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LATTICE POLYGONS AND GREEN'S THEOREM

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ABSTRACT. Associated to an n-dimensional integral convex polytope P is a toric variety X and divisor D, such that the integral points of P represent $H^0(\mathcal{O}_X(D))$. We study the free resolution of the homogeneous coordinate ring $\bigoplus_{m\in\mathbb{Z}} H^0(mD)$ as a module over $Sym(H^0(\mathcal{O}_X(D)))$. It turns out that a simple application of Green's theorem yields good bounds for the linear syzygies of a projective toric surface. In particular, for a planar polytope $P = H^0(\mathcal{O}_X(D))$, D satisfies Green's condition N_p if ∂P contains at least $p+3$ lattice points.

1. Green's theorem and hyperplane sections

For a curve C of genus g, a divisor D of degree $d \geq 2g+1$ is very ample, so gives an embedding of C into projective space. In fact, when $d \geq 2g+1$, work of Castelnuovo, Mattuck and Mumford shows that the embedding is *projectively normal*, which means that $S = Sym(H^0(\mathcal{O}_X(D)))$ surjects onto $\bigoplus_{m\in\mathbb{Z}} H^0(mD) = R$. When $d \geq 2g+2$, results of Fujita and St. Donat show that the homogeneous ideal of I_C is generated by quadrics. Let $F_\bullet$ be a minimal free resolution of R over S. A very ample divisor is said to satisfy property N_p if $F_0 = S$ and $F_q \simeq \bigoplus S(-q-1)$ for all $q \in \{1,\ldots,p\}$. Thus, N_0 means projectively normal, N_1 means that the homogeneous ideal is generated by quadrics, N_2 means that the minimal syzygies on the quadrics are linear, and so on. In [7], Green used Koszul cohomology to give a beautiful generalization of the classical results above: if $\deg(D) \geq 2g+p+1$, then D satisfies N_p.

In this brief note, we investigate the N_p property for toric varieties. For any divisor D and variety X such that R is arithmetically Cohen-Macaulay, it is natural to slice with hyperplanes until X has been reduced to a curve, and then apply Green's theorem. Results of Hochster [8] show that projectively normal toric varieties are always arithmetically Cohen-Macaulay. So it makes sense to apply the technique in this setting. In [4], Ewald and Wessels prove that if D is an ample divisor on a toric variety of dimension n, then $(n-1)D$ is very ample and satisfies N_0. Bruns, Gubeladze and Trung [2] give another proof and also show that nD satisfies property N_1. While it is often difficult to determine if a given divisor satisfies N_0, for a lattice polygon P and corresponding divisor on a toric surface, the property N_0 holds "for free".

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In [6] Gallego and Purnaprajna give criteria for the N_p property for smooth rational surfaces. Toric varieties are rational, and in the case of smooth surfaces the result we obtain is a toric restatement of the result in [6]. However, the proof is simpler in the toric case, applies to singular surfaces, and extends several results in the toric literature. For example, in [10] Koelman proves that a toric surface defined by P satisfies N_1 iff ∂P contains at least four lattice points, and Ewald and Schmeink [3] prove that certain polytopes associated to smooth toric varieties with $Pic(X) = 2$ satisfy N_1.

Theorem 1.1. *Let P be an n-dimensional lattice polytope, and X, D the associated projective toric variety and ample divisor; so $P = H^0(\mathcal{O}_X(D))$. If D satisfies N_0, then D satisfies N_p if P satisfies*

$$\sum_{facets F_i} vol(F_i) \geq n(n-2)vol(P) + \frac{p+3}{(n-1)!}.$$

Proof. Hochster's results mentioned earlier show that R is arithmetically Cohen-Macaulay. In [9], Khovanskii shows that a toric variety X defined by a lattice polytope P is *normal* iff the Hilbert polynomial of X and the Ehrhart polynomial of P agree. Projective normality implies normality, and so X is normal. Hence, the singular locus of X is of codimension at least two. So a general member of $|D|$ is smooth. Slicing with $n-1$ general hyperplanes, we obtain a smooth curve C with the same minimal free resolution as X. By Khovanskii's result,

$$\chi(\mathcal{O}_X(mD)) = |mP \cap \mathbb{Z}^n| = am^n + bm^{n-1} + \cdots .$$

After slicing with $n-1$ general hyperplanes, the resulting curve C has

$$\chi(\mathcal{O}_C(m)) = n!am + (n-1)!b - (n-1)!\binom{n}{2}a.$$

The first two coefficients of the Ehrhart polynomial are

$$\begin{array}{rcl} a & = & vol(P), \\ b & = & \frac{1}{2}\sum\limits_{facets F_i} vol(F_i). \end{array}$$

Thus, applying Green's theorem, the divisor D associated to P satisfies N_p if

$$\sum_{facets F_i} vol(F_i) \geq n(n-2)vol(P) + \frac{p+3}{(n-1)!}.$$

□

2. Applications

In [12], Wills shows that an n-dimensional lattice polytope P that contains an interior point satisfies $n \cdot vol(P) \geq \sum_{facets} vol(F_i)$. So at first glance the bound above seems useless. However, when $n = 2$ the term $n(n-2)vol(P)$ vanishes, and by [4] the divisor associated to a lattice polygon P satisfies N_0. So we obtain:

Corollary 2.1. *The divisor D associated to a lattice polygon P satisfies N_p if*

$$\# \text{ integral points in } \partial P \geq p + 3.$$

Example 2.2. If P is the unit lattice two-simplex, then dP defines the d-uple Veronese embedding of $\mathbb{P}^2$. By Corollary 2.1, dP satisfies N_p if $p \leq 3d-3$, recovering a result of [1]. In fact, Ottaviani and Paoletti [11] show that this bound is tight.

Example 2.3. The ideal sheaf of a projective toric surface X is two-regular iff N_p holds for all $p \leq codim(X)$. By Corollary 2.1, this is true if P has no interior points. In this case R is level with a-invariant -2, which gives half of Theorem 1.27 of [2]. If P has no interior points, then the corresponding divisor has arithmetic genus zero ([5], p. 91). Thus X is a surface of minimal degree. So if X is smooth, then it must be a rational normal scroll or the Veronese surface in $\mathbb{P}^5$.

If P is three-dimensional, then P satisfies N_p if $2\sum vol(F_i) - 6vol(P) - 3 \geq p$ and N_0 holds. In order to obtain a useful bound, we require that P have no interior points, so that the Ehrhart polynomial evaluated at -1 is zero. For such a polytope, this implies that $\sum vol(F_i) = \#$ integral points in $P - 2$, which yields:

Corollary 2.4. *A lattice three-polytope P with no interior points satisfies N_p if D is projectively normal and $\#$ integral points in $P \geq 3vol(P) + \frac{p+7}{2}$.*

Example 2.5. Polytopes corresponding to smooth torics with $Pic(X) = 2$ are studied in [3]; for threefolds there are only two families. Ewald and Schmeinck show that the polytopes below satisfy N_1:

$$\begin{array}{rcl} P_1(a) & = & conv\{\mathbf{0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_1} + (a+1)\mathbf{e_3}, \mathbf{e_1} + (a+1)\mathbf{e_2}\}, \\ P_2(a,b) & = & conv\{\mathbf{0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_1} + (a+1)\mathbf{e_3}, \mathbf{e_2} + (b+1)\mathbf{e_3}\}. \end{array}$$

A calculation shows that

$$\begin{array}{rclrcl} vol(P_1(a)) & = & \frac{a^2+3a+3}{6}, & \#\text{ integral points in } P & = & \frac{a^2+5a+12}{2}, \\ vol(P_2(a,b)) & = & \frac{a+b+3}{6}, & \#\text{ integral points in } P & = & a+b+6. \end{array}$$

Thus, $P_1(a)$ satisfies N_p if $p \leq 2a+2$, and $P_2(a,b)$ satisfies N_p if $p \leq a+b+2$.

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