LATTICE THEORY OF GENERALIZED PARTITIONS

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1. Introduction. In **(1)** the lattice of all equivalence relations on a set S was studied and many important properties were established. In **(2)** and **(3)** the lattice of all geometries on a set S was studied and it was shown to be a universal[1] lattice which shares many properties with the lattice of equivalence relations on S. In this paper we shall give the definition of a partition of type n and investigate the lattice formed by all partitions of type n on a fixed set S. It will be seen that a partition of type one on S can be considered as an equivalence relation on S and similarly a partition of type two on S can be considered as a geometry on S as defined in **(2)**. Thus we shall obtain a unified theory of lattices of equivalence relations, lattices of geometries and partition lattices of higher types. We shall observe that most of the properties which hold for the partition lattices of type one and two hold for partition lattices of any type. We shall first show that a partition lattice of type n on a set S is a complete point lattice which is isomorphic to the lattice of subspaces of a suitably chosen geometry. A characterization of the lattice of equivalence relations on S was given in **(1)**. We shall give a similar characterization of the lattice of all geometries on S (that is, the lattice of partitions of type two on S) by characterizing the geometries whose lattices of subspaces are isomorphic to the lattice of geometries. We shall then show that the lattices of partitions of any type are complemented and special properties of these complements will be investigated. It shall further be shown that these lattices are simple and the groups of automorphisms will be constructed. Finally we shall investigate the complete homomorphisms of lattices of subspaces of geometries and characterize them in terms of polygons in the geometry. We shall conclude by stating some unsolved problems.

2. Generalized partitions.

Definition 1. A partition of type $n, n \geqslant 1$, on the set S consisting of n or more elements is a collection of subsets of S such that any n distinct elements of S are contained in exactly one subset and every subset contains at least n distinct elements.

It can be seen that the subsets of a partition of type one on S define an equivalence relation on S and vice versa. Similarly a geometry on S is equivalent to a partition of type two on S if we consider the lines defining the geometry as the subsets which form the partition.

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[1]Any finite lattice is isomorphic to a sublattice of the lattice of all geometries on some finite set.

We recall that a subset T of S is said to be a subspace of a geometry G on S if with any two distinct elements of T the line containing these elements is in T. Thus the set of all subspaces of a geometry G on S ordered under set inclusion forms a complete lattice.

We shall refer to the subsets defining a partition as blocks. A block of a partition of type n is said to be non-trivial if it consists of at least $n + 1$ distinct elements. Otherwise we shall call the block trivial. We shall represent a partition P by the set of its non-trivial blocks, $P = \{S_\alpha\}$.

Let us now order the set of all partitions of type n on S by defining $P_1 \leqslant P_2$ if and only if every block of P_1 is contained in a block of P_2. Under this ordering it is a partially ordered set which is closed under arbitrary intersections. To see this we just have to note that if $\{P_\alpha \mid \alpha \in A\}$ is any set of partitions of type n on S then the partition, whose blocks containing any n prescribed points $x_1, x_2, \ldots, x_n$ of S are obtained by intersecting the corresponding blocks of P_α, $\alpha \in A$, is the g.l.b. of $\{P_\alpha \mid \alpha \in A\}$. Since this partially ordered set of partitions has a unit element we conclude that it is a complete lattice. We shall denote it by $L\,P_n(S)$. To simplify the statements of the theorems we shall first agree to define $L\,P_n(S)$ to consist of a single element if S contains less than n elements, secondly, let $L\,P_0(S)$ denote the Boolean algebra of all subsets of S. It can be seen that $L\,P_n(S)$, $|S| > n \geqslant 1$, is a point lattice and that the points are partitions consisting of only one non-trivial block and this block contains $n + 1$ distinct elements.

THEOREM 1. *$L\,P_n(S)$ is isomorphic to the lattice of subspaces of some geometry.*

Proof. The result holds for $n = 0$, since $L\,P_0(S)$ is isomorphic to the lattice of subspaces of the geometry on S in which every line consists of exactly two points. By our previous convention the result holds trivially for $L\,P_n(S)$, $n \geqslant 1$, if S consists of n or less elements. If S consists of more than n elements then the union in $L\,P_n(S)$ of any two distinct points $P_1 = \{(a_1, a_2, \ldots, a_{n+1})\}$ and $P_2 = \{(b_1, b_2, \ldots, b_{n+1})\}$ is either a partition with only one non-trivial block and then this block consists of $n + 2$ elements, or it is a partition consisting of two non-trivial blocks, that is, $P_1 \cup P_2 = \{(a_1, a_2, \ldots, a_{n+1}), (b_1, b_2, \ldots, b_{n+1})\}$. In either case $P_1 \cup P_2$ covers P_1 and P_2. This implies that the sets of points of $L\,P_n(S)$, which are contained in unions of two distinct points, form the set of lines for a geometry on the set of points of $L\,P_n(S)$. We now observe that if T is a subspace of this geometry and if for a point Q of $L\,P_n(S)$ we have that $Q \leqslant \bigcup \{P | P \in T\}$, then $Q \in T$. This implies that $L\,P_n(S)$ is isomorphic to the lattice of subspaces of the geometry defined by the unions of pairs of points of $L\,P_n(S)$.

3. Characterization of LG(S). We shall now characterize the geometries whose lattices of subspaces are isomorphic to the partition lattices of type two on S.

We shall introduce some concepts which are essential for the following theory.

Definition 2. Two distinct points p and q of a geometry G are said to be *related* if the line defined by p and q is non-trivial.

Definition 3. The points p and q of a geometry G are said to be *close* if p is equal to q, p is related to q, or there exists a point t of G such that p is related to t and t is related to q.

Definition 4. Let the line l of G consist of the four distinct points p_1, p_2, p_3, p_4, and let π denote the set consisting of the three collinear points p_1, p_2, and p_3. Then a point q of G is said to be *close* to π if one of the two following conditions holds:

(i) q is equal to p_1, p_2, or p_3,

(ii) q is close to p_1, p_2, p_3; q is distinct from p_4 and not related to p_4.

For further discussion π will denote a triplet of distinct collinear points of G.

Definition 5. π_1 is said to be *close* to π_2 if every point in π_1 is close to π_2.

THEOREM 2. *Let L be the lattice of subspaces of a geometry G on W and let W consist of four or more points. Then L is isomorphic to the lattice of all geometries on some set S if and only if G satisfies the following five axioms*:

Axiom 1. The non-trivial lines of G consist of four points and every point is contained in at least one non-trivial line.

Axiom 2. If a point p is close to π_1 and π_1 is close to π_2 then p is close to π_2.

Axiom 3. If π_1 is close to π_2 then π_2 is close to π_1.

Axiom 4. If π_1, π_2, π_3 are distinct then there exists a point p such that p is close to π_1, π_2, π_3.

Axiom 5. Let l be a non-trivial line and let p be a point which is not on this line but is close to every point on this line, then p is related to exactly two points of l.

To show that L is isomorphic to the lattice of all geometries on some set S we have to show that there exists a one-to-one mapping of the set W onto the set of points of $LG(S)$ ($LG(S) = L\,P_2(S)$) and that this mapping preserves lines. To do this we shall introduce the concept of a *star* of G. Let π be a triplet of distinct collinear points of G. Then the set of all points which are close to π will be called the star of G defined by π. This set will be denoted by $\Delta(\pi)$. We shall show that the lattice of all geometries on the set of stars of G is isomorphic to L. We know that a point of $LG(S)$ is a geometry with only one non-trivial line and this line consists of three points. Thus every point of $LG(S)$ is characterized by the three elements of S which are contained in its non-trivial line. Therefore we first have to establish a one-to-one mapping of the set W onto the set of all triplets consisting of distinct stars of G. We shall do this by showing that every point of G is contained in exactly three distinct stars and that any three distinct stars have exactly one point of G in common. The proof consists of the following lemmas.

LEMMA 1. *If the point p is close to π_1 then there exists π_2 such that p is contained in π_2 and π_2 is close to π_1.*

Proof. Let π_1 consist of the distinct points p_1, p_2, p_3 and let the fourth collinear point of this line l be p_4. If p is contained in l then p must be equal to p_1, p_2, or p_3 since by Definition 4 the fourth collinear point p_4 is not close to π_1. Thus we may set $\pi_2 = \pi_1$ since then p is contained in π_2 and by Definition 4 and Definition 5 we see that π_1 is close to π_1. Let us now assume that p is not on the line l but is related to a point of π_1. Let this point be p_1 as indicated in Figure 1. Then p_4 is related to p_1 and p_1 is related to every point of the

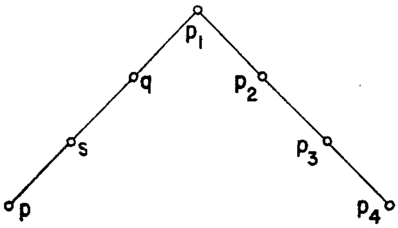

FIGURE 1

line defined by p and p_1. Thus by Definition 3, p_4 is close to every point on this line and therefore by Axiom 5, p_4 is related to exactly two of the four distinct points of this line. We know that p_4 is related to p_1 and let us denote the second point to which it is related by s. Since the line defined by p and p_1 consists of four points there must exist a point q on this line which is distinct from p and is not related to p_4. Let π_2 consist of p, p_1, and q. We see that p is contained in π_2 and we shall show that π_2 is close to π_1. By Definition 5 we have to show that every point of π_2 is close to π_1. p_1 is contained in π_1 and therefore by Definition 4 close to π_1. p and q are close to every point in π_1 and not related to p_4. Thus by Definition 4 they are close to π_1. It follows that π_2 is close to π_1. We may now assume that p is not related to any point on the line l. Since p is close to p_1 there exists a point t of G such that p is related to t and t is related to p_1 as shown in Figure 2. Using the result of the previous case we know that there exists a triplet π_3 on the line defined by p_1 and t such that t is contained in π_3 and π_3 is close to π_1. Let us denote

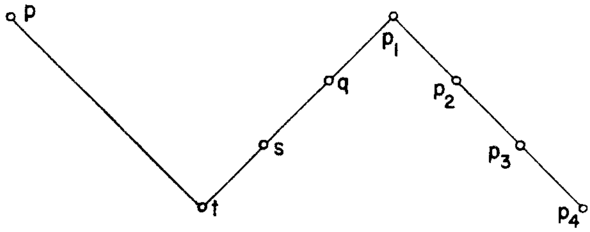

FIGURE 2

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the point of this line which is not contained in π_3 by s as we did in the previous case. We recall that s is related to p_4. Thus p cannot be related to s since otherwise p is related to s and s related to p_4 which implies that p is close to p_4 and therefore close to every point on l. From this we would conclude that p is related to exactly two points on the line l, contrary to assumption. But then p is close to π_3 and is related to t which is contained in π_3. Using again the result of the previous case there exists a triplet π_2 on the line defined by p and t such that p is contained in π_2 and π_2 is close to π_3. Now we have that p is contained in π_2, π_2 is close to π_3, and π_3 is close to π_1. Thus by Axiom 2 we conclude that π_2 is close to π_1. This completes the proof of Lemma 1.

LEMMA 2. *Any there distinct collinear points contained in a star define the star.*

Proof. Let π_1 be contained in $\Delta(\pi_2)$. Then every point of π_1 is close to π_2 and therefore π_1 is close to π_2. By Axion 3, π_2 is close to π_1 and therefore by Axiom 2 every point which is close to π_1 is close to π_2 and vice versa. From this it follows that $\Delta(\pi_1) = \Delta(\pi_2)$.

LEMMA 3. *Every point p of G is contained in at least three distinct stars.*

Proof. By Axiom 1 a point p of G is contained in at least one non-trivial line l and this line consists of four points. There are exactly three distinct triplets π_1, π_2, and π_3 of l which contain p. The fourth collinear point of l which is not contained in π_1 is by Definition 4 not contained in $\Delta(\pi_1)$. But this point is contained in π_2 and π_3 and therefore it is contained in $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Thus $\Delta(\pi_1)$ is distinct from $\Delta(\pi_2)$ and $\Delta(\pi_3)$. Similarly we show that $\Delta(\pi_2)$ is distinct from $\Delta(\pi_3)$. This shows that there are at least three distinct stars which contain p.

LEMMA 4. *There are exactly three distinct stars contining every point p of G.*

Proof. Let p be contained in a non-trivial line l and let π_1, π_2, and π_3 be the distinct triplets of l which contain p. By the previous result we know that the stars $\Delta(\pi_1)$, $\Delta(\pi_2)$, and $\Delta(\pi_3)$ are distinct. Let p be contained in some star $\Delta(\pi)$; we shall show that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$, or $\Delta(\pi_3)$. Note that if p is contained in $\Delta(\pi)$ then by Lemma 1 there exists a triplet π' such that p is contained in π' and π' is close to π. If π' is contained in the line l then it must be equal to π_1, π_2, or π_3 and therefore by Axiom 3 and Axiom 2 we conclude that $\Delta(\pi)$ is equal to $\Delta(\pi_1)$, $\Delta(\pi_2)$, or $\Delta(\pi_3)$. Thus we may assume that π' is contained in a non-trivial line l' and that l' is distinct from l. Let us denote the point of l' which is not contained in π' by q. Since p is contained in π' and therefore in l' we see that q is related to p and p is related to every element of the line l. Thus q is close to every point on the line l and therefore related to exactly two points of l. We know that q is related to p. Let the second point to which p is related be denoted by s. One of the triplets π_1, π_2, or π_3 does not contain the point s, say π_1. Then π_1 is close to π' since p

is contained in π_1 and the two remaining points of π_1 are close to every point in π' and not related or equal to q. From the fact that π_1 is close to π' it follows by Axiom 3 and Axiom 2 that $\Delta(\pi_1) = \Delta(\pi')$; this proves Lemma 4.

By Axiom 4 any three distinct stars have at least one point in common. The next lemma will show that there cannot be more than one such point.

LEMMA 5. *Any three distinct stars have exactly one point in common.*

Proof. Let p and q be distinct points of L. We shall show that the three distinct stars which contain p cannot all contain q. Let p be contained in the non-trivial line l and let π_1, π_2, and π_3 be the distinct triplets of l which contain p. These triplets define the three distinct stars which contain p. If q is also contained in the line l then q is not contained in one of these triplets. Let this triplet be π_1. Then q is not contained in the star $\Delta(\pi_1)$ since q is the point of l which is not contained in π_1. Thus we may assume that p and q are not related, which implies that q cannot be close to π_1, π_2, and π_3. Since if q would be close to π_1, π_2, and π_3 then q would be close to every point on l and therefore q would be related to exactly two elements of l. But then q would be related to an element of l which is not contained in one of the triplets π_1, π_2, or π_3 and therefore q would not be close to one of these triplets, contrary to the assumption. Thus q is not contained in one of the three stars which contain p. This proves Lemma 5.

So far we have shown that there exists a one-to-one mapping of the set W onto the set of points of $LG(S)$, where S is the set of stars of G. Let us denote this mapping by θ.

LEMMA 6. *The mapping θ preserves lines.*

Proof. Let l be a non-trivial line of G. Then l contains four distinct triplets π_1, π_2. π_3, π_4 and these triplets define the four distinct stars Δ_1, Δ_2, Δ_3, Δ_4, respectively. Every point of the line l is contained in three of these triplets and therefore in three of these stars. Under the mapping θ the line l is mapped into the line l' of $LG(S)$ which consists of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$, and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. This shows that every point on the line l is mapped into a point of the corresponding line l' of $LG(S)$. Conversely, let l' be a non-trivial line of $LG(S)$ and let this line consist of the four points $\{(\Delta_1, \Delta_2, \Delta_3)\}$, $\{(\Delta_1, \Delta_2, \Delta_4)\}$, $\{(\Delta_1, \Delta_3, \Delta_4)\}$ and $\{(\Delta_2, \Delta_3, \Delta_4)\}$. Let $\{(\Delta_1, \Delta_2, \Delta_3)\}$ and $\{(\Delta_1, \Delta_2, \Delta_4)\}$ be mapped onto the points p and q respectively. Let p be contained in a non-trivial line l of G. We know that the triplets π_1, π_2, π_3 of l which contain p define the stars Δ_1, Δ_2, Δ_3, respectively. Then q is contained in Δ_1 and Δ_2 and therefore q is close to every point on l. Thus q is related to exactly two points of l. Thus q has to be related to p since otherwise one of the triplets π_1 or π_2 would not contain a point of l to which q is related and therefore q would not be close to π_1 or π_2, contrary to assumption. Thus p and q are related. Without loss of generality we may assume that p and q are contained in l and that π_1, π_2, π_3, π_4 are the distinct triplets of l. Then π_1, π_2, π_3, and π_4 define the stars Δ_1, Δ_2, Δ_3, and Δ_4, respectively. Clearly every three distinct triplets of the set π_1, π_2, π_3, π_4 have a point in common and this point is contained in l. This shows that every point of the line l' is mapped into a point on the corresponding line l. Thus the mapping preserves lines and we conclude that L is isomorphic to $LG(S)$.

A straightforward computation shows that the five axioms of Theorem 2 hold in $LG(S)$ if S consists of four or more elements.

Thus we have completed the proof.

4. Lattice theoretic properties of $L\,P_n(S)$. We shall now study the lattice theoretic properties of $L\,P_n(S)$.

THEOREM 3. *For each given point P of $L\,P_n(S)$ and a given integer m, $n \geqslant m \geqslant 0$, there exists a complete sublattice L of $L\,P_n(S)$ such that*

(i) *L is isomorphic to $L\,P_m(S - \{a_1 \vee a_2 \vee \ldots \vee a_{n-m}\})$,*

(ii) *L contains the point P,*

(iii) *the unit and zero elements of L and $L\,P_n(S)$ coincide.*

Proof. Let a point $P = \{a_1 \vee a_2 \vee \ldots \vee a_{n+1}\}$ of $L\,P_n(S)$ be given. If $m = n$ then the theorem is trivially satisfied. Otherwise we let M denote the set $\{a_1, a_2, \ldots, a_{n+1}\}$. Let us now consider all the elements of $L\,P_n(S)$ whose non-trivial blocks contain M and let us denote this set by L. It can be seen that P, O, and I are in L and that L is closed under arbitrary intersections in $L\,P_n(S)$. Furthermore for any subset $\{A_\alpha\}$ of L and C in $L\,P_n(S)$ such that $A_\alpha \leqslant C$ there exists C' in L such that $A_\alpha \leqslant C' \leqslant C$. To obtain C' from C we just remove all the non-trivial blocks of C which do not contain the set M and replace them by the necessary trivial blocks. From this we conclude that $\bigcap\{A \mid A \geqslant A_\alpha\} = \bigcup\{A_\alpha\}$ is an element of L, which shows that L is a complete sublattice of $L\,P_n(S)$.

To show that L is isomorphic to a partition lattice of type n-m on S-M we observe that after the removal of the set M any two blocks of an element in L can have almost $n - m - 1$ points in common. Thus any element of L can be considered as a partition of type $n - m$ on $S - M$ and to every partition of this type there corresponds a unique element of $L\,P_n(S)$. Since this one-to-one correspondence is order-preserving we conclude that L is isomorphic to $L\,P_{n-m}(S - M)$.

THEOREM 4. *$L\,P_n(S)$ is complemented.*

Proof. $L\,P_0(S)$ is known to be complemented. We shall give a general proof for $n \geqslant 1$ which will include the cases previously proven for $n = 1$ and 2. To construct a complement for a partition P of $L\,P_n(S)$, $P \neq I, O$, we let A be a subset of S such that A has at most n points in common with any block of P. The collection of all such sets forms a partially ordered set under set inclusion and by the Maximal Principle it follows that there exists a maximal element in this partially ordered set. Let us denote this maximal set by M and let us consider the partition $\{M\}$ whose only non-trivial block is M. We first observe that $\{M\} \cap P = 0$. We shall now show that $\{M\} \cup P = I$. Let the block of $\{M\} \cup P$ which contains M be denoted by K. Then $\{M\} \cup P$ consists of K and the blocks of P which are not contained in K. But this will be shown to imply that $K = S$. If there should exist an x in S and not in M then, recalling that M was a maximal set, there must exist $n + 1$ distinct elements $x, x_1, x_2, \ldots, x_n$ which are in $M \vee \{x\}$ and some block B of P. But then $x_1, x_2, \ldots, x_n$ are in K and B and therefore $B \subseteq K$. Thus x is in K and we conclude that $K = S$. Which proves that $L\,P_n(S)$ is complemented.

We observe that the complement of P which was constructed in the previous proof has only one non-trivial block and that we can construct this block so that it contains any n prescribed points of S. Thus if P is not the zero or the unit element of $L\,P_n(S)$ it contains a block which contains at least $n + 1$ distinct elements. We can pick two distinct sets of n elements from this block and for each set construct a complement of P whose non-trivial block contains this set. It can be seen that these complements are distinct. Thus we have proven the following result.

COROLLARY 1. *If P in $L\,P_n(S)$, $n \geqslant 1$, has a unique complement then $P = I$ or 0.*

COROLLARY 2. *$L\,P_n(S)$ contains a sublattice isomorphic to a Boolean algebra and every element of $L\,P_n(S)$ has a complement in this sublattice.*

Proof. The corollary holds for $n = 0$. Otherwise we know from Theorem 3 that the set of elements of $L\,P_n(S)$ whose non-trivial blocks contain the fixed set consisting of n distinct elements $a_1, a_2, \ldots, a_n$ forms a sublattice L which is isomorphic to a Boolean algebra. On the other hand, we know from the preceding remarks that every element of $L\,P_n(S)$ has a complement in this sublattice. This completes the proof.

Let us now investigate the homomorphisms of $L\,P_n(S)$, $n \geqslant 1$.

THEOREM 5. *There are only trivial homomorphisms of $L\,P_n(S)$, $n \geqslant 1$.*

Proof. In **(1)** Ore showed that $L\,P_1(S)$ has only trivial homomorphisms. This will be shown to imply that $L\,P_n(S)$, $n \geqslant 1$, has only trivial homomorphisms. To see this we recall that if θ is a homomorphism on a point lattice which identifies at least two distinct elements then at least one point has to be mapped into the zero element. Thus a point P of $L\,P_n(S)$ has to be mapped into the zero element. On the other hand, by Theorem 3 we know that there exists a sublattice L of $L\,P_n(S)$ which is isomorphic to $L\,P_1(S)$ and which contains P. But then θ identifies two distinct elements of $L \cong L\,P_1(S - M)$ which implies that all elements of L are identified. Thus, since the zero and unit elements of L and $L\,P_n(S)$ coincide, we conclude that all elements of $L\,P_n(S)$ are identified.

THEOREM 6. *The group of automorphisms of $L\,P_n(S)$ is isomorphic to the symmetric group on S if S consists of more than n elements.*

Proof. The result has been shown to hold for $n = 0, 1, 2$. Our proof will hold for $n \geqslant 1$. We note that any automorphism of $L\,P_n(S)$ has to map an element of the form $\{S - p\}$, p in S, onto some other element of the same type. This induces a permutation on the set S and clearly to every permutation on the set S there corresponds an automorphism of $L\,P_n(S)$. Furthermore we know that every element of $L\,P_n(S)$ can be written as a union of intersections of elements of the form $\{S - p\}$. Thus we conclude that the group of automorphisms of $L\,P_n(S)$ is isomorphic to the symmetric group on S.

In previous research and in this paper, it was seen that the concept of a geometry and the lattice of subspaces of this geometry does appear in many mathematical investigations. We shall now show the connection between the non-trivial polygons in the geometry G and the complete homomorphisms of the lattice of subspaces of G.

We shall call a line l of G non-trivial if l consists of more than two points. Let us call a set of non-trivial lines $l_1, l_2, \ldots, l_n$ a non-trivial polygon if, considering $\{l_1\}, \{l_2\}, \ldots, \{l_n\}$ as partitions of type one on S, we have $\{l_1\} \cup \{l_2\} \cup \ldots \cup \{l_n\} = \{l_1 \vee l_2 \vee \ldots \vee l_n\}$. It can be seen that the non-trivial polygons induce an equivalence relation on the points of the geometry G if we define a and b to be in the same equivalence class if and only if a and b can be connected by a non-trivial polygon.

THEOREM 7. *There is a one-to-one order preserving correspondence between the complete homomorphisms of the lattice of G and the subsets of the equivalence classes defined in G by the non-trivial polygons.*

Proof. Let S consist of three or more elements. Then the lattice of subspaces of the geometry G on S, which has only one line $l = S$ has only trivial homomorphisms. Note that if $S = \{p_1, p_2, \ldots, p_n\}$ then the lattices of subspaces is $\{S > p_1, p_2, \ldots, p_n > \phi\}$, but this lattice is known to have only trivial homomorphisms. This implies that if a point p of any geometry G on S is mapped into the zero element by a homomorphism θ then the points on any non-trivial line which contains p are mapped by θ into the zero element. But then all the points contained in the equivalence class, induced by the non-trivial polygons of G, which contains p are identified with the zero element. Thus every homomorphism θ has to map all the points contained in a subset of the equivalence classes into the zero element. Conversely, to every subset of the equivalence classes there corresponds a homomorphism which identifies all the points in these equivalence classes with the zero element. Since every complete homomorphism θ on a complete point lattice is uniquely defined by the set of points which θ maps into the zero element we see that we have established a one-to-one order preserving mapping between the complete homomorphisms and the subsets of the equivalence classes.

The previous proof implies further the following result.

COROLLARY 3. *The lattice of complete homomorphisms of the lattice of subspaces of G on S is isomorphic to the Boolean algebra on the set of equivalence classes on S induced by the polygons of G.*

Finally we observe that any two complete homomorphisms on the lattice of subspaces of a geometry G permute.

So far we have characterized the complete homomorphisms of the lattices of subspaces of geometries. It remains an unsolved problem whether there are any incomplete homomorphisms in these lattices and if so how can these geometries be characterized. Furthermore, an interesting problem is to determine which geometries have complemented lattices of subspaces. Certainly one of the most interesting unsolved problems in lattice theory is Birkhoff's **(4)** problem number 48 which can be stated as follows: Is every finite lattice isomorphic to a sublattice of $L\,P_1(S)$ for some finite set S? So far we know by **(2)** and Theorem 3 that every finite lattice is isomorphic to a sublattice of $L\,P_n(S)$, S finite, $n \geqslant 2$.

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