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Lattice Substitution Systems and Model Sets

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Learning is but an adjunct to ourself
And where we are our learning likewise is.
W. Shakespeare

Abstract. This paper studies ways in which the sets of a partition of a lattice in $\mathbb{R}^n$ become regular model sets. The main theorem gives equivalent conditions which assure that a matrix substitution system on a lattice in $\mathbb{R}^n$ gives rise to regular model sets (based on p-adic-like internal spaces), and hence to pure point diffractive sets. The methods developed here are used to show that the n-dimensional chair tiling and the sphinx tiling are pure point diffractive.

1. Introduction

There have been two very successful approaches to building discrete mathematical structures with long-range aperiodic order. These are the substitution methods, notably symbolic substitutions and tiling substitutions, and the cut and project method. In the first case the structure is typically generated by successive substitution from a finite starting configuration. In the second it typically appears in one shot as the (partial) projection of a periodic structure in some "higher" dimensional embedding space [1].

The principal focus in this paper is the relationship between matrix substitution systems on a *lattice* and naturally related cut and project formalisms. We start with a partition of a lattice L in $\mathbb{R}^n$ into a finite number of point sets $\tilde{U} = (U_1, \ldots, U_m)$ and a finite set of substitution rules Φ which are affine inflations and under which $\tilde{U}$ is invariant. The main theorem (Theorem 3) provides conditions on Φ which are equivalent to $U_1, \ldots, U_m$ being regular model sets (i.e. cut and project sets). One of the characterizations (modular coincidence) affords a simple computational approach to testing for model sets. In a later section we go beyond the context of substitution systems and provide an alternative characterization (Theorem 4) of model sets. We use both types of characterization in showing that the sphinx and n-dimensional chair tilings can be realized as model sets.

The connection between substitution systems and cut and project sets is nothing new, e.g. the Fibonacci chain is often described in terms of a cut by a strip through $\mathbb{Z}^2$, and the klotz construction of [2] is a sophisticated elaboration of the same idea. Nonetheless, substitution systems and cut and project sets are not different formulations of the same thing, and the relationship between them remains inadequately understood.

In the early study of aperiodic order, the cut and project formalism was always based on projection into $\mathbb{R}^n$ from a lattice in some higher space $\mathbb{R}^n \times \mathbb{R}^p$, the projection being controlled by a compact set $W \subset \mathbb{R}^p$. However, it was already implicit in the much earlier work of Meyer [14] that $\mathbb{R}^p$ can be replaced by any locally compact abelian group H and $W \subset H$ by any compact set with non-empty interior, and the projection method still produces discrete aperiodic sets with diffractive properties (hence long-range order). Meyer's terminology for such sets was "model sets" and we use it here in deference to its priority and to emphasize the greater generality of the internal space H. Model sets have been studied in detail in [12] and [15]–[19]. The relevance of more general internal spaces to tiling theory and symbolic substitutions was made explicit in [5] where p-adic and mixed p-adic and real spaces naturally appear.

One of the important features of making the connection to model sets is that once it is established, pure point diffractivity is assured (see Theorem 2 for a precise statement of this). This type of information is generally quite difficult to obtain. For example, our results shown here prove that the n-dimensional chair tiling and the two-dimensional sphinx tiling are pure point diffractive. The former has been established for $n = 2$ previously [5], [20]. The latter is claimed in [20] as being provable by a geometric form of "coincidence" established there (see below for more on the concept of coincidence).

The p-adic type internal spaces occur when the aperiodic set in question is based on the points of a lattice and its sublattices in $\mathbb{R}^n$. An important class of examples of this type arises from the *equal length* symbolic substitution systems. Suppose that $A = \{a_1, \dots, a_m\}$ is a finite alphabet with associated monoid of words A^*, and we are given a primitive substitution $\sigma\colon A \longrightarrow A^*$ for which the length l of each of the words $\sigma(a_i)$ is the same. This substitution leads to a tiling of $\mathbb{R}$ of tiles of equal length, say equal to 1. Matching the coordinate of the left end of each tile with its tile type a_i, we obtain a partition $U_1 \cup \cdots \cup U_m$ of $\mathbb{Z}$, and σ may be viewed as comprised of a set of affine mappings of the form $x \mapsto lx + v$ where $v \in \mathbb{Z}$.

A lot more is known about equal length substitutions than arbitrary ones, a particularly important example of this being Dekking's criterion for diffraction [6]. An equal length aperiodic tiling has pure point dynamical spectrum if and only if it admits a coincidence (σ is said to admit a coincidence if there is a k, $1 \leq k \leq l^n$, for which the kth letter of each word $\sigma^n(a_i)$ for some n is the same).

In this paper we prove a related result, but this time the dimension is not restricted. Namely, there is again a notion of coincidence (in fact there are two such notions) and either of these is equivalent to the sets $U_1, \dots, U_m$ being regular model sets. One of the criteria for coincidence that we give is a straightforward algorithm and thus in principle is computable.

As we have already pointed out, a consequence of our result is that coincidence implies the pure point diffractivity of each of the sets $U_1, \dots, U_m$; that is to say, the Fourier transform of the volume averaged autocorrelation is a pure point measure. We do not know yet to what extent the condition is equivalent to pure point diffractivity.

The setting of the paper is entirely at the level of point sets, so necessarily the strong conditions implicit in the tiling situation are replaced here by a corresponding algebraic condition on the matrix substitution system: the Perron–Frobenius eigenvalue of the substitution matrix should equal its inflation constant. This is in fact a compatibility condition which is necessary for the model set connection to exist in our situation. This condition, not surprisingly, has occurred elsewhere in the literature (see, for instance, [13] and [20]). The important result that gets the process off the ground is Theorem 1, which is largely due to Martin Schlottmann and uses ideas from [10].

Matrix substitution systems, treated at the level of point sets, have recently appeared in [13] under the name of self-replicating Delone sets. In that paper, point sets X are not restricted to lattices and the principal question revolves around the interesting question of the existence of tilings of $\mathbb{R}^n$ by translations of certain prototiles for which the points of X are the appropriate translational vectors. Also related to our paper is the study of sets of affine mappings in the context of lattice tilings (see [21] for a nice recent survey on this). In relation to our paper, the situation there corresponds to the 1×1 matrix substitution systems and the problems become entirely different. Since the tilings there are lattice tilings, the whole issue of model sets and diffraction is trivial, and the issues lie more around the complex nature of the tiles themselves.

2. Definitions and Notation

Let X be a non-empty set. For $m \in \mathbb{Z}_+$, an $m \times m$ *matrix function system* (MFS) on X is an $m \times m$ matrix $\Phi = (\Phi_{ij})$, where each Φ_{ij} is a set (possibly empty) of mappings X to X.

The corresponding matrix $S(\Phi) := (\mathrm{card}(\Phi_{ij}))_{ij}$ is called the *substitution matrix* of Φ. The MFS is *primitive* if $S(\Phi)$ is primitive, i.e. there is an $l > 0$ for which $S(\Phi)^l$ has no zero entries.

In this paper we deal only with MFSs which are *finite* in the sense that $\mathrm{card}(\Phi_{ij}) < \infty$ for all i, j. Of particular importance are the Perron–Frobenius (PF) eigenvalue and the corresponding PF-eigenvector (unique up to a scalar factor) of $S(\Phi)$. We will also have use for the *incidence matrix* $I(\Phi)$ of Φ, which is defined by

$$(I(\Phi))_{ij} = \begin{cases} 1 & \text{if} \quad \mathrm{card}(\Phi_{ij}) \neq 0, \\ 0 & \text{else.} \end{cases}$$

Let $P(X)$ be the set of subsets of X. Any MFS induces a mapping on $P(X)^m$ by

$$\Phi \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} \bigcup_j \bigcup_{f \in \Phi_{1j}} f(U_j) \\ \vdots \\ \bigcup_j \bigcup_{f \in \Phi_{mj}} f(U_j) \end{bmatrix}, \tag{1}$$

which we call the *substitution* determined by Φ. We sometimes write $\Phi_{ij}(U_j)$ to mean $\bigcup_{f \in \Phi_{ij}} f(U_j)$.

In what follows, X is a lattice L in $\mathbb{R}^n$ and the mappings of Φ are always affine linear mappings of the form $x \mapsto Qx + a$, where $Q \in \mathrm{End}_{\mathbb{Z}}(L)$ is the *same* for all the maps. Such maps extend to $\mathbb{R}^n$.

Let Φ, Ψ be $m \times m$ MFSs on X. Then we can compose them:

$$\Psi \circ \Phi = ((\Psi \circ \Phi)_{ij}), \tag{2}$$

where

$$(\Psi\circ\Phi)_{ij} = \bigcup_{k=1}^{m} \Psi_{ik}\circ\Phi_{kj} \qquad \text{and} \qquad \Psi_{ik}\circ\Phi_{kj} := \begin{cases} \{g \circ f \mid g \in \Psi_{ik}, f \in \Phi_{kj}\}, \\ \emptyset \quad \text{if} \quad \Psi_{ik} = \emptyset \quad \text{or} \quad \Phi_{kj} = \emptyset. \end{cases}$$

Evidently, $S(\Psi \circ \Phi) \leq S(\Psi)S(\Phi)$ (see (10) for the definition of the partial order).

For an $m \times m$ MFS Φ, we say that $\tilde{U} := [U_1, \ldots, U_m]^T \in P(X)^m$ is a *fixed point* of Φ if $\Phi\tilde{U} = \tilde{U}$.

An affine lattice substitution system on L with inflation Q is a pair $(\tilde{U}, \Phi)$ consisting of disjoint subsets $\{U_i\}_{i=1}^m$ of L and an $m \times m$ MFS Φ on L for which $\tilde{U} = [U_1, \ldots, U_m]^T$ is a fixed point of Φ, i.e.

$$U_i = \bigcup_{j=1}^{m} \bigcup_{f\in\Phi_{ij}} f(U_j), \qquad i = 1, \ldots, m, \tag{3}$$

where the maps of Φ are affine mappings of the form $x \mapsto Qx + a$, $a \in L$, and in which the unions in (3) are *disjoint.*[1] In this paper all our matrix substitution systems are composed of affine mappings on a lattice and we often drop the words "affine lattice," speaking simply of substitution systems.

We say that the substitution system $(\tilde{U}, \Phi)$ is *primitive* if Φ is primitive. A second substitution system $(\tilde{U}', \Psi)$ is called *equivalent* to $(\tilde{U}, \Phi)$ if $\tilde{U}' = \tilde{U}$, Ψ and Φ have the same inflation, and $S(\Psi)$, $S(\Phi)$ have the same PF-eigenvalue and right PF-eigenvector (up to scalar factor).

For any affine mapping $f\colon x \mapsto Qx + b$ on L we denote the translational part, b, of f by $t(f)$. We say that $f, g \in \Phi$ are *congruent* mod QL if $t(f) \equiv t(g) \bmod QL$. This equivalence relation partitions Φ into congruence classes. For $a \in L$, $\Phi[a] := \{f \in \bigcup_{i,j} \Phi_{ij} \mid t(f) \equiv a \bmod QL\}$.

We say that Φ *admits a coincidence* if there is an i, $1 \leq i \leq m$, for which $\bigcap_{j=1}^m \Phi_{ij} \neq \emptyset$, i.e. the same map appears in every set of the ith row for some i. Furthermore, if $\Phi[a]$ is contained entirely in one row of the MFS (Φ) for some $a \in L$, then we say that $(\tilde{U}, \Phi)$ admits a *modular coincidence*.

3. Substitution Systems on Lattices and Properties

Let L be a lattice in $\mathbb{R}^n$. A mapping $Q \in \mathrm{End}_{\mathbb{Z}}(L)$ is an *inflation* for L if $\det Q \neq 0$ and

$$\bigcap_{k=0}^{\infty} Q^k L = \{0\}. \tag{4}$$

[1] In the case that one has unions (3) which are not disjoint there arises the natural question of the mulitplicities of points, or more generally densities of points which are invariant under the substitution. For more on this see [3] and [13].

Let Q be an inflation. Then $q := |\det Q| = [L : QL] > 1$. We define the Q-adic completion

$$\overline{L} = \overline{L_Q} = \lim_{\leftarrow k} L/Q^k L \tag{5}$$

of L. $\overline{L}$ will be supplied with the usual topology of a profinite group. In particular, the cosets $a + Q^k\overline{L}$, $a \in L$, $k = 0, 1, 2, \dots$, form a basis of open sets of $\overline{L}$ and each of these cosets is both open and closed. When we use the word *coset* in this paper, we mean either a coset of the form $a + Q^k\overline{L}$ in $\overline{L}$ or $a + Q^k L$ in L, according to the context. An important observation is that any two cosets in $\overline{L}$ are either disjoint or one is contained in the other. The same applies to cosets of L.

We let μ denote the Haar measure on $\overline{L}$, normalized so that $\mu(\overline{L}) = 1$. Thus for cosets,

$$\mu(a + Q^k\overline{L}) = \frac{1}{|\det Q|^k} = \frac{1}{q^k}. \tag{6}$$

We also have need of the metric d on $\overline{L}$ defined via the standard metric:

$$\|x\| := \frac{1}{q^k} \qquad \text{if} \quad x \in Q^k\overline{L}\backslash Q^{k+1}\overline{L}, \qquad \|0\| = 0. \tag{7}$$

From $\bigcap_{k=0}^{\infty} Q^k L = \{0\}$, we conclude that the mapping $x \mapsto \{x \bmod Q^k L\}_k$ embeds L in $\overline{L}$. We identify L with its image in $\overline{L}$. Note that $\overline{L}$ is the closure of L, whence the notation.

Let $(\tilde{U}, \Phi)$ be a substitution system on L. Identifying L as a dense subgroup of $\overline{L}$, we have a unique extension of Φ to an MFS on $\overline{L}$ in the obvious way. Thus if $f \in \Phi_{ij}$ and $f\colon x \mapsto Qx + a$, then this formula defines a mapping on $\overline{L}$, to which we give the same name. Note that f is a *contraction* on $\overline{L}$, since $\|Qx\| = (1/q)\|x\|$ for all $x \in \overline{L}$. Thus Φ determines a multi-component iterated function system on $\overline{L}$. Furthermore, defining the compact subsets

$$W_i := \overline{U_i}, \qquad i = 1, \dots, m, \tag{8}$$

and using (3) and the continuity of the mapping, we have

$$W_i = \bigcup_{j=1}^{m} \bigcup_{f \in \Phi_{ij}} f(W_j), \qquad i = 1, \dots, m, \tag{9}$$

which shows that $\tilde{W} = [W_1, \dots, W_m]^T$ is the unique *attractor* of Φ (see [3] and [9]).

We call $(\tilde{W}, \Phi)$ the *associated Q-adic system*. We cannot expect in general that the decomposition in (9) will be disjoint, so we do *not* call $(\tilde{W}, \Phi)$ a substitution system.

For $X, Y \in \mathbb{R}^n$, we write

$$\begin{aligned} X \le Y &\quad \text{if} \quad X_i \le Y_i \quad \text{for all} \quad 1 \le i \le n,\\ X < Y &\quad \text{if} \quad X_i < Y_i \quad \text{for all} \quad 1 \le i \le n. \end{aligned}$$

Similarly, for $A, B \in M_n(\mathbb{R})$,

$$\begin{aligned} A \le B &\quad \text{if} \quad A_{ij} \le B_{ij} \quad \text{for all} \quad 1 \le i, j \le n,\\ A < B &\quad \text{if} \quad A_{ij} < B_{ij} \quad \text{for all} \quad 1 \le i, j \le n. \end{aligned} \tag{10}$$

We begin by recalling a couple of results from the Perron–Frobenius theory.

Lemma 1. *Let A be a non-negative primitive matrix with PF-eigenvalue λ. If $0 \le \lambda X \le AX$, then $AX = \lambda X$.*

Proof. We can assume $X \ne 0$. Since $0 \le \lambda X$ and $\lambda > 0$, $X \ge 0$. Let $X' > 0$ be a PF right-eigenvector of A. Let $\alpha = \max\{(X_i/X_i') \mid 1 \le i \le m\}$. Then $X \le \alpha X'$ and X is not strictly less than $\alpha X'$. Claim $X = \alpha X'$. If $X \ne \alpha X'$, then $0 < A^N(\alpha X' - X) = \alpha\lambda^N X' - A^N X$ for some N, since A is primitive. So $\lambda^N X \le A^N X < \alpha\lambda^N X'$, i.e. $X < \alpha X'$. This is a contradiction. Therefore $AX = \lambda X$. □

Lemma 2. *Let λ be the PF-eigenvalue of the non-negative primitive matrix A and let μ be an eigenvalue of a matrix B where $0 \le B \le A$. If $A \ne B$, then $|\mu| < \lambda$.*

Proof. Let Y be a right eigenvector for eigenvalue μ of B, with $Y = [Y_1, \ldots, Y_m]^T$. Let $\overline{Y} = [|Y_1|, \ldots, |Y_m|]^T \ne 0$. Then $|\mu|\overline{Y} \le B\overline{Y} \le A\overline{Y}$. Let $\overline{X}^T$ be a positive left eigenvector for A with PF-eigenvalue λ. So $|\mu|\overline{X}^T\overline{Y} \le \overline{X}^T B\overline{Y} \le \overline{X}^T A\overline{Y} = \lambda\overline{X}^T\overline{Y}$. This shows that $|\mu| \le \lambda$. If $|\mu| = \lambda$, then $\lambda\overline{Y} \le A\overline{Y}$. By Lemma 1, $\lambda\overline{Y} = A\overline{Y}$. Since A is a primitive matrix, $\lambda^m\overline{Y} = A^m\overline{Y} > 0$ for some m. So $\overline{Y} > 0$. From $\lambda\overline{Y} \le B\overline{Y} \le A\overline{Y} = \lambda\overline{Y}$, we have $A\overline{Y} = B\overline{Y}$. Therefore $A = B$. □

Lemma 3. *Let $(\tilde{U}, \Phi)$ be a primitive substitution system. Then for all $l \in \mathbb{N}$, $(\tilde{U}, \Phi^l)$ is a primitive substitution system.*

Proof. Let $i, j, k \in \{1, 2, \ldots, m\}$. All the maps $g \in \Phi_{ik}$ have domain U_k and disjoint images in U_i. Moreover, all the mappings g are injective. Likewise all the maps f of Φ_{kj} have domain U_j and disjoint images in U_k. Thus all the maps $g \circ f \in \Phi_{ik} \circ \Phi_{kj}$ have domain U_j and disjoint images in U_i. Furthermore, $\Phi^2\tilde{U} = \Phi(\Phi\tilde{U}) = \Phi(\tilde{U}) = \tilde{U}$. So $(\tilde{U}, \Phi^2)$ is a substitution system. The argument extends in the same way to $(\tilde{U}, \Phi^l)$. The statement on primitivity is clear. □

Theorem 1. *Let $(\tilde{U}, \Phi)$ be a primitive substitution system with inflation Q on L. Let $(\tilde{W}, \Phi)$ be the corresponding associated Q-adic system. Suppose that the PF-eigenvalue of $S(\Phi)$ is $|\det Q|$ and $\overline{L} = \bigcup_{i=1}^m W_i$. Then*

(i) $S(\Phi^r) = (S(\Phi))^r, r \ge 1$;
(ii) $\mu(W_i) = (1/q^r)\sum_{j=1}^m (S(\Phi^r))_{ij}\mu(W_j)$, *for all* $i = 1, \ldots, m, r \ge 1$;
(iii) *for all* $i = 1, \ldots, m$, $\overset{\circ}{W_i} \ne \emptyset$ *and* $\mu(\partial W_i) = 0$.

Proof. For every measurable set $E \subset \overline{L}$ and all $f \in \Phi_{ij}$,

$$\mu(f(E)) = \mu(Q(E) + a) = \frac{1}{|\det Q|}\mu(E),$$

where $f\colon x \mapsto Qx + a$. In particular, $\mu(f(W_j)) = (1/q)w_j$, where $w_j := \mu(W_j)$ and $q = |\det Q|$. We obtain

$$w_i \le \sum_{j=1}^{m} \frac{1}{q^r}\,\mathrm{card}((\Phi^r)_{ij})w_j$$

from (9).

Let $w = [w_1, \ldots, w_m]^T$. Since $\bigcup_{i=1}^m W_i = \overline{L}$, the Baire category theorem assures us that for at least one i,

$$\overset{\circ}{W}_i \ne \emptyset \tag{11}$$

and then the primitivity gives this for all i. So $w > 0$ and

$$w \le \frac{1}{q^r} S(\Phi^r) w \le \frac{1}{q^r} S(\Phi)^r w, \qquad \text{for any} \quad r \ge 1. \tag{12}$$

Since the PF-eigenvalue of $S(\Phi)^r$ is $q^r = |\det Q|^r$ and $S(\Phi)^r$ is primitive, we have from Lemma 1 that

$$w = \frac{1}{q^r} S(\Phi^r) w = \frac{1}{q^r} S(\Phi)^r w, \qquad \text{for any} \quad r \ge 1. \tag{13}$$

The positivity of w together with $S(\Phi^r) \le S(\Phi)^r$ shows that $S(\Phi^r) = S(\Phi)^r$. This proves (i) and (ii).

Fix any $i \in \{1, \ldots, m\}$, let $\overset{\circ}{W}_i$ contain a basis open set $a + Q^r\overline{L}$ with some $r \in \mathbb{Z}_{\ge 0}$ by (11). Since $(\tilde{U}, \Phi^r)$ is a substitution system, $a + Q^r\overline{L} \subset \overset{\circ}{W}_i \subset W_i = \bigcup_{j=1}^m (\Phi^r)_{ij} W_j$. In particular, $(a + Q^r\overline{L}) \cap g(W_k) \ne \emptyset$ for some $k \in \{1, \ldots, m\}$ and some $g \in (\Phi^r)_{ik}$. However, $g(\overline{L}) = b + Q^r\overline{L}$ for some $b \in L$, so $(a + Q^r\overline{L}) \cap (b + Q^r\overline{L}) \ne \emptyset$. This means $a + Q^r\overline{L} = b + Q^r\overline{L}$. Thus

$$g(W_k) \subset g(\overline{L}) = a + Q^r\overline{L} \subset \overset{\circ}{W}_i\,. \tag{14}$$

For all $f \in (\Phi^r)_{ij}$, $j \in \{1, 2, \ldots, m\}$, f is clearly an open map, so $\bigcup_{j=1}^m (\Phi^r)_{ij}(\overset{\circ}{W}_j) \subset \overset{\circ}{W}_i$. Thus

$$\begin{aligned} \partial W_i = W_i \setminus \overset{\circ}{W}_i &= \left(\bigcup_{j=1}^m (\Phi^r)_{ij}(W_j)\right) \Big\backslash\, \overset{\circ}{W}_i\,. \\ &\subset \bigcup_{j=1}^m ((\Phi^r)_{ij}(W_j) \setminus (\Phi^r)_{ij}(\overset{\circ}{W}_j)) \\ &\subset \bigcup_{j=1}^m (\Phi^r)_{ij}(\partial W_j). \end{aligned} \tag{15}$$

Note that due to (14) at least one g in $(\Phi^r)_{ik}$ does not contribute to the relation (15).

Let $v_i := \mu(\partial W_i)$, $i = 1, \ldots, m$ and $v := [v_1, \ldots, v_m]^T$. So $v \le (1/q^r)S(\Phi^r)v$. Actually, by what we just said,

$$0 \le v \le \frac{1}{q^r} S' v \le \frac{1}{q^r} S(\Phi^r) v = \frac{1}{q^r} S(\Phi)^r v, \tag{16}$$

where $S' \leq S(\Phi)^r$, $S' \neq S(\Phi)^r$. Now applying Lemma 1 again we obtain equality throughout (16). However, by Lemma 2 the eigenvalues of $(1/q^r)S'$ are strictly less in absolute value than the PF-eigenvalue of $(1/q^r)S(\Phi)^r$, which is 1. This forces $v = 0$, and hence $\mu(\partial W_i) = 0$, $i = 1, \ldots, m$. □

In what follows, the central concern is to relate the sets U_i and the sets $\Lambda_i := W_i \cap L$. Clearly, $\Lambda_i \supset U_i$. The next lemma groups a circle of ideas that relate this question to the boundaries and interiors of the W_i.

Lemma 4. *Let $U_i, i = 1, \ldots, m$, be point sets of the lattice L in $\mathbb{R}^n$. Let Q be an inflation of L and identify L with its image in its Q-adic completion $\overline{L}$. Define $W_i := \overline{U_i}$ in $\overline{L}$ and $\Lambda_i := W_i \cap L$.*

(i) *If $U_1, \ldots, U_m$ are disjoint and $\mu(\overline{\Lambda_i \backslash U_i}) = 0$ for all $i = 1, \ldots, m$, then $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} = \emptyset$ for all $i \neq j$.*

(ii) *If $L = \bigcup_{i=1}^m U_i$ and $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} = \emptyset$ for all $i \neq j$, where $i, j \in \{1, \ldots, m\}$, then $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$ for all $i = 1, \ldots, m$.*

(iii) *If $\mu(\partial W_j) = 0$ for all $j = 1, \ldots, m$ and $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$, then $\mu(\overline{\Lambda_i \backslash U_i}) = 0$.*

Proof. (i) Suppose there are $i, j \in \{1, \ldots, m\}$ with $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} \neq \emptyset$. We can choose $a \in (\overset{\circ}{W_i} \cap \overset{\circ}{W_j}) \cap L$, since L is dense in $\overline{L}$ and $\overset{\circ}{W_i} \cap \overset{\circ}{W_j}$ is open. Choose $k \in \mathbb{Z}_+$ so that $a + Q^k\overline{L} \subset \overset{\circ}{W_i} \cap \overset{\circ}{W_j}$. Note that $a + Q^kL \subset \Lambda_i \cap \Lambda_j$. Then

$$\begin{aligned}\bigcup_{l=1}^{m}(\Lambda_l \backslash U_l) &\supseteq \big((a + Q^kL)\backslash U_i\big) \cup \big((a + Q^kL)\backslash U_j\big)\\ &= (a + Q^kL)\backslash(U_i \cap U_j)\\ &= a + Q^kL, \qquad \text{since the } U_i, i = 1, \ldots, m, \text{ are disjoint.}\end{aligned}$$

So

$$\begin{aligned}\sum_{i=1}^{m} \mu(\overline{\Lambda_i \backslash U_i}) &\geq \mu\left(\bigcup_{i=1}^{m}(\overline{\Lambda_i \backslash U_i})\right)\\ &\geq \mu(a + Q^k\overline{L})\\ &> 0,\end{aligned}$$

contrary to assumption.

(ii) Assume $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} = \emptyset$ for all $i \neq j$. For any $i \in \{1, \ldots, m\}$,

$$\begin{aligned}(\Lambda_i \backslash U_i) &\subset \left(\bigcup_{j \neq i} U_j\right) \cap W_i \qquad \left(\text{since } L = \bigcup_{i=1}^{m} U_i\right)\\ &\subset \bigcup_{j \neq i}(W_j \cap W_i) \subset \bigcup_{j=1}^{m} \partial W_j \qquad (\text{since } \overset{\circ}{W_i} \cap \overset{\circ}{W_j} = \emptyset \quad \text{for all } i \neq j).\end{aligned}$$

(iii) Obvious. □

4. Model Sets

We recall the notion of a model set (or cut and project set) [17]. A *cut and project scheme* (CPS) consists of a collection of spaces and mappings as follows:

$$\begin{array}{ccccc} \mathbb{R}^n & \stackrel{\pi_1}{\longleftarrow} & \mathbb{R}^n \times G & \stackrel{\pi_2}{\longrightarrow} & G \\ & & \cup & & \\ & & \tilde{L} & & \end{array} \tag{17}$$

where $\mathbb{R}^n$ is a real Euclidean space, G is some locally compact Abelian group, and $\tilde{L} \subset \mathbb{R}^n \times G$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^n \times G)/\tilde{L}$ is compact. Furthermore, we assume that $\pi_1|_{\tilde{L}}$ is injective and $\pi_2(\tilde{L})$ is dense in G.

A *model set* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which, up to translation, is of the form $\Lambda(V) = \{\pi_1(x) \mid x \in \tilde{L}, \pi_2(x) \in V\}$ for some cut and project scheme as above, where $V \subset G$ has non-empty interior and compact closure (relatively compact). When we need to be more precise we explicitly mention the cut and project scheme from which a model set arises. This is quite important in some of the theorems below. Model sets are always Delone subsets of $\mathbb{R}^n$, that is to say, they are relatively dense and uniformly discrete.

We call the model set $\Lambda(V)$ *regular* if the boundary $\partial V = \overline{V} \backslash \overset{\circ}{V}$ of V is of (Haar) measure 0. We also find it convenient to consider certain degenerate types of model sets. A *weak* model set is a set in $\mathbb{R}^n$ of the form $\Lambda(V)$ where we assume only that V is relatively compact, but not that it has a non-empty interior. When V has no interior, $\Lambda(V)$ is not necessarily relatively dense in $\mathbb{R}^n$ but regularity still means that the boundary of V is of measure 0.

Theorem 2 [19]. *If $\Lambda = \Lambda(V)$ is a regular model set, then Λ is a pure point diffractive set, i.e. the Fourier transform of its volume-averaged autocorrelation measure is a pure point measure.*

This theorem was established for real internal spaces by [8] and its full generality, as stated here, in [19]. For a new simpler proof of this result see [4]. It is this theorem that is a prime motivation for finding criteria for sets to be model sets.

Now let $(\tilde{U}, \Phi)$ be a substitution system with inflation Q on a lattice L of $\mathbb{R}^n$ and let $\overline{L}$ be the Q-adic completion of L. This gives rise to the cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R}^n & \stackrel{\pi_1}{\longleftarrow} & \mathbb{R}^n \times \overline{L} & \stackrel{\pi_2}{\longrightarrow} & \overline{L} \\ & & \cup & & \\ L & \longleftarrow & \tilde{L} & \longrightarrow & L \\ t & \longleftarrow & (t,t) & \longrightarrow & t \end{array} \tag{18}$$

where $\tilde{L} := \{(t,t) \mid t \in L\} \subset \mathbb{R}^n \times \overline{L}$.

We claim that $(\mathbb{R}^n \times \overline{L})/\tilde{L}$ is compact. $\tilde{L}$ is clearly discrete and closed in $\mathbb{R}^n \times \overline{L}$. Since $(\mathbb{R}^n \times \overline{L})/\tilde{L}$ is Hausdorff and satisfies the first axiom of countability, it is enough to show that it is sequentially compact [11]. If $\{(x_i, z_i) + \tilde{L}\}$ is a countable sequence in $(\mathbb{R}^n \times \overline{L})/\tilde{L}$, then there is a subsequence $\{(x_i, z_i) + \tilde{L}\}_S$ with $\{x_i + L\}_S$ a convergent sequence, since $\mathbb{R}^n/L$ is compact. We can rewrite $\{(x_i, z_i) + \tilde{L}\}_S$ as $\{(x_i', z_i') + \tilde{L}\}_S$, where

$\{x_i'\}_{i\in S}$ converges to x in $\mathbb{R}^n$. Since $\overline{L}$ is compact, there is a convergent subsequence $\{z_i'\}_{S'}$ to some z in $\overline{L}$. Thus $\{(x_i', z_i')\}_{S'}$ converges to (x, z) in $\mathbb{R}^n \times \overline{L}$. Therefore $(\mathbb{R}^n \times \overline{L})/\tilde{L}$ is sequentially compact.

Note also that $\pi_1|_{\tilde{L}}$ is injective and $\pi_2(\tilde{L})$ is dense in $\overline{L}$.

Lemma 5. *Let U_i, $i = 1, \ldots, m$, be disjoint point sets of the lattice L in $\mathbb{R}^n$. Identify L and its image in $\overline{L}$. Let $W_i := \overline{U_i}$ in $\overline{L}$ and $\Lambda_i := W_i \cap L$. Suppose that $\mu(\partial W_i) = 0$ for all $i = 1, \ldots, m$.*

(i) *If $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$, then, relative to the CPS (18), U_i is a regular weak model set when $\overset{\circ}{W_i}$ is empty, and U_i is a regular model set when $\overset{\circ}{W_i}$ is non-empty.*

(ii) *If $L = \bigcup_{j=1}^m U_j$ and each U_i is a regular model set, then $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$ for all $i = 1, \ldots, m$.*

Proof. (i) Assume that $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$ for all $i = 1, \ldots, m$. Since $\mu(\partial W_i) = 0$ for all $i = 1, \ldots, m$,

$$\mu(W_i) = \mu(\overset{\circ}{W_i}) = \mu\left(\overset{\circ}{W_i} \Bigg\backslash \bigcup_{j=1}^m \partial W_j\right). \tag{19}$$

Since $\Lambda_i = W_i \cap L$, $U_i = V_i \cap L$ where $V_i := W_i \backslash (\Lambda_i \backslash U_i)$. Now $V_i \supset \overset{\circ}{W_i} \setminus \bigcup_{j=1}^m \partial W_j$. From $\overset{\circ}{W_i} \setminus \bigcup_{j=1}^m \partial W_j \subset \overset{\circ}{V_i} \subset V_i \subset \overline{V_i} = W_i$ and (19), $\mu(\overline{V_i} \setminus \overset{\circ}{V_i}) = 0$. So U_i is regular. If $\overset{\circ}{W_i} = \emptyset$, then $\overset{\circ}{V_i} = \emptyset$ also. Then U_i is a regular weak model set. On the other hand, for any i with $\overset{\circ}{W_i} \neq \emptyset$, $\overset{\circ}{V_i} \neq \emptyset$ and $\overline{V_i}$ is compact. Then it follows that $U_i = \Lambda(V_i)$ is a regular model set for the CPS (18).

(ii) Suppose that $\overset{\circ}{V_i} \neq \emptyset$, $\mu(\overline{V_i} \setminus \overset{\circ}{V_i}) = 0$, where $U_i = V_i \cap L$, and $L = \bigcup_{j=1}^m U_j$. Then from $\overline{\Lambda_i \backslash U_i} = \overline{\Lambda(W_i) \backslash \Lambda(V_i)} \subset \overline{W_i \backslash V_i} \subset W_i \setminus \overset{\circ}{V_i} = \overline{V_i} \setminus \overset{\circ}{V_i}$, we have $\mu(\overline{\Lambda_i \backslash U_i}) = 0$ for all $i = 1, \ldots, m$. By Lemma 4(i) and (ii), $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} = 0$ for all $i \neq j$ and $\Lambda_i \backslash U_i \subset \bigcup_{j=1}^m \partial W_j$. □

Theorem 3. *Let $(\tilde{U}, \Phi)$ be a primitive substitution system with inflation Q on the lattice L in $\mathbb{R}^n$. Suppose that the PF-eigenvalue of the substitution matrix $S(\Phi)$ is equal to $|\det Q|$ and $L = \bigcup_{i=1}^m U_i$. Then the following are equivalent:*

(i) *There is a primitive matrix function system Ψ admitting a coincidence, where $(\tilde{U}, \Psi)$ is equivalent to $(\tilde{U}, \Phi^M)$ for some $M \geq 1$.*

(ii) *The sets U_i, $i = 1, \ldots, m$, of $\tilde{U}$ are model sets for the CPS (18).*

(iii) *For at least one i, U_i contains a coset $a + Q^M L$ for some $M \geq 1$.*

(iv) *$(\tilde{U}, \Phi^M)$ admits a modular coincidence for some $M \geq 1$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $(\tilde{U}, \Psi)$ admits a coincidence and is equivalent to $(\tilde{U}, \Phi^M)$. Fix $i \in \{1, \ldots, m\}$ with $\bigcap_{j=1}^m \Psi_{ij} \neq \emptyset$ and let g be in this intersection.

Recalling (9), and in view of the choice of g, we have

$$\mu(W_i) \leq \left(\sum_{j=1}^{m} \sum_{f \in \Psi_{ij}} \mu(f(W_j))\right) - \mu(g(W_k) \cap g(W_l)),$$

for any $k, l \in \{1, \ldots, m\}$ with $k \neq l$. On the other hand, from Theorem 1(ii)

$$\mu(W_i) = \frac{1}{q^M} \sum_{j=1}^{m} (S(\Psi))_{ij} \mu(W_j) = \sum_{j=1}^{m} \sum_{f \in \Psi_{ij}} \mu(f(W_j)). \tag{20}$$

Thus, in fact, $\mu(g(W_k) \cap g(W_l)) = 0$ whenever $k \neq l$. It follows at once that $\overset{\circ}{W_k} \cap \overset{\circ}{W_l} = \emptyset$ for all $k \neq l$, since the measure of any open set is larger than 0.

Recall that $\overset{\circ}{W_i} \neq \emptyset$ and $\mu(\partial W_i) = 0$ for all $i = 1, \ldots, m$. Then by Lemmas 4(ii) and 5, $U_i, i = 1, \ldots, m$, are model sets in CPS (18).

(ii) $\Rightarrow$ (iii) Assume that $U_i, i = 1, \ldots, m$, are model sets in CPS (18), i.e. $U_i = \Lambda(V_i) = V_i \cap L$ for some V_i with $\overset{\circ}{V_i} \neq \emptyset$. Thus there is a coset $a + Q^M \overline{L} \subset \overset{\circ}{V_i}$ and, since we can always choose the coset representative from the dense lattice L, we can arrange that $a + Q^M L \subset U_i$.

(iii) $\Rightarrow$ (iv) Assume that for at least one i, U_i contains a coset $a + Q^M L$. Fix i. Iterate Φ M-times. Then each function f in the matrix function system Φ^M has the form $f\colon x \mapsto Q^M x + b$. For each j, let $G_j := \{f \in (\Phi^M)_{ij} \mid t(f) \equiv a \bmod Q^M L\}$. (Recall that $t(f)$ is the translational part of f.) From $U_i = \bigcup_{j=1}^{m} \bigcup_{f \in (\Phi^M)_{ij}} f(U_j)$, we obtain $a + Q^M L \subset \bigcup_{j=1}^{m} \bigcup_{f \in G_j} f(U_j)$. In fact,

$$a + Q^M L = \bigcup_{j=1}^{m} \bigcup_{f \in G_j} f(U_j), \tag{21}$$

since the right-hand side is clearly inside $a + Q^M L$. From the fact $a + Q^M L \subset U_i$, we get $\Phi^M[a] = \bigcup_{j=1}^{m} G_j \subset \bigcup_{j=1}^{m} (\Phi^M)_{ij}$. Therefore Φ^M has a row containing an entire congruence class $\Phi^M[a]$.

(iv) $\Rightarrow$ (i) Assume Φ^M has a row, say the ith row, containing an entire congruence class $\Phi^M[a]$. Let $G_j := \Phi^M[a] \cap (\Phi^M)_{ij}$. Then $\bigcup_{j=1}^{m} \bigcup_{f \in G_j} f(U_j) \subset a + Q^M L$. Recall that $\bigcup_{j=1}^{m} U_j = L$ and $\tilde{U} = \Phi^M(\tilde{U})$. It follows that the elements of $a + Q^M L$ can be obtained from the matrix function system Φ^M *only* from the mappings of $\Phi^M[a]$, and indeed they must *all* appear as images of the mappings of $\Phi^M[a]$. Thus

$$a + Q^M L = \bigcup_{j=1}^{m} \bigcup_{f \in G_j} f(U_j) \subset U_i. \tag{22}$$

On the other hand,

$$a + Q^M L = \bigcup_{j=1}^{m} Q^M(U_j) + a, \tag{23}$$

which is a disjoint union.

We now alter our matrix function system Φ^M as follows: Define $g\colon L \to L$ by $g(x) = Q^M x + a$. We may, by restriction of the domain, consider g as a function on U_j, $j = 1, \dots, m$. We define Ψ by

$$\begin{cases} \Psi_{ij} = ((\Phi^M)_{ij} \backslash G_j) \cup \{g\}, \\ \Psi_{kj} = (\Phi^M)_{kj} \qquad \text{if} \quad k \neq i, \end{cases}$$

for all j. From (22) and (23), the Ψ_{ij}, $j = 1, \dots, m$, consist of maps from U_j to U_i and have the same total effect on U_i as the $(\Phi^M)_{ij}$, $j = 1, \dots, m$. Thus $(\tilde{U}, \Psi)$ is a substitution system admitting a coincidence.

Since $S(\Phi^M)$ is primitive, the incidence matrix $I(\Phi^M)$ is primitive. Then $I(\Psi)$ is also primitive, since $I(\Phi^M) \leq I(\Psi)$. So Ψ is primitive. In addition, Ψ has the inflation Q^M for L which is the inflation in Φ^M.

We claim that $S(\Psi)$, $S(\Phi^M)$ have the same PF-eigenvalue and right PF-eigenvector. Then $(\tilde{U}, \Psi)$ is equivalent to $(\tilde{U}, \Phi^M)$.

We verify first that $\overset{\circ}{W_k} \cap \overset{\circ}{W_j} = \emptyset$ for all $k, j \in \{1, \dots, m\}$ for which $k \neq j$. We can assume that $m > 1$, since there is nothing to prove when $m = 1$. With i fixed as above, let $g_1 \in G_l = (\Phi^M)_{il}[a] \neq \emptyset$ for some l. Take any $k \in \{1, \dots, m\}$. There is $M_0 \in \mathbb{Z}_+$ for which $(\Phi^{M_0})_{lk} \neq \emptyset$. Choose $f \in (\Phi^{M_0})_{lk}$. Let $g_1\colon x \mapsto Q^M x + a_1$, where $a_1 \equiv a \bmod Q^M L$, and $f\colon x \mapsto Q^{M_0} x + b$ with $b \in L$. Then $g_1 \circ f\colon x \mapsto Q^{M+M_0} x + Q^M b + a_1$. So $g_1 \circ f \in (\Phi^{M+M_0})_{ik}[a_1 + Q^M b]$. Furthermore, $(a_1 + Q^M b) + Q^{M+M_0}(L) \subset a_1 + Q^M L \subset U_i$.

Let $N := M + M_0$, $c := a_1 + Q^M b$, and $p := g_1 \circ f$. Note that

$$c + Q^N L = \bigcup_{j=1}^{m} \bigcup_{h \in H_j} h(U_j), \tag{24}$$

where $H_j = (\Phi^N)_{ij}[c]$.

There are at least two functions in $\bigcup_{j=1}^m H_j$, since, for all, j, $U_j \neq L$. We can write $c + Q^N L$ in the form

$$c + Q^N L = \bigcup \{Q^N U_j + Q^N \alpha_h + c \mid j \in \{1, \dots, m\}, h \in H_j, \alpha_h \in L\}, \tag{25}$$

where we have used the explicit form of each of the mappings $h \in H_j$. This union is disjoint, and as a consequence the elements $\alpha_h \in L$ for h in any single H_j are all distinct. In particular we have α_p coming from H_k. From (25) we have

$$L = \bigcup_{j=1}^{m} \bigcup_{h \in H_j} (U_j + \alpha_h) \tag{26}$$

and separating off U_k,

$$L = U_k \cup \bigcup_{j=1}^{m} \bigcup_{h \in H'_j} (U_j + \alpha_h - \alpha_p), \tag{27}$$

where $H'_j := H_j$ if $j \neq k$ and $H'_k := H_k \backslash \{p\}$. Again these decompositions are disjoint. However, we also know that U_k and $\bigcup_{\substack{j=1 \\ j \neq k}}^{m} U_j$ are disjoint, and it follows that

$$\bigcup_{\substack{j=1 \\ j \neq k}}^{m} U_j \subset \bigcup_{j=1}^{m} \bigcup_{h \in H'_j} (U_j + \alpha_h - \alpha_p).$$

Taking closures,

$$\bigcup_{\substack{j=1\\ j\neq k}}^{m} W_j \subset \bigcup_{j=1}^{m} \bigcup_{h\in H_j'} (W_j + \alpha_h - \alpha_p). \tag{28}$$

On the other hand, if we apply Theorem 1(ii) to Φ^N and look at (24) we see that

$$\mu(c + Q^N\overline{L}) = \sum_{j=1}^{m} \sum_{h\in H_j} \mu(h(W_j)) = \sum_{j=1}^{m} \sum_{h\in H_j} \mu(Q^N(W_j + \alpha_h) + c),$$

and hence

$$\mu(\overline{L}) = \sum_{j=1}^{m} \sum_{h\in H_j} \mu(W_j + \alpha_h) = \sum_{j=1}^{m} \sum_{h\in H_j} \mu(W_j + \alpha_h - \alpha_p).$$

Thus

$$\mu(\overline{L}) = \mu(W_k) + \left(\sum_{j=1}^{m} \sum_{h\in H_j'} \mu(W_j + \alpha_h - \alpha_p) \right)$$

which, after taking closures in (27), gives

$$\mu\left(W_k \cap \left(\bigcup_{j=1}^{m} \bigcup_{h\in H_j'} (W_j + \alpha_h - \alpha_p) \right) \right) = 0. \tag{29}$$

Finally, from (28) and (29) we obtain

$$\mu\left(W_k \cap \left(\bigcup_{\substack{j=1\\ j\neq k}}^{m} W_j \right) \right) = 0,$$

from which $\mathring{W}_k \cap \mathring{W}_j = \emptyset$ for all $j \in \{1, \ldots, m\}$ for which $j \neq k$. Since k is arbitrary in $\{1, \ldots, m\}$, this establishes the first verification.

Now

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{m} g(W_j) \right) &= \frac{1}{|\det Q^M|} \mu\left(\bigcup_{j=1}^{m} W_j \right) \\ &= \frac{1}{|\det Q^M|} \sum_{j=1}^{m} \mu(W_j) \\ &\quad \text{(from } \mu(\partial W_j) = 0,\ \mathring{W}_i \cap \mathring{W}_j = \emptyset \text{ for all } i \neq j) \\ &= \sum_{j=1}^{m} \mu(g(W_j)). \end{aligned} \tag{30}$$

Again using Theorem 1(ii), this time for Φ^M, we obtain

$$w = \frac{1}{|\det Q^M|} S(\Phi^M) w,$$

where $w = [\mu(W_1), \ldots, \mu(W_m)]^T$. The part of this relation in W_i which pertains to the coset $a + Q^M \overline{L}$ is

$$\mu(a + Q^M \overline{L}) = \sum_{j=1}^{m} \sum_{f \in G_j} \mu(f(W_j)). \tag{31}$$

However, from (23),

$$\mu(a + Q^M \overline{L}) = \mu \left(\bigcup_{j=1}^{m} g(W_j) \right). \tag{32}$$

Together, (30), (31), and (32) show

$$w = \frac{1}{|\det Q^M|} S(\Psi) w.$$

Since $w > 0$ and $S(\Psi)$ is primitive, $S(\Psi)$ has PF-eigenvalue $|\det Q^M|$ and PF-eigenvector w as required. □

Remark. Let $A = \{a_1, \ldots, a_m\}$ be an alphabet of m symbols and let σ be a primitive equal-length alphabetic substitution system on A, that is,

(i) $\sigma\colon A \longrightarrow A^q$ for some $q \in \mathbb{Z}_+$;
(ii) the $m \times m$ matrix $S = (S_{ij})$, whose (i, j) entry is the number of appearances of a_i in $\sigma(a_j)$, is primitive.

According to Gottschalk [7], for some iteration σ^k of σ, there is a word $w \in A^{\mathbb{Z}}$ which is fixed by σ in the sense that

$$\begin{aligned} \sigma^k(w_0 w_1 \ldots) &= w_0 w_1 \ldots, \\ \sigma^k(\ldots w_{-2} w_{-1}) &= \ldots w_{-2} w_{-1}. \end{aligned} \tag{33}$$

Replacing σ^k by σ and q^k by q if necessary we can suppose that $k = 1$, and assume then that $\sigma(w) = w$.

We can view w as a tiling of $\mathbb{R}$ by tiles of types $a_1, \ldots, a_m$, all of the same length 1. If we coordinatize each tile by its left-hand endpoint so that w_l gets coordinate l, then we obtain a partition $U_1 \cup \cdots \cup U_m$ of $\mathbb{Z}$ and an $m \times m$ matrix function system Φ of q-affine mappings derived directly from σ: namely, $\sigma a_j = a_{i_1} \cdots a_{i_q}$ gives rise to the mappings $(x \mapsto qx + l - 1) \in \Phi_{i_l j}$, $l = 1, \ldots, q$.

We take as our cut and project scheme

$$\begin{array}{ccccc} \mathbb{R} & \longleftarrow & \mathbb{R} \times \mathbb{Z}_q & \longrightarrow & \mathbb{Z}_q \\ & & \cup & & \\ \mathbb{Z} & \longleftarrow & \tilde{\mathbb{Z}} & \longrightarrow & \mathbb{Z} \\ z & \longleftarrow & (z, z) & \longrightarrow & z \end{array} \tag{34}$$

(see (18)), where $\mathbb{Z}_q$ is the q-adic completion of $\mathbb{Z}$.

According to Theorem 3, the U_i's are model sets for (34) if and only if for some iteration σ^M of σ, there is a $k \in \mathbb{Z}$ for which all the mappings f_l: $x \mapsto q^M x + l$ with $l \equiv k \bmod q^M$ lie in one row of Φ^M.

Since $\sigma^M a_j$ has q^M letters in it, there are q^M mappings in the jth column of Φ^M. Furthermore, since the letters $\sigma^M a_j$ are represented by contiguous tiles, their coordinates fall in a range of consecutive integers, and so the mappings of the jth column of Φ^M are the maps f_l, where $0 \le l < q^M$, in some order. In particular, all of the mappings in Φ^M are of this restricted form. It follows that modular coincidence is equivalent to the existence of a row of Φ^M, say the ith row, and a k, $0 \le k < q^M$, so that f_k belongs to *each* of $\Phi^M_{i1}, \ldots, \Phi^M_{im}$.

This condition precisely says that there is a k so that the kth position of $\sigma^M(a_j)$ contains the same letter a_i for all j. This is the well-known coincidence condition of Dekking [6], and he has proved that for non-periodic primitive equal-length substitutions, this condition is equivalent to pure point diffractivity of the dynamical spectrum. It is straighforward to show that $S(\Phi)$ has its PF-eigenvalue equal to $|\det Q|$. Thus we have

Corollary 1. *Let σ be a primitive equal-length $(= q)$ alphabetic substitution with a fixed bi-infinite word w. Let $h(\sigma) = \max\{n \ge 1 \mid \gcd(n, q) = 1,\ n$ divides $\gcd\{i \mid w_i = w_0\}\}$ where $w = (w_i)_{i \in \mathbb{Z}}$. Assume $h(\sigma) = 1$. Let Φ be the corresponding matrix function system and let $\mathbb{Z} = U_1 \cup \cdots \cup U_m$ be the corresponding partition of $\mathbb{Z}$. Then the following are equivalent*:

(i) *there is an M so that σ^M has a coincidence in the sense of Dekking*;
(ii) *$(\tilde{U}, \Phi)$, where $\tilde{U} = [U_1, \ldots, U_m]$, has a modular coincidence*;
(iii) *the U_i's are model sets for* (34);
(iv) *the U_i's are pure point diffractive.*

We note that this interesting *equivalence* of model sets for (34) and pure point diffractivity is more than we can yet prove in the higher dimensional substitution systems.

Remark. The condition height $= h(\sigma) = 1$ in Corollary 1 should be stressed. For example, consider the substitution σ such that $\sigma(a) = aba$, $\sigma(b) = bab$, which is of height 2. There is no coincidence in any σ^M, so the points of type a (or b) do not form model sets for (34). However, the set of a-points (and likewise the set of b-points) does form a translate of a lattice and hence is pure point diffractive.

5. Sphinx Tiling

Long we sought the wayward lynx
And bowed before the subtle sphinx
But solved we not the cryptic sphinx
Before we found the wayward links.

Anon

In this section we take up sphinx tiling. This is a substitution tiling whose subdivision rule is shown in Figs. 1 and 2.

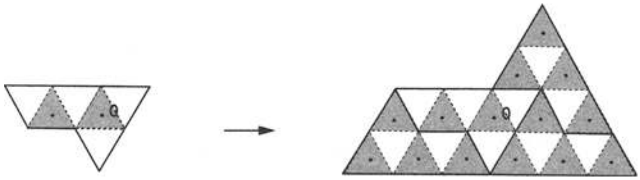

Fig. 1. Sphinx inflation [Type 1].

It has 12 sphinx-like tiles (up to translation). If we choose a single point in the same way in each sphinx, then we arrive at 12 sets of points. We wish to show that each of these sets is a regular model set. Actually we make a slight alteration to this, choosing several points from each tile, but this is equivalent to our original problem.

Each sphinx can be viewed as consisting of six equilateral triangles of two orientations. In this way, any sphinx tiling determines a tessellation of the plane by equilateral triangles. We consider the centre points of the triangles of one orientation. These clearly form a lattice L, once we have chosen one of them as the origin. Note that some sphinxes have two points and others have four points in L. We give names to each tile and the points in it as shown in Fig. 3. Then the 12 types of sphinx partition L into 36 subsets forming a matrix function system. We show that these are model sets for a 2-adic-like cut and project scheme of the form of (18).

With the origin as shown, the coordinates are chosen so that in the standard rectangular system $(1, 0)$ is the lattice point directly to the right of $(0, 0)$. It is more convenient to replace this by an oblique coordinate system: $L = \{ae + bw \mid a, b \in \mathbb{Z}\}$, where $e = (1, 0)$, $w = (\frac{1}{2}, \sqrt{3}/2)$ in the standard rectangular system and relative to this basis we can identify L and $\mathbb{Z}^2$ and denote $ae + bw$ by (a, b). The basic inflation shown in Fig. 1 gives rise to the map

$$T\colon x \mapsto 2Rx + (1, 0),$$

where R is a reflection in $\mathbb{R}^2$ through the x-axis, i.e. in the new coordinates, $R(1, 0) = (1, 0)$, $R(0, 1) = (1, -1)$.

The various types of points are designated by letter pairs $i\alpha$, where $i \in \{1, \ldots, 12\}$ and $\alpha \in \{a, \ldots, d\}$ (of which only 36 actually occur). Let $U_{i\alpha}$ be the set of points of type $i\alpha$. On the basis of this we can make mappings of each point set to other point sets.

Using Figs. 1–3, we consider the following mappings:

$$\text{the } 1a\text{-points} \longrightarrow \text{the } 9a, 9b, 9c, 9d, 4a, 4b, 4c, 4d\text{-points},$$

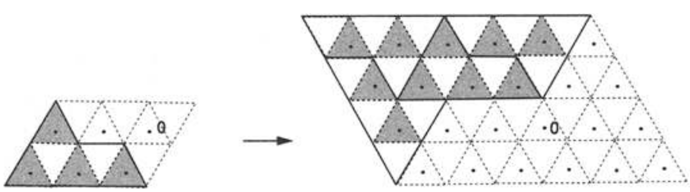

Fig. 2. Sphinx inflation [Type 2].

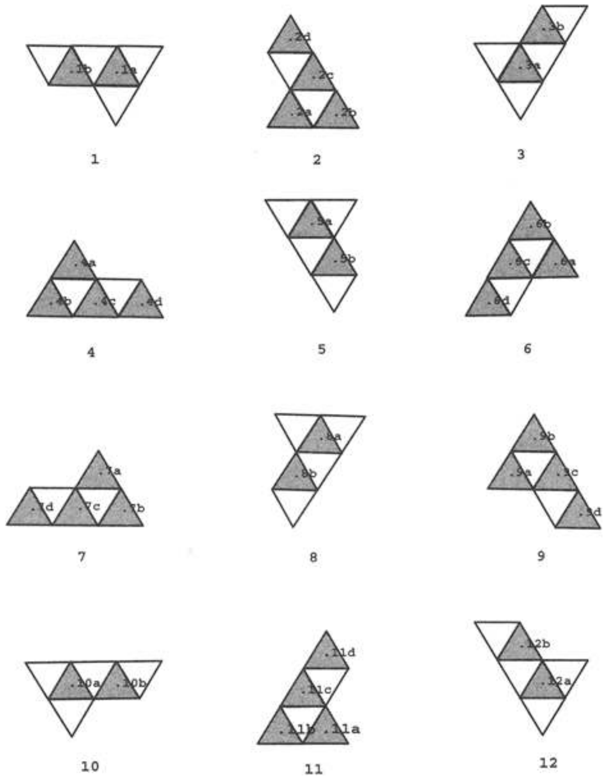

Fig. 3. Twelve sphinx tiles.

the $1b$-points $\longrightarrow$ the $1a$, $1b$, $4a$, $4b$, $4c$, $4d$-points,
the $4a$-points $\longrightarrow$ the $12a$, $12b$, $4a$, $4b$, $4c$, $4d$-points,
the $4b$-points $\longrightarrow$ the $1a$, $1b$-points,
the $4c$-points $\longrightarrow$ the $4a$, $4b$, $4c$, $4d$-points,
the $4d$-points $\longrightarrow$ the $1a$, $1b$-points.

Define

$$h_1\colon x \mapsto Tx + (0,0), \qquad h_2\colon x \mapsto Tx + (1,0),$$
$$h_3\colon x \mapsto Tx + (0,1), \qquad h_4\colon x \mapsto Tx + (-1,1),$$
$$h_5\colon x \mapsto Tx + (-1,0), \qquad h_6\colon x \mapsto Tx + (0,-1),$$
$$h_7\colon x \mapsto Tx + (1,-1), \qquad h_8\colon x \mapsto Tx + (2,-1),$$
$$h_9\colon x \mapsto Tx + (-1,2), \qquad h_{10}\colon x \mapsto Tx + (-1,-1).$$

Let $f_{i\alpha j\beta}$ be the function of the form $x \mapsto Tx + b$ which maps the $j\beta$-point into the $i\alpha$-point. Then each mapping can be described by [Type 1] and [Type 2].

[Type 1]

$$f_{9a1a} = h_4\colon x \mapsto Tx + (-1,1), \qquad f_{1a1b} = h_2\colon x \mapsto Tx + (1,0),$$

$$f_{9b1a} = h_9\colon x \mapsto Tx + (-1, 2), \qquad f_{1b1b} = h_1\colon x \mapsto Tx + (0, 0),$$
$$f_{9c1a} = h_3\colon x \mapsto Tx + (0, 1), \qquad f_{4a1b} = h_5\colon x \mapsto Tx + (-1, 0),$$
$$f_{9d1a} = h_2\colon x \mapsto Tx + (1, 0), \qquad f_{4b1b} = h_{10}\colon x \mapsto Tx + (-1, -1),$$
$$f_{4a1a} = h_1\colon x \mapsto Tx + (0, 0), \qquad f_{4c1b} = h_6\colon x \mapsto Tx + (0, -1),$$
$$f_{4b1a} = h_6\colon x \mapsto Tx + (0, -1), \qquad f_{4d1b} = h_7\colon x \mapsto Tx + (1, -1),$$
$$f_{4c1a} = h_7\colon x \mapsto Tx + (1, -1),$$
$$f_{4d1a} = h_8\colon x \mapsto Tx + (2, -1).$$

[Type 2]

$$f_{12a4a} = h_1\colon x \mapsto Tx + (0, 0), \qquad f_{1a4b} = h_2\colon x \mapsto Tx + (1, 0),$$
$$f_{12b4a} = h_4\colon x \mapsto Tx + (-1, 1), \qquad f_{1b4b} = h_1\colon x \mapsto Tx + (0, 0).$$

$$f_{4a4c} = h_1\colon x \mapsto Tx + (0, 0), \qquad f_{1a4d} = h_1\colon x \mapsto Tx + (0, 0),$$
$$f_{4b4c} = h_6\colon x \mapsto Tx + (0, -1), \qquad f_{1b4d} = h_5\colon x \mapsto Tx + (-1, 0),$$
$$f_{4c4c} = h_7\colon x \mapsto Tx + (1, -1),$$
$$f_{4d4c} = h_8\colon x \mapsto Tx + (2, -1).$$

All points in a sphinx having 2-points in it are mapped as in [Type 1] changing the translation part according to the orientation and reflection of the sphinx relative to sphinx 1. Likewise, all points in a sphinx having 4-points in it are mapped as in [Type 2] relative to sphinx 4.

Now we can list the 36×36 matrix(Φ) of affine mappings that make up our substitution system (Table 1).

We can check that $S(\Phi)$ has PF-eigenvalue 4 and is a primitive matrix and the union of point sets is L. We used Mathematica to check that property (iv) in Theorem 3 is satisfied in Φ^8 (it may actually be satisfied at some lower power). Certainly in Φ^8 there are a large number of modular coincidences. Theorems 1 and 3 say that all 36 point sets are regular model sets in CPS (18).

6. The Total Index and Model Sets

In this section we derive another criterion for determining when a partition of a lattice is a partition into Q-adic model sets, the difference this time being that there is no substitution system involved.

We assume that we are given a lattice L in $\mathbb{R}^n$ and an inflation Q on L as in (4). The notation remains the same as before. The main ingredient is a non-negative sub-additive function called the total index which is defined on the subsets of L and its Q-adic completion $\overline{L}$.

For any subset V of L the *coset part* of V is defined as

$$\mathcal{C}(V) := \bigcup \{C \mid C \text{ is a coset in } V\}. \tag{35}$$

The key point to remember in what follows is that two cosets in L or $\overline{L}$) are either disjoint or one of them is contained in the other. If $C = a + Q^k L$ is a coset, then we write $[L : C]$ for the index of the subgroup $Q^k L$ in L.

Table 1. Sphinx matrix function system (Φ).

$$\left(\begin{array}{*{36}{c}}
\emptyset & h_2 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_2 & \emptyset & h_1 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_1 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
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\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_9 & h_3 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_9 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_3 & \emptyset \\
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h_1 & h_5 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_1 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_5 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
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h_7 & h_6 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_7 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_6 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
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h_4 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_1 & h_4 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_1 & \emptyset & \emptyset & \emptyset \\
h_9 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_3 & h_9 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_3 & \emptyset & \emptyset & \emptyset \\
h_3 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_2 & h_3 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_2 & \emptyset & \emptyset & \emptyset \\
h_2 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_8 & h_2 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_8 & \emptyset & \emptyset & \emptyset \\
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\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_{10} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_5 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_5 & h_{10} \\
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\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_1 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_7 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & h_7 & \emptyset & h_1 & \emptyset & \emptyset \\
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\end{array}\right)$$

Lemma 6. *The coset part of V can be written as a disjoint union of cosets in V.*

Proof. If V contains no cosets, then the result is clear. Suppose V contains cosets. Let $C_1 = a_1 + Q^{k_1}L$ be a coset in V with k_1 minimal. Consider $V \backslash C_1$. No coset can be partly in C_1 and partly in $V \backslash C_1$. Thus, if $V \backslash C_1$ contains no cosets, then $\mathcal{C}(V) = C_1$. Otherwise let C_2 be a coset $a_2 + Q^{k_2}L$ with k_2 minimal in $V \backslash C_1$. Then $\mathcal{C}(V) \supset C_1 \dot{\cup} C_2$. We continue this process. Since there are only finitely many cosets for $Q^k L$ in L, either we obtain $\mathcal{C}(V) = C_1 \dot{\cup} \cdots \dot{\cup} C_r$ for some r or $\mathcal{C}(V) \supset C_1 \dot{\cup} C_2 \dot{\cup} \cdots$, where $k_1 \leq k_2 \leq \cdots$ is infinite and unbounded. In the latter case, $\mathcal{C}(V) = \bigcup_{i=1}^{\infty} C_i$ is our required decompostion. If not, there is a coset $C = a + Q^k L$ in V such that $C \not\subset \bigcup_{i=1}^{\infty} C_i$. Then there is C_i with $k_{i-1} \leq k < k_i$. This contradicts the choice of C_i. □

For $V \subset L$, we call a decomposition $\mathcal{C}(V) = \bigcup_i C_i$ of $\mathcal{C}(V)$ into mutually disjoint cosets using the algorithm of Lemma 6, an *efficient* decomposition of V into cosets. In this case we call $c(V) := \sum_i [L : C_i]^{-1}$ the *total index* of V. Since any coset is an efficient decomposition of itself, we have $c(V) = \sum_i c(C_i)$. We will see shortly that the total index is finite.

It is useful to note that an efficient decomposition of $\mathcal{C}(V) = \bigcup_i C_i$ of $\mathcal{C}(V)$ into cosets has the following special property: if D is any coset of V, then necessarily $D \subset C_i$ for some i.

Lemma 7. *Any two efficient decompositions of $\mathcal{C}(V)$ are the same up to rearrangement of the order of the cosets. In particular the total index is well-defined.*

Proof. Let $\mathcal{C}(V) = \bigcup C'$ be a second decomposition of $\mathcal{C}(V)$ determined by the same algorithm as in Lemma 6. Then with k_1 as in the lemma, let $D_1, \ldots, D_r$ be all the cosets of V of the form $a + Q^{k_1}L$. These are all disjoint and by the algorithm all of them must be chosen in the decomposition of $\mathcal{C}(V)$, and they all occur before all the others. Thus $C_1, \ldots, C_r$ and $C'_1, \ldots, C'_r$ are $D_1, \ldots, D_r$ in some order. Removing these and continuing in the same way the result is clear. □

We have similar concepts in $\overline{L}$. For $W \subset \overline{L}$ we have the coset part $\mathcal{C}^*(W)$ of W and $\mathcal{C}^*(W)$ can be written as a disjoint union of cosets in W. Let $\mathcal{C}^*(W) = \bigcup_i D_i$ where $D_i, i = 1, 2, \ldots,$ are mutually disjoint cosets in W. We call $c^*(W) := \sum_i [\overline{L} : D_i]^{-1}$ the *total index* of W. This time we do not need to be careful about the way in which the decomposition is obtained since the total index is nothing else than the measure $\mu(\mathcal{C}^*(W))$ of $\mathcal{C}^*(W)$.

Given an efficient decomposition $\mathcal{C}(V) = \bigcup_{i=1} C_i$ into disjoint cosets in L, we define $\overline{\mathcal{C}}(V) := \bigcup_{i=1} \overline{C_i} \subset \overline{L}$. This is actually an open set in $\overline{L}$. Since $[L : C] = [\overline{L} : \overline{C}]$ we see that $c(V) = c^*(\overline{\mathcal{C}}(V))$. In particular, it follows that the total index of any subset V of L is finite and bounded by $\mu(\overline{\mathcal{C}}(V))$.

Lemma 8. *For $X, Y \subset L$ and $X \subset Y$, and any decomposition $\mathcal{C}(X) = \bigcup_i C_i$ into disjoint cosets, $\sum_i c(C_i) \le c(Y)$. In particular, $c(X) \le c(Y)$.*

Proof. Assume first that Y is a single coset C. Then

$$\sum_i c(C_i) = \sum_i c^*(\overline{C}_i) = \sum_i \mu(\overline{C}_i) \le \mu(\overline{C}) = c^*(\overline{C}) = c(C), \tag{36}$$

since the cosets remain distinct after closing them in $\overline{L}$.

In the general case, let $\mathcal{C}(Y) = \bigcup_{j=1} C'_j$ be an efficient decomposition of Y. Since $X \subset Y$, each $C_i \subset Y$. In view of the remark above about efficient decompositions, there is for each i a unique j for which $C_i \subset C'_j$. Thus we can arrange the C_i's so that

$$\mathcal{C}(X) = \bigcup_{j=1}^{\infty} \bigcup_{i \in A_j} C_i, \tag{37}$$

where $A_j := \{i \mid C_i \subset C'_j\}$. Now $\bigcup_{i \in A_j} C_i \subset C'_j$, so by the first part of the proof, $\sum_{i \in A_j} c(C_i) \le c(C'_j)$. Finally,

$$c(X) = \sum_j \sum_{i \in A_j} c(C_i) \le \sum c(C'_j) = c(Y). \quad \square \tag{38}$$

Lemma 9. *Let U_i, $i = 1, \ldots, m$, be disjoint point sets of the lattice L in $\mathbb{R}^n$. Let $\Lambda_i = \overline{U_i} \cap L$ and $\mathcal{C}(U_i)$ be the coset part in U_i. Then $\bigcup_{i=1}^m (\Lambda_i \backslash U_i) \subset L \backslash \bigcup_{i=1}^m \mathcal{C}(U_i)$, with equality if $L = \bigcup_{i=1}^m U_i$.*

Proof. For $x \in \bigcup_{i=1}^m \mathcal{C}(U_i)$ there is a coset $C \subset \mathcal{C}(U_i)$ for which $x \in C \subset U_i$. Let $C = a + Q^k L$, $a \in L$. Suppose x is a limit point of U_j in $\overline{L}$ for some $j \neq i$. Then, since $a + Q^k\overline{L}$ is an open neighbourhood of x, $(a + Q^k\overline{L}) \cap U_j \neq \emptyset$, i.e. $(a + Q^k L) \cap U_j \neq \emptyset$. However, then $U_i \cap U_j \neq \emptyset$, contrary to the assumption. This means $x \notin \bigcup_{i=1}^m (\Lambda_i \backslash U_i)$, proving the first part.

Suppose that $L = \bigcup_{i=1}^m U_i$ and $x \in L$ but $x \notin \bigcup_{i=1}^m \mathcal{C}(U_i)$. Then $x \in U_i$ for some U_i but there is no coset in U_i which contains x. For any $k \in \mathbb{Z}_+$, $B_k(x) := x + Q^k\overline{L}$ is an open neighbourhood of x in $\overline{L}$ and $L \cap B_k(x) \not\subset U_i$, by assumption. Since $L = \bigcup_{i=1}^m U_i$, $(L \cap B_k(x)) \cap U_j \neq \emptyset$ for some $j \neq i$. So we can choose $x_k^j \in (L \cap B_k(x)) \cap U_j$. Then we get a sequence $\{x_k^j\}$ convergent to x as $k \to \infty$. Choosing a subsequence lying entirely in one U_j shows that $x \in \Lambda_j$ for some $j \neq i$. Since $x \in U_i$, and U_i, U_j are disjoint, $x \in \Lambda_j \backslash U_j$. □

Theorem 4. *Let U_i, $i = 1, \ldots, m$, be disjoint non-empty point sets of the lattice L in $\mathbb{R}^n$. Let $\mathcal{C}(U_i)$ be the coset part in U_i, let $c(U_i)$ be the total index of U_i, and let W_i be the closure of U_i in $\overline{L}$. Then $\sum_{i=1}^m c(U_i) = 1$ if and only if the sets U_i, $i = 1, \ldots, m$, are regular weak model sets in the CPS* (18) *and* $\overline{L} = \bigcup_{i=1}^m W_i$.

Proof. ($\Rightarrow$) Assume that $\sum_{i=1}^m c(U_i) = 1$. Let $U_{m+1} := L \backslash \bigcup_{i=1}^m U_m$. Using Lemma 8 and the fact that $c(L) = 1$, we see that $c(U_{m+1}) = 0$ and $\sum_{i=1}^{m+1} c(U_i) = 1$. For this reason we can assume, in proving that the U_i are weak model sets, that $\bigcup_{i=1}^m U_i = L$ in the first place.

For $j \neq k$ the cosets of $\mathcal{C}(U_j)$ (of which there may be none!) and those of $\mathcal{C}(U_k)$ are disjoint from one another, and the same applies to $\overline{\mathcal{C}}(U_j)$ and $\overline{\mathcal{C}}(U_k)$. Thus

$$\mu\left(\bigcup_{i=1}^m \overline{\mathcal{C}}(U_i)\right) = \sum_{i=1}^m \mu(\overline{\mathcal{C}}(U_i)) = \sum_{i=1}^m c(U_i) = 1$$

and

$$\mu\left(\overline{L} \Big\backslash \left(\bigcup_{i=1}^m \overline{\mathcal{C}}(U_i)\right)\right) = 0. \tag{39}$$

Now note that $\partial W_j \cap \bigcup_{i=1}^m \overline{\mathcal{C}}(U_i) = \emptyset$ for any j. If not let $a \in \partial W_j \cap \overline{\mathcal{C}}(U_k)$ for some k. Since $\overline{\mathcal{C}}(U_k) \subset \mathring{W}_k$, we see that $j \neq k$. However, $a \in W_j$, so a is a limit point of U_j, and $\overline{\mathcal{C}}(U_k)$ is an open neighbourhood of a, so $U_j \cap \mathcal{C}(U_k) \neq \emptyset$. This violates the disjointness of the U_i's. We conclude that $\partial W_j \subset \overline{L} \backslash (\bigcup_{i=1}^m \overline{\mathcal{C}}(U_i))$ and hence that

$$\mu(\partial W_j) = 0, \tag{40}$$

for all $j = 1, \ldots, m$. Note also that

$$\begin{aligned} \Lambda_i \backslash U_i &\subseteq \bigcup_{j=1}^m (\Lambda_j \backslash U_j) \\ &\subset L \backslash \bigcup_{j=1}^m \mathcal{C}(U_j) \qquad \text{(by Lemma 9)} \end{aligned}$$

$$= L\backslash \bigcup_{j=1}^{m} (\overline{C}(U_j) \cap L) = L\backslash \bigcup_{j=1}^{m} \overline{C}(U_j).$$

This shows that

$$\mu(\overline{\Lambda_i \backslash U_i}) \leq \mu\left(\overline{L \Big\backslash \left(\bigcup_{i=1}^{m} \overline{C}(U_i)\right)}\right) \leq \mu\left(\overline{L} \Big\backslash \bigcup_{i=1}^{m} \overline{C}(U_i)\right) = 0. \tag{41}$$

By Lemma 4(i) and (ii), $\overset{\circ}{W_i} \cap \overset{\circ}{W_j} = \emptyset$ for all $i \neq j$ and $(\Lambda_i \backslash U_i) \subset \bigcup_{j=1}^{m} \partial W_j$ for all $i = 1, \dots, m$.

Using Lemma 5(i) we obtain that the sets U_i, $i = 1, \dots, m$, are regular weak model sets in the CPS (18).[2]

Since $\bigcup_{i=1}^{m} \overline{C}(U_i) \subset \bigcup_{i=1}^{m} W_i$, $\mu(\bigcup_{i=1}^{m} W_i) = 1$. Thus $\overline{L} \backslash \bigcup_{i=1}^{m} W_i$ is open of measure 0 and $\overline{L} = \bigcup_{i=1}^{m} W_i$. This last argument does not require that $\bigcup_{i=1}^{m} U_i = L$.

($\Leftarrow$) Assume that $U_i = \Lambda(V_i) = V_i \cap L$ where $\overline{V_i} \backslash \overset{\circ}{V_i}$ has measure 0 and $\overline{L} = \bigcup_{i=1}^{m} W_i$. Thus $U_i \subset V_i$ and $W_i := \overline{U_i} \subset \overline{V_i}$. Since L is dense in $\overline{L}$ and for $x \in \overset{\circ}{V_i}$ every open ball around x contains points of $\overset{\circ}{V_i} \cap L \subset U_i$, it follows that $\overline{U_i} \supset \overset{\circ}{V_i}$. This proves that $\overset{\circ}{V_i} \subset W_i \subset \overline{V_i}$. So $\mu(\overset{\circ}{V_i}) = \mu(W_i)$ and $\mu(W_i \backslash \overset{\circ}{V_i}) = 0$. Now

$$\bigcup_{i=1}^{m} W_i = \left(\bigcup_{i=1}^{m} \overset{\circ}{V_i}\right) \cup \left(\bigcup_{i=1}^{m} (W_i \backslash \overset{\circ}{V_i})\right).$$

So $\mu(\bigcup_{i=1}^{m} W_i) = \mu(\bigcup_{i=1}^{m} \overset{\circ}{V_i})$. Also the disjointness of the U_i gives $\overset{\circ}{V_i} \cap \overset{\circ}{V_j} = \emptyset$ for $i \neq j$ (since L is dense in $\overline{L}$). Finally,

$$\begin{aligned} 1 &= \mu\left(\bigcup_{i=1}^{m} W_i\right) = \mu\left(\bigcup_{i=1}^{m} \overset{\circ}{V_i}\right) = \sum_{i=1}^{m} \mu(\overset{\circ}{V_i}) \\ &= \sum_{i=1}^{m} c^*(\overset{\circ}{V_i}) \leq \sum_{i=1}^{m} c(\overset{\circ}{V_i} \cap L) \leq \sum c(U_i) \leq 1. \end{aligned}$$

□

Corollary 2. *Let* $(\tilde{U}, \Phi)$ *be a primitive substitution system with inflation* Q *on the lattice* L *in* $\mathbb{R}^n$. *Suppose that the PF-eigenvalue of the substitution matrix* $S(\Phi)$ *is equal to* $|det\, Q|$ *and* $L = \bigcup_{i=1}^{m} U_i$. *Then* $\sum_{i=1}^{m} c(U_i) = 1$, *where* $c(U_i)$ *is the total index of* U_i, *if and only if the sets* U_i, $i = 1, \dots, m$, *are model sets in CPS* (18).

Proof. Use Theorem 1 to determine that for all i, $\overset{\circ}{W_i} \neq \emptyset$. Now use Theorem 4 and footnote 2. □

Corollary 2 gives us another criterion to be model sets under the same circumstances as in Theorem 3.

[2] Whenever $\overset{\circ}{W_i} \neq \emptyset$, U_i is actually a regular model set.

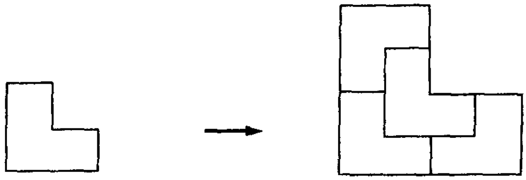

Fig. 4. Two-dimensional chair tiling inflation.

7. Chair Tiling

The two-dimensional chair tiling is generated by the inflation rule shown in Fig. 4. There are four orientations of the chairs in any chair tiling. In [5] it was shown that chair tiling has an interpretation in terms of model sets based on the lattice $\mathbb{Z}^2$ and its 2-adic completion as internal space.

In this section we generalize this result to n-dimensional chair tiling using the results of the last section (see Figure 6 for an example of the three-dimensional chair). To make things clearer we begin with the case $n = 2$.

I. *Chair Tiling in* $\mathbb{R}^2$

The starting point is to replace each tile by three oriented squares. Figure 5 shows the inflation rule, for one chair, in terms of oriented squares. The resulting tiling is a square tiling of the plane in which each of the squares has one of four orientations. The centre points of each square form a square lattice which we identify with $\mathbb{Z}^2$ by assigning coordinates as shown.

Let U_i be the set of centre points corresponding to squares of orientation (i) as given in Fig. 5. We start out from a basic generating set $A_2 := \{(x_1, x_2) \mid x_i \in \{0, -1\}\}$ and determine the precise maps for the substitution rules of Fig. 5.

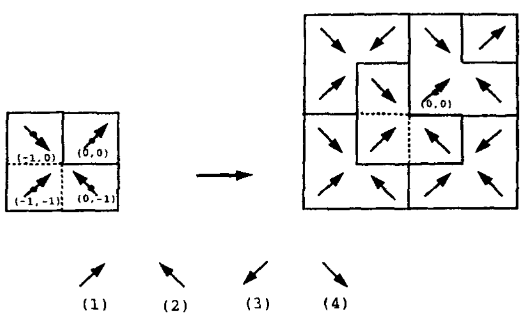

Fig. 5. Two-dimensional chair tiling substitution.

Letting $e_1 := (0,0)$, $e_2 := (1,0)$, $e_3 := (1,1)$, and $e_4 := (0,1)$, these maps are defined as

$$f_{j,i}\colon U_i \to U_j \quad \text{by} \quad (x,i) \mapsto (2x + e_j, j) \qquad \text{if} \quad j \neq i \pm 2,$$
$$f_{i,i}^{(2)}\colon U_i \to U_i \quad \text{by} \quad (x,i) \mapsto (2x + e_j, i) \qquad \text{if} \quad j = i \pm 2,$$

where

$$i, j \in \{1,2,3,4\}, \qquad x \in \mathbb{Z}^2, \qquad i \pm 2 := \begin{cases} i+2 & \text{if} \quad i \leq 2, \\ i-2 & \text{if} \quad i > 2. \end{cases}$$

These are the maps of an affine substitution system Φ. In fact, if we define

$$h_1\colon x \mapsto 2x + e_1, \quad h_2\colon x \mapsto 2x + e_2, \quad h_3\colon x \mapsto 2x + e_3, \quad h_4\colon x \mapsto 2x + e_4,$$

then

$$\begin{array}{llll} f_{1,1} = h_1, & f_{1,2} = h_1, & f_{1,1}^{(2)} = h_3, & f_{1,4} = h_1, \\ f_{2,1} = h_2, & f_{2,2} = h_2, & f_{2,3} = h_2, & f_{2,2}^{(2)} = h_4, \\ f_{3,3}^{(2)} = h_1, & f_{3,2} = h_3, & f_{3,3} = h_3, & f_{3,4} = h_3, \\ f_{4,1} = h_4, & f_{4,4}^{(2)} = h_2, & f_{4,3} = h_4, & f_{4,4} = h_4, \end{array}$$

and

$$\Phi = \begin{pmatrix} \{h_1, h_3\} & \{h_1\} & \{\} & \{h_1\} \\ \{h_2\} & \{h_2, h_4\} & \{h_2\} & \{\} \\ \{\} & \{h_3\} & \{h_3, h_1\} & \{h_3\} \\ \{h_4\} & \{\} & \{h_4\} & \{h_4, h_2\} \end{pmatrix}.$$

Inflating A_2 by the substitutions above we generate the four point sets $U_i, i = 1,2,3,4$. The precise description of U_i is the following:

$$\begin{aligned} U_1 &= \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((0,0) + 2^k(2,0) + t(1,1) + 2^k \cdot 4\mathbb{Z}^2) \\ &\quad \cup \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((0,0) + 2^k(0,2) + t(1,1) + 2^k \cdot 4\mathbb{Z}^2) \cup \bigcup_{t=-\infty}^{\infty} \{t(1,1)\}, \\ U_2 &= \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-1,0) + 2^k(2,0) + t(-1,1) + 2^k \cdot 4\mathbb{Z}^2) \\ &\quad \cup \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-1,0) + 2^k(0,2) + t(-1,1) + 2^k \cdot 4\mathbb{Z}^2) \\ &\quad \cup \bigcup_{t=0}^{\infty} \{(0,-1) + t(1,-1)\}, \\ U_3 &= \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-1,-1) + 2^k(2,0) + t(-1,-1) + 2^k \cdot 4\mathbb{Z}^2) \end{aligned}$$

$$\cup \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-1,-1) + 2^k(0,2) + t(-1,-1) + 2^k \cdot 4\mathbb{Z}^2),$$

$$U_4 = \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((0,-1) + 2^k(2,0) + t(1,-1) + 2^k \cdot 4\mathbb{Z}^2)$$

$$\cup \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((0,-1) + 2^k(0,2) + t(1,-1) + 2^k \cdot 4\mathbb{Z}^2)$$

$$\cup \bigcup_{t=0}^{\infty} \{(-1,0) + t(-1,1)\}.$$

Each of these decompositions is basically into cosets, with the exception of three trailing sets in types 1, 2, 4 which we designate by V_1, V_2, V_4, respectively.

We can prove the correctness of this as follows:

Let U_1', U_2', U_3', U_4' be the sets on the right-hand sides, respectively. Note that

(i) The generating set A_2 is contained in U_i' adequately, i.e.

$$(0,0) \in U_1', \quad (0,-1) \in U_2', \quad (-1,-1) \in U_1', \quad (-1,0) \in U_4'.$$

(ii) Claim that $U_i' \supset \bigcup_{j=1}^4 \Phi_{ij} U_j'$, $i = 1,2,3,4$. Check that for any i,

$$\begin{aligned} h_i(U_i') &\subset \bigcup_{\substack{j=1 \\ j \neq i, i\pm 2}}^{4} \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-e_i) + 2^{k+1}(2(e_i - e_j)) + 2t(e_{i\pm 2} - e_i) \\ &\qquad + 2^{k+1} \cdot 4\mathbb{Z}^2) \cup V_i \\ &\subset U_i', \\ h_{i\pm 2}(U_i') &\subset \bigcup_{\substack{j=1 \\ j \neq i, i\pm 2}}^{4} \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-e_i) + 2^{k+1}(2(e_i - e_j)) + (2t+1)(e_{i\pm 2} - e_i) \\ &\qquad + 2^{k+1} \cdot 4\mathbb{Z}^2) \cup V_i \\ &\subset U_i', \\ h_i(U_l') &\subset (-2e_l + e_i + 4\mathbb{Z}^2) \cup (-2e_{l\pm 2} + e_i + 4\mathbb{Z}^2) \\ &\subset U_i', \quad \text{where} \quad l \neq i, \quad i \pm 2, \quad l \in \{1,2,3,4\}. \end{aligned}$$

(iii) U_i', $i = 1,2,3,4$, are all disjoint. Two cosets or non-coset sets chosen from U_i' and U_j', where $j \neq i, i \pm 2$, cannot intersect, since they are different modulo 2. Futhermore, two of the cosets or non-coset sets chosen from U_i' and $U_{i\pm 2}'$ cannot intersect either, since for $a + 2^k \cdot 4\mathbb{Z}^2 \subset U_i'$, $b + 2^l \cdot 4\mathbb{Z}^2 \subset U_{i\pm 2}'$ with $k \leq l$, $a - b \neq 0 \bmod 2^k \cdot 4\mathbb{Z}^2$.

Now since U_1', U_2', U_3', U_4' are generated from A_2 by Φ, $U_i \subset U_i'$ for all $i = 1,2,3,4$. Also from $\bigcup U_i = \mathbb{Z}^2$, we get $\bigcup U_i' = \mathbb{Z}^2$. Since all U_i', $i = 1,2,3,4$, are disjoint, $U_i = U_i'$ for all $i = 1,2,3,4$.

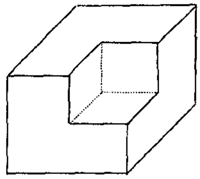

Fig. 6. Three-dimensional chair tile.

Finally, for any $i = 1, 2, 3, 4$, the cosets within U_i are clearly disjoint, so

$$c(U_i) \geq 2 \cdot \sum_{k=0}^{\infty} \sum_{t=0}^{2^k-1} \frac{1}{(2^k \cdot 4)^2} = 2 \cdot \sum_{k=0}^{\infty} \frac{2^k}{16 \cdot (2^k)^2} = \frac{1}{4}.$$

Thus $\sum_{i=1}^{4} c(U_i) = 1$. Theorem 4 shows that U_i, $i = 1, 2, 3, 4$, are regular model sets.

II. *Chair Tiling in* $\mathbb{R}^n$

In this section we generalize the foregoing to the n-dimensional chair tilings for all $n \geq 2$. The n-chair is an n-cube with a corner taken out of it, Fig. 6 The inflation rule, which we spell out algebraically below, is geometrically the obvious generalization of the two-dimensional case.

We transform the geometry by replacing each chair by a $2^n - 1$ oriented cube, as before, and coordinatize the lattice formed by the centres of the cubes, starting from the basic generating set $A_n := \{(x_1, \ldots, x_n) \mid x_i \in \{0, -1\}\}$. There are 2^n orientations of cubes and hence 2^n *types* of points (but only $2^n - 1$ of these types appear in the starting set A_n).

For each $k \geq 0$ let $\beta(k)$ be the binary expansion $\varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \cdots$ of k, $\varepsilon_l \in \{0, 1\}$. We define the basic orientation vectors $e_1, \ldots, e_{2^n}$ by

$$e_i := \begin{cases} (\varepsilon_0, \ldots, \varepsilon_{n-1}) \text{ the binary digits of } \beta(i-1) & \text{if } i \leq 2^{n-1}, \\ (1, \ldots, 1) - e_{i-2^{n-1}} & \text{if } i > 2^{n-1}. \end{cases}$$

We determine the sets U_i, $i = 1, \ldots, 2^n$, of all i-type points in $\mathbb{Z}^n$ from the points of the basic generating set A_n, using the inflation rules below.

The types of the points of A_n are as follows: for $x = (x_1, \ldots, x_n) \in A_n$,

when $x_n = -1$;
  $x \in U_i$, for which $\beta(i-1) = (1, \ldots, 1) + x$,
when $x_n = 0$;
  if $x = (0, \ldots, 0)$, $x \in U_1$
  otherwise, $x \in U_{i+2^{n-1}}$, for which $\beta(i-1) = (1, \ldots, 1) - ((1, \ldots, 1) + x)$.

The idea of considering our vectors in the form $(1, \ldots, 1) + x$ is to make it easy to compare them with the basic orientation vectors.

This conforms with what happens when $n = 2$: there are $2^n - 1$ types in the basic starting set that are in $2^{n-1} - 1$ complementary pairs and one pair of vectors $(0, \ldots, 0)$ and $(-1, \ldots, -1)$ of the same type, namely of type 1.

Define

$$f_{j,i}\colon U_i \to U_j \quad \text{by} \quad (x,i) \mapsto (2x+e_j, j) \qquad \text{if} \quad j \neq i \pm 2^{n-1},$$
$$f^{(2)}_{i,i}\colon U_i \to U_i \quad \text{by} \quad (x,i) \mapsto (2x+e_j, i) \qquad \text{if} \quad j = i \pm 2^{n-1},$$

where

$$i, j \in \{1, \dots, 2^n\}, \qquad x \in \mathbb{Z}^n, \qquad i \pm 2^{n-1} := \begin{cases} i + 2^{n-1} & \text{if} \quad i \leq 2^{n-1}, \\ i - 2^{n-1} & \text{if} \quad i > 2^{n-1}. \end{cases}$$

Let Φ be the matrix function system. Define $h_i\colon x \mapsto 2x + e_i$, $i \in \{1, \dots, 2^n\}$,

$$\Phi = \begin{pmatrix} \{h_1, h_{1+2^{(n-1)}}\}\{h_1\} \cdots \overset{\overset{1+2^{n-1}}{\downarrow}}{\{\}} \cdots \{h_1\} \\ \vdots \\ \{h_{2^n}\}\{h_{2^n}\} \cdots \overset{\overset{2^n-2^{n-1}}{\downarrow}}{\{\}} \cdots \{h_{2^n-2^{n-1}}, h_{2^n}\} \end{pmatrix}.$$

Inflating A_n by the maps, we get the precise description of U_i:

$$U_i = \bigcup_{\substack{j=1 \\ j \neq i, i\pm 2^{n-1}}}^{2^n} \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} \left((-e_i) + 2^k(2(e_i - e_j)) + t(e_{i\pm 2^{n-1}} - e_i) + 2^k \cdot 4\mathbb{Z}^n\right) \cup V_i,$$

where

$$V_i = \begin{cases} \bigcup_{t=-\infty}^{\infty} \{t(e_{1\pm 2^{n-1}} - e_1)\} & \text{if} \quad i = 1, \\ \bigcup_{t=0}^{\infty} \{t(e_i - e_{i\pm 2^{n-1}}) + (-e_{i\pm 2^{n-1}})\} & \text{if} \quad i \neq 1, \quad 1 \pm 2^{n-1}, \\ \emptyset & \text{if} \quad i = 1 + 2^{n-1}. \end{cases} \tag{42}$$

The equalities can be proved in the same way as in the two-dimensional case.

Let U'_i be the set of the right-hand side in (42). Note that:

(i) The generating set A_n is contained in U'_i adequately, i.e.

$$\begin{aligned} e_1 &\in U'_1 && \text{if} \quad i = 1, \\ -e_{i\pm 2^{n-1}} &\in U'_i && \text{if} \quad i \neq 1, \quad 1 \pm 2^{n-1}, \\ -e_{1\pm 2^{n-1}} &\in U'_1 && \text{if} \quad i = 1 \pm 2^{n-1}. \end{aligned}$$

(ii) Claim that $U'_i \supset \bigcup_{j=1}^{2^n} \Phi_{ij} U'_j$, $i = 1, \dots, 2^n$. Indeed for $i \in \{1, \dots, 2^n\}$,

$$\begin{aligned} h_i(U'_i) \subset & \bigcup_{\substack{j=1 \\ j \neq i, i\pm 2^{n-1}}}^{2^n} \bigcup_{k=0}^{\infty} \bigcup_{t=0}^{2^k-1} ((-e_i) + 2^{k+1}(2(e_i - e_j)) + 2t(e_{i\pm 2^{n-1}} - e_i) \\ & \qquad + 2^{k+1} \cdot 4\mathbb{Z}^n) \cup V_i \\ \subset & \; U'_i, \end{aligned}$$

$$\begin{aligned} h_{i\pm 2^{n-1}}(U_i') &\subset \bigcup_{\substack{j=1\\ j\neq i,i\pm 2^{n-1}}}^{2^n} \bigcup_{k=0}^{\infty}\bigcup_{t=0}^{2^k-1}((-e_i)+2^{k+1}(2(e_i-e_j))+(2t+1)\\ &\qquad \times (e_{i\pm 2^{n-1}}-e_i)+2^{k+1}\cdot 4\mathbb{Z}^n)\cup V_i\\ &\subset U_i',\\ h_i(U_l') &\subset (-2e_l+e_i+4\mathbb{Z}^n)\cup(-2e_{l\pm 2^{n-1}}+e_i+4\mathbb{Z}^n)\\ &\subset U_i', \qquad \text{where} \quad l\neq i, \quad i\pm 2^{n-1}, \quad l\in\{1,\dots,2^n\}. \end{aligned}$$

(iii) $U_i', i=1,\dots,2^n$, are disjoint. Two cosets or non-coset sets chosen from U_i' and U_j', where $j\neq i, i\pm 2^{n-1}$, cannot intersect, since they are different modulo 2. Futhermore, two of the cosets or non-coset sets chosen from U_i' and $U'_{i\pm 2^{n-1}}$ cannot intersect either, since for $a+2^k\cdot 4\mathbb{Z}^n\subset U_i', b+2^l\cdot 4\mathbb{Z}^n\subset U'_{i\pm 2^{n-1}}$ with $k\le l, a-b\neq 0 \bmod 2^k\cdot 4\mathbb{Z}^n$.

Now since $U_i', i=1,\dots,2^n$, are generated from A_n by Φ, $U_i\subset U_i'$ for all $i=1,\dots,2^n$. Also from $\bigcup_{i=1}^{2^n}U_i=\mathbb{Z}^n$, $\bigcup_{i=1}^{2^n}U_i'=\mathbb{Z}^n$. Since all $U_i', i=1,\dots,2^n$, are disjoint, $U_i=U_i'$ for all $i=1,\dots,2^n$.

For any $i=1,\dots,2^n$, all the cosets in U_i' are disjoint, so

$$c(U_i)\ge (2^n-2)\cdot\sum_{k=0}^{\infty}\sum_{t=0}^{2^k-1}\frac{1}{(2^k\cdot 4)^n}=(2^n-2)\cdot\sum_{k=0}^{\infty}\frac{2^k}{2^{2n}\cdot(2^k)^n}=\frac{1}{2^n}.$$

Thus $\sum_{i=1}^{2^n}c(U_i)=1$. Theorem 4 shows that $U_i, i=1,\dots,2^n$, are regular model sets.

To get a model set interpretation of the chair tiling itself we proceed as follows. We observe that every arrow points to the inner corner of exactly one chair. We label each chair by its inner corner point which is at the tip of exactly 2^n-1 arrows. These corner points give us 2^n sets $X_1,\dots,X_{2^n}$ according to the type, and all lie in the shift $L'=(\frac{1}{2},\dots,\frac{1}{2})+\mathbb{Z}^n$ of our lattice $\mathbb{Z}^n$. Let $f_i, i=1,\dots,2^n$, be $(\frac{1}{2},\dots,\frac{1}{2})-e_i$, respectively. Then U_i+f_i is the set of tips of all arrows of type i and $U_i+f_i=L'\cap(V_i+f_i)$, for some $V_i\subset\mathbb{Z}_2^n$ for which $\overline{V_i}$ is compact, $\overset{\circ}{V_i}\neq\emptyset$, and $\mu(\partial V_i)=0$. Now

$$X_i=L'\cap\left(\bigcap_{j\neq i\pm 2^{n-1}}(V_j+f_j)\right)$$

which is the required regular model set description of X_i, since

$$\partial\left(\bigcap_{j\neq i\pm 2^{n-1}}(V_j+f_j)\right)\subset\bigcup_{j\neq i\pm 2^{n-1}}\partial(V_j+f_j)$$

and $\mu(\partial(V_j+f_j))=0$ for all $j=1,\dots,2^n$.

From this result we can show that if we mark each chair with a single point in a consistent way, then the set of points obtained from all the chairs of any one type also forms a regular model set, and hence a pure point diffractive set.

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