Lattices for Distributed Source Coding: Jointly Gaussian Sources and Reconstruction of a Linear Function

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Abstract. Consider a pair of correlated Gaussian sources (X_1, X_2). Two separate encoders observe the two components and communicate compressed versions of their observations to a common decoder. The decoder is interested in reconstructing a linear combination of X_1 and X_2 to within a mean-square distortion of D. We obtain an inner bound to the optimal rate-distortion region for this problem. A portion of this inner bound is achieved by a scheme that reconstructs the linear function directly rather than reconstructing the individual components X_1 and X_2 first. This results in a better rate region for certain parameter values. Our coding scheme relies on lattice coding techniques in contrast to more prevalent random coding arguments used to demonstrate achievable rate regions in information theory. We then consider the case of linear reconstruction of K sources and provide an inner bound to the optimal rate-distortion region. Some parts of the inner bound are achieved using the following coding structure: lattice vector quantization followed by "correlated" lattice-structured binning.

1 Introduction

In this work, we present a coding scheme for distributed coding of a pair of jointly Gaussian sources. The encoders each observe a different component of the source and communicate compressed versions of their observations to a common decoder through rate-constrained noiseless channels. The decoder is interested in reconstructing a linear function of the sources to within a mean squared error distortion of D.

The problem of distributed source coding to reconstruct a function of the sources losslessly was considered in [1]. An inner bound was obtained for the performance limit which was shown to be optimal if the sources are conditionally independent given the function. In [2], the performance limit is given for the case of lossless reconstruction of the modulo-2 sum of two correlated binary

* This work was supported by NSF grant (CAREER) CCF-0448115.

sources and was shown to be tight for the symmetric case. This has been extended to several cases in [3] (see Problem 23 on page 400) and [4]. An improved inner bound was provided for this case in [5]. The key point to note is that the performance limits given in [2, 4, 5] are outside the inner bound provided in [1]. While [1] employs random vector quantization followed by independent random binning , the coding schemes of [2, 4, 5] instead use structured random binning based on linear codes on finite fields. Further, the binning operation of the quantizers of the sources are "correlated". This incorporation of structure in binning appears to give improvements in rates especially for those cases that involve reconstruction of a function of the sources.

With this as motivation, in this paper we consider a lossy distributed coding problem with K jointly Gaussian sources with one reconstruction. The decoder wishes to reconstruct a linear function of the sources with squared error as fidelity criterion. We consider a coding scheme with the following structure: sources are quantized using structured vector quantizers followed by "correlated" structured binning. The structure used in this process is given by lattice codes. We provide an inner bound to the optimal rate-distortion region. We show that the proposed inner bound is better for certain parameter values than an inner bound that can be obtained by using a coding scheme that uses random vector quantizers following by independent random binning. For this purpose we use the machinery developed by [9–12] for the Wyner-Ziv problem in the quadratic Gaussian case.

The paper is organized as follows. In Section 2, we give a concise overview of the asymptotic properties of high-dimensional lattices that are known in the literature and we use these properties in the rest of the paper. In Section 3, we define the problem formally for the case of two sources and present an inner bound to the optimal rate-distortion region given by a coding structure involving structured quantizers followed by "correlated" structured binning. Further, we also present another inner bound achieved by a scheme that is based on the Berger-Tung inner bound. Then we present our lattice based coding scheme and prove achievability of the inner bound. In Section 4, we consider a generalization of the problem that involves reconstruction of a linear function of an arbitrary finite number of sources. In Section 5, we provide a set of numerical results for the two-source case that demonstrate the conditions under which the lattice based scheme performs better than the Berger-Tung based scheme. We conclude with some comments in Section 6. We use the following notation throughout this paper. Variables with superscript n denote an n-length random vector whose components are mutually independent. However, random vectors whose components are not independent are denoted without the use of the superscript. The dimension of such random vectors will be clear from the context.

2 Preliminaries on high-dimensional Lattices

2.1 Overview of Lattice Codes

Lattice codes play the same role in Euclidean space that linear codes play in Hamming space. Introduction to lattices and to coding schemes that employ lattice codes can be found in [9–11]. In the rest of this section, we will briefly review some properties of lattice codes that are relevant to our coding scheme. We use the same notation as in [10] for these quantities.

An n-dimensional lattice Λ is composed of all integer combinations of the columns of an $n \times n$ matrix G called the generator matrix of the lattice. Associated with every lattice Λ is a natural quantizer namely one that associates with every point in $\mathbb{R}^n$ its nearest lattice point. This quantizer can be described by the function $Q_\Lambda(x)$. The quantization error associated with the quantizer $Q_\Lambda(\cdot)$ is defined by $x \bmod \Lambda = x - Q_\Lambda(x)$. This operation satisfies the useful distribution property

$$((x \bmod \Lambda) + y) \bmod \Lambda = (x + y) \bmod \Lambda \quad \forall\, x, y. \tag{1}$$

The basic Voronoi region $\mathcal{V}_0(\Lambda)$ of the lattice Λ is the set of all points closer to the origin than to any other lattice point. Let $V(\Lambda)$ denote the volume of the Voronoi region of Λ. The second moment of a lattice Λ is the expected value per dimension of the norm of a random vector uniformly distributed over $\mathcal{V}_0(\Lambda)$ and is given by

$$\sigma^2(\Lambda) = \frac{1}{n} \frac{\int_{\mathcal{V}_0(\Lambda)} \parallel x \parallel^2 \mathrm{dx}}{\int_{\mathcal{V}_0(\Lambda)} \mathrm{dx}} \tag{2}$$

The normalized second moment is defined as $G(\Lambda) \triangleq \sigma^2(\Lambda)/V^{2/n}(\Lambda)$.

In [12], the existence of high dimensional lattices that are "good" for quantization and for coding is discussed. The criteria used therein to define goodness are as follows:

– A sequence of lattices $\Lambda^{(n)}$ (indexed by the dimension n) is said to be a good channel σ_Z^2-code sequence if $\forall \epsilon > 0$, $\exists N(\epsilon)$ such that for all $n > N(\epsilon)$ the following conditions are satisfied for some $E(\epsilon) > 0$:

$$V(\Lambda^{(n)}) < 2^{n(\frac{1}{2}\log(2\pi e \sigma_Z^2)+\epsilon)} \quad \text{and} \quad P_e(\Lambda^{(n)}, \sigma_Z^2) < 2^{-nE(\epsilon)}. \tag{3}$$

  Here P_e is the probability of decoding error when the lattice points of $\Lambda^{(n)}$ are used as codewords in the problem of coding for the unconstrained AWGN channel with noise variance σ_Z^2 as considered by Poltyrev [13].
– A sequence of lattices $\Lambda^{(n)}$ (indexed by the dimension n) is said to be a good source D-code sequence if $\forall \epsilon > 0, \exists N(\epsilon)$ such that for all $n > N(\epsilon)$ the following conditions are satisfied:

$$\log(2\pi e G(\Lambda^{(n)})) < \epsilon \quad \text{and} \quad \sigma^2(\Lambda^{(n)}) = D. \tag{4}$$

2.2 Nested Lattice Codes

For lossy coding problems involving side-information at the encoder/decoder, it is natural to consider nested codes [10]. We review the properties of nested lattice codes here. Further details can be found in [10].

A pair of n-dimensional lattices (Λ_1, Λ_2) is nested, i.e., $\Lambda_2 \subset \Lambda_1$, if their corresponding generating matrices G_1, G_2 satisfy $G_2 = G_1 \cdot J$ where J is an $n \times n$ integer matrix with determinant greater than one. Λ_1 is referred to as the fine lattice while Λ_2 is the coarse lattice. In many applications of nested lattice codes, we require the lattices involved to be a good source code and/or a good channel code. We term a nested lattice (Λ_1, Λ_2) good if (a) the fine lattice Λ_1 is both a good δ_1-source code and a good δ_1-channel code and (b) the coarse lattice Λ_2 is both a good δ_2-source code and a δ_2-channel code. The existence of good lattice codes and good nested lattice codes (for various notions of goodness) has been studied in [11, 12, 14] which use the random coding method of [15]. Using the results of [11, 12], it was shown in [14] that good nested lattices in the sense described above do exist.

3 Distributed source coding for the two-source case

3.1 Problem Statement and Main Result

In this section we consider a distributed source coding problem for the reconstruction of the linear function $Z \triangleq F(X_1, X_2) = X_1 - cX_2$. Consideration of this function is enough to infer the behavior of any linear function $c_1X_1 + c_2X_2$ and has the advantage of fewer variables.

Consider a pair of correlated jointly Gaussian sources (X_1, X_2) with a given joint distribution $p_{X_1X_2}(x_1, x_2)$. The source sequence (X_1^n, X_2^n) is independent over time and has the product distribution $\prod_{i=1}^{n} p_{X_1X_2}(x_{1i}, x_{2i})$. The fidelity criterion used is average squared error. Given such a jointly Gaussian distribution $p_{X_1X_2}$, we are interested in the optimal rate-distortion region which is defined as the set of all achievable tuples (R_1, R_2, D) where achievability is defined in the usual Shannon sense. Here D is the mean squared error between the function and its reconstruction at the decoder. Without loss of generality, the sources can be assumed to have unit variance and let the correlation coefficient $\rho > 0$. In this case, $\sigma_Z^2 \triangleq \mathrm{Var}(Z) = 1 + c^2 - 2\rho c$.

We present the rate region of our scheme below.

Theorem 3.1. *The set of all tuples of rates and distortion* (R_1, R_2, D) *that satisfy*

$$2^{-2R_1} + 2^{-2R_2} \leq \left(\frac{\sigma_Z^2}{D}\right)^{-1} \tag{5}$$

are achievable.

Proof: See Section 3.2.

We also present an achievable rate region based on ideas similar to Berger-Tung coding scheme [6, 7].

Theorem 3.2. *Let the region $\mathcal{RD}_{in}$ be defined as follows.*

$$\mathcal{RD}_{in} = \bigcup_{(q_1,q_2)\in\mathbb{R}_+^2} \Bigg\{(R_1, R_2, D) \colon R_1 \geq \frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_1(1+q_2)},$$

$$R_2 \geq \frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_2(1+q_1)},\ R_1+R_2 \geq \frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_1 q_2}$$

$$D \geq \frac{q_1\alpha + q_2 c^2\alpha + q_1 q_2 \sigma_Z^2}{(1+q_1)(1+q_2)-\rho^2}\Bigg\}. \tag{6}$$

where $\alpha \triangleq 1-\rho^2$ and $\mathbb{R}_+$ is the set of positive reals. Then the rate distortion tuples (R_1, R_2, D) which belong to $\mathcal{RD}_{in}^*$ are achievable where * denotes convex closure.

Proof: Follows directly from the application of Berger-Tung inner bound with the auxiliary random variables involved being Gaussian.

For certain values of ρ, c and D, the sum-rate given by Theorem 3.1 is better than that given in Theorem 3.2. This implies that each rate region contains rate points which are not contained in the other. Thus, an overall achievable rate region for the coding problem can be obtained as the convex closure of the union of all rate distortion tuples (R_1, R_2, D) given in Theorems 3.1 and 3.2. A further comparison of the two schemes is presented in Section 5. Note that for $c < 0$, it has been shown in [8] that the rate region given in Theorem 3.2 is tight.

3.2 The Coding Scheme

In this section, we present a lattice based coding scheme for the problem of reconstructing the above linear function of two jointly Gaussian sources whose performance approaches the inner bound given in Theorem 3.1. In what follows, a nested lattice code is taken to mean a sequence of nested lattice codes indexed by the lattice dimension n.

We will require nested lattice codes $(\Lambda_{11}, \Lambda_{12}, \Lambda_2)$ where $\Lambda_2 \subset \Lambda_{11}$ and $\Lambda_2 \subset \Lambda_{12}$. We need the fine lattices Λ_{11} and Λ_{12} to be good source codes (of appropriate second moment) and the coarse lattice Λ_2 to be a good channel code. The proof of the existence of such nested lattices was shown in [14]. The parameters of the nested lattice are chosen to be

$$\sigma^2(\Lambda_{11}) = q_1,\ \sigma^2(\Lambda_{12}) = \frac{D\sigma_Z^2}{\sigma_Z^2 - D} - q_1, \quad \text{and } \sigma^2(\Lambda_2) = \frac{\sigma_Z^4}{\sigma_Z^2 - D} \tag{7}$$

where $0 < q_1 < D\sigma_Z^2/(\sigma_Z^2 - D)$. The coding problem is non-trivial only for $D < \sigma_Z^2$ and in this range, $D\sigma_Z^2/(\sigma_Z^2 - D) < \sigma^2(\Lambda_2)$ and therefore $\Lambda_2 \subset \Lambda_{11}$ and $\Lambda_2 \subset \Lambda_{12}$ indeed.

Let U_1 and U_2 be random vectors (dithers) that are independent of each other and of the source pair (X_1, X_2). Let U_i be uniformly distributed over the basic Voronoi region $\mathcal{V}_{0,1i}$ of the fine lattices Λ_{1i} for $i = 1, 2$. The decoder is assumed to share this randomness with the encoders. The source encoders use these nested lattices to quantize X_1 and cX_2 respectively according to equation

$$S_1 = (Q_{\Lambda_{11}}(X_1^n + U_1)) \mod \Lambda_2\,,\; S_2 = (Q_{\Lambda_{12}}(cX_2^n + U_2)) \mod \Lambda_2. \quad (8)$$

Note that the second encoder scales the source X_2 before encoding it. The decoder receives the indices S_1 and S_2 and reconstructs

$$\hat{Z} = \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)\left([(S_1 - U_1) - (S_2 - U_2)] \mod \Lambda_2\right). \quad (9)$$

In general, the rate of a nested lattice encoder (Λ_1, Λ_2) with $\Lambda_2 \subset \Lambda_1$ is given by $R = \frac{1}{2}\log\frac{\sigma^2(\Lambda_2)}{\sigma^2(\Lambda_1)}$. Thus, the rates of the two encoders are given by

$$R_1 = \frac{1}{2}\log\frac{\sigma_Z^4}{q_1(\sigma_Z^2 - D)} \quad \text{and} \quad R_2 = \frac{1}{2}\log\frac{\sigma_Z^4}{D\sigma_Z^2 - q_1(\sigma_Z^2 - D)} \quad (10)$$

Clearly, for a fixed choice of q_1 all rates greater than those given in equation (10) are achievable. The union of all achievable rate-distortion tuples (R_1, R_2, D) over all choices of q_1 gives us an achievable region. Eliminating q_1 between the two rate equations gives the rate region claimed in Theorem 3.1. It remains to show that this scheme indeed reconstructs the function Z to within a distortion D. We show this in the following.

Using the distributive property of lattices described in equation (1), we can reduce the coding scheme to a simpler equivalent scheme by eliminating the first mod-Λ_2 operation in both the signal paths. The decoder can now be described by the equation

$$\hat{Z} = \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)\left([(X_1^n + e_{q_1}) - (cX_2^n + e_{q_2})] \mod \Lambda_2\right) \quad (11)$$

$$= \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)\left([Z^n + e_{q_1} - e_{q_2}] \mod \Lambda_2\right) \quad (12)$$

where e_{q_1} and e_{q_2} are dithered lattice quantization noises given by

$$e_{q_1} = Q_{\Lambda_{11}}(X_1^n + U_1) - (X_1^n + U_1)\,,\; e_{q_2} = Q_{\Lambda_{12}}(cX_2^n + U_2) - (cX_2^n + U_2). \quad (13)$$

The subtractive dither quantization noise e_{q_i} is independent of both sources X_1 and X_2 and has the same distribution as $-U_i$ for $i = 1, 2$ [10]. Since the dithers U_1 and U_2 are independent and for a fixed choice of the nested lattice e_{q_i} is a function of U_i alone, e_{q_1} and e_{q_2} are independent as well. Let $e_q = e_{q_1} - e_{q_2}$ be the effective dither quantization noise. The decoder reconstruction in equation (12) can be simplified as

$$\hat{Z} = \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)\left([Z^n + e_q] \mod \Lambda_2\right) \stackrel{c.d}{=} \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)(Z^n + e_q) \quad (14)$$

$$= Z^n + \left(\left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)e_q - \frac{D}{\sigma_Z^2}Z^n\right) \triangleq Z^n + N. \quad (15)$$

The $\stackrel{c.d}{=}$ in equation (14) stands for equality under the assumption of correct decoding. Decoding error occurs if equation (14) doesn't hold. Let P_e be the probability of decoding error. Assuming correct decoding, the distortion achieved by this scheme is the second moment per dimension[1] of the random vector N in equation (15). This can be expressed as

$$\frac{\mathbb{E} \parallel N \parallel^2}{n} = \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)^2 \frac{\mathbb{E} \parallel e_q \parallel^2}{n} + \left(\frac{D}{\sigma_Z^2}\right)^2 \frac{\mathbb{E} \parallel Z^n \parallel^2}{n} \tag{16}$$

where we have used the independence of e_{q_1} and e_{q_2} to each other and to the sources X_1 and X_2 (and therefore to $Z = X_1 - cX_2$). Since e_{q_i} has the same distribution as $-U_i$, their expected norm per dimension is just the second moment of the corresponding lattice $\sigma^2(\Lambda_{1i})$. Hence the effective distortion achieved by the scheme is

$$\frac{1}{n}\mathbb{E}\|Z^n - \hat{Z}\|^2 = \left(\frac{\sigma_Z^2 - D}{\sigma_Z^2}\right)^2 \left(\frac{D\sigma_Z^2}{\sigma_Z^2 - D}\right) + \frac{D^2\sigma_Z^2}{\sigma_Z^4} = D. \tag{17}$$

Hence, the proposed scheme achieves the desired distortion provided correct decoding occurs at equation (14). Let us now prove that equation (14) indeed holds with high probability for an optimal choice of the nested lattice, i.e., there exists a nested lattice code for which $P_e \to 0$ as $n \to \infty$ where, $P_e = Pr\left((Z^n + e_q) \bmod \Lambda_2 \neq (Z^n + e_q)\right)$.

To this end, let us first compute the normalized second moment of $(Z^n + e_q)$.

$$\frac{1}{n}\mathbb{E} \parallel Z^n + e_q \parallel^2 = \sigma_Z^2 + q_1 + \frac{\sigma_Z^2 D}{\sigma_Z^2 - D} - q_1 = \sigma^2(\Lambda_2). \tag{18}$$

It was shown in [9] that as $n \to \infty$, the quantization noises e_{q_i} tend to a white Gaussian noise for an optimal choice of the nested lattice. It can be shown that, under these conditions, e_q also tends to a white Gaussian noise of the same variance as e_q. The proof involves entropy power inequality and is omitted.

We choose Λ_2 to be an exponentially good channel code in the sense defined in Section 2.1 (also see [10]). For such lattices, the probability of decoding error $P_e \to 0$ exponentially fast if $(Z^n + e_q)$ is Gaussian. The analysis in [11] showed that if $(Z^n + e_q)$ tends to a white Gaussian noise vector, the effect on P_e of the deviation from Gaussianity is sub-exponential and the overall error behavior is asymptotically the same. This implies that the reconstruction error $Z^n - \hat{Z}$ tends in probability to the random vector N defined in equation (15). Since all random vectors involved have finite normalized second moment, this convergence in probability implies convergence in second moment as well, i.e., $\frac{1}{n}\mathbb{E} \parallel Z^n - \hat{Z} \parallel^2 \to D$. Averaged over the random dithers U_1 and U_2, we have shown that the appropriate distortion is achieved. Hence there must exist a pair of deterministic dithers that also achieve distortion D and we have proved the claim of Theorem 3.1.

[1] We refer to this quantity also as the normalized second moment of the random vector N. This should not be confused with the normalized second moment of a lattice as defined in Section 2.1.

4 Distributed source coding for the K source case

In this section, we consider the case of reconstructing a linear function of an arbitrary number of sources. In the case of two sources, the two strategies used in Theorems 3.1 and 3.2 were direct reconstruction of the function Z and estimating the function from noisy versions of the sources respectively. In the presence of more than two sources, a host of strategies which are a combination of these two strategies become available. Some sets of sources might use the "correlated" binning strategy of Theorem 3.1 while others might use the "independent" binning strategy of Theorem 3.2. The union of the rate-distortion tuples achieved by all such schemes gives an achievable rate region for the problem.

Let the sources be given by $X_1, X_2, \ldots, X_K$ which are jointly Gaussian. The decoder wishes to reconstruct a linear function given by $Z = \sum_{i=1}^{K} c_i X_i$ with squared error fidelity criterion. The performance limit $\mathcal{RD}$ is given by the set of all rate-distortion tuples $(R_1, R_2, \ldots, R_K, D)$ that are achievable in the sense defined in Section 3.

For any set $A \subset \{1, \ldots, K\}$, let X_A denote those sources whose indices are in A, i.e., $X_A \triangleq \{X_i : i \in A\}$. Let Z_A be defined as $\sum_{i \in A} c_i X_i$. Let Θ be a partition of $\{1, \ldots, K\}$ with $\theta = |\Theta|$. Let $\pi_\Theta : \Theta \to \{1, \ldots, \theta\}$ be a permutation. One can think of π_Θ as ordering the elements of Θ. Each set of sources $X_A, A \in \Theta$ are decoded simultaneously at the decoder with the objective of reconstructing Z_A. The order of decoding is given by $\pi_\Theta(A)$ with the lower ranked sets of sources decoded earlier. Let $\mathcal{Q} = (q_1, \ldots, q_K) \in \mathbb{R}_+^K$ be a tuple of positive reals. For any partition Θ and ordering π_Θ, let us define recursively a positive-valued function $\sigma_\Theta^2 : \Theta \to \mathbb{R}^+$ as $\sigma_\Theta^2(A) = \mathbb{E}\left[(Z_A - f_A(S_A))^2\right]$ where $f_A(S_A) = \mathbb{E}(Z_A|S_A)$, $S_A = \{Z_B + Q_B : B \in \Theta, \pi_\Theta(B) < \pi_\Theta(A)\}$ and $\{Q_A : A \in \Theta\}$ is a collection of $|\Theta|$ independent zero-mean Gaussian random variables with variances given by $q_A = \mathrm{Var}(Q_A) \triangleq \sum_{i \in A} q_i$, and this collection is independent of the sources. Let $f(\{Z_A + Q_A : A \in \Theta\}) \triangleq \mathbb{E}\left(Z | \{Z_A + Q_A : A \in \Theta\}\right)$.

Theorem 4.1. *For a given tuple of sources $X_1, \ldots, X_K$ and tuple of real numbers $(c_1, c_2, \ldots, c_K)$, we have $\mathcal{RD}_{in}^* \subset \mathcal{RD}$, where * denotes convex closure and*

$$\mathcal{RD}_{in} = \bigcup_{\Theta, \pi_\Theta, \mathcal{Q}} \Bigg\{ (R_1, \ldots, R_K, D) : R_i \geq \frac{1}{2} \log \frac{\sigma_\Theta^2(A) + q_A}{q_i} \text{ for } i \in A,$$

$$D \geq \mathbb{E}\left[(Z - f(\{Z_A + Q_A : A \in \Theta\}))^2\right] \Bigg\} \tag{19}$$

Proof: This inner bound to the optimal rate region can be proved by demonstrating a coding scheme that achieves the rates given. As in Section 3.2, we use "correlated" binning based on lattice codes. The basic idea of the proof is to use high dimensional lattices to mimic the Gaussian test channels used in the description of Theorem 4.1. The details are omitted. We remark that the general K-user rate region described above can be used to re-derive Theorems 3.1 and 3.2 by appropriate choices of the partition Θ.

5 Comparison of the Rate Regions

In this section, we compare the rate regions of the lattice based coding scheme given in Theorem 3.1 and the Berger-Tung based coding scheme given in Theorem 3.2 for the case of two users. The function under consideration is $Z = X_1 - cX_2$. To demonstrate the performance of the lattice binning scheme, we choose the sum rate of the two encoders as the performance metric.

In Fig. 1, we compare the sum-rates of the two schemes for $\rho = 0.8$ and $c = 0.8$. Fig. 1 shows that for small distortion values, the lattice scheme achieves a smaller sum rate than the Berger-Tung based scheme. We observe that the lattice based scheme performs better than the Berger-Tung based scheme for small distortions provided ρ is sufficiently high and c lies in a certain interval. Fig. 2 is a contour plot that illustrates this in detail. The contour labeled R encloses that region in which the pair (ρ, c) should lie for the lattice binning scheme to achieve a sum rate that is at least R units less than the sum rate of the Berger-Tung scheme for some distortion D. Observe that we get improvements only for $c > 0$.

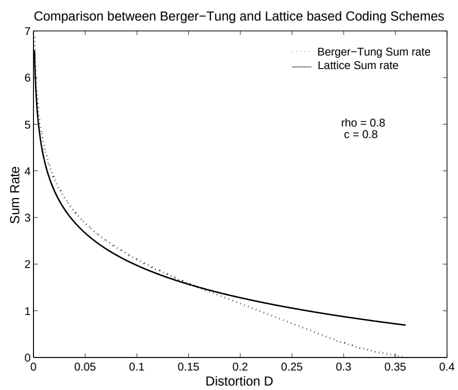

Fig. 1. Comparison of the sum-rates

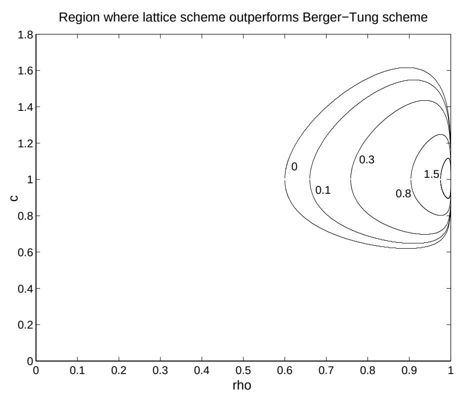

Fig. 2. (ρ, c) region for lower sum rate

6 Conclusion

We have thus demonstrated a lattice based coding scheme that directly encodes the linear function that the decoder is interested in instead of encoding the sources separately and estimating the function at the decoder. For the case of two users, it is seen that the lattice based coding scheme gives a lower sum-rate for certain values of ρ, c, D. Hence, using a combination of the lattice based and the Berger-Tung based coding schemes results in a better rate-region than using any one scheme alone. For the case of reconstructing a linear function of K sources, we have extended this concept to provide an inner bound to the optimal rate-distortion function. Some parts of the inner bound are achieved using a coding scheme that has the following structure: lattice vector quantization followed by "correlated" lattice-structured binning.

Acknowledgements

The authors would like to thank Dr. Ram Zamir and Dr. Uri Erez of Tel Aviv University for helpful discussions.

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