Lattice Strategies for the Dirty Multiple Access Channel †

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Abstract

In Costa's dirty-paper channel, *Gaussian random binning* is able to eliminate the effect of interference which is known at the transmitter, and thus achieve capacity. We examine a generalization of the dirty-paper problem to a multiple access channel setup, where *structured (lattice-based) binning* seems to be necessary to achieve capacity. In the dirty-MAC, two additive interference signals are present, one known to each transmitter but none to the receiver. The achievable rates using Costa's Gaussian binning vanish if both interference signals are strong. In contrast, it is shown that lattice-strategies ("lattice precoding") can achieve positive rates, independent of the interference power. Furthermore, in some cases - which depend on the noise variance and power constraints - high-dimensional lattice strategies are in fact optimal. In particular, they are optimal in the limit of high SNR - where the capacity region of the dirty MAC approaches that of a clean MAC whose power is governed by the minimum of the users' powers rather than their sum. The rate gap at high SNR between lattice-strategies and optimum (rather than Gaussian) random binning is conjectured to be $\frac{1}{2}\log_2(\pi e/6) \approx 0.254$ bit. Thus, the doubly-dirty MAC is another instance of a network setting, like the Körner-Marton problem, where (linear) structured coding is potentially better than random binning. Finally, it is shown that lattice strategies are at most 0.167 bit from the capacity region for all SNR. The results are also compared and contrasted to the *single dirt* multiple access channel case (considered by other researchers), where lattice strategies and Gaussian random binning have similar performance.

Index Terms

Dirty paper coding, multiple access channel, channel state information, lattice-strategies, interference cancellation, interference alignment, interference concentration.

I. Introduction

A subclass of multiple-access channels (MAC) with side information (SI) known at the transmitters is considered. Figure 1 depicts the problem of interest, a two-user Gaussian MAC with two known interferences. The channel output is given by

$$Y = X_1 + X_2 + S_1 + S_2 + Z, \tag{1}$$

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where Z is an additive white Gaussian noise, i.e. $Z \sim \mathcal{N}(0, N)$, and X_1 and X_2 are the channel inputs from user 1 and user 2, respectively, which must satisfy the power constraints P_1 and P_2. The interference signals S_1 and S_2 are known non-causally to the transmitters of user 1 and user 2, respectively, but unknown to the receiver. We call this setup the *doubly-dirty MAC*.

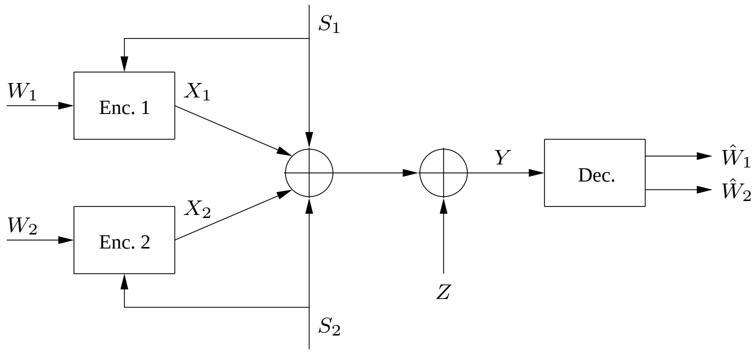

Fig. 1: Doubly-dirty MAC.

This channel model generalizes Costa's dirty-paper channel [1] to a multiple access setup. In [1], Costa considered the single-user case,

$$Y = X + S + Z, \tag{2}$$

where the interference is assumed to be i.i.d. Gaussian, i.e., $S \sim \mathcal{N}(0, Q)$. He showed that the capacity of this channel is $\frac{1}{2}\log_2(1 + \mathrm{SNR})$, where $\mathrm{SNR} = \frac{\mathrm{P}}{\mathrm{N}}$, independent of the interference power Q. Thus, the capacity is the same as that of the "clean" (interference-free AWGN) channel and no loss is incurred by the presence of the the interference. We will compare (and contrast) this well-known result with effect of known interference on the capacity region of the doubly-dirty MAC as well as some other related scenarios.

The proof of Costa [1] uses the general capacity formula derived by Gel'fand and Pinsker [2] for channels with (non-causal) side information at the transmitter. Their technique falls in the framework of *random binning* which is widely used in the analysis of multi-terminal source and channel coding problems. Using random binning for the direct coding theorem, they obtained a *single letter* capacity expression (originally derived for the discrete channel case) which involves an auxiliary random variable U:

$$C_{GP} = \max_{p(u,x|s)} \{I(U;Y) - I(U;S)\} \tag{3}$$

where the maximization is over all joint distributions of the form $p(u, s, y, x) = p(s)p(u, x|s)p(y|x, s)$. Selecting the auxiliary random variable U to be

$$U = X + \alpha S, \tag{4}$$

where $X \sim \mathcal{N}(0, P)$ is independent of S, and taking $\alpha = \frac{P}{P+N}$, maximizes (3), and the associated random binning scheme is capacity achieving[1].

[1]Although (3) was originally derived for the case of discrete memoryless channel, it holds also for continues signals.

A special case of the dirty MAC (1) was considered by Gel'fand and Pinsker in [3]. They showed that in the noiseless case ($N = 0$), arbitrary large rate pairs (R_1, R_2) are achievable. For the general ($N > 0$) case and independent Gaussian interferences, they conjectured that the capacity region is the same as that of a "clean" MAC, i.e., the standard Gaussian MAC with no interference. The outer bound in Section IV shows that the capacity region is in fact smaller.

An interesting observation we make in this work is that in the limit when both interference signals are strong, Gaussian binning (i.e., the extension of Costa's solution (4) to the two-user case) is *unable* to achieve *positive rates* over the doubly-dirty MAC of Fig. 1 (see Proposition 1 in Section III). This is in contrast not only to Costa's problem, but also to the "single dirt" MAC case (with one interference known to one user) and the common interference case (one interference known to both users), where Gaussian binning was shown to be optimal (or nearly optimal) [4], [5], [3], [6]. Nevertheless, as we show in this work, lattice-strategies [7] achieve positive rates over the doubly-dirty MAC by employing *interference concentration* and *alignment*.

One-dimensional lattice-strategies provide a positive - though still sub-optimal - *single-letter* solution for the rate region. We conjecture that this is, in fact, the *best* single-letter solution for the doubly-dirty MAC when the interference is strong and the SNR is high.[2] High-dimensional lattice strategies - which can be regarded as a special case of a multi letter solution - are strictly better; as we show, they are in fact asymptotically optimal for this problem, i.e., capacity achieving, under certain conditions (e.g., high SNR).

The sum-rate gap between the one-dimensional and the high-dimensional lattice schemes is the *shaping gain* [8] $\frac{1}{2}\log_2(2\pi e/12) \approx 0.254$ bit. Thus, the doubly-dirty MAC is an instance where linear codes (lattices) are strictly better than any known single letter solution, i.e., better than any random binning technique; see [9] for an extensive discussion on this issue. A similar phenomenon was observed by Körner and Marton [10] in a distributed lossless source coding problem (the modulo-two sum problem), where they showed that the rate region achievable using linear codes is optimal, and is superior to the "best known single letter characterization" for the rate region.

Beyond the the central role that linearity plays in coding for the doubly-dirty MAC channel, we will observe that the capacity region itself exhibits some interesting characteristics. First, there is an inherent "power loss" with respect to the clean MAC channel, i.e., the sum rate is governed by the minimum (rather than the sum) of the encoders' powers. This follows from the outer bound presented in Section IV. A second phenomenon, which is manifested at least in the *achievable* region derived in Section VI, is the further (partial) loss of the "1" in the capacity expressions. More specifically, the "1" is replaced by a factor of $1/K$, where K is the number of users. While this observation is only based on our coding approach and achievability results, we conjecture that this loss is in fact inherent.

The paper is organized as follows. Section II defines the doubly-dirty MAC, the MAC with a single dirty user and the MAC with common interference. Section III gives a brief overview of the main concepts and insights developed in the paper. Section IV derives outer bounds for the capacity region of the doubly-dirty MAC and

[2] This approach may be interpreted as a degenerate form of random binning, as we shall discuss in Section VI.

for the MAC with a single dirty user for the case of strong-interference. A brief review of lattice codes, and a lattice-alignment transmission scheme are presented in Section V. The main result of this work, the near-optimality of lattice strategies for the doubly-dirty MAC, is presented in Section VI. In Sections VII and VIII we study the single dirt variants (MAC with a single dirty user and MAC with common interference) which were previously treated in [5], [4], [3], [6]. Using the lattice strategies approach, we extend these previously derived results (which assumed Gaussian interference of known power) to the case of an arbitrary interference. Other extensions of these problems are considered in Section IX. Section X concludes the paper.

II. Problem Formulation

A. The General Memoryless Model

The channel model in (1) is a special case of a memoryless MAC with two channel states $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$, which are known non-causally at the transmitters of user 1 and user 2, respectively. The states S_1 and S_2 are memoryless and independent with distributions $p(s_1)$ and $p(s_2)$, respectively. The channel transition probability is $p(y|x_1, x_2, s_1, s_2)$, where $X_1 \in \mathcal{X}_1$ and $X_2 \in \mathcal{X}_2$ are the channel inputs, and $Y \in \mathcal{Y}$ is the channel output. The channel is memoryless i.e.,

$$p(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2, \mathbf{s}_1, \mathbf{s}_2) = \prod_{i=1}^{n} p(y_i|x_{1i}, x_{2i}, s_{1i}, s_{2i}), \tag{5}$$

where bold face indicates vectors (of length n). The encoder outputs of user 1 and user 2 are given by

$$\mathbf{x}_i = f_i(w_i, \mathbf{s}_i) \quad \text{for, i} = 1, 2,$$

where $w_i \in \mathcal{W}_i$ are the transmitted messages. The achievable rates are denoted by R_1 and R_2 where $|\mathcal{W}_1| = 2^{nR_1}$ and $|\mathcal{W}_2| = 2^{nR_2}$. The decoder reconstructs the transmitted messages w_1, w_2 from the channel output, hence

$$(\hat{w}_1, \hat{w}_2) = g(\mathbf{y}).$$

A single letter characterization for the capacity region is not known; see [9], [11] for a more detailed discussion. The best known achievable rate region for this channel, based on the random binning technique, was presented by Jafar in [12], and it is given by the convex hull of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\le I(U_1; Y|U_2) - I(U_1; S_1) \\ R_2 &\le I(U_2; Y|U_1) - I(U_2; S_2) \\ R_1 + R_2 &\le I(U_1, U_2; Y) - I(U_1; S_1) - I(U_2; S_2) \end{aligned} \tag{6}$$

for some $p(u_1, u_2, x_1, x_2|s_1, s_2) = p(u_1, x_1|s_1)p(u_2, x_2|s_2)$.[3] The case where there is only a single state S_1 known to user 1 was treated by Kotagiri and Laneman in [5]. In this case, the single letter expression (6) reduces to the

[3]If the channel inputs and states have finite alphabets, then it is enough to use in (6) auxiliary random variables with alphabets whose cardinality is bounded by $|\mathcal{U}_i| \le |\mathcal{X}_i| + |\mathcal{S}_i|$ for $i = 1, 2$.

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for some $p(u_1, x_1, x_2|s_1) = p(x_2)p(u_1, x_1|s_1)$. The common message capacity ($W_1 = W_2$) was solved by Somekh-Baruch et al. in [4]. Furthermore, the capacity region for the case of degraded messages was derived in [13], [14].

B. The Gaussian Model

We now turn to the Gaussian channel case which is the focus of the paper. Specifically, consider the following models:

1) Doubly-dirty MAC:

$$Y = X_1 + X_2 + S_1 + S_2 + Z, \tag{8}$$

where $Z \sim \mathcal{N}(0, N)$ is independent of X_1, X_2, S_1, S_2, and where user 1 and user 2 must satisfy the power constraints $\frac{1}{n}\sum_{i=1}^{n} x_{1_i}^2 \leq P_1$ and $\frac{1}{n}\sum_{i=1}^{n} x_{2_i}^2 \leq P_2$, respectively; see Fig. 1. The interferences S_1 and S_2 are known non-causally to the transmitters of user 1 and user 2, respectively. The signal-to-noise ratio for each user is defined as $\mathrm{SNR}_1 = \frac{\mathrm{P}_1}{\mathrm{N}}$ and $\mathrm{SNR}_2 = \frac{\mathrm{P}_2}{\mathrm{N}}$. We consider the case of *strong interferences*, i.e., the interferences are assumed to be either *arbitrary sequences*, or independent Gaussian variables with unbounded variances:

$$S_i \sim \mathcal{N}(0, Q_i),\ i = 1, 2, \qquad Q_1, Q_2 \rightarrow \infty. \tag{9}$$

Ideally, we wish to be able to cancel the effect of S_1 and S_2 regardless of their strength - just as in Costa's single-user case (2). However, as we shall see, this is not always possible.

2) MAC with a single dirty user and the "helper problem":

$$Y = X_1 + X_2 + S_1 + Z. \tag{10}$$

In this asymmetric case, shown in Fig. 2, user 1 knows the interference S_1 (informed user) and user 2 is not aware of the interference (uninformed user) [4].

The "helper problem" is a special case of (10), where the informed user does not send any information, and its sole role is to help the uninformed user.

3) MAC with common Interference [3], [6]:

$$Y = X_1 + X_2 + S_c + Z. \tag{11}$$

In this case, there is a single interference S_c which is known non-causally to both encoders, as shown in Fig. 3.

[4]Note that under the strong interference assumption, (10) is not a special case of (8) because we cannot set $S_2 = 0$ in (8). Indeed, the fact that only a single interference is present allows us to derive in Section VII a better achievable rate region than for the doubly-dirty case.

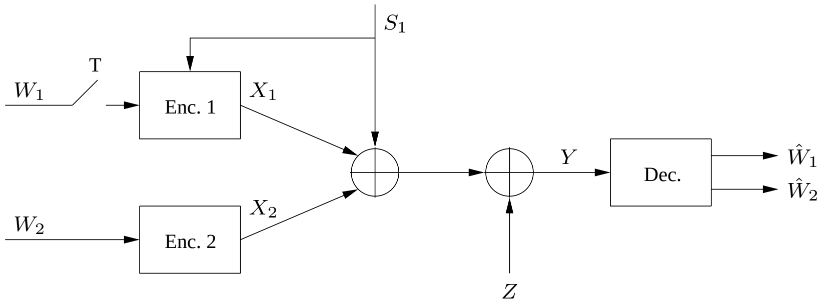


Fig. 2: MAC with a single dirty user (T open corresponds to the helper problem).

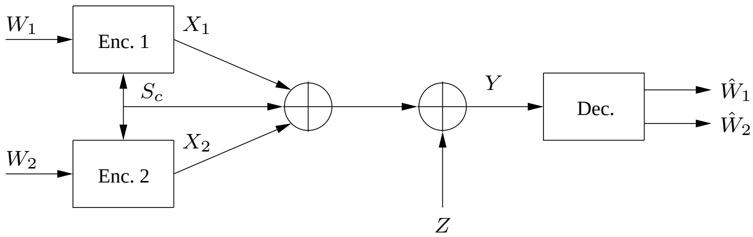


Fig. 3: MAC with common interference.

Remark: The above models can also be extended by allowing common randomness (dither signals) at the encoders and decoder.

III. Overview of the Key Concepts

In this section we introduce the main ideas in a nutshell for some special cases. For simplicity, we assume throughout the limit of high SNR, $\frac{P}{N} \to \infty$, in addition to the strong interference assumption (9). We begin with a simple interpretation of some known techniques for the single-user dirty paper channel.

A. Single-User Dirty Paper Channel

The capacity of the dirty paper channel can be achieved using random binning. The single-letter expression for the capacity is given in (3) which is maximized by the auxiliary variable U in (4) . At high SNR, this choice of U is given by $U = X + S$. Hence, the achievable rate using random binning is given by

$$\begin{aligned} R &= I(U;Y) - I(U;S) \\ &= h(U|S) - h(U|Y) \\ &= h(X) - h(X+S|Y) \\ &= h(X) - h(Z|Y) \\ &\approx h(X) - h(Z) \\ &= \frac{1}{2}\log_2\left(\frac{P}{N}\right) \end{aligned}$$

where the approximation $\approx$ is due to the strong interference assumption $Q \to \infty$. We call this solution "Costa strategy" or "Gaussian random binning".

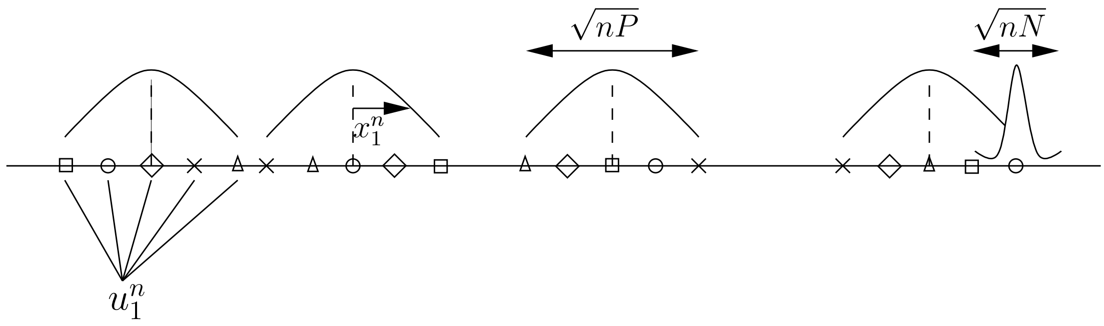


Fig. 4: Single-user: geometric view of random binning.

Random Binning: To translate the auxiliary variable U above into a random binning scheme (see, e.g., [2]), we select $\approx 2^{nI(U;Y)}$ vectors u_1^n i.i.d. according to the distribution of U, and partition them evenly into 2^{nR} bins, where $R \approx I(U;Y) - I(U;S)$ (i.e., there are approximately $2^{nI(U;S)}$ vectors u_1^n in each bin). Each bin represents a message V, and the encoder selects a vector u_1^n in bin V (i.e., in the message's bin) which is jointly typical with the side-information s_1^n. With high probability there exists at least one such u_1^n (for large n). This u_1^n induces a channel input x_1^n which in turn induces a channel output y_1^n. The decoder decodes the message (bin) V by looking for a vector u_1^n which is jointly typical with y_1^n. With high probability there exists one and only one such u_1^n which is the true one (for large n).

Since in our case the auxiliary variable is $U = X + S$, the channel output is given by $Y = U + Z$. Thus, the selected u_1^n is in the vicinity (for large enough n) of the channel output vector y_1^n within a distance of $\sqrt{nN}$, and to the interference vector s_1^n within a distance of $\sqrt{nP}$, where the transmitted vector x_1^n is the latter difference: $x_1^n = u_1^n - s_1^n$.

Let $Q_V(s_1^n)$ denote the vector u_1^n selected by the encoder to transmit the message $V \in \{1, \ldots, 2^{nR}\}$. Bin V thus consists of all possible values that $Q_V(\cdot)$ can take for different s_1^n vectors. We can think of $Q_V(\cdot)$ as a *quantizer* for S_1^n with average "distortion" nP. The transmitted vector x_1^n,

$$x_1^n = Q_V(s_1^n) - s_1^n,$$

can thus be interpreted as the *quantization error*; while the channel output,

$$y_1^n = Q_V(s_1^n) + z_1^n, \tag{12}$$

is the superposition of the noise over the quantized value.

Fig. 4 describes the random binning technique in a qualitative manner. The x-axis describes the collection of the vectors u_1^n. Due to the randomness of the binning scheme, the points are not necessarily located on a uniform grid. Each of the symbols $\square, \circ, \times, \diamondsuit, \triangle$ represents a different bin. Again, due to the randomness of the scheme, each bin has a possibly different pattern of points on the x-axis. The set of typical u_1^n's for a given vector s_1^n is represented by a bell shape of standard deviation $\sqrt{nP}$, while for a given vector y_1^n - by a bell shape of standard deviation $\sqrt{nN}$.

Interference Concentration: Willems [15] proposed the technique of *interference concentration* for the causal dirty paper ("dirty tape") channel. Although the scheme is sub-optimal even at high SNR, it conveys the main idea of canceling the interference using a structured coding scheme. Willems suggested to dedicate half of the input power to mitigate the interference effect and half of the power to send the information. Specifically, the transmitted signal is given by

$$X = V - [S \bmod \Delta], \tag{13}$$

where V is now a real number, and $S \bmod \Delta = \mathrm{S} - \mathrm{Q}(\mathrm{S})$ where $Q(S)$ is a uniform quantizer with step size Δ, i.e., $Q(S) = \Delta \cdot \left\lfloor \frac{S}{\Delta} \right\rfloor$ where $\lfloor \cdot \rfloor$ is the floor operation. The input power P is divided between the information signal V which is uniformly distributed over Δ, and the interference concentration operation $S \bmod \Delta$. Therefore, the input power and step size are related by $\Delta = \sqrt{6P}$. The channel output is given by

$$\begin{aligned} Y &= V + Q(S) + Z && (14) \\ &= V + Z + i\Delta && (15) \end{aligned}$$

for some integer i. The interference is thus concentrated on a discrete and uniform grid with step size Δ, i.e., on the one-dimensional lattice $\Lambda = \Delta \cdot \mathbb{Z}$. By restricting the information-bearing signal V to an interval of size Δ, it can be reconstructed from Y as if the channel was *interference-free*. Since only half the power is used for carrying the information, the achievable rate at high SNR is given by

$$R \approx \frac{1}{2} \log_2 \left(\frac{P}{2N} \right) - \frac{1}{2} \log \left(\frac{2\pi e}{12} \right) \tag{16}$$

where the second term is the loss of the shaping gain due to the channel input being uniform rather than Gaussian.

Fig. 5 describes the interference concentration technique. The center of each cell is denoted by $Q(\cdot)$, where this time they are located on a *uniform grid* Λ. The symbols $\Box, \circ, \times, \Diamond, \triangle$ represent modulation of $Q(S)$ by different values of V.[5]

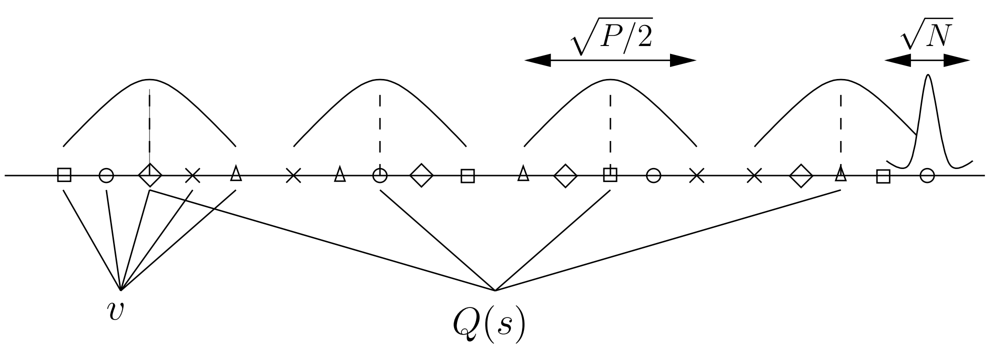


Fig. 5: Single-user: geometric view of interference concentration scheme.

[5]The modulation signal V can in general depend on $Q(S)$, although for ease of exposition it is not shown in (13).

Lattice Strategies: In [16], [7] the idea of lattice strategies was presented. It was shown how the transmitted power can be exploited such that all the power goes effectively to the information signal. Specifically, using the same notation as in (13), the transmitted signal is

$$X = [V - S] \bmod \Delta, \tag{17}$$

where $\Delta = \sqrt{12P}$. Since V is distributed uniformly over Δ, the transmitted signal uses the full power P. In this case, the channel output is given by

$$Y = V - Q(V - S) + Z \tag{18}$$

$$= V + Z + i\Delta \tag{19}$$

for some integer i. Again, the residual interference is concentrated on the discrete set of values Λ, and it can be completely eliminated if we restrict V to an interval of size Δ. Furthermore, it was shown in [7] that using high-dimensional lattice vector quantizers, and a suitable choice of V, the full (non-causal) dirty-paper channel capacity - $\frac{1}{2}\log\left(1+\frac{P}{N}\right)$ - is achieved.

Fig. 6 illustrates the lattice strategies technique. The center of each cell are again located on a uniform grid Λ, as in interference concentration. Each of the information-bearing symbols $\square, \diamond, \circ.\triangle$, however, corresponds now to a *shift* $\Lambda - V$ of the uniform grid:

$$Q_V(S) = Q(V + S) - V \tag{20}$$

for some fixed value $V = v$, and it can be decoded from (Y modulo Δ) - the channel output modulo the grid step size. Thus, lattice strategies amount to a *structured* form of the random binning technique discussed earlier: each bin is a *lattice shift*, and all bins are shifts of the *same* lattice.

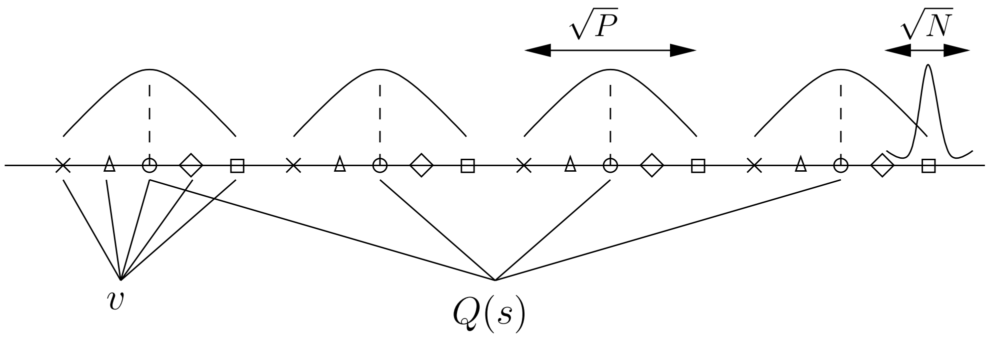


Fig. 6: Single-user: geometric view of lattice strategies.

B. MAC with a Single Dirty User ($S_2 = 0$)

Taking the Costa strategy for user 1 (the informed user), the auxiliary random variable U_1 is given by $U_1 = X_1 + S_1$, where $X_1 \sim \mathcal{N}(0, P_1)$ is independent of S_1. For user 2 (uninformed user), the natural choice is $U_2 =$

$X_2 \sim \mathcal{N}(0, P_2)$, independent of X_1 and S_1. Substituting in (6), and noting that $Y = U_1 + X_2 + Z$, we get that the sum rate is given by

$$
\begin{aligned}
R_1 + R_2 &= I(U_1, X_2; Y) - I(U_1; S_1) \\
&= \big[h(Y) - h(Z)\big] - \big[h(U_1) - h(X_1)\big] \\
&\approx h(X_1) - h(Z) \\
&= \frac{1}{2}\log_2\left(\frac{P_1}{N}\right)
\end{aligned}
$$

where the approximation follows since $h(Y) \approx h(U_1) \approx h(S_1)$ for strong Gaussian interference ($Q_1 \to \infty$). The individual bounds in (6) imply also $R_1 \le \frac{1}{2}\log_2\left(\frac{P_1}{N}\right)$ and (for high SNR) $R_2 \le \frac{1}{2}\log_2\left(\frac{P_2}{N}\right)$. Hence, if user 1 (the informed user) serves as a helper, then

$$
R_2 \approx \frac{1}{2}\log_2\left(\frac{\min(P_1, P_2)}{N}\right)
$$

is achievable at high SNR.

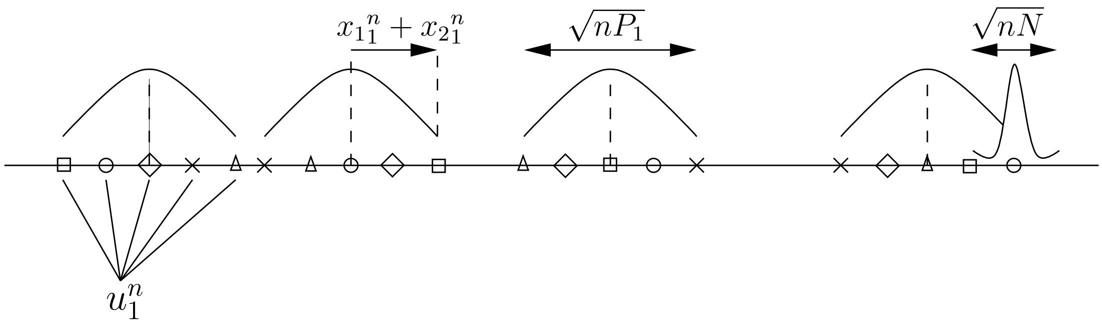


Fig. 7: Two Users with a Singe Dirt ($S2 = 0$): geometric view of random binning.

Random binning for the MAC with single dirt can be thought of as a superposition of clean-paper transmission, X_2, over dirty-paper transmission. The latter can be written as (setting $n = 1$ for ease of notation) $X_1 = Q_V(S_1) - S_1$, where $Q_V(\cdot)$ is a quantizer with "distortion" P_1 for S_1. See Fig. 7.

We can equivalently use lattice strategies instead of random binning, in which case a bin is a lattice shift $\Lambda - V$, as in (20). In the helper case V degenerates, and we have only one bin which is the uniform grid (or lattice) Λ. In this case, *lattice strategies reduce to interference concentration.*

C. Doubly Dirty MAC

In all problems we have seen so far, capacity can be achieved using either the random binning technique [1] or lattice-strategies [7]. In the doubly-dirty MAC, however, lattice structure is essential to achieve or approach capacity.

Consider the doubly-dirty MAC (8), where $S_1 \sim \mathcal{N}(0, Q_1)$ and $S_2 \sim \mathcal{N}(0, Q_2)$ are independent. We shall first show that Costa's strategy (4) is not efficient in the limit of strong interference and high SNR. We substitute

$$\begin{aligned} U_1 &= X_1 + S_1 \\ U_2 &= X_2 + S_2 \end{aligned} \tag{21}$$

in Jafar's inner bound (6), where $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 \sim \mathcal{N}(0, P_2)$ are independent.

Proposition 1 (Costa's strategies in Jafar's inner bound): The sum-rate of (6) for the auxiliary random variables (21) is bounded from above by

$$R_1 + R_2 \leq \big[h(S_1 + S_2) - h(S_1) - h(S_2) + \Gamma + o(1)\big]^+ \underset{Q_1, Q_2 \to \infty}{\longrightarrow} 0 \tag{22}$$

where $\Gamma \triangleq \frac{1}{2}\log_2(2\pi e \frac{P_1 P_2}{N})$, and $o(1) \to 0$ as $Q_1, Q_2 \to \infty$.

Proof: From (6) we get that

$$\begin{aligned} R_1 + R_2 &= \big[I(U_1, U_2; Y) - I(U_1; S_1) - I(U_2; S_2)\big]^+ && (23)\\ &= \big[h(Y) - h(Y|U_1, U_2) - h(U_1) + h(U_1|S_1) - h(U_2) + h(U_2|S_2)\big]^+ && (24)\\ &= \big[h(Y) - h(Z) - h(U_1) - h(U_2) + h(X_1) + h(X_2)\big]^+ && (25)\\ &\leq \big[h(Y) - h(S_1) - h(S_2) + h(X_1) + h(X_2) - h(Z)\big]^+ && (26)\\ &= \big[h(Y) - h(S_1) - h(S_2) + \Gamma\big]^+ && (27)\\ &\leq \big[h(S_1 + S_2) - h(S_1) - h(S_2) + \Gamma + o(1)\big]^+ && (28) \end{aligned}$$

where (25) follows since $Y = U_1 + U_2 + Z$ and since $h(U_i|S_i) = h(X_i)$ for $i = 1, 2$; (26) follows since $h(S_i) \geq h(U_i)$; (27) follows from the definition of the constant $\Gamma \triangleq \frac{1}{2}\log_2(2\pi e \frac{P_1 P_2}{N})$; (28) follows since $h(Y) \leq h(S_1 + S_2) + o(1)$ as $Q_1, Q_2 \to \infty$. The lema follows since $h(S_1 + S_2) - h(S_1) - h(S_2) \to -\infty$ as $Q_1, Q_2 \to \infty$. □

Thus, the random binning scheme corresponding to this choice of U_1 and U_2 *does not achieve any positive rate*. To understand this failure, observe that the channel output (12) can be written as (setting again $n = 1$ for simplicity of notation)

$$Y = Q_{V_1}(S_1) + Q_{V_2}(S_2) + Z$$

where $Q_{V_1}(S_1) = U_1$ and $Q_{V_2}(S_2) = U_2$. If the bins $Q_{V_1}(\cdot)$ and $Q_{V_2}(\cdot)$ have no structure, and if they are spread over a large region in the interference domain (since the interference is strong), then the range of their set sum $Q_{V_1}(\cdot) + Q_{V_2}(\cdot)$ tends to be dense. See Fig. 8 for the bin labeled by $\circ$. Fig. 9 further illustrates the effect of increasing the size of bin $\circ$. Thus the immunity to noise is lost, and the bins cannot be decoded from the channel output.

To overcome this failure, we would like to have the property that

$$Q_{V_1}(S_1) + Q_{V_2}(S_2) = Q_V(S_1 + S_2)$$

for some V, i.e., that the order of quantization and summation can be exchanged. In other words, we require the range of $Q_{V_1}(\cdot)$ and $Q_{V_2}(\cdot)$ to be a lattice - a set which is closed under addition.

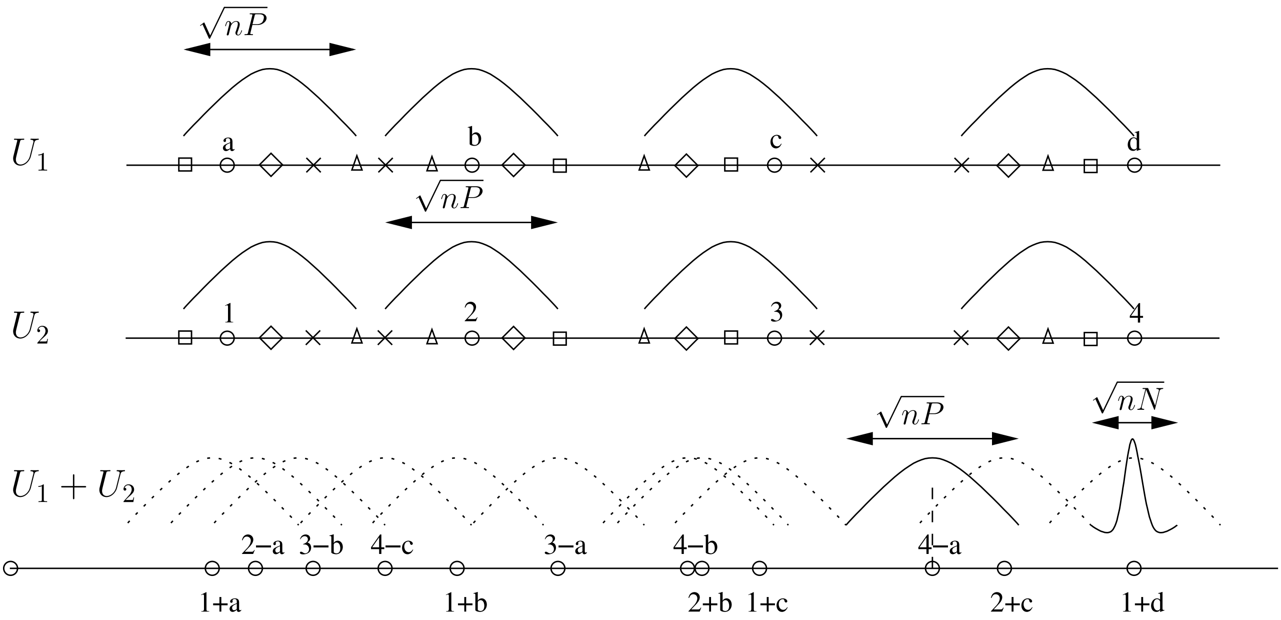


Fig. 8: Doubly dirty MAC: bottom axis shows the reflection of bin $\circ$ of the two users on the decoder.

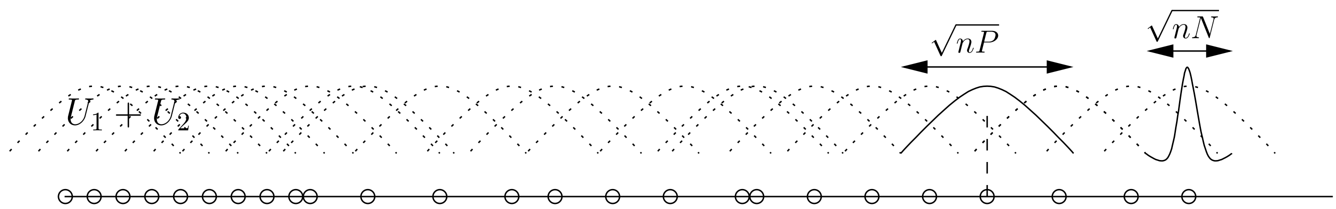


Fig. 9: Doubly dirty MAC: geometric view of random binning - increasing the range of bin $\circ$.

With this motivation in mind, consider now lattice strategies for the doubly-dirty MAC where only user 1 carries information and user 2 serves as a helper. Both transmitters use the *same lattice* where $P_1 = P_2 = P$. The encoders send

$$X_1 = [V_1 - S_1 \bmod \Delta] \tag{29}$$

$$X_2 = [-S_2 \bmod \Delta], \tag{30}$$

where $\Delta = \sqrt{12P}$. Thus, user 2 (helper) performs interference concentration (with respect to its known interference) while user 1 uses lattice strategies. In this case, the channel output is given by

$$Y = V_1 - Q(V_1 - S_1) - Q(-S_2) + Z = V_1 + Z + i\Delta \tag{31}$$

for some integer i. since the sum of two uniform grids is a uniform grid, the residual interference is concentrated *and aligned* on the same set of discrete values (Λ) as shown in Fig. 10.

As in the point-to-point case, if V is restricted to an interval of size Δ then the interference is completely eliminated; if we use high dimensional lattices instead of a scalar lattice, then a rate of

$$\frac{1}{2}\log_2\left(\frac{P}{N}\right) \tag{32}$$

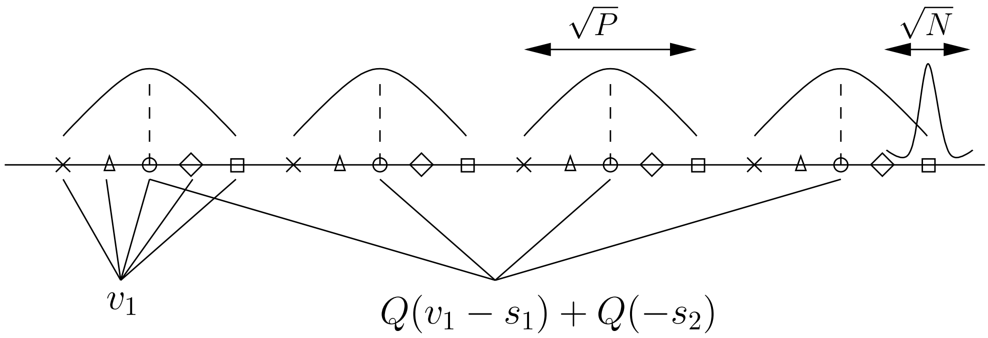


Fig. 10: Doubly-dirty MAC: geometric view of lattice strategies.

can be achieved. Note that this is almost the full capacity of the *clean* MAC: we loose only the (non-coherent) summation of the powers of the two transmitters.

Can random binning approach this rate? Indeed, if we substitute in Jafar's inner bound (6) the auxiliary random variables

$$U_i = [X_i + S_i] \bmod \Delta_i, \quad i = 1, 2 \tag{33}$$

where $\Delta_i = \sqrt{12P_i}$, then we obtain the rates corresponding to *one-dimensional* lattice strategies. This amounts to the capacity in (32), up to a loss of *shaping gain* [8]:

$$\frac{1}{2}\log_2\left(\frac{2\pi e}{12}\right).$$

We conjecture that this loss of the single-letter expression (6) is unavoidable at high SNR.

To summarize, we have seen that structured (linear) coding plays a key role in the doubly-dirty MAC channel. A formal derivation based on multi-dimensional lattices (some background on which is given in Section V) is carried out in Section VI. In Section VI we also extend the analysis to general SNR and discuss the (conjectured) loss of the "1" in the capacity expression, which was mentioned in the Introduction.

We have also seen that the capacity of the doubly-dirty MAC channel (as well as that of the uninformed user in the MAC with a single dirty user) is governed by the power of the weaker of the users. This observation is substantiated in the next Section where it is proved that the "power loss" is unavoidable.

IV. Outer Bounds for the Dirty MAC

We establish an outer bound for the capacity region of the Gaussian MAC with a single dirty user (10), and then this result is used to obtain an outer bound for the doubly-dirty MAC (8).

Theorem 1 (Outer bound for single dirty user[6]): In the limit of strong interference, the capacity region of the MAC with a single dirty user (user 1) (10) is contained in the following region:

$$\begin{aligned} R_2 &\leq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right) \\ R_1 + R_2 &\leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right) \end{aligned} \tag{34}$$

The outer bound in Theorem 1 indicates that in the limit of strong interference, the sum-rate of Gaussian MAC with a single dirty user is limited by the power of the informed user P_1, where in the clean MAC the optimal scheme gains the sum of the users powers, that is $P_1 + P_2$. In the sequel we show that in the limit of strong interference the Gaussian doubly-dirty MAC is limited by $\min(P_1, P_2)$.

For the case of Gaussian interference, the outer bound (34) can also be derived by taking the limit ($Q_1 \rightarrow \infty$) of the common message capacity in [4]. To keep the paper self-contained, we provide below a direct proof of the outer bound, based on Lemma 1 below.

Consider the MAC with a single dirty user (10), with Gaussian interference S_1 with finite variance, i.e $S_1 \sim \mathcal{N}(0, Q_1)$. For this case, an outer bound for the capacity region is given in the following lemma.

Lemma 1 (Outer bound for the single dirty user with Gaussian interference): For finite Gaussian interference $S_1 \sim \mathcal{N}(0, Q_1)$, the capacity region of the MAC with a single dirty user (user 1) (10) is contained in the following region:

$$\begin{aligned} R_2 &\leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right) \\ R_1 + R_2 &\leq \frac{1}{2}\log_2\left(\frac{(N + (\sqrt{P_1} + \sqrt{P_2} + \sqrt{Q_1})^2)}{Q_1} \cdot \frac{(P_1 + N)}{N}\right) \end{aligned} \tag{35}$$

Proof: The proof is given in Appendix I. □

We note that the outer bound still holds if we let encoder 1 and the decoder share common randomness (dither). Clearly, the outer bound (35) for the *individual rate* of user 2 can not be exceeded by applying common randomness. Additionally, since common randomness does not result in a greater capacity for *fixed probabilistic* channels with SI at the transmitter [17], also the outer bound for the *sum-rate* can not be exceeded by using common randomness.

Assume that the interference has an infinite variance, i.e., $Q_1 \rightarrow \infty$. We have that $\frac{1}{2}\log_2(N + (\sqrt{P_1} + \sqrt{P_2} + \sqrt{Q_1})^2) \leq \left(\frac{1}{2}\log_2 Q_1 + o(1)\right)$ where $o(1) \rightarrow 0$ as $Q_1 \rightarrow \infty$ for fixed P_1, P_2. Hence, in this case the outer bound for the sum-rate (35) becomes

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right) + o(1).$$

As a consequence, the individual rate for user 2 is bounded from above by $R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right) + o(1)$.

The outer bound of Theorem 1 now follows since the capacity region for an arbitrary interference cannot be greater than the capacity region with Gaussian interference of unbounded variance. This is because arbitrary interference contains, as a special case, the set of typical sequences of Gaussian interference (of any variance).

The outer bound is depicted in Fig. 11 and in Fig. 12 for $P_1 \leq P_2$ and for $P_1 > P_2$, respectively, where $C(x) \triangleq \frac{1}{2} \cdot \log_2(1 + x)$. In Fig. 12, the corner point (R_1^c, R_2^c) is given by

$$\begin{aligned} R_1^c &= \frac{1}{2}\log_2\left(\frac{P_1 + N}{P_2 + N}\right) \\ R_2^c &= \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right). \end{aligned} \tag{36}$$

The outer bound in Theorem 1 is specialized to the *helper* problem in the following corollary.

Corollary 1 (Outer bound for the helper problem): If only user 2 (the uninformed user) sends a message (i.e., $R_1 = 0$) in the single dirty user model (10), then for strong interference, an upper bound for the rate R_2 is given

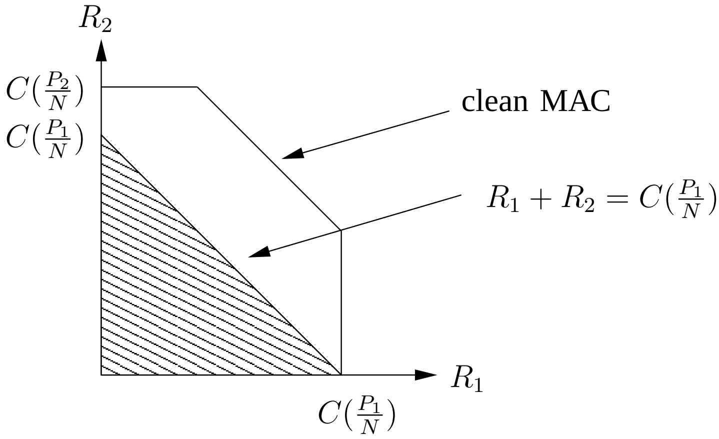


Fig. 11: Outer bound for MAC with a single dirty user (user 1) for $P_1 \leq P_2$.

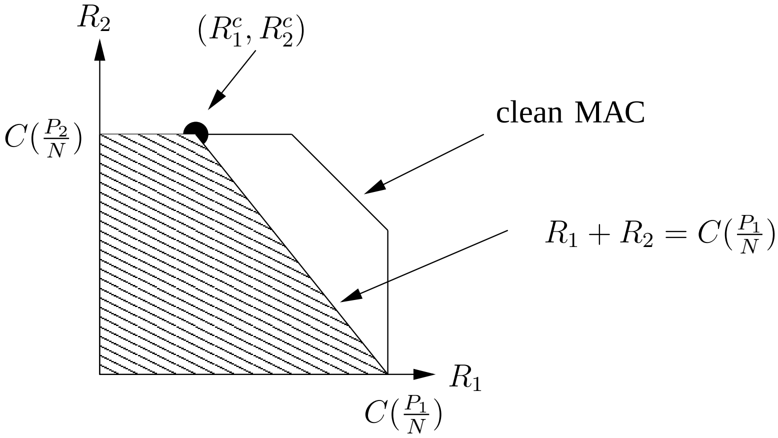


Fig. 12: Outer bound for MAC with a single dirty user (user 1) for $P_1 > P_2$.

by

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right). \tag{37}$$

The outer bound (34) for the single dirty user case is also an outer bound for the doubly-dirty MAC, provided that S_1 and S_2 are strong interferences. Clearly, the intersection of the outer bounds for a MAC with a single interference S_1 known to user 1 (34), and a MAC with a single interference S_2 known to user 2 (where P_1 and P_2 switch roles in (34)) gives the following tighter outer bound for the doubly-dirty MAC.

Corollary 2 (Outer bound for the doubly-dirty MAC): For strong interferences, the capacity region of the doubly-dirty MAC (8) with S_1 and S_2 independent is contained in the following region:

$$R_1 + R_2 \quad \leq \quad \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right). \tag{38}$$

From Theorem 1, the outer bound for the doubly dirty MAC holds also for the case that encoder 1, encoder 2 and the decoder share a dither signal. In Figure 13, the outer bound for the doubly-dirty MAC region is plotted.

Gel'fand and Pinsker in [3] showed that in the noiseless case ($N = 0$), arbitrarily large rate pairs (R_1, R_2) are achievable. For the general case ($N > 0$) and independent Gaussian interferences, they conjectured that the capacity

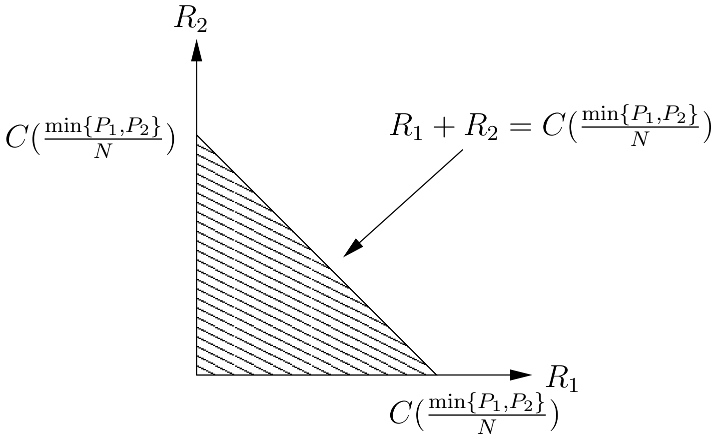


Fig. 13: Outer bound for the doubly-dirty MAC in Fig. 1.

region is the same as that of the MAC with no interference (clean MAC). The outer bound for the doubly-dirty MAC (38) as shown in Figure 13 disproves their conjecture. The sum capacity of the clean MAC is given by $\frac{1}{2}\log_2(1+\frac{P_1+P_2}{N})$. For the case that $P_1 = P_2$ the loss of the doubly-dirty MAC is at least 3 dB with respect to the clean MAC.

V. Lattice Alignment

A. Preliminary: Lattices

An n-dimensional lattice Λ is a discrete group in the Euclidian space $\mathbb{R}^n$ which is closed with the respect to the addition and reflection operations (over $\mathbb{R}$). The lattice may be specified by

$$\Lambda = \{\lambda = G \cdot \mathbf{i} \ : \ \mathbf{i} \in \mathbb{Z}^n\}, \tag{39}$$

where G is an $n \times n$ real valued matrix called the lattice generator matrix. A coset of the lattice is any translation of the original lattice $\mathbf{a} + \Lambda$ where $\mathbf{a} \in \mathbb{R}^n$.

The nearest neighbor quantizer $Q_\Lambda(\cdot)$ associated with Λ is defined by

$$Q_\Lambda(\mathbf{x}) = \lambda \in \Lambda \quad \text{if } ||\mathbf{x} - \lambda|| \leq ||\mathbf{x} - \lambda'||, \quad \forall \lambda' \in \Lambda, \tag{40}$$

where $||\cdot||$ denotes Euclidian norm. The Voronoi region of a lattice point λ is the set of all points in $\mathbb{R}^n$ that are closer (in Euclidian distance) to λ than to any other lattice point. Specifically, the fundamental Voronoi region is defined as the set of all points that are closest to the origin

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^n : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}, \tag{41}$$

where ties are broken arbitrarily. The modulo lattice operation with respect to Λ is defined as

$$\mathbf{x} \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}). \tag{42}$$

The modulo lattice operation satisfies the following distributive property

$$[\mathbf{x} \bmod \Lambda + \mathbf{y}] \bmod \Lambda = [\mathbf{x} + \mathbf{y}] \bmod \Lambda. \quad (43)$$

The second moment of a lattice Λ is given by

$$\sigma_\Lambda^2 = \frac{\frac{1}{n}\int_{\mathcal{V}_0} ||\mathbf{x}||^2 \mathbf{dx}}{V}, \quad (44)$$

where V is the volume of the fundamental Voronoi region, i.e., $V = \int_{\mathcal{V}_0} \mathbf{dx}$ (the same for all Voronoi regions of Λ). The normalized second moment is given by

$$G(\Lambda) = \frac{\sigma_\Lambda^2}{V^{2/n}}. \quad (45)$$

The normalized second moment is always greater than $1/2\pi e$. It is known [18] that for sufficiently large dimension there exist lattices that are good for quantization (these lattices are also known as good lattices for shaping [19]), in the sense that for any $\epsilon > 0$

$$\log_2(2\pi e G(\Lambda)) < \epsilon, \quad (46)$$

for large enough n. In addition, there exist lattices with second moment P that are *good for AWGN channel coding*, satisfying [19]

$$\Pr(\mathbf{X} \notin \mathcal{V}) < \epsilon, \text{ where } \mathbf{X} \sim \mathcal{N}(\mathbf{0}, (\mathrm{P}-\epsilon)\mathrm{I_n}), \ \forall \epsilon > 0, \quad (47)$$

where I_n is an $n \times n$ identity matrix.

The differential entropy of an n-dimensional random vector $\mathbf{D}$ which is distributed uniformly over the fundamental Voronoi cell, i.e., $\mathbf{D} \sim \mathrm{Unif}(\mathcal{V})$ is given by [18]

$$\begin{aligned} h(\mathbf{D}) &= \log_2(V) \\ &= \log_2\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right)^{n/2} \\ &= \frac{n}{2}\log_2\left(\frac{\sigma_\Lambda^2}{G(\Lambda)}\right) \\ &\approx \frac{n}{2}\log_2\left(2\pi e \sigma_\Lambda^2\right), \end{aligned}$$

where the last (approximate) equality holds for lattices that are good for quantization.

B. Lattice-Alignment Transmission Scheme

We present a general lattice-based transmission scheme which will be specialized to the Gaussian doubly-dirty MAC (in Section VI) and for the MAC with a single dirty user (in Section VII).

In the following transmission scheme, encoder 1 and encoder 2 use the lattices Λ_1 and Λ_2, with second moments P_1 and P_2, and fundamental Voronoi regions $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. We further require that the two lattices are *identical up to scaling*. That is,

$$\Lambda_1 = \kappa_1 \Lambda \tag{48}$$

$$\Lambda_2 = \kappa_2 \Lambda \tag{49}$$

for some real numbers κ_1 and κ_2 to be specified.

The encoders transmit the following signals as shown in Fig. 14:

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1] \text{ mod } \Lambda_1 \\ \mathbf{X}_2 &= [\mathbf{V}_2 - \alpha_2 \mathbf{S}_2 + \mathbf{D}_2] \text{ mod } \Lambda_2, \end{aligned} \tag{50}$$

where $\alpha_1, \alpha_2 \in [0,1]$; $\mathbf{V}_1 \in \text{Unif}(\mathcal{V}_1)$ and $\mathbf{V}_2 \in \text{Unif}(\mathcal{V}_2)$ are independent and carry the information of user 1 and user 2, respectively. The encoders use independent (pseudo-random) dither signals $\mathbf{D}_1 \sim \text{Unif}(\mathcal{V}_1)$ and $\mathbf{D}_2 \sim \text{Unif}(\mathcal{V}_2)$, where $\mathbf{D}_1$ is known to encoder 1 and to the decoder, and $\mathbf{D}_2$ is known to encoder 2 and to the decoder, as shown in Fig. 14. From the dithered quantization property [18],

$$\mathbf{X}_i \sim \text{Unif}(\mathcal{V}_\text{i}) \text{ for any } \mathbf{V}_\text{i} = \mathbf{v}_\text{i}, \text{ for i} = 1,2 \tag{51}$$

where $\mathbf{X}_i$ independent of $\mathbf{V}_i$, and hence the power constraints are satisfied.

The decoder uses a lattice $\Lambda_r = \kappa_r \Lambda$, which is another scaled version of Λ, and reduces modulo-Λ_r the term $\alpha_r \mathbf{Y} - \gamma \mathbf{D}_1 - \beta \mathbf{D}_2$, i.e.,

$$\mathbf{Y}' = [\alpha_r \mathbf{Y} - \gamma \mathbf{D}_1 - \beta \mathbf{D}_2] \text{ mod } \Lambda_r. \tag{52}$$

The scalars $\alpha_1, \alpha_2, \alpha_r, \kappa_1, \kappa_2, \kappa_r, \beta, \gamma$ and the basic lattice Λ will be determined in each scenario in the sequel.

The main advantage of the lattice-alignment transmission above is its robustness. Unlike in the random binning technique, the achievable rates of the lattice-alignment scheme are oblivious to the exact distributions of the interferences. Hence, this scheme remains applicable for arbitrary interference sequences.

In the above lattice-alignment transmission scheme, it is assumed that the information-bearing signals $\mathbf{V}_1, \mathbf{V}_2$ are uniformly distributed over the basic cell of the appropriate shaping lattice (also known as coarse lattice [20]). Of course, it is possible to use a nested lattice structure as in [20] where $\mathbf{V}_1, \mathbf{V}_2$ belong to fine lattices and the coarse lattices are nested in these fine lattices, i.e., we have a nested lattice chain with two nesting ratios.

VI. The Doubly-Dirty MAC

In this section we present lattice-alignment transmission scheme of Section V for the Gaussian doubly-dirty MAC (8). We derive conditions for optimality as well as (when the conditions do not apply) a uniform bound for the gap-to-capacity. The results formalize the presentation in Section III, as well as utilize multi-dimensional lattices and extend the scope to general SNR.

As discussed in Section III, while the capacity of the single-user dirty paper can be achieved both by using random binning [1] or using lattice-strategies [7], in the doubly-dirty MAC, random binning results in a strictly smaller achievable rate region with respect to that obtained using lattice strategies.

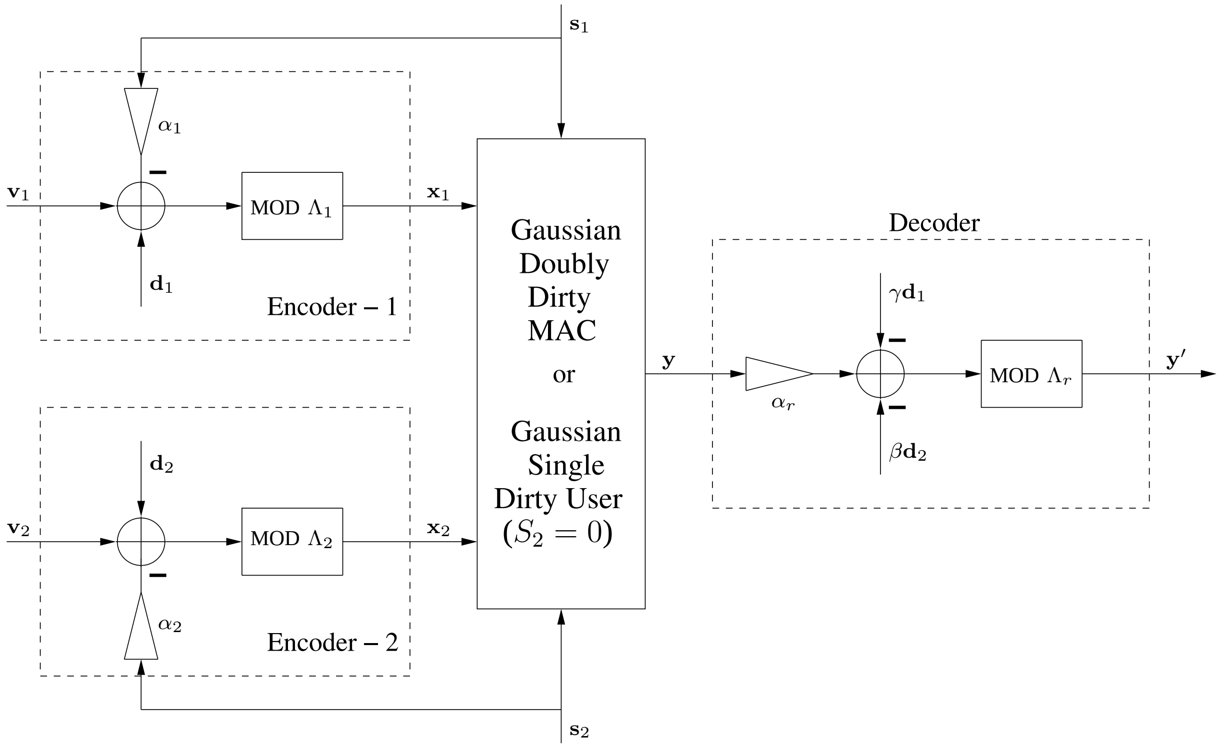


Fig. 14: Lattice-alignment transmission scheme.

It turns out that the gap in SNR between the two users plays a central role in the analysis. More specifically, when the SNR gap is large ("imbalanced case"), i.e., $\sqrt{\mathrm{SNR}_1\mathrm{SNR}_2} - \min(\mathrm{SNR}_1, \mathrm{SNR}_2) \geq 1$, the capacity region is fully determined. A natural extension is the high SNR regime where the capacity is also fully characterized. For the "nearly balanced" case, i.e., when $\sqrt{\mathrm{SNR}_1\mathrm{SNR}_2} - \min(\mathrm{SNR}_1, \mathrm{SNR}_2) < 1$, we obtain achievable regions using lattice-alignment transmission schemes, and derive a universal bound on the gap to capacity. In this case the lattice-alignment scheme looses the "1" in the capacity expression, due to the accumulation of two self noise components (rather than one self noise in the single-user dirty paper case [7]). This loss is avoided in the "imbalanced case" by *pre-inflating* the lattice of the user with the redundant power. We shall begin with the latter case.

A. Imbalanced Doubly-Dirty MAC

In the following theorem, we provide conditions under which lattice-strategies are optimal.

Theorem 2 (Imbalanced SNRs): Suppose that $N \leq \sqrt{P_1P_2} - \min(P_1, P_2)$ for $P_1 \neq P_2$. The capacity region of the doubly-dirty MAC (8) in the limit of strong interferences meets the outer bound of Corollary 2, and is given by the set of all rate pairs (R_1, R_2) satisfying

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right).$$

Proof: The converse part has been proved in Corollary 2. In this proof we show achievability for the case where user 1 is a helper for user 2, i.e., for the point

$$(R_1, R_2) = \left(0, \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right)\right), \tag{53}$$

where $N \leq \sqrt{P_1 P_2} - \min(P_1, P_2)$ and $P_1 \neq P_2$. We present here the achievability of (53) for the case where $P_2\left(\frac{P_2+N}{P_2}\right)^2 \leq P_1$. While the achievability of (53) for the case where $P_1\left(\frac{P_1+N}{N}\right)^2 \leq P_2$ is proved similarly and is given in Appendix II.

Clearly from the symmetric between P_1 and P_2 in (53) also the point

$$(R_1, R_2) = \left(\frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right), 0\right), \tag{54}$$

can be achieved. In view of the outer bound (38) in Corollary 2, the theorem follows by time sharing between (53) and (54).

In order to achieve (53) where $P_2\left(\frac{P_2+N}{P_2}\right)^2 \leq P_1$, we apply the lattice-alignment transmission scheme of Section V-B. Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda$ and $\Lambda_2 = \Lambda_r = \alpha_2\Lambda$ for some Λ (that is $\kappa_1 = 1$ and $\kappa_2 = \kappa_r = \alpha_2$). The second moments of the lattices Λ_1 and Λ_2 are $\sigma_1^2 = P_1$ and $\sigma_2^2 = \alpha_2^2 P_1$, respectively, where α_2 will be determined later. We set $\mathbf{V}_1 = \mathbf{0}$, $\alpha_1 = \beta = 1$ and $\alpha_r = \gamma = \alpha_2$, hence the encoders send

$$\mathbf{X}_1 = [-\mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \tag{55}$$

$$\mathbf{X}_2 = [\mathbf{V}_2 - \alpha_2\mathbf{S}_2 + \mathbf{D}_2] \bmod \Lambda_2, \tag{56}$$

where $\mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ carries the information of user 2; $\mathbf{D}_1$ and $\mathbf{D}_2$ are the dithers signal where $\mathbf{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ and $\mathbf{D}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$. User 1 mitigates the influence of the interference signal $\mathbf{S}_1$ by quantizing $\mathbf{S}_1$ with respect to the shifted lattice $\Lambda_1 + \mathbf{D}_1$. It is equivalent to using the *concentration* technique originally proposed by Willems [15].

The receiver calculates $\mathbf{Y}' = [\alpha_2(\mathbf{Y} - \mathbf{D}_1) - \mathbf{D}_2] \bmod \Lambda_2$. The equivalent channel from $\mathbf{V}_2$ to $\mathbf{Y}'$ is given by

$$\mathbf{Y}' = \Big[\alpha_2(\mathbf{X}_1 + \mathbf{S}_1 + \mathbf{X}_2 + \mathbf{S}_2 + \mathbf{Z} - \mathbf{D}_1) - \mathbf{D}_2\Big] \bmod \Lambda_2 \tag{57}$$

$$= \Big[\alpha_2[\mathbf{X}_2 + \mathbf{S}_2 + \mathbf{Z}] - \mathbf{D}_2 - \alpha_2 Q_{\Lambda_1}(-\mathbf{S}_1 + \mathbf{D}_1)\Big] \bmod \Lambda_2 \tag{58}$$

$$= \Big[\mathbf{V}_2 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z} - \alpha_2 Q_{\Lambda_1}(-\mathbf{S}_1 + \mathbf{D}_1)\Big] \bmod \Lambda_2, \tag{59}$$

where (58) follows from (55); (59) follows from (56).

Since $\Lambda_1 = \Lambda$ and $\Lambda_2 = \alpha_2\Lambda$ (scaled lattices), we have that $\alpha_2 Q_{\Lambda_1}(-\mathbf{S}_1 + \mathbf{D}_1) \in \Lambda_2$ i.e., the interference signal is aligned with Λ_2. Hence, the element $\alpha_2 Q_{\Lambda_1}(-\mathbf{S}_1 + \mathbf{D}_1)$ disappears after the modulo-Λ_2 operation. In this case, the equivalent channel is given by

$$\mathbf{Y}' = \Big[\mathbf{V}_2 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z}\Big] \bmod \Lambda_2. \tag{60}$$

From the dithered quantization property (51), $\mathbf{V}_2$ and $\mathbf{X}_2$ are independent. The term $(1-\alpha_2)\mathbf{X}_2$ is known as the *self noise* [7] which is due to user 2. The rate achieved by user 2 is given by

$$\begin{aligned} R_2 &= \frac{1}{n}I(\mathbf{V}_2;\mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_2)\right\} \\ &= \frac{1}{n}\left\{h(\mathbf{Y}') - h([(1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z}] \bmod \Lambda_2)\right\} \\ &\geq \frac{1}{2}\log_2\left(\frac{P_2}{G(\Lambda_2)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_2)^2 P_2 + \alpha_2^2 N\right)\right) \end{aligned}$$

where in the last inequality we used the fact that $\mathbf{V}_2$ is unform over $\mathcal{V}_2$ then $\mathbf{Y}'$ is also uniform over $\mathcal{V}_2$, and since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment.

For $P_1 = P_2\left(\frac{P_2+N}{P_2}\right)^2$, using the optimal MMSE factor for user 2, i.e., $\alpha_2 = \frac{P_2}{P_2+N}$, and for lattice that is good for quantization (46), i.e., $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, we get that any rate

$$R_2 \quad \leq \quad \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right), \tag{61}$$

is achievable. Clearly, for $P_1 = P_2\left(\frac{P_2+N}{P_2}\right)^2$ the inner bound meets the outer bound (38). Likewise, for $P_2(\frac{P_2+N}{P_2})^2 \leq P_1$, the outer bound (38) remains $\frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right)$, thus the outer bound is also achievable.

The proof is completed in Appendix II for the case that $P_1\left(\frac{P_1+N}{N}\right)^2 \leq P_2$.

□

In the above lattice-alignment scheme, the "strong user" (the user with higher power constraint) effectively uses $\alpha = 1$ (the scalar factor which multiplies the interference at the encoder (55)). Therefore, this user performs interference concentration which does not contribute an additional self noise term in (60). This technique can be viewed as *pre-inflated* lattice transmission by the strong user.

Furthermore, an additional property of the above scheme is that the users use the *same* lattice Λ (up to scaling), and therefore the residual interferences are aligned, and can in turn be eliminated. Hence, the lattice-base transmission simultaneously accomplishes *interference concentration* and *interference alignment*.

B. Nearly Balanced Doubly-Dirty MAC

We now derive an inner bound for the "nearly balanced" case, where $N > \sqrt{P_1 P_2} - \min(P_1, P_2)$. For simplicity, we first consider the symmetric ("exactly balanced") case, i.e., $P_1 = P_2 = P$ for any N.

Using the lattice-alignment transmission scheme of Section V-B with $\Lambda_1 = \Lambda_2 = \Lambda_r = \Lambda$ (that is $\kappa_1 = \kappa_2 = \kappa_r = 1$) where $\alpha_1 = \alpha_2 = \alpha_r = \alpha$ and $\beta = \gamma = 1$, the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \alpha\mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda \tag{62}$$

$$\mathbf{X}_2 = [\mathbf{V}_2 - \alpha\mathbf{S}_2 + \mathbf{D}_2] \bmod \Lambda, \tag{63}$$

where $\mathbf{V}_1, \mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V})$ are independent and carry the information of user 1 and user 2, respectively. Since $\mathbf{D}_1, \mathbf{D}_2 \sim \mathrm{Unif}(\mathcal{V})$ are independent dither signals, from the dither property $\mathbf{X}_1, \mathbf{X}_2 \sim \mathrm{Unif}(\mathcal{V})$, and hence the power constraints are satisfied. In this case, the decoder is given by

$$\mathbf{Y}' = [\alpha \mathbf{Y} - \mathbf{D}_1 - \mathbf{D}_2] \bmod \Lambda. \tag{64}$$

The equivalent mod $-\ \Lambda$ MAC is given in the following lemma.

Lemma 2 (The equivalent $\bmod \boldsymbol{\Lambda}$ *MAC):* The equivalent channel using the encoders (62) and (63) and the decoder (64) is given by

$$\mathbf{Y}' = \Big[\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{Z}_{eq}\Big] \bmod \Lambda, \tag{65}$$

where

$$\mathbf{Z}_{eq} = \Big[-(1-\alpha)\mathbf{X}_1 - (1-\alpha)\mathbf{X}_2 + \alpha \mathbf{Z}\Big] \bmod \Lambda, \tag{66}$$

and $\mathbf{Z}_{eq}$ is independent of $\mathbf{V_1}$ and $\mathbf{V_2}$, where $\mathbf{X}_1$, $\mathbf{X}_2$ are the self noises which are mutually independent, and independent of $\mathbf{Z}, \mathbf{V_1}, \mathbf{V_2}$

Proof: The equivalent channel is given by

$$\mathbf{Y}' = \Big[\alpha(\mathbf{X}_1 + \mathbf{S}_1 + \mathbf{X}_2 + \mathbf{S}_2 + \mathbf{Z}) - \mathbf{D}_1 - \mathbf{D}_2\Big] \bmod \Lambda \tag{67}$$

$$= \Big[\mathbf{V}_1 + \mathbf{V}_2 - (1-\alpha)\mathbf{X}_1 - (1-\alpha)\mathbf{X}_2 + \alpha\mathbf{Z}\Big] \bmod \Lambda, \tag{68}$$

where (67) follows since $Y = X_1 + S_1 + X_2 + S_2 + Z$; and (68) follows from (62) and (63). Due to the dithers, the vectors $\mathbf{V_1}$, $\mathbf{V_2}$, $\mathbf{X}_1$, $\mathbf{X}_2$ are independent, and also independent of $\mathbf{Z}$. Therefore, $\mathbf{Z}_{eq}$ is independent of $\mathbf{V_1}$ and $\mathbf{V_2}$. □

From the modulo-Λ equivalent channel (65) and (66), the achievable sum-rate is given by

$$R_1 + R_2 = \frac{1}{n} I(\mathbf{V}_1, \mathbf{V}_2; \mathbf{Y}') \tag{69}$$

$$= \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_1, \mathbf{V}_2)\right\} \tag{70}$$

$$= \frac{1}{n}\left\{h(\mathbf{Y}') - h([(1-\alpha)\mathbf{X}_1 + (1-\alpha)\mathbf{X}_2 + \alpha\mathbf{Z}] \bmod \Lambda)\right\} \tag{71}$$

$$\geq \left[\frac{1}{2}\log_2\left(\frac{P}{G(\Lambda)}\right) - \frac{1}{2}\log_2\left(2\pi e(\alpha^2 N + 2(1-\alpha)^2 P)\right)\right]^+ \tag{72}$$

$$= \left[\frac{1}{2}\log_2\left(\frac{P}{\alpha^2 N + 2(1-\alpha)^2 P}\right) - \frac{1}{2}\log_2\left(2\pi e G(\Lambda)\right)\right]^+ \tag{73}$$

where (72) follows since $\mathbf{Y}'$ has uniform distribution over $\mathcal{V}$, and since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment.

Like in the single-user case [7], the problem of finding the optimal α when the lattice dimension goes to infinity amounts to finding the value of α that minimizes the mean squared error of the effective noise term, i.e., of $-(1-\alpha)\mathbf{X}_1 - (1-\alpha)\mathbf{X}_2 + \alpha\mathbf{Z}$, hence

$$\alpha^{opt} = \alpha^{MMSE} = \frac{2P}{2P+N}, \tag{74}$$

For the optimal α and for a lattice that is good for quantization, i.e., for which $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, we get that any rate pair satisfying

$$R_1 + R_2 \le \left[\frac{1}{2}\log_2\left(\frac{1}{2} + \frac{P}{N}\right)\right]^+$$

is achievable, where $[x]^+ \triangleq \max(x, 0)$. Clearly, using a time sharing argument the following rates can be achieved

$$R_1 + R_2 \le u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{1}{2} + \frac{P}{N}\right)\right]^+\right\}, \tag{75}$$

where $u.c.e$ is the upper convex envelope with respect to $\frac{P}{N}$. Compared to the outer bound (38), the partial loss of the "1" inside the logarithmic function (instead of one) is due to the presence of *two* independent self noises $\mathbf{X}_1$ and $\mathbf{X}_2$ that we have in the equivalent channel model as shown in Lemma 2. Nonetheless, this technique is asymptotically optimal at high SNR, since $\log\left(\frac{1}{2} + \frac{P}{N}\right) \approx \log\left(\frac{P}{N}\right)$ as $\frac{P}{N} \to \infty$.

At low SNR, i.e., $\text{SNR} \le 1/2$ (-3dB), *pure* (infinite dimensional) lattice-strategies cannot achieve any positive rates as shown in Fig. 15. Hence, time sharing is required between the point $\text{SNR} = 0$ and SNR^*, which is a solution of the following equation

$$\frac{df(\text{SNR})}{d\text{SNR}} = \frac{f(\text{SNR})}{\text{SNR}},$$

where $f(x) = \frac{1}{2}\log_2(\frac{1}{2} + x)$. Numerical evaluation gives that $\text{SNR}^* \approx 1.655$. At low SNR, i.e., $\text{SNR} \to 0$ the inner bound is given by $R_1 + R_2 \simeq 0.425\frac{P}{N}$, while the outer bound is given by $R_1 + R_2 \approx 0.721\frac{P}{N}$, hence the gap between the outer bound and the inner bound is bounded by approximately 2.3 dB. In Fig. 15, we also evaluate numerically the achievable rates for one dimensional lattice strategies (the dashed curve), which is given in (71) where Λ is a scalar lattice with $G(\Lambda) = \frac{1}{12}$ using the optimal α for each SNR (which is not necessarily the MMSE factor). Like for the infinite dimensional case, time sharing also improves the achievable rates of pure one dimensional lattice strategies. Clearly, the achievable rates of infinite dimensional lattice strategies are strictly higher than one dimensional lattice strategies when applying time sharing as shown in Fig. 15.

We now return to consider the general "nearly-balanced" case, where $N > \sqrt{P_1 P_2} - \min(P_1, P_2)$ for general P_1, P_2.

Theorem 3 (Nearly-balanced SNRs): Suppose that $N \ge \sqrt{P_1 P_2} - \min(P_1, P_2)$. An achievable region for the doubly-dirty MAC (8) is given for any interferences by the set of rate pairs (R_1, R_2) satisfying

$$R_1 + R_2 \le u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{P_1 + P_2 + N}{2N + (\sqrt{P_1} - \sqrt{P_2})^2}\right)\right]^+\right\}, \tag{76}$$

where the upper convex envelope is with respect to P_1 and P_2.

Proof: The proof is given in Appendix III □

For the symmetric case, i.e., $P_1 = P_2$, the region becomes

$$R_1 + R_2 = u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{2P + N}{2N}\right)\right]^+\right\} = u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{1}{2} + \frac{P}{N}\right)\right]^+\right\}, \tag{77}$$

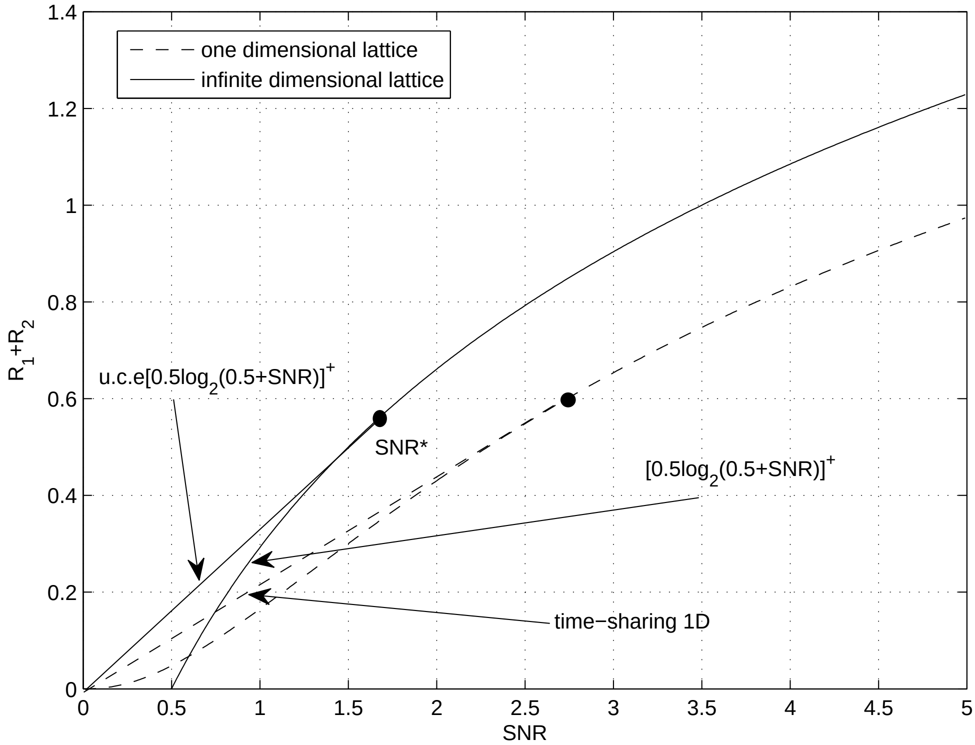


Fig. 15: Achievable sum-rate for $P_1 = P_2$.

which coincides with that in (75). For $N = \sqrt{P_1 P_2} - \min(P_1, P_2)$ the expression in (77) coincides with that in Theorem 2.

Unfortunately, as can be seen from Theorem 3, there is a gap between the inner bound and the outer bound for the "nearly balanced" case. We now derive a uniform bound on this gap. For $N > \sqrt{P_1 P_2} - \min(P_1, P_2)$, the gap between the outer bound (38) and the inner bound (77) is defined as

$$\zeta(P_1, P_2, N) \triangleq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right) - u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{P_1 + P_2 + N}{2N + (\sqrt{P_1} - \sqrt{P_2})^2}\right)\right]^+\right\}. \tag{78}$$

The following lemma provides a uniform upper bound for $\zeta(P_1, P_2, N)$.

Lemma 3: Let x^* be the solution of the equation $\frac{x}{x+1/2} = \log_e(x+1/2)$. For any P_1, P_2, N, the gap $\zeta(P_1, P_2, N)$ is bounded by

$$\zeta(P_1, P_2, N) \leq \frac{\log_2\left(\frac{1}{2} + x^*\right)}{4x^*} \approx 0.167 \text{ bit}, \tag{79}$$

where equality holds for $P_1 = P_2 = P$, and $\frac{P}{N} = x^* - 0.5 \approx 1.155$.

Proof: The proof is given in Appendix IV □

The solution x^* is evaluated numerically and it is equal to 1.655.

From the proof of Lemma 3, the gap is bounded by the symmetric case, i.e., $\zeta(P_1, P_2, N) \leq \zeta(P_{\min}, P_{\min}, N)$ where $P_{\min} = \min(P_1, P_2)$. In Fig. 16, the upper bound for the gap $\zeta(P, P, N)$ is depicted with respect to $\mathrm{SNR} = \frac{\mathrm{P}}{\mathrm{N}}$.

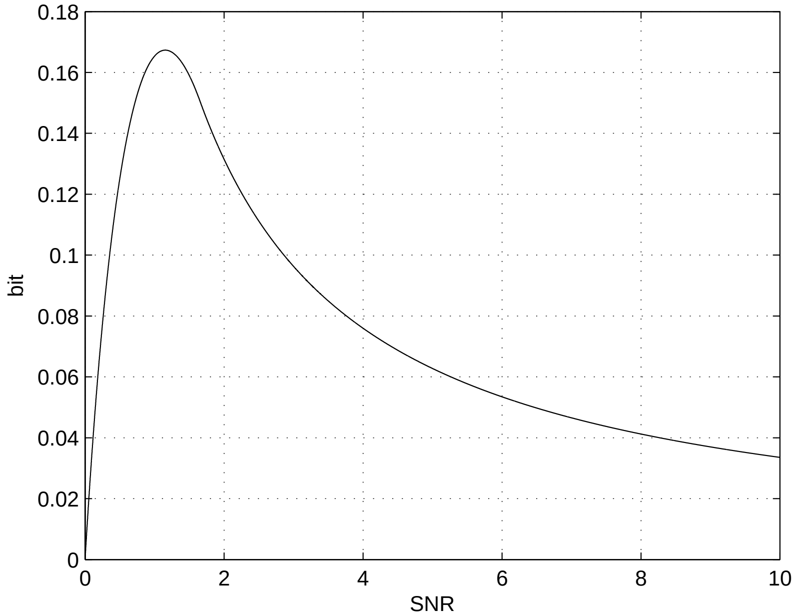


Fig. 16: Outer bound for $\zeta(P,P,N)$.

The above bound describes a uniform outer bound for the gap $\zeta(P_1,P_2,N)$ which is tight for the case that $P_1=P_2$. A tighter outer bound for the gap in the asymmetric case, i.e., $P_1\neq P_2$ can be derived [21]. Let us define

$$P_{\max}\triangleq \max(P_1,P_2) \tag{80}$$

$$P_{\min}\triangleq \min(P_1,P_2), \tag{81}$$

and $\mu^2\triangleq P_{\max}/P_{\min}$, hence $\mu\geq 1$. The bound find the worst gap for fixed power ratio μ, hence there is such a ratio that the bound is tight. The outer bound for the gap is shown in Fig. 17 with respect to μ^2. For $\mu=1$, i.e., $P_1=P_2$, the gap is equal to 0.167 bit. The following lemma is due to Mustafa Kesal.

*Lemma 4 (**Kesal** [21]):* For any P_1 and P_2, the gap $\zeta(P_1,P_2,N)$ is upper bounded by

$$\zeta(P_1,P_2,N)\leq C^*\log_2(e)-\frac{1}{2}\log_2(eC^*)-\frac{1}{2}, \tag{82}$$

where C^* is defined as follow:

$$C^*=\frac{d}{d\theta}f(\theta)|_{\theta=\theta^*} \tag{83}$$

$$\theta^*=\frac{f(\theta^*)}{C^*} \tag{84}$$

$$f(\theta)\triangleq\frac{1}{2}\log_e\left(\frac{(\mu^2+1)\theta+1}{(\mu-1)^2\theta+2}\right) \tag{85}$$

where $\theta\triangleq\frac{P_{\min}}{N}$. For any μ equality in (82) holds for $\theta=\frac{1}{2C^*}-1$.

Proof: The proof can be found in [22] □

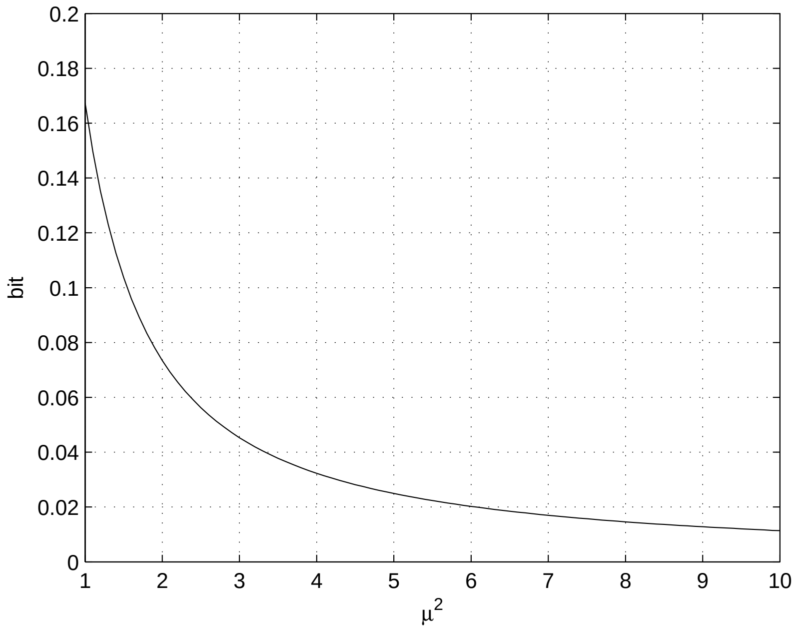


Fig. 17: Outer bound for the gap $\zeta(P_1, P_2, N)$.

C. Doubly-Dirty MAC at High SNR

We now observe that although there is a gap for the "nearly balanced" case between the inner and outer bounds, the gap vanishes at high SNR and hence the capacity region is completely determined in this limit. Indeed, for fixed P_1, P_2 which are not equal, if we take the noise power N to zero, we enter (eventually) the imbalanced regime. We next formally show that the outer bound is indeed tight at high SNR (even when $P_1 = P_2$) as a direct corollary to Lemma 2.

Corollary 3: At high SNR and in the limit of strong interferences, the capacity region of the doubly-dirty MAC (8), is given by the set of all rate pairs (R_1, R_2) satisfying

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right) - o(1), \tag{86}$$

where $o(1) \to 0$ as $\min(P_1, P_2) \to \infty$.

Proof: Using Lemma 2 with $\alpha = 1$ and taking Λ to be a lattice (that is good for quantization) with second moment equal to $\min(P_1, P_2)$, we get the equivalent channel

$$\mathbf{Y}' = \Big[\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{Z}\Big] \bmod \Lambda, \tag{87}$$

for which the sum rate $\frac{1}{2}\log_2\left(\frac{\min(P_1,P_2)}{N}\right)$ is achievable, and hence (86) holds at high SNR. □

VII. MAC WITH A SINGLE DIRTY USER

In this section, a lattice-based transmission scheme is presented for the Gaussian dirty MAC with a single dirty user (10), see Fig. 2. Clearly we could apply the scheme for the doubly-dirty case as presented in the previous section. However, we will see that some of the "loss of the 1", see (75), may be avoided in the present case, taking advantage of the presence of a *single* interference. This in turn translates to a single self-noise component (rather than two as in the doubly-dirty case).

For the case of Gaussian interference, the results obtained in this section coincide with previous works [4], [5], [23], which were based on random binning. We extend the results to arbitrary interference (rather than Gaussian with known variance).

The results in this section are derived using the lattice-alignment transmission scheme of Section V. However, here the requirement that Λ_1 and Λ_2 are equal up to scaling is not necessary. Furthermore, the informed user could use any code that is good for both quantization and channel coding, while the uniformed user could use any code that is good for channel coding (for instance, a Gaussian codebook).

As in the previous section, the tightness of the results depends on the gap between the SNRs, i.e., on how "balanced" the SNRs are. The precise conditions on the gap are different from the previous section. This difference is due to the non-presence of the second interference which reduces the constraints on the transmission scheme.

We now say that the SNR gap is large ("imbalanced case") when $|\mathrm{SNR}_1 - \mathrm{SNR}_2| \geq 1$, in which case the capacity region is fully determined. The "nearly balanced" case is now defined by $|\mathrm{SNR}_1 - \mathrm{SNR}_2| < 1$, for which we obtain achievable regions using lattice-alignment transmission schemes, and derive a universal bound on the gap to capacity which is *tighter* than the one obtained for the doubly-dirty MAC scenario above. We begin by treating the helper problem where only the uninformed user has a message to transmit, and then consider the full rate region.

A. The Helper Problem

We now consider the *helper problem*, where only user 2, the uninformed user, has a message to send and the informed user (user 1) helps user 2 to transmit at the highest possible rate, i.e., a rate pair of the form $(0, R_2)$ is considered. The upper bound for this case is given in corollary 1. In the following theorem, we present the capacity for the helper problem, for the "nearly-balanced" case where $N \leq |P_1 - P_2|$.

Theorem 4 (Imbalanced SNRs): Suppose that $N \leq |P_1 - P_2|$ in a MAC with a single dirty user (10). In the limit of strong interference, the capacity of the helper problem is given by

$$C_{\text{helper}}(P_1, P_2) = \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right). \tag{88}$$

Proof: The proof is given in Appendix VI. □

For $|P_1 - P_2| < N$, we derive the following inner bound.

Lemma 5 (Nearly-balanced SNRs): Suppose that $|P_1 - P_2| < N$. The capacity of the *helper problem* satisfies

$$C_{\text{helper}}(P_1, P_2, N) \geq u.c.e\left\{\frac{1}{2}\log_2\left(1 + \frac{4P_1P_2}{(P_2 - P_1 + N)^2 + 4P_1N}\right)\right\}, \tag{89}$$

where the upper convex envelope is with respect to P_1 and P_2. For $P_1 = P_2 = P$, this inner bound reduces to

$$C_{\text{helper}}(\mathrm{SNR}) \geq \mathrm{u.c.e}\left\{\frac{1}{2}\log_2\left(1 + \mathrm{SNR}\left(\frac{4\mathrm{SNR}}{4\mathrm{SNR}+1}\right)\right)\right\}, \tag{90}$$

where the upper convex envelope is with respect to $\mathrm{SNR} \triangleq \frac{\mathrm{P}}{\mathrm{N}}$.

Proof: The proof is given in Appendix VII. □

Although the function inside the upper convex envelope operation in (89) is non-negative, by examining its Hessian matrix [24] it can be shown that this function is not convex-∩ for any P_1 and P_2 (also in (90) the function inside the upper convex envelope operation is not convex-∩ for any SNR).

The above inner bound can be also expressed in terms of $\mathrm{SNR}_{\min} \triangleq \min(\mathrm{SNR}_1, \mathrm{SNR}_2)$ and $\Delta\mathrm{SNR} \triangleq |\mathrm{SNR}_1 - \mathrm{SNR}_2|$, in this case we have that

$$C_{\text{helper}}(\mathrm{SNR}_{\min}, \Delta\mathrm{SNR}) \geq \text{u.c.e}\left\{\frac{1}{2}\log_2\left(1 + \frac{4\mathrm{SNR}_{\min}(\mathrm{SNR}_{\min} + \Delta\mathrm{SNR})}{(\Delta\mathrm{SNR}+1)^2 + 4\mathrm{SNR}_{\min}}\right)\right\}. \tag{91}$$

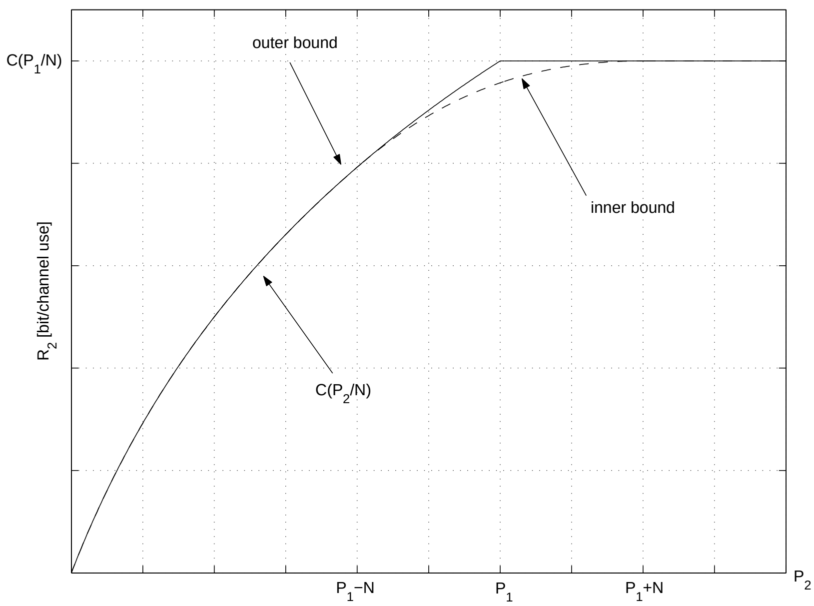


Fig. 18: Inner bound versus outer bound for the helper problem for $P_1 > N$.

In Fig. 18, the outer bound and the inner bound for the capacity of the helper problem are depicted for various values of P_1, P_2, N. As indicated in Lemma 5, there is a gap between the inner bound (89) and the outer bound (37) for $|P_1 - P_2| < N$. This gap is defined as

$$\eta(P_1, P_2, N) \triangleq \frac{1}{2}\log_2\left(1 + \frac{\min(P_1, P_2)}{N}\right) - u.c.e\left\{\frac{1}{2}\log_2\left(1 + \frac{4P_1P_2}{(P_2 - P_1 + N)^2 + 4P_1N}\right)\right\}. \tag{92}$$

In the following lemma a uniform upper bound for the gap $\zeta(P_1, P_2, N)$ is derived.

Lemma 6: For $|P_1 - P_2| < N$, the gap $\eta(P_1, P_2, N)$ (92) is upper bounded by

$$\eta(P_1, P_2, N) \leq \eta(P_{\min}, P_{\min}, N) < \log_2(3) - \frac{3}{2} \approx 0.085 \text{ bit},$$

where $P_{\min} = \min(P_1, P_2)$.

Proof: The proof is given in Appendix VIII. □

We now show that at high SNR, i.e., $P_1, P_2 \gg N$ and for $|P_1 - P_2| < N$, the achievable rate R_{helper} (89) meets asymptotically the outer bound (37).

Lemma 7: In the limit of strong interference, the capacity of the helper problem at high SNR is given by

$$C_{\text{helper}} = \frac{1}{2}\log_2\left(1+\frac{\min(P_1,P_2)}{N}\right) - o(1), \tag{93}$$

where $o(1) \to 0$ as $P_1, P_2 \to \infty$ for fixed N.

Proof: The lemma trivially follows by combining the outer bound given in Corollary 1 and noticing that (93) is achievable by Corollary 3. □

The *pure* lattice-strategies approach is not optimal at low SNR in the helper problem, i.e. the upper convex envelope strictly increases the achievable rate in the helper problem. In order to see that, consider the case of $P_1 = P_2 = P$. We now observe that time sharing can achieve higher rates than pure lattice-strategies transmission (the expression inside the upper convex envelope in (90)). Assume that the users coordinate their transmissions only for $1/\delta$ of the time ($\delta \geq 1$), while the rest of the time the users stay silent. During the transmission period ($1/\delta$), user 2 transmits with power δP, while user 1 transmits during half of the transmission period ($\frac{1}{2\delta}$), with power $\delta P - N$, and during the rest of the time, with $\delta P + N$. In this way, the users satisfy the power constraints. The achievable rate of user 2 is given by

$$\begin{aligned} R_2 &= \frac{1}{2\delta}\cdot\frac{1}{2}\log_2\left(1+\frac{\delta P}{N}\right) + \frac{1}{2\delta}\cdot\frac{1}{2}\log_2\left(1+\frac{\delta P - N}{N}\right) \\ &= \frac{1}{4\delta}\log_2\left(\delta\frac{P}{N}\left(1+\delta\frac{P}{N}\right)\right). \end{aligned}$$

Numerical evaluation shows that this expression is maximized for $\delta = 1.832\frac{N}{P}$, and the rate is given by $R_2 = 0.324\cdot\text{SNR}$, which is higher than achievable rate using pure lattice-strategies in (90) as shown in Fig. 19. However, this scheme is feasible only for $\text{SNR} \leq 1.832$ since $\delta \geq 1$.

For $\text{SNR} \to 0$, this inner bound behaves like $O(\text{SNR})$, while the inner bound in (90) behaves like $O(\text{SNR}^2)$. On the other hand, the outer bound (37) for $\text{SNR} \to 0$ is $\lim_{\text{SNR}\to 0}\frac{1}{2}\log_2(1+\text{SNR}) \approx 0.721\cdot\text{SNR}$ which behaves like $O(\text{SNR})$ as the inner bound.

B. Capacity Region at High SNR

While the capacity region for the MAC with a single dirty user (10) is not known in general, the following theorem determines the capacity region at high SNR, i.e., when $P_1, P_2 \gg N$.

Lemma 8: In the limit of strong interference, the capacity region of dirty MAC with a single dirty user (10) and high SNR, is given by

$$\begin{aligned} R_2 &\leq \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right) - o(1) \\ R_1 + R_2 &\leq \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right) - o(1), \end{aligned} \tag{94}$$

where $o(1) \to 0$ as $P_1, P_2 \to \infty$.

Proof: When $P_1 \leq P_2$, the lemma follows by combining the outer bound given in Theorem 1 and noticing that (94) is achievable by Corollary 3. The proof for the case $P_1 > P_2$ is given in Appendix IX. □

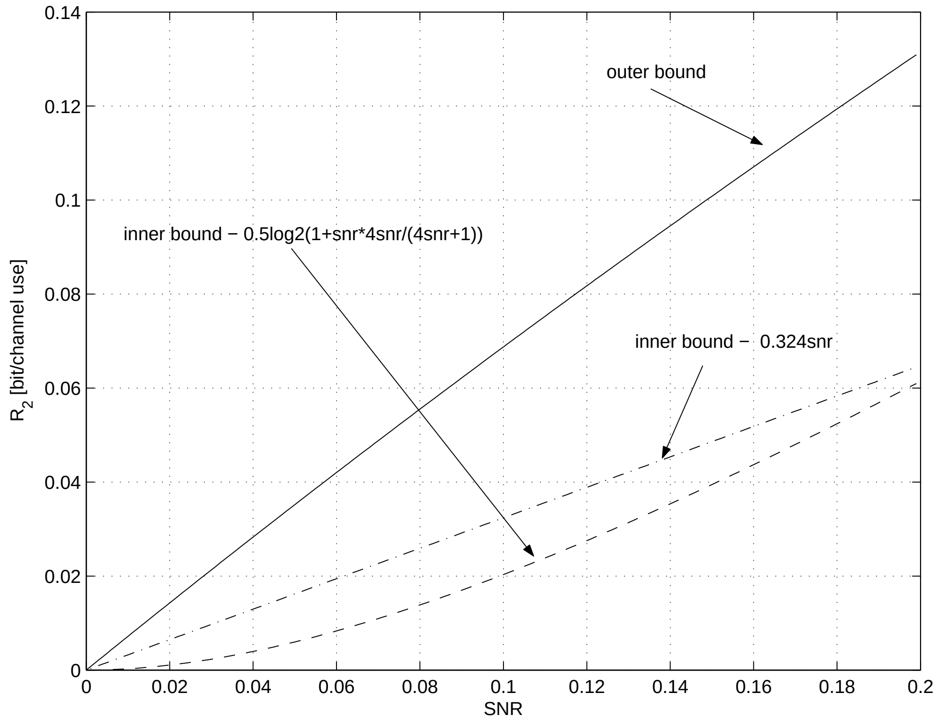


Fig. 19: Inner bounds and outer bound for helper problem at low SNR.

C. Achievable Rate Region

We now derive an achievable rate region using lattice-based transmission for any P_1, P_2, N. The same region was derived using random binning in [5].

Lemma 9: An achievable rate region for the MAC with a single dirty user (10) is given by

$$\mathcal{R} = \text{cl conv} \left\{ \bigcup_{\alpha_1 \in [0,1]} \mathcal{R}(\alpha_1) \right\}, \tag{95}$$

and

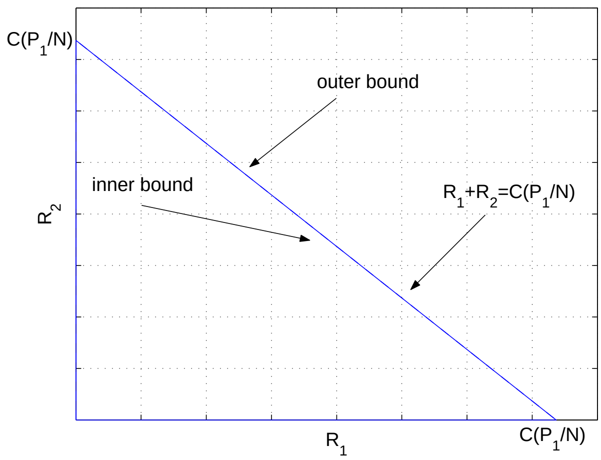
$$\mathcal{R}(\alpha_1) = \left\{ (R_1, R_2) : \quad \begin{aligned} R_1 &\leq \frac{1}{2} \log_2 \left(\frac{P1}{\min(P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2 (N + P_2))} \right) \\ R_2 &\leq \frac{1}{2} \log_2 \left(\frac{\min(P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2 (P_2 + N))}{(1-\alpha_1)^2 P_1 + \alpha_1^2 N} \right) \end{aligned} \right\} \tag{96}$$

where cl and conv are the closure and the convex hull operations, respectively.

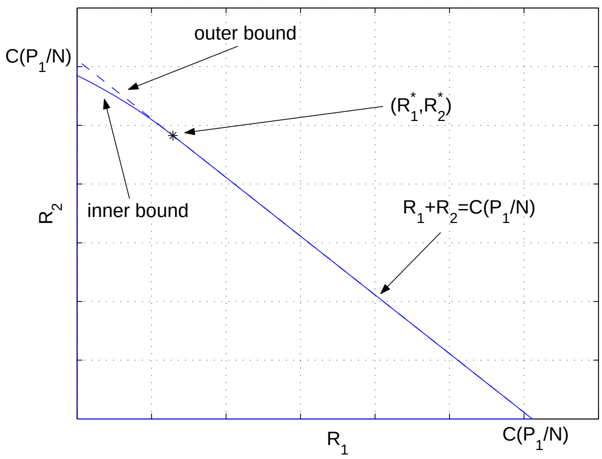
Proof: The proof is given in Appendix X □

This expression is a general form which describes the achievable rate region of the MAC with a single dirty user (10). It includes the achievable rate of the helper problem, i.e., the point $(0, R_2)$ for any P_1, P_2, N, and also the capacity region at high SNR.

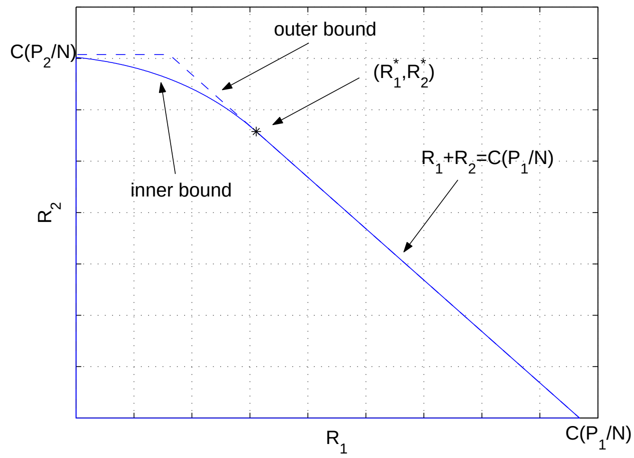
We now explore the behavior of the achievable rate region specified in Lemma 9 for several cases with respect to P_1, P_2, N:



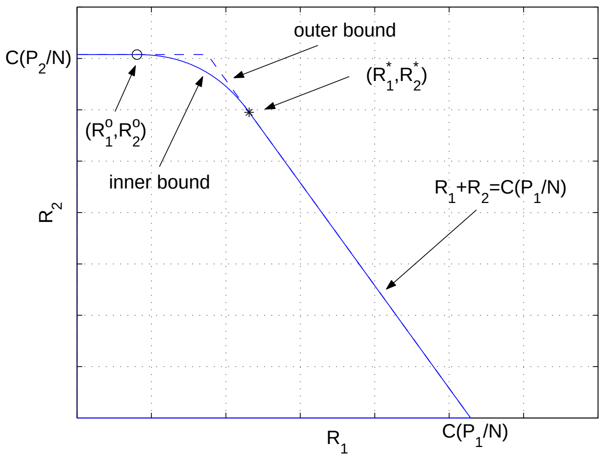
(a) $P_1 < P_2 - N$



(b) $P_2 - N < P_1 \leq P_2$



(c) $P_2 < P_1 \leq P_2 + N$



(d) $P_2 + N < P_1$

Fig. 20: Inner bound versus outer bound in the MAC with a single dirty user.

a) For $P_1 \leq P_2 - N$: It is easily verified that the point $(R_1 = \frac{1}{2}\log_2(1+P_1/N), 0)$ can be achieved when user 2 is silent, i.e., $X_2 = 0$ while user 1 performs point-to-point dirty-paper coding (DPC), which can be implemented using lattice-strategies precoding. Furthermore, in Theorem 4 it was shown that for $P_1 \leq P_2 - N$, user 2 can achieve the rate $R_2 = \frac{1}{2}\log_2(1 + P_1/N)$, and thus the point $(0, R_2 = \frac{1}{2}\log_2(1 + P_1/N))$ is also achievable. Therefore, time sharing between these two points achieves the outer bound (34) as shown in Fig. 20a.

Corollary 4: In the limit of strong interference, for $P_1 \leq P_2 - N$ the capacity region of the MAC with a single dirty user (10), is given by

$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right). \tag{97}$$

b) For $P_1 > P_2 - N$: This case refers to Fig. 20b-20d. We define the following rate pair

$$R_1^* \triangleq \frac{1}{2}\log_2\left(\frac{P_1+N}{N+\frac{P_1P_2}{P_1+N}}\right)$$
$$R_2^* \triangleq \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\cdot\frac{P_1}{P_1+N}\right).$$

This rate pair is located on the outer bound (34) as shown in Fig. 20b-20d. To see that, it can be verified that $R_1^*+R_2^* = \frac{1}{2}\log_2(1+P_1/N)$ and $R_2^* < \frac{1}{2}\log_2(1+\min(P_1,P_2)/N)$. On the other hand, using $\alpha_1 = \frac{P_1}{P_1+N}$ in (96) (Lemma 9), this rate pair can be achieved. Therefore, the rate pair (R_1^*, R_2^*) belongs to the boundary of the capacity region.

Corollary 5: In the limit of strong interference, and for $P_1 > P_2 - N$, the rate pair (R_1^*, R_2^*) belongs to the boundary of the capacity region in MAC with a single dirty user (10).

The rate pair (R_1^*, R_2^*) corresponds to the vertex point where the inner bound and the outer bound depart from each other as shown in Fig. 20b-20d. The behavior of the achievable region versus the outer bound is shown in Fig. 20b for $P_2 - N < P_1 \leq P_2$. In this case, the gap between the inner bound and the outer bound is maximal for the helper problem, i.e., the point $(0, R_2)$, which is bounded by $\log_2(3) - 3/2 \approx 0.085$ bit (Lemma 6). In Fig. 20c, the inner bound and the outer bound for $P_2 < P_1 \leq P_2 + N$ are depicted.

c) For $P_2 + N < P_1$: We define the following rate pair

$$R_1^o \triangleq \frac{1}{2}\log_2\left(\frac{P_1}{P_2+N}\right)$$
$$R_2^o \triangleq \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right).$$

Clearly, this rate pair is located on the boundary of the outer bound (34). On the other hand, using $\alpha_1 = 1$ in (96) (Lemma 9), this rate pair can be achieved, as shown in Fig. 20d. In fact, it is the maximal achievable rate that user 1 can transmit while user 2 transmits at its highest rate $R_2 = \frac{1}{2}\cdot\log_2(1+P_2/N)$.

Corollary 6: In the limit of strong interference, and for $P_2 + N < P_1$ the rate pair (R_1^o, R_2^o) belongs to the boundary of the capacity region in MAC with a single dirty user (10).

VIII. MAC WITH COMMON INTERFERENCE

In this section we consider the MAC with common interference (11). The state S_c is known non-causally to both users. The channel model is given by

$$Y = X_1 + X_2 + S_c + Z, \tag{98}$$

where $Z \sim \mathcal{N}(0, N)$. The power constraints are $\frac{1}{n}\sum_{i=1}^{n} x_{1i}^2 \leq P_i$ for $i = 1, 2$. In [3], it was shown that as in the point-to-point writing on dirty paper problem, the capacity region of the dirty MAC is the same as that of the interference-free Gaussian MAC (clean MAC), i.e, the capacity region is a pentagonal region [25]. This is unlike the MAC with a single dirty user problem (Section VII), where the capacity of the uninformed user is limited by the minimum power between the users.

The corner point $(R_1, R_2) = (\frac{1}{2} \cdot \log_2(1 + \frac{P_1}{P_2+N}), \frac{1}{2} \cdot \log_2(1 + P_2/N))$ of the pentagon is achieved by applying DPC twice for each user [6]. As in the point-to-point case, the auxiliary random variables are set to $U_1 = X_1 + \alpha_1 S_c$ where X_1 and S_1 are independent, and $U_2 = X_2 + \alpha_2 \tilde{S}_c$ where $\tilde{S}_c = (1 - \alpha_1)S_c$, and X_2 and S_2 are independent.

a) Writing on dirty paper for user 1 - the channel is given by

$$Y = X_1 + S_c + Z_{eq}, \tag{99}$$

where $Z_{eq} = X_2 + Z$, thus Z_{eq} is independent of X_1 and S_c. Using $\alpha_1 = \frac{P_1}{P_1+P_2+N}$, user 1 can achieve $R_1 = \frac{1}{2} \cdot \log_2(1 + \frac{P_1}{P_2+N})$.

b) Writing on dirty paper for user 2 - the equivalent channel is given by

$$Y' = Y - U_1 = X_2 + \tilde{S}_c + Z, \tag{100}$$

where $\tilde{S}_c = (1 - \alpha_1)S_c$. Using $\alpha_2 = \frac{P_2}{P_2+N}$ user 2 can achieve $R_2 = \frac{1}{2} \cdot \log_2(1 + P_2/N)$.

We now present how to achieve the capacity region of Gaussian MAC with common interference (98) using lattice-strategies. Specifically, we derive a transmission scheme for the corner point of the pentagon $(R_1, R_2) = (\frac{1}{2} \cdot \log_2(1 + \frac{P_1}{P_2+N}), \frac{1}{2} \cdot \log_2(1 + P_2/N))$ using lattice-strategies. User 1 and user 2 use the lattices Λ_1 and Λ_2 with second moments P_1 and P_2, respectively. Specifically, the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \alpha_1 \mathbf{S}_c + \mathbf{D}_1] \bmod \Lambda_1 \tag{101}$$

$$\mathbf{X}_2 = [\mathbf{V}_2 - \alpha_2 \tilde{\mathbf{S}}_c + \mathbf{D}_2] \bmod \Lambda_2, \tag{102}$$

where $\tilde{\mathbf{S}}_c = (1 - \alpha_1)\mathbf{S}_c$. The vectors $\mathbf{V}_i \sim U(\mathcal{V}_i)$ carries the information of user i for $i = 1, 2$. The dither signals $\mathbf{D}_1$ and $\mathbf{D}_2$ are independent, where $\mathbf{D}_1 \sim U(\mathcal{V}_1)$ is known at the encoder of user 1 and to the decoder, and $\mathbf{D}_2 \sim U(\mathcal{V}_2)$ is known at the encoder of user 2 and to the decoder as well. From the dither quantization property the power constraints are satisfied.

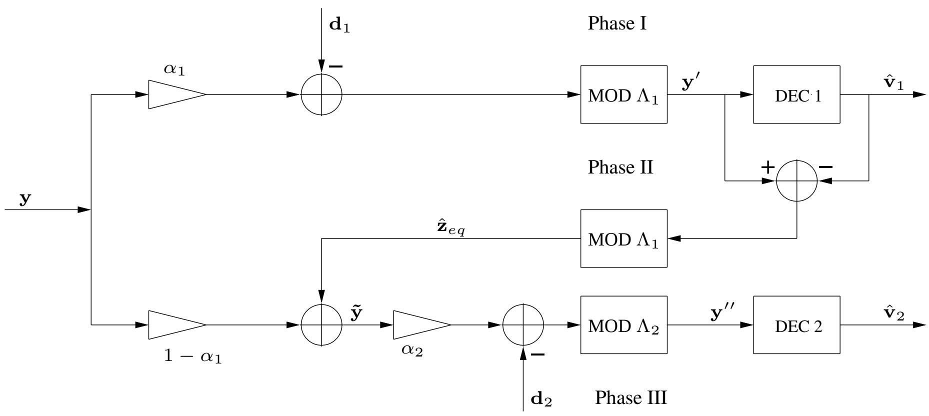


Fig. 21: Decoder for MAC with common interference.

The information-bearing signals, $\mathbf{V}_1$ and $\mathbf{V}_2$, are reconstructed using a three-stage decoder as shown in Fig. 21:

a) *Stage I*: The decoder calculates $\mathbf{Y}' = [\alpha_1 \mathbf{Y} - \mathbf{D}_1] \bmod \Lambda_1$. The equivalent channel is given by

$$\begin{aligned}\mathbf{Y}' &= \Big[\alpha_1(\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{S}_c + \mathbf{Z}) - \mathbf{D}_1\Big] \bmod \Lambda_1 \\ &= \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1(\mathbf{X}_2 + \mathbf{Z})\Big] \bmod \Lambda_1.\end{aligned}$$

From the dither quantization property, $\mathbf{V}_1$ and $\mathbf{X}_1$ are independent. The rate achieved by user 1 is given by

$$\begin{aligned}R_1 &= \frac{1}{n} I(\mathbf{V}_1; \mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_1))\right\} \\ &= \frac{1}{n}\left\{h(\mathbf{Y}') - h([(1-\alpha_1)\mathbf{X}_1 + \alpha_1(\mathbf{X}_2 + \mathbf{Z})] \bmod \Lambda_1)\right\} \\ &\geq \frac{1}{2}\log_2\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_1)^2 P_1 + \alpha_1^2(P_2+N)\right)\right).\end{aligned}$$

Using $\alpha_1 = \frac{P_1}{P_1+P_2+N}$ and lattices that are good for quantization, i.e., $G(\Lambda_1) \to 1/2\pi e$ as $n \to \infty$, any rate R_1 such that

$$R_1 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{P_2+N}\right) \tag{103}$$

is achievable. As a consequence, the decoder can reconstruct $\mathbf{V}_1$ with high probability.

b) *Stage II*: The decoder reconstructs the effective noise, i.e.,

$$\begin{aligned}\hat{\mathbf{Z}}_{eq} &= [\mathbf{Y}' - \hat{\mathbf{V}}_1] \bmod \Lambda_1 \\ &= \Big[-(1-\alpha_1)\mathbf{X}_1 + \alpha_1(\mathbf{X}_2 + \mathbf{Z})\Big] \bmod \Lambda_1.\end{aligned}$$

Furthermore, with high probability we have that $\hat{\mathbf{Z}}_{eq} = -(1-\alpha_1)\mathbf{X}_1 + \alpha_1(\mathbf{X}_2 + \mathbf{Z})$, since $\frac{1}{n}E\{|| -(1-\alpha_1)\mathbf{X}_1 + \alpha_1(\mathbf{X}_2 + \mathbf{Z})||^2\} = \frac{P_1(P_2+N)}{P_1+P_2+N} < P_1$.

The decoder now calculates $\mathbf{Y}_1 = \mathbf{Y} + \beta\hat{\mathbf{Z}}_{eq}$, thus

$$\begin{aligned}\mathbf{Y}_1 &= \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{S}_c + \mathbf{Z} - \beta(1-\alpha_1)\mathbf{X}_1 + \beta\alpha_1(\mathbf{X}_2 + \mathbf{Z}) \\ &= (1 - \beta(1-\alpha_1))\mathbf{X}_1 + (1+\beta\alpha_1)\mathbf{X}_2 + \mathbf{S}_c + \mathbf{Z}(1+\beta\alpha_1).\end{aligned}$$

For $\beta = \frac{1}{1-\alpha_1}$, we have that

$$\mathbf{Y}_1 = \frac{1}{1-\alpha_1}\mathbf{X}_2 + \mathbf{S}_c + \frac{1}{1-\alpha_1}\mathbf{Z}.$$

The receiver calculates $\tilde{\mathbf{Y}} = (1-\alpha_1)\mathbf{Y}_1$, and hence

$$\tilde{\mathbf{Y}} = \mathbf{X}_2 + \tilde{\mathbf{S}}_c + \mathbf{Z},$$

where $\tilde{\mathbf{S}}_c = (1-\alpha_1)\mathbf{S}_c$.

c) *Stage III*: The decoder calculates $\mathbf{Y}'' = [\alpha_2\tilde{\mathbf{Y}} - \mathbf{D}_2] \bmod \Lambda_2$. The equivalent channel is given by

$$\begin{aligned}\mathbf{Y}'' &= \Big[\alpha_2(\mathbf{X}_2 + \tilde{\mathbf{S}}_c + \mathbf{Z}) - \mathbf{D}_2\Big] \bmod \Lambda_2 \\ &= \Big[\mathbf{V}_2 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z}\Big] \bmod \Lambda_2.\end{aligned}$$

Again $\mathbf{V}_2$ and $\mathbf{X}_2$ are independent. The rate achieved by user 2 is given by

$$\begin{aligned} R_2 &= \frac{1}{n} I(\mathbf{V}_2;\mathbf{Y}'') = \frac{1}{n}\left\{h(\mathbf{Y}'') - h(\mathbf{Y}''|\mathbf{V}_2))\right\} \\ &= \frac{1}{n}\left\{h(\mathbf{Y}'') - h([(1-\alpha_2)\mathbf{X}_2 + \alpha_2 \mathbf{Z}] \bmod \Lambda_2)\right\} \\ &\geq \frac{1}{2}\log_2\left(\frac{P_2}{G(\Lambda_2)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_2)^2 P_2 + \alpha_2^2 N\right)\right). \end{aligned}$$

Using $\alpha_2 = \frac{P_2}{P_2+N}$ and a lattices that are good for quantization, any rate R_2 such that

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right) \tag{104}$$

is achievable.

From symmetry, the achievability of the second corner point $(\frac{1}{2}\cdot\log_2(1+P_1/N), \frac{1}{2}\cdot\log_2(1+\frac{P_2}{P_1+N}))$ is achieved by first decoding user 2 and then decoding user 1. The capacity region follows by using time sharing of these corner points.

IX. Extensions

A. Strong Correlated Interferences

In this section we consider a generalized scenario for the doubly-dirty MAC (8), where the interference signals are correlated. Specifically, the channel model is given by

$$Y = X_1 + X_2 + \tilde{S}_1 + \tilde{S}_2 + Z, \tag{105}$$

where $\tilde{S}_1$ and $\tilde{S}_2$ are interference signals with a joint Gaussian distribution, i.e.,

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \tilde{\sigma}_{s_1}^2 & \rho\tilde{\sigma}_{s_1}\tilde{\sigma}_{s_2} \\ \rho\tilde{\sigma}_{s_1}\tilde{\sigma}_{s_2} & \tilde{\sigma}_{s_2}^2 \end{pmatrix}\right) \tag{106}$$

where $|\rho| < 1$ is the correlation coefficient, and $\tilde{\sigma}_{s_1}^2$ and $\tilde{\sigma}_{s_1}^2$ are the variances of $\tilde{S}_1$ and $\tilde{S}_2$, respectively. For any $\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, \rho$, the capacity region of (105) is denoted by $\mathcal{C}_{COR}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, \rho)$. The capacity region of the doubly-dirty MAC (8) with independent Gaussian interferences S_1 and S_2 is denoted by $\mathcal{C}_{DMAC}(\sigma_{s_1}, \sigma_{s_2})$. Clearly, we have that $\mathcal{C}_{DMAC}(\sigma_{s_1}, \sigma_{s_2}) \equiv \mathcal{C}_{COR}(\sigma_{s_1}, \sigma_{s_2}, 0)$.

Generally, any joint Gaussian variables can be decomposed as

$$\tilde{S}_1 = S_1 + \beta_1 S_0 \tag{107}$$

$$\tilde{S}_2 = S_2 + \beta_2 S_0 \tag{108}$$

where $S_0 \sim \mathcal{N}(0, \sigma_{s_0}^2)$, $S_1 \sim \mathcal{N}(0, \sigma_{s_1}^2)$ and $S_2 \sim \mathcal{N}(0, \sigma_{s_2}^2)$ are independent Gaussian variables, and $\beta_1 = \text{sign}(\rho)\sqrt{|\rho|}$, $\beta_2 = \frac{\tilde{\sigma}_{s_2}}{\tilde{\sigma}_{s_1}}\sqrt{|\rho|}$ and $\sigma_{s_0}^2 = \tilde{\sigma}_{s_1}^2$. In this case, we have that

$$\sigma_{s_1}^2 = \tilde{\sigma}_{s_1}^2(1-|\rho|) \tag{109}$$

$$\sigma_{s_2}^2 = \tilde{\sigma}_{s_2}^2(1-|\rho|). \tag{110}$$

The channel output can be expressed as

$$Y = X_1 + X_2 + S_1 + \beta_1 S_0 + S_2 + \beta_2 S_0 + Z \tag{111}$$

$$= X_1 + X_2 + S_1 + S_2 + S_c + Z, \tag{112}$$

where $S_c \triangleq (\beta_1 + \beta_2) S_0$, hence S_1, S_2, S_c are Gaussian independent random variables. $\mathcal{C}_{COM}(\sigma_{s_1}, \sigma_{s_2}, \sigma_{s_c})$ is denoted to be the capacity region for the case that (S_1, S_c) are known non-causally at encoder 1, and (S_2, S_c) are known non-causally at encoder 2. Clearly, we have that $\mathcal{C}_{COM}(\sigma_{s_1}, \sigma_{s_2}, \sigma_{s_c}) = \mathcal{C}_{COR}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, \rho)$

Lemma 10: For $|\rho| < 1$, in the limit of $\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2} \to \infty$, we have that

$$\mathcal{C}_{COR}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, \rho) = \mathcal{C}_{COM}(\sigma_{s_1}, \sigma_{s_2}, \sigma_{s_c}) = \mathcal{C}_{DMAC}(\sigma_{s_1}, \sigma_{s_2}), \tag{113}$$

where $\sigma_{s_i}^2 = \tilde{\sigma}_{s_i}^2 (1 - |\rho|)$ for $i = 1, 2$.

Proof: For any $\tilde{\sigma}_{s_1}^2, \tilde{\sigma}_{s_2}^2$, we have that

$$\mathcal{C}_{DMAC}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}) = \mathcal{C}_{COR}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, 0) \tag{114}$$

$$\subseteq \mathcal{C}_{COR}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2}, \rho) \tag{115}$$

$$= \mathcal{C}_{COM}(\sigma_{s_1}, \sigma_{s_2}, \sigma_{s_c}) \tag{116}$$

$$\subseteq \mathcal{C}_{COM}(\sigma_{s_1}, \sigma_{s_2}, 0) \tag{117}$$

$$= \mathcal{C}_{DMAC}(\sigma_{s_1}, \sigma_{s_2}), \tag{118}$$

where (115) follows since correlation between the interferences can only increase the capacity region; (117) follows since the capacity region increases for $S_c = 0$. The proof follows since for $\tilde{\sigma}_{s_1}^2, \tilde{\sigma}_{s_2}^2 \to \infty$, also $\sigma_{s_1}^2, \sigma_{s_2}^2 \to \infty$, and hence $\mathcal{C}_{DMAC}(\sigma_{s_1}, \sigma_{s_2}) = \mathcal{C}_{DMAC}(\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2})$. □

Lemma 10 implies that for jointly Gaussian $\tilde{S}_1$ and $\tilde{S}_2$ with $|\rho| < 1$ where $\tilde{\sigma}_{s_1}, \tilde{\sigma}_{s_2} \to \infty$, the capacity region is independent of the correlation between the interferences. Therefore, the channel model in (105) is equivalent to the "standard" doubly-dirty MAC (8) with uncorrelated S_1 and S_2. Furthermore from Lemma 10, the case that we have in addition to S_1 and S_2, a common interference S_c which is known non-causally to both encoders, as shown in Fig. 22, is also equivalent to doubly-dirty MAC (8) in the limit of strong interferences S_1 and S_2.

B. K-User Case

The results in Section VI can be extended to the K-user case. For simplicity, we consider only the symmetric case, i.e., all the users have equal power constraints. The channel model is given by

$$Y = \sum_{i=1}^{K} X_i + \sum_{i=1}^{K} S_i + Z, \tag{119}$$

where $Z \sim \mathcal{N}(0, N)$, and the power constraint for each user is P. The interferences $\{S_i\}_{i=1}^K$ are strong and independent, where the i-th interference is known non-causally only to the encoder of user i. Since the derivation is a straightforward extension of the two-user case, only the final results are stated.

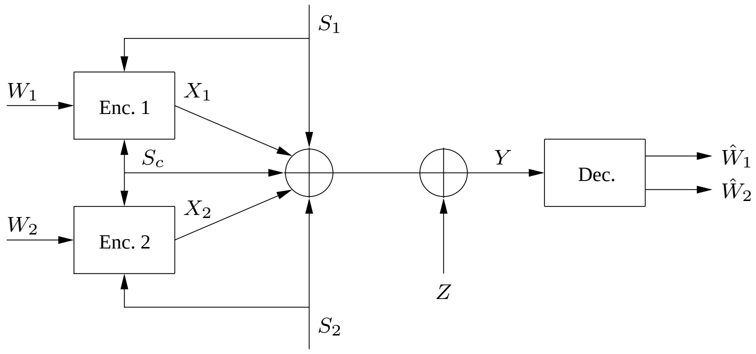


Fig. 22: MAC with private and common interferences.

Corollary 7: In the limit of strong interference, the capacity region of (119) is contained in the following region:

$$\sum_{i=1}^{K} R_i \leq \frac{1}{2}\log_2\left(1+\frac{P}{N}\right).$$

An achievable region for (119) is given by the set of all the rates satisfying

$$\sum_{i=1}^{K} R_i \leq u.c.e\left[\frac{1}{2}\log_2\left(\frac{1}{K}+\frac{P}{N}\right)\right]^{+}.$$

As in the two-user case (Lemma 2), the factor of $1/K$ inside the logarithm function stems from the K independent self noises that result in this case. As a consequence, the rate loss between the outer bound and the inner bound increases with respect to K, yet the rate loss is bounded by $1/2$ bit for any K.

X. SUMMARY

In this work the Gaussian doubly-dirty MAC was introduced, where each interference is known to a different transmitter. An outer bound for the capacity region was derived and sufficient conditions were found under which lattice-strategies meet the outer bound. It was shown that a scheme based on lattice strategies accomplishes simultaneously the interference concentration and interference alignment to achieve these rates.

The *additive* doubly-dirty MAC is a special case of channels with distributed knowledge of the channel state information among several transmitters. Unlike the special case treated in this paper, however, the rate loss with respect to full knowledge of the channel state at the receiver may in general be large. For example, consider the *additive-multiplicative* model:

$$Y = X_1 + X_2 + S_1 \cdot S_2 + Z$$

where S_1 and S_2 are known to the transmitters of user 1 and user 2, respectively. In this case, for strong interferences, the uncertainty at the decoder cannot be resolved for any choice of encoders, which indicates that the capacity tends to zero (while a fully informed receiver can clearly achieve the clean MAC capacity).

The asymmetric case was also considered, i.e., the Gaussian MAC with a single dirty user. In particular, for the helper problem, sufficient conditions were found under which lattice-strategies are optimal.

We also provide a lattice-based transmission scheme, which achieves the capacity region of the Gaussian MAC with common interference.

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APPENDIX I
PROOF OF LEMMA 1 - OUTER BOUND FOR SINGLE DIRTY USER WITH GAUSSIAN INTERFERENCE

The bound for R_2 trivially follows by revealing S_1 to the decoder.

For the sum-rate bound, we assume that a genie reveals the message of user 1 to user 2 and vice versa, implying that, in fact, both users intend to transmit a common message W. An upper bound on the rate of this message clearly upper bounds $R_1 + R_2$ for the independent messages case ($W_1 \neq W_2$). Applying Fano's inequality to the common message rate R we have,

$$nR \leq H(W) = H(W|Y^n) + I(W;Y^n) \leq n\epsilon_n + I(W;Y^n),$$

where $\epsilon_n \to 0$ as the error probability ($P_e^{(n)}$) goes to zero. The following chain of inequalities can be easily verified.

$$\begin{aligned} I(W;Y^n) &= h(Y^n) - h(Y^n|W) \\ &\leq h(Y^n) - h(Y^n|W, X_2^n) && (120) \\ &= h(Y^n) - h(Y^n|W, X_2^n, S_1^n) - I(S_1^n; Y^n|W, X_2^n) && (121) \\ &\leq h(Y^n) - h(Z^n) - I(S_1^n; Y^n|W, X_2^n) && (122) \\ &= h(Y^n) - h(Z^n) - h(S_1^n) + h(S_1^n|W, X_2^n, Y^n) && (123) \\ &= h(Y^n) - h(Z^n) - h(S_1^n) + h(X_1^n + Z^n|W, X_2^n, Y^n) && (124) \\ &\leq h(Y^n) - h(Z^n) - h(S_1^n) + h(X_1^n + Z^n), && (125) \end{aligned}$$

where the equality in (123) follows from the fact that S_1^n is independent of (X_2^n, W) and the three inequalities are a consequence of the fact that conditioning reduces differential entropy. The lemma follows since $S_1 \sim \mathcal{N}(0, Q_1)$, we have by the Cauchy-Schwarz inequality that $h(Y^n) \leq \frac{n}{2}\log_2 2\pi e(N + (\sqrt{P_1} + \sqrt{P_2} + \sqrt{Q_1})^2)$, and $h(X_1^n + Z^n) \leq \frac{n}{2}\log_2 2\pi e(N + P_1)$.

APPENDIX II
PROOF OF THEOREM 2 - DOUBLY-DIRTY MAC FOR IMBALANCED SNRS(FOR $P_1\left(\frac{P_1+N}{P_1}\right)^2 \leq P_2$)

Here we complete the proof of Theorem 2 for the case that $P_1\left(\frac{P_1+N}{P_1}\right)^2 \leq P_2$. We show achievability for the point

$$(R_1, R_2) = \left(0, \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right)\right) \qquad (126)$$

Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \alpha_1\Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \alpha_1$ and $\kappa_2 = 1$). The second moments of the lattices Λ_1 and Λ_2 are $\sigma_1^2 = \alpha_1^2 P_2$ and $\sigma_2^2 = P_2$, respectively, where α_1 will be determined later. We set $\mathbf{V}_1 = \mathbf{0}$, $\mathbf{D}_2 = \mathbf{0}$, $\alpha_2 = \gamma = 1$, $\beta = 0$ and $\alpha_r = \alpha_1$, hence the encoders send

$$\mathbf{X}_1 = [-\alpha_1\mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \quad (127)$$

$$\mathbf{X}_2 = [\mathbf{V}_2 - \mathbf{S}_2] \bmod \Lambda_2, \quad (128)$$

where $\mathbf{V}_2 \sim \text{Unif}(\mathcal{V}_2)$ carries the information of user 2; $\mathbf{D}_1 \sim \text{Unif}(\mathcal{V}_1)$ is the dither signal. The receiver calculates $\mathbf{Y}' = [\alpha_1\mathbf{Y} - \mathbf{D}_1] \bmod \Lambda_1$. The equivalent channel is given by

$$\mathbf{Y}' = \Big[\alpha_1(\mathbf{X}_1 + \mathbf{S}_1 + \mathbf{X}_2 + \mathbf{S}_2 + \mathbf{z}) - \mathbf{D}_1\Big] \bmod \Lambda_1 \quad (129)$$

$$= \Big[\alpha_1\mathbf{V}_2 + \alpha_1(\mathbf{X}_1 + \mathbf{S}_1) + \alpha_1\mathbf{Z} - \alpha_1 Q_{\Lambda_2}(\mathbf{V}_2 - \mathbf{S}_2) - \mathbf{D}_1\Big] \bmod \Lambda_1 \quad (130)$$

$$= \Big[\alpha_1\mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} - \alpha_1 Q_{\Lambda_2}(\mathbf{V}_2 - \mathbf{S}_2)\Big] \bmod \Lambda_1, \quad (131)$$

where (130) follows from (128); (131) follows from (127).

Since $\Lambda_1 = \alpha_1\Lambda$ and $\Lambda_2 = \Lambda$ (scaled lattices), we have that $\alpha_1 Q_{\Lambda_2}(\mathbf{V}_2 - \mathbf{S}_2) \in \Lambda_1$, i.e., the interference signal is aligned with Λ_1. Hence, the element $\alpha_1 Q_{\Lambda_2}(\mathbf{V}_2 - \mathbf{S}_2)$ disappears after the modulo-Λ_1 operation. In this case, the equivalent channel is given by

$$\mathbf{Y}' = [\alpha_1\mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \bmod \Lambda_1, \quad (132)$$

where $\alpha_1\mathbf{V}_2 \sim \text{Unif}(\mathcal{V}_1)$. Since $\mathbf{V}_2$ and $\mathbf{X}_1$ are independent, hence the rate achieved by user 2 is given by

$$R_2 = \frac{1}{n} I(\mathbf{V}_2;\mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_2)\right\}$$
$$= \frac{1}{n}\left\{h(\mathbf{Y}') - h([(1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \bmod \Lambda_1)\right\}$$
$$\geq \frac{1}{2}\log_2\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_1)^2 P_1 + {\alpha_1}^2 N\right)\right) \quad (133)$$

where in the last inequality we used the fact that $\alpha_1\mathbf{V}_2$ has uniform distribution over $\mathcal{V}_1$ then $\mathbf{Y}'$ is also uniform over $\mathcal{V}_1$, and since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment.

For $P_2 = P_1\left(\frac{P_1+N}{P_1}\right)^2$, using the optimal MMSE factor, i.e., $\alpha_1 = \frac{P_1}{P_1+N}$, and for lattice that is good for quantization (46), i.e., $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, we get that any rate

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right), \quad (134)$$

is achievable. Clearly, for $P_2 = P_1\left(\frac{P_1+N}{P_1}\right)^2$ the inner bound meets the outer bound (38). Likewise, for $P_1(\frac{P_1+N}{P_1})^2 \leq P_2$, the outer bound (38) remains $\frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right)$, thus the outer bound is also achievable.

From (134) and (61), the following rate is achievable for the point $(0, R_2)$ where

$$R_2 = \begin{cases} \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right), & P_1\left(\frac{P_1+N}{P_1}\right)^2 \leq P_2 \\ \frac{1}{2}\log_2\left(1 + \frac{P_2}{N}\right), & P_2\left(\frac{P_2+N}{P_2}\right)^2 \leq P_1 \end{cases} \quad (135)$$

As discussed at the beginning of the proof, also the point $(R_1, 0)$ where

$$R_1 = \begin{cases} \frac{1}{2}\log_2\left(1+\frac{P_2}{N}\right), & P_2\left(\frac{P_2+N}{P_2}\right)^2 \leq P_1 \\ \frac{1}{2}\log_2\left(1+\frac{P_1}{N}\right), & P_1\left(\frac{P_1+N}{P_1}\right)^2 \leq P_2 \end{cases} \tag{136}$$

is achievable. The theorem follows since any rate pair in the straight line $R_1 + R_2 = \frac{1}{2}\log_2\left(1+\frac{\min(P_1,P_2)}{N}\right)$ is achievable using time sharing between (135) and (136) for $N \leq \sqrt{P_1P_2} - \min(P_1, P_2)$ and $P_1 \neq P_2$.

APPENDIX III

PROOF OF THEOREM 3 - DOUBLY-DIRTY MAC FOR NEARLY-BALANCED SNRS

Clearly, it is only required to show the achievable region inside the upper convex envelope operation in (77), since the region including the upper convex envelope can be achieved using time sharing.

We first consider the case that $P_1 \leq P_2 \leq P_1\left(\frac{P_1+N}{N}\right)^2$, and we show achievability for the rate pair $(R_1, 0)$ where

$$R_1 = \frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_1}-\sqrt{P_2})^2}\right).$$

Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \frac{\alpha_1}{\alpha_2}\Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \frac{\alpha_1}{\alpha_2}$ and $\kappa_2 = 1$). The second moments of the lattices Λ_1 and Λ_2 are $\sigma_1^2 = \frac{\alpha_1^2}{\alpha_2^2}P_2$ and $\sigma_2^2 = P_2$, respectively, where α_1 and α_2 will be determined later. We set $\mathbf{V}_2 = \mathbf{0}$, $\gamma = 1$, $\beta = \frac{\alpha_1}{\alpha_2}$ and $\alpha_r = \alpha_1$, hence the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \tag{137}$$

$$\mathbf{X}_2 = [-\alpha_2\mathbf{S}_2 + \mathbf{D}_2] \bmod \Lambda_2, \tag{138}$$

where $\mathbf{V}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ carries the information of user 1; $\mathbf{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ and $\mathbf{D}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ are the dither signals. The receiver calculates $\mathbf{Y}' = [\alpha_1\mathbf{Y} - \mathbf{D}_1 - \beta\mathbf{D}_2] \bmod \Lambda_1$. The equivalent channel is given by

$$\mathbf{Y}' = \Big[\alpha_1(\mathbf{X}_1 + \mathbf{S}_1 + \mathbf{X}_2 + \mathbf{S}_2 + \mathbf{Z}) - \mathbf{D}_1 - \beta\mathbf{D}_2\Big] \bmod \Lambda_1 \tag{139}$$

$$= \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} + \alpha_1(\mathbf{X}_2 + \mathbf{S}_2) - \beta\mathbf{D}_2\Big] \bmod \Lambda_1 \tag{140}$$

$$= \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} + \alpha_1(1-\alpha_2)\mathbf{S}_2 - (\beta-\alpha_1)\mathbf{D}_2 - \alpha_1 Q_{\Lambda_2}(-\alpha_2\mathbf{S}_2 + \mathbf{D}_2)\Big] \bmod \Lambda_1 \tag{141}$$

$$= \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)[-\alpha_2\mathbf{S}_2 + \mathbf{D}_2 - Q_{\Lambda_2}(-\alpha_2\mathbf{S}_2 + \mathbf{D}_2)] - \frac{\alpha_1}{\alpha_2}Q_{\Lambda_2}(-\alpha_2\mathbf{s}_2 + \mathbf{d}_2)\Big] \bmod \Lambda_1 \tag{142}$$

$$= \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)\mathbf{X}_2 - \frac{\alpha_1}{\alpha_2}Q_{\Lambda_2}(-\alpha_2\mathbf{S}_2 + \mathbf{D}_2)\Big] \bmod \Lambda_1, \tag{143}$$

where (140) follows from (137); (141) follows from (138); (142) follows since $\beta = \frac{\alpha_1}{\alpha_2}$; (143) follows from (138).

Since $\Lambda_1 = \frac{\alpha_1}{\alpha_2}\Lambda$ and $\Lambda_2 = \Lambda$ (scaled lattices), we have that $\frac{\alpha_1}{\alpha_2}Q_{\Lambda_2}(-\alpha_2\mathbf{S}_2 + \mathbf{D}_2) \in \Lambda_1$, i.e., the interference signal is aligned with Λ_1. Hence the element $\frac{\alpha_1}{\alpha_2}Q_{\Lambda_2}(-\alpha_2\mathbf{S}_2 + \mathbf{D}_2)$ disappears after the modulo-Λ_1 operation. In this case, the equivalent channel is given by

$$\mathbf{Y}' = \Big[\mathbf{V}_1 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z} - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)\mathbf{X}_2\Big] \bmod \Lambda_1, \tag{144}$$

From the dithered quantization property (51), $\mathbf{V}_1$ and $\mathbf{X}_1$ are independent. Furthermore, $\mathbf{X}_2$ is independent of $\mathbf{V}_1$ and $\mathbf{X}_1$, hence the rate achieved by user 2 is given by

$$\begin{aligned} R_1 &= \frac{1}{n} I(\mathbf{V}_1; \mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_1)\right\} \\ &= \frac{1}{n}\left\{h(\mathbf{Y}') - h\left(\left[(1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z} - \frac{\alpha_1}{\alpha_2}(1-\alpha_2)\mathbf{X}_2\right] \bmod \Lambda_1\right)\right\} \\ &\geq \left[\frac{1}{2}\log_2\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_1)^2 P_1 + {\alpha_1}^2 N + \left(\frac{\alpha_1}{\alpha_2}\right)^2 (1-\alpha_2)^2 P_2\right)\right)\right]^+ \end{aligned}$$

where in the last inequality we used the fact that $\mathbf{V}_1$ has uniform distribution over $\mathcal{V}_1$ then $\mathbf{Y}'$ has also uniform distribution over $\mathcal{V}_1$, and since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment.

For $\frac{\alpha_1}{\alpha_2} = \sqrt{\frac{P_1}{P_2}}$ and using lattices that are good for quantization (46), i.e., $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, the optimal α_1 that maximizes R_1 is given by $\alpha_1 = \frac{\sqrt{P_1}(\sqrt{P_1}+\sqrt{P_2})}{P_1+P_2+N}$, in this case we get that any rate

$$R_1 \leq \left[\frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_2}-\sqrt{P_1})^2}\right)\right]^+ \tag{145}$$

is achievable.

We now consider the case that $P_2 \leq P_1 \leq P_2\left(\frac{P_2+N}{N}\right)^2$. Again, we show achievability for the rate pair $(R_1, 0)$ where

$$R_1 = \frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_1}-\sqrt{P_2})^2}\right).$$

Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda$ and $\Lambda_2 = \Lambda_r = \frac{\alpha_2}{\alpha_1}\Lambda$ (that is $\kappa_1 = 1$ and $\kappa_2 = \kappa_r = \frac{\alpha_2}{\alpha_1}$). The second moments of the lattices Λ_1 and Λ_2 are $\sigma_1^2 = P_1$ and $\sigma_2^2 = \frac{\alpha_2^2}{\alpha_1^2}P_1$, respectively, where α_1 and α_2 will be determined later. We set $\mathbf{V}_2 = \mathbf{0}$, $\beta = 1$, $\gamma = \frac{\alpha_2}{\alpha_1}$ and $\alpha_r = \alpha_2$, hence the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \tag{146}$$

$$\mathbf{X}_2 = [-\alpha_2 \mathbf{S}_2 + \mathbf{D}_2] \bmod \Lambda_2, \tag{147}$$

where $\mathbf{V}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ carries the information of user 1; $\mathbf{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ and $\mathbf{D}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ are the dither signals. The receiver calculates $\mathbf{Y}' = [\alpha_2 \mathbf{y} - \mathbf{D}_2 - \gamma \mathbf{D}_1] \bmod \Lambda_2$. The equivalent channel is given by

$$\mathbf{Y}' = \left[\alpha_2(\mathbf{x}_1 + \mathbf{S}_1 + \mathbf{X}_2 + \mathbf{s}_2 + \mathbf{Z}) - \mathbf{D}_2 - \gamma \mathbf{D}_1\right] \bmod \Lambda_2 \tag{148}$$

$$= \left[(1-\alpha_2)\mathbf{X}_2 + \alpha_2 \mathbf{Z} + \alpha_2(\mathbf{X}_1 + \mathbf{s}_1) - \gamma \mathbf{D}_1\right] \bmod \Lambda_2 \tag{149}$$

$$= \left[-(1-\alpha_2)\mathbf{X}_2 + \alpha_2 \mathbf{Z} + \alpha_2[\mathbf{V}_1 + (1-\alpha_1)\mathbf{S}_1 - Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] - (\gamma - \alpha_2)\mathbf{D}_1\right] \bmod \Lambda_2 \tag{150}$$

$$\begin{aligned} &= \Big[\frac{\alpha_2}{\alpha_1}\mathbf{V}_1 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2 \mathbf{Z} - \frac{\alpha_2}{\alpha_1}(1-\alpha_1)[\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1 - Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] \\ &\quad - \frac{\alpha_2}{\alpha_1} Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1)\Big] \bmod \Lambda_2 \end{aligned} \tag{151}$$

$$= \left[\frac{\alpha_2}{\alpha_1}\mathbf{V}_1 - \frac{\alpha_2}{\alpha_1}(1-\alpha_1)\mathbf{X}_1 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2 \mathbf{Z} - \frac{\alpha_2}{\alpha_1} Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1 \mathbf{S}_1 + \mathbf{D}_1)\right] \bmod \Lambda_2 \tag{152}$$

where (149) follows from (147); (150) follows from (146); (151) follows since $\gamma = \frac{\alpha_2}{\alpha_1}$; (152) follows from (146).

Since $\Lambda_1 = \Lambda$ and $\Lambda_2 = \frac{\alpha_2}{\alpha_1}\Lambda$ (scaled lattices), we have that $\frac{\alpha_2}{\alpha_1}Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1) \in \Lambda_2$, i.e., the interference is aligned with Λ_2. Hence the element $\frac{\alpha_2}{\alpha_1}Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1)$ disappears after the modulo-Λ_2 operation. In this case, the equivalent channel is given by

$$\mathbf{Y}' = \left[\frac{\alpha_2}{\alpha_1}\mathbf{V}_1 - \frac{\alpha_2}{\alpha_1}(1-\alpha_1)\mathbf{X}_1 - (1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z}\right] \bmod \Lambda_2, \tag{153}$$

where $\frac{\alpha_2}{\alpha_1}\mathbf{V}_1 \sim \mathrm{Unif}(\mathcal{V}_2)$. From the dithered quantization property (51), $\mathbf{V}_1$ and $\mathbf{X}_1$ are independent. Furthermore, $\mathbf{X}_2$ is independent of $\mathbf{V}_1$ and $\mathbf{X}_1$, hence the rate achieved by user 2 is given by

$$\begin{aligned} R_1 &= \frac{1}{n}I(\mathbf{V}_1;\mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_1)\right\} \\ &= \frac{1}{n}\left\{h(\mathbf{Y}') - h\left(\left[(1-\alpha_2)\mathbf{X}_2 + \alpha_2\mathbf{Z} - \frac{\alpha_2}{\alpha_1}(1-\alpha_1)\mathbf{X}_1\right] \bmod \Lambda_2\right)\right\} \\ &\geq \left[\frac{1}{2}\log_2\left(\frac{P_2}{G(\Lambda_2)}\right) - \frac{1}{2}\log_2\left(2\pi e\left((1-\alpha_2)^2P_2 + {\alpha_2}^2N + \left(\frac{\alpha_2}{\alpha_1}\right)^2(1-\alpha_1)^2P_1\right)\right)\right]^+ \end{aligned}$$

where in the last inequality we used the fact that $\frac{\alpha_2}{\alpha_1}\mathbf{V}_1$ has uniform distribution over $\mathcal{V}_2$ then $\mathbf{Y}'$ has also uniform distribution over $\mathcal{V}_2$, and since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment.

For $\frac{\alpha_2}{\alpha_1} = \sqrt{\frac{P_2}{P_1}}$ and using lattices that are good for quantization (46), i.e., $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, the optimal α_2 that maximizes R_1 is given by $\alpha_2 = \frac{\sqrt{P_2}(\sqrt{P_1}+\sqrt{P_2})}{P_1+P_2+N}$, in this case we get that any rate

$$R_1 \leq \left[\frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_1}-\sqrt{P_2})^2}\right)\right]^+ \tag{154}$$

is achievable, which is identical to the case that $P_1 \leq P_2 \leq P_1\left(\frac{P_1+N}{N}\right)^2$ (145). Therefore, the achievable rate of the point $(R_1, 0)$ for $N \geq \sqrt{P_1P_2} - \min(P_1, P_2)$ is given by.

$$(R_1, 0) = \left(\left[\frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_1}-\sqrt{P_2})^2}\right)\right]^+, 0\right). \tag{155}$$

Due to the symmetry, it can be shown that the achievable rate of the point $(0, R_2)$ for $N \geq \sqrt{P_1P_2} - \min(P_1, P_2)$ is given by

$$(0, R_2) = \left(0, \left[\frac{1}{2}\log_2\left(\frac{P_1+P_2+N}{2N+(\sqrt{P_1}-\sqrt{P_2})^2}\right)\right]^+\right). \tag{156}$$

The theorem follows by using a time sharing between the achievable rate pairs in (155) and (156).

APPENDIX IV

PROOF OF LEMMA 3 - A UNIFORM OUTER BOUND FOR THE GAP ζ

For any P_1, P_2, N, the gap is upper bounded by

$$\zeta(P_1, P_2, N) \leq \zeta(P_{\min}, P_{\min}, N). \tag{157}$$

where $P_{\min} = \min(P_1, P_2)$, i.e., the symmetric case where $P_1 = P_2$ is the worst case. To see this, we fix P_1 and vary P_2 such that $P_2 \geq P_1$. The second term on the RHS of (78) is increasing in P_2, while the first term is a constant. Therefore, we get that the gap $\zeta(P_1, P_2, N)$ is maximized for $P_1 = P_2$. Of course, for the opposite condition, that is $P_1 \leq P_2$, the maximum occurs again for $P_1 = P_2$.

Without loss of generality it can be assumed that $P_1 \leq P_2$ where $N > \sqrt{P_1 P_2} - \min(P_1, P_2)$. From (157), we have that

$$\zeta(P_1, P_2, N) \leq \zeta(P_1, P_1, N) = \frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right) - u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{1}{2} + \frac{P_1}{N}\right)\right]^+\right\}. \tag{158}$$

Let us define that $x \triangleq \frac{P_1}{N}$, thus

$$\zeta(P_1, P_1, N) = \frac{1}{2}\log_2(1 + x) - u.c.e\left\{\left[\frac{1}{2}\log_2\left(\frac{1}{2} + x\right)\right]^+\right\} \triangleq \tilde{\zeta}(x), \tag{159}$$

where the upper convex envelope is with respect to x. We also define the following function

$$f(x) \triangleq \frac{1}{2}\log_2\left(\frac{1}{2} + x\right). \tag{160}$$

The function $[f(x)]^+$ is not a convex - $\cap$ function with respect to x. The point x^* is defined such that the upper convex envelope of $[f(x)]^+$ is achieved by time-sharing between the points $x = 0$ and $x = x^*$, therefore we have that

$$\frac{\partial f(x = x^*)}{\partial x} = \frac{\frac{1}{2}\log_2(e)}{\frac{1}{2} + x^*} = \frac{\frac{1}{2}\log_2\left(\frac{1}{2} + x^*\right)}{x^*} \tag{161}$$

Therefore,

$$u.c.e\left\{[f(x)]^+\right\} = \begin{cases} \frac{1}{2}\log_2\left(\frac{1}{2} + x\right), & x \geq x^* \\ C^* x, & 0 \leq x \leq x^* \end{cases} \tag{162}$$

where $C^* \triangleq \frac{\frac{1}{2}\log_2(e)}{\frac{1}{2}+x^*}$. The value of x^* can be evaluated (numerically) from the equation $C^* x^* = \frac{1}{2}\log_2\left(\frac{1}{2} + x^*\right)$, which results that $x^* \approx 1.655$.

a) For $x \geq x^*$: $\tilde{\zeta}(x)$ is given by

$$\tilde{\zeta}(x) = \frac{1}{2}\log_2\left(\frac{1 + x}{\frac{1}{2} + x}\right) = \frac{1}{2}\log_2\left(1 + \frac{\frac{1}{2}}{1 + x}\right). \tag{163}$$

Since $\tilde{\zeta}(x)$ is decreasing with respect to x, hence $\tilde{\zeta}(x)$ is maximized for $x = x^*$.

b) For $0 \leq x \leq x^*$: $\tilde{\zeta}(x)$ is given by

$$\tilde{\zeta}(x) = \frac{1}{2}\log_2(1 + x) - C^* x. \tag{164}$$

The maximum of $\tilde{\zeta}(x)$ occurs at $x^* - \frac{1}{2}$, hence we get that

$$\tilde{\zeta}(x) \leq \tilde{\zeta}\left(x^* - \frac{1}{2}\right) = \frac{\frac{1}{2}\log_2\left(\frac{1}{2} + x^*\right)}{2x^*}. \tag{165}$$

The lemma follows since $\tilde{\zeta}(x^*) \leq \tilde{\zeta}(x^* - 1/2)$.

APPENDIX V
LEMMA 11

The following lemma is useful in characterizing the entropy of the effective noise in lattice transmission schemes. (46) and for AWGN channel decoding (47).

Lemma 11: Assume a sequence of lattices Λ_n with second moment P, that are simultaneously good for quantization (46) (and covering) and for AWGN channel coding (47). Let $\mathbf{U} \sim \mathrm{Unif}(\kappa\mathcal{V})$ independent of $\mathbf{Z} \sim \mathcal{N}(0, NI_n)$, where I_n is an $n \times n$ identity matrix. For any $N < P$ and κ such that $\kappa^2 P + N = P - \epsilon$, for some $\epsilon > 0$, we have that

$$\lim_{n\to\infty} \frac{1}{n} h\Big([\mathbf{U} + \mathbf{Z}] \bmod \Lambda_n\Big) = \frac{1}{2}\log_2(2\pi e P) - \epsilon', \tag{166}$$

where ϵ' may be made arbitrarily small by taking ϵ to be sufficiently small.

Proof: Clearly $\frac{1}{2}\log_2(2\pi e P)$ is an upper bound since a white Gaussian random vector maximizes the differential entropy under a power constraint. On the other hand, the the entropy of the l.h.s. of (166) satisfies

$$h(\mathbf{U} + \mathbf{Z} \bmod \Lambda_n) \geq h(\mathbf{U} + \mathbf{Z} | Q_{\Lambda_n}(\mathbf{U} + \mathbf{Z})) \tag{167}$$

$$= h(\mathbf{U} + \mathbf{Z}) - H(Q_{\Lambda_n}(\mathbf{U} + \mathbf{Z})). \tag{168}$$

Now, since Λ_n is good for both channel coding and covering, it follows that $p_0 = \Pr(Q_{\Lambda_n} = 0) \to 1$ as $n \to \infty$ and furthermore that $\frac{1}{n}H(Q_{\Lambda_n}) \to 0$.[7] Moreover, by the entropy-power inequality [25], we have that

$$\begin{aligned}
\frac{1}{n}h(\mathbf{U} + \mathbf{Z}) &\geq \frac{1}{2}\log_2\left(2^{\frac{2}{n}h(\mathbf{U})} + 2^{\frac{2}{n}h(\mathbf{Z})}\right) \\
&= \frac{1}{2}\log_2\left(2^{\log_2\left(\frac{\kappa^2 P}{G(\Lambda_n)}\right)} + 2^{\log_2(2\pi e N)}\right) \\
&= \frac{1}{2}\log_2\left(\frac{\kappa^2 P}{G(\Lambda_n)} + 2\pi e N\right).
\end{aligned}$$

The lemma now follows since $G(\Lambda_n) \to \frac{1}{2\pi e}$ as $n \to \infty$. □

APPENDIX VI
PROOF OF THEOREM 4 - IMBALANCED SNRS FOR THE MAC WITH A SINGLE DIRTY USER

The converse part has been proved in corollary 1. In view of the outer bound (37) in corollary 1, it is sufficient to show achievability for $P_2 = P_1 + N$ and $P_1 = P_2 + N$.

We consider the case that $P_2 = P_1 + N$. Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \sqrt{\frac{P_1}{P_2}}\Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \sqrt{\frac{P_1}{P_2}}$ and $\kappa_2 = 1$). The lattice Λ is both good for quantization (46) and good for AWGN channel coding (47). The second moments of the lattices

[7] For any $\epsilon > 0$, since the covering diameter of the cells grows as $\sqrt{n}$ (but no faster), there exists r large enough such that the contribution to $\frac{1}{n}H(Q_{\Lambda_n})$ of cells outside a radius of $r\sqrt{n}$ is negligible for all n. On the other hand, inside this ball, the number of cells is exponentially equal to $(r^2/P)^{n/2}$. Thus, $\frac{1}{n}H(Q_{\Lambda_n}) \leq \frac{1}{n}\left(-p_0 \log p_0 + (1-p_0)\log(r^2/P)^{n/2}\right) + \epsilon$.

Λ_1 and Λ_2 are P_1 and P_2, respectively. We set $\mathbf{V}_1 = \mathbf{0}$, $\mathbf{D}_2 = \mathbf{0}$, $\gamma = 1$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1$ where α_1 will be determined later, hence the encoders send

$$\begin{aligned} \mathbf{X}_1 &= [-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \\ \mathbf{X}_2 &= \mathbf{V}_2, \end{aligned} \tag{169}$$

where $\mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ carries the information of user 2; $\mathbf{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ is the dither signal. From the dithered quantization property (51), the transmitted signal has uniform distribution over $\mathcal{V}_1$, i.e., $\mathbf{X}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$. The receiver calculates $\mathbf{Y}' = [\alpha_1 \mathbf{Y} - \mathbf{D}_1] \bmod \Lambda_1$. The equivalent channel is given by

$$\begin{aligned} \mathbf{Y}' &= [\alpha_1(\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{S}_1 + \mathbf{Z}) - \mathbf{D}_1] \bmod \Lambda_1 \\ &= [\alpha_1 \mathbf{V}_2 - (1-\alpha_1)[-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1 - Q_{\Lambda_1}(-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] + \alpha_1 \mathbf{Z} - Q_{\Lambda_1}(-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] \bmod \Lambda_1 \\ &= [\alpha_1 \mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}] \bmod \Lambda_1, \end{aligned} \tag{170}$$

where $\mathbf{X}_1$ and $\mathbf{V}_2$ are independent and $\alpha_1 \mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V}_1)$. The scalar α_1 is determined to be the optimal MMSE factor, i.e., $\alpha_1 = \frac{P_1}{P_1+N} = \frac{P_1}{P_2}$, hence

$$E\Big\{[\alpha_1 \mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}]^2\Big\} = P_1.$$

For lattice Λ that is both good for quantization (46) and for AWGN channel coding (47), the rate achieved by user 2 is given by

$$R_2 = \frac{1}{n} I(\mathbf{V}_2; \mathbf{Y}') = \frac{1}{n} \left\{ h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_2)) \right\} \tag{171}$$

$$= \frac{1}{n} \left\{ h(\mathbf{Y}') - h([(1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}] \bmod \Lambda_1) \right\} \tag{172}$$

$$\geq \frac{1}{2} \log_2 \left(2\pi e P_1 \right) - \frac{1}{2} \log_2 \left(2\pi e \left((1-\alpha_1)^2 P_1 + {\alpha_1}^2 N \right) \right) - \epsilon \tag{173}$$

$$= \frac{1}{2} \log_2 \left(1 + \frac{P_1}{N} \right) - \epsilon. \tag{174}$$

where (173) follows since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment, and from Lemma 11 where $\epsilon \to 0$ for $n \to \infty$.

Therefore, for $P_2 = P_1 + N$ the inner bound meets the outer bound (37). Likewise for $P_2 \geq P_1 + N$, the outer bound (37) remains $\frac{1}{2} \log_2 \left(1 + \frac{P_1}{N}\right)$, thus the outer bound is also achievable.

We consider the case $P_1 = P_2 + N$. The same transmission scheme as in (169) is used, where now $\alpha_1 = \alpha_r = 1$. From (170), the equivalent channel is given by

$$\mathbf{Y}' = [\mathbf{V}_2 + \mathbf{Z}] \bmod \Lambda_1. \tag{175}$$

In this case we have that

$$E\{[\mathbf{V}_2 + \mathbf{Z}]^2\} = P_2 + N = P_1.$$

For lattice Λ that is both good for quantization (46) and for AWGN channel coding (47), the rate achieved by user 2 is given by

$$R_2 = \frac{1}{n} I(\mathbf{V}_2; \mathbf{Y}') = \frac{1}{n} \left\{ h(\mathbf{Y}') - h(\mathbf{Z} \bmod \Lambda_1) \right\} \tag{176}$$

$$\geq \frac{1}{2} \log_2 (2\pi e P_1) - \frac{1}{2} \log_2 (2\pi e N) - \epsilon \tag{177}$$

$$= \frac{1}{2} \log_2 \left(\frac{P_1}{N} \right) - \epsilon \tag{178}$$

$$= \frac{1}{2} \log_2 \left(1 + \frac{P_2}{N} \right) - \epsilon, \tag{179}$$

where (177) follows since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment, and from Lemma 11 where $\epsilon \to 0$ for $n \to \infty$.

Therefore for $P_1 = P_2 + N$, the inner bound meets the outer bound (37). Likewise for $P_1 \geq P_2 + N$, the outer bound (37) remains $\frac{1}{2} \log_2 \left(1 + \frac{P_2}{N}\right)$, thus the outer bound is also achievable.

APPENDIX VII

PROOF OF LEMMA 5 - NEARLY-BALANCED SNRS FOR THE MAC WITH A SINGLE DIRTY USER

Clearly, it is only required to prove the achievable rate inside the upper convex envelope operation (89), since the region including the upper convex envelope may be achieved using time sharing.

Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \sqrt{\frac{P_1}{P_2}} \Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \sqrt{\frac{P_1}{P_2}}$ and $\kappa_2 = 1$). The lattice Λ is both good for quantization (46) and good for AWGN channel coding (47). The second moments of the lattices Λ_1 and Λ_2 are P_1 and P_2, respectively. We set $\mathbf{V}_1 = \mathbf{0}$, $\mathbf{D}_2 = \mathbf{0}$, $\gamma = 1$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1$ where α_1 will be determined later, hence the encoders send

$$\begin{aligned} \mathbf{X}_1 &= [-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1 \\ \mathbf{X}_2 &= \mathbf{V}_2, \end{aligned} \tag{180}$$

where $\mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ carries the information of user 2; $\mathbf{D}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ is the dither signal. The receiver calculates $\mathbf{Y}' = [\alpha_1 \mathbf{Y} - \mathbf{D}_1] \bmod \Lambda_1$. The equivalent channel is given by

$$\begin{aligned} \mathbf{Y}' &= [\alpha_1 (\mathbf{x}_1 + \mathbf{X}_2 + \mathbf{S}_1 + \mathbf{Z}) - \mathbf{D}_1] \bmod \Lambda_1 \\ &= [\alpha_1 \mathbf{V}_2 - (1-\alpha_1)[-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1 - Q_{\Lambda_1}(-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] + \alpha_1 \mathbf{Z} - Q_{\Lambda_1}(-\alpha_1 \mathbf{S}_1 + \mathbf{D}_1)] \bmod \Lambda_1 \\ &= [\alpha_1 \mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}] \bmod \Lambda_1, \end{aligned} \tag{181}$$

The scalar α_1 is determined such that the second moment of $\alpha_1 \mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}$ will be P_1, hence $\alpha_1^2 (P_2 + N) + (1-\alpha_1)^2 P1 = P1$, where

$$\alpha_1 = \frac{2P1}{P_1 + P_2 + N} \triangleq \alpha_1^*. \tag{182}$$

For lattice Λ that is both good for quantization (46) and for AWGN channel coding (47), the rate achieved by user 2 is given by

$$R_2 = \frac{1}{n}I(\mathbf{V}_2;\mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h\left([(1-\alpha_1^*)\mathbf{X}_1 + \alpha_1^*\mathbf{Z}] \mod \Lambda_1\right)\right\} \tag{183}$$

$$\geq \frac{1}{2}\log_2(2\pi e P_1) - \frac{1}{2}\log_2\left(2\pi e((1-\alpha_1^*)^2 P_1 + \alpha_1^{*2} N)\right) + \epsilon \tag{184}$$

$$= \frac{1}{2}\log_2\left(\frac{P_1}{\frac{P_1(P_2-P_1+N)^2+4P_1^2N}{(P_1+P_2+N)^2}}\right) + \epsilon \tag{185}$$

$$= \frac{1}{2}\log_2\left(1 + \frac{4P_1P_2}{(P_2-P_1+N)^2+4P_1N}\right) + \epsilon \tag{186}$$

where (184) follows since modulo operation reduces the second moment and Gaussian distribution maximizes the entropy for fixed second moment, and from Lemma 11 where $\epsilon \to 0$ for $n \to \infty$.

Appendix VIII

Proof of Lemma 6 - A uniform outer bound for the gap η

For given P_1 and $P_1 \leq P_2$, the gap $\eta(P_1,P_2,N)$ is decreasing with respect to P_2. Therefore, $\eta(P_1,P_2,N) \leq \eta(P_1,P_1,N)$. In the same way, it can be shown that for given P_2 and $P_1 \geq P_2$, $\eta(P_1,P_2,N) \leq \eta(P_2,P_2,N)$. As a consequence, we have that

$$\eta(P_1,P_2,N) \leq \eta(P_{\min},P_{\min},N), \tag{187}$$

where $P_{\min} = \min(P_1,P_2)$.

Since the upper convex envelope in (78) can only decrease the gap, we have that

$$\eta(P_{\min},P_{\min},N) \leq \frac{1}{2}\log_2\left(1+\frac{P_{\min}}{N}\right) - \frac{1}{2}\log_2\left(1+\frac{4P_{\min}^2}{N^2+4P_{\min}N}\right) \tag{188}$$

$$\leq \max_{P_{\min},N}\frac{1}{2}\log_2\left(\frac{P_{\min}+N}{N}\cdot\frac{4P_{\min}N+N^2+4P_{\min}^2}{N^2+4P_{\min}N}\right) \tag{189}$$

$$= \max_{P_{\min},N}\frac{1}{2}\log_2\left(\frac{(P_{\min}+N)(4P_{\min}+N)}{(2P_{\min}+N)^2}\right) \tag{190}$$

$$= \max_{P_{\min},N}\frac{1}{2}\log_2\left(\frac{(1+P_{\min}/N)(1+4P_{\min}/N)}{(1+2P_{\min}/N)^2}\right). \tag{191}$$

The proof follows since the maximum of the function $f(x) = \frac{(1+x)(1+4x)}{(1+2x)^2}$ occurs at $x^* = 1/2$, and $f(x^*) = 9/8$.

Appendix IX

Proof of Lemma 8 - Capacity region of MAC with a single dirty user at high SNR

We consider here the case that $P_1 > P_2$. Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \sqrt{\frac{P_1}{P_2}}\Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \sqrt{\frac{P_1}{P_2}}$ and $\kappa_2 = 1$). The lattice Λ is both good for quantization (46) and good for AWGN channel coding (47). The second moments of the lattices

Λ_1 and Λ_2 are P_1 and P_2, respectively. We set $\mathbf{D}_1 = \mathbf{0}$, $\mathbf{D}_2 = \mathbf{0}$ $\gamma = 0$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1 = 1$, hence the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \mathbf{S}_1] \bmod \Lambda_1 \tag{192}$$

$$\mathbf{X}_2 = \mathbf{V}_2, \tag{193}$$

where $\mathbf{V}_1 \sim \mathrm{Unif}(\mathcal{V}_1)$ and $\mathbf{V}_2 \sim \mathrm{Unif}(\mathcal{V}_2)$ carry the information of user 1 and user 2, respectively. The receiver calculates $\mathbf{Y}' = \mathbf{Y} \bmod \Lambda_1$. The equivalent channel is given by

$$\mathbf{Y}' = [\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{Z} - Q_{\Lambda_1}(\mathbf{V}_1 - \mathbf{S}_1)] \bmod \Lambda_1 \tag{194}$$

$$= [\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{Z}] \bmod \Lambda_1. \tag{195}$$

The decoder uses successive decoding to reconstruct $\mathbf{V}_1$ and $\mathbf{V}_2$ in (195). First the decoder decodes $\mathbf{V}_1$ where $\mathbf{V}_2$ acts as a noise, in this case we get that

$$R_1 = \frac{1}{2}\log_2\left(\frac{P_1}{P_2 + N}\right),$$

is achievable. Then, the decoder subtracts the reconstruction of $\mathbf{V}_1$ and reduces the result modulo-Λ_2, in this case the equivalent channel is given by

$$\mathbf{Y}'' = [\mathbf{V}_2 + \mathbf{Z}] \bmod \Lambda_2.$$

Hence, we get that

$$R_2 = \frac{1}{2}\log_2\left(\frac{P_2}{N}\right),$$

is achievable.

Clearly at high SNR, i.e., for $P_1, P_2 \gg N$, this achievable rate pair coincides with the point (R_1^c, R_2^c) (36). From Lemma 7, the rate pair $(0, R_2) = (0, \frac{1}{2}\log_2\left(\frac{P_1}{N}\right) - o(1))$ is also achievable at high SNR. Likewise, the point $(R_1, 0) = (\frac{1}{2}\log_2\left(1 + \frac{P_1}{N}\right), 0)$ is achievable for any SNR. The theorem follows since the region defined by the time sharing between these three points coincides with the outer bound (34) at high SNR .

Appendix X
Proof of Lemma 9 - Achievable rate region of MAC with a single dirty user

Using the lattice-alignment transmission scheme of Section V-B, Λ_1 and Λ_2 are scaled lattices, i.e., $\Lambda_1 = \Lambda_r = \sqrt{\frac{P_1}{P_2}}\Lambda$ and $\Lambda_2 = \Lambda$ (that is $\kappa_1 = \kappa_r = \sqrt{\frac{P_1}{P_2}}$ and $\kappa_2 = 1$). The lattice Λ is both good for quantization (46) and good for AWGN channel coding (47). The second moments of the lattices Λ_1 and Λ_2 are P_1 and P_2, respectively. We set $\mathbf{D}_2 = \mathbf{0}$, $\gamma = 1$, $\alpha_2 = 0$, $\beta = 0$ and $\alpha_r = \alpha_1$ where α_1 will be determined later, hence the encoders send

$$\mathbf{X}_1 = [\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1] \bmod \Lambda_1$$

$$\mathbf{X}_2 = \mathbf{V}_2, \tag{196}$$

where $\mathbf{V}_1 \sim \text{Unif}(\mathcal{V}_1)$ and $\mathbf{V}_2 \sim \text{Unif}(\mathcal{V}_2)$ are independent and carry the information of user 1 and user 2 respectively; $\mathbf{D}_1 \sim \text{Unif}(\mathcal{V}_1)$ is the dither signal. The receiver calculates $\mathbf{Y}' = [\alpha_1 \mathbf{Y} - \mathbf{D}_1] \bmod \Lambda_1$. The equivalent channel is given by

$$\begin{aligned}
\mathbf{Y}' &= [\alpha_1(\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{S}_1 + \mathbf{Z}) - \mathbf{D}_1] \mod \Lambda_1 \\
&= [\mathbf{V}_1 + \alpha_1\mathbf{V}_2 - (1-\alpha_1)[\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1 - Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1)] + \alpha_1\mathbf{Z} - Q_{\Lambda_1}(\mathbf{V}_1 - \alpha_1\mathbf{S}_1 + \mathbf{D}_1)] \mod \Lambda_1 \\
&= [\mathbf{V}_1 + \alpha_1\mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \mod \Lambda_1,
\end{aligned}$$

The rate achieved by user 1 is given by

$$\begin{aligned}
R_1 &= \frac{1}{n} I(\mathbf{V}_1; \mathbf{Y}') = \frac{1}{n}\left\{h(\mathbf{Y}') - h(\mathbf{Y}'|\mathbf{V}_1))\right\} && (197)\\
&= \frac{1}{n}\left\{h(\mathbf{Y}') - h([\alpha_1\mathbf{V}_2 + (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \bmod \Lambda_1)\right\} && (198)\\
&\geq \frac{1}{n}\left\{h(\mathbf{Y}') - \min\left\{\frac{1}{2}\log_2(2\pi e P_1), h(\alpha_1\mathbf{V}_2 + (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z})\right\}\right\} && (199)\\
&\geq \frac{1}{2}\log_2\left(\frac{P_1}{G(\Lambda_1)}\right) - \frac{1}{2}\log_2\left(2\pi e \cdot \min\left\{P_1, \alpha_1^2 P_2 + (1-\alpha_1)^2 P_1 + \alpha_1^2 N\right\}\right) && (200)\\
&= \frac{1}{2}\log_2\left(\frac{P_1}{\min\left\{P_1, \alpha_1^2 P_2 + (1-\alpha_1)^2 P_1 + \alpha_1^2 N\right\}}\right) - \frac{1}{2}\log_2\left(2\pi e G(\Lambda_1)\right), && (201)
\end{aligned}$$

where (199) follows since $h(\mathbf{U} \bmod \Lambda_1) \leq \min(\frac{n}{2}\log_2(2\pi e P_1), h(\mathbf{U}))$ for any random vector $\mathbf{U}$; (200) follows since $\mathbf{Y}' \sim \text{Unif}(\mathcal{V}_1)$ thus $h(\mathbf{Y}') = \frac{1}{2}\log_2\left(\frac{P_1}{G_n(\Lambda_1)}\right)$, and since Gaussian distribution maximizes the entropy for fixed variance. Since lattice Λ is good for quantization, i.e., $G(\Lambda) \to 1/2\pi e$ as $n \to \infty$, we get that any rate

$$R_1 \leq \frac{1}{2}\log_2\left(\frac{P_1}{\min\left\{P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2(N + P_2)\right\}}\right) \tag{202}$$

is achievable. Since $\mathbf{V}_1$ is reconstructed at the decoder with high probability, we can subtract $\hat{\mathbf{V}}_1$ from $\mathbf{Y}'$. i.e

$$\begin{aligned}
\tilde{\mathbf{Y}} &= [\mathbf{Y}' - \hat{\mathbf{V}}_1] \bmod \Lambda_1 && (203)\\
&= [\alpha_1\mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \mod \Lambda_1. && (204)
\end{aligned}$$

In order to reconstruct $\mathbf{V}_2$, the receiver calculates $\mathbf{Y}'' = [\tilde{\mathbf{Y}}] \bmod \Lambda'_r$, where the lattice Λ'_r has a second moment $\rho^2 P_1$, and $\rho = \sqrt{\frac{\min(P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2(N+P_2))}{P_1}}$. The lattice Λ_1 is a sub-lattice of Λ'_r, i.e., Λ_1 and Λ'_r are nested lattices. The equivalent channel is given by

$$\mathbf{Y}'' = [\alpha_1\mathbf{V}_2 - (1-\alpha_1)\mathbf{X}_1 + \alpha_1\mathbf{Z}] \mod \Lambda'_r. \tag{205}$$

Since the lattice Λ is both good for quantization (46) and good for AWGN channel coding (47), hence Λ'_r is both good for quantization and for AWGN channel coding as well. Therefore, the rate achieved by user 2 is given by

$$R_2 = \frac{1}{n} I(\mathbf{V}_2; \mathbf{Y}'') = \frac{1}{n} \left\{ h(\mathbf{Y}'') - h(\mathbf{Y}''|\mathbf{V}_2)) \right\} \tag{206}$$

$$= \frac{1}{n} \left\{ h(\mathbf{Y}'') - h([(1-\alpha_1)\mathbf{X}_1 + \alpha_1 \mathbf{Z}] \bmod \Lambda'_r) \right\} \tag{207}$$

$$\geq \frac{1}{n} h(\mathbf{Y}'') - \frac{1}{2} \log_2 \left(2\pi e \left((1-\alpha_1)^2 P_1 + \alpha_1^2 N \right) \right) \tag{208}$$

$$\geq \frac{1}{2} \log_2 \left(2\pi e \cdot \min(P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2 (P_2 + N)) \right) - \frac{1}{2} \log_2 \left(2\pi e \left((1-\alpha_1)^2 P_1 + \alpha_1^2 N \right) \right) - \epsilon \tag{209}$$

$$= \frac{1}{2} \log_2 \left(\frac{\min(P_1, (1-\alpha_1)^2 P_1 + \alpha_1^2 (P_2 + N))}{(1-\alpha_1)^2 P_1 + \alpha_1^2 N} \right) - \epsilon, \tag{210}$$

where (209) follows from Lemma 11, and $\epsilon \to 0$ as $n \to \infty$.

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