\beta_j'$, $G = M^*/L^*$. Then we have (41) and (42).

Proof. Set $V_1 = V$, $M_1 = M$, $L_1 = \bigoplus_{j=0}^n \mathbf{Z}\gamma_j$, where $\gamma_j = \beta_j/p_j$; and $V_2 = V_1^*$, $M_2 = M_1^*$, $L_2 = L_1^*$. Since $\langle \gamma_j, \beta_k' \rangle = \delta_{jk}$, we have $L_2 = L^*$. We then have a natural pairing $V_1 \times V_2 \to \mathbf{R}$ such that $M_1 \times M_2 \to \mathbf{Z}$ and $L_1 \times L_2 \to \mathbf{Z}$. Keep in mind that $l = n$; choose a basis $\{\eta_k\}_{k=0}^n$ of M and its dual basis $\{\eta_k' := \eta_k^*\}_{k=0}^n$ of M^*, i.e., $\langle \eta_j, \eta_k' \rangle = \delta_{jk}$; we then obtain the conclusions of Lemma 3.6.

Now we set $L = L_1$, $V' = V_2$, $M' = M^*$, $L' = L_2$. Then all conditions of Theorem 3.4 on V, V', M, M', L, L' are satisfied and $G = M'/L' = M^*/L^*$. We thus have (41) and (42). $\square$

It is not easy to list the elements of the group $G = M^*/L^*$. However, the group M^*/L^* has fewer elements than other groups that will be given later. In the following we apply Theorem 3.4 to find some more convenient groups to express the generating functions in terms of hyperbolic cotangent functions. Recall the lattice basis $\{\xi_0, \xi_1, \ldots, \xi_N\}$ of the

lattice Λ_1 in $\mathbf{R}^{N+1}$; there is a canonical inner product on $\mathbf{R}^{N+1}$: for $\alpha = \sum_{k=0}^{N} u_k \xi_k$ and $\beta = \sum_{k=0}^{N} v_k \xi_k$, define

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{N} u_k v_k. \tag{43}$$

Example 3.8. Let $C = \bigoplus_{j=0}^{n} \mathbf{R}_{\geq 0} \beta_j$; and let $\beta_j' \in \Lambda_1$ be a normal vector of the facet of C without β_j such that $\langle \beta_j, \beta_j' \rangle = p_j$ is positive, i.e., $\langle \beta_j, \beta_k' \rangle = p_j \delta_{jk}$, $p_j > 0$. Set $p = \mathrm{lcm}\{p_j\}_{j=0}^{n}$, $b_k = \gcd\{\langle \beta_j, \xi_k \rangle\}_{j=0}^{n}$, $q_k = p/\gcd\{p, b_k\}$; $G = \prod_{k=0}^{N} \mathbf{Z}/q_k \mathbf{Z}$. Then we have (41) and (42).

Proof. Note that p_j are integers because both β_j and β_j' are points of Λ_1. Set $V_1 = V_2 = \mathbf{R}^{N+1}$, $M_1 = M_2 = \Lambda_1$, $L_1 = \bigoplus_{j=0}^{n} \mathbf{Z} \gamma_j$, $L_2 = \bigoplus_{k=0}^{N} \gamma_k'$, where $\gamma_j = \beta_j/p_j$, $\gamma_k' = q_k \xi_k$; the inner product (43) gives a natural pairing $V_1 \times V_2 \to \mathbf{R}$. Keep in mind that $l = N$, $\eta_k = \eta_k' = \xi_k$; then all conditions of Lemma 3.6 on $V_1, V_2, M_1, M_2, L_1, L_2$ are satisfied.

Now we set $V = \bigoplus_{j=0}^{n} \mathbf{R} \beta_j$, $M = V \cap \Lambda_1$, $L = L_1$, $V' = V_2$, $M' = M_2$, $L' = L_2$. Obviously, the conditions of Theorem 3.4 on V, V', M, M', L, L' are all satisfied; and M'/L' can be identified to $G = \prod_{k=0}^{N} \mathbf{Z}/q_k \mathbf{Z}$. We thus have (41) and (42). $\qquad\square$

Example 3.9. Let B be the $(N+1) \times (n+1)$ matrix whose columns are the coordinate vectors of $\beta_0, \beta_1, \ldots, \beta_n$, respectively, under the lattice basis $\{\xi_k\}_{k=0}^{N}$. Set

$$a_j = \text{gcd of the } j\text{th row of } \mathrm{adj}(B^{\mathrm{T}}B)B^{\mathrm{T}},$$

$$b_k = \text{gcd of the } k\text{th row of } B \, \mathrm{diag}[a_0, a_1, \ldots, a_n],$$

$$p_j = \frac{|\det B^{\mathrm{T}}B|}{a_j},$$

$$q_k = \frac{|\det B^{\mathrm{T}}B|}{\gcd(b_k, \det B^{\mathrm{T}}B)};$$

and $G = \prod_{k=0}^{N} \mathbf{Z}/q_k \mathbf{Z}$. Then we have (41) and (42).

Proof. Write the integral matrix B as

$$B = \begin{pmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{N0} & b_{N1} & \cdots & b_{Nn} \end{pmatrix}.$$

Let B^{T} denote the transpose of B. Since $\beta_0, \beta_1, \ldots, \beta_n$ are linearly independent, the $(n+1) \times (n+1)$ matrix $B^{\mathrm{T}}B$ is non-singular. From the elementary theory of linear algebra, we have

$$\mathrm{adj}(B^{\mathrm{T}}B)(B^{\mathrm{T}}B) = \mathrm{adj}(B^{\mathrm{T}}B)B^{\mathrm{T}}B = (\det B^{\mathrm{T}}B)I_{n+1}, \tag{44}$$

where I_{n+1} is the $(n+1) \times (n+1)$ identity matrix. Without loss of generality we assume that $\det B^T B$ is positive.

Set $V_1 = V_2 = \mathbf{R}^{N+1}$, $M_1 = M_2 = \Lambda_1$, $L_1 = \bigoplus_{j=0}^{n} \mathbf{Z}\gamma_j$, $L_2 = \bigoplus_{k=0}^{N} \mathbf{Z}\gamma_k'$, where $\gamma_j = \beta_j/p_j$, $\gamma_k' = q_k\xi_k$; the inner product (43) gives a natural pairing $V_1 \times V_2 \to \mathbf{R}$. Let β_j' be the jth row of $\mathrm{adj}(B^T B)B^T$ divided by a_j. Then (44) shows that $\langle \beta_j, \beta_k' \rangle = p_j\delta_{jk}$. Since both β_j and β_j' are points of Λ_1, it implies that p_j are integers. Keep in mind that $l = N$, $\eta_k = \eta_k' = \xi_k$; we then obtain conclusions (i) and (ii) of Lemma 3.6. Note that

$$\langle \gamma_j, \gamma_k' \rangle = \frac{|\det B^T B|}{p_j \gcd(b_k, \det B^T B)} \langle \beta_j, \xi_k \rangle = \frac{a_j b_{kj}}{\gcd(b_k, \det B^T B)} \in \mathbf{Z}.$$

Then $L_1 \times L_2 \to \mathbf{Z}$.

Now we set $V = \bigoplus_{j=0}^{n} \mathbf{R}\beta_j$, $M = V \cap \Lambda_1$, $L = L_1$, $V' = V_2$, $M' = M_2$, $L' = L_2$. Obviously, all conditions of Theorem 3.4 on V, V', M, M', L, L' are satisfied; and M'/L' can be identified to $G = \prod_{k=0}^{N} \mathbf{Z}/q_k\mathbf{Z}$. We thus have (41) and (42). $\square$

Example 3.10. Let A be the $N \times n$ matrix whose columns are the coordinate vectors of $\alpha_1 - \alpha_0, \ldots, \alpha_n - \alpha_0$, respectively, under the lattice basis $\{\xi_k\}_{k=1}^{N}$; and let B be the $(N+1) \times (n+1)$ matrix whose columns are the coordinate vectors of $\beta_0, \beta_1, \ldots, \beta_n$ respectively under the lattice basis $\{\xi_k\}_{k=0}^{N}$. Set $p = \mathrm{tor}A$; then

$$\sum_{m=0}^{\infty} L(\sigma, \Lambda, m)e^{-2\pi m s} = \frac{1}{2^{n+1}p^{N+1}}$$
$$\times \sum_{\substack{h_0, h_1, \ldots, h_N=0 \\ h=(h_0, h_1, \ldots, h_N)}}^{p-1} \prod_{j=0}^{n} \left(1 + \coth\frac{\pi}{p}(s + \iota\langle\beta_j, h\rangle)\right); \quad (45)$$

$$\sum_{m=1}^{\infty} L(\sigma^0, \Lambda, m)e^{-2\pi m s} = \frac{(-1)^{(n+1)}}{2^{n+1}p^{N+1}}$$
$$\times \sum_{\substack{h_0, h_1, \ldots, h_N=0 \\ h=(h_0, h_1, \ldots, h_N)}}^{p-1} \prod_{j=0}^{n} \left(1 - \coth\frac{\pi}{p}(s + \iota\langle\beta_j, h\rangle)\right). \quad (46)$$

Proof. Set $V = \bigoplus_{j=0}^{n} \mathbf{R}\beta_j$, $V' = \mathbf{R}^{N+1}$, $M = V \cap \Lambda_1$, $M' = \Lambda_1$, $C = \bigoplus_{j=0}^{n} \mathbf{R}_{\geq 0}\beta_j$; the inner product (43) gives a pairing $V \times V' \to \mathbf{R}$ such that $M \times M' \to \mathbf{Z}$. Set $\gamma_j = \beta_j/p$, $\gamma_k' = p\xi_k$, $L = \bigoplus_{j=0}^{n} \mathbf{Z}\gamma_j$, $L' = \bigoplus_{k=0}^{N} \mathbf{Z}\gamma_k'$. By Theorem 2.6, $C \cap M \subset C \cap L$. Obviously, $\langle \gamma_j, \gamma_k' \rangle = \langle \beta_j, \xi_k \rangle \in \mathbf{Z}$; this shows that $L \times L' \to \mathbf{Z}$. For $\alpha \in C \cap (L - M)$, write $\alpha = \sum_{k=0}^{N} u_k\xi_k$; since $\alpha \notin M$, some u_k must be non-integral, say $u_{k'}$; we then have $\xi_{k'} \in M'$ and $\langle \alpha, \xi_{k'} \rangle = u_{k'} \notin \mathbf{Z}$. Finally, $G = M'/L'$ can be identified to the set $[0, p)^{N+1} \cap \mathbf{Z}^{N+1}$. Now all conditions of Theorem 3.4 are satisfied. We thus have (45) and (46). $\square$

Example 3.11. Let σ be an n-simplex of $\mathbf{R}^n$ with the vertex set $\{\alpha_j\}_{j=0}^n \subset \mathbf{Z}^n$; and let A be the matrix whose columns are the vectors $\{\alpha_j - \alpha_0\}_{j=1}^n$. Set

$$a_j = \text{gcd of the } j\text{th row of adj } B,$$

$$b_j = \text{gcd of the } j\text{th row of } B \text{ diag}[a_0, a_1, \ldots, a_n],$$

$$p_j = \frac{|\det B|}{a_j},$$

$$q_j = \frac{|\det B|}{\gcd(b_j, \det B)},$$

and $G = \prod_{j=0}^n \mathbf{Z}/q_j\mathbf{Z}$. Then we have (41) and (42).

Proof. Without loss of generality we assume that $\det B$ is positive. Set $V_1 = V_2 = \mathbf{R}^{n+1}$, $M_1 = M_2 = \mathbf{Z}^{n+1}$, $L_1 = \bigoplus_{j=0}^n \mathbf{Z}\gamma_j$, $L_2 = \bigoplus_{j=0}^n \mathbf{Z}\gamma_j'$, where $\gamma_j = \beta_j/p_j$, $\gamma_j' = q_j e_j$. The standard inner product of $\mathbf{R}^{n+1}$ gives a natural pairing $V_1 \times V_2 \to \mathbf{R}$ such that $\langle \eta_j, \eta_k' \rangle = \delta_{jk}$, where $\eta_j = \eta_j' = e_j$. Let β_j' denote the jth row vector of adj B divided by a_j. The identity $(\text{adj } B)B = (\det B)I_n$ shows that $\langle \beta_j, \beta_k' \rangle = p_j\delta_{jk}$. Since both β_j and β_j' are integral points, p_j must be an integer. Then we have conclusions (i) and (ii) of Lemma 3.6. Note that

$$\langle \gamma_j, \gamma_k' \rangle = \frac{|\det B|}{p_j \gcd(b_k, \det B)} \langle \beta_j, e_k \rangle = \frac{a_j b_{kj}}{\gcd(b_k, \det B)} \in \mathbf{Z},$$

where b_{kj} is the kth coordinate of β_j. Then $L_1 \times L_2 \to \mathbf{Z}$.

Now we set $V = V' = \mathbf{R}^{n+1}$, $M = M' = \Lambda_1$, $L = L_1$, $L' = L_2$. Then all conditions of Theorem 3.4 are satisfied, and $G = M'/L' = \prod_{j=0}^n \mathbf{Z}/q_j\mathbf{Z}$. We thus have (41) and (42). $\square$

As shown in [8], the coefficients of Ehrhart polynomials can be computed from any of such generating functions. In fact, let $L(\sigma, m) = \sum_{j=0}^n c_j(\sigma)m^j$. Then

$$\begin{aligned}
\sum_{m=0}^\infty L(\sigma, m)e^{-2\pi ms} &= \sum_{m=0}^\infty \sum_{j=0}^n c_j(\sigma)m^j e^{-2\pi ms} \\
&= \sum_{j=0}^n \frac{c_j(\sigma)}{(-2\pi)^j} \sum_{m=0}^\infty \frac{d^j}{ds^j} e^{-2\pi ms} \\
&= \sum_{j=0}^n \frac{c_j(\sigma)}{(-2\pi)^j} \cdot \frac{d^j}{ds^j}\left(\frac{1}{1 - e^{-2\pi s}}\right) \\
&= \sum_{j=0}^n \frac{c_j(\sigma)}{(-2\pi)^j} \cdot \frac{d^j}{ds^j}\left(\frac{1}{2\pi s} + \cdots\right) \\
&= \sum_{j=0}^n \frac{j!c_j(\sigma)}{(2\pi)^{j+1}}\left(\frac{1}{s^{j+1}} + \cdots\right),
\end{aligned}$$

where the dots $\cdots$ mean the holomorphic terms near $s = 0$. So we have the following proposition on the coefficients of Ehrhart polynomials in the generating functions.

Proposition 3.12. *The jth coefficient of the Ehrhart polynomial $L(\sigma, m)$ is the coefficient of the term $1/s^{j+1}$ in the Laurent expansion at $s = 0$ of the function*

$$\frac{(2\pi)^{j+1}}{j!} \sum_{m=0}^{\infty} L(\sigma, m) e^{-2\pi m s}.$$

4.   Applications and Dedekind Sums

It is not surprising from (29) and (30) that the coefficients of Ehrhart polynomials are related to the function $((t))$, defined by

$$((t)) = \begin{cases} t - \lceil t \rceil + \frac{1}{2} & \text{for} \quad t \notin \mathbf{Z}, \\ 0 & \text{for} \quad t \in \mathbf{Z}, \end{cases}$$

thereby related to Dedekind sums. Recall the *Dedekind sum $s(q, p)$* of coprime positive integers p and q is defined by

$$s(q, p) = \sum_{k=1}^{p-1} \left(\left(\frac{qk}{p} \right) \right) \left(\left(\frac{k}{p} \right) \right).$$

Since $((t))$ is a periodic odd function of period 1, it is routine to check the following properties (see [19]):

$$\sum_{k=0}^{p-1} \left(\left(\frac{k+t}{p} \right) \right) = ((t));$$

$$\sum_{k=0}^{p-1} \left(\left(\frac{k}{p} \right) \right)^l = 0 \qquad \text{for odd } l;$$

$$\sum_{k=0}^{p-1} \left(\left(\frac{k}{p} \right) \right)^2 = \frac{p}{12} + \frac{1}{6p} - \frac{1}{4}.$$

Let $\sigma(a_1, \ldots, a_n)$ be the lattice n-simplex of $\mathbf{R}^n$ whose vertices are $(0, \ldots, 0, a_j, 0, \ldots, 0)$ and the origin; a_j are positive integers. We apply (29) to compute explicitly the coefficient c_{n-2} of the Ehrhart polynomial $L(\sigma^0(a_1, \ldots, a_n), t)$. For simplicity we assume that a_j are pairwise coprime and write $a = a_1 a_2 \cdots a_n$. From (29), c_{n-2} can be written as

$$c_{n-2} = \frac{1}{n!} \sum_{j_1=0}^{a_1-1} \cdots \sum_{j_n=0}^{a_n-1} s_2 \left(\left\lceil \frac{j_1}{a_1} + \cdots + \frac{j_n}{a_n} \right\rceil - 1, \cdots, \left\lceil \frac{j_1}{a_1} + \cdots + \frac{j_n}{a_n} \right\rceil - n \right).$$

If $j_1/a_1 + \cdots + j_n/a_n = k$ is an integer, then $(a/a_1)j_1 + \cdots + (a/a_n)j_n = ka$ implies $j_i \equiv 0 \pmod{a_i}$ for all $1 \le i \le n$; so $j_1 = \cdots = j_n = 0$. Thus if

$(j_1, \ldots, j_n) \neq (0, \ldots, 0)$,

$$\left\lceil \frac{j_1}{a_1} + \cdots + \frac{j_n}{a_n} \right\rceil = \left(\frac{j_1}{a_1} + \cdots + \frac{j_n}{a_n} \right) - \left(\left(\frac{j_1}{a_1} + \cdots + \frac{j_n}{a_n} \right) \right) + \frac{1}{2}.$$

Write $u = j_1/a_1 + \cdots + j_n/a_n$. For positive integers $1 \leq k < l \leq n$, we have

$$f(u) := \sum_{k<l} \left(u - ((u)) - k + \tfrac{1}{2} \right) \left(u - ((u)) - l + \tfrac{1}{2} \right)$$

$$= \sum_{k<l} \left(u^2 + ((u))^2 - 2u((u)) \right) - \left(\sum_{k<l} (k + l - 1) \right) (u - ((u)))$$

$$+ \sum_{k<l} \left(kl - \frac{k+l}{2} + \frac{1}{4} \right).$$

We understand that the summation $\sum_{j_1, \ldots, j_n} f(u)$ is taken over $0 \leq j_1 \leq a_1 - 1, \ldots, 0 \leq j_n \leq a_n - 1$ and so forth in the following. Notice the difference between c_{n-2} and $\sum_{j_1, \ldots, j_n} f(u)$ at $j_1 = \cdots = j_n = 0$. It follows that

$$c_{n-2} = \frac{1}{n!} \left[\sum_{j_1, \ldots, j_n} f(u) + s_2(-1, \ldots, -n) - \sum_{k<l} \left(kl - \frac{k+l}{2} + \frac{1}{4} \right) \right].$$

An elementary careful calculation shows that

$$\sum_{j_1, \ldots, j_n} u^2 = \frac{n(3n+1)a}{12} + \frac{a}{2} \sum_{k=1}^{n} \left(-\frac{n}{a_k} + \frac{1}{3a_k^2} \right) + \frac{a}{2} \sum_{k<l} \frac{1}{a_k a_l};$$

$$\sum_{j_1, \ldots, j_n} ((u))^2 = \frac{a}{12} + \frac{1}{6a} - \frac{1}{4};$$

$$\sum_{j_1, \ldots, j_n} u((u)) = \sum_{k=1}^{n} s\left(\frac{a}{a_k}, a_k \right);$$

$$\sum_{j_1, \ldots, j_n} u = \frac{a}{2} \left(n - \sum_{k=1}^{n} \frac{1}{a_k} \right);$$

$$\sum_{k<l} (k + l - 1) = \frac{(n-1)n^2}{2};$$

$$\sum_{k<l} \left(kl - \frac{k+l}{2} + \frac{1}{4} \right) = \frac{(n-1)n(3n^2 - n - 1)}{24};$$

$$s_2(-1, \ldots, -n) = \frac{(n-1)n(n+1)(3n+2)}{24}.$$

Proposition 4.1. *Let $a_1, \ldots, a_n$ be pairwise coprime positive integers. Set $a = a_1 a_2 \cdots a_n$. Then the coefficient c_{n-2} of the Ehrhart polynomial $L(\sigma^0(a_1, \ldots, a_n), t)$ is*

given by

$$c_{n-2} = \frac{1}{(n-2)!} \left[\frac{n}{4} + \frac{1}{12a} + \sum_{j=1}^{n} \left(\frac{a}{12a_j^2} - s\left(\frac{a}{a_j}, a_j\right) \right) + \sum_{1 \le k < l \le n} \frac{a}{4a_k a_l} \right]. \quad (47)$$

Take $n = 2$, then c_0 is the Euler characteristic of a relatively open triangle so that $c_0 = 1$. We thus obtain the following well-known relation on Dedekind sums; see [19].

Corollary 4.2 (Reciprocity Law of Dedekind Sums).   *For any coprime positive integers a_1 and a_2,*

$$s(a_1, a_2) + s(a_2, a_1) = -\frac{1}{4} + \frac{1}{12} \left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1} \right).$$

Take $n = 3$, then we have the coefficient c_1 for the Ehrhart polynomial of the closed tetrahedron with vertices $(0, 0, 0)$, $(a_1, 0, 0)$, $(0, a_2, 0)$, $(0, 0, a_3)$, where a_1, a_2, a_3 are pairwise coprime, as given by

$$\begin{aligned} c_1 &= \frac{3}{4} + \frac{1}{12} \left(\frac{1}{a_1 a_2 a_3} + \frac{a_1 a_2}{a_3} + \frac{a_1 a_3}{a_2} + \frac{a_2 a_3}{a_1} \right) + \tfrac{1}{4}(a_1 + a_2 + a_3) \\ &\quad - s(a_1 a_2, a_3) - s(a_1 a_3, a_2) - s(a_2 a_3, a_1). \end{aligned} \quad (48)$$

Adding up the coefficients $c_0 = 1$, c_1, $c_2 = \tfrac{1}{4}(a_1 a_2 + a_1 a_3 + a_2 a_3 + 1)$, and $c_3 = a_1 a_2 a_3/6$, one obtains Mordell's formula [15] on the number of lattice points of the closed tetrahedron.

The other coefficients of $L(\sigma(a_1, \ldots, a_n), t)$ can be calculated in a similar way, using Zagier's higher-dimensional Dedekind sums [10], [21], but the calculation is too complicated to be presented here. However, it is not clear whether the coefficients of the Ehrhart polynomial for an arbitrary lattice simplex can be fully expressed in terms of the higher-dimensional Dedekind sums. Since $c_0 = (-1)^n$ for any relatively open n-simplex, it reduces to the reciprocity law for higher-dimensional Dedekind sums. In order to express fully the coefficients of Ehrhart polynomials in terms of certain cotangent expansions, it seems that one needs to generalize the Dedekind sums further. We introduce a *matrix Dedekind sum* in the following.

Definition 4.3.   Let A be an $m \times n$ integral matrix and let B be an $l \times n$ integral matrix whose row vectors are denoted $\beta_1, \ldots, \beta_l$. Let $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_n)$, and $r = (r_1, \ldots, r_l)$ be integral vectors with positive entries. The matrix Dedekind sum is

$$s(A, B; p, q, r) = \sum_{\substack{0 \le h < q \\ h \in \mathbf{Z}^n \\ p \mid Ah}} \prod_{j=1}^{l} \left(\left(\frac{\beta_j h}{r_j} \right) \right), \quad (49)$$

where $0 \le h < q$ means that all the entries of h are non-negative and strictly less than the corresponding entries of q; $p \mid Ah$ means that each entry of p divides the corresponding entry of Ah.

The idea to define the matrix Dedekind sum (49) comes from formulas (29) and (50) for coefficients c_j and the calculation of c_{n-2}. Theorem 2.6 is also important to imagine what sum should be taken over. The right side of (29) can be divided into two parts: Part 1, a sum over $|\gamma| \notin \mathbf{Z}$; and Part 2, a sum over $|\gamma| \in \mathbf{Z}$. In Part 1, since $\lceil |\gamma| \rceil = |\gamma| - ((|\gamma|)) + \frac{1}{2}$, the sum can be expanded into a collection of sums

$$\sum a(k, l) |\gamma|^k ((|\gamma|))^l, \qquad k + l \leq n - j.$$

These sums can be further reduced to a collection of sums of the form (49), some of them may be in rational forms without involving the symbol $((\cdot))$. In Part 2, the sum may be written as counting the number of lattice points of some lower-dimensional lattice polytopes. By induction on dimension, they can be reduced in principle to a collection of sums of the form (49). However, one may still doubt the program with (49) until it is actually done explicitly. Since $c_0(\sigma^0) = (-1)^n$ for any lattice n-simplex σ, it is sure that the program will give certain reciprocity laws on certain generalized Dedekind sums (may not be discovered yet) when the right side of (29) is treated correctly in arithmetic ways. Study of this matrix Dedekind sum in the special case $p_1 = \cdots = p_m = r_1 = \cdots = r_l$ is particularly wanted.

Theorem 4.4. *Let σ be an n-simplex of $\mathbf{R}^N$ with the vertex set $\{\alpha_j\}_{j=0}^n \subset \mathbf{Z}^N$; and let A be the $N \times n$ matrix whose columns are the vectors $\{\alpha_j - \alpha_0\}_{j=0}^n$. Let a_j be the gcd of the jth row of $\mathrm{adj}(A^\mathrm{T} A) A^\mathrm{T}$; $p_j = |\det A^\mathrm{T} A| / a_j$. If p is a positive integer such that either $\mathrm{lcm}\{p_j\}_{j=0}^n \mid p$ or $(\mathrm{tor} A) \mid p$, then the jth coefficient of the Ehrhart polynomial $L(\sigma^0, t)$ is given by*

$$c_j(\sigma^0) = \frac{1}{n!} \sum_{\substack{h_1, \dots, h_n = 0 \\ h = (h_1, \dots, h_n) \\ p \mid Ah}}^{p-1} s_{n-j} \left(\left\lceil \frac{h_1 + \cdots + h_n}{p} \right\rceil - 1, \cdots, \left\lceil \frac{h_1 + \cdots + h_n}{p} \right\rceil - n \right),$$

(50)

where $p \mid Ah$ means that p divides each entry of the integral vector Ah.

Proof. This is an immediate consequence of Theorm 2.6 and the coefficient formula (29). $\qquad \square$

Formula (50) shows directly that $c_j(\sigma^0)$ is invariant under unimodular $N \times N$ matrices. Write $N = m$ and $\dim \sigma = n$; expanding the symmetric functions in the right side of (50) and using $c_0(\sigma^0) = (-1)^{\dim \sigma}$, we have the following corollary.

Corollary 4.5 (Reciprocity Law). *Let A be an $m \times n$ integral matrix with $\mathrm{rank}\, A = n$ and $p = \mathrm{tor}\, A$. Then*

$$\sum_{j=0}^n (-1)^j s_{n-j}(1, 2, \dots, n) \sum_{\substack{h_1, \dots, h_n = 0 \\ h = (h_1, \dots, h_n) \\ p \mid Ah}}^{p-1} \left\lceil \frac{h_1 + \cdots + h_n}{p} \right\rceil^j = n!.$$

(51)

Let P be a bounded lattice polyhedron of $\mathbf{R}^N$ whose vertices are points of the integral lattice $\mathbf{Z}^N$, decomposed into a collection $\mathcal{F}$ of disjoint relatively open lattice simplices.

The subcollection of n-simplices are denoted $\mathcal{F}(n)$. Then the jth coefficient of the Ehrhart polynomial $L(P, t)$ can be written as

$$c_j(P) = \sum_{\alpha \in \mathcal{F}(0)} \sum_{n=0}^{\dim P} \frac{1}{(n+1)!} \sum_{\substack{\alpha \leq \sigma \in \mathcal{F}(n) \\ \gamma \in D(\sigma, \alpha)}} s_{n-j}(\lceil |\gamma| \rceil - 1, \ldots, \lceil |\gamma| \rceil - n). \quad (52)$$

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References

1. A. I. Barvinok, Computing the Ehrhart polynomial of a convex lattice polytope, *Discrete Comput. Geom.* **12** (1994), 35–48.
2. M. Brion and M. Vergne, Lattice points in simple polytopes, *J. Amer. Math. Soc.* **10** (1997), 371–392.
3. S. E. Cappell and J. L. Shaneson, Genera of algebraic varieties and counting lattice points, *Bull. Amer. Math. Soc.* **30** (1994), 62–69.
4. B. Chen, On the Euler characteristics of finite unions of convex sets, *Discrete Comput. Geom.* **10** (1993), 79–93.
5. B. Chen, Weight functions, double reciprocity laws, and volume formulas for lattice polyhedra, *Proc. Nat. Acad. Sci. USA*, **95** (1998), 9093–9098.
6. B. Chen and V. Tauraev, Counting lattice points of rational polyhedra, *Adv. in Math.* **155** (2000), 84–97.
7. V. I. Danilov, The geometry of toric varieties, *Russian Math. Surveys* **33** (1978), 97–154.
8. R. Diaz and S. Robins, The Ehrhart polynomial of a lattice polytope, *Ann. of Math.* **135** (1997), 503–518.
9. E. Ehrhart, Sur un problème de geómétrie diophantienne linéaire II, *J. Reine Angew. Math.* **227** (1967), 25–49.
10. F. Hirzebruch and D. Zagier, *The Atiyah-Singer Index Theorem and Elementary Number Theory*, Publish or Perish, Boston, MA, 1974.
11. J. M. Kantor and A. G. Khovanskii, Une application du Théoréme de Riemann–Roch combinatorie au polynôme d'Ehrhart des polytopes entiers de $\mathbf{R}^n$, *C. R. Acad. Sci. Paris*, *Sér. I* **317** (1993), 501–507.
12. D. A. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, Cambridge, 1997.
13. I. G. Macdonald, The volume of a lattice polyhedron, *Math. Proc. Cambridge Philos. Soc.* **59** (1963), 719–726.
14. I. G. Macdonald, Polynomials associated with finite cell complexes, *J. London Math. Soc.* (2) **4** (1971), 181–192.
15. L. J. Mordell, Lattice points in a tetrahedron and Dedekind sums, *J. Indian Math.* **15** (1951), 41–46.
16. R. Morelli, Pick's theorem and the Todd class of toric variety, *Adv. in Math.* **100** (1993), 183–231.
17. M. Newmans, *Integral Matrices*, Pure and Applied Mathematics, vol. 45, Academic Press, New York, 1972.
18. J. Pommersheim, Toric varieties, lattice points and Dedekind sums, *Math. Ann.* **295** (1993), 1–24.
19. H. Rademacher and E. Grosswald, *Dedekind Sums*, The Carus Mathematical Monographs, vol. 16, The Mathematical Association of America, Washington, DC, 1972.
20. R. P. Stanley, *Enumerative Combinatorics*, Vol. I, Cambridge University Press, Cambridge, 1997.
21. D. Zagier, Higher dimensional Dedekind sums, *Math. Ann.* **202** (1973), 149–172.