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LATTICE-VALUED BOREL MEASURES

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ABSTRACT. A Riesz representation type theorem is proved for measures on locally compact spaces, taking values in some ordered vector spaces.

In a series of papers ([4], [5], [6]), J. M. Maitland Wright has established, among other things, some Riesz representation type theorems for positive linear mappings from $C(X)$ to E, X being a compact Hausdorff space and E a complete (or σ-complete) vector-lattice. In this paper we prove these results (Theorem 4) by using the properties of order convergence in vector lattices.

We shall use the notations of ([2], [3]). For a compact Hausdorff space X, we denote by $C(X)$ the vector space of all continuous real-valued functions on X with sup norm, by $L(X)$ and $M(X)$ the dual and bidual of $C(X)$, respectively, and by $\beta(X)$ and $\beta_1(X)$ the sets of all bounded Borel and Baire measurable real-valued functions on X, respectively. In the natural order $C(X)$ is a vector lattice and $\beta(X)$ and $\beta_1(X)$ are boundedly σ-complete lattices. Also $L(X)$ and $M(X)$ are boundedly complete vector lattices and $C(X)$ is a sublattice of $M(X)$. Let $S(X)$ be the subspace of $M(X)$ generated by those elements of $M(X)$ which are suprema of bounded subsets of $C(X)$.

Let E be a vector lattice (always assumed to be over the field of real numbers). Order convergence, order closure (σ-closure), order continuity (σ-continuity) in vector lattices are taken in the usual sense ([1], [2], [3]). If A is a subset of E, let A_1 be the set of order limits, in E, of sequences in A, A_2 be the set of order limits of sequences in $A \cup A_1$ $(= A_1)$, and so on. Continuing this process transfinitely, if necessary, and taking the union of all these subsets, we get the order σ-closure of the set A. A vector subspace B of E we shall call monotone order closed (σ-closed), if for any net (sequence) $\{x_\alpha\}$, such that $x_\alpha \uparrow x$ in E, $x \in B$ ($x_\alpha \uparrow x$ means $\{x_\alpha\}$ is increasing and its sup is x). Now if A is a vector sublattice of a boundedly σ-complete vector lattice E, E_1 a monotone order σ-closed vector subspace of E, and $E_1 \supset A$, then $E_1 \supset A_1$ (A_1 as defined above); since A_1 is also a vector sublattice of E, $E_1 \supset A_2$, and so continuing this (transfinitely if necessary) we get $E_1 \supset$ order σ-closure of A. This result will be needed later. Monotone order continuity (σ-continuity) can be defined between ordered vector spaces in a similar way. For any real-valued function on a topological space Z, $\sup f=$ closure of $\{z \in Z : f(z) \neq 0\}$ in Z.

The order σ-closure of $S(X)$ ($C(X)$), in $M(X)$, will be denoted by Bo(X) (Ba(X)). We denote by $B(X)$ the set of all bounded real-valued functions on X with the natural point-wise order. $\beta_1(X)$ is the order σ-closure of $C(X)$ in $B(X)$, and $\beta(X)$ is the order σ-closure of the vector space generated by bounded lower semicontinuous functions on X. If X is Stonian (σ-Stonian), $C(X)$ is a boundedly complete (σ-complete) vector lattice, and in this case $H = \{f \in B(X) : \exists g \in C(X)$ such that $f = g$ except on a meagre subset of $X\} \supset \beta(X)$ ($\beta_1(X)$); this gives a mapping $\psi : \beta(X) \to C(X)$ ($\psi_1 : \beta_1(X) \to C(X)$). We prove first the following simple lemmas.

Lemma 1. *There exists a* 1-1, *onto, linear, both way positive, mapping* $\varphi : \mathrm{Bo}(X) \to \beta(X)$ ($\varphi_1 : \mathrm{Ba}(X) \to \beta_1(X)$), *such that*

(i) $\varphi(f) = f$ ($\varphi_1(f) = f$), $\forall f \in C(X)$;

(ii) $\varphi, \varphi^{-1}, \varphi_1, \varphi_1^{-1}$ *are all order σ-continuous;*

(iii) *for any increasing net* $\{f_\alpha\}$ *in* $C(X)$, $\varphi(\sup f_\alpha) = \sup \varphi(f_\alpha)$, *and* $\varphi^{-1}(\sup f_\alpha) = \sup \varphi^{-1}(f_\alpha)$.

Proof. On $B(X)$, the space of all bounded, real-valued functions on X, we take the topology of point-wise convergence. Since the identity map $i : (C(X), \|\circ\|) \to B(X)$ is a weakly compact linear operator, its second adjoint $i'' : (M(X), \sigma(M(X), L(X))) \to B(X)$ is continuous, and so is order continuous, since order convergence in $M(X)$ implies $\sigma(M(X), L(X))$-convergence. This means that for an increasing net $\{f_\alpha\}$ in $C(X)$, $\varphi(\sup\{f_\alpha\}$ — in $M(X)) = \sup\{f_\alpha\}$ — in $B(X)$. This proves that $i''^{-1}(\beta(X)) \supset S(X)$ and is order σ-closed, and so $i''^{-1}(\beta(X)) \supset$ Bo(X); similar results hold for $\beta_1(X)$. Let $\varphi = i'' \mid \mathrm{Bo}(X)$ ($\varphi_1 = i'' \mid \mathrm{Ba}(X)$). Then $\varphi : \mathrm{Bo}(X) \to \beta(X)$ ($\varphi_1 : \mathrm{Ba}(X) \to \beta_1(X)$). If $f \in \mathrm{Bo}(X)$ and $f \geqq 0$, then there exists a net $\{f_\alpha\} \subset C(X)$ such that $f_\alpha \xrightarrow{o} f$ (i.e., order converges to f in $M(X)$) [2]. This means $f_\alpha^+ \xrightarrow{o} f$, and so $f_\alpha^+ \xrightarrow{o} \varphi(f)$ which means that $\varphi(f) \geqq 0$. Now suppose that for some $f \in \mathrm{Bo}(X)$ $\varphi(f) = 0$. Take $\{f_\alpha\} \subset C(X)$ such that $f_\alpha \xrightarrow{o} f$ and so $f_\alpha \xrightarrow{o} \varphi(f) = 0$. This means $f_\alpha(x) \to 0$, $\forall x \in X$, and so $\langle f, \epsilon_x \rangle = 0$, $\forall$ point measure ϵ_x in $L(x)$, which proves that $f = 0$ ([2], p. 83), and thus φ is 1-1. To prove that φ^{-1} is positive, take $f \in \mathrm{Bo}(X)$, such that $\varphi(f) \geqq 0$. There exists a net $\{f_\alpha\} \subset C(X)$ such that $f_\alpha \xrightarrow{o} f$ in M, which means that $f_\alpha^+ \xrightarrow{o} f^+$ and $f_\alpha^- \xrightarrow{o} f^-$. This gives that $\lim f_\alpha^-(x) = 0$, $\forall x \in X$, and so $\langle f^-, \epsilon_x \rangle = 0$ for any point measure ϵ_x in $L(X)$ which means $f^- = 0$ ([2], p. 83). This proves φ^{-1} is positive. To prove that φ is onto, take a lower semi-continuous function f in $B(X)$. Then there exists an increasing net $\{f_\alpha\}$ in $C(X)$ such that $f_\alpha \uparrow f$. Taking

$g = \sup\{f_\alpha^-\}$ in $M(X)$, we get $f = \varphi(g)$. Also if an increasing sequence $h_n \uparrow h$ in $B(X)$, by positivity of φ^{-1}, $g_n = \varphi^{-1}(h_n)$ is increasing in $M(X$; and so $\varphi(g) = h$, where $g = \sup\{g_n$, in $M(X)$. This proves $\varphi$is onto. The order σ-continuity and other properties of φ^{-1} are easily verified. Similar arguments prove the corresponding results for φ_1. This completes the proof.

LEMMA 2. *If X is Stonian (σ-Stonian), the mapping $\psi : \beta(X) \to C(X)$ ($\psi_1 : \beta_1(X) \to C(X)$) is a positive order σ-continuous linear mapping. Also if $\{f_\alpha\}$ is an increasing net in $C(X)$ such that $\sup f_\alpha = f$ in $\beta(X)$, then $\psi(f) = \sup \psi(f_\alpha)$, if X is Stonian.*

PROOF. The linearity and positivity of ψ are obvious. Also if $\{f_\alpha\}$ is an increasing net in $C(X)$, then pointwise $\sup\{f_\alpha\} = f$ and $\sup\{f_\alpha\} = h^-$ in $C(X)$ are equal except on a meagre subset of X [4], and so $\psi(f) = h = \sup \psi(f_\alpha)$. If $\{h_n\}$ is an increasing sequence in $\beta(X)$ such that $h_n = f_n \in C(X)$ on $X \backslash A_n$, A_n being meagre for every n, and $h_n \uparrow h$ in $\beta(X)$, then $f = \sup f_n^-$ in $C(X)$, and $g =$ pointwise $\sup\{f_n\}$ are equal on $X \backslash A$, A being a meagre subset of X. This proves $\psi(h_n) \uparrow \psi(h)$, and so ψ is order σ-continuous. The corresponding results for ψ_1 can be proved in a similar way.

LEMMA 3. *Let X and S be compact Hausdorff spaces with S also a Stonian (σ-Stonian) space, and $\mu : C(X) \to C(S)$ a positive linear mapping. Then μ can be uniquely extended to a positive linear mapping $\bar{\mu} : \beta(X) \to C(S)$ ($\bar{\mu} : \beta_1(X) \to C(S)$), satisfying the following conditions.*

(i) *$\bar{\mu}$ is order σ-continuous;*

(ii) *for any increasing net $\{f_\alpha\} \subset C(X)$, with $\sup f_\alpha = f$ in $\beta(X)$, $\bar{\mu}(f) = \sup \bar{\mu}(f_\alpha)$, in case X is Stonian.*

PROOF. Assume first that S is Stonian. The second adjoint of $\mu : C(X) \to C(S)$, $\mu'' : M(X) \to M(S)$, is an order-continuous positive linear mapping ([3], p. 525), and so μ''^{-1} (Bo(S)) $\supset$ Bo(X). Using Lemmas 2 and 3 we get a mapping $\bar{\mu} : \beta(X) \to C(S)$, satisfying the conditions of the lemma. If ν is another extension satisfying the conditions of the theorem, then $\bar{\mu}$ and ν are equal on the subspace generated by l.s.c. bounded functions on X, and so by order σ-continuity, they are equal on $\beta(X)$. The σ-Stonian case can be dealt with in a similar way.

Let Y be a locally compact Hausdorff space, $\beta'(Y)$ ($\beta_1'(Y)$) all bounded Borel (Baire) measurable functions with compact supports, $B'(Y)$ all bounded real-valued functions on Y with compact supports, and $K(Y)$ all continuous real-valued functions on Y with compact supports. For any open (open F_σ) relatively compact subset $V \subset Y$, let

$\beta'(Y, V) = \{f \in \beta'(Y) : f \equiv 0 \text{ on } Y \setminus V\}$ ($\beta_1'(Y, V) = \{f \in \beta_1'(Y), f \equiv 0$ on $Y \setminus V\}$). If $K(Y, V) = \{f \in K(Y), \sup f \subset V\}$ and $S'(Y, V)$ is the subspace of $B'(Y)$ generated by $\{f \in B'(Y) : \exists$ an increasing net $\{f_\alpha\} \subset K(Y, V)$, with $\sup f_\alpha = f\}$, then $\beta'(Y, V) =$ order σ-closure of $S'(Y, V)$ ($\beta_1'(Y, V) =$ order σ-closure of $K(Y, V)$). Also $\beta'(Y) = \bigcup\{\beta'(Y, V) : V$ open relatively compact in $Y\}$ ($\beta_1'(Y, V) = \bigcup\{\beta_1'(Y, V) : V$ open F_σ, relatively compact in $Y\}$).

THEOREM 4. *Let E be a boundedly monotone complete (σ-complete) ordered vector and $\mu : K(Y) \to E$ a positive linear map. Then μ can be uniquely extended to $\bar{\mu} : \beta'(Y) \to E$ ($\bar{\mu} : \beta_1'(Y) \to E$) with the properties that* (i) *$\bar{\mu}$ is monotone order σ-continuous,* (ii) *for any increasing net $\{f_\alpha\}$ in $K(Y)$ with $\sup f_\alpha = f$ in $\beta'(Y)$, $\bar{\mu}(f) = \sup \mu(f_\alpha)$, in case E is boundedly monotone complete.*

PROOF. Let V be an open relatively compact subset of Y. Take $\{g_\alpha\}$ ($\alpha \in I$), an increasing net in $K(Y)$, with $\sup g_\alpha \subset V$, $0 \leqq g_\alpha \leqq 1$, $\forall\, \alpha$, and $\sup\{g_\alpha\} = \chi_V$. Also take $g \in K(Y)$, $0 \leqq g \leqq 1$, and $g = 1$ on V. Assuming E to be boundedly monotone complete, let $e = \sup\{\mu(g_\alpha) : \alpha \in I\}$ (note $\mu(g_\alpha)$ is increasing and $\mu(g_\alpha) \leqq \mu(g)$, $\forall\, \alpha$). For any $f \in K(Y)$, with $\operatorname{supp} f \subset V$, and $f \leqq \chi_V$ ($= \sup g_\alpha$), we first prove that $\mu(f) \leqq e$. Let $C = \sup f \subset V$, n any positive intéger and $V_\alpha = \{x \in V : f(x) < g_\alpha(x) + 1/n\}$. Using the facts that $\{V_\alpha\}$ is increasing and $\bigcup V_\alpha \supset C$, a compact set, we get $V_{\alpha(n)} \supset C$, for some $\alpha(n) \in I$. Thus $f < g_{\alpha(n)} + (1/n)g$ and so $\mu(f) \leqq e + (1/n)\, \mu(g)$, $\forall\, n$, which gives $\mu(f) \leqq e$, since $\inf\{(1/n)\mu(g) : n$, a positive integer$\} = 0$, (note $\mu(g) \geqq 0$).

Let $E_0 = \{p \in E : -\lambda e \leqq p \leqq \lambda e$, for some real $\lambda > 0\}$. Then E_0 is a boundedly monotone complete, directed, integrally closed, ordered vector subspace of E ([1], p. 290; to prove the integral closedness of E_0, we need the boundedly monotone σ-completeness of E). Thus the completion by non-void cuts of E_0, say E_1, will be a boundedly complete vector lattice ([1], Theorem 9, p. 357). Let $E_2 = \{p \in E_1 : -\lambda e \leqq p \leqq \lambda e$, for some $\lambda > 0\}$. This means E_2 is a boundedly complete vector lattice with a strong unit e and so there exists a compact Hausdorff Stonian space S, such that E_2 and $C(S)$ are vector lattice isomorphic (i.e., there exists a 1-1, onto, both-way positive linear map from E_2 to $C(S)$ which preserves arbitrary suprema and infima). Let $V' = V \cup \{x_0\}$ be the Alexandroff one point compactification of the locally compact space V (if V is compact, we take $V' = V$), and A the subspace of $C(V')$ generated by constant functions and $K(Y, V)$. Any element of A can be uniquely written in the form $\lambda + f$, where $\lambda \in R$,

$f \in K(Y, V)$. Define a linear mapping $\mu_0 : A \to C(S)$ as $\mu_0(\lambda + f) = \lambda e + \mu(f)$. We first prove that μ_0 is positive. Suppose first that $\lambda > 0$ and $\lambda + f \geqq 0$ on V'. This gives $1 + (1/\lambda)f \geqq 0$, and so $-(1/\lambda)f \leqq \sup g_\alpha$. From what is proved above it follows that $-(1/\lambda)\mu(f) \leqq e$, and so $\lambda e + \mu(f) \geqq 0$. If $\lambda = 0$, there is nothing to prove. If $\lambda < 0$ and V is not compact, take $x \in V \setminus \sup f$. Then $f(x) = 0$ and $\lambda + f(x) < 0$, a contradiction. If $\lambda < 0$ and V is compact, then $\chi_V \in K(Y, V)$, $\chi_V \geqq g_\alpha$, $\forall\, \alpha$, and $\chi_V \leqq \sup g_\alpha$. So from what is proved above it follows that $\mu(\chi_V) = e$. Now $\lambda + f \geqq 0$ on $V = V'$ implies that $\lambda\chi_V + f \geqq 0$, and so $\mu(\lambda\chi_V + f) \geqq 0$, which gives $\lambda e + \mu(f) \geqq 0$. This proves μ_0 is positive. Also considering A as a subspace of $C(V')$, with sup norm topology, μ_0 is also continuous and as such has a unique extension $\mu_V : C(V') \to C(S)$, since, by the Stone-Weierstrass approximation theorem, A is dense in $C(V')$. It is easy to verify that this extension is also a positive linear operator. By Lemma 3, μ_V can be uniquely extended to $\bar{\mu}_V : \beta(V') \to E_1$ which is order σ-continuous, and if $\{f_\alpha\}$ is an increasing net in $C(V')$ with $\sup f_\alpha = f \in \beta(V')$, then $\bar{\mu}_V(f) = \sup \mu_V(f_\alpha)$. It immediately follows that $\bar{\mu}_V\{\chi_{\{x_0\}} = 0$, i.e., $\bar{\mu}_V(f_1) = \bar{\mu}_V(f_2)$ if $f_i \in \beta(V')$ $(i = 1, 2)$ and $f_{1|V} = f_{2|V}$. We define $\bar{\bar{\mu}}_V : \beta'(Y, V) \to E_1$ as: for any $f \in \beta'(Y, V)$, $\bar{\bar{\mu}}_V(f) = \bar{\mu}_V(f')$, where $f = f'$ on V, and $f'(x_0) = 0$: this mapping is positive, linear and order σ-continuous and has the property that for any increasing net $\{f_\alpha\} \subset K(Y, V)$ with $\sup f_\alpha = f \in \beta'(Y, V)$, $\bar{\bar{\mu}}_V(f) = \sup \bar{\bar{\mu}}_V(f_\alpha)$. Now $\bar{\bar{\mu}}_V{}^{-1}(E_0) \supset K(Y, V)$, and so, by bounded monotone completeness at E_0, $\bar{\bar{\mu}}_V{}^{-1}(E_0) \supset S(Y, V)$: Since $\bar{\bar{\mu}}_V{}^{-1}(E_0)$ is a boundedly monotone order σ-closed (since $\bar{\bar{\mu}}_V$ is order σ-continuous) subspace of $B'(Y)$, and $S(Y, V)$ is a vector sublattice of $B'(Y)$, $\bar{\bar{\mu}}^{-1}(E_0) \supset$ order σ-closure of $S'(Y, Y) = \beta'(Y, V)$. Thus $\bar{\bar{\mu}}_V : \beta'(Y, V) \to E$ $(E \supset E_0)$ is a positive, linear, and monotone order σ-continuous map, and for any increasing net $\{f_\alpha\} \subset K(Y, V)$ with $\sup f_\alpha = f \in \beta'(Y, V)$, $\bar{\bar{\mu}}_V(f) = \sup \bar{\bar{\mu}}_V(f_\alpha)$. Now define $\bar{\mu} : \beta'(Y) \to E$ as: For any $f \in \beta'(Y)$, $f \in \beta'(Y, V)$ for some open relatively compact subset V of Y. We define $\bar{\mu}(f) = \bar{\bar{\mu}}_V(f)$. To see that this mapping is well-defined, let $f \in \beta'(Y, V_i)$ $(i = 1, 2)$; this means $f \in \beta'(Y, V_1 \cap V_2)$. Since $\bar{\bar{\mu}}_{V_1} = \bar{\bar{\mu}}_{V_2}$ on $K(Y, V_1 \cap V_2)$, they are equal on $S'(Y, V_1 \cap V_2)$ and so are equal on $\beta'(Y, V_1 \cap V_2) = \beta'(Y, V_1) \cap \beta'(Y, V_2)$ (using σ-continuity of these measures). This proves $\bar{\mu}$ is well-defined. Also it is easily verified that μ is linear, positive, monotone order σ-continuous, and for any increasing net $\{f_\alpha\}$ in $K(Y)$ with $\sup f_\alpha = f$ in $\beta'(Y)$, $\bar{\mu}(f) = \sup \bar{\mu}(f_\alpha)$. Uniqueness of $\bar{\mu}$ is easily verified. Also the case when E is boundedly monotone σ-complete can be proved in a similar way. This completes the proof.

REMARK. For compact Y, this result is proved in [6] by an entirely different method.

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