Lattices of intermediate and cylindric modal logics

Bezhanishvili, N.

Publication date
2006

Document Version
Final published version

Lattices of intermediate and cylindric modal logics

Nick Bezhanishvili

Lattices of intermediate and cylindric modal logics

ILLC Dissertation Series DS-2006-02

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

For further information about ILLC-publications, please contact

Institute for Logic, Language and Computation
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
phone: +31-20-525 6051
fax: +31-20-525 5206
e-mail: illc@science.uva.nl
homepage: http://www.illc.uva.nl/

Lattices of intermediate and cylindric modal logics

Academisch Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof.mr. P.F. van der Heijden
ten overstaan van een door het college voor
promoties ingestelde commissie, in het openbaar
te verdedigen in de Aula der Universiteit
op vrijdag 17 maart 2006, te 12.00 uur

door

Nikoloz Bezhanishvili

geboren te Tbilisi, Georgië.

Promotor: Prof.dr. D.H.J. de Jongh
Co-promotor: Dr. Y. Venema

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Printed and bound by PrintPartners Ipskamp
ISBN: 90–5776–147–5

Contents

Acknowledgments

First of all, I would like to thank my promotor Dick de Jongh for all the help and support he has given me throughout my time as a PhD student. Dick's door was always open for me, and he was always ready to share his knowledge and insights. I've learned a lot from Dick. Our long discussions would always result in clarifying things that looked vague to me beforehand. I very much enjoyed working with Dick, and his ideas always inspired me. I'd like to thank him for all his care during these four years and also for his help in writing this thesis.

Second, I'd like to thank my co-promotor Yde Venema who perfectly combined his roles as an excellent supervisor and a good friend. Yde was always there when I needed his help. His interesting ideas and remarkably clear way of expressing them were always encouraging. Yde's comments on drafts of this thesis were very helpful. I also have fond memories of all the trips organized by Yde to different parts of the Netherlands.

Next, I'd like to mention Leo Esakia. The long seminars that we had at Leo's place in Tbilisi are among my very fondest memories. I'd like to thank Leo for teaching me logic. His unique way of explaining complicated things in a simple way with a twist of humor was very inspiring. It is thanks to Leo that I ever got interested in logic. I'd also like to thank the other members of the logic group in Georgia, in particular, Dimitri Pataraya, Mamuka Jibladze, and, especially, Revaz Grigolia for many interesting conversations concerning various parts of this thesis. I also learned a lot from enlightening meetings that I had with Lazo Zambakhidze. Each of these meetings could be compared to a full course in topology. I thank David Gabelaia for his detailed comments on a draft of this thesis, as well as for the fun we had in London, Amsterdam, and New Mexico, including the cooking of Georgian dinners.

Discussions with Johan van Benthem were always very inspiring. Thanks to these conversations I approach many topics differently. The tennis match that my brother and I had with Lucas and Johan is among most enjoyable moments of my stay in Amsterdam. The very thorough comments of Ian Hodkinson significantly improved this thesis. Ian's suggestions helped me to fill in gaps in many proofs. It was also a pleasure to work with Ian on a paper on which parts of Chapter 8 are based. Ian keeps impressing me with his original ideas and deep insights. I'd also like to thank Maarten Marx. Maarten's enthusiasm is really remarkable. It is always a pleasure to talk to Maarten about various issues and to hear his interesting ideas. It was also an exciting experience to work with Maarten on a paper on which other parts of Chapter 8 are based. I am also very appreciative of Albert Visser for sharing with me his thoughts on various aspects of logic.

I very much enjoyed sharing an office with Clemens Kupke. Clemens was an ideal officemate. When I needed his help in math, he would help me to prove theorems, and when I needed to relax, he would come up with a joke that would give me a good laugh. Time at ILLC would not have been as enjoyable as it was if I hadn't had such a good officemate and friend. I would also like to thank Balder ten Cate for many enjoyable and fruitful chats that we had and for the work that we did together. I also have pleasant memories of the time that I spent with Balder and Dana at the Batumi conference in Georgia. Thanks also to Benedikt Löwe for interesting discussions and for always being available to give good advice.

The staff members of the ILLC managed to create a very warm and friendly environment on the third floor of Euclides. Thanks to Ingrid van Loon, Marjan Veldhuisen, Tanja Kassenar and Jessica Pogorzelski, and also to René Goedman, whose friendly greeting was always a good start to the day.

Hanging out with my fellow PhD students and postdocs from Euclides is always fun and relaxing for which I should thank Merlijn Sevenster, Aline Honingh, Yoav Seginer, Reut Tsarfaty, Fenrong Liu, Ulle Endriss, Leigh Smith, Eric Pacuit, Brian Semmes, Olivia Ladinig, and Levan Uridia among others. Special thanks to Joost Joosten whose absence for the past year has been very noticeable. "Rondje Nieuwmarkt", organized by Joost, is another thing that I'll remember for a long time. I'll definitely miss our "gezellig" lunch breaks and Friday evenings in Kriterion. Special thanks also to Olivier Roy and Jelle Zuidema. The many dinners that I had together with Olivier and Jelle are among my most pleasant memories of this period. I'd also like to thank Olivier for helping me design the cover of this thesis.

Thanks to David Ahn for being such a good roommate and also for patiently proofreading and correcting my writing, including parts of this thesis. I also have fond memories of our philosophical discussions that would last till early morning. Be Birchall provided many helpful comments on parts of this thesis. Special thanks go to Gaëlle Fontaine for her careful reading of Part I and for helping me fix many gaps, and also for her famous *mousse au chocolat*.

I definitely benefited from the entertaining discussions and meetings that I had during various occasions with Lev Beklemishev, Rosalie Iemhoff, Lex Hendriks, Michael Zakharyaschev, Szabolcz Mikulás, Alexander Kurz, Larry Moss, Silvio Ghilardi, Tadeusz Litak, Felix Bou, John Harding and Mai Gehrke. I'd also like to thank Mai and Hilary Priestley for their hospitality and for the interesting seminars that we had during my visit to Oxford.

Thanks to my brother Guram for all his help and support over the years. Starting from my days as an undergraduate he has spent lots of time clarifying to me various mathematical ideas. He has guided me throughout my first steps in research. Without his support and (sometimes critical) comments, I would probably never have managed to do research in logic. I'd also like to thank Guram for his comments on a draft of this thesis.

Finally, I'd like to thank my father Michael Bezhanishvili, who motivated my brother and me to become logicians. That we both decided to do research is a result of his spending time helping us with our high school math and also explaining to us ideas from philosophy, mathematics and logic. Last of all, I'd like to mention my mother Tata Tvaltchrelidze, who would have been extremely happy to see this thesis. Her life was always an example for me. I'd like to dedicate this book to my parents.

Nick Bezhanishvili
Amsterdam, January 26, 2006.

Chapter 1

Introduction

Brief history

In this thesis we investigate classes of intuitionistic and modal logics. The origins of intuitionistic logic and modal logic go back to the beginning of the 20th century. Intuitionistic logic was introduced by Heyting [61] as a formalization of Brouwer's ideas about intuitionism and constructive mathematics. Investigations into modal logics started with the work of Lewis [86], who introduced the modal systems **S1**–**S5**. Lewis' original goal was to axiomatize the so-called strict implication and thus provide alternatives to material implication. The first systematic semantics for intuitionistic and modal logics was provided by McKinsey and Tarski [96, 97, 98, 119]. (The precursor to this semantics was the semantics based on the so-called Jaśkowski matrices [66].) McKinsey and Tarski interpreted the intuitionistic propositional calculus **IPC** and the modal logic **S4** in topological spaces. Their work can also be seen as the beginning of an algebraic approach towards intuitionistic and modal logics. Moreover, McKinsey and Tarski were the first who treated intuitionistic and modal logics in a single framework. They showed that the modal logic **S4** is complete with respect to the class of closure algebras (one might say: the algebras of topological spaces) and that the intuitionistic propositional calculus is complete with respect to the class of Heyting algebras[1], which basically consists of the open elements of closure algebras. This topological semantics works nicely for intuitionistic logic and the modal logic **S4**. However, it becomes less transparent when applied to other logics. In contrast, closure algebras can be very naturally generalized to Boolean algebras with operators (BAOs, for short). There is a class (a variety) of BAOs that corresponds to every modal logic, and every modal logic is complete with respect to this class. Thus, before Kripke's discovery of relational semantics for intuitionistic and modal logics [76, 77, 78], algebraic semantics was the main tool

[1] In fact, McKinsey and Tarski studied the Brouwerian algebras that are the order duals of Heyting algebras.

for investigating these logics.

After the introduction of relational semantics, interest shifted from the algebraic semantics of intuitionistic and modal logics to Kripke semantics. But researchers continued to investigate these logics using algebraic methods and the field remained active. We mention a few important contributions of this early period which are directly related to the subject of this thesis. Tarski and his students developed the theory of cylindric algebras [60], which provide an algebraic semantics for the classical first-order logic, Halmos studied monadic and polyadic algebras [58], Jankov introduced characteristic formulas for finite Heyting algebras and used them to prove that there are continuum many logics between the classical propositional calculus **CPC** and intuitionistic propositional calculus **IPC** [64, 65]. These logics are nowadays called "intermediate logics" or "superintuitionistic logics". Independently, de Jongh [69] introduced similar formulas and used them to characterize intuitionistic logic, applying a mix of algebraic and relational semantics. Rieger [106] described the one-generated free Heyting algebra and showed that it is infinite. Independently, Nishimura [102] obtained the same result using proof-theoretic methods. Kuznetsov [80, 81, 82] began a systematic study of intermediate logics using algebraic methods. It turned out that most logical notions can be translated into statements about varieties of algebras. Therefore, a whole range of techniques of universal algebra can be applied to problems of intermediate and modal logics. For example we consider the well-known property of interpolation, which is purely syntactical. It was shown by Maksimova [89, 91] that an intermediate or modal logic has the interpolation property if and only if the corresponding variety of algebras has the superamalgamation property. This directly links the interpolation property with a purely algebraic property concerning varieties of Heyting algebras and BAOs. The field of logic that studies logic via algebraic methods is nowadays called *algebraic logic*.

There were two observations that made algebraic logic even more attractive. First, in the '70s a number of Kripke-incomplete logics were discovered. Thomason [120] constructed a Kripke incomplete temporal logic. Fine [40] and van Benthem [5] found examples of Kripke incomplete modal logics. Shehtman [114] constructed an incomplete intermediate logic. Therefore, there are logics that cannot be investigated using only Kripke semantics. In contrast to this, every intermediate and modal logic is complete with respect to its algebraic semantics.

The second main observation is that algebraic and Kripke semantics are, in fact, very closely related. They are in a sense dual to each other. This connection goes through the Stone duality. There is a one-to-one correspondence between algebraic models of intuitionistic and modal logics and Kripke frames augmented with a special topology, the so-called Stone topology. This correspondence can be extended to a duality between varieties of algebras and categories of these topological Kripke frames. For Heyting algebras and closure algebras this duality was discovered by Esakia [38]. Goldblatt [51, 52] worked it out for BAOs and descriptive frames. However, the idea of a duality between Boolean algebras with operators and Kripke frames equipped with a special structure can be traced all the way back to the important work of Jónsson and Tarski [71]. Note that the duality between Heyting algebras and intuitionistic descriptive frames, on the one hand, and the duality between BAOs and modal descriptive frames, on the other, imply that every intermediate and modal logic is complete with respect to a class of descriptive frames. This duality allows us to approach problems in intermediate and modal logics from different perspectives. As we already mentioned, properties of a logic can be translated into algebraic terms. Now, using the duality between algebras and descriptive frames these properties can be translated into terms of descriptive frames. The interpolation property again provides us with a good example. As we mentioned above, an intermediate or modal logic has the interpolation property if and only if the corresponding variety of algebras has the superamalgamation property. However, as is shown in [90], the easiest way to either prove or refute the superamalgamation property is to translate it into terms of descriptive frames and then use order-topological techniques. Thus, we have three powerful tools for studying intermediate and modal logics: purely logical (syntactical), algebraic, and order-topological. Our investigations throughout this thesis will be based on algebraic and order-topological techniques and on the correspondence between them.

We continue by mentioning some other important contributions to the field of algebraic logic. Rautenberg [105] and Blok [21] started a systematic investigation of the lattices of varieties of BAOs. They thoroughly studied the splitting varieties of BAOs. Blok [20] also defined and investigated the degree of incompleteness of modal logics. In [19] Blok constructed an embedding of the lattice of intermediate logics into the lattice of normal extensions of the modal logic **S4**. Blok's proof of this theorem used only algebraic methods. On the other hand, Esakia [34] independently arrived at the same embedding using the duality between Heyting algebras and topological Kripke frames.

The next important step was made by Zakharyaschev [132, 133, 134] who generalized the notion of Jankov's characteristic formula. Zakharyaschev defined canonical formulas for intermediate and transitive modal logics and showed that every such logic is axiomatizable by canonical formulas. The technique of Zakharyaschev was again based on a duality between descriptive frames and their corresponding Heyting algebras and BAOs. Wolter [129, 130] and Kracht [73, 74] studied tense logics, extensions of basic modal logic **K** and various intermediate and modal logics using the splitting technique.

Finally, we mention yet another important line of research in algebraic logic. This is the theory of canonicity and canonical extensions. These topics will not be considered in this thesis at all, so we will only give a few important references: Sahlqvist [109], Ghilardi and Meloni [49], Goldblatt [53], Gehrke and Jónsson [47], Gehrke, Harding, Venema [45], Goldblatt, Hodkinson, Venema [54]. For a systematic overview of these results as well as other useful material on algebraic logic see Venema [126].

Main results

Now that we have briefly discussed the main techniques of our investigations in this thesis, we turn to the type of questions that we are going to study. As we mentioned in our short historical overview, the investigation of intuitionistic logic and modal logics started with a study of particular systems. Later on this study was extended to the investigation of classes of intermediate and modal logics, often all extensions of a particular interesting logic. This approach provides us with a uniform perspective on the field. It usually gives a better understanding of why a logical system does or does not have a particular property. There are many such examples, of which we mention only a few here. Segerberg [112] showed that every transitive modal logic of finite depth has the finite model property, Fine [42] proved that every transitive logic of finite width is Kripke complete. Therefore, instead of proving the finite model property and Kripke completeness for every given logic of finite depth or width we simply apply these general results. Sahlqvist's theorem [109] (see also [18, §3.6], [24, §10.3]) provides us with a different general completeness result, which says that if a logic is axiomatized by the formulas of some particular shape, then it is Kripke complete. Again, this theorem gives us for free a Kripke completeness result for large classes of logics. Maksimova's characterization of all intermediate logics with the interpolation property can be seen as a general result of a similar nature. In this thesis we follow this "global" approach to intermediate and modal logics. The precursors of this approach were Scroggs [111], who studied all extensions of **S5**, Dummett and Lemmon [31], who investigated modal logics between **S4** and **S5**, and Bull [22], Fine [39], and later Hemaspaandra [118], who showed that all extensions of **S4.3** have the finite model property, are finitely axiomatizable, and are NP-complete, respectively. Segerberg [112] investigated various classes of modal logics, Blok [19] and Esakia [34] studied isomorphisms of lattices of modal and intermediate logics, and Fine [41, 42] and Zakharyaschev [132, 133, 134, 135] investigated the classes of subframe and cofinal subframe logics, to name only a few; see [131] for an overview of these results.

The results in this thesis should be seen as a continuation of this line of research. We also concentrate on the classes of extensions of some particular logics. In this thesis we investigate:

1. The intermediate logic **RN** of the Rieger-Nishimura ladder and its extensions.

2. Cylindric modal logics. In particular:

    (a) The two-dimensional cylindric modal logic $\mathbf{S5}^2$ (without the diagonal).

    (b) The two-dimensional cylindric modal logic $\mathbf{CML}_2$ (with the diagonal).

We first discuss these two topics and then concentrate on the particular questions that we are going to address in this thesis.

The Rieger-Nishimura ladder is the dual frame of the one-generated free Heyting algebra described by Rieger [106] and Nishimura [102]. We study the intermediate logic **RN** of the Rieger-Nishimura ladder. This logic is the greatest 1-conservative extension of **IPC**. It was studied earlier by Kuznetsov and Gerciu [83], Gerciu [48] and Kracht [73]. We provide a systematic analysis of this system and its extensions. We also study an intermediate logic **KG**, introduced by Kuznetsov and Gerciu. It is closely related to **RN** and will play an important role in our investigations. The logic **RN** is a proper extension of **KG**. By studying extensions of **KG** and **RN** we introduce some general techniques. For example, we give a systematic method for constructing intermediate logics without the finite model property, we give a method for constructing infinite antichains of finite Kripke frames that implies the existence of a continuum of logics with and without the finite model property. We also introduce a gluing technique for proving the finite model property for large classes of logics.

Cylindric modal logics are the direct logical analogues of Tarski's cylindric algebras. The theory of cylindric algebras was originally introduced and developed by Tarski and his collaborators in an attempt to algebraize the classical first-order logic **FOL** [60]. Finite-dimensional cylindric algebras provide algebraic models for the finite variable fragments of **FOL**, and so finite-dimensional cylindric algebras give an "approximation" of **FOL**.

Cylindric modal logics were first formulated explicitly in [125]. They are closely related to n-dimensional products of the well-known modal logic **S5**. The lattice of extensions of **S5**, i.e., the lattice of extensions of the one-dimensional cylindric modal logic, is very simple: every extension of **S5** is finitely axiomatizable and decidable. Moreover, every proper extension of **S5** is complete with respect to a single finite frame. In contrast to this, the lattice of extensions of the three-dimensional cylindric modal logic is very complicated. The three-dimensional cylindric modal logic is undecidable and has continuum many undecidable extensions. In this thesis we concentrate on two-dimensional cylindric modal logics. We consider two similarity types: two-dimensional cylindric modal logics with and without diagonal. Cylindric modal logic with the diagonal corresponds to the full two-variable fragment of **FOL** and the cylindric modal logic without the diagonal corresponds to the two-variable substitution-free fragment of **FOL**. We study the lattices of two-dimensional cylindric modal logics.

There is a two-fold connection between these two themes of the thesis. First, for all these systems, we investigate the same properties of axiomatization, finite model property, local tabularity, etc. Second, in both cases we use the same techniques. Our main tools are algebras and their dual frames. In the intuitionistic case we use the duality between Heyting algebras and intuitionistic descriptive frames (resp. ordered topological spaces). In the modal case we use the duality between Boolean algebras with operators and modal descriptive frames (resp. Stone spaces with point-closed and clopen relations). As we pointed out above, we approach the problems of intermediate and modal logics both from an algebraic and from frame-theoretic, (or rather order-topological) perspective and jump back and forth between these two frameworks at our convenience.

Our investigations mostly concern the following topics:

- *Axiomatization*. Our main tools for obtaining positive or negative results concerning axiomatization of intermediate and modal logics are the so-called frame-based formulas. In particular, the Jankov-de Jongh formulas for intermediate logics, the Jankov-Fine formulas for modal logics, and subframe and cofinal subframe formulas for intermediate and modal logics. In Chapter 3 we put all these formulas into a unified framework. We use these formulas for showing that **RN** is finitely axiomatizable. We also prove that every normal extension of $\mathbf{S5}^2$ is finitely axiomatizable, and that there are non-finitely axiomatizable extensions of $\mathbf{CML}_2$.

- *The finite model property*. Using the technique of gluing models we prove that every extension of the logic **RN** of the Rieger-Nishimura ladder has the finite model property. Using the Jankov-de Jongh formulas we develop a systematic method for constructing intermediate logics without the finite model property. We also prove that every normal extension of $\mathbf{S5}^2$ has the finite model property. We leave it as an open problem whether every extension of $\mathbf{CML}_2$ has the finite model property.

- *Local tabularity*. This property is especially useful since every locally tabular logic has the finite model property. We derive a criterion for recognizing when an extension of **RN**, **KG**, $\mathbf{S5}^2$, or $\mathbf{CML}_2$ is locally tabular.

- *Pre-P-properties*. Let P be a property of logics. A logic L has a pre-P-property if L lacks P but every proper extension of L has P. We characterize the only extension of **KG** that has the pre-finite model property. We also describe all pre-tabular and all pre-locally tabular extensions of **KG**, $\mathbf{S5}^2$ and $\mathbf{CML}_2$.

- *Decidability/complexity*. In Chapter 8 we prove that every proper normal extension of $\mathbf{S5}^2$ is decidable and has an NP-complete satisfiability problem. This result together with the finite model property and finite axiomatization of normal extensions of $\mathbf{S5}^2$ gives us the analogue of the Bull-Fine-Hemaspaandra theorem for normal extensions of $\mathbf{S5}^2$.

Contents

This thesis has two parts. First we describe the contents of Part I. It is a well-known result of universal algebra that every variety of algebras is generated by its finitely generated members. Therefore, an understanding of the structure of finitely generated algebras of a given variety provides the key for understanding this variety. That is why we start our investigation of intermediate logics with an investigation of finitely generated Heyting algebras. Many facts about these algebras are known. However, these results are scattered in the literature. Our aim is to give a coherent exposition of finitely generated Heyting algebras. We show that their dual frames can be seen as "icebergs" consisting of the upper part (the tip of the iceberg) and the lower part. We give a full description of the upper part of these frames.

We also discuss the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas in a uniform framework of frame-based formulas. We define subframe formulas and cofinal subframe formulas in a new way which connects them with the NNIL formulas of [127]. We give a general criterion for an intermediate logic to be axiomatized by frame-based formulas and show that in general not every logic is axiomatized by frame-based formulas. This gives another explanation of why we need to enrich these formulas with an additional parameter as in Zakharyaschev's canonical formulas.

Next we use finitely generated Heyting algebras, the Jankov-de Jongh formulas and subframe formulas in the study of the lattice of extensions of one particular intermediate logic, the logic of the Rieger-Nishimura ladder. We will see that the complicated construction of finitely generated Heyting algebras becomes surprisingly simple in this case. We define the n-scheme logics of **IPC** and n-conservative extensions of **IPC**. We show that the logic of the Rieger-Nishimura ladder is the 1-scheme logic of **IPC** and, by virtue of that, the greatest 1-conservative extension of **IPC**. We show that every extension of **RN** has the finite model property. We also study the Kuznetsov-Gerciu logic **KG**. The logic **RN** is a proper extension of **KG**, but in contrast to **RN**, the logic **KG** has continuum many extensions without the finite model property. Finally, we give a criterion of local tabularity in extensions of **RN** and **KG**.

In Part II we investigate in detail lattices of the two-dimensional cylindric modal logics. Cylindric modal logic without the diagonal is the two-dimensional product of **S5**, which we denote by $\mathbf{S5}^2$. It is well-known that $\mathbf{S5}^2$ is finitely axiomatizable, has the finite model property, is decidable [60] and has a NEXPTIME-complete satisfiability problem [93]. We show that every proper normal extension of $\mathbf{S5}^2$ is also finitely axiomatizable, has the finite model property, and is decidable. Moreover, we prove that in contrast to $\mathbf{S5}^2$, every proper normal extensions of $\mathbf{S5}^2$ has an NP-complete satisfiability problem. We also show that the situation for cylindric modal logics with the diagonal is different. There are continuum many non-finitely axiomatizable extensions of the cylindric modal logic $\mathbf{CML}_2$.

We leave it as an open problem whether all of them have the finite model property. We also give a criterion of local tabularity for two-dimensional cylindric modal logics with and without diagonal and characterize pre-tabular cylindric modal logics.

The thesis is organized as follows. In Chapter 2 we discuss the Kripke, algebraic and order-topological semantics of the intuitionistic propositional calculus. In Chapter 3 we give a systematic overview of finitely generated Heyting algebras, universal models for intuitionistic logic, and of frame-based formulas. Chapter 4 investigates in detail the lattice of extensions of the logic **RN** of the Rieger-Nishimura ladder, and the lattice of extensions of the Kuznetsov-Gerciu logic **KG**. In Chapter 5 we introduce the basic notions of cylindric modal logic and define cylindric algebras. Chapter 6 investigates the lattice of normal extensions of $\mathbf{S5}^2$—the two-dimensional cylindric modal logic without the diagonal. In Chapter 7 we study the lattice of normal extensions of $\mathbf{CML}_2$—the two-dimensional cylindric modal logic with the diagonal. Finally, in Chapter 8 we prove that every proper normal extension of $\mathbf{S5}^2$ is finitely axiomatizable, has the poly-size model property and has an NP-complete satisfiability problem.

We close the introduction by mentioning prior work on which some of the chapters are based. Chapter 3 is partially based on [13]. Chapter 4 is based on joint work with Dick de Jongh and Guram Bezhanishvili [8]. Chapters 5 and 6 are based on [12], Chapter 7 is based on [14], and Chapter 8 is based on joint work with Maarten Marx [17] and Ian Hodkinson [16].

Part I

Lattices of intermediate logics

Chapter 2

Algebraic semantics for intuitionistic logic

In this chapter we give an overview of the basic facts about intuitionistic logic and its extensions. In particular, we recall their Kripke, algebraic and general frame semantics, and the duality between Heyting algebras and descriptive frames.

2.1 Intuitionistic logic and intermediate logics

2.1.1 Syntax and semantics

Let $\mathcal{L}$ denote a *propositional language* consisting of

- infinitely many propositional variables (letters) $p_0, p_1, \ldots$,
- propositional connectives $\wedge$, $\vee$, $\rightarrow$,
- a propositional constant $\bot$.

We denote by Prop the set of all propositional variables. Formulas in $\mathcal{L}$ are defined as usual. Denote by Form($\mathcal{L}$) (or simply by Form) the set of all well-formed formulas in the language $\mathcal{L}$. We assume that $p, q, r, \ldots$ range over propositional variables and $\phi, \psi, \chi, \ldots$ range over arbitrary formulas. For every formula ϕ and ψ we let $\neg\phi$ abbreviate $\phi \rightarrow \bot$ and $\phi \leftrightarrow \psi$ abbreviate $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. We also let $\top$ abbreviate $\neg\bot$. First we recall the definition of intuitionistic propositional calculus.

2.1.1. Definition. *Intuitionistic propositional calculus* **IPC** is the smallest set of formulas containing the axioms:

1. $p \rightarrow (q \rightarrow p)$,
2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

3. $p \wedge q \to p$,

4. $p \wedge q \to q$,

5. $p \to p \vee q$,

6. $q \to p \vee q$,

7. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r)))$,

8. $\bot \to p$.

and closed under the inference rules:

Modus Ponens (MP) : from ϕ and $\phi \to \psi$ infer ψ,

Substitution (Subst) : from $\phi(p_1, \ldots, p_n)$ infer $\phi(\psi_1, \ldots, \psi_n)$.

For an introduction to *intuitionism* and the connection between intuitionistic logic and intuitionism we refer to [62], [28], [123] and [15].

2.1.2. Definition. Let **CPC** denote *classical propositional calculus.*

It is well known (see e.g., [24, §2.3]) that **CPC** properly contains **IPC**. Indeed, we have $p \vee \neg p, \neg\neg p \to p \in$ **CPC**, but $p \vee \neg p, \neg\neg p \to p \notin$ **IPC**. In fact, we have the following theorem; see e.g., [24, §2.6].

2.1.3. Theorem.

1. **CPC** *is the smallest set of formulas that contains* **IPC**, *the formula* $p \vee \neg p$, *and is closed under* (MP) *and* (Subst).

2. **CPC** *is the smallest set of formulas that contains* **IPC**, *the formula* $\neg\neg p \to p$, *and is closed under* (MP) *and* (Subst).

2.1.4. Definition. A set of formulas $L \subseteq$ Form closed under (MP) and (Subst) is called an *intermediate logic* if **IPC** $\subseteq L \subseteq$ **CPC**.

Thus, the intermediate logics are "intermediate" between classical and intuitionistic propositional logics. Next we introduce a class containing all the intermediate logics.

2.1.5. Definition. A set of formulas $L \subseteq$ Form closed under (MP) and (Subst) is called a *superintuitionistic* logic if $L \supseteq$ **IPC**.

A superintuitionistic logic L is said to be *consistent* if $\bot \notin L$, and *inconsistent* if $\bot \in L$. By (8) and (MP), L is inconsistent iff $L = \textsc{Form}$. We will use the notation $L \vdash \phi$ to denote $\phi \in L$. The next proposition tells us that not only every intermediate logic is superintuitionistic, but that for consistent logics, the converse obtains as well. For a proof see, e.g., [24, Theorem 4.1].

2.1.6. Proposition. *For every consistent superintuitionistic logic $L \subsetneq \textsc{Form}$ we have $L \subseteq \mathbf{CPC}$. That is, L is intermediate.*

Therefore, every consistent superintuitionistic logic is intermediate and vice versa. From now on we will use the term "intermediate logic" only. Let L_1 and L_2 be intermediate logics. We say that L_2 is an *extension* of L_1 if $L_1 \subseteq L_2$.

2.1.7. Remark. In contrast to the propositional case, not every extension of the intuitionistic first-order logic is contained in the classical first-order logic. Indeed, it is known that the classical first-order logic has continuum many extensions. Every one of these is an extension of the intuitionistic first-order logic not contained in the classical first-order logic. Thus, the notions of superintuitionistic and intermediate logics do not coincide in the first-order case.

For every intermediate logic L and a formula ϕ, let $L + \phi$ denote the smallest intermediate logic containing $L \cup \{\phi\}$. Then we can reformulate Theorem 2.1.3 as:

$$\mathbf{CPC} = \mathbf{IPC} + (p \vee \neg p) = \mathbf{IPC} + (\neg\neg p \to p).$$

Now we recall the Kripke semantics for intuitionistic logic. Let R be a binary relation on a set W. For every $w, v \in W$ we write wRv if $(w, v) \in R$ and we write $\neg(wRv)$ if $(w, v) \notin R$.

2.1.8. Definition.

1. An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and R is a partial order; that is, a reflexive, transitive and anti-symmetric relation on W.

2. An *intuitionistic Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is an intuitionistic Kripke frame and V is an *intuitionistic valuation*; that is, a map $V : \textsc{Prop} \to \mathcal{P}(W)$,[1] satisfying the condition:

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

[1]By $\mathcal{P}(W)$ we denote the powerset of W.

All the Kripke frames and Kripke models that we consider in Part I of this thesis are intuitionistic. So, we will simply call them Kripke fames and Kripke models or just frames and models.

Let $\mathfrak{M} = (W, R, V)$ be an intuitionistic Kripke model, $w \in W$ and $\phi \in$ FORM. The following provides an inductive definition of $\mathfrak{M}, w \models \phi$.

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$,
2. $\mathfrak{M}, w \models \phi \wedge \psi$ iff $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$,
3. $\mathfrak{M}, w \models \phi \vee \psi$ iff $\mathfrak{M}, w \models \phi$ or $\mathfrak{M}, w \models \psi$,
4. $\mathfrak{M}, w \models \phi \to \psi$ iff for all v with wRv, if $\mathfrak{M}, v \models \phi$ then $\mathfrak{M}, v \models \psi$,
5. $\mathfrak{M}, w \not\models \bot$.

If $\mathfrak{M}, w \models \phi$, we say "$\phi$ is true at w" or "w satisfies the formula ϕ in $\mathfrak{M}$". We write $w \models \phi$ instead of $\mathfrak{M}, w \models \phi$ if the model $\mathfrak{M}$ is clear from the context. Since $\neg\phi$ abbreviates $\phi \to \bot$, we can spell out the truth definitions of formulas with negation as follows:

- $\mathfrak{M}, w \models \neg\phi$ iff $\mathfrak{M}, v \not\models \phi$ for all v with wRv,
- $\mathfrak{M}, w \models \neg\neg\phi$ iff for all v with wRv there exists u such that vRu and $\mathfrak{M}, u \models \phi$.

2.1.9. DEFINITION. Let $\phi \in$ FORM, $\mathfrak{F}$ be a Kripke frame, $\mathfrak{M}$ be a model on $\mathfrak{F}$, and K be a class of Kripke frames.

1. We say that ϕ is *true* in $\mathfrak{M}$, and write $\mathfrak{M} \models \phi$, if $\mathfrak{M}, w \models \phi$ for every $w \in W$.
2. We say that ϕ is *valid* in $\mathfrak{F}$, and write $\mathfrak{F} \models \phi$, if for every valuation V on $\mathfrak{F}$ we have that $\mathfrak{M} \models \phi$, where $\mathfrak{M} = (\mathfrak{F}, V)$.
3. We say that ϕ is *valid* in K, and write $\mathsf{K} \models \phi$, if $\mathfrak{F} \models \phi$ for every $\mathfrak{F} \in \mathsf{K}$.

For every intermediate logic L let $\mathbf{Fr}(L)$ be the class of Kripke frames that validate all the formulas in L. We call $\mathbf{Fr}(L)$ the class defined by L.

2.1.10. DEFINITION.

1. For every Kripke frame $\mathfrak{F}$ let $Log(\mathfrak{F})$ denote the set of all formulas that are valid in $\mathfrak{F}$, i.e., $Log(\mathfrak{F}) = \{\phi : \mathfrak{F} \models \phi\}$.
2. For a class K of Kripke frames, let $Log(\mathsf{K}) = \bigcap\{Log(\mathfrak{F}) : \mathfrak{F} \in \mathsf{K}\}$.

3. An intermediate logic L is called *Kripke complete* if there exists a class K of Kripke frames such that $L = Log(\mathsf{K})$. In such a case we say that L is *complete with respect to* K.

It is easy to check that for every frame $\mathfrak{F}$ the set $Log(\mathfrak{F})$ is an intermediate logic. We call it the *logic of* $\mathfrak{F}$. Then $Log(\mathsf{K})$ is an intermediate logic which we call the *logic of* K. It is easy to see that if an intermediate logic L is Kripke complete, then $L = Log(\mathbf{Fr}(L))$.

It is well known that **IPC** and **CPC** are Kripke complete. The proof of the following theorem is standard and uses the so-called canonical model argument. See, e.g., [24, Theorems 1.16 and 5.12], [28], [15].

2.1.11. THEOREM. *The following holds.*

1. **IPC** *is complete with respect to the class of all partially ordered frames.*

2. **CPC** *is complete with respect to the frame consisting of one reflexive point.*

Next we recall the main operations on Kripke frames and models.

GENERATED SUBFRAMES AND GENERATED SUBMODELS. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A subset $U \subseteq W$ is called an *upset of* $\mathfrak{F}$ if for every $w, v \in W$ we have that $w \in U$ and wRv imply $v \in U$. A frame $\mathfrak{F}' = (U, R')$ is called a *generated subframe of* $\mathfrak{F}$ if $U \subseteq W$, U is an upset of $\mathfrak{F}$ and R' is the restriction of R to U, i.e., $R' = R \cap U^2$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a Kripke model. A model $\mathfrak{M}' = (\mathfrak{F}', V')$ is called a *generated submodel of* $\mathfrak{M}$ if $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and V' is the restriction of V to U, i.e., $V'(p) = V(p) \cap U$. Let $\mathfrak{F} = (W, R)$ be a Kripke frame and let $w \in W$. Let the *subframe of* $\mathfrak{F}$ *generated by* w be the frame $\mathfrak{F}_w := (R(w), R')$, where $R(w) = \{v \in W : wRv\}$ and R' is the restriction of R to $R(w)$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a Kripke model and $w \in W$. The *submodel of* $\mathfrak{M}$ *generated by* w is the model $\mathfrak{M}_w := (\mathfrak{F}_w, V')$, where $\mathfrak{F}_w$ is the subframe of $\mathfrak{F}$ generated by w and V' is the restriction of V to $R(w)$.

p-MORPHISMS. Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ is called a *p-morphism between* $\mathfrak{F}$ *and* $\mathfrak{F}'$ if for every $w, v \in W$ and $w' \in W'$:

1. wRv implies $f(w)R'f(v)$,

2. $f(w)R'w'$ implies that there exists $u \in W$ such that wRu and $f(u) = w'$.

Some authors call such maps *bounded morphisms*; see, e.g., [18]. We call the conditions (1) and (2) the "forth" and "back" conditions, respectively. We say that f is *monotone* if it satisfies the forth condition. If f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, then $\mathfrak{F}'$ is called a *p-morphic image of* $\mathfrak{F}$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and

$\mathfrak{M}' = (\mathfrak{F}', V')$ be Kripke models. A map $f : W \to W'$ is called a *p-morphism between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if f is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}'$ and for every $w \in W$ and $p \in \text{PROP}$:

$$\mathfrak{M}, w \models p \text{ iff } \mathfrak{M}', f(w) \models p.$$

If f is surjective, then $\mathfrak{M}$ is called a *p-morphic image of* $\mathfrak{M}'$. p-morphic images are also called *reductions*; see, e.g., [24].

DISJOINT UNIONS. Let $\{\mathfrak{F}_i\}_{i\in I}$ be a family of Kripke frames, where $\mathfrak{F}_i = (W_i, R_i)$, for every $i \in I$. The *disjoint union of* $\{\mathfrak{F}_i\}_{i\in I}$ is the frame $\biguplus_{i\in I} \mathfrak{F}_i := (\biguplus_{i\in I} W_i, R)$ such that $\biguplus_{i\in I} W_i$ is the disjoint union of W_i's and R is defined by

$$wRv \text{ iff there exists } i \in I \text{ such that } w, v \in W_i \text{ and } wR_iv.$$

Let $\{\mathfrak{M}_i\}_{i\in I}$ be a family of Kripke models, where $\mathfrak{M}_i = (\mathfrak{F}_i, V_i)$, for every $i \in I$. The *disjoint union of* $\{\mathfrak{M}_i\}_{i\in I}$ is the model $\biguplus_{i\in I} \mathfrak{M}_i := (\biguplus_{i\in I} \mathfrak{F}_i, V)$ such that $\biguplus_{i\in I} \mathfrak{F}_i$ is the disjoint union of $\mathfrak{F}_i$'s and $V(p) = \bigcup_{i\in I} V_i(p)$.

Now we formulate the truth-preserving properties of these operations. For a proof we refer to [24, §2.3].

2.1.12. THEOREM.

1. *If a model* $\mathfrak{M}' = (W', R', V')$ *is a generated submodel of a model* $\mathfrak{M} = (W, R, V)$*, then for every* $\phi \in \text{FORM}$ *and* $v \in W'$ *we have*

$$\mathfrak{M}, v \models \phi \textit{ iff } \mathfrak{M}', v \models \phi.$$

2. *If a model* $\mathfrak{M}' = (W', R', V')$ *is a p-morphic image of a model* $\mathfrak{M} = (W, R, V)$ *via* f*, then for every* $\phi \in \text{FORM}$ *and* $w \in W$ *we have*

$$\mathfrak{M}, w \models \phi \textit{ iff } \mathfrak{M}', f(w) \models \phi.$$

3. *Let* $\{\mathfrak{M}_i\}_{i\in I}$ *be a family of Kripke models, where* $\mathfrak{M}_i = (W_i, R_i, V_i)$*, for every* $i \in I$*. Let* $\phi \in \text{FORM}$ *and* $w \in W_i$ *for some* $i \in I$*. Then*

$$\biguplus_{i\in I} \mathfrak{M}_i, w \models \phi \textit{ iff } \mathfrak{M}_i, w \models \phi.$$

Now we formulate the truth-preserving properties for frames.

2.1.13. THEOREM.

1. *If a frame* $\mathfrak{F}'$ *is a generated subframe of a frame* $\mathfrak{F}$*, then for every* $\phi \in \text{FORM}$ *we have*

$$\mathfrak{F} \models \phi \textit{ implies } \mathfrak{F}' \models \phi.$$

2. *If a frame $\mathfrak{F}'$ is a p-morphic image of a frame $\mathfrak{F}$ via f, then for every $\phi \in$* Form *we have*

$$\mathfrak{F} \models \phi \textit{ implies } \mathfrak{F}' \models \phi.$$

3. *Let $\{\mathfrak{F}_i\}_{i \in I}$ be a family of Kripke frames and let $\phi \in$* Form*. Then*

$$\biguplus_{i \in I} \mathfrak{F}_i \models \phi \textit{ iff } \mathfrak{F}_i \models \phi \textit{ for all } i \in I.$$

2.1.14. Definition. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. $\mathfrak{F}$ is called *rooted* if there exists $w \in W$ such that for every $v \in W$ we have wRv.

Then Theorem 2.1.13 entails the following useful corollary; see, e.g., [24, Theorem 8.58].

2.1.15. Corollary. *If an intermediate logic L is Kripke complete, then L is Kripke complete with respect to the class of its rooted frames.*

This means that we can restrict ourselves to rooted Kripke frames.

2.1.2 Basic properties of intermediate logics

Next we look at the important properties of intermediate logics that we will be concerned with in this thesis.

The fmp. First we recall the definition of the finite model property.

2.1.16. Definition. An intermediate logic L is said to have the *finite model property*, the fmp for short, if there exists a class K of finite Kripke frames such that $L = Log(\mathsf{K})$.[2]

Recall that a Kripke frame $\mathfrak{F} = (W, R)$ is a *chain* if for every $w, v \in W$ we have wRv or vRw. Also recall that a *finite tree* is a finite rooted Kripke frame $\mathfrak{F}$ such that the predecessors of every point of $\mathfrak{F}$ form a chain [24, p.32]. A standard argument using the techniques of filtration and unraveling shows that the following theorem holds. For the proof see, e.g., [24, Corollary 2.33].

2.1.17. Theorem. **IPC** *has the finite model property with respect to rooted partial orders. Moreover,* **IPC** *is complete with respect to the class of finite trees.*[3]

[2] Some authors define the finite model property in the following way: L has the fmp iff there is a class $\mathbf{M}$ of finite models such that for every formula ϕ, we have $\phi \in L \Leftrightarrow \mathfrak{M} \models \phi$ for every $\mathfrak{M} \in \mathbf{M}$. The property defined in Definition 6.1.1 is then called the *finite frame property*. It can be shown that for intermediate logics these two properties coincide; see, e.g., [24, Theorem 8.47].

[3] This result can be improved by considering the so-called Jaškowski frames, which are a special kind of finite trees [24, p.56].

Clearly every logic that has the finite model property is complete. The converse, in general, does not hold. In the next chapter we will see examples of complete logics that lack the fmp.

Tabularity. Let L be an intermediate logic. If L has the fmp, then it is complete with respect to a class K of finite frames. Clearly K can be very big. Now we define a very restricted notion of the fmp.

2.1.18. Definition. A logic L is called *tabular* if there exists a finite (not necessarily rooted) frame $\mathfrak{F}$ such that $L = Log(\mathfrak{F})$.

Obviously, if L is tabular, then L has the fmp. However, there are logics with the fmp that are not tabular. In particular, **IPC** enjoys the fmp but is not tabular [24, Theorem 2.56]. The best known example of a tabular logic is the classical propositional calculus **CPC**, which is the logic of a frame consisting of a single reflexive point.

Local tabularity. We say that two formulas ϕ and ψ are *L-equivalent* if $L \vdash \phi \leftrightarrow \psi$.

2.1.19. Definition. A logic L is called *locally tabular* if for every $n \in \omega$ there are only finitely many pairwise non-L-equivalent formulas in n variables.

Every tabular logic is locally tabular. Therefore, **CPC** is locally tabular. However, there are locally tabular logics that are not tabular.

2.1.20. Definition. Let $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$. **LC** is called the *linear calculus* or *Dummett's logic.*

For the proof of the next theorem consult, e.g., [24, Theorems 5.33 and 12.15 and §12.4, p.428].

2.1.21. Theorem. *The following holds.*

1. **LC** *is complete with respect to the class of all finite chains.*
2. **LC** *is not tabular.*
3. **LC** *is locally tabular.*

The fact that **LC** is locally tabular and has the fmp is not a pure coincidence. The following theorem explains this connection; see, e.g., [23, Theorem 10.15].

2.1.22. Theorem. *If a logic L is locally tabular, then L enjoys the finite model property.*

The intuitionistic propositional calculus **IPC** provides a counter-example to the converse of Theorem 2.1.22. As we mentioned above, **IPC** has the finite model property, but as we will see in Chapter 3, it is not locally tabular.

Finite axiomatization. Now we recall the notion of finite axiomatization.

2.1.23. Definition. An intermediate logic L is called *finitely axiomatizable* or *finitely axiomatized* if there exist finitely many formulas $\phi_1, \ldots, \phi_n$ such that $L = \mathbf{IPC} + \phi_1 + \ldots + \phi_n$.[4]

Even though most of the well-known logics are finitely axiomatizable, there are also non-finitely axiomatizable logics. In Chapter 4 we will construct non-finitely axiomatizable intermediate logics.

Decidability. One of the most crucial properties of logics is decidability.

2.1.24. Definition. A logic L is called *decidable* if for every given formula ϕ there exists an algorithm deciding whether $\phi \in L$.

It is well known that every finitely axiomatizable logic that has the fmp is decidable. This result is due to Harrop; see, e.g., [24, Theorem 16.13]. Therefore, **CPC**, **IPC** and **LC** are decidable. There are also undecidable intermediate logics [24, §16.5].

Finally, notice that we can define lattice-theoretic operations on the class of intermediate logics. Suppose $\{L_i\}_{i\in I}$ is a set of intermediate logics. Let $\bigwedge_{i\in I} L_i := \bigcap_{i\in I} L_i$ and $\bigvee_{i\in I} L_i$ be the smallest intermediate logic containing $\bigcup_{i\in I} L_i$. For every intermediate logic L, let $\Lambda(L)$ be the set of all intermediate logics containing L. Then $(\Lambda(L), \bigvee, \bigwedge, L, \mathbf{CPC})$ is a complete lattice. In fact, as we will see below, it is a Heyting algebra.[5], The greatest element of $(\Lambda(L), \bigvee, \bigwedge, L, \mathbf{CPC})$ is **CPC** and the least element is L. If we do not restrict ourselves to consistent logics then the greatest element of $\Lambda(L)$ is the inconsistent logic Form. For every intermediate logic L, we call $(\Lambda(L), \bigvee, \bigwedge, L, \mathbf{CPC})$ the *lattice of extensions of* L. From now on we will use the shorthand $\Lambda(L)$ for $(\Lambda(L), \bigvee, \bigwedge, \mathbf{CPC}, L)$.

2.2 Heyting algebras

In this section we define Heyting algebras, formulate algebraic completeness of intermediate logics, and spell out the connection between Heyting algebras and Kripke frames.

2.2.1 Lattices, distributive lattices and Heyting algebras

Kripke semantics, discussed in the previous section, provides a very intuitive semantics for intermediate logics. However, there are intermediate logics that are

[4]Clearly, we can substitute for $\phi_1 + \ldots + \phi_n$ one formula $\phi = \bigwedge_{i=1}^{n} \phi_i$. Therefore, if an intermediate logic is finitely axiomatizable, then it is axiomatizable by adding one extra axiom to **IPC**.

[5]For a definition of a complete lattice and a Heyting algebra consult the next section.

not Kripke complete [24, §6]. So we cannot restrict the study of intermediate logics to the study of their Kripke semantics. In this section we recall an algebraic semantics of **IPC**. As we will see below, an attractive feature of algebraic semantics is that every intermediate logic is complete with respect to its algebraic models.

We begin by introducing some basic notions. A partially ordered set $(A, \leq)$ is called a *lattice* if every two element subset of A has a least upper bound and a greatest lower bound. Let $(A, \leq)$ be a lattice. For $a, b \in A$ let $a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$. We assume that every lattice is bounded, i.e., it has a least and greatest element denoted by 0 and 1, respectively. The next proposition shows that lattices can also be defined axiomatically, see, e.g., [2, Theorem 1, p.44] and [23, p.8].

2.2.1. PROPOSITION. *A structure $(A, \vee, \wedge, 0, 1)$, where $A \neq \emptyset$, $\vee$ and $\wedge$ are binary operations and 0 and 1 are elements of A, is a bounded lattice iff for every $a, b, c \in A$ the following holds:*

1. $a \vee a = a$, $\quad a \wedge a = a$,
2. $a \vee b = b \vee a$, $\quad a \wedge b = b \wedge a$,
3. $a \vee (b \vee c) = (a \vee b) \vee c$, $\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
4. $a \vee 0 = a$, $\quad a \wedge 1 = a$,
5. $a \vee (b \wedge a) = a$, $\quad a \wedge (b \vee a) = a$.

Proof. It is a matter of routine checking that every lattice satisfies the axioms 1–5. Now suppose $(A, \vee, \wedge, 0, 1)$ satisfies the axioms 1–5. We say that $a \leq b$ if $a \vee b = b$ or equivalently if $a \wedge b = a$. Checking that $(A, \leq)$ is a lattice with least and greatest elements 0 and 1, respectively, is routine. □

From now on we let $(A, \vee, \wedge, 0, 1)$ denote a bounded lattice. We say that a lattice $(A, \vee, \wedge, 0, 1)$ is *complete* if for every subset $X \subseteq A$ there exist $\bigvee X = sup(X)$ and $\bigwedge X = inf(X)$.

2.2.2. DEFINITION. A bounded lattice $(A, \vee, \wedge, 0, 1)$ is called *distributive* if it satisfies the *distributivity laws*[6]:

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Note that the lattices shown in Figure 2.1 are not distributive. The next theorem, due to Birkhoff, shows that, in fact, these are typical examples of non-distributive lattices. For the proof the reader is referred to [2, Theorem 9, p.51] and [23, Theorem 3.6].

[6] In fact, each of these two axioms implies the other. Nevertheless, we list them both.

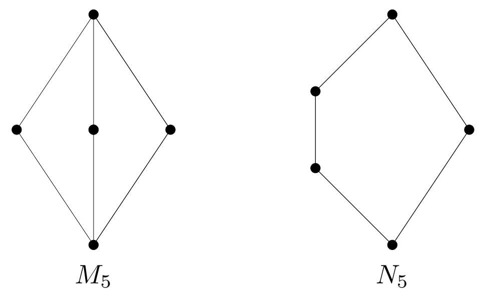

Figure 2.1: Non-distributive lattices M_5 and N_5

2.2.3. Theorem. *A lattice* $(A, \vee, \wedge, 0, 1)$ *is distributive iff* M_5 *and* N_5 *are not sublattices of* $(A, \vee, \wedge, 0, 1)$.

We are ready to define the main notion of this section.

2.2.4. Definition. A distributive lattice $(A, \vee, \wedge, 0, 1)$ is said to be a *Heyting algebra* if for every $a, b \in A$ there exists an element $a \to b$ such that for every $c \in A$ we have:

$$c \leq a \to b \text{ iff } a \wedge c \leq b.$$

We call $\to$ a *Heyting implication* or simply an *implication*. For every element a of a Heyting algebra, let $\neg a := a \to 0$.

2.2.5. Remark. It is easy to see that if $\mathfrak{A}$ is a Heyting algebra, then $\to$ is a binary operation on $\mathfrak{A}$, as follows from Proposition 2.2.7(1). Therefore, we should add $\to$ to the signature of Heyting algebras. Note also that $0 \to 0 = 1$. Hence, we can exclude 1 from the signature of Heyting algebras. From now on we will let $(A, \vee, \wedge, \to, 0)$ denote a Heyting algebra.

Similarly to the case of lattices, Heyting algebras can be defined in a purely axiomatic way; see, e.g., [68, Lemma 1.10].

2.2.6. Theorem. *A distributive lattice*[7] $\mathfrak{A} = (A, \vee, \wedge, 0, 1)$ *is a Heyting algebra iff there is a binary operation* $\to$ *on* A *such that for every* $a, b, c \in A$*:*

1. $a \to a = 1$,
2. $a \wedge (a \to b) = a \wedge b$,
3. $b \wedge (a \to b) = b$,

[7] In fact, it is not necessary to state that $\mathfrak{A}$ is distributive. Every lattice satisfying conditions 1–4 of Theorem 2.2.6 is automatically distributive [68, Lemma 1.11(i)].

4. *$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.*

Proof. Suppose $\mathfrak{A}$ satisfies the conditions 1–4. Assume $c \leq a \to b$. Then by (2), $c \wedge a \leq (a \to b) \wedge a = a \wedge b \leq b$. For the other direction we first show that for every $a \in A$ the map $(a \to \cdot)$ is monotone, i.e., if $b_1 \leq b_2$ then $a \to b_1 \leq a \to b_2$. Indeed, since $b_1 \leq b_2$ we have $b_1 \wedge b_2 = b_1$. Hence, by (4), $(a \to b_1) \wedge (a \to b_2) = a \to (b_1 \wedge b_2) = a \to b_1$. Thus, $a \to b_1 \leq a \to b_2$. Now suppose $c \wedge a \leq b$. By (3), $c = c \wedge (a \to c) \leq 1 \wedge (a \to c)$. By (1) and (4), $1 \wedge (a \to c) = (a \to a) \wedge (a \to c) = a \to (a \wedge c)$. Finally, since $(a \to \cdot)$ is monotone, we obtain that $a \to (a \wedge c) \leq a \to b$ and therefore $c \leq a \to b$.

It is easy to check that $\to$ from Definition 2.2.4 satisfies the conditions 1–4. We skip the proof. □

For the next proposition consult [68, Theorem 4.2] and [35].

2.2.7. Proposition.

1. *In every Heyting algebra $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ we have that for every $a, b \in A$:*

$$a \to b = \bigvee \{c \in A : a \wedge c \leq b\}.$$

2. *A complete distributive lattice $(A, \wedge, \vee, 0, 1)$ is a Heyting algebra iff it satisfies the infinite distributive law*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

for every $a, b_i \in A$, $i \in I$.

Proof. (1) Clearly $a \to b \leq a \to b$. Hence, $a \wedge (a \to b) \leq b$. So, $a \to b \leq \bigvee\{c \in A : a \wedge c \leq b\}$. On the other hand, if c is such that $c \wedge a \leq b$, then $c \leq a \to b$. Therefore, $\bigvee\{c \in A : a \wedge c \leq b\} \leq a \to b$.

(2) Suppose $\mathfrak{A}$ is a Heyting algebra. For every $i \in I$ we have that $a \wedge b_i \leq a \wedge \bigvee_{i \in I} b_i$. Hence, $\bigvee_{i \in I}(a \wedge b_i) \leq a \wedge \bigvee_{i \in I} b_i$. Now let $c \in A$ be such that $\bigvee_{i \in I}(a \wedge b_i) \leq c$. Then $a \wedge b_i \leq c$ for every $i \in I$. Therefore, $b_i \leq a \to c$ for every $i \in I$. This implies that $\bigvee_{i \in I} b_i \leq a \to c$, which gives us that $a \wedge \bigvee_{i \in I} b_i \leq c$. Thus, taking $\bigvee_{i \in I}(a \wedge b_i)$ as c we obtain $a \wedge \bigvee_{i \in I} b_i \leq \bigvee_{i \in I}(a \wedge b_i)$.

Conversely, suppose that a complete distributive lattice satisfies the infinite distributive law. Then we put $a \to b = \bigvee\{c \in A : a \wedge c \leq b\}$. It is now easy to see that $\to$ is a Heyting implication. □

Next we will give a few examples of Heyting algebras.

2.2.8. Example.

1. Every finite distributive lattice is a Heyting algebra. This immediately follows from Proposition 2.2.7(2), since every finite distributive lattice is complete and satisfies the infinite distributive law.

2. Every chain $\mathfrak{C}$ with a least and greatest element is a Heyting algebra and for every $a, b \in \mathfrak{C}$ we have

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

3. Every Boolean algebra $\mathfrak{B}$ is a Heyting algebra, where for every $a, b \in \mathfrak{B}$ we have

$$a \to b = \neg a \vee b$$

The next proposition characterizes those Heyting algebras that are Boolean algebras. For the proof see, e.g., [68, Lemma 1.11(ii)].

2.2.9. Proposition. *Let $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ be a Heyting algebra. Then the following three conditions are equivalent:*

1. *$\mathfrak{A}$ is a Boolean algebra,*

2. *$a \vee \neg a = 1$ for every $a \in A$,*

3. *$\neg\neg a = a$ for every $a \in A$.*

2.2.2 Algebraic completeness of IPC and its extensions

In this section we discuss the connection between intuitionistic logic and Heyting algebras. We first recall the definition of basic algebraic operations.

2.2.10. Definition. Let $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ and $\mathfrak{A}' = (A', \vee', \wedge', \to', 0')$ be Heyting algebras. A map $h : A \to A'$ is called a *Heyting homomorphism* or simply a *homomorphism* if

- $h(a \vee b) = h(a) \vee' h(b)$,
- $h(a \wedge b) = h(a) \wedge' h(b)$,
- $h(a \to b) = h(a) \to' h(b)$,
- $h(0) = 0'$.

A Heyting algebra $\mathfrak{A}'$ is called a *homomorphic* image of $\mathfrak{A}$ if there exists a Heyting homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

2.2.11. DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two Heyting algebras. We say that an algebra $\mathfrak{A}' = (A', \vee', \wedge', \to', 0')$ is a *subalgebra* of $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ if $A' \subseteq A$, the operations $\vee'$, $\wedge'$, $\to'$ are the restrictions of $\vee$, $\wedge$, $\to$ to A' and $0' = 0$.

It is easy to see that if $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$, then for every $a, b \in A'$ we have $a \vee b, a \wedge b, a \to b, 0 \in A'$. Next we define products of Heyting algebras.

2.2.12. DEFINITION.

1. Let $\mathfrak{A}_1 = (A_1, \vee_1, \wedge_1, \to_1, 0_1)$ and $\mathfrak{A}_2 = (A_2, \vee_2, \wedge_2, \to_2, 0_2)$ be Heyting algebras. The *product* of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the algebra $\mathfrak{A}_1 \times \mathfrak{A}_2 := (A_1 \times A_2, \vee, \wedge, \to, 0)$, where

   - $(a_1, a_2) \vee (b_1, b_2) := (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$,
   - $(a_1, a_2) \wedge (b_1, b_2) := (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$,
   - $(a_1, a_2) \to (b_1, b_2) := (a_1 \to_1 b_1, a_2 \to_2 b_2)$,
   - $0 := (0_1, 0_2)$.

2. More generally, let $\{\mathfrak{A}_i\}_{i \in I}$ be a family of Heyting algebras, where $\mathfrak{A}_i = (A_i, \vee_i, \wedge_i, \to_i, 0_i)$. The *product* of $\{\mathfrak{A}_i\}_{i \in I}$ is the Heyting algebra $\prod_{i \in I} \mathfrak{A}_i := (\prod_{i \in I} A_i, \vee, \wedge, \to, 0)$, where for every $f_1, f_2 \in \prod_{i \in I} A_i$, i.e., maps $f_1, f_2 : I \to \bigcup_{i \in I} A_i$ such that $f_1(i), f_2(i) \in A_i$, we have:

   - $(f_1 \vee f_2)(i) := f_1(i) \vee_i f_2(i)$,
   - $(f_1 \wedge f_2)(i) := f_1(i) \wedge_i f_2(i)$,
   - $(f_1 \to f_2)(i) := f_1(i) \to_i f_2(i)$,
   - $0(i) := 0_i$.

Let K be a class of algebras of the same signature. We say that K is a *variety* if K is closed under homomorphic images, subalgebras and products. It can be shown that K is a variety iff $\mathsf{K} = \mathbf{HSP}(\mathsf{K})$, where $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ are the operations of taking homomorphic images, subalgebras and products, respectively. The next theorem, due to Birkhoff, gives another characterization of varieties. For the proof we refer to any textbook in universal algebra, e.g., Burris and Sankappanavar [23, Theorem 11.9] or Grätzer [56, Theorem 3, p.171].

2.2.13. THEOREM. *A class of algebras forms a variety iff it is equationally definable.*

Let $\mathcal{HA}$ denote the class of all Heyting algebras.

2.2.14. COROLLARY. *$\mathcal{HA}$ is a variety.*

Proof. The result follows immediately from Theorems 2.2.1, 2.2.6 and 2.2.13. □

We are now ready to spell out the connection between Heyting algebras and intuitionistic logic and state an algebraic completeness result for **IPC**.

2.2.15. Definition. Let $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ be a Heyting algebra. A function $v : \textsc{Prop} \to A$ is called a *valuation* into the Heyting algebra $\mathfrak{A}$. We extend the valuation from $\textsc{Prop}$ to the whole of $\textsc{Form}$ via the recursive definition:

- $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
- $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,
- $v(\phi \to \psi) = v(\phi) \to v(\psi)$,
- $v(\bot) = 0$.

A formula ϕ is *true* in $\mathfrak{A}$ under v if $v(\phi) = 1$; ϕ is *valid* into $\mathfrak{A}$ if ϕ is true for every valuation in $\mathfrak{A}$. Using the well-known Lindenbaum-Tarski construction (which is very similar to the canonical model construction) we obtain algebraic completeness of **IPC**, see, e.g., [24, Theorem 7.21].

2.2.16. Theorem. **IPC** $\vdash \phi$ *iff* ϕ *is valid in every Heyting algebra.*

We also recall algebraic completeness of classical propositional calculus; see e.g., [24, Theorem 7.22].

2.2.17. Theorem. **CPC** $\vdash \phi$ *iff* ϕ *is valid in every Boolean algebra.*

We can extend the algebraic semantics of **IPC** to all intermediate logics. With every intermediate logic $L \supseteq \mathbf{IPC}$ we associate the class $\mathbf{V}_L$ of Heyting algebras in which all the theorems of L are valid. It follows from Theorem 2.2.13 that $\mathbf{V}_L$ is a variety. For example $\mathbf{V}_{\mathbf{IPC}} = \mathcal{HA}$ and $\mathbf{V}_{\mathbf{CPC}} = \mathcal{BA}$, where $\mathcal{BA}$ denotes the variety of all Boolean algebras. For every variety $\mathbf{V} \subseteq \mathcal{HA}$ let $L_{\mathbf{V}}$ be the logic of all formulas valid in $\mathbf{V}$. Note that $L_{\mathcal{HA}} = \mathbf{IPC}$ and $L_{\mathcal{BA}} = \mathbf{CPC}$. The Lindenbaum-Tarski construction shows that every intermediate logic is complete with respect to its algebraic semantics, see, e.g., [24, Theorem 7.73(iv)].

2.2.18. Theorem. *Every extension L of* **IPC** *is sound and complete with respect to* $\mathbf{V}_L$.

The connection between varieties of Heyting algebras and intermediate logics which we described above is one-to-one. That is, $L_{\mathbf{V}_L} = L$ and $\mathbf{V}_{L_{\mathbf{V}}} = \mathbf{V}$. For every family $\{\mathbf{V}_i\}_{i \in I}$ of subvarieties of $\mathbf{V}$ we have $\bigwedge_{i \in I} \mathbf{V}_i := \bigcap_{i \in I} \mathbf{V}_i$ and $\bigvee_{i \in I} \mathbf{V}_i := \mathbf{HSP}(\bigcup_{i \in I} \mathbf{V}_i)$, i.e., the smallest variety containing all $\mathbf{V}_i$'s. For every variety $\mathbf{V}$ of algebras the set of its subvarieties forms a complete lattice which we denote by $(\Lambda(\mathbf{V}), \bigvee, \bigwedge, \mathcal{BA}, \mathbf{V})$. The variety $\mathcal{BA}$ of all Boolean algebras is the least element of this lattice and $\mathbf{V}$ is the greatest element. Moreover, it can be shown that $(\Lambda(\mathbf{V}), \bigvee, \bigwedge, \mathcal{BA}, \mathbf{V})$ satisfies the infinite distributive law and hence by Proposition 2.2.7, $(\Lambda(\mathbf{V}), \bigvee, \bigwedge, \mathcal{BA}, \mathbf{V})$ is a Heyting algebra. However, if we also consider the trivial variety **Triv** generated by the one element Heyting algebra, then **Triv** will be the least element of $\Lambda(\mathbf{V})$. From now on we will use the shorthand $\Lambda(\mathbf{V})$ for $(\Lambda(\mathbf{V}), \bigvee, \bigwedge, \mathcal{BA}, \mathbf{V})$.

We have that for every $L_1, L_2, \supseteq$ **IPC**, $L_1 \subseteq L_2$ iff $\mathbf{V}_{L_1} \supseteq \mathbf{V}_{L_2}$ and moreover this correspondence is a lattice anti-isomorphism; see, e.g., [24, Theorem 7.56(ii)].

2.2.19. THEOREM. *The lattice of extensions of* **IPC** *is anti-isomorphic to the lattice of subvarieties of* $\mathcal{HA}$.

2.2.3 Heyting algebras and Kripke frames

Next we spell out in detail a connection between Kripke frames and Heyting algebras. Let $\mathfrak{F} = (W, R)$ be a partially ordered set (i.e., an intuitionistic Kripke frame). For every $w \in W$ and $U \subseteq W$ let

$$R(w) = \{v \in W : wRv\},$$

$$R^{-1}(w) = \{v \in W : vRw\},$$

$$R(U) = \bigcup_{w \in U} R(w),$$

$$R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w).$$

Recall that a subset $U \subseteq W$ is an *upset* if $w \in U$ and wRv imply $v \in U$. Let $Up(\mathfrak{F})$ be the set of all upsets of $\mathfrak{F}$. Then $(Up(\mathfrak{F}), \cup, \cap, \to, \emptyset)$ forms a Heyting algebra, where

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \setminus R^{-1}(U_1 \setminus U_2).$$

For example the Heyting algebra shown in Figure 2.2(b) corresponds to the 2-fork frame shown in Figure 2.2(a). Now we show how to construct a Kripke frame from a Heyting algebra.

2.2.20. DEFINITION. Let $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ be a Heyting algebra. A proper subset F of A is called a *filter* if

- $a, b \in F$ imply $a \wedge b \in F$
- $a \in F$ and $a \leq b$ imply $b \in F$

A filter F is called *prime* if

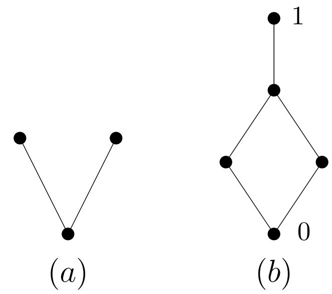

Figure 2.2: A Kripke frame and the corresponding Heyting algebra

- $a \vee b \in F$ implies $a \in F$ or $b \in F$

In a Boolean algebra every prime filter is maximal. However, this is not the case for Heyting algebras. For instance, the unit filter $\{1\}$ of the Heyting algebra shown in Figure 2.2(b) is a prime filter but is not maximal.

Now let

$$W_{\mathfrak{A}} := \{F : F \text{ is a prime filter of } \mathfrak{A}\}.$$

For $F, F' \in W_{\mathfrak{A}}$ we put

$$F R_{\mathfrak{A}} F' \text{ if } F \subseteq F'.$$

It is clear that $R_{\mathfrak{A}}$ is a partial order and hence $(W_{\mathfrak{A}}, R_{\mathfrak{A}})$ is an intuitionistic Kripke frame.

This correspondence is one-to-one for finite Heyting algebras and Kripke frames. For the proof see, e.g., [24, Theorem 7.30].

2.2.21. Theorem. *For every finite Heyting algebra $\mathfrak{A}$ there exists a Kripke frame $\mathfrak{F}$ such that $\mathfrak{A}$ is isomorphic to $Up(\mathfrak{F})$.*

However, in the infinite case the situation is more complicated. Not every Heyting algebra arises from a Kripke frame and vice versa, not every Kripke frame can be obtained from a Heyting algebra. We will give a simple argument why not every Heyting algebra can be obtained from a Kripke frame. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. Then the lattice $Up(\mathfrak{F})$ is complete. To see this, first observe that arbitrary unions and intersections of upsets are upsets again. Now it is routine to check that for every $\{U_i\}_{i \in I} \subseteq Up(\mathfrak{F})$, we have that $\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i$ and $\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i$. Hence, a non-complete Heyting algebra (for instance any Heyting algebra based on a non-complete linear order with a least and greatest elements) cannot be obtained from a Kripke frame. For a purely algebraic characterization of the Heyting algebras that arise from Kripke frames see [29], [46] or [6]. As we will see in Theorem 2.3.24, the Kripke frames that arise from Heyting algebras have maximal elements. Therefore, every Kripke frame without maximal elements (for example, the set of natural numbers with the standard ordering) is an example of a Kripke frame that cannot be obtained from a Heyting algebra.

2.3 Duality for Heyting algebras

Next we generalize the notion of a Kripke frame to that of a descriptive frame (resp. Esakia space) and illustrate the duality between descriptive frames (resp. Esakia spaces) and Heyting algebras.

2.3.1 Descriptive frames

In this section we discuss the duality between Heyting algebras and descriptive frames. We first recall from [24, §8.1 and 8.4] the definitions of general frames and descriptive frames.

2.3.1. DEFINITION. An *intuitionistic general frame* or simply a *general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$, where (W, R) is an intuitionistic Kripke frame and $\mathcal{P}$ is a set of upsets, i.e., $\mathcal{P} \subseteq Up(\mathfrak{F})$ such that $\emptyset$ and W belong to $\mathcal{P}$, and $\mathcal{P}$ is closed under $\cup$, $\cap$ and $\to$ defined by

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \setminus R^{-1}(U_1 \setminus U_2).$$

Every Kripke frame can be seen as a general frame where $\mathcal{P}$ is the set of all upsets of $\mathfrak{F}$.

2.3.2. DEFINITION. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a general frame.

1. We call $\mathfrak{F}$ *refined* if for every $w, v \in W$: $\neg(wRv)$ implies that there is $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$.
2. We call $\mathfrak{F}$ *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \setminus U : U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the *finite intersection property* (that is, every intersection of finitely many elements of $\mathcal{X} \cup \mathcal{Y}$ is nonempty) then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.
3. We call $\mathfrak{F}$ *descriptive* if it is refined and compact.

We call the elements of $\mathcal{P}$ *admissible sets*.

Note that if $\mathfrak{F} = (W, R, \mathcal{P})$ is a descriptive frame, then $(\mathcal{P}, \cup, \cap, \to, \emptyset)$ is a Heyting subalgebra of $(Up(\mathfrak{F}), \cup, \cap, \to, \emptyset)$. Moreover, as follows from the next theorem, every Heyting algebra can be obtained in such a way. For the proof see, e.g., [24, Theorem 8.18].

2.3.3. THEOREM. *For every Heyting algebra $\mathfrak{A}$ there exists an intuitionistic descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ such that $\mathfrak{A}$ is isomorphic to $(\mathcal{P}, \cup, \cap, \to, \emptyset)$.*

Proof. (Sketch) The construction of $\mathfrak{F}$ is similar to the one defined in the previous section. We take the frame $(W_{\mathfrak{A}}, R_{\mathfrak{A}})$ of all prime filters of $\mathfrak{A}$ ordered by inclusion and put $\mathcal{P}_{\mathfrak{A}} = \{\widehat{a} : a \in A\}$, where $\widehat{a} = \{w \in W_{\mathfrak{A}} : a \in w\}$. Then $(W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}})$ is a descriptive frame and $\mathfrak{A}$ is isomorphic to $(\mathcal{P}_{\mathfrak{A}}, \cup, \cap, \to, \emptyset)$. $\square$

For every Heyting algebra $\mathfrak{A}$, let $\mathfrak{A}_*$ denote the descriptive frame of all prime filters of $\mathfrak{A}$. For every descriptive frame $\mathfrak{F}$, let $\mathfrak{F}^*$ denote the Heyting algebra of all admissible sets of $\mathfrak{F}$. Then we have the following duality [24, §8.4].

2.3.4. THEOREM. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathfrak{F}$ be a descriptive frame. Then*

1. $\mathfrak{A} \simeq (\mathfrak{A}_*)^*$.

2. $\mathfrak{F} \simeq (\mathfrak{F}^*)_*$

For every Heyting algebra $\mathfrak{A}$, we call $\mathfrak{A}_*$ the *dual* of $\mathfrak{A}$ or the *descriptive frame corresponding to* $\mathfrak{A}$; and for every descriptive frame $\mathfrak{F}$, we call $\mathfrak{F}^*$ the *dual* of $\mathfrak{F}$ or the *Heyting algebra corresponding to* $\mathfrak{F}$.

2.3.5. DEFINITION. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A *descriptive valuation* is a map $V : \text{PROP} \to \mathcal{P}$. A pair $(\mathfrak{F}, V)$ where V is a descriptive valuation is called a *descriptive model.*

Validity of formulas in a descriptive frame (model) is defined in exactly the same way as for Kripke frames (models).

Note that in the same way descriptive frames correspond to Heyting algebras, descriptive models correspond to Heyting algebras with valuations, where a *Heyting algebra with a valuation* is a pair $(\mathfrak{A}, v)$ such that $v : \text{PROP} \to A$.

Next we recall the definitions of generated subframes, p-morphisms, and disjoint unions of descriptive frames.

2.3.6. DEFINITION.

1. A descriptive frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is called a *generated subframe* of a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ if (W', R') is a generated subframe of (W, R) and $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$.

2. A map $f : W \to W'$ is called a *p-morphism* between $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ if f is a p-morphism between (W, R) and (W', R') and for every $U' \in \mathcal{P}'$ we have $f^{-1}(U') \in \mathcal{P}$ and $W \setminus f^{-1}(W \setminus U') \in \mathcal{P}$.[8]

3. Let $\{\mathfrak{F}_i\}_{i=1}^n$ be a finite set of descriptive frames.[9] The *disjoint union* of $\{\mathfrak{F}_i\}_{i=1}^n$ is a descriptive frame $\biguplus_{i=1}^n \mathfrak{F}_i = (\biguplus W_i, R, \mathcal{P})$, where $(\biguplus_{i=1}^n W_i, R)$ is a disjoint union of $\{(W_i, R_i)\}_{i=1}^n$ and $\mathcal{P} = \bigcup_{i=1}^n \mathcal{P}_i$.

[8] The motivation for this definition is to make sure that p-morphisms preserve the validity of formulas. Moreover, this definition guarantees that f^{-1} is a Heyting algebra homomorphism between $\mathcal{P}'$ and $\mathcal{P}$, see Theorems 2.3.7 and 2.3.25.

[9] The disjoint union of infinitely many descriptive frames is not a descriptive frame (it is not compact). This is the reason why we define disjoint unions only for finitely many descriptive frames.

Generated submodels, p-morphisms between descriptive models, and finite disjoint unions of descriptive models are defined as in the case of Kripke semantics. The analogues of Theorems 2.1.13 and 2.1.12 also hold for descriptive frames and models. We will not formulate them here. All one needs to do is simply to replace everywhere "Kripke frames" with "descriptive frames".

The next theorem spells out the connection between homomorphisms, subalgebras and products with generated subframes, p-morphisms and disjoint unions. For the proof the reader is referred to [24, §8.5]. Theorem 2.3.7 for finite Heyting algebras and finite Kripke frames was first established by de Jongh and Troelstra [70].

2.3.7. THEOREM. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras and $\mathfrak{F}$ and $\mathfrak{G}$ be descriptive frames. Let also $\{\mathfrak{A}_i\}_{i=1}^n$ and $\{\mathfrak{F}_i\}_{i=1}^n$ be the sets of Heyting algebras and descriptive frames, respectively. Then*

1. *(a) $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$ iff $\mathfrak{A}_*$ is isomorphic to a generated subframe of $\mathfrak{B}_*$.*

   (b) $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ iff $\mathfrak{A}_*$ is isomorphic to a p-morphic image of $\mathfrak{B}_*$.

   (c) $(\prod_{i=1}^n \mathfrak{A}_i)_*$ is isomorphic to the disjoint union $\biguplus_{i=1}^n (\mathfrak{A}_i)_*$, for any $n \in \omega$.

2. *(a) $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{G}$ iff $\mathfrak{F}^*$ is a homomorphic image of $\mathfrak{G}^*$.*

   (b) $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$ iff $\mathfrak{F}^*$ is isomorphic to a subalgebra of $\mathfrak{G}^*$.

   (c) $(\biguplus_{i=1}^n \mathfrak{F}_i)^*$ is isomorphic to $\prod_{i=1}^n \mathfrak{F}_i^*$, for any $n \in \omega$.

Note that every surjective p-morphism f from $\mathfrak{F} = (W, R, \mathcal{P})$ onto $\mathfrak{F}' = (W', R', \mathcal{P}')$ gives rise to an equivalence relation E_f on $\mathfrak{F}$ defined by

$$wE_fv \text{ iff } f(w) = f(v).$$

Then for every $w \in W$ we have that $E_fR(w) \subseteq RE_f(w)$ and non-E_f-equivalent points can be separated by an element of $\mathcal{P}$. On the other hand, with any equivalence relation E on $\mathfrak{F}$ we can associate a *quotient frame* $\mathfrak{F}/E = (W/E, R', \mathcal{P}_E)$ such that

$$W_E := \{E(w) : w \in W\}, \text{ where } \ E(w) = \{v \in W : wEv\},$$

$$E(w)R'E(v) \text{ iff } w'Rv' \text{ for some } w' \in E(w) \text{ and } v' \in E(v),$$

and

$$\mathcal{P}_E := \{U \in \mathcal{P} : E(U) = U\}.$$

We define a map $f_E : W \to W/E$ by

$$f_E(w) = E(w).$$

Then if $ER(w) \subseteq RE(w)$ and non-E-equivalent points can be separated by an element of $\mathcal{P}$, then f_E is a p-morphism. We now look at the connection between p-morphisms and these equivalence relations in more detail.

2.3.8. Definition. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. An equivalence relation E on W is called a *bisimulation equivalence*[10] on $\mathfrak{F}$ if the following two conditions are satisfied:

1. For every $w, v, u \in W$, wEv and vRu imply that there is $z \in W$ such that wRz and zEu. In other words, $RE(w) \subseteq ER(w)$ for every $w \in W$.

2. For every $w, v \in W$ If $\neg(wEv)$ then w and v are *separated* by an E-saturated admissible upset. That is, there exists $U \in \mathcal{P}$ such that $E(U) = U$ and either $w \in U$ and $v \notin U$ or $w \notin U$ and $v \in U$.

For a full proof of the next theorem we refer to [35] and [6].

2.3.9. Theorem. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. Then there is a one-to-one correspondence between bisimulation equivalences on $\mathfrak{F}$ and p-morphic images of $\mathfrak{F}$.*

Proof. Suppose $f : W \to W'$ is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, where $\mathfrak{F}' = (W', R', \mathcal{P}')$. Define E_f on W by

$$wE_f v \text{ iff } f(w) = f(v).$$

Let $wE_f v$ and vRu. Then $f(w) = f(v)$ and therefore $f(w)Rf(u)$. Since f is a p-morphism there exists $z \in W$ such that wRz and $f(z) = f(u)$, which means that $zE_f u$. Now suppose that $\neg(wE_f v)$. Then $f(w) \neq f(v)$. This means that $\neg(f(w)Rf(v))$ or $\neg(f(v)Rf(w))$. Without loss of generality we may assume that $\neg(f(w)Rf(v))$. Since $\mathfrak{F}'$ is a descriptive frame, there exists $U \in \mathcal{P}'$ such that $f(w) \in U$ and $f(v) \notin U$. As f is a p-morphism, we have $f^{-1}(U) \in \mathcal{P}$ and clearly $w \in f^{-1}(U)$ and $v \notin f^{-1}(U)$.

For the converse we need to check that if E is a bisimulation equivalence, then $f_E : W \to W/E$ defined by $f_E(w) = E(w)$ is a p-morphism. We will sketch the proof. That f_E is monotone follows from the definition of R'. That f_E satisfies the "back" condition is implied by Definition 2.3.8(1). Therefore, f_E is a p-morphism between the Kripke frames. Finally, f_E is a p-morphism between descriptive frames since E satisfies Condition (2) of Definition 2.3.8. $\square$

The next theorem was first established by Esakia [38] (see also [6]).

[10] Some authors call such equivalence relations *correct partitions* [35], [6].

2.3.10. COROLLARY. *Let $\mathfrak{A}$ be a Heyting algebra. There is a one-to-one correspondence between the subalgebras of $\mathfrak{A}$ and the bisimulation equivalences of $\mathfrak{A}_*$.*[11]

Proof. The result follows immediately from Theorems 2.3.7 and 2.3.9. Nevertheless, since we will use this theorem in subsequent sections, we briefly sketch the main idea of a direct proof.

With any subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ we associate an equivalence relation $E_{\mathfrak{A}'}$ on $\mathfrak{A}_* = (W, R, \mathcal{P})$ defined by

$$wE_{\mathfrak{A}'}v \text{ iff } w \cap A' = v \cap A'.$$

It is routine to check that $E_{\mathfrak{A}'}$ is a bisimulation equivalence.

Conversely, with every bisimulation equivalence E of $\mathfrak{A}_*$ we associate the algebra $\mathcal{P}_E$ of all E-saturated elements of $\mathcal{P}$, i.e., those $U \in \mathcal{P}$ that satisfy $E(U) = U$. It is again easy to show that $\mathcal{P}_E$ is a Heyting subalgebra of $\mathcal{P}$ and that this correspondence is one-to-one.

□

2.3.2 Subdirectly irreducible Heyting algebras

As in the case of Boolean algebras, for Heyting algebras there exists a one-to-one correspondence between congruences (that is, equivalence relations preserving the operations $\vee$, $\wedge$, $\to$ and 0) and filters.[12] For the proof of the next theorem see, e.g., [2, Lemma 4, p. 178] and [24, Theorem 8.57].

2.3.11. THEOREM. *Let $\mathfrak{A}$ be a Heyting algebra. There exists a one-to-one correspondence between:*

1. *congruences of $\mathfrak{A}$,*
2. *filters of $\mathfrak{A}$,*
3. *generated subframes of $\mathfrak{A}_*$.*

2.3.12. DEFINITION. An algebra $\mathfrak{A}$ is said to be *subdirectly irreducible*, s.i. for short, if among its non-trivial congruence relations there exists the least one.

Subdirectly irreducible algebras play a crucial role in investigating varieties because of the next theorem due to Birkhoff. For the proof see, e.g., [23, Theorem 8.6 and Corollary 9.7] and [56, Theorem 3, p.124]. For every class of algebras K, let $\mathrm{SI}(\mathsf{K})$ denote the class of all s.i. members of K.

[11] In fact, there is a lattice anti-isomorphism between the lattice of subalgebras of $\mathfrak{A}$ and the lattice of bisimulation equivalences of $\mathfrak{A}_*$.

[12] Note that in contrast to Boolean algebras, for Heyting algebras there is no one-to-one correspondence between congruences and ideals.

2.3.13. THEOREM. *Let* $\mathbf{V}$ *be a variety of algebras. Then* $\mathbf{V} = \mathbf{HSP}(\mathrm{SI}(\mathbf{V}))$.

Therefore, every variety is generated by its subdirectly irreducible algebras. The following characterization of s.i. Heyting algebras was first established by Jankov [65]. For the proof see, e.g., [2, Theorem 5, p.179].

2.3.14. THEOREM. *Let* $\mathfrak{A}$ *be a Heyting algebra. Then the following conditions are equivalent.*

1. $\mathfrak{A}$ *is subdirectly irreducible,*
2. $\mathfrak{A}$ *contains a least prime filter* (*least with respect to the inclusion relation*),
3. $\mathfrak{A}$ *has a second greatest element.*

To obtain the dual characterization of subdirectly irreducible Heyting algebras we need to extend the definition of rooted Kripke frames to descriptive frames.

2.3.15. DEFINITION. A descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *rooted* if (W, R) is a rooted Kripke frame and $W \setminus \{r\} \in \mathcal{P}$, where r is the root of $\mathfrak{F}$.

The following theorem is due to Esakia [35] (see also [6]).

2.3.16. THEOREM. *Let* $\mathfrak{A}$ *be a Heyting algebra.* $\mathfrak{A}$ *is subdirectly irreducible iff* $\mathfrak{A}_*$ *is a rooted descriptive frame.*

We will use this characterization of s.i. Heyting algebras throughout this thesis.

2.3.3 Order-topological duality

Here we will sketch the so-called Priestley-Esakia duality between Heyting algebras and descriptive frames in terms of order and topology. First we recall some basic definitions from general topology.

2.3.17. DEFINITION. A pair $\mathcal{X} = (X, \mathcal{O})$ is called a *topological space* if $X \neq \emptyset$ and $\mathcal{O}$ is a set of subsets of X such that

1. $X, \emptyset \in \mathcal{O}$,
2. If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$,
3. If $U_i \in \mathcal{O}$ for every $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{O}$.

Elements of $\mathcal{O}$ are called *open sets* and their complements are called *closed sets*. Let $\mathcal{X} = (X, \mathcal{O})$ be a topological space.

- $\mathcal{X}$ is called *Hausdorff* if for every $x, y \in X$, $x \neq y$ implies there are $U_1, U_2 \in \mathcal{O}$ such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.
- $\mathcal{X}$ is called *compact* if for every family $\mathcal{F}$ of closed sets with the finite intersection property (see Definition 2.3.2(2)) we have $\bigcap \mathcal{F} \neq \emptyset$.
- $\mathcal{X}$ is called 0-dimensional if every $U \in \mathcal{O}$ is the union of *clopens*, i.e., sets that are simultaneously closed and open.

2.3.18. Definition.

- A topological space $\mathcal{X} = (X, \mathcal{O})$ is called a *Stone space* if it is 0-dimensional, compact and Hausdorff.
- For every Stone space $\mathcal{X} = (X, \mathcal{O})$ let $\mathcal{CP}(X)$ denote the Boolean algebra of all clopens of $\mathcal{X}$.

Then the celebrated Stone representation theorem states that:

2.3.19. Theorem. *For every Boolean algebra $\mathfrak{B}$ there exists a Stone space $\mathcal{X} = (X, \mathcal{O})$ such that $\mathfrak{B}$ is isomorphic to $\mathcal{CP}(X)$.*

2.3.20. Definition. Let $\mathcal{X} = (X, \mathcal{O}, R)$ be such that $\mathcal{X} = (X, \mathcal{O})$ is a Stone space and R is a partial order on X.

1. R satisfies the *Priestley separation axiom* if for every $x, y \in X$:

   $\neg(xRy)$ implies there is a clopen upset U such that $x \in U$ and $y \notin U$.

2. R is called *point-closed* if $R(x)$ is closed for every $x \in X$.
3. R is called *clopen* if $R^{-1}(U)$ is clopen for every clopen set U.
4. $\mathcal{X} = (X, \mathcal{O}, R)$ is said to be a *Priestley space* if X is a Stone space and R satisfies the Priestley separation axiom.
5. $\mathcal{X}$ is called an *Esakia space* if $(X, \mathcal{O}, R)$ is a Priestley space and R is a clopen relation.

Esakia spaces can be characterized by avoiding the Priestley separation axiom. For item (1) of the next proposition consult Esakia [35] and for (2) see Priestley [103].

2.3.21. Proposition.

1. *$\mathcal{X} = (X, \mathcal{O}, R)$ is an Esakia space iff $(X, \mathcal{O})$ is a Stone space and R is a point-closed and clopen partial order.*

2. *For every Priestley space $\mathcal{X} = (X, \mathcal{O}, R)$, the relation R is point-closed and for every $x \in X$ the set $R^{-1}(x)$ is closed.*

Next we spell out the connection between descriptive frames and Esakia spaces. Let $\mathcal{X} = (X, \mathcal{O}, R)$ be an Esakia space and $\mathcal{P}_{\mathcal{X}} = \{U \subseteq X : U$ is a clopen upset$\}$. Then $(X, R, \mathcal{P}_{\mathcal{X}})$ is a descriptive frame.

Conversely, let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. Let $-\mathcal{P}$ denote the set $\{W \setminus U : U \in \mathcal{P}\}$. Define a topology on W by declaring $\mathcal{P}' = \mathcal{P} \cup -\mathcal{P}$ as a sub-basis. That is, we define the topology $\mathcal{O}_{\mathcal{P}}$ such that $U \in \mathcal{O}_{\mathcal{P}}$ iff U is a union of finite intersections of elements of $\mathcal{P}'$. (In the literature $\mathcal{O}_{\mathcal{P}}$ is called the patch topology; see, e.g., [68].) Then one can show that $\mathfrak{F} = (W, \mathcal{O}_{\mathcal{P}}, R)$ is an Esakia space. Moreover, every clopen of $\mathfrak{F}$ is a finite union of finite intersections of elements of $\mathcal{P}'$. Therefore, we can formulate the representation theorem of Heyting algebras in terms of Esakia spaces.[13]

2.3.22. Theorem. *For every Heyting algebra $\mathfrak{A}$ there exists an Esakia space $\mathcal{X}$ such that $\mathfrak{A}$ is isomorphic to the Heyting algebra of all clopen upsets of $\mathcal{X}$.*

Now we reformulate the notions of generated subframes, p-morphisms and disjoint unions of descriptive frames in topological terms.

Let $\mathcal{X} = (X, \mathcal{O}, R)$ and $\mathcal{X}' = (X', \mathcal{O}', R')$ be Esakia spaces.

- $\mathcal{X}'$ is a generated subframe of $\mathcal{X}$ iff (X', R') is a generated subframe of (X, R) and $(X', \mathcal{O}')$ is a (topologically) closed subspace of $(X, \mathcal{O})$.

- A map $f : X \to X'$ is a p-morphism iff it is a p-morphism between (X, R) and (X', R') and is continuous, i.e., $f^{-1}(U)$ is an open set of $\mathcal{X}$ ($f^{-1}(U) \in \mathcal{O}$) for every open set U of $\mathcal{X}'$ ($U \in \mathcal{O}'$).

- Let $\{\mathcal{X}_i\}_{i=1}^n$ be a finite set of Esakia spaces, where $\mathcal{X}_i = (X_i, \mathcal{O}_i, R_i)$ for every $i = 1, \ldots, n$. The *disjoint union* of $\{\mathcal{X}_i\}_{i=1}^n$ is the Esakia space $\biguplus_{i=1}^n \mathcal{X}_i = (X, \mathcal{O}, R)$, where (X, R) is the disjoint union $\biguplus_{i=1}^n (X_i, R_i)$ of the (X_i, R_i), and $(X, \mathcal{O})$ is the topological sum of the $(X_i, \mathcal{O}_i)$.

From now on we will move "back and forth" between descriptive frames and Esakia spaces at our convenience.

We illustrate the usefulness of the topological approach by showing that every Esakia space (descriptive frame) has a nonempty maximum. In fact, we will show more: that for every point x there is a maximal point y such that xRy.

[13]Note that the representation theorem for Heyting algebras was first proved in [38] and formulated in topological terms as in Theorem 2.3.22. The representation of distributive lattices in terms of Priestley spaces was proved in [103].

2.3.23. Definition. Let $\mathfrak{F} = (W, R)$ be a (descriptive or Kripke) frame.

- Call a point w of $\mathfrak{F}$ *maximal* (*minimal*) if for every $v \in W$ we have that wRv (vRw) implies $w = v$.
- For every frame $\mathfrak{F}$ let $max(\mathfrak{F})$ and $min(\mathfrak{F})$ denote the sets of all maximal and minimal points of $\mathfrak{F}$, respectively.

The next theorem is due to Esakia [35].

2.3.24. Theorem. *Let $\mathcal{X} = (X, \mathcal{O}, R)$ be an Esakia space.*

1. *For every $x \in X$ there exists $y \in max(\mathcal{X})$ such that xRy.*
2. *For every $x \in X$ there exists $z \in min(\mathcal{X})$ such that zRx.*

Proof. (1) Let C be an arbitrary R-chain of X. Consider the family $\mathcal{F} = \{R(x) : x \in C\}$. The fact that C is a chain implies that $\mathcal{F}$ has the finite intersection property. Since R is point-closed, the elements of $\mathcal{F}$ are closed. Hence, by compactness, $\bigcap \mathcal{F} \neq \emptyset$ and every element $x \in \bigcap \mathcal{F}$ is greater than every element in C. Therefore, every chain in $\mathcal{X}$ has an upper bound. By Zorn's lemma,[14] $\mathcal{X}$ has a maximal element. Now if we do the same for a generated subframe of $\mathcal{X}$ based on the set $R(x)$ we obtain that for every point $x \in X$ there is $y \in max(\mathcal{X})$ such that xRy.

(2) The proof is analogous to that of (1) and uses the fact, stated in Proposition 2.3.21(2), that $R^{-1}(x)$ is a closed set for every $x \in X$. □

Note that in this proof we only used compactness of $\mathcal{X}$ and the fact that R is point-closed. Hence, it also holds in every Priestley space. However, as follows from [35], in every Esakia space $\mathcal{X}$ the set $max(\mathcal{X})$ is always topologically closed, which need not be the case for Priestley spaces.

2.3.4 Duality of categories

In this section we extend the correspondence between Heyting algebras and descriptive frames (resp. Esakia spaces) to the duality of the corresponding categories.[15] These results will not be used subsequently, but we include this material for the sake of completeness.

Let $\mathcal{HA}$ be the category of Heyting algebras and Heyting homomorphisms, **DF** be the category of descriptive frames and descriptive p-morphisms, and let **ES** be the category of Esakia spaces and continuous p-morphisms. The next fact was first established by Esakia [38].

[14] Recall that Zorn's lemma is equivalent to the axiom of choice and states that if in a partially ordered set every chain has an upper bound, then this partial order has a maximal element.

[15] We assume that the reader is familiar with the very basic notions of category theory, such as a category and (covariant and contravariant) functor. For basic facts about category theory the reader is referred to [87].

2.3.25. THEOREM.

1. *$\mathcal{HA}$ is dually equivalent to* **DF**.

2. *$\mathcal{HA}$ is dually equivalent to* **ES**.

Proof. (1) (Sketch) We will define contravariant functors $\Phi : \mathcal{HA} \to \mathbf{DF}$ and $\Psi : \mathbf{DF} \to \mathcal{HA}$. For every Heyting algebra $\mathfrak{A}$ let $\Phi(\mathfrak{A})$ be $\mathfrak{A}_*$. For a homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ define $\Phi(h) : \Phi(\mathfrak{A}') \to \Phi(\mathfrak{A})$ by $\Phi(h) = h^{-1}$; that is, for every element $F \in \Phi(\mathfrak{A}')$ (a prime filter of $\mathfrak{A}'$) we let $\Phi(h)(F) = h^{-1}(F)$. Then $\Phi(h)$ is a well-defined descriptive p-morphism and Φ is a contravariant functor.

We now define a functor $\Psi : \mathbf{DF} \to \mathcal{HA}$. For every descriptive frame $\mathfrak{F}$ let $\Psi(\mathfrak{F}) = \mathfrak{F}^*$. If $f : \mathfrak{F} \to \mathfrak{F}'$ is a descriptive p-morphism, then define $\Psi(f) : \Psi(\mathfrak{F}') \to \Psi(\mathfrak{F})$ by $\Psi(f) = f^{-1}$; that is, for every element of $U \in \Psi(\mathfrak{F}')$ (an upset of $\mathfrak{F}'$) we let $\Psi(f)(U) = f^{-1}(U)$.

Then $\Psi(f)$ is a well-defined Heyting homomorphism and Ψ is a contravariant functor. Then it can be shown that the functors Φ and Ψ establish a duality between $\mathcal{HA}$ and **DF**.

(2) The proof is similar to (1). $\square$

2.3.5 Properties of logics and algebras

In this section we discuss the algebraic counterparts of the logical properties that we introduced in Section 2.1.2. We say that a class K *generates* a variety $\mathbf{V}$ if $\mathbf{V} = \mathbf{HSP}(\mathsf{K})$. Now we recall the basic definitions from universal algebra; see, e.g., [23, Definitions 9.4, 10.14] and [56, §60].

2.3.26. DEFINITION. Let $\mathbf{V}$ be a variety of algebras.

1. $\mathbf{V}$ is *finitely approximable* if is generated by its finite members,

2. $\mathbf{V}$ is *finitely generated* if it is generated by a single finite algebra, i.e., if there is a finite algebra $\mathfrak{A}$ such that $\mathbf{V} = \mathbf{HSP}(\mathfrak{A})$,

3. $\mathbf{V}$ is *locally finite* if every finitely generated algebra in $\mathbf{V}$ is finite,

4. $\mathbf{V}$ is *finitely axiomatizable*[16] if $\mathbf{V}$ is defined by finitely many equations.

Then we have the following correspondence between the logical and algebraic notions, which we will use throughout this thesis. It was first observed by Kuznetsov [81].

2.3.27. THEOREM. *Let L be an intermediate logic and $\mathbf{V}_L$ be the corresponding variety of Heyting algebras.*

[16]Finitely axiomatizable varieties are also called finitely based; see, e.g., [56].

1. *L has the finite model property iff $\mathbf{V}_L$ is finitely approximable.*

2. *L is tabular iff $\mathbf{V}_L$ is finitely generated.*

3. *L is locally tabular iff $\mathbf{V}_L$ is locally finite.*

4. *L is finitely axiomatizable iff $\mathbf{V}_L$ is finitely axiomatizable.*

5. *L is decidable iff the equational theory of $\mathbf{V}_L$ is decidable.*

Throughout this thesis we will jump back and forth between algebraic and logical notions at our convenience.

This finishes the introductory chapter. In the next chapters we will apply this framework in studying some intermediate and modal logics.

Chapter 3

Universal models and frame-based formulas

In this chapter we provide a unified treatment of finitely generated Heyting algebras, their dual descriptive frames, and the frame-based formulas. Many results and constructions related to these topics are scattered throughout the literature. Here, we give a coherent overview of these topics. We discuss in detail the structure of Henkin models and universal models of **IPC** and their connection with free Heyting algebras. We introduce the Jankov-de Jongh formulas, subframe formulas, and cofinal subframe formulas. The subframe formulas and cofinal subframe formulas are defined in a new way which connects them with the NNIL formulas of [127]. We apply Jankov-de Jongh formulas and (cofinal) subframe formulas to axiomatize large classes of intermediate logics. We also show how to place these formulas in a unified framework of frame-based formulas. The results presented in this chapter are formulated for intermediate logics, but they can be generalized to transitive modal logics.

The chapter is organized as follows. In the first section we discuss finitely generated Heyting algebras. In Section 3.2 we define n-universal models for **IPC** and prove that these form the upper parts of the n-Henkin models of **IPC**. Section 3.3 introduces the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas. In the final section we show how to axiomatize some intermediate logics using these formulas, define the precise notion of frame-based formulas and show how this notion unifies the previously defined formulas.

3.1 Finitely generated Heyting algebras

We start by recalling the definition of finitely generated algebras; see, e.g., [23, Definition 3.4].

3.1.1. Definition. Let $\mathfrak{A}$ be an algebra and let X be a set of elements of $\mathfrak{A}$. We say that X *generates* $\mathfrak{A}$ if there is no proper subalgebra of $\mathfrak{A}$ that contains

X. The elements of X are called the *generators* of $\mathfrak{A}$. We say that $\mathfrak{A}$ is *finitely generated* if it has a finite set of generators. $\mathfrak{A}$ is called α*-generated*, for some cardinal α, if $\mathfrak{A}$ is generated by X and $|X| = \alpha$.

In other words, $\mathfrak{A}$ is finitely generated if there are elements $g_1, \ldots, g_n$ of $\mathfrak{A}$ such that for every element a of $\mathfrak{A}$, we have $a = P(g_1, \ldots, g_n)$, where P is a polynomial over $\mathfrak{A}$. Finitely generated algebras play a crucial role in investigating varieties of universal algebras because of the following theorem; see, e.g., [56, Lemma 3, p.130, Theorem 4, p.137] and [23, Corollary 11.5].

3.1.2. THEOREM. *Every variety of algebras is generated by its finitely generated members.*

Below we will study the structure of finitely generated Heyting algebras and their dual descriptive frames.

3.1.3. DEFINITION. Let $\mathfrak{A}$ be a Heyting algebra and $\mathfrak{F}$ be its corresponding descriptive frame. $\mathfrak{F}$ is said to be *finitely generated* if $\mathfrak{A}$ is a finitely generated Heyting algebra. We call $\mathfrak{F}$ α*-generated* if $\mathfrak{A}$ is an α-generated Heyting algebra.

For each $n \in \omega$ let PROP_n denote the set $\{p_1, \ldots, p_n\}$ of propositional variables. Let $\mathfrak{A}$ be a Heyting algebra, and $\mathfrak{F}$ be its dual descriptive frame. Fix $g_1, \ldots, g_n$ in $\mathfrak{A}$. Then we can think of $\mathfrak{A}$ together with these fixed elements as a Heyting algebra with a valuation $v : \text{PROP}_n \to \mathfrak{A}$ such that $v(p_i) = g_i$, for $i = 1, \ldots, n$. From now on we will not distinguish between a Heyting algebra $\mathfrak{A}$ with fixed elements $g_1, \ldots, g_n$ and $\mathfrak{A}$ with the valuation defined above. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be the descriptive model corresponding to $(\mathfrak{A}, v)$.

3.1.4. DEFINITION. With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \ldots i_n$ such that for $k = 1, \ldots, n$:

$$i_k = \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k. \end{cases}$$

We call the sequence $i_1 \ldots i_n$ associated with w the *color* of w and denote it by $col(w)$.

Let W be a non-empty set and E, an equivalence relation on W. E is called *proper* if there are distinct points $w, v \in W$ such that wEv. A subset U of W is called E*-saturated* or simply *saturated* if $E(U) = U$. A map $f : W \to W'$ is called *proper* if there exist distinct $w, v \in W$ such that $f(w) = f(v)$.

Now we are ready to give a criterion for recognizing whether $\mathfrak{A}$ is generated by $g_1, \ldots, g_n$. This criterion was first established in [37].

3.1.5. Theorem. (Coloring Theorem) *Let $\mathfrak{A}$ be a Heyting algebra, $g_1, \dots, g_n$ be fixed elements of $\mathfrak{A}$, and $(\mathfrak{F}, V)$ be the corresponding descriptive model. Then the following conditions are equivalent:*

1. *$\mathfrak{A}$ is generated by $g_1, \dots, g_n$.*
2. *For every proper onto p-morphism $f : \mathfrak{F} \to \mathfrak{F}'$, there exist points u and v in $\mathfrak{F}$ such that $f(u) = f(v)$ and $col(u) \neq col(v)$.*
3. *For every proper bisimulation equivalence E of $\mathfrak{F}$, there exists an E-equivalence class containing points of different colors.*

Proof. (2) $\Leftrightarrow$ (3) follows from Theorem 2.3.9. We show that (1) $\Leftrightarrow$ (3). Suppose $\mathfrak{A}$ is generated by $g_1, \dots, g_n$, and E be a proper bisimulation equivalence on $\mathfrak{F}$. Let $\mathfrak{A}_E$ be the Heyting algebra corresponding to E, i.e., the algebra of all E-saturated admissible subsets of $\mathfrak{F}$. Since E is proper, $\mathfrak{A}_E$ is a proper subalgebra of $\mathfrak{A}$. As $\mathfrak{A}$ is generated by $g_1, \dots, g_n$, there is $i \leq n$ such that g_i does not belong to $\mathfrak{A}_E$. This means that $V(p_i)$ (where p_i is such that $v(p_i) = g_i$) is not E-saturated, i.e., $E(V(p_i)) \not\subseteq V(p_i)$. Therefore, there are two elements u, v in $\mathfrak{F}$ such that uEv, $u \in V(p_i)$ and $v \notin V(p_i)$, which implies that $col(u) \neq col(v)$.

Conversely, suppose $\mathfrak{A}$ is not generated by $g_1, \dots, g_n$. Denote by $\mathfrak{A}'$ the subalgebra generated by $g_1, \dots, g_n$. Obviously, $\mathfrak{A}'$ is a proper subalgebra of $\mathfrak{A}$. Let $E_{\mathfrak{A}'}$ be the proper bisimulation equivalence of $\mathfrak{F}$ corresponding to $\mathfrak{A}'$. Since every g_i belongs to $\mathfrak{A}'$, we have that every $V(p_i)$ is $E_{\mathfrak{A}'}$-saturated. Therefore, every $E_{\mathfrak{A}'}$-equivalence class contains points of the same color. $\square$

Next we will recall from [70] two lemmas about p-morphisms that will enable us to decide quickly whether there exists a p-morphism between two finite rooted frames.

For a frame $\mathfrak{F} = (W, R)$ and $w, v \in W$, we say that a point w is an *immediate successor* of a point v if vRw, $w \neq v$, and there are no intervening points, i.e., for every $u \in W$ such that vRu and uRw we have $u = v$ or $u = w$. We call v an *immediate predecessor* of w if w is an immediate successor of v.

3.1.6. Lemma. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame and $w, v \in W$.*

1. *Suppose $R(w) \setminus \{w\} = R(v)$ (i.e., v is the only immediate successor of w). Let E be the smallest equivalence relation that identifies w and v, i.e., $E = \{(u, u) : u \in W\} \cup \{(w, v), (v, w)\}$. Then E is a bisimulation equivalence. We call the corresponding map $f_E : W \to W/E$ an α-*reduction.
2. *Suppose $R(w) \setminus \{v\} = R(v) \setminus \{w\}$ (i.e., the set of immediate successors of w and v coincide). Let E be the smallest equivalence relation that identifies w and v. Then E is a bisimulation equivalence. We call the corresponding map $f_E : W \to W/E$ a β-*reduction.

Proof. The proof is a routine check. □

3.1.7. LEMMA. *Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (W', R')$ be finite frames. Suppose $f : W \to W'$ is a proper p-morphism. Then there exists a sequence $f_1, \cdots, f_n$ of α- and β-reductions such that $f = f_1 \circ \cdots \circ f_n$.*

Proof. Let f be a proper p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$. Let w be a maximal point of $\mathfrak{G}$ that is the image under f of at least two distinct points of $\mathfrak{F}$. Let $u, v \in max(f^{-1}(w))$. Then, by the conditions on a p-morphism, the sets of successors of u and v in $\mathfrak{F}$, disregarding u and v themselves, are the same. There are two possibilities:

Case 1. u and v are incomparable in $\mathfrak{F}$. Let E be the smallest equivalence relation that identifies u and v. Then there exists a β-reduction $f_E : W \to W/E$ from $\mathfrak{F}$ onto $\mathfrak{F}/E = (W/E, R_E)$. It suffices to construct a p-morphism g from $\mathfrak{F}/E$ onto $\mathfrak{G}$ such that $g \circ f_E = f$ (and apply induction on the number of points that are identified by f). We define $g : W/E \to W'$ by

$$g(E(x)) = f(x),$$

for every $E(x) \in W/E$. Checking that g satisfies the definition of p-morphism is trivial.

Case 2. u is the unique immediate successor of v or v is the unique immediate successor of u. We do exactly the same as in Case 1 (i.e., we consider the smallest equivalence relation E that identifies the points u and v), except that the map $f_E : W \to W/E$ is now an α-reduction.

□

We now begin our investigation of the structure of finitely generated descriptive frames.

3.1.8. THEOREM. *Let $\mathfrak{A}$ be a Heyting algebra generated by $g_1, \ldots, g_n$ and let $\mathfrak{F} = (W, R, \mathcal{P})$ be the corresponding descriptive frame. Then $max(\mathfrak{F})$ is a finite admissible subset of $\mathfrak{F}$ of size at most 2^n.*

Proof. Let $v : \text{PROP}_n \to \mathfrak{A}$ be such that $v(p_i) = g_i$, for every $i = 1, \ldots, n$. Therefore, we can assume that we have a coloring of $\mathfrak{F}$. First we show that for every $w, v \in max(\mathfrak{F})$, if $u \neq v$, then $col(u) \neq col(v)$. Suppose there exist distinct points $u, v \in max(\mathfrak{F})$ such that $col(u) = col(v)$. We consider the smallest equivalence relation E on W that identifies the points u and v. By Lemma 3.1.6(2), E is a bisimulation equivalence. By the Coloring Theorem, this implies that $\mathfrak{A}$ is not generated by $g_1, \ldots, g_n$, which is a contradiction. Therefore, distinct maximal points have different colors. There are 2^n different colors. Thus, there are at most 2^n points in $max(\mathfrak{F})$.

Now consider the formula

$$\tau := \bigwedge_{i=1}^{n}(p_i \vee \neg p_i)$$

We will prove that $V(\tau) = \{w \in W : w \models \tau\}$ is equal to $max(\mathfrak{F})$. It is easy to check that if $w \in max(\mathfrak{F})$, then $w \models p_i \vee \neg p_i$, for each $i = 1, \ldots, n$. Hence, $w \models \tau$. For the other direction suppose a point w is such that $w \models \tau$. We show that $w \in max(\mathfrak{F})$. Let $J = \{p_i : w \models p_i\}$ and $J' = \{\neg p_i : w \not\models p_i\}$, where $i = 1, \ldots, n$. Let also $\xi := \bigwedge J \wedge \bigwedge J'$ and $V(\xi) = \{u \in W : u \models \xi\}$. Obviously, $V(\xi)$ is an admissible upset, and by definition of ξ every point of $V(\xi)$ has the same color as w. We show that $w \in V(\xi)$. It is clear that $w \models \bigwedge J$. On the other hand, $w \not\models p_i$ and $w \models \tau$ imply that $w \models \neg p_i$. It follows that $w \models \bigwedge J'$ and therefore $w \models \xi$. Now consider the smallest equivalence relation E that identifies points in $V(\xi)$. In other words let

$$E = \{(z, z) : z \in W\} \cup \{(u, v) : u, v \in V(\xi)\}.$$

We show that E is a bisimulation equivalence. That E satisfies Definition 2.3.8(1) follows from the fact that $V(\xi)$ is an upset. Indeed, if zEv and $z \neq v$, then $z, v \in V(\xi)$. Now suppose vRu. Then as $V(\xi)$ is an upset and $v \in V(\xi)$, we have $u \in V(\xi)$, and so zEu. To show that E satisfies Definition 2.3.8(2) assume that $\neg(zEv)$. If $z \in V(\xi)$ and $v \notin V(\xi)$, then $V(\xi)$ is an E-saturated admissible upset that separates z and v. In case $z, v \notin V(\xi)$, we have $\neg(zRv)$ or $\neg(vRz)$. Therefore, by the definition of a descriptive frame, there exists an admissible upset U that separates z and v. If $U \cap V(\xi) = \emptyset$, then U is E-saturated. If $U \cap V(\xi) \neq \emptyset$, then $U \cup V(\xi) = V(\xi) \cup (U \setminus V(\xi))$. By the definition of E, both $U \setminus V(\xi)$ and $V(\xi)$ are E-saturated. Therefore, $U \cup V(\xi)$ is an E-saturated admissible upset that separates z and v. Note that, by the definition of E, if there are at least two distinct points in $V(\xi)$, then E is proper. Since $V(\xi)$ is an upset, $V(\xi)$ is a singleton set iff $V(\xi)$ consists of one maximal point of $\mathfrak{F}$. Therefore we have:

$$E \text{ is not proper iff } V(\xi) = \{w\} \text{ and } w \in max(\mathfrak{F}).$$

If E is proper, then by the Coloring Theorem, $\mathfrak{A}$ is not generated by $g_1, \ldots, g_n$, which is a contradiction. Therefore, E is not proper and $w \in max(\mathfrak{F})$. Hence, $V(\tau) = max(\mathfrak{F})$, which implies that $max(\mathfrak{F}) \in \mathcal{P}$. Thus, $max(\mathfrak{F})$ is admissible and $|max(\mathfrak{F})| \leq 2^n$. $\square$

Next we give a rough description of the structure of finitely generated descriptive frames.

3.1.9. DEFINITION. Let $\mathfrak{F}$ be a (descriptive or Kripke) frame.

1. We say that $\mathfrak{F}$ is of *depth* $n < \omega$, denoted $d(\mathfrak{F}) = n$, if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points. The frame $\mathfrak{F}$ is of finite depth if $d(\mathfrak{F}) < \omega$.

2. We say that $\mathfrak{F}$ is of an *infinite depth*, denoted $d(\mathfrak{F}) = \omega$, if for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of n points.

3. The *depth* of a point $w \in W$ is the depth of $\mathfrak{F}_w$, i.e., the depth of the subframe of $\mathfrak{F}$ generated by w. We denote the depth of w by $d(w)$.

For a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$, let $Upper(\mathfrak{F}) = \{w \in W : d(w) < \omega\}$, and $Lower(\mathfrak{F}) = \{w \in W : d(w) = \omega\}$. Clearly, $W = Upper(\mathfrak{F}) \cup Lower(\mathfrak{F})$ and $Upper(\mathfrak{F}) \cap Lower(\mathfrak{F}) = \emptyset$. If $\mathfrak{F}$ has finite depth, then $Lower(\mathfrak{F}) = \emptyset$. Note that because of Theorem 2.3.24, we have that $Upper(\mathfrak{F}) \neq \emptyset$. For every $m \in \omega$, let $D_m = \{w \in W : d(w) = m\}$ and $D_{\leq m} = \{w \in W : d(w) \leq m\}$. We call D_m the *m-th layer* of $\mathfrak{F}$. The next theorem gives an intuitive description of the structure of finitely generated descriptive frames. They are built layer by layer from the points of finite depth. Moreover, every point of an infinite depth is related to infinitely many points of finite depth.

3.1.10. THEOREM. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a finitely generated infinite descriptive frame. Then*

1. *For every $m \in \omega$, the set D_m is finite.*

2. *For every $m \in \omega$, the set $D_{\leq m}$ is admissible.*

3. *$Upper(\mathfrak{F}) = \bigcup_{m \in \omega} D_m$, and $D_m \cap D_k = \emptyset$ for $m \neq k$.*

4. *For every $x \in Lower(\mathfrak{F})$ and $m \in \omega$, there is a point $y \in D_m$ such that xRy.*

Proof. Let $\mathfrak{A}$ be the Heyting algebra corresponding to $\mathfrak{F}$ and let $g_1, \ldots, g_n$ be the generators of $\mathfrak{A}$. We define $v : \mathrm{P{\small ROP}}_n \to \mathfrak{A}$ by $v(p_i) = g_i$ for every $i = 1, \ldots, n$. This defines a coloring of $\mathfrak{F}$. We first prove (1) and (2) by an induction on $m \geq 1$. The case when $m = 1$ is given by Theorem 3.1.8. Now we assume that (1) and (2) hold for some $m > 1$ and show that they also hold for $m + 1$.

Let $W_m = W \setminus D_{\leq m}$ and let $\mathfrak{F}_m = (W_m, R_m, \mathcal{P}_m)$ where R_m is the restriction of R to W_m and $\mathcal{P}_m = \{U \cap W_m : U \in \mathcal{P})$. In other words, $\mathfrak{F}_m$ is the frame obtained from $\mathfrak{F}$ by cutting out the first m layers of $\mathfrak{F}$. Then $\mathfrak{F}_m$ is also a descriptive frame.[1]

[1] The simplest argument for this claim is topological. Since $D_{\leq m}$ is admissible, it is a clopen subset of an Esakia space. Therefore, W_m is also clopen, and thus an Esakia space, see [35].

Since $D_{\leq m}$, is admissible there is a formula τ_m that defines $D_{\leq m}$. Moreover, since $D_{\leq m}$ is finite, we have that every upset U of $\mathfrak{F}$ that is contained in $D_{\leq m}$, is also admissible. Let $\phi_1, \ldots, \phi_k$ be the formulas that define these upsets. (These formulas are called the *de Jongh* formulas. In Section 3.3.2 we will define them explicitly.)

3.1.11. CLAIM. *$\mathfrak{F}_m$ is finitely generated.*[2]

Proof. Consider the following elements of $\mathfrak{A}$:

$$g'_1 = v(\tau_m \vee p_1), \ldots, g'_n = v(\tau_m \vee p_n),$$

$$g'_{n+1} = v(\tau_m \vee (\tau_m \to \phi_1)), \ldots, g'_{n+k} = v(\tau_m \vee (\tau_m \to \phi_k)).$$

The elements $g'_1, \ldots, g'_{n+k}$ provide a new coloring of $\mathfrak{F}$, and hence of $\mathfrak{F}_m$. Let $g''_1, \ldots, g''_{n+k}$ be the elements of $\mathfrak{A}_m$ corresponding to this new coloring. We show that $\mathfrak{A}_m$ is generated by $g''_1, \ldots, g''_{n+k}$. For every $w \in W$ let $col(w)$ denote the color of w according to the old coloring, and let $col_N(w)$ denote the color of w according to the new coloring. It is easy to see that for every $w, v \in W_m$, if $col_N(w) = col_N(v)$, then $col(w) = col(v)$.

Now suppose $\mathfrak{A}_m$ is not generated by $g''_1, \ldots, g''_{n+k}$. By the Coloring Theorem, there exists a proper bisimulation equivalence E of $\mathfrak{F}_m$ such that for every $x, y \in W_m$, if $E(x) = E(y)$, then $col_N(x) = col_N(y)$. Define Q on W by

$$Q = E \cup \{(w, w) : w \in D_{\leq m}\}.$$

As E is proper, Q is also proper. We show that Q is a bisimulation equivalence of $\mathfrak{F}$. Let $\neg(xQy)$. Then there are two cases:

Case 1.1. $x \in D_{\leq m}$ or $y \in D_{\leq m}$. Then $\neg(xQy)$ implies $x \neq y$. Without loss of generality we may assume that $\neg(xRy)$ and also that $x \in D_{\leq m}$. Then $R(x) \subseteq D_{\leq m}$ is a finite upset. Therefore, it is admissible. Moreover, $R(x)$ is Q-saturated since, by the definition of Q, every subset of $D_{\leq m}$ is Q-saturated. Thus, we found an admissible upset of $\mathfrak{F}$ that separates x and y.

Case 1.2. $x, y \in W_m$. Then we have $\neg(xEy)$. Therefore, as E is a bisimulation equivalence of $\mathfrak{F}_m$, there exists an E-saturated admissible upset U of $\mathfrak{F}_m$ that separates x and y. Then it is easy to see that $U \cup D_{\leq m}$ is a Q-saturated admissible upset of $\mathfrak{F}$ that separates x and y.

Thus, Q satisfies Definition 2.3.8(2). Next we prove that Q satisfies Definition 2.3.8(1). Suppose $x, y, z \in W$ are such that xQy and yRz. If $x, y \in D_{\leq m}$, then xQy implies $x = y$, and so xRz. Thus, we may assume $x, y \in W_m$ and xEy. Then two cases are possible:

[2] This claim was first proved by Kuznetsov using an algebraic technique [80]; see also [26] and [11, Lemma 2.2(3)]. Our proof uses the Coloring Theorem.

Case 2.1. $R(x)\cap D_{\leq m} \neq R(y)\cap D_{\leq m}$. Without loss of generality we may assume that $R(x) \cap D_{\leq m} \not\subseteq R(y) \cap D_{\leq m}$. Then there is $t \in R(x) \cap D_{\leq m}$ and a formula ϕ_i, for some $i = 1, \ldots, k$, such that for every $u \in R(y) \cap D_{\leq m}$ we have $u \models \phi_i$ and $t \not\models \phi_i$. Then $x \not\models \tau_m \to \phi_i$ and $y \models \tau_m \to \phi_i$. This means that $col_N(x) \neq col_N(y)$, which is a contradiction.

Case 2.2. $R(x) \cap D_{\leq m} = R(y) \cap D_{\leq m}$. If $z \in D_{\leq m}$, then xRz. And if $z \in W_m$, as E is a bisimulation equivalence of $\mathfrak{F}_m$, there is a point $u \in W_m$ such that xRu and uEz. Thus, there exists u such that xRu and uQz.

Consequently, Definition 2.3.8(1) is satisfied and Q is a bisimulation equivalence of $\mathfrak{F}$. Now since $col_N(x) = col_N(y)$, implies $col(x) = col(y)$ we obtain that Q is a proper bisimulation equivalence of $\mathfrak{F}$ such that every Q-equivalence class has the same (old) color. By the Coloring Theorem, $\mathfrak{A}$ is not generated by $g_1, \ldots, g_n$. This contradiction finishes the proof of the claim. □

Continuing the proof of Theorem 3.1.10, by Theorem 3.1.8, $max(\mathfrak{F}_m) = D_{m+1}$, is a finite admissible subset of $\mathfrak{F}_m$. In topological terms this means that D_{m+1} is a clopen upset of $\mathfrak{F}_m$, and so D_{m+1} is a clopen subset of $\mathfrak{F}$. By the induction hypothesis, $D_{\leq m}$ is also clopen in $\mathfrak{F}$. Thus, $D_{\leq m+1} = D_{\leq m} \cup D_{m+1}$ is a clopen upset of $\mathfrak{F}$, which means that $D_{\leq m+1}$ is admissible.

(3) follows immediately from the definition of $Upper(\mathfrak{F})$.

(4) follows from Claim 3.1.11 and Theorem 2.3.24. □

3.2 Free Heyting algebras and n-universal models

In this section we define the n-universal models of **IPC** and spell out in detail the connection between n-universal models and finitely generated free Heyting algebras. In particular, we show that universal models are the upper parts of n-Henkin models—the dual descriptive frames of n-generated free Heyting algebras.

3.2.1 n-universal models

For $n \in \omega$ let $\mathcal{L}_n$ be the propositional language built on a finite set of propositional letters $\textsc{Prop}_n = \{p_1, \ldots, p_n\}$. Let $\textsc{Form}_n$ denote the set of all formulas of $\mathcal{L}_n$. Let $\mathfrak{M}$ be an intuitionistic Kripke model. As we mentioned in the previous section, with every point w of $\mathfrak{M}$, we associate the color $col(w)$.

3.2.1. Definition. Let $i_1 \ldots i_n$ and $j_1 \ldots j_n$ be two colors. We write

$$i_1 \ldots i_n \leq j_1 \ldots j_n \text{ iff } i_k \leq j_k \text{ for each } k = 1, \ldots, n.$$

We also write $i_1 \ldots i_n < j_1 \ldots j_n$ if $i_1 \ldots i_n \leq j_1 \ldots j_n$ and $i_1 \ldots i_n \neq j_1 \ldots j_n$.

Thus, the set of colors of length n ordered by $\leq$ forms a 2^n-element Boolean algebra. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. We say that a set $A \subseteq W$ *totally covers* a point v and write $v \prec A$ if A is the set of all immediate successors of v. Note that $\prec$ is a relation relating points and sets. We will use the shorthand $v \prec w$ for $v \prec \{w\}$. Thus, $v \prec w$ means not only that w is an immediate successor of v, but that w is the only immediate successor of w. It is easy to see that if every point of W has only finitely many successors, then R is the reflexive and transitive closure of the immediate successor relation. Therefore, if (W, R) is such that every point of W has only finitely many successors, then R is uniquely defined by the immediate successor relation and vice versa. Thus, to define such a frame (W, R), it is sufficient to define the relation $\prec$. A set $A \subseteq W$ is called an *anti-chain* if $|A| > 1$ and for each $w, v \in A$, $w \neq v$ implies $\neg(wRv)$ and $\neg(vRw)$.

Now we are ready to construct the n-universal model of **IPC** for each $n \in \omega$. As we mentioned above, to define $\mathcal{U}(n) = (U(n), R, V)$, it is sufficient to define the set $U(n)$, the relation $\prec$ relating points and sets, and the valuation V on $U(n)$. Let P be a property of Kripke models. We say that a model $\mathfrak{M}$ is the *minimal model with property* P if $\mathfrak{M}$ satisfies P and no proper submodel of $\mathfrak{M}$ satisfies P.

3.2.2. Theorem.

1. *For every $n \in \omega$ there exists a minimal model $\mathcal{U}(n)$ satisfying the following three conditions.*

   (a) *$max(\mathcal{U}(n))$ consists of 2^n points of distinct colors.*

   (b) *For every $w \in U(n)$ and every color $i_1 \dots i_n < col(w)$, there exists a unique $v \in U(n)$ such that $v \prec w$ and $col(v) = i_1 \dots i_n$.*

   (c) *For every finite anti-chain A in $U(n)$ and every color $i_1 \dots i_n$ with $i_1 \dots i_n \leq col(u)$ for all $u \in A$, there exists a unique $v \in U(n)$ such that $v \prec A$ and $col(v) = i_1 \dots i_n$.*

2. *For every $n \in \omega$ a minimal model satisfying conditions $(a), (b), (c)$ is unique up to isomorphism.*

Proof. (1) For every $n \in \omega$ we construct $\mathcal{U}(n)$ by induction on layers. We start with 2^n points $x_1, \dots, x_{2^n}$ of different color such that $R(x_i) = \{x_i\}$. For every point w of depth m and each color $i_1 \dots i_n < col(w)$ we add to the model a unique point v such that $R(v) = R(w) \cup \{v\}$ and $col(v) = i_1 \dots i_n$. For every finite anti-chain A of points of depth $\leq m$ with at least one point of depth m, and each color $i_1 \dots i_n$ with $i_1 \dots i_n \leq col(u)$ for all $u \in A$ we add to the model a unique point v such that $R(v) = R(A) \cup \{v\}$ and $col(v) = i_1 \dots i_n$. It is now easy to see that the model constructed in such a way is a minimal model satisfying Conditions (a)–(c).

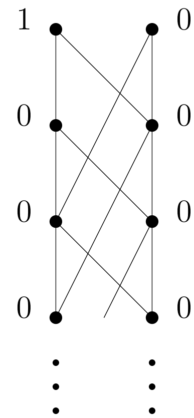

Figure 3.1: The 1-universal model

(2) Let $\mathcal{W}(n)$ be a minimal model satisfying Conditions (a)–(c). Then every point of $\mathcal{W}(n)$ has finite depth. We prove by induction on the number of layers of $\mathcal{W}(n)$ that $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are isomorphic. By Condition (a), $max(\mathcal{U}(n))$ and $max(\mathcal{W}(n))$ are isomorphic Kripke models. Now assume that first m layers of $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are isomorphic. Then by the minimality of $\mathcal{W}(n)$ and Conditions (b) and (c), it follows that the first $m+1$ layers of $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are also isomorphic, which finishes the proof of the proposition. $\square$

3.2.3. Definition. The *n-universal model* $\mathcal{U}(n)$ is the minimal model satisfying the following three conditions.

1. $max(\mathcal{U}(n))$ consists of 2^n points of distinct colors.
2. For every $w \in U(n)$ and every color $i_1 \dots i_n < col(w)$, there exists a unique $v \in U(n)$ such that $v \prec w$ and $col(v) = i_1 \dots i_n$.
3. For every finite anti-chain A in $U(n)$ and every color $i_1 \dots i_n$ with $i_1 \dots i_n \leq col(u)$ for all $u \in A$, there exists a unique $v \in U(n)$ such that $v \prec A$ and $col(v) = i_1 \dots i_n$.

By Theorem 3.2.2 for every $n \in \omega$ the n-universal model of **IPC** exists and is unique up to isomorphism. The 1-universal model of **IPC** is shown in Figure 3.1. The 1-universal model is often called the *Rieger-Nishimura ladder* (for more information on the Rieger-Nishimura ladder, see Chapter 4). More generally, for each $n > 1$, one can think of the n-universal model of **IPC** as it is shown in Figure 3.2.

3.2.4. Definition. We call the underlying frame $\mathbb{U}(n) = (U(n), R)$ of $\mathcal{U}(n)$ the *n-universal frame.*

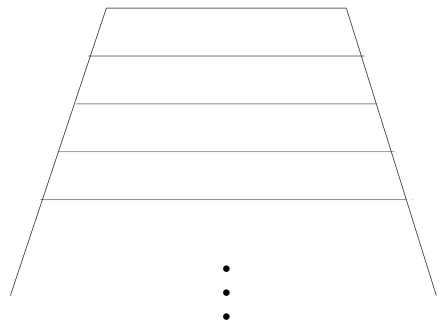

Figure 3.2: The n-universal model

3.2.5. Lemma. *For every $m, n \in \omega$, the frame $\mathfrak{G}_m = (D_{\leq m}, R \restriction D_{\leq m})$ consisting of the first m-layers of $\mathcal{U}(n)$ is n-generated.*

Proof. Let V' be the restriction of the valuation V of $\mathcal{U}(n)$ to $\mathfrak{G}_m$. Suppose $f : \mathfrak{G}_m \to \mathfrak{F}'$ is a proper onto p-morphism, where $\mathfrak{F}'$ is some finite frame. Then by Lemma 3.1.7, f is a composition of finitely many α- and β-reductions. It follows from the construction of $\mathcal{U}(n)$ that any α- or β-reduction of $\mathfrak{G}_m$ identifies points of different colors. Therefore, by the Coloring Theorem, $\mathfrak{G}_m$ is n-generated. $\square$

3.2.2 Free Heyting algebras

In this section we show that universal models constitute the upper part of the dual frames of finitely generated free Heyting algebras. First we recall the definition of free algebras; see, e.g., [23, Definition 10.9].

3.2.6. Definition. Let $\mathbf{V}$ be a variety of algebras. For every set X, the free X-generated $\mathbf{V}$-algebra, denoted $F(X)$, is the $\mathbf{V}$-algebra containing X and satisfying the following property: for every $\mathbf{V}$-algebra $\mathfrak{A}$, every map $f : X \to \mathfrak{A}$ can be extended uniquely to a homomorphism $h : F(X) \to \mathfrak{A}$.

There is a close connection between free Heyting algebras and canonical or Henkin models of intuitionistic logic. In fact, the descriptive frame dual to the n-generated free Heyting algebra is isomorphic to the n-Henkin frame of intuitionistic logic; see, e.g., [24, §7].

3.2.7. Definition.

1. Let $F(n)$ be the free n-generated Heyting algebra. Let $\mathbb{H}(n)$ denote the descriptive frame of $F(n)$. We call $\mathbb{H}(n)$ the *n-Henkin frame of* **IPC**.

2. Let $g_1, \ldots, g_n$ be the generators of $F(n)$. These generators define a coloring of $\mathbb{H}(n)$. We call the n-Henkin frame with this coloring the *n-Henkin model* and denote it by $\mathcal{H}(n) = (\mathbb{H}(n), V)$.[3]

3.2.8. LEMMA. *Let $\mathfrak{A}$ be a Heyting algebra generated by $g'_1, \ldots, g'_n$, for some $n \in \omega$, and let $(\mathfrak{F}, V')$ be the corresponding descriptive model. Then $(\mathfrak{F}, V')$ is up to isomorphism a generated submodel of $\mathcal{H}(n)$.*

Proof. Let $g_1, \ldots, g_n$ be the generators of $F(n)$. Then there exists an onto homomorphism $h : F(n) \to \mathfrak{A}$ such that $h(g_i) = g'_i$ for every $i = 1, \ldots, n$. Therefore, by Theorem 2.3.7(1), $\mathfrak{F}$ is a generated subframe of $\mathbb{H}(n)$. Let $\mathfrak{F} = (W, R, \mathcal{P})$. Then $h(g_i) = g'_i$, for every $i = 1, \ldots, n$, implies that $V'(p_i) = V(p_i) \cap W$, where V is the valuation of $\mathcal{H}(n)$. Thus, $(\mathfrak{F}, V')$ is a generated submodel of $\mathcal{H}(n)$. $\square$

For the next theorem consult either of [24, Sections 8.6 and 8.7], [57, §2], [4], [116] and [108].

3.2.9. THEOREM. *The generated submodel of $\mathcal{H}(n)$ consisting of all the points of finite depth is isomorphic to the universal model $\mathcal{U}(n)$; that is, $Upper(\mathcal{H}(n))$ is isomorphic to $\mathcal{U}(n)$.*

Proof. By Theorem 3.1.10, $Upper(\mathcal{H}(n)) = \bigcup_{m\in\omega} D_m$, where $D_m \cap D_k = \emptyset$, for $m \neq k$. By Lemmas 3.2.5 and 3.2.8, the generated submodel $max(\mathcal{U}(n))$ of $\mathcal{U}(n)$ consisting of the maximal points of $\mathcal{U}(n)$ is isomorphic to a generated submodel of $\mathcal{H}(n)$. Moreover, by Definition 3.2.3(1) and Theorem 3.1.8, $|max(\mathcal{U}(n))| = 2^n$ and $|max(\mathcal{H}(n))| \leq 2^n$. Therefore, $max(\mathcal{H}(n))$ and $max(\mathcal{U}(n))$ are isomorphic.

Now assume that for each $k \in \omega$, the first k layers of $\mathcal{H}(n)$ and $\mathcal{U}(n)$ are isomorphic. We will prove that the first $k+1$ layers of $\mathcal{U}(n)$ and $\mathcal{H}(n)$ are isomorphic as well. By Lemmas 3.2.5 and 3.2.8 we know that the model $\mathfrak{M}_{k+1}$ consisting of the first $k+1$ layers of $\mathcal{U}(n)$ is n-generated and is isomorphic to a generated submodel of $\mathcal{H}(n)$. (We identify $\mathfrak{M}_{k+1}$ with the generated submodel of $\mathcal{H}(n)$ that it is isomorphic to.) Now suppose there is u in $\mathcal{H}(n)$ of depth $k+1$ such that u does not belong to $\mathfrak{M}_{k+1}$. Let $\{u_1, \ldots, u_m\}$ be the set of immediate successors of u. By the induction hypothesis, each u_i belongs to $\mathfrak{M}_{k+1}$. By Theorem 3.1.10(1), $\{u_1, \ldots, u_m\}$ is finite. Moreover, $m > 0$ as u is not a maximal point. If $m = 1$, two cases are possible:

Case 1. $col(u) = col(u_1)$; see Figure 3.3(a). In this case we consider the α-reduction that identifies u and u_1.

[3] As we mentioned above Henkin frames and Henkin models are also called canonical frames and canonical models.

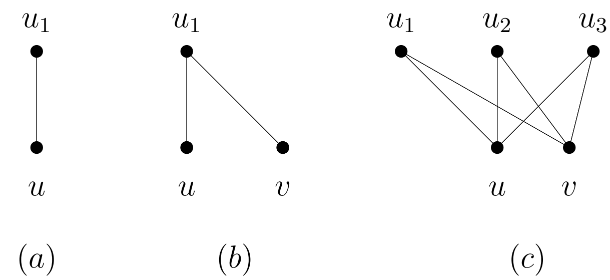

Figure 3.3: The α- and β-reductions

Case 2. $col(u) < col(u_1)$. In this case, by the construction of the n-universal model, there is v in $\mathfrak{M}_{k+1}$ such that v is totally covered by u_1 and $col(v) = col(u)$; see Figure 3.3(b). Then we consider the β-reduction that identifies u and v.

In either case the Coloring Theorem ensures that $F(n)$ is not generated by $g_1, \ldots, g_n$, which is a contradiction.

If $m > 1$ we have that $col(u) \leq col(u_i)$ for every $i = 1, \ldots, m$. Again, by the construction of $\mathcal{U}(n)$, there exists a point v of $\mathfrak{M}_{k+1}$ that is totally covered by $\{u_1, \ldots, u_m\}$, and $col(u) = col(v)$; see Figure 3.3(c), where $m = 3$. Consider the β-reduction that identifies u and v. The Coloring Theorem ensures that $F(n)$ is not generated by $g_1, \ldots, g_n$, which is again a contradiction. Therefore, the first $k + 1$ layers of $\mathcal{H}(n)$ and $\mathcal{U}(n)$ are isomorphic. Thus, by induction, $\mathcal{U}(n)$ is isomorphic to $Upper(\mathcal{H}(n))$. □

From now on we will identify $\mathcal{U}(n)$ with $Upper(\mathcal{H}(n))$. For every intermediate logic L, let $\mathcal{H}_L(n)$ be defined by replacing **IPC** by L in Definition 3.2.7. It is well known that every logic is characterized by its n-Henkin models; see, e.g., [24, Theorem 5.5]:

3.2.10. Theorem. *Let L be an intermediate logic. Then for every $n \in \omega$ and every formula ϕ in n variables, we have*

$$L \vdash \phi \textit{ iff } \mathcal{H}_L(n) \models \phi.$$

Next we recall the definition of the disjunction property for intermediate logics; see, e.g., [24, p.19 and p.471].

3.2.11. Definition. An intermediate logic L has the *disjunction property* if $L \vdash \phi \vee \psi$ implies $L \vdash \phi$ or $L \vdash \psi$.

The following theorem can be found with a different proof in [24, Theorem 15.5(ii)].

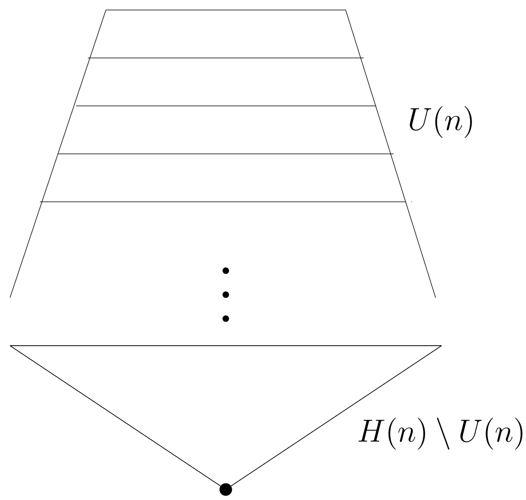

Figure 3.4: The n-Henkin model

3.2.12. THEOREM. *An intermediate logic L has the disjunction property iff the n-Henkin model $\mathcal{H}_L(n)$ of L is rooted, for every $n \in \omega$.*

Proof. Suppose L has the disjunction property. Let $F_L(n)$ be the n-generated free algebra dual to $\mathcal{H}_L(n)$. We show that the filter $\{1\}$ is prime. Recall that elements of $F_L(n)$ are the equivalence classes of the relation $\equiv$ defined on FORM_n by

$$\phi \equiv \psi \text{ iff } L \vdash \phi \leftrightarrow \psi.$$

Suppose $[\phi] \vee [\psi] = 1$ for some $[\phi], [\psi] \in F_L(n)$. Then $L \vdash \phi \vee \psi$. Since L has the disjunction property, we have that $L \vdash \phi$ or $L \vdash \psi$. Therefore, $[\phi] = 1$ or $[\psi] = 1$. Thus, $\{1\}$ is a prime filter. This proves that $\{1\}$ is a prime filter. Clearly, for every filter F of $F_L(n)$ we have $\{1\} \subseteq F$. Therefore, $\{1\}$ is the root of $\mathcal{H}_L(n)$.

Conversely, suppose $\mathcal{H}_L(n)$ is rooted for every $n \in \omega$, and $L \vdash \phi \vee \psi$. Let n be the number of distinct variables occurring in ϕ and ψ. Then, by Theorem 3.2.10, $\mathcal{H}_L(n), r \models \phi \vee \psi$, where r is the root of $\mathcal{H}_L(n)$. Thus $\mathcal{H}_L(n), r \models \phi$ or $\mathcal{H}_L(n), r \models \psi$, which by Theorem 3.2.10, shows that $L \vdash \phi$ or $L \vdash \psi$. Therefore, L has the disjunction property. $\square$

Since **IPC** has the disjunction property its n-Henkin models are rooted. Therefore, we can think of $\mathcal{H}(n)$ as it is shown in Figure 3.4. It is rooted and its upper part is isomorphic to $\mathcal{U}(n)$. We will see in the next section that for $n > 1$, the cardinality of $H(n) \setminus U(n)$ is that of the continuum (see Theorem 3.4.21).

3.2.13. THEOREM.

1. *$H(n) \setminus U(n) \neq \emptyset$, for every $n \geq 1$.*

2. *$H(1) \setminus U(1)$ is a singleton set. Therefore, $\mathcal{H}(1)$ is isomorphic to the model shown in Figure 3.5.*

Proof. (1) Suppose $H(n) \setminus U(n) = \emptyset$. Then $\mathbb{H}(n)$ is isomorphic to $\mathbb{U}(n)$. This implies that $\mathbb{U}(n)$ is a descriptive frame. Therefore, by Theorem 2.3.24(2) every point of $\mathbb{U}(n)$ is seen by some minimal point. This is a contradiction because, by the construction of $\mathbb{U}(n)$, we have $min(\mathbb{U}(n)) = \emptyset$.

(2) By (1), $H(1) \setminus U(1) \neq \emptyset$. By Theorems 3.2.9 and 3.1.10, for every $w \in H(1) \setminus U(1)$ and $m \in \omega$, there is a point v of depth m such that wRv. Looking at the coloring of $\mathcal{U}(1)$, (see Figure 3.1) we see that for every $v \in U(1)$ with $d(v) > 1$ we have $col(v) = 0$. By Theorem 3.2.9, for every $w \in H(1) \setminus U(1)$ there exists $v \in U(1)$ such that wRv and $col(v) = 0$. Then $col(w) \leq col(v)$ and therefore $col(w) = 0$. Consider an equivalence relation E on $H(1)$ such that

$$E = \{(w, w) : w \in U(1)\} \cup \{(w, v) : w, v \in H(1) \setminus U(1)\}.$$

Then E is a bisimulation equivalence. If E is proper, then by the Coloring Theorem, $\mathcal{H}(1)$ is not 1-generated, which is a contradiction. Thus, E is not proper, which means that $H(1) \setminus U(1)$ is a singleton set. $\square$

3.2.14. Remark. We point out on some topological properties of the Esakia space corresponding to $\mathcal{H}(n)$. One can show that $U(n)$ is an open subset of $\mathcal{H}(n)$ consisting of all the points that are topologically isolated, and that the topological closure of $U(n)$ is equal to $H(n)$. Since $U(n)$ is open, the set $H(n) \setminus U(n)$ is closed. Therefore, it is also an Esakia space. Thus, by Theorem 2.3.24(1), every point in $H(n) \setminus U(n)$ sees some maximal point of $H(n) \setminus U(n)$. In fact, $H(n)$ is an order compactification of $U(n)$ with the discrete topology.

In the remainder of this section we state some properties of the n-universal and n-Henkin models that will be used subsequently. These results have previously appeared in [24, Sections 8.6 and 8.7], [57], [4], [116] and [108].

3.2.15. Lemma.

1. *Let $\mathfrak{A}$ be a Heyting algebra and $v : \mathrm{Prop}_m \to \mathfrak{A}$ be a valuation on $\mathfrak{A}$. Then for every $n \in \omega$, there exist a subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ and a valuation $v' : \mathrm{Prop}_n \to \mathfrak{A}'$ such that $\mathfrak{A}'$ is generated by $\{v'(p) : p \in \mathrm{Prop}_n\}$, and $v'(p) = v(p)$ for every $p \in \mathrm{Prop}_k$, where $k = min(m, n)$.*

2. *For every descriptive model $\mathfrak{M} = (\mathfrak{F}, V)$ and $n \in \omega$ there exists a generated submodel $\mathfrak{M}' = (\mathfrak{F}', V')$ of $\mathcal{H}(n)$ such that $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$.*

Proof. (1) Suppose $n > m$. Then we let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{A}$ generated by $\{v(p) : p \in \mathrm{Prop}_m\}$, we let $v'(p) = v(p)$ for all $p \in \mathrm{Prop}_m$ and $v'(p) = v(p_1)$

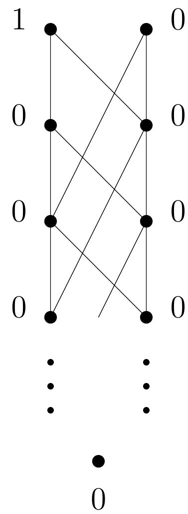

Figure 3.5: The 1-Henkin model

for all other $p \in \text{Prop}_n$. Now suppose $n \leq m$ then we let $\mathfrak{A}'$ be the subalgebra generated by $\{v(p) : p \in \text{Prop}_n\}$ and we let v' be the restriction of v to Prop_n.

(2) follows from (1) and the duality between Heyting algebras and descriptive frames. □

3.2.16. Theorem. *For every finite frame $\mathfrak{F}$, there exist a valuation V and $n \leq |\mathfrak{F}|$ such that $\mathfrak{M} = (\mathfrak{F}, V)$ is a generated submodel of $\mathcal{U}(n)$.*

Proof. The result follows immediately from the fact that every finite algebra is finitely generated and hence is a homomorphic image of $F(n)$ for some $n \leq |\mathfrak{F}|$. One can observe this directly too. For every point w of $\mathfrak{F}$ introduce a new propositional variable p_w and define a valuation V on $\mathfrak{F}$ by putting $V(p_w) = R(w)$. It is easy to see that the model $(\mathfrak{F}, V)$ is a generated submodel of the $|\mathfrak{F}|$-universal model.[4] □

Recall that $\mathcal{L}_n$ is the propositional language built from $\text{Prop}_n = \{p_1, \ldots, p_n\}$.

3.2.17. Corollary. *For every formula ϕ in the language $\mathcal{L}_n$, we have*

$$\mathbf{IPC} \vdash \phi \quad \textit{iff} \quad \mathcal{U}(n) \models \phi.$$

Proof. It is clear that if $\mathbf{IPC} \vdash \phi$, then $\mathcal{U}(n) \models \phi$. Conversely, suppose $\mathbf{IPC} \nvdash \phi$. Then by Theorems 2.1.17 and 2.3.27, there exists a finite Heyting algebra $\mathfrak{A}$ with a valuation $v : \text{Prop}_n \to \mathfrak{A}$ such that $v(\phi) \neq 1_{\mathfrak{A}}$. Let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{A}$ generated by the elements $v(p_1), \ldots, v(p_n)$. Then $\mathfrak{A}'$ is finite, n-generated and

[4]However, in most cases n may be taken much smaller than $|\mathfrak{F}|$.

$v(\phi) \neq 1_{\mathfrak{A}'}$. Therefore, $\mathfrak{A}'$ is a homomorphic image of $F(n)$. This, by Lemma 3.2.8, means that the corresponding model $\mathfrak{M}$ is a generated submodel of $\mathcal{H}(n)$. Since $\mathfrak{M}$ is finite, $\mathfrak{M}$ is a generated submodel of $\mathcal{U}(n)$. This implies that $\mathcal{U}(n) \not\models \phi$. □

3.2.18. DEFINITION. We call a set $U \subseteq U(n)$ *definable* if there is a formula $\phi(p_1, \ldots, p_n)$ such that $U = \{w \in U(n) : w \models \phi\}$. In other words, a subset U of $U(n)$ is definable if there exists a formula ϕ such that $U = V(\phi) \cap U(n)$, where V is the valuation of $\mathcal{H}(n)$.

3.2.19. THEOREM.

1. *For every $n > 1$, the set $Z(n) := \{w \in U(n) : col(w) > \underbrace{0 \ldots 0}_{n\ times}\}$ is infinite.*

2. *For every $n > 1$, there are continuum many distinct upsets of $\mathcal{U}(n)$.*

Proof. (1) Consider the maximal points w and v of $\mathcal{U}(n)$ such that $col(w) > col(v) > \underbrace{0 \ldots 0}_{n\ times}$. It is easy to see that if $n > 1$, such w and v always exist (if $n = 2$ we can take the points w and v such that $col(w) = 11$ and $col(v) = 10$). Let $\mathfrak{M}$ be the model obtained from the 1-universal model $\mathcal{U}(1)$ (shown in Figure 3.1) by replacing everywhere the color 0 by $col(v)$ and the color 1 by $col(w)$. Then it follows from Definition 3.2.3 that $\mathfrak{M}$ is a generated submodel of $\mathcal{U}(n)$. Every point of $\mathfrak{M}$ belongs to $Z(n)$. Therefore, $Z(n)$ is infinite.

(2) We will construct an infinite antichain of points of $\mathcal{U}(n)$. By the construction of $\mathcal{U}(n)$, for every $v \in Z(n)$ there exists u such that $u \prec v$ (that is, v totally covers u) and $col(u) = \underbrace{0 \ldots 0}_{n\ times}$. Let $T(n)$ be the set of all such u's. Now we show that $T(n)$ forms an antichain. Suppose $u_1, u_2 \in T(n)$, $u_1 \neq u_2$ and $u_1 R u_2$. Let $u_1' \in Z(n)$ be the point that totally covers u_1. Then, we have $u_1' R u_2$ and $col(u_1') \leq col(u_2)$. This is a contradiction since $col(u_1') > col(u_1) = col(u_2)$. Therefore, $T(n)$ is an antichain. This implies that for every $U, U' \subseteq T(n)$, if $U \neq U'$, then $R(U) \neq R(U')$. By (1), $Z(n)$ is countably infinite. Thus, $T(n)$ is also infinite, and so there are continuum many distinct upsets of $\mathcal{U}(n)$.

□

By Theorem 3.2.19(2), there are continuum many upsets of $\mathcal{U}(n)$, whereas there are only countably many formulas in n variables. Therefore, not every upset of $\mathcal{U}(n)$ is definable.

3.2.20. THEOREM. *The Heyting algebra of all definable upsets of the n-universal model is isomorphic to the free n-generated Heyting algebra.*

Proof. Because of Theorem 3.2.9, all we need to show is that for all formulas ϕ and ψ in n variables, if $V(\phi) \neq V(\psi)$ in $\mathcal{H}(n)$, then $V(\phi) \cap U(n) \neq V(\psi) \cap U(n)$, where V is the valuation of $\mathcal{H}(n)$. If $V(\phi) \cap U(n) = V(\psi) \cap U(n)$, then $\mathcal{U}(n) \models \phi \leftrightarrow \psi$. This by Corollary 3.2.17, implies **IPC** $\vdash \phi \leftrightarrow \psi$. Thus, $\mathcal{H}(n) \models \phi \leftrightarrow \psi$, which means that $V(\phi) = V(\psi)$. $\square$

3.3 The Jankov-de Jongh and subframe formulas

Next we discuss three types of frame based formulas. We define the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas. In subsequent sections we show how to use these formulas to axiomatize large classes of intermediate logics.

3.3.1 Formulas characterizing point generated subsets

In this section we introduce the so-called de Jongh formulas and prove that they characterize point-generated submodels of n-Henkin models. We also show that the de Jongh formulas do the same job as Jankov's characteristic formulas for **IPC**. The de Jongh formulas were introduced in [69, §4], see also [59, §2.5].

3.3.1. DEFINITION. Let w be a point in the n-universal model (a point of finite depth in the n-Henkin model). We inductively define formulas ϕ_w and ψ_w. If $d(w) = 1$, then let

$$\phi_w := \bigwedge\{p_k : w \models p_k\} \wedge \bigwedge\{\neg p_j : w \not\models p_j\} \text{ for each } k, j = 1, \ldots, n$$

and

$$\psi_w := \neg\phi_w.$$

If $d(w) > 1$, then let $\{w_1, \ldots, w_m\}$ be the set of all immediate successors of w. We let

$$prop(w) := \{p_k : w \models p_k\}$$

and

$$newprop(w) := \{p_k : w \not\models p_k \text{ and } w_i \models p_k \text{ for each } i \text{ such that } 1 \leq i \leq m\}.$$

We define ϕ_w and ψ_w by

$$\phi_w := \bigwedge prop(w) \wedge \left((\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}) \to \bigvee_{i=1}^{m} \phi_{w_i} \right)$$

and

$$\psi_w := \phi_w \to \bigvee_{i=1}^{m} \phi_{w_i}$$

We call ϕ_w and ψ_w the *de Jongh formulas.*

3.3.2. THEOREM. *For every $w \in U(n)$ ($w \in H(n)$ such that $d(w)$ is finite) we have that:*

- $R(w) = \{v \in H(n) : v \models \phi_w\}$, *i.e.,* $V(\phi_w) = R(w)$.
- $H(n) \setminus R^{-1}(w) = \{v \in H(n) : v \models \psi_w\}$, *i.e.,* $V(\psi_w) = H(n) \setminus R^{-1}(w)$.

Proof. We prove the theorem by induction on the depth of w. Let the depth of w be 1. This means, that w belongs to the maximum of $\mathcal{H}(n)$. By Definition 3.2.3(1) for every $v \in max(\mathcal{H}(n))$ such that $w \neq v$ we have $col(v) \neq col(w)$ and thus $v \not\models \phi_w$. Therefore, if $u \in H(n)$ is such that uRv for some maximal point v of $\mathcal{H}(n)$ distinct from w, then $u \not\models \phi_w$. Finally, assume that vRw and v is not related to any other maximal point. By Definition 3.2.3(2) and (3), this implies that $col(v) < col(w)$. Therefore, $v \not\models \phi_w$, and so $v \models \phi_w$ iff $v = w$. Thus, $V(\phi_w) = \{w\}$. Consequently, by the definition of the intuitionistic negation, we have that $V(\psi_w) = V(\neg\phi_w) = H(n) \setminus R^{-1}(V(\phi_w)) = H(n) \setminus R^{-1}(w)$.

Now suppose the depth of w is greater than 1 and the theorem holds for the points with depth strictly less than $d(w)$. This means that the theorem holds for every immediate successor w_i of w, i.e., for each $i = 1, \ldots, m$ we have $V(\phi_{w_i}) = R(w_i)$ and $V(\psi_{w_i}) = H(n) \setminus R^{-1}(w_i)$.

First note that, by the induction hypothesis, $w \not\models \bigvee_{i=1}^{m} \psi_{w_i}$; hence, by the definition of $newprop(w)$, we have $w \not\models \bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}$. Therefore, $w \models \phi_w$, and so, by the persistence of intuitionistic valuations, $v \models \phi_w$ for every $v \in R(w)$.

Now let $v \notin R(w)$. First assume that $v \in U(n)$. If $v \not\models \bigwedge prop(w)$, then $v \not\models \phi_w$. Thus, suppose $v \models \bigwedge prop(w)$. This means that $col(v) \geq col(w)$. Then two cases are possible:

Case 1. $v \in \bigcup_{i=1}^{m} H(n) \setminus R^{-1}(w_i)$. Then by the induction hypothesis, $v \models \bigvee_{i=1}^{m} \psi_{w_i}$ and since $v \notin R(w)$, we have $v \not\models \bigvee_{i=1}^{m} \phi_{w_i}$. Therefore, $v \not\models \phi_w$.

Case 2. $v \notin \bigcup_{i=1}^{m} H(n) \setminus R^{-1}(w_i)$. Then vRw_i for every $i = 1, \ldots, m$. If vRv' and $v' \in \bigcup_{i=1}^{m} H(n) \setminus R^{-1}(w_i)$, then, by Case 1, $v' \not\models \phi_w$, and so $v \not\models \phi_w$. Now assume that for every $v' \in U(n)$, vRv' implies $v' \notin \bigcup_{i=1}^{m} H(n) \setminus R^{-1}(w_i)$. By the construction of $\mathcal{U}(n)$ (see Definition 3.2.3(3)), there exists a point $u \in U(n)$ such that $u \prec \{w_1, \ldots, w_m\}$ and vRu. We again specify two cases.

Case 2.1. $u = w$. Then there exists $t \in U(n)$ such that $t \prec w$ and vRt. So, $col(v) \leq col(t)$ and by Definition 3.2.3(2), $col(t) < col(w)$, which is a contradiction.

Case 2.2. $u \neq w$. Since vRu and $col(v) \geq col(w)$, we have $col(u) \geq col(v) \geq col(w)$. If $col(u) > col(w)$, then there exists p_j, for some $j = 1, \ldots, n$, such that $u \models p_j$ and $w \not\models p_j$. Then $w_i \models p_j$, for every $i = 1, \ldots, m$, and hence $p_j \in newprop(w)$. Therefore, $u \models \bigvee newprop(w)$ and $u \not\models \bigvee_{i=1}^m \phi_{w_i}$. Thus, $u \not\models \phi_w$ and so $v \not\models \phi_w$. Now suppose $col(u) = col(w)$. Then by Definition 3.2.3(3), $u = w$ which is a contradiction.

Therefore, for every point v of $U(n)$ we have:

$$v \models \phi_w \text{ iff } wRv.$$

Finally, if $v \in H(n) \setminus U(n)$, by Theorem 3.2.9, v sees a point $v' \in U(n)$ of depth greater than $d(w)$. Then, $v' \not\models \phi_w$. Therefore $v \not\models \phi_w$ and $V(\phi_w) = R(w)$.

Now we show that ψ_w defines $H(n) \setminus R^{-1}(w)$. For every $v \in H(n)$, $v \not\models \psi_w$ iff there exists $u \in H(n)$ such that vRu and $u \models \phi_w$ and $u \not\models \bigvee_{i=1}^m \phi_{w_i}$, which holds iff $u \in R(w)$ and $u \notin \bigcup_{i=1}^m R(w_i)$, which, in turn, holds iff $u = w$. Hence, $v \not\models \psi_w$ iff $v \in R^{-1}(w)$. This finishes the proof of the theorem. $\square$

3.3.2 The Jankov-de Jongh theorem

In [64] Jankov introduced the so-called characteristic formulas and proved Theorem 3.3.3 formulated below. In this subsection we show that the de Jongh formulas do the same job as Jankov's characteristic formulas. We first state the Jankov-de Jongh theorem. Note that Jankov's original result was formulated in terms of Heyting algebras. We will formulate it in logical terms. Most of the results in this and subsequent sections have their natural algebraic counterparts but we will not discuss these here. For an algebraic treatment of the Jankov formulas we refer to [107, §5.2] and [121]. Note that analogues of these formulas for transitive modal logic were introduced by Fine [41]. In modal logic these formulas are called the Jankov-Fine formulas (see Chapter 8, for the details). Now we formulate the Jankov-de Jongh theorem; see [64], [69] and [24, Proposition 9.41].

3.3.3. THEOREM. *For every finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$ such that for every descriptive frame $\mathfrak{G}$:*

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \quad \textit{iff } \mathfrak{F} \textit{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Here we give a proof of Theorem 3.3.3 using the de Jongh formulas. An alternative proof is given in [24, §9.4], where *Jankov formulas* are treated as particular instances of more general "canonical formulas". First we prove one additional lemma.

3.3.4. LEMMA. *A descriptive frame $\mathfrak{F}$ is a p-morphic image of a generated subframe of a descriptive frame $\mathfrak{G}$ iff $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$.*

Proof. The proof follows from Theorem 2.3.7 and a result in universal algebra which says that if a variety $\mathbf{V}$ has the congruence extension property, then for every algebra $\mathfrak{A} \in \mathbf{V}$ we have $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$. It is well known that the variety of Heyting algebras has the congruence extension property [2, §4, p. 178]. The result now follows from the duality established in Theorem 2.3.7. □

Proof of Theorem 3.3.3

Suppose $\mathfrak{F}$ is a finite rooted frame. By Theorem 3.2.16, there exists an $n \in \omega$ and a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V)$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$. Let $w \in U(n)$ be the root of $\mathfrak{F}$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_w$. We show that we can take ψ_w as $\chi(\mathfrak{F})$. By Lemma 3.3.4, for proving Theorem 3.3.3 it is sufficient to show that for every frame $\mathfrak{G}$:

$$\mathfrak{G} \not\models \psi_w \text{ iff } \mathfrak{F}_w \text{ is a generated subframe of a } p\text{-morphic image of } \mathfrak{G}.$$

Suppose $\mathfrak{F}_w$ is a generated subframe of a p-morphic image of $\mathfrak{G}$. Clearly, $w \not\models \psi_w$. Therefore, $\mathfrak{F}_w \not\models \psi_w$, and since p-morphisms preserve the validity of formulas, $\mathfrak{G} \not\models \psi_w$.

Now suppose $\mathfrak{G} \not\models \psi_w$. Then, there exists a model $\mathfrak{M} = (\mathfrak{G}, V_1)$ such that $\mathfrak{M} \not\models \psi_w$. By Lemma 3.2.15(2), there exists a generated submodel $\mathfrak{M}' = (\mathfrak{G}', V')$ of $\mathcal{H}(n)$ such that $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$. This implies that $\mathfrak{M}' \not\models \psi_w$. Now, $\mathfrak{M}' \not\models \psi_w$ iff there exists v in $\mathfrak{G}'$ such that vRw, which holds iff w belongs to $\mathfrak{G}'$. Therefore, w is in $\mathfrak{G}'$, and $\mathfrak{F}_w$ is a generated subframe of $\mathfrak{G}'$. Thus, $\mathfrak{F}_w$ is a generated subframe of a p-morphic image of $\mathfrak{G}$. □

3.3.5. Remark. We point out one essential difference between the Jankov formulas and the de Jongh formulas: the number of propositional variables used in the Jankov formula depends on the cardinality of $\mathfrak{F}$, whereas the number of variables in the de Jongh formula is the smallest n such that $\mathcal{U}(n)$ contains $\mathfrak{F}$ as a generated subframe. Therefore, in general, the de Jongh formula contains fewer variables than the Jankov formula. From now on we will use the general term "the Jankov-de Jongh formula" to refer to the formulas having the property formulated in Theorem 3.3.3 and denote them by $\chi(\mathfrak{F})$.

3.3.3 Subframes, subframe and cofinal subframe formulas

In this section we introduce subframe formulas and cofinal subframe formulas. The subframe formulas for modal logic were first defined by Fine [42]. Subframe formulas for intuitionistic logic were introduced by Zakharyaschev [133]. Zakharyaschev also defined cofinal subframe formulas for intuitionistic and transitive modal logic [135]. For an overview of these results see [24, §9.4]. We define the subframe and cofinal subframe formulas differently and connect them to the NNIL formulas of [127], i.e., the formulas that are preserved under submodels. For an algebraic approach to subframe formulas we refer to [9].

3.3.6. Definition.

1. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A frame $\mathfrak{F}' = (W', R')$ is called a *subframe* of $\mathfrak{F}$ if $W' \subseteq W$ and R' is the restriction of R to W'.

2. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A descriptive frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is called a subframe of $\mathfrak{F}$ if (W', R') is a subframe of (W, R), $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$ and the following condition, which we call the *topo-subframe condition*, is satisfied:

   For every $U \subseteq W'$ such that $W' \setminus U \in \mathcal{P}'$ we have $W \setminus R^{-1}(U) \in \mathcal{P}$.

In topological terms the formulation becomes simpler. An Esakia space $\mathcal{X}' = (X', \mathcal{O}', R')$ is a *subframe* of an Esakia space $\mathcal{X} = (X, \mathcal{O}, R)$ if (X', R') is a subframe of (X, R), and $(X', \mathcal{O}')$ is a subspace of $(X, \mathcal{O})$,[5] and

For every clopen U of $\mathcal{X}'$ we have that $R^{-1}(U)$ is a clopen subset of $\mathcal{X}$.

3.3.7. Remark. The reason for adding the topo-subframe condition to the definition of subframes of descriptive frames is explained by the next proposition. The topo-subframe condition allows us to extend a descriptive valuation V' defined on a subframe $\mathfrak{F}'$ of a descriptive frame $\mathfrak{F}$ to a descriptive valuation V of $\mathfrak{F}$ such that the restriction of V to $\mathfrak{F}'$ is equal to V'. A correspondence between subframes and nuclei (special operations on Heyting algebras) is established in [9]. This correspondence gives another motivation for defining the subframes of descriptive frames in this way.

Now we prove one of the main properties of subframes. Note that the proof makes essential use of the topo-subframe condition.

3.3.8. Proposition. *Let $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ be descriptive frames. If $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, then for every descriptive valuation V' on $\mathfrak{F}'$ there exists a descriptive valuation V on $\mathfrak{F}$ such that the restriction of V to W' is V'.*

Proof. For every $p \in$ Prop let $V(p) = W \setminus R^{-1}(W' \setminus V'(p))$. By the topo-subframe condition, $V(p) \in \mathcal{P}$. Now suppose $x \in W'$. Then $x \notin V(p)$ iff $x \in R^{-1}(W' \setminus V'(p))$ iff (there is $y \in W'$ such that $y \notin V'(p)$ and xRy) iff $x \notin V'(p)$, since $V'(p)$ is an upset of $\mathfrak{F}'$. Therefore, $V(p) \cap W' = V'(p)$. □

Next we introduce cofinal subframes.

[5] Since a compact subset of a Hausdorff space is closed (see e.g., [32]) every subframe of an Esakia space is topologically closed.

3.3.9. Definition.

1. Let (W, R) and (W', R') be Kripke frames. $\mathfrak{F}'$ is called a *cofinal subframe* of $\mathfrak{F}$ if $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$ and $R(W') \subseteq R^{-1}(W')$, that is, for every $w, v \in W$ if $w \in W'$ and wRv, there exists $u \in W'$ such that vRu.

2. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A subframe $\mathfrak{F}' = (W', R', \mathcal{P}')$ of $\mathfrak{F}$ is called a *cofinal subframe* if (W', R') is a cofinal subframe of (W, R).

We extend the notion of subframes and cofinal subframes to descriptive models and Kripke models.

3.3.10. Definition. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$ be (descriptive or Kripke) models. We say that $\mathfrak{M}'$ is a (*cofinal*) *submodel* of $\mathfrak{M}$ if $\mathfrak{F}'$ is a (cofinal) subframe of $\mathfrak{F}$ and V' is the restriction of V.

Let $\mathfrak{F}$ be a finite rooted frame. For every point w of $\mathfrak{F}$ we introduce a propositional letter p_w and let V be such that $V(p_w) = R(w)$. We denote by $\mathfrak{M}$ the model $(\mathfrak{F}, V)$. It is easy to see that $\mathfrak{M}$ is isomorphic to a generated submodel of the n-Henkin model, where $n = |\mathfrak{F}|$ (see Theorem 3.2.16).

3.3.11. Proposition. *Let $(\mathfrak{F}, V)$ be as above. Then for every $w, v \in W$ we have:*

1. *$w \neq v$ and wRv iff $col(w) < col(v)$,*

2. *$w = v$ iff $col(w) = col(v)$.*

Proof. The proof is just spelling out the definitions. □

Next we inductively define the subframe formula $\beta(\mathfrak{F})$. Note that this definition is different from that of [24, §9.4].

For every $v \in W$ let

$$notprop(v) := \{p_k : v \not\models p_k, k \leq n\}.$$

3.3.12. Definition. We define $\beta(\mathfrak{F})$ by induction. If v is a maximal point of $\mathfrak{M}$ then let

$$\beta(v) := \bigwedge prop(v) \to \bigvee notprop(v)$$

Let w be a point in $\mathfrak{M}$ and let $w_1, \ldots, w_m$ be all the immediate successors of w. We assume that $\beta(w_i)$ is already defined, for every w_i. We define $\beta(w)$ by

$$\beta(w) := \bigwedge prop(w) \to \Big(\bigvee notprop(w) \vee \bigvee_{i=1}^{m} \beta(w_i)\Big).$$

Let r be the root of $\mathfrak{F}$. We define $\beta(\mathfrak{F})$ by

$$\beta(\mathfrak{F}) := \beta(r).$$

We call $\beta(\mathfrak{F})$ the *subframe formula* of $\mathfrak{F}$.

We will need the next three lemmas for establishing the crucial property of subframe formulas and cofinal subframe formulas.

3.3.13. LEMMA. *Let $\mathfrak{F} = (W, R)$ be a finite rooted frame and let V be defined as above. Let $\mathfrak{M}' = (W', R', V')$ be an arbitrary (descriptive or Kripke) model. For every $w, v \in W$ and $x \in W'$, if wRv, then*

$$\mathfrak{M}', x \not\models \beta(w) \text{ implies } \mathfrak{M}', x \not\models \beta(v).$$

Proof. The proof is a simple induction on the depth of v. If $d(v) = d(w) - 1$ and wRv, then v is an immediate successor of w. Then $\mathfrak{M}', x \not\models \beta(w)$ implies $\mathfrak{M}', x \not\models \beta(v)$, by the definition of $\beta(w)$. Now suppose $d(v) = d(w) - (k+1)$ and the lemma is true for every u such that wRu and $d(u) = d(w) - k$, for every k. Let u' be an immediate predecessor of v such that wRu'. Such a point clearly exists since we have wRv. Then $d(u') = d(w) - k$ and by the induction hypothesis $\mathfrak{M}, x \not\models \beta(u')$. This, by definition of $\beta(u')$, means that $\mathfrak{M}', x \not\models \beta(v)$. □

3.3.14. LEMMA. *Let $\mathfrak{M}_1 = (W_1, R_1, \mathcal{P}_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, \mathcal{P}_2, V_2)$ be descriptive models. Let $\mathfrak{M}_2$ be a submodel of $\mathfrak{M}_1$. Then for every finite rooted frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{M}_2 \not\models \beta(\mathfrak{F})$ implies $\mathfrak{M}_1 \not\models \beta(\mathfrak{F})$.*

Proof. We prove the lemma by induction on the depth of $\mathfrak{F}$. If the depth of $\mathfrak{F}$ is 1, i.e., it is a reflexive point, then the lemma clearly holds. Now assume that it holds for every rooted frame of depth less than the depth of $\mathfrak{F}$. Let r be the root of $\mathfrak{F}$. Then $\mathfrak{M}_2 \not\models \beta(\mathfrak{F})$ means that there is a point $t \in W_2$ such that $\mathfrak{M}_2, t \models \bigwedge prop(r)$, $\mathfrak{M}_2, t \not\models \bigvee notprop(r)$ and $\mathfrak{M}_2, t \not\models \beta(r')$, for every immediate successor r' of r. By the induction hypothesis, we get that $\mathfrak{M}_1, t \not\models \beta(r')$. Since $V_2(p) = V_1(p) \cap W_2$ we also have $\mathfrak{M}_1, t \not\models \bigvee notprop(r)$ and $\mathfrak{M}_1, t \models \bigwedge prop(r)$. Therefore, $\mathfrak{M}_1, t \not\models \beta(\mathfrak{F})$. □

Subsequently we will use the following auxiliary lemma.

3.3.15. LEMMA. *Let $\mathfrak{F} = (W, R, \mathcal{P}, V)$ be a descriptive model and let $\mathcal{X} = (X, \mathcal{O}, R, V)$ be an Esakia space with a valuation.*

1. *For every color $c = i_1 \ldots i_n$ the set $C = \{w \in W : col(w) = c\}$ is a finite intersection of elements of $\mathcal{P} \cup -\mathcal{P}$, where $-\mathcal{P} = \{W \setminus U : U \in \mathcal{P}\}$.*

2. *For every color $c = i_1 \ldots i_n$ the set $C = \{x \in X : col(x) = c\}$ is a clopen of $\mathcal{X}$.*

Proof. (1) It is a easy to see that $C = \bigcap_{k=1}^{n} I^{\epsilon_k}$, where

$$I^{\epsilon_k} = \begin{cases} V(p_k) & \text{if } \epsilon_k = 1, \\ W \setminus V(p_k) & \text{if } \epsilon_k = 0. \end{cases}$$

(2) The result follows from (1) and the duality between descriptive frames and Esakia spaces, see Section 2.3.3. □

The next theorem states the crucial property of subframe formulas.

3.3.16. THEOREM. *Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame and let $\mathfrak{F} = (W, R)$ be a finite rooted frame. Then*

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}.$$

Proof. Suppose $\mathfrak{G} \not\models \beta(\mathfrak{F})$. Then there exists a valuation V' on $\mathfrak{G}$ such that $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$. For every $w \in W$, let $\{w_1, \dots w_m\}$ denote the set of all immediate successors of w. Let $p_1, \dots, p_n$ be the propositional variables occurring in $\beta(\mathfrak{F})$ (in fact $n = |W|$). Therefore, V' defines a coloring of $\mathfrak{G}$. Let

$$P_w := \{x \in W' : col(x) = col(w) \text{ and } x \not\models \bigvee_{i=1}^{m} \beta(w_i)\}.$$

Let $Y := \bigcup_{w \in W} P_w$ and let $\mathfrak{H} := (Y, S, \mathcal{Q})$, where S is the restriction of R' to Y and $\mathcal{Q} = \{U' \cap Y : U' \in \mathcal{P}'\}$. We show that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ and $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$.

First we show that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$. The definition of $\mathfrak{H}$ ensures that (Y, S) is a subframe of (W', R'). We need to show that $\mathfrak{H}$ satisfies the topo-subframe condition. To simplify the proof we will use the topological terminology. First note that for every $w \in W$, $P_w = C_w \cap D_w$, where $C_w = \{x \in W' : col(x) = col(w)\}$ and $D_w = \{x \in W' : x \not\models \bigvee_{i=1}^{m} \beta(w_i)\}$. By Lemma 3.3.15, C_w is a clopen set. For every $w \in W$ we have $D_w \in -\mathcal{P}'$, i.e., $W \setminus D_w \in \mathcal{P}'$. This means that D_w is also clopen. Hence P_w is an intersection of two clopens and thus is again a clopen. Then Y is a finite union of clopens and therefore is also a clopen. Thus, every clopen subset U of $\mathfrak{H}$ is a clopen subset of $\mathfrak{G}$ and by Definition 2.3.20(5), $R^{-1}(U)$ is clopen. Therefore, $\mathfrak{H}$ satisfies the topo-subframe condition and $\mathfrak{H}$ is a subframe of $\mathfrak{G}$.

Define a map $f : Y \to W$ by

$$f(x) = w \text{ if } x \in P_w.$$

We show that f is a well-defined onto p-morphism. By Proposition 3.3.11, distinct points of W have distinct colors. Therefore, $P_w \cap P_{w'} = \emptyset$ if $w \neq w'$. This means that f is well defined.

Now we prove that f is onto. By the definition of f, it is sufficient to show that $P_w \neq \emptyset$ for every $w \in W$. If r is the root of $\mathfrak{F}$, then since $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$, there exists a point $x \in W'$ such that $x \models \bigwedge prop(r)$ and $x \not\models \bigvee notprop(r)$ and $x \not\models \bigvee_{i=1}^{m} \beta(r_i)$. This means that $x \in P_r$. If w is not the root of $\mathfrak{F}$ then we have rRw. Therefore, by Lemma 3.3.13, we have $x \not\models \beta(w)$. This means that there is a successor y of x such that $y \models \bigwedge prop(w)$, $y \not\models \bigvee notprop(w)$ and $y \not\models \beta(w_i)$, for every immediate successor w_i of w. Therefore, $y \in P_w$ and f is surjective.

Next assume that $x, y \in Y$ and xSy. Note that by the definition of f, for every $t \in Y$ we have

$$col(t) = col(f(t)).$$

Obviously, xSy implies $col(x) \leq col(y)$. Therefore, $col(f(x)) = col(x) \leq col(y) = col(f(y))$. By Proposition 3.3.11, this yields $f(x)Rf(y)$. Now suppose $f(x)Rf(y)$. Then by the definition of f we have that $x \not\models \beta(f(x))$ and by Lemma 3.3.13, $x \not\models \beta(f(y))$. This means that there is $z \in W'$ such that $xR'z$, $col(z) = col(f(y))$, and $z \not\models \beta(u)$, for every immediate successor u of $f(y)$. Thus, $z \in P_{f(y)}$ and $f(z) = f(y)$. Therefore, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$.

Conversely, suppose $\mathfrak{H}$ is a subframe of a descriptive frame $\mathfrak{G}$ and $f : \mathfrak{H} \to \mathfrak{F}$ is a p-morphism. Clearly, $\mathfrak{F} \not\models \beta(\mathfrak{F})$ and since f is a p-morphism, we have that $\mathfrak{H} \not\models \beta(\mathfrak{F})$. This means that there is a valuation V' on $\mathfrak{H}$ such that $(\mathfrak{H}, V') \not\models \beta(\mathfrak{F})$. By Proposition 3.3.8, V' can be extended to a valuation V on $\mathfrak{G}$ such that the restriction of V to $\mathfrak{G}'$ is equal to V'. This, by Lemma 3.3.14, implies that $\mathfrak{G} \not\models \beta(\mathfrak{F})$. $\square$

3.3.17. Remark. We remark on a close connection between subframe formulas and NNIL formulas introduced in [127]. NNIL formulas are the formulas without nestings of implications to the left. In [127] it is proved that NNIL formulas are exactly those formulas that are preserved under taking submodels, and therefore they are also preserved under taking subframes. It is easy to see that every $\beta(\mathfrak{F})$ is a NNIL formula. It will follow from Theorem 3.4.16 that every subframe logic is axiomatized by NNIL formulas.

Next we define cofinal subframe formulas in a fashion similar to subframe formulas. Let $\mathfrak{F}$ be a finite rooted frame. For every point w of $\mathfrak{F}$ introduce a propositional letter p_w and let V be such that $V(p_w) = R(w)$. For the root r of $\mathfrak{F}$ let $r_1, \ldots, r_m$ be the immediate successors of r and $u_1, \ldots, u_k$ be the maximal points of $\mathfrak{F}$. For every $w \in W$ let $\beta(w)$ be as in Definition 3.3.12. Let

$$\mu(\mathfrak{F}) := \neg\neg\Big((\bigwedge prop(u_1) \wedge \neg \bigvee notprop(u_1)) \vee \ldots \vee$$

$$(\bigwedge prop(u_k) \wedge \neg \bigvee notprop(u_k))\Big).$$

We are now ready to define cofinal subframe formulas.

3.3.18. Definition. The formula

$$\gamma(\mathfrak{F}) := \Big(\bigwedge prop(r) \wedge \mu(\mathfrak{F})\Big) \to \Big(\bigvee notprop(r) \vee \bigvee_{i=1}^{m} \beta(r_i)\Big)$$

is called the *cofinal subframe formula of* $\mathfrak{F}$.

3.3.19. Theorem. *Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame and $\mathfrak{F} = (W, R)$ a finite rooted frame. Then*

$$\mathfrak{G} \not\models \gamma(\mathfrak{F}) \quad \textit{iff } \mathfrak{F} \textit{ is a p-morphic image of a cofinal subframe of } \mathfrak{G}.$$

Proof. The proof is similar to the proof of Theorem 3.3.16. We follow the notations of the proof of Theorem 3.3.16. For every $w \in W$ we define

$$P_w := \{\ x \in W' : col(x) = col(w) \text{ and } x \not\models \bigvee_{i=1}^{k} \beta(w_i) \text{ and } x \models \mu(\mathfrak{F})\ \}.$$

We proceed as in the proof of Theorem 3.3.16. Define Y as the union of all P_w, for $w \in W$. The frame $\mathfrak{H}$ is obtained by restricting to Y the valuation and the order of $\mathfrak{G}$. Exactly the same argument as in the proof of Theorem 3.3.16 shows that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ and that $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$. All we need to show is that in this case, $\mathfrak{H}$ is a cofinal subframe of $\mathfrak{G}$. Let $x \in Y$ and $xR'y$. We need to find $z \in Y$ such that $yR'z$. By Theorem 2.3.24, there exists $z \in max(\mathfrak{G})$ such that $yR'z$. We show that $z \in Y$. Since $(\mathfrak{G}, V'), x \models \mu(\mathfrak{F})$, we have $z \models \mu(\mathfrak{F})$ and moreover $z \models \Big(\bigwedge prop(u_1) \wedge \neg \bigvee notprop(u_1)) \vee \ldots \vee (\bigwedge prop(u_k) \wedge \neg \bigvee notprop(u_k)\Big)$, (for the truth definition of the formulas with double negations consult Section 2.1). This means that $z \models \mu(\mathfrak{F})$ and there exists a maximal point u_i of $\mathfrak{F}$, for some $i = 1, \ldots, k$, such that $col(u_i) = col(z)$. Thus, $z \in P_{u_i}$ and $z \in Y$. Therefore, $\mathfrak{H}$ is a cofinal subframe of $\mathfrak{G}$. $\square$

3.4 Frame-based formulas

In this section we will treat the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas in a uniform framework. This will enable us to get simple proofs of some old results and also derive some new results. We give a definition of frame-based formulas and show that these three types of formulas are particular cases of frame-based formulas. We prove a criterion for recognizing whether an intermediate logic is axiomatized by frame-based formulas. Using this criterion we show that every locally tabular intermediate logic is axiomatized by the Jankov-de Jongh formulas and that every tabular logic is finitely axiomatized by these formulas. We also recall the definitions of subframe logics and cofinal subframe logics and show that every subframe logic is axiomatized by subframe formulas and every cofinal subframe logic is axiomatized by cofinal subframe formulas. At the end of the section we show that there are intermediate logics that are not axiomatized by frame-based formulas. We first recall some basic definitions and results.

3.4.1. Definition. Let L be an intermediate logic.

1. A descriptive frame $\mathfrak{F}$ is called *an L-frame* if $\mathfrak{F}$ validates all the theorems of L.

2. Let $\mathbb{FG}(L)$ denote the set of all finitely generated rooted descriptive L-frames modulo isomorphism.

3. Let $\mathbf{F}_L$ denote the set of all finite rooted L-frames modulo isomorphism.

Then $\mathbf{F}_{\mathbf{IPC}}$ is the set of all finite rooted frames modulo isomorphism. As we mentioned in the beginning of this chapter, every variety of algebras is generated by its finitely generated members. This result can be extended to finitely generated subdirectly irreducible algebras; see, e.g., [23].

3.4.2. Theorem. *Every variety of algebras is generated by its finitely generated subdirectly irreducible algebras.*

Translating this theorem in terms of intermediate logics we obtain the following corollary.

3.4.3. Corollary. *Every intermediate logic L is complete with respect to its finitely generated rooted descriptive frames, i.e., L is complete with respect to $\mathbb{FG}(L)$.*

Next we define three relations on descriptive frames.

3.4.4. Definition. Let $\mathfrak{F}$ and $\mathfrak{G}$ be descriptive frames. We say that

1. $\mathfrak{F} \leq \mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.[6]

2. $\mathfrak{F} \preccurlyeq \mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{G}$.

3. $\mathfrak{F} \preccurlyeq' \mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a cofinal subframe of $\mathfrak{G}$.

We write $\mathfrak{F} < \mathfrak{G}$, $\mathfrak{F} \prec \mathfrak{G}$ and $\mathfrak{F} \prec' \mathfrak{G}$ if $\mathfrak{F} \leq \mathfrak{G}$, $\mathfrak{F} \preccurlyeq \mathfrak{G}$ and $\mathfrak{F} \preccurlyeq' \mathfrak{G}$, respectively, and $\mathfrak{F}$ is not isomorphic to $\mathfrak{G}$.

The next proposition discusses some basic properties of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$. The proof is simple and we will skip it.

3.4.5. Proposition.

1. *Each of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ is reflexive and transitive.*

2. *If we restrict ourselves to finite frames, then each of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ is a partial order.*

3. *In the infinite case none of $\leq$, $\preccurlyeq$, $\preccurlyeq'$ is in general anti-symmetric.*

4. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two finite rooted frames. Let $\mathfrak{G}$ be an arbitrary descriptive frame. Then*

[6] By Lemma 3.3.4, this is equivalent to saying that $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$.

(a) $\mathfrak{F} \leq \mathfrak{F}'$ *and* $\mathfrak{G} \models \chi(\mathfrak{F})$ *imply* $\mathfrak{G} \models \chi(\mathfrak{F}')$.

(b) $\mathfrak{F} \preccurlyeq \mathfrak{F}'$ *and* $\mathfrak{G} \models \beta(\mathfrak{F})$ *imply* $\mathfrak{G} \models \beta(\mathfrak{F}')$.

(c) $\mathfrak{F} \preccurlyeq' \mathfrak{F}'$ *and* $\mathfrak{G} \models \gamma(\mathfrak{F})$ *imply* $\mathfrak{G} \models \gamma(\mathfrak{F}')$.

Note that Theorems 3.3.3, 3.3.16 and 3.3.19 can be formulated in terms of the relations $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ as follows:

3.4.6. THEOREM. *For every finite rooted frame $\mathfrak{F}$ there exist formulas $\chi(\mathfrak{F})$, $\beta(\mathfrak{F})$ and $\gamma(\mathfrak{F})$ such that for every descriptive frame $\mathfrak{G}$:*

1. $\mathfrak{G} \not\models \chi(\mathfrak{F})$ *iff* $\mathfrak{F} \leq \mathfrak{G}$.

2. $\mathfrak{G} \not\models \beta(\mathfrak{F})$ *iff* $\mathfrak{F} \preccurlyeq \mathfrak{G}$.

3. $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *iff* $\mathfrak{F} \preccurlyeq' \mathfrak{G}$.

Proposition 3.4.5 and Theorem 3.4.6 clearly indicate that these three types of formulas can be treated in a uniform framework. Next we give a general definition of frame-based formulas and show that the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas are particular cases of frame-based formulas. Let $\trianglelefteq$ be a relation on $\mathbb{FG}(L)$. We write $\mathfrak{F} \triangleleft \mathfrak{G}$ if $\mathfrak{F} \trianglelefteq \mathfrak{G}$ and $\mathfrak{F}$ and $\mathfrak{G}$ are not isomorphic.

3.4.7. DEFINITION. Call a reflexive and transitive relation $\trianglelefteq$ on $\mathbb{FG}(\mathbf{IPC})$ a *frame order* if the following two conditions are satisfied:

1. For every $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(L)$, $\mathfrak{G} \in \mathbf{F_{IPC}}$ and $\mathfrak{F} \triangleleft \mathfrak{G}$ imply $|\mathfrak{F}| < |\mathfrak{G}|$.

2. For every finite rooted frame $\mathfrak{F}$ there exists a formula $\alpha(\mathfrak{F})$ such that for every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}) \quad \text{iff} \quad \mathfrak{F} \trianglelefteq \mathfrak{G}.$$

We call the formula $\alpha(\mathfrak{F})$ the *frame-based formula for* $\trianglelefteq$ or simply the *α-formula* of $\mathfrak{F}$.

Obviously, the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas are frame-based formulas for $\leq$, $\preccurlyeq$ and $\preccurlyeq'$, respectively.

3.4.8. LEMMA.

1. *The restriction of $\trianglelefteq$ to $\mathbf{F_{IPC}}$ is a partial order.*

2. $\mathbf{F_{IPC}}$ *is a $\trianglelefteq$-downset, i.e., $\mathfrak{F} \in \mathbf{F_{IPC}}$ and $\mathfrak{F}' \trianglelefteq \mathfrak{F}$ imply $\mathfrak{F}' \in \mathbf{F_{IPC}}$.*

Proof. The relation $\trianglelefteq$ is reflexive and transitive by definition. That the restriction of $\trianglelefteq$ is anti-symmetric on finite frames follows from Definition 3.4.7(1). That $\mathbf{F_{IPC}}$ is a $\trianglelefteq$-downset, also follows immediately from Definition 3.4.7(1). $\square$

3.4.9. LEMMA. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be finite rooted frames.*

$$\textit{If } \mathfrak{F}\trianglelefteq\mathfrak{F}', \textit{ then } \mathbf{IPC} + \alpha(\mathfrak{F}) \vdash \alpha(\mathfrak{F}').$$

Proof. Let $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$ and $\mathfrak{G} \not\models \alpha(\mathfrak{F}')$, then $\mathfrak{F}'\trianglelefteq\mathfrak{G}$. By the transitivity of $\trianglelefteq$ we then have that $\mathfrak{F}\trianglelefteq\mathfrak{G}$ and $\mathfrak{G} \not\models \alpha(\mathfrak{F})$. By Corollary 3.4.3 we get that $\mathbf{IPC} + \alpha(\mathfrak{F}) \vdash \alpha(\mathfrak{F}')$. $\square$

3.4.10. DEFINITION. Let L be an intermediate logic and let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. We say that L is *axiomatized* by frame-based formulas for $\trianglelefteq$ if there exists a family $\{\mathfrak{F}_i\}_{i\in I}$ of finite rooted frames such that $L = \{\alpha(\mathfrak{F}_i) : i \in I\}$.

For every subset U of $\mathbb{FG}(L)$ let $min_{\trianglelefteq}(U)$ denote the set of the $\trianglelefteq$-minimal elements of U.

3.4.11. DEFINITION. Let L be an intermediate logic. We let

$$\mathbf{M}(L, \trianglelefteq) := min_{\trianglelefteq}(\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L))$$

We give a criterion recognizing whether an intermediate logic is axiomatized by frame-based formulas.

3.4.12. THEOREM. *Let L be an intermediate logic and let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. Then L is axiomatized by frame-based formulas for $\trianglelefteq$ iff the following two conditions are satisfied.*

1. *$\mathbb{FG}(L)$ is a $\trianglelefteq$-downset. That is, for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$, if $\mathfrak{G} \in \mathbb{FG}(L)$ and $\mathfrak{F} \trianglelefteq \mathfrak{G}$, then $\mathfrak{F} \in \mathbb{FG}(L)$.*
2. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

Proof. Suppose L is axiomatized by frame-based formulas for $\trianglelefteq$. Then $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}_i) : i \in I\}$, for some family $\{\mathfrak{F}_i\}_{i\in I}$ of finite rooted frames. First we show that $\mathbb{FG}(L)$ is $\trianglelefteq$-downset. Suppose, for some $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$ we have $\mathfrak{G} \in \mathbb{FG}(L)$ and $\mathfrak{F}\trianglelefteq\mathfrak{G}$. Assume that $\mathfrak{F} \notin \mathbb{FG}(L)$. Then there exists $i \in I$ such that $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i)$. Therefore, by Definition 3.4.7(2), $\mathfrak{F}_i\trianglelefteq\mathfrak{F}$. By the transitivity of $\trianglelefteq$, we have that $\mathfrak{F}_i\trianglelefteq\mathfrak{G}$, which implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i)$, a contradiction. Thus, $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset.

Suppose there exist $i, j \in I$ such that $i \neq j$ and $\mathfrak{F}_i\trianglelefteq\mathfrak{F}_j$. Then by Lemma 3.4.9, $\mathbf{IPC} + \alpha(\mathfrak{F}_i) \vdash \alpha(\mathfrak{F}_j)$. Therefore, we can exclude $\alpha(\mathfrak{F}_j)$ from the axiomatization of L. So it is sufficient to consider only $\trianglelefteq$-minimal elements of $\{\mathfrak{F}_i\}_{i\in I}$. (By Definition 3.4.7(1), the set of $\trianglelefteq$-minimal elements of an infinite set of finite rooted frames is non-empty.) Thus, without loss of generality we may assume that $\neg(\mathfrak{F}_i \trianglelefteq \mathfrak{F}_j)$, for $i \neq j$. To verify the second condition suppose $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Then $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i)$ for some $i \in I$, which implies $\mathfrak{F}_i \trianglelefteq \mathfrak{G}$. Hence, if we show that $\mathfrak{F}_i \in \mathbf{M}(L, \trianglelefteq)$, then Condition (2) of the theorem is satisfied.

We now prove that every $\mathfrak{F}_i$ belongs to $\mathbf{M}(L, \trianglelefteq)$. By the reflexivity of $\trianglelefteq$, we have $\mathfrak{F}_i \not\models \alpha(\mathfrak{F}_i)$ for every $i \in I$. Therefore, $\mathfrak{F}_i \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Now suppose $\mathfrak{F} \triangleleft \mathfrak{F}_i$. By Definition 3.4.7(1), $|\mathfrak{F}| < |\mathfrak{F}'|$ implying that $\mathfrak{F}$ is finite. By Lemma 3.4.8, $\trianglelefteq$ is anti-symmetric on finite frames, hence $\neg(\mathfrak{F}_i \trianglelefteq \mathfrak{F})$. If $\mathfrak{F}_j \trianglelefteq \mathfrak{F}$, for some $j \in I$ and $j \neq i$, then by the transitivity of $\trianglelefteq$ we have $\mathfrak{F}_j \trianglelefteq \mathfrak{F}_i$, which is a contradiction. Therefore, $\neg(\mathfrak{F}_j \trianglelefteq \mathfrak{F})$, for every $j \in I$. Thus, $\mathfrak{F} \models \alpha(\mathfrak{F}_j)$, for every $j \in I$, which implies that $\mathfrak{F} \in \mathbb{FG}(L)$ and that $\mathfrak{F}_i$ is a minimal element of $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Thus, $\mathfrak{F}_i \in \mathbf{M}(L, \trianglelefteq)$ and Condition (2) is satisfied.

For the right to left direction, first note that, by our assumption, $\mathbf{M}(L, \trianglelefteq)$ consists of only finite frames. We show that $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. We prove this by showing that the finitely generated rooted descriptive frames of L and of $\mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$ coincide. Let $\mathfrak{G} \in \mathbb{FG}(L)$, then since $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset, for every $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ we have that $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$ and hence $\mathfrak{G} \models \alpha(\mathfrak{F})$. On the other hand, if $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, then by our assumption there exists $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$. Therefore, $\mathfrak{G} \not\models \alpha(\mathfrak{F})$ and $\mathfrak{G}$ is not a frame for $\mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. Since every intermediate logic is complete with respect to its finitely generated rooted descriptive frames (see Corollary 3.4.3), we obtain that $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. □

Next we apply this criterion to the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas.

3.4.13. THEOREM. *Let L be an intermediate logic.*

1. *$\mathbb{FG}(L)$ is a $\leq$-downset.*
2. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \preccurlyeq)$ such that $\mathfrak{F} \preccurlyeq \mathfrak{G}$.*
3. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \preccurlyeq')$ such that $\mathfrak{F} \preccurlyeq' \mathfrak{G}$.*

Proof. (1) is trivial since generated subframes and p-morphisms preserve the validity of formulas. The proofs of (2) and (3) are quite involved, we will skip them here. For the proofs we refer to [24, Theorem 11.15]. □

These results allow us to obtain the following criterion.

3.4.14. COROLLARY. *Let L be an intermediate logic.*

1. *L is axiomatized by the Jankov-de Jongh formulas iff for every frame $\mathfrak{G}$ in $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$.*

2. *L is axiomatized by subframe formulas iff $\mathbb{FG}(L)$ is a $\preccurlyeq$-downset.*

3. *L is axiomatized by cofinal subframe formulas iff $\mathbb{FG}(L)$ is a $\preccurlyeq'$-downset.*

Proof. The result is an immediate consequence of Theorems 3.4.12 and 3.4.13. □

3.4.15. DEFINITION. Let L be an intermediate logic.

1. L is called a *subframe logic* if for every L-frame $\mathfrak{G}$, every subframe $\mathfrak{G}'$ of $\mathfrak{G}$ is also an L-frame.

2. L is called a *cofinal subframe logic* if for every L-frame $\mathfrak{G}$, every cofinal subframe $\mathfrak{G}'$ of $\mathfrak{G}$ is also an L-frame.

For the next theorem consult [24, Theorem 11.21].

3.4.16. COROLLARY. *Let L be an intermediate logic.*

1. *L is axiomatized by subframe formulas iff L is a subframe logic.*

2. *L is axiomatized by cofinal subframe formulas iff L is a cofinal subframe logic.*

Proof. Since every intermediate logic L is complete with respect to $\mathbb{FG}(L)$, it is easy to see that L is a subframe logic iff $\mathbb{FG}(L)$ is a $\preccurlyeq$-downset and L is a cofinal subframe logic iff $\mathbb{FG}(L)$ is a $\preccurlyeq'$-downset. The proof now follows from Theorem 3.4.13. □

Next we mention yet another general result about subframe logics and cofinal subframe logics; see [24, Theorem 11.20]. An algebraic proof of the result can be found in [9].

3.4.17. THEOREM. *All subframe logics and cofinal subframe logics enjoy the finite model property.*

Proof. We prove the theorem for subframe logics only. The proof for cofinal subframe logics is identical. Let L be a subframe logic. Suppose $L \nvdash \phi$. Then there exists $\mathfrak{F} \in \mathbb{FG}(L)$ such that $\mathfrak{F} \not\models \phi$. Consider $L+\phi$. If it is inconsistent then every finite L-frame refutes ϕ. Thus, assume $L + \phi$ is consistent. By Proposition 2.1.6, it is an intermediate logic. Then by Theorem 3.4.13(2), there is $\mathfrak{F}' \in \mathbf{M}(L + \phi, \preccurlyeq)$ such that $\mathfrak{F}' \preccurlyeq \mathfrak{F}$. Since $\mathfrak{F}' \in \mathbf{M}(L + \phi, \preccurlyeq)$ we have $\mathfrak{F}' \not\models \phi$ and as L is a subframe logic, by Corollary 3.4.14(2), $\mathbb{FG}(L)$ is a $\preccurlyeq$-downset. Therefore, $\mathfrak{F}' \in \mathbf{F}_L$ and L has the fmp. □

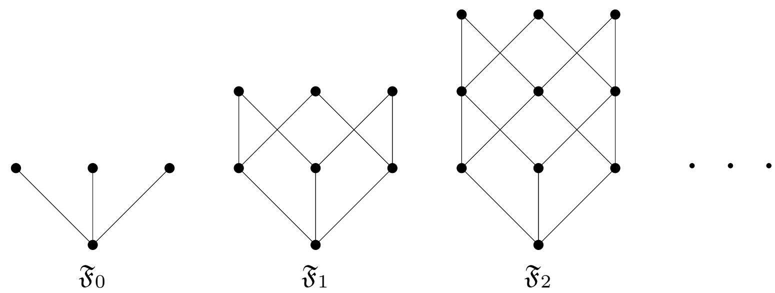

Figure 3.6: The sequence Δ

Next we show that every locally tabular intermediate logic is axiomatized by the Jankov-de Jongh formulas, and that every tabular logic is finitely axiomatized by the Jankov-de Jongh formulas. We also construct intermediate logics that can be axiomatized by Jankov-de Jongh formulas but not by subframe and cofinal subframe formulas, and vice versa. First we show that there are continuum many intermediate logics.

We discuss a method for constructing continuum many intermediate logics. Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. A set of frames Δ is called an $\trianglelefteq$-*antichain* if for every distinct $\mathfrak{F}, \mathfrak{G} \in \Delta$ we have $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$ and $\neg(\mathfrak{G} \trianglelefteq \mathfrak{F})$.

3.4.18. Theorem. *Let $\Delta = \{\mathfrak{F}_i\}_{i\in\omega}$ be an $\trianglelefteq$-antichain. For every $\Gamma_1, \Gamma_2 \subseteq \Delta$, if $\Gamma_1 \neq \Gamma_2$, then $Log(\Gamma_1) \neq Log(\Gamma_2)$.*

Proof. Without loss of generality assume that $\Gamma_1 \not\subseteq \Gamma_2$. This means that there is $\mathfrak{F} \in \Gamma_1$ such that $\mathfrak{F} \notin \Gamma_2$. Consider the α-formula $\alpha(\mathfrak{F})$. Then, by the reflexivity of $\trianglelefteq$, we have $\mathfrak{F} \not\models \alpha(\mathfrak{F})$. Hence, $\Gamma_1 \not\models \alpha(\mathfrak{F})$ and $\alpha(\mathfrak{F}) \notin Log(\Gamma_1)$. Now we show that $\alpha(\mathfrak{F}) \in Log(\Gamma_2)$. Suppose $\alpha(\mathfrak{F}) \notin Log(\Gamma_2)$. Then there is $\mathfrak{G} \in \Gamma_2$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F})$. This means that $\mathfrak{F} \trianglelefteq \mathfrak{G}$, which contradicts the fact that Δ forms an $\trianglelefteq$-antichain. Therefore, $\alpha(\mathfrak{F}) \notin Log(\Gamma_1)$ and $\alpha(\mathfrak{F}) \in Log(\Gamma_2)$. Thus, $Log(\Gamma_1) \neq Log(\Gamma_2)$. $\square$

Now we construct an infinite $\leq$-antichain. Consider the sequence Δ of finite rooted frames shown in Figure 3.6.

3.4.19. Lemma. *Δ forms an $\leq$-antichain.*

Proof. Suppose there are distinct frames $\mathfrak{F}, \mathfrak{G} \in \Delta$ such that $\mathfrak{F} \leq \mathfrak{G}$. Then there is a generated subframe $\mathfrak{G}'$ of $\mathfrak{G}$ and an onto p-morphism $f : \mathfrak{G}' \to \mathfrak{F}$. By Proposition 3.1.7, there are finitely many α- and β-reductions $f_1, \ldots, f_n$ such that $f = f_n \circ \cdots \circ f_1$. Looking at the structure of $\mathfrak{G}$ (see Figure 3.6) we see that there is no point that has a unique immediate successor and that the only points w and v such that $R(w) \setminus \{w\} = R(v) \setminus \{v\}$ are the maximal points. Therefore, f_1 can only be the β-reduction identifying two maximal points of $\mathfrak{G}'$. Thus, $f(\mathfrak{G}')$ cannot be isomorphic to $\mathfrak{F}$. $\square$

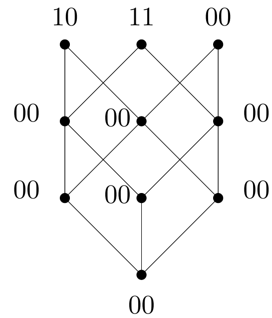

Figure 3.7: The coloring of $\mathfrak{F}_2$

In the next chapter we construct more antichains of finite rooted frames. We have the following corollary of Theorem 3.4.18 and Lemma 3.4.19 first observed by Jankov [65].

3.4.20. Corollary. *There are continuum many intermediate logics.*

Proof. Consider the countable sequence Δ of finite rooted frames. Then by Lemma 3.4.19, Δ forms an $\leq$-antichain. By Theorem 3.4.18, this implies that there are continuum many intermediate logics. $\square$

For the examples of infinite $\preccurlyeq$ and $\preccurlyeq'$-antichains of finite rooted frames consult [24, Lemma 11.18 and Theorem 11.19]. Now we determine the size of $\mathcal{H}(n)$ using the Jankov-de Jongh formulas.

3.4.21. Theorem. *The cardinality of $\mathcal{H}(n)$, for every $n > 1$ is that of the continuum.*

Proof. (Sketch) We first show that if there is a sequence of formulas $\{\phi_i\}_{i\in\omega}$ in n variables such that for every finite $\Phi, \Psi \subsetneq \{\phi_i\}_{i\in\omega}$ we have $\mathbf{IPC} \nvdash \bigwedge \Phi \to \bigvee \Psi$, then the cardinality of $\mathcal{H}(n)$ is that of continuum. Obviously, the n-generated free Heyting algebra $F(n)$ is countable; there are only countably many formulas in n variables. Therefore, there are at most continuum many prime filters of $F(n)$ and the cardinality of $\mathcal{H}(n)$ is at most continuum. For every subset $I \subseteq \omega$ consider $\{\phi_i\}_{i\in I}$ and let F_I be the filter generated by $\{\phi_i\}_{i\in I}$. Then $\phi_i \in F_I$ iff $i \in I$. Now using the standard Lindenbaum construction (see e.g., [24, Lemma 5.1]) we extend F_I to a prime filter F_I' such that $\phi_j \notin F_I'$ for every $j \notin I$. Now let $I, J \subseteq \omega$ and $I \neq J$. Then w.l.o.g. there is $i \in I$ such that $i \notin J$. It follows that $\phi_i \in F_I \subseteq F_I'$ and $\phi_i \notin F_J'$. Therefore, $F_I' \neq F_J'$, for every $I, J \subseteq \omega$ and $I \neq J$.

Therefore, all we need to do is to construct such a sequence of formulas. Let Δ be the sequence of frames shown in Figure 3.6. Then every $\mathfrak{F} \in \Delta$ is finitely generated. To see this, consider the coloring shown in Figure 3.7. Now it is easy to see that every $\mathfrak{F} \in \Delta$ with this coloring is a generated submodel of $\mathcal{U}(2)$. Indeed, the maximal points of $\mathfrak{F}$ have different colors. No point is totally covered by a singleton set and if a point is totally covered by an antichain then there is no other point that is totally covered by the same antichain. This guarantees that $\mathfrak{F}$ with this coloring is a generated submodel of $\mathcal{U}(2)$.

Therefore, $\{\chi(\mathfrak{F}_i) : i > 1 \text{ and } \mathfrak{F}_i \in \Delta\}$ is a sequence of formulas in two variables. Finally, we will sketch the proof of $\mathbf{IPC} \not\vdash \bigwedge \Phi \to \bigvee \Psi$ for $\Phi, \Psi \subseteq \{\chi(\mathfrak{F}_i) : i > 1 \text{ and } \mathfrak{F}_i \in \Delta\}$. Let $\Phi = \{\chi(\mathfrak{F}_{i_1}), \dots, \chi(\mathfrak{F}_{i_k})\}$ and let $\Psi = \{\chi(\mathfrak{F}_{j_1}), \dots, \chi(\mathfrak{F}_{j_m})\}$. Let $\mathfrak{F}$ be the frame obtained by adjoining a new root to the disjoint union of $\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_m}$. Obviously, every $\mathfrak{F}_{j_s}$ is a generated subframe of $\mathfrak{F}$. So $\mathfrak{F} \not\models \chi(\mathfrak{F}_{j_s})$, which implies $\mathfrak{F} \not\models \bigvee \Psi$. Moreover, we can show that for every $j > 1$ and $j \notin \{j_1, \dots, j_m\}$ we have $\mathfrak{F}_j \not\leq \mathfrak{F}$. It follows that $\mathfrak{F} \models \bigwedge \Phi$. Thus, $\mathfrak{F} \not\models \bigwedge \Phi \to \bigvee \Psi$, which finishes the proof of the theorem. □

Next we axiomatize some intermediate logics using the Jankov-de Jongh formulas. Intuitively speaking the Jankov-de Jongh formula of a frame $\mathfrak{F}$ axiomatizes the least logic that does not have $\mathfrak{F}$ as its frame.

3.4.22. LEMMA. *Let L be an intermediate logic. Then*

1. *$(\mathbf{F}_L, \leq)$ is well-founded.*
2. *For every finite rooted frame $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, there exists a finite rooted $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$.*

Proof. (1) The proof follows immediately from the fact that if $\mathfrak{F}, \mathfrak{G} \in \mathbf{F}_L$ then $\mathfrak{F} < \mathfrak{G}$ implies $|\mathfrak{F}| < |\mathfrak{G}|$.

(2) The proof is similar to the proof of (1). □

To prove that every locally tabular intermediate logic is axiomatized by the Jankov-de Jongh formulas, we use the following criterion of local tabularity established by G. Bezhanishvili [7].

3.4.23. THEOREM. *A logic L is locally tabular iff the class of rooted descriptive L-frames is uniformly locally tabular. That is, for every natural number n there exists a natural number $M(n)$ such that for every n-generated rooted descriptive L-frame $\mathfrak{F}$ we have $|\mathfrak{F}| \leq M(n)$.*

3.4.24. THEOREM. *Every locally tabular intermediate logic is axiomatized by Jankov-de Jongh formulas.*

Proof. Let L be a locally tabular intermediate logic. By Corollary 3.4.14(1), we need to show that for every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$. Suppose $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. If $\mathfrak{G}$ is finite, then by Lemma 3.4.22(2), there exists a finite rooted $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$. Now assume that $\mathfrak{G}$ is infinite. Let $\mathfrak{G}'$ be a finite rooted frame such that $\mathfrak{G}' < \mathfrak{G}$. If $\mathfrak{G}' \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, then by Lemma 3.4.22(2), there exists $\mathfrak{F} \in \mathbf{M}(L, \leq)$ with $\mathfrak{F} \leq \mathfrak{G}'$. Since $\leq$ is transitive, we have $\mathfrak{F} \leq \mathfrak{G}$. Now suppose, for every finite rooted $\mathfrak{G}'$ such that $\mathfrak{G}' < \mathfrak{G}$ we have $\mathfrak{G}' \in \mathbb{FG}(L)$. By Theorem 3.1.10, for every $i \in \omega$ there exists a point x_i of $\mathfrak{G}$ of depth i. Let $\mathfrak{H}_i$ be the x_i-generated subframe of $\mathfrak{G}$. Then $\mathfrak{H}_i$ is finite and n-generated (since $\mathfrak{H}_i$ is a generated subframe of $\mathfrak{G}$). Moreover, $sup\{|\mathfrak{H}_i| : i \in \omega\} = \omega$. Therefore, the set of all rooted finitely generated descriptive L-frames is not uniformly locally finite. By Theorem 3.4.23, L is not locally tabular, which is a contradiction. Thus, by Corollary 3.4.14(1) L is axiomatized by the Jankov-de Jongh formulas. $\square$

Since every tabular logic is locally tabular, it follows from Theorem 3.4.24 that every tabular logic is also axiomatized by the Jankov-de Jongh formulas. Next we show that every tabular logic is in fact finitely axiomatized by the Jankov-de Jongh formulas. For an alternative proof of the theorem consult [24, Theorem 12.4]. First we prove two auxiliary lemmas.

3.4.25. LEMMA. *For every finite rooted frame $\mathfrak{F}$, consisting of at least two points, there exists a frame $\mathfrak{G}$ and $f : \mathfrak{F} \to \mathfrak{G}$ such that f is an α- or β-reduction.*

Proof. If $max(\mathfrak{F})$ contains more than one point, consider the β-reduction that identifies two distinct maximal points of $\mathfrak{F}$. If $max(\mathfrak{F})$ is a singleton set, we consider the second layer of $\mathfrak{F}$. By our assumption the second layer is not empty. If the second layer of $\mathfrak{F}$ consists of one point, then consider the α-reduction that identifies the point of the second layer with the maximal point. If the second layer of $\mathfrak{F}$ consists of at least two points, we consider a β-reduction that identifies two points from the second layer. $\square$

3.4.26. LEMMA. *Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. Suppose that $\mathfrak{F}$ is a finite rooted L-frame, where $L = Log(\mathfrak{G})$ for some $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$. Then $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

Proof. Suppose $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$. Then $\mathfrak{G} \models \alpha(\mathfrak{F})$, where $\alpha(\mathfrak{F})$ is the frame-based formula for $\trianglelefteq$. Therefore, since $\mathfrak{F}$ is an L-frame, $\mathfrak{F} \models \alpha(\mathfrak{F})$. This is a contradiction since $\trianglelefteq$ is reflexive. $\square$

3.4.27. THEOREM. *Every tabular logic is finitely axiomatizable by Jankov-de Jongh formulas.*

Proof. Let L be tabular. Then $L = Log(\mathfrak{F})$ for some finite frame $\mathfrak{F}$. By Lemma 3.4.26, for every rooted L-frame $\mathfrak{F}'$ we have $\mathfrak{F}' \leq \mathfrak{F}$. Therefore, if $\mathfrak{F}' \in \mathbf{F}_L$, then $|\mathfrak{F}'| \leq |\mathfrak{F}|$. Hence, every finite rooted L-frame contains at most $|\mathfrak{F}|$ points. We will show that $\mathbf{M}(L, \leq)$ is finite.

3.4.28. Claim. *For every $\mathfrak{H} \in \mathbf{M}(L, \leq)$ we have $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$.*

Proof. Assume $\mathfrak{H} \in \mathbf{M}(L, \leq)$. If $|\mathfrak{H}| = 1$, then trivially $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$. Now suppose $\mathfrak{H}$ is such that $|\mathfrak{H}| > 1$. Then by Lemma 3.4.25, there exists a frame $\mathfrak{H}'$ such that $\mathfrak{H}' < \mathfrak{H}$. If $\mathfrak{H}' \notin \mathbf{F}_L$, then $\mathfrak{H}$ is not a minimal element of $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, that is, $\mathfrak{H} \notin \mathbf{M}(L, \leq)$, which is a contradiction. If $\mathfrak{H}' \in \mathbf{F}_L$, then since α- and β-reductions identify only two points, $|\mathfrak{H}| = |\mathfrak{H}'| + 1$. As $\mathfrak{H}'$ is an L-frame, $|\mathfrak{H}'| \leq |\mathfrak{F}|$. Thus, $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$. □

There are only finitely many non-isomorphic frames consisting of m points for $m \in \omega$. Therefore, $\mathbf{M}(L, \leq)$ is finite. Let $\mathbf{M}(L, \leq) = \{\mathfrak{G}_1, \ldots, \mathfrak{G}_k\}$. Then, by the proof of Theorem 3.4.12, we have $L(\mathfrak{F}) = \mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$. □

However, not every intermediate logic is axiomatized by Jankov-de Jongh formulas. We construct a subframe logic that is not axiomatized by Jankov-de Jongh formulas. We first introduced the notion of width of an intermediate logic. For modal logics this notion was defined by Fine [42] and for intermediate logics by Sobolev [117].

3.4.29. Definition. Let $\mathfrak{F}$ be a rooted (descriptive or Kripke) frame. We say that

1. $\mathfrak{F}$ has (*cofinal*) *width* n if there is an antichain of n points in $\mathfrak{F}$ (in $max(\mathfrak{F})$) and no other antichain in $\mathfrak{F}$ (in $max(\mathfrak{F})$) contains more than n points.

2. An intermediate logic $L \supseteq \mathbf{IPC}$ has *width* (*cofinal width*) $n \in \omega$ if every descriptive rooted L-frame has width (cofinal width) $\leq n$.

We denote by $w(\mathfrak{F})$ the width of $\mathfrak{F}$ and by $w_c(\mathfrak{F})$ the cofinal width of $\mathfrak{F}$.

3.4.30. Definition. For every $n \in \omega$ let

1. $L_w(n) := Log(\Gamma_n)$, where $\Gamma_n = \{\mathfrak{F} : |\mathfrak{F}| < \omega \text{ and } w(\mathfrak{F}) \leq n\}$.

2. $L'_w(n) := Log(\Gamma'_n)$, where $\Gamma'_n = \{\mathfrak{F} : |\mathfrak{F}| < \omega \text{ and } w_c(\mathfrak{F}) \leq n\}$.

It can be shown that $L_w(n)$ is the least logic of width n and $L'_w(n)$ is the least logic of cofinal width n.

We sketch a proof that $L'_w(5)$ is not axiomatizable by the Jankov-de Jongh formulas. For the details we refer to [24, Proposition 9.50].

3.4.31. Theorem. *$L'_w(5)$ is not axiomatizable by Jankov-de Jongh formulas.*

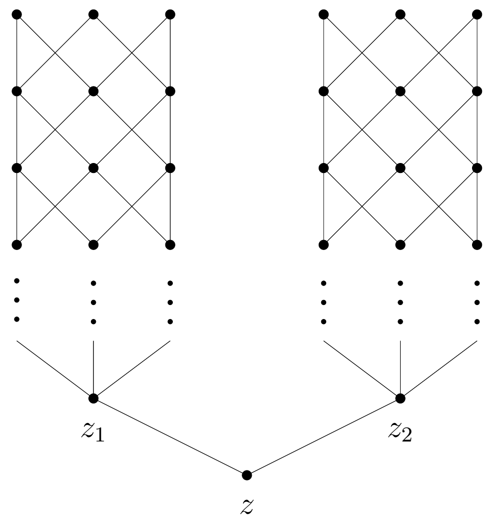

Figure 3.8: The frame $\mathfrak{G}$

Proof. (Sketch) By Corollary 3.4.14, it is sufficient to construct a finitely generated rooted descriptive frame $\mathfrak{G}$ such that $\mathfrak{G}$ is not an $L'_w(5)$-frame and if a finite rooted $\mathfrak{F}$ is such that $\mathfrak{F} \leq \mathfrak{G}$, then $\mathfrak{F}$ is an $L'_w(5)$-frame.

We will modify the example used in [24, Proposition 9.50]. Consider the frame $\mathfrak{G} = (W, R, \mathcal{P})$ shown in Figure 3.8, where $\mathcal{P} = \{R(z_1) \cup R(z_2), W, \emptyset, U, U \cup R(z_i) : U$ is a finite upset of $\mathfrak{G}, i = 1, 2\}$. Then it can be shown that $\mathfrak{G}$ is a finitely generated descriptive frame. It is obvious that $\mathfrak{G}$ has width 6 and hence is not an $L'_w(5)$-frame.

The main idea of the proof is that every finite rooted generated subframe of $\mathfrak{G}$ has width ≤ 5 and every p-morphism identifies at least two maximal points of $\mathfrak{G}$. Therefore, for every finite $\mathfrak{F} < \mathfrak{G}$ we have $w_c(\mathfrak{F}) \leq 5$ and $\mathfrak{F}$ is an $L'_w(5)$-frame. By Corollary 3.4.14(1), this means that $L'_w(5)$ is not axiomatized by the Jankov-de Jongh formulas. We skip the details. $\square$

3.4.32. THEOREM. *For every $n \in \omega$ the following holds.*

1. *$L_w(n)$ is axiomatized by subframe formulas.*

2. *$L'_w(n)$ is axiomatized by cofinal subframe formulas.*

Proof. By Corollary 3.4.14, it is sufficient to observe that for every frame $\mathfrak{F}$ of width $\leq n$ every subframe and cofinal subframe of $\mathfrak{F}$ also has width $\leq n$. Thus $L_w(n)$ is a subframe logic and $L'_w(n)$ is a cofinal subframe logic and therefore by Corollary 3.4.16, they are axiomatizable by subframe formulas and cofinal subframe formulas, respectively. $\square$

Now we prove the converse of Theorem 3.4.31.

3.4.33. THEOREM. *There are intermediate logics that are axiomatized by Jankov-de Jongh formulas but not axiomatized by subframe formulas or by cofinal subframe formulas.*

Proof. Let Δ be as in Lemma 3.4.19. Consider $\mathfrak{F}_i \in \Delta$ such that $i > 0$. Then $L = Log(\mathfrak{F}_i)$ is tabular and by Theorem 3.4.27, L is finitely axiomatized by the Jankov-de Jongh formulas. Now we show that L is neither a subframe nor a cofinal subframe logic. It is easy to see that $\mathfrak{F}_0$ is a subframe of $\mathfrak{F}_i$, moreover it is a cofinal subframe. By Lemma 3.4.26, if $\mathfrak{F}_0$ is an L-frame, then $\mathfrak{F}_0 \leq \mathfrak{F}_i$. This is a contradiction because by Theorem 3.4.18, Δ is an $\leq$-antichain. Therefore L is neither a subframe nor a cofinal subframe logic and by Corollary 3.4.16, it is not axiomatized by subframe formulas. $\square$

We will close this section by showing that there are intermediate logics that are not axiomatized by frame-based formulas. Note that this proof is very non-constructive.

3.4.34. THEOREM. *For every frame order $\trianglelefteq$ on $\mathbb{FG}(\mathbf{IPC})$ there are intermediate logics that are not axiomatized by frame-based formulas for $\trianglelefteq$.*

Proof. Suppose every intermediate logic is axiomatized by frame-based formulas for $\trianglelefteq$. We show that this implies that every intermediate logic has the fmp, which contradicts the fact that there are continuum many intermediate logics without the fmp, e.g., [24, Theorem 6.3], see also Chapter 4. Let L be an intermediate logic. Suppose $L \not\vdash \phi$. Then there exists a finitely generated rooted L-frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models \phi$. Consider the logic $L + \phi$. If $L + \phi$ is inconsistent, then every finite L-frame refutes ϕ. So, assume that $L + \phi$ is consistent. By our assumption, $L + \phi$ is also axiomatized by frame-based formulas for $\trianglelefteq$. Then $\mathfrak{G}$ is not an $(L + \phi)$-frame and by applying Theorem 3.4.12 to the logic $L + \phi$, we obtain that there exists a frame $\mathfrak{H} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{H} \trianglelefteq \mathfrak{G}$. Since $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset, $\mathfrak{H}$ is an L-frame. Since $\mathfrak{H} \in \mathbf{M}(L + \phi, \trianglelefteq)$ we have that $\mathfrak{H} \not\models \phi$. Therefore, we found a finite L-frame that refutes ϕ. This means that L has the fmp. This contradiction finishes the proof of the theorem.

$\square$

Thus, it is impossible to axiomatize all the intermediate logics by frame-based formulas only. In order to axiomatize all intermediate logics by formulas arising from finite frames one has to generalize frame-based formulas by introducing a new parameter. Zakharyaschev's canonical formulas are extensions of the Jankov-de Jongh formulas and (cofinal) subframe formulas with a new parameter. Instead of considering just finite rooted frame $\mathfrak{F}$ we need to consider a pair $(\mathfrak{F}, \mathfrak{D})$, where $\mathfrak{D}$ is some set of antichains of $\mathfrak{F}$. We would also need to modify the definition of $\trianglelefteq$

to take this parameter into account. Formulas arising from such pairs are called "canonical formulas". They provide axiomatizations of all intermediate logics. We do not discuss canonical formulas here. For a systematic study of canonical formulas the reader is referred to [24, §9].

Chapter 4

The logic of the Rieger-Nishimura ladder

In this chapter, which is based on [8], we apply the tools and techniques developed in the previous chapter to the logic **RN** of the Rieger-Nishimura ladder. The logic **RN** was first studied by Kuznetsov and Gerciu [83], Gerciu [48], and independently by Kracht [73]. Kuznetsov and Gerciu [83] introduced an intermediate logic **KG** of which **RN** is a proper extension. This logic will play an important role in our investigations. We show that the structure of finitely generated **KG** and **RN**-frames is quite simple. These frames are the finite sums of 1-generated descriptive frames.

We apply the technique of frame-based formulas in two ways. Firstly, using the Jankov-de Jongh formulas we construct a continuum of extensions of **KG** that do not have the finite model property. Secondly, we give a simple axiomatization of **RN** using subframe formulas and the Jankov-de Jongh formulas. In contrast to the extensions of **KG**, every extension of **RN** does have the finite model property. This result was first proved by Gerciu [48], and independently by Kracht [73]. However, both proofs contain some gaps. We will develop the technique of gluing models and provide a rather simple proof of this theorem.

Finally, we show that $\mathbf{RN}.\mathbf{KC} = \mathbf{RN} + (\neg p \vee \neg\neg p)$ is the unique pre-locally tabular extension of **KG**. It follows that an extension L of **KG** (**RN**) is not locally tabular iff $L \subseteq \mathbf{RN}.\mathbf{KC}$. For extensions of **RN** we establish another criterion of local tabularity. For every $L \supseteq \mathbf{RN}$ we define the internal depth of L and prove that L is locally tabular iff its internal depth is finite.

This chapter is organized as follows: in the first section we introduce **RN**, define the n-scheme logics over **IPC** and n-conservative extensions of **IPC**. We prove that **RN** is the 1-scheme logic over **IPC** and the greatest 1-conservative extension of **IPC**. In Section 4.2 we describe the finite rooted frames of **RN**. The next section introduces the logic **KG** and characterizes the finitely generated descriptive frames of **KG**. In Section 4.4 we prove that every extension of **RN** has the fmp. In Section 4.5, continuum many extensions of **KG** without the finite model property are constructed. In the last two section we give an axiomatization of **RN** using the Jankov-de Jongh formulas and subframe formulas and investigate locally tabular extensions of **KG** and **RN**.

4.1 n-conservative extensions, linear and vertical sums

In this section we recall the structure of the 1-generated free Heyting algebra and its dual 1-Henkin frame. We call them the Rieger-Nishimura lattice and ladder respectively. We will also introduce the n-conservative and the n-scheme logics over **IPC** and show that the logic of the n-Henkin model is the n-scheme logic over **IPC** and the greatest n-conservative extension of **IPC**. In the last section we define the linear and vertical sums of descriptive frames and Heyting algebras and prove that these operations are dual to each other.

4.1.1 The Rieger-Nishimura lattice and ladder

In the previous chapter we discussed finitely generated free Heyting algebras and their dual Henkin models. In this chapter we will take a closer look at the simplest finitely generated free Heyting algebra, namely, the 1-generated free Heyting algebra. The 1-generated free Heyting algebra was described independently by Rieger [106] and Nishimura [102] and is called the Rieger-Nishimura lattice after them. Recall that by Theorem 3.2.13(2), the 1-Henkin model of **IPC** is isomorphic to the model shown in Figure 4.1, where $V(p) = \{w_0\}$.

4.1.1. Definition.

1. Denote by $\mathfrak{L}$ the 1-Henkin frame. We call $\mathfrak{L}$ the *Rieger-Nishimura ladder*. We also let $\mathfrak{L}_0$ denote the upper part of $\mathfrak{L}$, i.e., the frame $\mathfrak{L} \setminus \{\omega\}$.

2. Denote by $\mathfrak{N}$ the 1-generated free Heyting algebra. We call $\mathfrak{N}$ the *Rieger-Nishimura lattice*.

By Theorem 3.2.9, $\mathfrak{L}_0$ is isomorphic to the 1-universal frame. By Theorem 3.3.2, every finite upset of $\mathfrak{L}$ is admissible. It is also easy to see that the carrier set of $\mathfrak{L}_0$ is not admissible. We will give a topological argument to this fact. Suppose the carrier set of $\mathfrak{L}_0$ is admissible. Then it is (topologically) closed. Every closed subset of a compact space is compact. Thus $\mathfrak{L}_0$ is compact, which is a contradiction; $\mathcal{F} = \{R^{-1}(w_i)\}_{i\in\omega}$ is a family of closed subsets of $\mathfrak{L}_0$ with the finite intersection property but $\bigcap \mathcal{F} = \emptyset$.

By the duality between descriptive frames and Heyting algebras, the Rieger-Nishimura lattice $\mathfrak{N}$ is isomorphic to the Heyting algebra of all admissible subsets of $\mathfrak{L}$. The generator of $\mathfrak{N}$ is the upset $V(p) = \{w_0\}$. It is easy to check that $\mathfrak{N}$ is

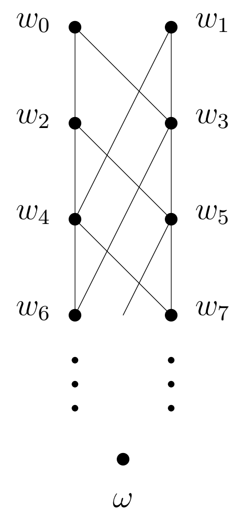

Figure 4.1: The Rieger-Nishimura ladder $\mathfrak{L}$

isomorphic to the lattice shown in Figure 4.2 and every element of $\mathfrak{N}$ is represented by one of the *Rieger-Nishimura polynomials*:

4.1.2. Definition. The *Rieger-Nishimura polynomials* are given by the following recursive definition:

1. $g_0(p) := p$,
2. $g_1(p) := \neg\, p$,
3. $f_1(p) := p \vee \neg p$,
4. $g_2(p) := \neg\,\neg\, p$,
5. $g_3(p) := \neg\,\neg\, p \rightarrow p$,
6. $g_{n+4}(p) := g_{n+3}(p) \rightarrow (g_n(p) \vee g_{n+1}(p))$,
7. $f_{n+2}(p) := g_{n+2}(p) \vee g_{n+1}(p)$.

Let $\mathfrak{A} = (A, \vee, \wedge, \rightarrow, 0)$ be a Heyting algebra. For every element $a \in A$ let

$$[a) = \{b \in A : a \leq b\}$$

and

$$(a] = \{b \in A : b \leq a\}.$$

$[a)$ and $(a]$ are called the *principal filter* and the *principal ideal generated by a*, respectively. It is obvious that the principal filters $[g_k(p))$ and $[f_k(p))$ are proper

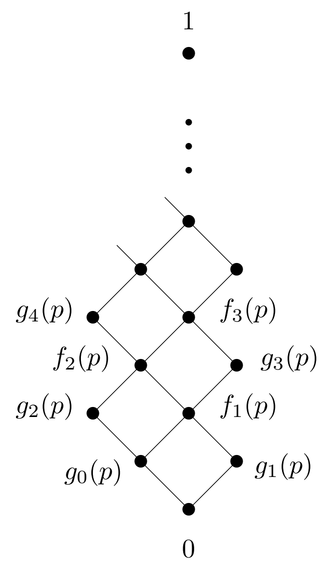

Figure 4.2: The Rieger-Nishimura lattice $\mathfrak{N}$

filters of $\mathfrak{N}$ for every $k \in \omega$. Moreover, it is obvious that the unit filter $\{1\}$ is a proper filter of $\mathfrak{N}$, and that every proper filter of $\mathfrak{N}$ is principal. Furthermore, $\{1\}$ and $[g_k(p))$, for every $k \in \omega$, are the only prime filters of $\mathfrak{N}$.

4.1.3. Definition. Let $\mathfrak{L}$ be labeled by w_k's as it is shown in Figure 4.1. For every $k \in \omega$:

1. Let $\mathfrak{L}_{g_k}$ denote the generated subframe of $\mathfrak{L}$ generated by the point w_k, i.e., $\mathfrak{L}_{g_k} = (R(w_k), R \restriction R(w_k))$,
2. Let $\mathfrak{L}_{f_k}$ denote the generated subframe of $\mathfrak{L}$ generated by the points w_k and w_{k-1}, i.e., $\mathfrak{L}_{f_k} = (R(w_k) \cup R(w_{k-1}), R \restriction R(w_k) \cup R(w_{k-1}))$,
3. Let $\mathfrak{N}_{g_k}$ denote the algebra corresponding to $\mathfrak{L}_{g_k}$,
4. Let $\mathfrak{N}_{f_k}$ denote the algebra corresponding to $\mathfrak{L}_{f_k}$.

The next proposition shows that $\mathfrak{L}_{g_k}$ and $\mathfrak{L}_{f_k}$ are precisely those generated subframes of $\mathfrak{L}$ that satisfy $g_k(p)$ and $f_k(p)$, respectively.

4.1.4. Proposition. *For every $k \in \omega$ we have:*

1. $R(w_k) = \{w \in \mathfrak{L} : w \models g_k(p)\}$,
2. $R(w_k) \cup R(w_{k-1}) = \{w \in \mathfrak{L} : w \models f_k(p)\}$.

Proof. The proof is a routine check. $\square$

Now we introduce the logic that we are going to study in this chapter.

4.1.5. Definition.

1. Let **RN** denote the logic of $\mathfrak{L}$, i.e., $\mathbf{RN} = Log(\mathfrak{L})$.
2. Let $\mathcal{RN}$ denote the variety generated by $\mathfrak{N}$, i.e., $\mathcal{RN} = \mathbf{HSP}(\mathfrak{N})$.

The rest of this chapter will be devoted to the investigation of **RN** ($\mathcal{RN}$) and other intermediate logics (varieties of Heyting algebras) related to **RN** (to $\mathcal{RN}$).

4.1.6. Remark. Before engaging into the technical details we mention one more example of a very natural "appearance" of the Rieger-Nishimura ladder from a different perspective. This fact was first observed by L. Esakia [36]. Consider the ordered set $(\mathbb{N}, \leq)$ of natural numbers. Define the relation R on $\mathbb{N}$ by putting: nRm if $n - m \geq 2$. It is now easy to check that the frame $(\mathbb{N}, R)$ is isomorphic to $\mathfrak{L}_0$, the upper part of the Rieger-Nishimura ladder.

4.1.2 n-conservative extensions and the n-scheme logics

In this section we describe some syntactic properties of **RN**. They will not be used subsequently but give some motivation for studying **RN**.

4.1.7. Definition. Suppose L and S are intermediate logics. We say that S is an *n-conservative extension* of L if $L \subseteq S$ and for every formula $\phi(p_1, \ldots, p_n)$ in n variables we have $L \vdash \phi$ iff $S \vdash \phi$.

Note that this definition, as well as the next one, apply not only to intermediate logics, but to any propositional logic. By a *propositional logic* we mean any set of formulas (not necessarily in the language $\mathcal{L}$), closed under (Subst).

4.1.8. Definition. Let L be an intermediate logic. A set of formulas $L(n)$ is called the *n-scheme logic* of L if for every $\psi(p_1, \ldots, p_k)$ and $k \in \omega$:

$$\psi(p_1, \ldots, p_k) \in L(n) \Leftrightarrow \text{for all } \chi_1(p_1, \ldots p_n), \ldots, \chi_k(p_1, \ldots, p_n)$$

$$\text{we have } L \vdash \psi(\chi_1, \ldots, \chi_k).$$

It is easy to see $L(n)$ is closed under (MP) and (Subst). Therefore, $L(n)$ is an intermediate logic, for every $n \in \omega$.

4.1.9. Proposition. *Let L be an intermediate logic.*

1. *L' is an n-conservative extension of L iff $L \subseteq L' \subseteq L(n)$.*
2. *$L(n)$ is the largest n-conservative extension of L.*

Proof. Suppose L' is n-conservative. Let $L' \vdash \psi(p_1, \ldots, p_k)$. Then for arbitrary $\chi_1(p_1, \ldots, p_n), \ldots, \chi_k(p_1, \ldots, p_n)$ we have that $L' \vdash \psi(\chi_1, \ldots, \chi_k)$. By n-conservativity $L \vdash \psi(\chi_1, \ldots, \chi_k)$. By the definition of the n-scheme logic $L(n) \vdash \psi(p_1, \ldots, p_k)$. Therefore, $L' \subseteq L(n)$.

For the converse it is sufficient to show that $L(n)$ is n-conservative over L. Let $L(n) \vdash \psi(p_1, \ldots, p_k)$. Then $L \vdash \psi(\psi_1, \ldots, \psi_k)$, for every $\psi_i(p_1, \ldots, p_n)$, $i \leq k$ and $k \in \omega$. This obviously holds for $\psi_i = p_i$, for $i \leq k$. Thus, $L \vdash \psi(p_1, \ldots, p_k)$.

(2) The result follows from (1). $\square$

The next theorem spells out the connection between the n-scheme logic of **IPC** and the n-universal and n-Henkin models. Recall from the previous chapter that for every $n \in \omega$ $\mathbb{H}(n) = (H(n), R, \mathcal{P})$ and $\mathbb{U}(n) = (U(n), R', \mathcal{P}')$ denote the n-Henkin frame and the n-universal frame, i.e., the underlying descriptive frames of the n-Henkin model $\mathcal{H}(n)$ and the n-universal model $\mathcal{U}(n)$, respectively. Recall also that for every frame $\mathfrak{F}$, we denote by $Log(\mathfrak{F})$ the set of formulas that are valid in $\mathfrak{F}$.

4.1.10. THEOREM.

1. $Log(\mathbb{U}(n))$ *is the greatest* n*-conservative extension of* **IPC**.
2. $Log(\mathbb{H}(n)) = Log(\mathbb{U}(n)) = \mathbf{IPC}(n)$.

Proof. (1) Clearly, $Log(\mathbb{U}(n))$ is an intermediate logic. Therefore, $\mathbf{IPC} \subseteq Log(\mathbb{U}(n))$. Now suppose $Log(\mathbb{U}(n)) \vdash \phi(p_1, \ldots, p_n)$, then ϕ is valid in the n-universal frame and hence it is valid in the n-universal model. Thus, by Theorem 3.2.17, $\mathbf{IPC} \vdash \phi$. Therefore, $Log(\mathbb{U}(n))$ is n-conservative over **IPC**.

Let L be an n-conservative extension of **IPC**. If $L \not\subseteq Log(\mathbb{U}(n))$, then there exists a formula ϕ such that $\phi \in L$ and $\phi \notin Log(\mathbb{U}(n))$. Therefore, there exists $x \in U(n)$ such that $x \not\models \phi$. Let $\mathfrak{F}$ be the rooted upset of $\mathbb{U}(n)$ generated by x. Then $\mathfrak{F}$ is finite and $\mathfrak{F} \not\models \phi$. Let $\chi(\mathfrak{F})$ be the de Jongh formula of $\mathfrak{F}$. By the definition of the de Jongh formulas $\chi(\mathfrak{F})$ is in n variables.[1] If $\chi(\mathfrak{F}) \notin L$, then $\mathfrak{F}$ is an L-frame refuting ϕ, which contradicts the assumption $\phi \in L$. Therefore, $\chi(\mathfrak{F}) \in L$. But then $\chi(\mathfrak{F}) \in \mathbf{IPC}$ as L is n-conservative over **IPC**, which is obviously false. Thus, $L \subseteq Log(\mathbb{U}(n))$ and $Log(\mathbb{U}(n))$ is the greatest n-conservative extension of **IPC**.

(2) That $Log(\mathbb{H}(n))$ is the greatest n-conservative extension of **IPC** is proved in a similar way as (1), using the fact that $\mathbb{H}(n)$ is completely determined by $\mathbb{U}(n)$. That is, $\mathbb{H}(n) \models \phi$ iff $\mathbb{U}(n) \models \phi$ The result now follows from (1). $\square$

4.1.11. COROLLARY. **RN** *is the* 1*-scheme logic of* **IPC** *and the greatest* 1*-conservative extension of* **IPC**.

Proof. The result is an immediate consequence of Theorem 4.1.10. $\square$

[1] Note that in this case it is essential that we take the de Jongh formula and not the Jankov formula. This ensures us that this formula is in n propositional variables.

4.1.3 Sums of Heyting algebras and descriptive frames

In this section we recall the constructions of the linear sum of descriptive frames and the vertical sum of Heyting algebras used subsequently in this chapter.

4.1.12. DEFINITION. (see e.g., [30, p.17 and p.179]) Let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$ be Kripke frames. The *linear sum* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is the Kripke frame $\mathfrak{F}_1 \oplus \mathfrak{F}_2 := (W_1 \uplus W_2, R)$ such that $W_1 \uplus W_2$ is a disjoint union of W_1 and W_2 and for every $w, v \in W_1 \uplus W_2$ we have

$$\begin{array}{lll} wRv \text{ iff} & & w, v \in W_1 \text{ and } wR_1v, \\ & \text{or} & w, v \in W_2 \text{ and } wR_2v, \\ & \text{or} & w \in W_2 \text{ and } v \in W_1. \end{array}$$

In other words, $R = R_1 \cup R_2 \cup (W_2 \times W_1)$.

Figuratively speaking, the operation $\oplus$ puts $\mathfrak{F}_1$ on top of $\mathfrak{F}_2$. Now we define the dual operation of $\oplus$ for Heyting algebras.

4.1.13. DEFINITION. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras. The *vertical sum* $\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is obtained from a linear sum of $\mathfrak{A}_2 \oplus \mathfrak{A}_1$ by identifying the greatest element of $\mathfrak{A}_1$ with the least element of $\mathfrak{A}_2$.

Figuratively speaking, $\overline{\oplus}$ puts $\mathfrak{A}_2$ on top of $\mathfrak{A}_1$. The next proposition was first observed by Troelstra [122].

4.1.14. PROPOSITION. *For every Heyting algebra $\mathfrak{A}_1$ and $\mathfrak{A}_2$ the vertical sum $\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2$ is also a Heyting algebra.*

Proof. The proof is just spelling out the definitions. □

Next we extend the definition of a linear sum to descriptive frames.

4.1.15. DEFINITION. Let $\mathfrak{F}_1 = (W_1, R_1, \mathcal{P}_1)$ and $\mathfrak{F}_2 = (W_2, R_2, \mathcal{P}_2)$ be descriptive frames. The *linear sum* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is the descriptive frame $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = (W, R, \mathcal{P})$, where (W, R) is the linear sum of (W_1, R_1) and (W_2, R_2) and $\mathcal{P}$ is such that

$$U \in \mathcal{P} \text{ iff } U \in \mathcal{P}_1 \text{ or } U = W_1 \cup S, \text{ where } S \in \mathcal{P}_2.$$ [2]

The operations of the vertical sum of Heyting algebras and the linear sum of descriptive frames are dual to each other.

4.1.16. THEOREM. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras and $\mathfrak{F}_1 = (W_1, R_1, \mathcal{P}_1)$ and $\mathfrak{F}_2 = (W_2, R_2, \mathcal{P}_2)$ be descriptive frames. Then*

[2] In topological terminology we take the linear sum of the Kripke frames and the topological sum of the corresponding topologies.

1. *$(\mathfrak{F}_1 \oplus \mathfrak{F}_2)^*$ is isomorphic to $\mathfrak{F}_1^* \overline{\oplus} \mathfrak{F}_2^*$.*

2. *$(\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_*$ is isomorphic to $(\mathfrak{A}_1)_* \oplus (\mathfrak{A}_2)_*$.*

Proof. (Sketch) (1) We define $h : (\mathfrak{F}_1 \oplus \mathfrak{F}_2)^* \to \mathfrak{F}_1^* \overline{\oplus} \mathfrak{F}_2^*$ by putting for every element of $(\mathfrak{F}_1 \oplus \mathfrak{F}_2)^*$, i.e., an admissible upset U of $\mathfrak{F}_1 \oplus \mathfrak{F}_2$:

$$h(U) = \begin{cases} U & \text{if } U \subseteq W_1, \\ U \cap W_2 & \text{otherwise.} \end{cases}$$

(2) We define $f : (\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_* \to (\mathfrak{A}_1)_* \oplus (\mathfrak{A}_2)_*$ by putting for every point of $(\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2)_*$, i.e., a prime filter F of $\mathfrak{A}_1 \overline{\oplus} \mathfrak{A}_2$:

$$f(F) = \begin{cases} F & \text{if } F \subsetneq A_2, \\ F \cap A_1 & \text{otherwise.} \end{cases}$$

We exclude the case $F = A_2$ in the definition of f, since in that case F is not a proper subset of A_2 and therefore is not a filter of $\mathfrak{A}_2$. It is not hard to see that f and h are isomorphisms.

□

Next we generalize the notions of the vertical sum of two Heyting algebras and the linear sum of two descriptive frames to countable sums; see [10].

4.1.17. DEFINITION. Let $\{\mathfrak{A}_i\}_{i\in\omega}$ be a countable family of Heyting algebras. The *vertical sum* of $\{\mathfrak{A}_i\}_{i\in\omega}$ is the partially ordered set $\overline{\bigoplus}_{i\in\omega}\mathfrak{A}_i = (\bigcup_{i\in\omega}\overline{A}_i \cup \{1\}, \leq)$, where $\overline{\mathfrak{A}}_i = (\overline{A}_i, \vee_i, \wedge_i, \to_i, 0_i)$ is an isomorphic copy of $\mathfrak{A}_i$, such that $\overline{A}_i \cap \overline{A}_{i+1} = \{1_i\} = \{0_{i+1}\}$. Let $\leq_i$ be the order of $\mathfrak{A}_i$. The order $\leq$ is defined by letting for every $a, b \in \bigcup_{i\in\omega}\overline{A}_i$:

$$\begin{array}{lll} a \leq b \text{ iff} & & a \in \overline{A}_i, b \in \overline{A}_j \text{ and } i < j, \\ & \text{or} & \text{there is } i \in \omega \text{ such that } a, b \in \overline{A}_i \text{ and } a \leq_i b, \\ & \text{or} & b = 1. \end{array}$$

Figuratively speaking, we form a tower from a countable family of Heyting algebras by putting all algebras on top of each other, and then adjoining a new top element. The reason that we adjoin a new top to the vertical sum of Heyting algebras is to make sure that the resulting object is again a Heyting algebra.

4.1.18. PROPOSITION. *A vertical sum of a countable family of Heyting algebras is also a Heyting algebra.*

Proof. The proof is a routine check. □

Note that the filter $\{1\}$ of $\overline{\oplus}_{i\in\omega}\mathfrak{A}_i$ is a prime filter, which implies that the corresponding descriptive frame should have a root. This is the motivation behind the following definition of the linear sum of a countable family of descriptive frames.

4.1.19. Definition. Let $\{\mathfrak{F}_i\}_{i\in\omega}$ be a countable family of descriptive frames, where $\mathfrak{F}_i = (W_i, R_i, \mathcal{P}_i)$ for every $i \in \omega$. The *linear sum* of $\{\mathfrak{F}_i\}_{i\in\omega}$ is a frame $\bigoplus_{i\in\omega}\mathfrak{F}_i = (\{\infty\} \cup \biguplus_{i\in\omega}, W_i, R, \mathcal{P})$ such that for every $w, v \in \biguplus_{i\in\omega} W_i$:

$$\begin{array}{lll} wRv \text{ iff} & & w \in W_i, v \in W_j \text{ and } i > j, \\ & \text{or} & \text{there is } i \in \omega \text{ such that } w, v \in W_i \text{ and } wR_iv, \\ & \text{or} & w = \infty. \end{array}$$

and $\mathcal{P}$ is such that

$$U \in \mathcal{P} \text{ iff } U \text{ is an upset, } U \neq \biguplus_{i\in\omega} W_i \text{ and } U \cap W_i \in \mathcal{P}_i, \text{ for every } i \in \omega.$$

Figuratively speaking, we form a tower from a countable family of descriptive frames by putting all frames below each other, and then adjoining a new root to it. Moreover, the complement of the root is not admissible.

4.1.20. Proposition. *A linear sum of a countable family of descriptive frames is also a descriptive frame.*

Proof. The proof is a routine check. □

We have the following infinite analogue of Theorem 4.1.16.

4.1.21. Theorem. *Let $\{\mathfrak{A}_i\}_{i\in\omega}$ be a family of Heyting algebras and $\{\mathfrak{F}_i\}_{i\in\omega}$ a family of descriptive frames. Then*

1. *$(\bigoplus_{i\in\omega}\mathfrak{F}_i)^*$ is isomorphic to $\overline{\bigoplus}_{i\in\omega}\mathfrak{F}_i^*$.*
2. *$(\overline{\bigoplus}_{i\in\omega}\mathfrak{A}_i)_*$ is isomorphic to $\bigoplus_{i\in\omega}(\mathfrak{A}_i)_*$.*

Proof. The proof is similar to the proof of Theorem 4.1.16. □

If each $\mathfrak{A}_i$ and $\mathfrak{F}_i$ is equal to $\mathfrak{A}$ or $\mathfrak{F}$ respectively, then we simply write $\overline{\bigoplus}_\omega\mathfrak{A}$ or $\bigoplus_\omega\mathfrak{F}$. Next we consider linear sums of finitely generated frames.

4.1.22. Theorem. *If a descriptive finitely generated frame $\mathfrak{F}$ is isomorphic to $\mathfrak{G}\oplus\mathfrak{H}$ and both $\mathfrak{G}$ and $\mathfrak{H}$ are descriptive, then $\mathfrak{G}$ and $\mathfrak{H}$ are also finitely generated.*

Proof. Let $\mathfrak{F}$ be n-generated, for some $n \in \omega$. This means that there is a valuation $V : \textsc{Prop}_n \to \mathfrak{F}$ such that the upsets $V(p_1), \ldots, V(p_n)$ generate $\mathfrak{F}^*$. As was shown in the previous chapter, V defines a coloring of $\mathfrak{F}$. Let V' be the restriction of V to $\mathfrak{G}$. We show that $V'(p_1), \ldots, V'(p_n)$ generate $\mathfrak{G}^*$, which implies that $\mathfrak{G}$ is finitely generated. Suppose $\mathfrak{G}^*$ is not generated by $V'(p_1), \ldots, V'(p_n)$. Then by the Coloring Theorem, (see Theorem 3.1.5) there exists a descriptive frame $\mathfrak{T}$ and a p-morphism $f : \mathfrak{G} \to \mathfrak{T}$ such that for every $u, v \in \mathfrak{G}$, $f(u) = f(v)$ implies $col(u) = col(v)$. Consider the frame $\mathfrak{T} \oplus \mathfrak{H}$ and let $\bar{f} : \mathfrak{G} \oplus \mathfrak{H} \to \mathfrak{T} \oplus \mathfrak{H}$ be such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathfrak{G}, \\ x & \text{if } x \in \mathfrak{H}. \end{cases}$$

Then it is easy to see that $\bar{f}$ is a p-morphism and for every $u, v \in \mathfrak{G} \oplus \mathfrak{H}$, $\bar{f}(u) = \bar{f}(v)$ implies $col(u) = col(v)$. By the Coloring Theorem, this means that $\mathfrak{G} \oplus \mathfrak{H}$ is not generated by $V(p_1), \ldots, V(p_n)$, which is a contradiction. Therefore, $\mathfrak{G}$ is n-generated. The proof that $\mathfrak{H}$ is n-generated is similar. $\square$

The next lemma shows that an n-generated descriptive frame cannot be a linear sum of more than $2n$ frames.

4.1.23. Lemma. *Suppose $\mathfrak{F}$ is an n-generated descriptive frame isomorphic to $\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_m$. Then $m \leq 2n$.*

Proof. Let $V : \textsc{Prop}_n \to \mathfrak{F}$ be such that $V(p_1), \ldots, V(p_n)$ generate the Heyting algebra $\mathfrak{F}^*$. Then V defines a coloring of $\mathfrak{F}$. Suppose $m > 2n$. Given the fact that every $V(p_i)$ is an upset of $\mathfrak{F}$, for each i there can be at most one j such that $\mathfrak{F}_j$ contains both points that make p_i true, and points that make p_i false. So, if $m > 2n$ there exists $j < m$ such that $col(x) = col(y)$ for every x, y in $\mathfrak{F}_j$ or $\mathfrak{F}_{j+1}$. Consider the smallest equivalence relation that identifies all the points in $\mathfrak{F}_j$ and $\mathfrak{F}_{j+1}$. Then E is a bisimulation equivalence and every E-equivalence class contains points of the same color. This, by the Coloring Theorem, implies that $\mathfrak{F}^*$ is not generated by $V(p_1), \ldots, V(p_n)$, which is a contradiction. $\square$

4.2 Finite frames of RN

This section is devoted to finite frames of **RN**. We characterize the finite rooted **RN**-frames and the finite subdirectly irreducible algebras of $\mathcal{RN}$ in terms of linear and vertical sums. First we characterize the generated subframes of $\mathfrak{L}$ and the homomorphic images of $\mathfrak{N}$.

4.2.1. Theorem.

1. *A descriptive frame $\mathfrak{F}$ is a generated subframe of $\mathfrak{L}$ iff $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$, $\mathfrak{L}_{g_k}$ or $\mathfrak{L}_{f_k}$ for some $k \in \omega$.*

2. *Every proper generated subframe of $\mathfrak{L}$ is finite.*

Proof. The proof is a routine verification. The only fact that needs to be pointed out is that since the carrier set of $\mathfrak{L}_0$ is not compact, see Section 4.1.1, $\mathfrak{L}_0$ is not a generated subframe of $\mathfrak{L}$. $\square$

4.2.2. Corollary.

1. *A Heyting algebra $\mathfrak{A}$ is a homomorphic image of $\mathfrak{N}$ iff $\mathfrak{A}$ is isomorphic to $\mathfrak{N}$, $\mathfrak{N}_{g_k}$ or $\mathfrak{N}_{f_k}$ for some $k \in \omega$.*

2. *Every proper homomorphic image of $\mathfrak{N}$ is finite.*

Proof. The theorem follows immediately from Theorem 4.2.1 and the duality between Heyting algebras and descriptive Kripke frames. $\square$

Similarly to Theorem 4.2.1 and Corollary 4.2.2 we can characterize the generated subframes of $\mathfrak{L}_{g_k}$ and $\mathfrak{L}_{f_k}$, and the homomorphic images of $\mathfrak{N}_{g_k}$ and $\mathfrak{N}_{f_k}$, respectively.

4.2.3. Theorem. *For every $k \in \omega$:*

1. *A frame $\mathfrak{F}$ is a generated subframe of $\mathfrak{L}_{g_k}$ iff $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{g_j}$ for some $j \leq k$ and $j \neq k-1$, or $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{f_j}$ for some $j \leq k-2$.*

2. *A frame $\mathfrak{F}$ is a generated subframe of $\mathfrak{L}_{f_k}$ iff $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{g_j}$ for some $j \leq k$, or $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{f_j}$ for some $j \leq k$.*

3. *A Heyting algebra $\mathfrak{A}$ is a homomorphic image of $\mathfrak{N}_{g_k}$ iff $\mathfrak{N}$ is isomorphic to $\mathfrak{N}_{g_j}$ for some $j \leq k$, and $j \neq k-1$ or $\mathfrak{N}$ is isomorphic to $\mathfrak{N}_{f_j}$ for some $j \leq k-2$.*

4. *A frame $\mathfrak{N}$ is a generated subframe of $\mathfrak{N}_{f_k}$ iff $\mathfrak{N}$ is isomorphic to $\mathfrak{N}_{g_j}$ for some $j \leq k$, or $\mathfrak{N}$ is isomorphic to $\mathfrak{N}_{f_j}$ for some $j \leq k$.*

Proof. The proof is a routine verification. $\square$

Next we characterize the p-morphic images of $\mathfrak{L}$ and the subalgebras of $\mathfrak{N}$. We will show that up to isomorphism there are three different types of p-morphic images of $\mathfrak{L}$ and subalgebras of $\mathfrak{N}$. In order to describe them, we will use the linear sums of descriptive frames and vertical sums of Heyting algebras.

4.2.4. Definition.

1. Let **2** denote the two-element Boolean algebra.

2. Let **4** denote the the four-element Boolean algebra.

4.2.5. PROPOSITION.

1. *The frame* $\mathbf{2}_*$ *consists of a single reflexive point.*

2. *The frame* $\mathbf{4}_*$ *is isomorphic to* $\mathbf{2}_* \uplus \mathbf{2}_*$.

Proof. The proof is easy. $\square$

The following result was first established by Kracht [73] using descriptive frames for **IPC**, see also [10]. Below we will give a purely algebraic proof, which in our opinion is the simplest one.

4.2.6. THEOREM. *A Heyting algebra* $\mathfrak{A}$ *is a subalgebra of* $\mathfrak{N}$ *iff* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{N}$, $\overline{\bigoplus}_{i\in\omega}\mathfrak{B}_i$, $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathbf{2}$, *or* $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathfrak{N}$, *for some* $n \in \omega$, *where each* $\mathfrak{B}_i$ *is isomorphic to* $\mathbf{2}$ *or* $\mathbf{4}$.

Proof. Suppose $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ is a subalgebra of $\mathfrak{N}$. If $\mathfrak{A}$ is a chain, then $\mathfrak{A}$ is isomorphic to $\overline{\bigoplus}_{i\in\alpha}\mathfrak{B}_i$, for $\alpha \leq \omega$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$. Now assume that $\mathfrak{A}$ is not isomorphic to a chain. Then there are at least two incomparable elements a and b in $\mathfrak{A}$. Since $\mathfrak{N}$ is well-founded we can assume that a and b are the least two incomparable elements of $\mathfrak{A}$; that is, the set $\{c \in A : c \leq a \text{ or } c \leq b\}$ is a chain. Then from the structure of $\mathfrak{N}$ it follows directly that there is $k \in \omega$ such that one of the following four cases holds:

1. $\{a, b\} = \{f_{2k}, g_{2k+1}\}$.

2. $\{a, b\} = \{g_{2k}, f_{2k-1}\}$.

3. $\{a, b\} = \{g_{2k}, g_{2k-1}\}$.

4. $\{a, b\} = \{g_{2k}, g_{2k+1}\}$.

Case 1. If $\{a, b\} = \{f_{2k}, g_{2k+1}\}$, then the element $f_{2k-1} = f_{2k} \wedge g_{2k+1}$ belongs to A. Looking at the filter $[f_{2k-1})$ we see that it is isomorphic to $\mathfrak{N}$. Moreover, in the same way as g_0 generates $\mathfrak{N}$, f_{2k} generates $[f_{2k-1})$. So the whole filter $[f_{2k-1})$ is contained in A. Now since a and b are the least two incomparable elements in $\mathfrak{A}$, we have that $A\backslash[f_{2k-1})$ is a chain. Therefore, $\mathfrak{A}$ is isomorphic to $\mathfrak{N}$ or $\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i\overline{\oplus}\mathfrak{N}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$.

Case 2. The proof is similar to the proof of Case 1. If $\{a, b\} = \{g_{2k}, f_{2k-1}\}$, then the element $f_{2k} = g_{2k} \wedge f_{2k-1}$ belongs to A, and so the whole filter $[f_{2k})$ is contained in A. Now $[f_{2k})$ is isomorphic to $\mathfrak{N}$, and $A\backslash[f_{2k})$ is a chain. Thus $\mathfrak{A}$ is isomorphic to $\mathfrak{N}$ or $\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i\overline{\oplus}\mathfrak{N}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$.

Case 3. If $\{a,b\} = \{g_{2k}, g_{2k-1}\}$, then $f_{2k-1} = g_{2k} \wedge g_{2k-1}$ and $f_{2k} = g_{2k} \vee g_{2k-1}$ belong to A. Since g_{2k} and g_{2k-1} are the least two incomparable elements, none of $g_{2(k-1)}, f_{2k-1}, f_{2k}, g_{2k+1}$ are in A. Therefore, every element of $\mathfrak{A}$ is below $a \wedge b$, above $a \vee b$, or in $\{a, b, a \wedge b, a \vee b\}$, which is isomorphic to $\mathbf{4}$. Moreover, $(a \wedge b]$ is a chain. If $[a \vee b)$ is also a chain, then $\mathfrak{A}$ is isomorphic to $\overline{\bigoplus}_{i\in\omega}\mathfrak{B}_i$ or $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathbf{2}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$ or $\mathbf{4}$. (In fact there will be exactly one $\mathfrak{B}_i$ isomorphic to $\mathbf{4}$.) If $[a \vee b)$ is not a chain, then let c and d be the least incomparable elements in $[a \vee b)$. Then one of the above four possibilities holds for $\{c,d\}$, and we are back in one of the four cases, but this time for $\{c,d\}$. Repeating this process we eventually obtain that $\mathfrak{A}$ is isomorphic to one of $\mathfrak{N}$, $\overline{\bigoplus}_{i\in\omega}\mathfrak{B}_i$, $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathbf{2}$, or $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathfrak{N}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$ or $\mathbf{4}$.

Case 4. The proof is similar to the proof of Case 3. If $\{a,b\} = \{g_{2k}, g_{2k+1}\}$, then $f_{2k} = g_{2k} \wedge g_{2k+1}$ and $f_{2k+1} = g_{2k} \vee g_{2k+1}$ are in A, and none of g_{2k-1}, f_{2k-1}, f_{2k}, and $g_{2(k+1)}$ belong to A. Therefore, every element of $\mathfrak{A}$ is either below $a \wedge b$, above $a \vee b$, or in $\{a, b, a \wedge b, a \vee b\}$, and $(a \wedge b]$ is a chain; and we proceed as in 3.

$\square$

4.2.7. Corollary. *A descriptive frame $\mathfrak{F}$ is a p-morphic image of $\mathfrak{L}$ iff $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$, $\bigoplus_{i\in\omega}\mathfrak{F}_i$, $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathbf{2}_*$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathfrak{L}$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$ or $\mathbf{4}_*$ and $n \in \omega$.*

Proof. Follows immediately from Theorem 4.2.6 and the duality between Heyting algebras and descriptive frames. $\square$

Theorem 4.2.1 and Corollary 4.2.7 enable us to characterize generated subframes of p-morphic images of $\mathfrak{L}$.

4.2.8. Theorem.

1. *An infinite descriptive frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ iff $\mathfrak{F}$ is isomorphic to $\bigoplus_{i\in\omega}\mathfrak{F}_i$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathfrak{L}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$ and $n \in \omega$.*

2. *A finite frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ iff $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathfrak{L}_{f_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$ and $k, n \in \omega$.*

3. *A finite rooted frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ iff $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$ and $k, n \in \omega$.*

Proof. (1) The right to left implication follows immediately from Corollary 4.2.7. Conversely, suppose an infinite descriptive frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$. Then there exists an infinite descriptive frame $\mathfrak{G}$ such that $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$ and $\mathfrak{G}$ is a p-morphic image of $\mathfrak{L}$. Then by Corollary 4.2.7, $\mathfrak{G}$ is isomorphic to $\bigoplus_{i\in\omega}\mathfrak{F}_i$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}$. It is easy to see that neither $\bigoplus_{i\in\omega}\mathfrak{F}_i$ nor $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}$ contains a proper infinite generated subframe. Therefore, $\mathfrak{F}$ is isomorphic to either $\bigoplus_{i\in\omega}\mathfrak{F}_i$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}$.

(2) The right to left implication again follows from Corollary 4.2.7. Conversely, suppose $\mathfrak{G}$ is a p-morphic image of $\mathfrak{L}$ and $\mathfrak{F}$ is a finite generated subframe of $\mathfrak{G}$. Then by Corollary 4.2.7, $\mathfrak{G}$ is isomorphic to $\mathfrak{L}$, $\bigoplus_{i\in\omega}\mathfrak{F}_i$, $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathbf{2}_*$, or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}$. Consequently, in the first case $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{g_k}$ or $\mathfrak{L}_{f_k}$, in the second and third cases $\mathfrak{F}$ is isomorphic to $\bigoplus_{i=1}^{n}\mathfrak{F}_i$, and in the fourth case $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$ or $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{f_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$.

(3) The result follows immediately from (2) since for every $k>0$ the frame $\mathfrak{L}_{f_k}$ is not rooted. $\square$

We recall that for a class of algebras K, $\mathbf{H}(K)$, $\mathbf{S}(K)$, and $\mathbf{P}(K)$ denote the classes of all homomorphic images, subalgebras, and direct products of the algebras from K, respectively.

4.2.9. Corollary.

1. *An infinite Heyting algebra $\mathfrak{A}$ belongs to $\mathbf{HS}(\mathfrak{N})$ iff $\mathfrak{A}$ is isomorphic to $\overline{\bigoplus}_{i\in\omega}\mathfrak{B}_i$ or $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathfrak{N}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$ or $\mathbf{4}$ and $k,n\in\omega$.*

2. *A finite Heyting algebra $\mathfrak{A}$ belongs to $\mathbf{HS}(\mathfrak{N})$ iff $\mathfrak{A}$ is isomorphic to $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)$ $\overline{\oplus}\mathfrak{N}_k$, where each $\mathfrak{B}_i$ is isomorphic to $\mathbf{2}$ or $\mathbf{4}$ and $k,n\in\omega$.*

Proof. The result follows immediately from Corollary 4.2.8 and the duality theory for Heyting algebras. $\square$

4.2.10. Corollary. *A finite rooted frame $\mathfrak{F}$ is an $\mathbf{RN}$-frame iff $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$ and $k,n\in\omega$.*

Proof. It is obvious that if a finite rooted frame $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}\oplus$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$, then $\mathfrak{F}$ is an $\mathbf{RN}$-frame. Conversely, suppose a finite rooted frame $\mathfrak{F}$ is an $\mathbf{RN}$-frame. Then $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$. To see this, note that if $\mathfrak{F}$ is not a generated subframe of a p-morphic image of $\mathfrak{L}$, by Theorem 3.3.3, we have $\mathfrak{L}\models\chi(\mathfrak{F})$. Then, as $\mathbf{RN}=Log(\mathfrak{L})$ and $\mathfrak{F}$ is an $\mathbf{RN}$-frame we have $\mathfrak{F}\models\chi(\mathfrak{F})$, which is a contradiction. Thus $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$. Therefore, by Theorem 4.2.8(3), $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$. $\square$

4.2.11. THEOREM. *A finite subdirectly irreducible algebra $\mathfrak{A}$ belongs to $\mathcal{RN}$ if $\mathfrak{A}$ is isomorphic to $(\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i)\overline{\oplus}\mathfrak{N}_{g_k}$, where each $\mathfrak{B}_i$ is isomorphic to* **2** *or* **4** *and $k, n \in \omega$.*

Proof. The theorem follows immediately from Theorem 4.2.10 and the duality theory for Heyting algebras. □

Similarly to Theorem 4.2.6 and Corollary 4.2.7 we can characterize subalgebras and p-morphic images of $\mathfrak{L}_{g_k}$'s and $\mathfrak{N}_{g_k}$'s.

4.2.12. THEOREM. *For every $k, n \in \omega$:*

1. *The frame $\bigoplus_{i=1}^{n}\mathfrak{F}_i \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_{(k+3n)}}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$.*
2. *The algebra $\overline{\bigoplus}_{i=1}^{n}\mathfrak{B}_i\overline{\oplus}\mathfrak{N}_{g_k}$, is a subalgebra of $\mathfrak{N}_{g_{(k+3n)}}$, where each $\mathfrak{B}_i$ is either empty of isomorphic to* **2** *or* **4**.

Proof. The proof is an adaptation of the proofs of Theorem 4.2.6 and Corollary 4.2.7. □

4.2.13. LEMMA. *$\mathfrak{L}_{g_k}$ is not a p-morphic image of $\mathfrak{L}_{g_m}$, for $m \neq k$.*

Proof. Suppose there exists a p-morphism $f : \mathfrak{L}_{g_m} \to \mathfrak{L}_{g_k}$, then by Proposition 3.1.7, f is a composition of α and β-reductions. It is easy to see that by applying α and β-reductions to $\mathfrak{L}_{g_m}$ we cannot obtain a frame isomorphic to $\mathfrak{L}_{g_k}$. □

4.3 The Kuznetsov-Gerciu logic

In this section we introduce the logic whose finitely generated frames are the finite linear sums of 1-generated frames. This logic and the corresponding variety were first introduced and studied by Kuznetsov and Gerciu [83].

4.3.1. DEFINITION. Let

$$\phi_{KG} := (p \to q) \vee (q \to r) \vee ((q \to r) \to r) \vee (r \to (p \vee q))$$

We call $\mathbf{IPC} + \phi_{KG}$ the Kuznetsov-Gerciu logic and denote it by **KG**. We denote the corresponding variety by $\mathcal{KG}$.

Our first task is to show that **KG** is a subframe logic. Consider the frames $\mathfrak{K}_1$, $\mathfrak{K}_2$, and $\mathfrak{K}_3$ shown in Figure 4.3.

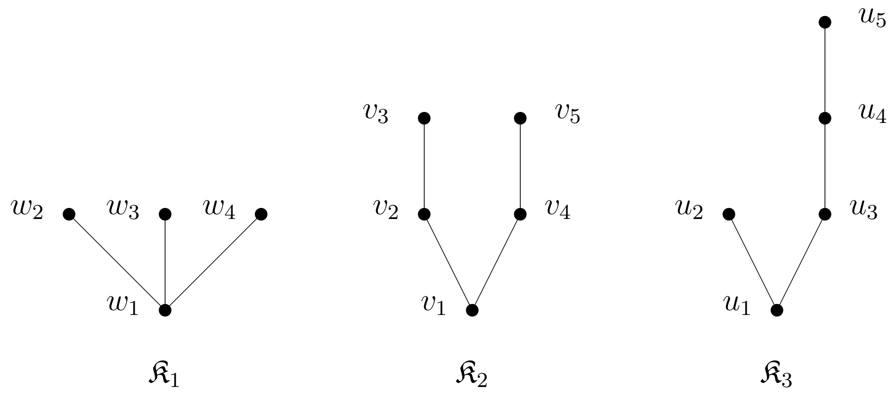

Figure 4.3: The frames $\mathfrak{K}_1, \mathfrak{K}_2$, and $\mathfrak{K}_3$

4.3.2. Lemma. *Suppose $\mathfrak{F} = (W, R, \mathcal{P})$ is a descriptive frame.*

1. *If either $\mathfrak{K}_1$, $\mathfrak{K}_2$ or $\mathfrak{K}_3$ is a p-morphic image of a subframe of $\mathfrak{F}$, then $\mathfrak{F} \not\models \phi_{KG}$.*

2. *If $\mathfrak{F} \models \phi_{KG}$, then $\mathfrak{F} \models \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$.*

Proof. (1) First we show that $\mathfrak{K}_i \not\models \phi_{KG}$ for every $i = 1, 2, 3$. In case $i = 1$ we let $V_1(p) = \{w_2\}$, $V_1(q) = \{w_3\}$ and $V_1(r) = \{w_4\}$. If $i = 2$, then let $V_2(p) = \{v_2, v_3\}$, $V_2(q) = \{v_3\}$ and $V_2(r) = \{v_5\}$. And if $i = 3$, then we put $V_3(p) = \{u_5\}$, $V_3(q) = \{u_2\}$ and $V_3(r) = \{u_4, u_5\}$. It is easy to check that $(\mathfrak{K}_i, V_i) \not\models \phi_{KG}$ for each $i = 1, 2, 3$. Now assume that $\mathfrak{G}$ is a subframe of $\mathfrak{F}$ such that $\mathfrak{K}_i$ is a p-morphic image of $\mathfrak{G}$ for some $i = 1, 2, 3$. Suppose $f : \mathfrak{G} \to \mathfrak{K}_i$ is this p-morphism. Let V' be a valuation on $\mathfrak{G}$ defined by

$$V'(p) = f^{-1}(V_i(p))$$

Then $(\mathfrak{G}, V') \not\models \phi_{KG}$. Now let us extend the valuation V' on $\mathfrak{G}$ to a valuation V on $\mathfrak{F}$ as in the proof of Lemma 3.3.14. That is, we put

$$V(p) = W \setminus R^{-1}(V'(p)).$$

Then it is easy to see that $(\mathfrak{F}, V) \not\models \phi_{KG}$.

(2) is an immediate consequence of (1). □

Next we prove the converse of Lemma 4.3.2 for Kripke frames. Consequently, we will axiomatize **KG** by the subframe formulas of $\mathfrak{K}_1, \mathfrak{K}_2$, and $\mathfrak{K}_3$.

4.3.3. Lemma. *Suppose $\mathfrak{F} = (W, R)$ is a rooted Kripke frame.*

1. *If neither $\mathfrak{K}_1$, $\mathfrak{K}_2$, nor $\mathfrak{K}_3$ is a subframe of $\mathfrak{F}$, then $\mathfrak{F} \models \phi_{KG}$.*

2. If $\mathfrak{F} \models \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$*, then* $\mathfrak{F} \models \phi_{KG}$.

Proof. (1) Suppose $\mathfrak{F} \not\models \phi_{KG}$. Let w_0 be the root of $\mathfrak{F}$. Then there exists a valuation V on W such that $\mathfrak{M}, w_0 \not\models \phi_{KG}$, where $\mathfrak{M} = (\mathfrak{F}, V)$. Therefore, there exist $w_1, w_2, w_3, w_4 \in R(w)$ such that $\mathfrak{M}, w_1 \models p$ and $\mathfrak{M}, w_1 \not\models q$, $\mathfrak{M}, w_2 \models q$ and $\mathfrak{M}, w_2 \not\models r$, $\mathfrak{M}, w_3 \models q \to r$ and $\mathfrak{M}, w_3 \not\models r$, and $\mathfrak{M}, w_4 \models r$, $\mathfrak{M}, w_4 \not\models p$ and $\mathfrak{M}, w_4 \not\models q$.

Let us assume that $\mathfrak{K}_1$ is not a subframe of $\mathfrak{F}$, and show that then either $\mathfrak{K}_2$ or $\mathfrak{K}_3$ is a subframe of $\mathfrak{F}$. Since $\mathfrak{M}, w_2 \models q$, $\mathfrak{M}, w_4 \not\models q$ and $\mathfrak{M}, w_4 \models r$, $\mathfrak{M}, w_2 \not\models r$, we have that w_2 and w_4 are incomparable. As $\mathfrak{M}, w_3 \models q \to r$ and $\mathfrak{M}, w_3 \not\models r$, it follows that $\mathfrak{M}, w_3 \not\models q$, which together with $\mathfrak{M}, w_2 \models q$ gives us $\neg(w_2 R w_3)$. Also since $\mathfrak{M}, w_3 \models q \to r$, $\mathfrak{M}, w_2 \models q$ and $\mathfrak{M}, w_2 \not\models r$, we have that $\neg(w_3 R w_2)$. Thus, w_2 and w_3 are incomparable as well. As $\mathfrak{M}, w_4 \models r$ and $\mathfrak{M}, w_3 \not\models r$, we also have that $\neg(w_4 R w_3)$. Therefore, as $\mathfrak{K}_1$ is not a subframe of $\mathfrak{F}$, we have that $w_3 R w_4$. Otherwise the subframe of $\mathfrak{F}$ based on $\{w_0, w_2, w_3, w_4\}$ would be isomorphic to $\mathfrak{F}_1$. Moreover, $\mathfrak{M}, w_2 \models q$ and $\mathfrak{M}, w_1 \not\models q$ give us $\neg(w_2 R w_1)$, and $\mathfrak{M}, w_1 \models p$ and $\mathfrak{M}, w_4 \not\models p$ give us that $\neg(w_1 R w_4)$, and hence that $\neg(w_1 R w_3)$. Since $\mathfrak{K}_1$ is not a subframe of $\mathfrak{F}$, we have that either $w_1 R w_2$ or $w_4 R w_1$. First suppose that $w_1 R w_2$. Then as w_3 and w_2 are incomparable we have that $\neg(w_3 R w_1)$ and $\neg(w_4 R w_1)$. Therefore, $\mathfrak{K}_2$ is a subframe of $\mathfrak{F}$. Now suppose that $w_4 R w_1$. Then as w_4 and w_2 are incomparable we have that $\neg(w_1 R w_2)$, which implies that $\mathfrak{K}_3$ is a subframe of $\mathfrak{F}$. Thus, if $\mathfrak{F} \not\models \phi_{KG}$, then either $\mathfrak{K}_1$, $\mathfrak{K}_2$ or $\mathfrak{K}_3$ is a subframe of $\mathfrak{F}$.

(2) is an immediate consequence of (1). □

4.3.4. THEOREM. $\mathbf{KG} = \mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$.

Proof. Suppose $\mathfrak{F}$ is a descriptive **KG**-frame. Then, by Lemma 4.3.2(2), $\mathfrak{F} \models \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$. Therefore, $\mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3) \subseteq \mathbf{KG}$. Now suppose $\mathfrak{F}$ is a Kripke frame such that $\mathfrak{F} \models \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$. Then, by Lemma 4.3.3(2), $\mathfrak{F}$ is a **KG**-frame. As $\mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$ is a subframe logic, see Corollary 3.4.16(2), it follows from Theorem 3.4.17 that $\mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$ has the finite model property, and hence is Kripke complete. Therefore, $\mathbf{KG} \subseteq \mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$, and our result follows. □

4.3.5. COROLLARY. $\mathcal{KG} = \mathcal{HA} + [\beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3) = 1]$.

Subsequently in the paper we will use the following shorthand.

4.3.6. DEFINITION. A frame $\mathfrak{F}$ is called *cyclic* if it is a 1-generated[3] descriptive frame.

Then we have the following immediate characterization of cyclic frames.

[3]Recall that n-generated frames were defined in Definition 3.1.3.

4.3.7. THEOREM. *A descriptive frame $\mathfrak{F}$ is cyclic iff $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$, $\mathfrak{L}_{g_k}$ or $\mathfrak{L}_{f_k}$, for some $k \in \omega$.*

Proof. Every 1-generated Heyting algebra is a homomorphic image of the 1-generated free Heyting algebra. By the duality, this means that every cyclic frame is a generated subframe of $\mathfrak{L}$.

By definition, $\mathfrak{F}$ is cyclic iff $\mathfrak{F}$ is a generated subframe of $\mathfrak{L}$. The result now follows from Theorem 4.2.1. □

Thus, every cyclic frame is descriptive, moreover except for $\mathfrak{L}$, every cyclic frame is finite. To characterize the finitely generated rooted **KG**-frames, we will need the following technical lemma.

4.3.8. LEMMA. *Let $\mathfrak{F}$ be a finitely generated rooted descriptive* **KG***-frame.*

1. *There exist descriptive frames $\mathfrak{G}'$ and $\mathfrak{H}'$ such that $\mathfrak{H}'$ is cyclic and $\mathfrak{F}$ is isomorphic to $\mathfrak{G}' \oplus \mathfrak{H}'$.*

2. *Suppose $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{G}$, where $\mathfrak{G}$ is a non-cyclic descriptive frame and $\mathfrak{H}$ is a cyclic descriptive frame. Then there exist descriptive frames $\mathfrak{G}'$ and $\mathfrak{H}'$ such that $\mathfrak{H}'$ is cyclic and $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{H}' \oplus \mathfrak{G}'$.*

Proof. (1) Let r be the root of $\mathfrak{F}$. As $\mathfrak{F}$ is a **KG**-frame, $\mathfrak{K}_1$ is not a subframe of $\mathfrak{F}$, implying that $|max(\mathfrak{F})| \leq 2$. If $max(\mathfrak{F}) = \{x\}$, then let $\mathfrak{H}' = (\{x\}, =)$, and let $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It is then obvious that $\mathfrak{H}'$ is a cyclic frame, and that $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. Moreover, by Theorem 3.1.10 (see also Claim 3.1.11), $\mathfrak{G}'$ is a descriptive frame. If $max(\mathfrak{F}) = \{x, y\}$, then two cases are possible: either the next layer of $\mathfrak{F}$ consists of a single point z, or the next layer of $\mathfrak{F}$ consists of two distinct points z and u.

Case 1. Suppose that the next layer of $\mathfrak{F}$ consists of a single point z, and that zRx and zRy. Then we put $\mathfrak{H}' = (\{x, y\}, =)$ and $\mathfrak{G}' = \mathfrak{G} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame. By Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and therefore $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. Now suppose that zRx and $\neg(zRy)$. Then we again have two cases: either the next layer of $\mathfrak{F}$ consists of a single point v, or the next layer of $\mathfrak{F}$ consists of two distinct points v and w.

Case 1a. Suppose the next layer of $\mathfrak{F}$ consists of a single point v. Then vRz. If $\neg(vRy)$, then $(\{r, v, z, x, y\}, R \restriction \{r, v, z, x, y\})$ is a subframe of $\mathfrak{F}$, isomorphic to $\mathfrak{K}_3$, which is a contradiction. Therefore, we have vRy, and we put $\mathfrak{H}' = (\{v, z, x, y\}, R \restriction \{v, z, x, y\})$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame. Again by Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and therefore $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}$.

Case 1b. Suppose the next layer of $\mathfrak{F}$ consists of two distinct points v and w. Then vRz and wRz. If $\neg(vRy)$ and $\neg(wRy)$, then $(\{r, v, w, y\}, R \restriction \{r, v, w, y\})$

is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_1$, which is a contradiction. Therefore, vRy or wRy. If $\neg(vRy)$ and wRy, then $(\{r, v, z, x, y\}, R \restriction \{r, v, z, x, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_3$; and if vRy and $\neg(wRy)$, then $(\{r, w, z, x, y\}, R \restriction \{r, w, z, x, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_3$. In both cases we arrive at a contradiction. Thus, vRy and wRy. But then we put $\mathfrak{H}' = (\{z, x, y\}, R \restriction \{z, x, y\})$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It follows that $\mathfrak{H}'$ is a cyclic frame. By Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$.

Case 2. Suppose the next layer of $\mathfrak{F}$ consists of two distinct points z and u. If zRx, uRx, $\neg(zRy)$ and $\neg(uRy)$, then $(\{r, z, u, y\}, R \restriction \{r, z, u, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_1$, which is a contradiction. If zRx, $\neg(zRy)$, uRy and $\neg(uRx)$, then $(\{r, z, u, x, y\}, R \restriction \{r, z, u, x, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_2$, which is also a contradiction. If zRx, zRy, uRx and uRy, then we put $\mathfrak{H}' = (\{x, y\}, =)$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame, by Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. Finally, if zRx, $\neg(zRy)$, uRx and uRy, then there are two possible cases: either the next layer of $\mathfrak{F}$ consists of a single point z_1, or the next layer of $\mathfrak{F}$ consists of two distinct points z_1 and u_1.

Case 2a. Suppose the next layer of $\mathfrak{F}$ consists of a single point z_1. If z_1Rz and z_1Ru, then we put $\mathfrak{H}' = (\{z, u, x, y\}, R \restriction \{z, u, x, y\})$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame, by Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. Otherwise we have that either z_1Rz and $\neg(z_1Ru)$, or z_1Ru and $\neg(z_1Rz)$. If z_1Ru and $\neg(z_1Rz)$, then $(\{r, z_1, z, x, y\}, R \restriction \{r, z_1, z, x, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_2$, which is a contradiction. So we can assume that z_1Rz and $\neg(z_1Ru)$. If $\neg(z_1Ry)$, then we have that $(\{r, z_1, z, x, y\}, R \restriction \{r, z_1, z, x, y\})$ is a subframe of $\mathfrak{F}$ isomorphic to $\mathfrak{K}_3$, which is a contradiction. Therefore z_1Ry, and we again have two cases: either the next layer of $\mathfrak{F}$ consists of a single point v_1, or the next layer of $\mathfrak{F}$ consists of two distinct points v_1 and w_1. In the former case, the same argument as in Case 1a gives us that v_1Rz_1 and v_1Ru. Thus we put $\mathfrak{H}' = (\{z_1, z, u, x, y\}, R \restriction \{z_1, z, u, x, y\})$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame (in fact $\mathfrak{H}'$ is isomorphic to $\mathfrak{L}_{g_5}$). By Theorem 3.1.10, $\mathfrak{G}'$ is a descriptive frame, and $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. In the latter case, the same argument as in Case 1b gives us that v_1Rz_1, w_1Rz_1, v_1Ru and w_1Ru. Thus, we put $\mathfrak{H}' = (\{z_1, z, u, x, y\}, R \restriction \{z_1, z, u, x, y\})$ and $\mathfrak{G}' = \mathfrak{F} \setminus \mathfrak{H}'$. It then follows that $\mathfrak{H}'$ is a cyclic frame, that $\mathfrak{G}'$ is a descriptive frame, and that $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$.

Case 2b. Suppose the next layer of $\mathfrak{F}$ consists of two distinct points z_1 and u_1. Then the same argument as in the beginning of Case 2 guarantees that z_1Rz, z_1Ry, u_1Rz and u_1Ru, and we move on to the next layer of $\mathfrak{F}$.

Continuing in this fashion, if our process terminates after finitely many steps, we obtain that either $\mathfrak{F}$ is isomorphic to a finite cyclic frame or that $\mathfrak{F}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$ with $\mathfrak{H}'$ cyclic and $\mathfrak{G}'$ descriptive. Otherwise we obtain that $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$ or that $\mathfrak{F}$ is isomorphic to $\mathfrak{L} \oplus \mathfrak{G}'$ with $\mathfrak{G}'$ descriptive. In either case, our result follows.

(2) Suppose $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{G}$, where $\mathfrak{G}$ is a non-cyclic descriptive frame and $\mathfrak{H}$ is a cyclic descriptive frame. Since $\mathfrak{F}$ is finitely generated, so is $\mathfrak{G}$ by Theorem 4.1.22. Therefore, by (1) there exist descriptive frames $\mathfrak{G}'$ and $\mathfrak{H}'$ such that $\mathfrak{H}'$ is cyclic and $\mathfrak{G}$ is isomorphic to $\mathfrak{H}' \oplus \mathfrak{G}'$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{H}' \oplus \mathfrak{G}'$. This finishes the proof of the lemma. □

Recall from Definition 2.3.15 that descriptive rooted frames are such rooted frames that the complement of the root is admissible.

4.3.9. COROLLARY. *A rooted descriptive* **KG***-frame $\mathfrak{F}$ is finitely generated iff $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is a cyclic frame and $k \in \omega$.*

Proof. It is obvious that if $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is a cyclic frame, then $\mathfrak{F}$ is finitely generated. Conversely, suppose $\mathfrak{F}$ is a finitely generated rooted descriptive **KG**-frame. If $\mathfrak{F}$ is cyclic, then we are done. Otherwise, by Lemma 4.3.8 (1), $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{G}$, where $\mathfrak{H}$ is cyclic and $\mathfrak{G}$ is descriptive. If $\mathfrak{G}$ is cyclic, then we are done. If not, by Lemma 4.3.8(2), $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{H}' \oplus \mathfrak{G}$, where $\mathfrak{H}'$ is cyclic and $\mathfrak{G}'$ is descriptive. Continuing this process we obtain that $\mathfrak{F}$ is isomorphic to $\bigoplus_{i \in \alpha} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is a cyclic frame. If $\omega \leq \alpha$, then by Lemma 4.1.23, $\bigoplus_{i \in \alpha} \mathfrak{F}_i$ is not finitely generated. Therefore, $\alpha < \omega$. This means that $min(\mathfrak{F}) = min(\mathfrak{F}_\alpha)$. Thus, if $\mathfrak{F}_\alpha$ is isomorphic to $\mathfrak{L}_{f_m}$ for some $m > 0$, then $\mathfrak{F}$ is not rooted. If $\mathfrak{F}_\alpha$ is isomorphic to $\mathfrak{L}$, then by Definition 4.1.15, the complement of the root of $\mathfrak{F}$ is not an admissible set. Therefore, by Definition 2.3.15, $\mathfrak{F}$ is not a rooted descriptive frame. Thus, we obtain that $\mathfrak{F}_\alpha$ is isomorphic to $\mathfrak{L}_{g_k}$ for some $k \in \omega$. □

4.3.10. COROLLARY. *A subdirectly irreducible algebra $\mathfrak{A} \in \mathcal{KG}$ is finitely generated iff $\mathfrak{A}$ is isomorphic to $(\overline{\bigoplus}_{i=1}^{n} \mathfrak{A}_i) \overline{\oplus} \mathfrak{N}_{g_k}$, where each $\mathfrak{A}_i$ is a cyclic Heyting algebra and $k \in \omega$.*

Proof. The result follows immediately from Corollary 4.3.9 and the duality theory for Heyting algebras. □

4.4 The finite model property in extensions of RN

In this section we characterize finitely generated rooted **RN**-frames and prove that every extension of **RN** has the finite model property. First we show that **RN** is a proper extension of **KG**.

4.4.1. THEOREM.

1. $\mathbf{RN} \supsetneq \mathbf{KG}$.

2. $\mathcal{RN} \subsetneq \mathcal{KG}$.

Proof. (1) That none of $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$ is a subframe of $\mathfrak{L}$ is routine to check. Therefore, by Theorem 4.3.4, $\mathfrak{L}$ is a $\mathbf{KG}$-frame. Hence, $Log(\mathfrak{L}) = \mathbf{RN} \supseteq \mathbf{KG}$. Now we show that $\mathbf{RN} \neq \mathbf{KG}$. Consider the frame $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$. By Theorem 4.3.4, $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ is a rooted $\mathbf{KG}$-frame. On the other hand, by Corollary 4.2.10, $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ is not an $\mathbf{RN}$-frame. Consider the Jankov-de Jongh formula $\chi(\mathfrak{L}_{g_4} \oplus \mathbf{2}_*)$. Then $\chi(\mathfrak{L}_{g_4} \oplus \mathbf{2}_*) \in \mathbf{RN}$ and $\chi(\mathfrak{L}_{g_4} \oplus \mathbf{2}_*) \notin \mathbf{KG}$. Therefore, $\mathbf{RN} \not\subseteq \mathbf{KG}$.

(2) is an immediate consequence of (1). □

Therefore, by Corollary 4.3.9 and Theorem 4.4.1, every finitely generated rooted descriptive $\mathbf{RN}$-frame is a finite linear sum of cyclic frames. In this section we characterize those finitely generated rooted $\mathbf{KG}$-frames that are also $\mathbf{RN}$-frames.

4.4.2. THEOREM. *Let $\mathfrak{F}$ be a finitely generated descriptive rooted* $\mathbf{KG}$*-frame and $\mathfrak{A}$ a subdirectly irreducible Heyting algebra in $\mathcal{KG}$.*

1. *If $\mathfrak{F}$ is an* $\mathbf{RN}$*-frame, then there exist $k, n \in \omega$ such that $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathbf{2}_*$ or $\mathbf{4}_*$.*

2. *If $\mathfrak{A}$ belongs to $\mathcal{RN}$, then there exist $k, n \in \omega$ such that $\mathfrak{A}$ is isomorphic to $(\overline{\bigoplus}_{i=1}^{n} \mathfrak{B}_i) \overline{\oplus} \mathfrak{N}_{g_k}$, where each $\mathfrak{B}_i$ is isomorphic to $\mathfrak{N}$, $\mathbf{2}$ or $\mathbf{4}$.*

Proof. By Theorems 4.4.1 and 4.3.9, $\mathfrak{F}$ is isomorphic to a linear sum $(\bigoplus_{k=1}^{n} \mathfrak{G}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{G}_i$ is a cyclic frame and $k \in \omega$. If for every $j \leq n$ we have that $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}$, $\mathbf{2}_*$ or $\mathbf{4}_*$, then $\mathfrak{F}$ satisfies the condition of the theorem. Therefore, assume that there exists $j \leq n$, such that $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}_{g_m}$ for some $m \geq 4$ or $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}_{f_l}$ for some $l \geq 2$. (For $m < 4$ and $l < 2$ the frames $\mathfrak{L}_{g_m}$ and $\mathfrak{L}_{f_l}$ are isomorphic to linear sums of $\mathbf{2}_*$'s and $\mathbf{4}_*$'s.) Let $j \leq n$ be the the least such j. We show that $\mathfrak{F}$ is not an $\mathbf{RN}$-frame.

We first discuss the idea of the proof. We show that there exists a finite rooted $\mathfrak{F}'$ such that $\mathfrak{F}'$ is not an $\mathbf{RN}$-frame and $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{F}$. Thus, if $\mathfrak{F}$ is an $\mathbf{RN}$-frame then so is $\mathfrak{F}'$, which is a contradiction. To construct this p-morphism we consider a bisimulation equivalence on $\mathfrak{F}$ which identifies: all the points above $\mathfrak{G}_j$, all the points below $\mathfrak{G}_j$ and leaves the points of $\mathfrak{G}_j$ untouched. Then the resulting rooted frame is a p-morphic image of $\mathfrak{F}$, but by our characterization of finite rooted $\mathbf{RN}$-frames (see Theorem 4.2.10) it is not an $\mathbf{RN}$-frame.

We will now make this more precise. We only consider the case when $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}_{g_m}$. The proof for the other case is similar. Thus, assume $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}_{g_m}$, for some $m \geq 4$. Then two cases are possible:

Case 1. $1 < j \leq n$. We define an equivalence relation E on $\mathfrak{F}$ by

- wEv if $w = v$ for every $w, v \in \mathfrak{G}_j$,
- wEv if $w, v \in \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_{j-1}$,
- wEv if $w, v \in \mathfrak{G}_{j+1} \oplus \ldots \oplus \mathfrak{G}_n \oplus \mathfrak{L}_{g_k}$.

It is easy to check that E is a bisimulation equivalence and that $\mathfrak{F}/E$ is isomorphic to $\mathbf{2}_* \oplus \mathfrak{G}_j \oplus \mathbf{2}_*$. Since $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}_{g_m}$, we obtain that $\mathbf{2}_* \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}$. This, by Theorem 4.2.10, is a contradiction.

Case 2. $j = 1$. The proof is similar to that of Case 1, except that, in this case we obtain that $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}$, which again contradicts to Theorem 4.2.10.

(2) follows form (1) by the duality theory of Heyting algebras. □

To show that the converse of Theorem 4.4.2 also holds we need to define a new operation on frames.

4.4.3. DEFINITION.

1. Let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$ be two Kripke frames and let $x \in min(\mathfrak{F}_1)$ and $y \in max(\mathfrak{F}_2)$. The *gluing sum* of the pairs $(\mathfrak{F}_1, x)$ and $(\mathfrak{F}_2, y)$ is a frame $(\mathfrak{F}_1, x)\widehat{\oplus}(\mathfrak{F}_2, y) = (W \uplus W', S)$ such that $W \uplus W'$ is the disjoint union of W and W' and

$$S := R_1 \cup R_2 \cup (W_2 \times W_1 \setminus \{(y, x)\}).$$

2. Let $\mathfrak{F}_1 = (W_1, R_1, \mathcal{P}_1)$ and $\mathfrak{F}_2 = (W_2, R_2, \mathcal{P}_2)$ be descriptive frames and let $x \in min(\mathfrak{F}_1)$ and $y \in max(\mathfrak{F}_2)$. The *gluing sum* of $(\mathfrak{F}_1, x)$ and $(\mathfrak{F}_2, y)$ is the descriptive frame $(\mathfrak{F}_1, x)\widehat{\oplus}(\mathfrak{F}_2, y) = (W_1 \uplus W_2, S, \mathcal{P})$ such that $(W_1 \uplus W_2, S)$ is the gluing sum of $((W_1, R_1), x)$ and $((W_2, R_2), y)$ and

$$\mathcal{P} := \{U \subseteq W_1 \uplus W_2 : U \text{ is an } S\text{-upset and } U \cap W_1 \in \mathcal{P}_1 \text{ and } U \cap W_2 \in \mathcal{P}_2\}.$$

Figuratively speaking, we take the linear sum of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ and erase an arrow going from y to x. This definition is motivated by the next lemma, which states that we can "glue" two cyclic frames together in such a way that the resulting frame is again a cyclic frame. We will need the operation of gluing models for proving the main theorem of this section that every extension of **RN** has the fmp.

4.4.4. PROPOSITION. *The gluing sum of two descriptive frames is again a descriptive frame.*

Proof. The proof is just spelling out the definitions. □

For every $k \in \omega$ we assume that $\mathfrak{L}_{g_k}$ and $\mathfrak{L}_{f_k}$ are labeled as in Figure 4.1.

4.4.5. LEMMA. *Suppose $k, m \in \omega$ and m is odd. Then the following holds.*

1. $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}, w_0)$ *is isomorphic to* $\mathfrak{L}$.
2. $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_{g_k}, w_0)$ *is isomorphic to* $\mathfrak{L}_{g_{k+m}}$.

Proof. The proof is a routine check. □

Next we recall the definition of the complexity of a formula.

4.4.6. DEFINITION. We define the *complexity* $c(\phi)$ of a formula ϕ as follows:

$$\begin{aligned} c(p) &= 0, \\ c(\bot) &= 0, \\ c(\phi \wedge \psi) &= max\{c(\phi), c(\psi)\}, \\ c(\phi \vee \psi) &= max\{c(\phi), c(\psi)\}, \\ c(\phi \to \psi) &= 1 + max\{c(\phi), c(\psi)\}. \end{aligned}$$

Recall from the previous chapter that for every point x of a frame $\mathfrak{F}$ the depth of x is denoted by $d(x)$. Let U be an upset of $\mathfrak{F}$, then the *depth* $d(U)$ of U is defined as

$$d(U) := sup\{d(x) : x \in U\}.$$

4.4.7. DEFINITION. Let $V : \text{PROP}_n \to \mathfrak{L}$ be a descriptive valuation on $\mathfrak{L}$.

1. The *rank of* V is the number

$$rank(V) := max\{d(V(p_i)) : V(p_i) \subsetneq \mathfrak{L}\}.$$

2. For every formula $\phi(p_1, \ldots, p_n)$, let

$$M_V(\phi) = rank(V) + c(\phi) + 1.$$

4.4.8. LEMMA. *Let V be any descriptive valuation on $\mathfrak{L}$. Then for an arbitrary formula $\phi(p_1, \ldots, p_n)$ and for every $x, y \in \mathfrak{L}$ such that $d(x), d(y) > M_V(\phi)$, we have*

$$x \models \phi \text{ iff } y \models \phi.$$

Proof. We will prove the lemma by induction on the complexity of ϕ. If $c(\phi) = 0$, that is, ϕ is either $\bot$ or a propositional letter then the the lemma obviously holds. Now assume that $c(\phi) = k$ and the lemma is correct for every formula ψ such that $c(\psi) < k$. The cases when $\phi = \psi_1 \wedge \psi_2$ and $\phi = \psi_1 \vee \psi_2$ are trivial. So, suppose $\phi = \psi_1 \to \psi_2$, for some formulas ψ_1 and ψ_2. Clearly, $c(\psi_1), c(\psi_2) < k$. Let $x, y \in \mathfrak{L}$ be such that $d(x), d(y) > M_V(\phi)$. Without loss of generality assume $x \not\models \phi$ and show that $y \not\models \phi$. Then $x \not\models \psi_1 \to \psi_2$ implies that there exists x' such that xRx', $x' \models \psi_1$ and $x' \not\models \psi_2$. If $d(x') < d(y) - 1$, because of the structure of $\mathfrak{L}$, we have yRx' and so $y \not\models \phi$. If $d(x') \geq d(y) - 1$, then $d(x') > M_V(\phi) - 1 = rank(V) + c(\phi) \geq rank(V) + c(\psi_i) + 1 = M(\psi_i)$, for each $i = 1, 2$. Thus, $d(x'), d(y) > M(\psi_i)$ and by the induction hypothesis $y \models \psi_1$ and $y \not\models \psi_2$, which again implies $y \not\models \phi$. $\square$

Observe that if $c(\phi) > c(\psi)$, then $M_V(\phi) > M_V(\psi)$. The analogue of Lemma 4.4.9(1) is proved in Kracht [75].

4.4.9. LEMMA. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two distinct isomorphic copies of $\mathfrak{L}$. For an arbitrary formula $\phi(p_1, \ldots, p_n)$ the following holds.*

1. *If $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \not\models \phi$, then $\mathfrak{L} \not\models \phi$.*
2. *If $\mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{G} \not\models \phi$, for some frame $\mathfrak{G}$, then $\mathfrak{L} \oplus \mathfrak{G} \not\models \phi$.*
3. *If $\mathfrak{F} \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \not\models \phi$, for some frame $\mathfrak{F}$, then $\mathfrak{F} \oplus \mathfrak{L} \not\models \phi$.*
4. *If $\mathfrak{F} \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{G} \not\models \phi$, for some frames $\mathfrak{F}$ and $\mathfrak{G}$, then $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{G} \not\models \phi$.*
5. *If for some $k \in \omega$, $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \phi$, then $\mathfrak{L}_{g_m} \not\models \phi$, for some $m \geq k$.*
6. *If for some $k \in \omega$, $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G} \not\models \phi$, then $\mathfrak{L}_{g_m} \oplus \mathfrak{G} \not\models \phi$, for some $m \geq k$.*
7. *If for some $k \in \omega$ and some frame $\mathfrak{F}$, $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \phi$, then $\mathfrak{F} \oplus \mathfrak{L}_{g_m} \not\models \phi$, for some $m \geq k$.*
8. *If for some $k \in \omega$ and some frames $\mathfrak{G}$ and $\mathfrak{F}$, $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{F} \not\models \phi$, then $\mathfrak{F} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G} \not\models \phi$, for some $m \geq k$.*

Proof. (1) Let V be a descriptive valuation on $\mathfrak{L}_1 \oplus \mathfrak{L}_2$ such that $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V) \not\models \phi$. Let V_1 and V_2 be the restrictions of V to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. That is, $V_i(p) = V(p) \cap \mathfrak{L}_i$ for each $i = 1, 2$. Let $M_1(\phi) = rank(V_1) + c(\phi) + 1$ and let $m := 2 \cdot M_1(\phi) + 1$. Assume that on $\mathfrak{L}_1$ and $\mathfrak{L}_2$ we have the labeling shown in Figure 4.1. Consider the gluing sum $(\mathfrak{L}_{f_m}, w_m) \widehat{\oplus} (\mathfrak{L}_2, w_0)$ and let V' be the restriction of V to $(\mathfrak{L}_{f_m}, w_m) \widehat{\oplus} (\mathfrak{L}_2, w_0)$. By Lemma 4.4.5, $(\mathfrak{L}_{f_m}, w_m) \widehat{\oplus} (\mathfrak{L}_2, w_0)$ is isomorphic to $\mathfrak{L}$. Thus, to finish the proof we only need to show that $((\mathfrak{L}_{f_m}, w_m) \widehat{\oplus} (\mathfrak{L}_2, w_0), V') \not\models \phi$. The next claim finishes the proof.

4.4.10. Claim. *$((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), V') \not\models \phi$.*

Proof. We prove the claim by induction on the complexity of ϕ. The cases when ϕ is either $\bot$, a propositional variable, a conjunction or disjunction of two formulas are simple. Now let $\phi = \psi \to \chi$. Then since $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V) \not\models \phi$, there exists y in $\mathfrak{L}_1 \oplus \mathfrak{L}_2$ such that $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V), y \models \psi$ and $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V), y \not\models \chi$. If y belongs to $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$ then we are done. If y does not belong to $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$, then we take a point y' in $\mathfrak{L}_{f_m}$ of depth $M_1(\phi)$. Since $c(\psi), c(\chi) < c(\phi)$ we have $M_1(\psi), M_1(\chi) < M_1(\phi)$ and it follows from Lemma 4.4.8, that $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V), y' \models \psi$ and $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, V), y' \not\models \chi$. Therefore, $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), V'), y' \models \psi$ and $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), V'), y' \not\models \chi$. Thus, $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), V'), y' \not\models \phi$.

$\square$

(2) The proof is similar to (1).

(3),(4) The proof is similar to (1) and (2) with the only difference that in these cases instead of $\mathfrak{L}_{f_m}$ we should consider $\mathfrak{F} \oplus \mathfrak{L}_{f_m}$.

(5) The proof is similar to (1). We take the upset $\mathfrak{F}'$ consisting of $M_V(\phi)$ layers of $\mathfrak{L}$ and then consider a gluing sum of this frame with $\mathfrak{L}_{g_k}$.

(6), (7) and (8) are similar to (5).

$\square$

4.4.11. Lemma. *For every* **RN**-*frame $\mathfrak{F}$ there exist $k, m \in \omega$ such that $\mathfrak{F}$ is a p-morphic image of $(\bigoplus_{i=1}^{n} \mathfrak{L}_i) \oplus \mathfrak{L}_{g_k}$, where $\mathfrak{L}_i$ is an isomorphic copy of $\mathfrak{L}$, for each $i = 1, \ldots, m$.*

Proof. By Theorem 4.4.2, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where every $\mathfrak{F}_i$ is isomorphic to either $\mathfrak{L}$, $\mathbf{2}_*$ or $\mathbf{4}_*$. Let m be the number of copies of $\mathfrak{L}$ occurring in $\bigoplus_{i=1}^{n} \mathfrak{F}_i$. Then $\mathfrak{F}$ is isomorphic to a frame $\bigoplus_{i=1}^{m}(\bigoplus_{j=1}^{m_i}(\mathfrak{G}_j \oplus \mathfrak{L}_i)) \oplus \bigoplus_{j=1}^{s} \mathfrak{G}_j \oplus \mathfrak{L}_{g_k}$, for some $k \in \omega$, where each $\mathfrak{G}_j$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$ and $m_i \in \omega$. By Corollary 4.2.7, the frame $(\bigoplus_{j=1}^{m_i} \mathfrak{G}_j) \oplus \mathfrak{L}_i$ is a p-morphic image of $\mathfrak{L}_i$. On the other hand, by Theorem 4.2.12, the frame $(\bigoplus_{j=1}^{s} \mathfrak{G}_j) \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_{k+3s}}$. Therefore, $\mathfrak{F}$ is a p-morphic image of the frame $(\bigoplus_{i=1}^{m} \mathfrak{L}_i) \oplus \mathfrak{L}_{g_{k+3s}}$, where each $\mathfrak{L}_i$ is an isomorphic copy of $\mathfrak{L}$ for every $i = 1, \ldots, m$. $\square$

We are now ready to characterize the finitely generated rooted descriptive **RN**-frames and subdirectly irreducible algebras in $\mathcal{RN}$.

4.4.12. Theorem.

1. *A finitely generated rooted descriptive* **KG**-*frame $\mathfrak{F}$ is an* **RN**-*frame iff $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathfrak{L}$, $\mathbf{2}_*$ or $\mathbf{4}_*$ and $k \in \omega$.*

2. *A finitely generated subdirectly irreducible $\mathcal{KG}$-algebra $\mathfrak{A}$ belongs to $\mathcal{RN}$ iff $\mathfrak{A}$ is isomorphic to $(\overline{\bigoplus}_{i=1}^{n} \mathfrak{B}_i)\overline{\oplus}\mathfrak{N}_{g_k}$, where each $\mathfrak{B}_i$ is isomorphic to either $\mathfrak{L}$, $\mathbf{2}$ or $\mathbf{4}$, and $k \in \omega$.*

Proof. (1) The direction from left to right is proved in Theorem 4.4.2. For the other direction, by Lemma 4.4.11, it is sufficient to show that if a frame $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{m} \mathfrak{L}_i) \oplus \mathfrak{L}_{g_k}$, where $\mathfrak{L}_i$ is an isomorphic copy of $\mathfrak{L}$, for each $i = 1, \ldots, m$ and $k, m \in \omega$, then $\mathfrak{F}$ is an **RN**-frame.

We will prove that for every formula ϕ, if $\phi \in \mathbf{RN}$ then $\mathfrak{F} \models \phi$. So, assume that $\mathfrak{F} \not\models \phi$. By applying Lemma 4.4.9 (2), $(m-1)$ times, we obtain that $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \phi$. By Lemma 4.4.9 (5), there is $m \geq k$ such that $\mathfrak{L}_{g_m} \not\models \phi$. Thus, we found an **RN**-frame that refutes ϕ. Since $\phi \in \mathbf{RN}$, this is a contradiction. Hence, $\mathfrak{F} \models \phi$ for every $\phi \in \mathbf{RN}$ and therefore $\mathfrak{F}$ is an **RN**-frame.

(2) The result follows immediately from (1) by the duality theory of Heyting algebras. □

The next result was proved independently by Gerciu [48] and Kracht [73]. However, both proofs contain some gaps. We will provide a simple proof of this result. Our technique is very similar to the one from [73]. However, [73] claims that every extension of **KG** has the fmp, which, as we will see in the next section, is not the case.

4.4.13. Theorem.

1. *Every extension of* **RN** *has the finite model property.*
2. *Every subvariety of* $\mathcal{RN}$ *is finitely approximable.*

Proof. (1) Suppose $L \supseteq \mathbf{RN}$ and let $\phi \notin L$. Then there exists a finitely generated rooted descriptive L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \phi$. By Theorem 4.4.12, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{G}_i) \oplus \mathfrak{L}_{g_k}$, where every $\mathfrak{G}_i$ is isomorphic to $\mathfrak{L}$, $\mathbf{2}_*$ or $\mathbf{4}_*$ for every $i = 1, \ldots, n$. Let $j \leq n$ be the least such that $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}$. If such j does not exist then $\mathfrak{F}$ is finite and there is nothing to prove. Denote by $\mathfrak{G}'$, the finite frame $\mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_{j-1}$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{G}' \oplus \mathfrak{G}_j \oplus \ldots \oplus \mathfrak{G}_n \oplus \mathfrak{L}_{g_k}$. By Lemma 4.4.11, $\mathfrak{G}_j \oplus \ldots \oplus \mathfrak{G}_n \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of the frame $\mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_s \oplus \mathfrak{L}_{g_m}$, where $\mathfrak{L}_i$ is an isomorphic copy of $\mathfrak{L}$ for each $i = 1, \ldots, s$ and $s, m \in \omega$. Therefore, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_s \oplus \mathfrak{L}_{g_m}$. Since p-morphisms preserve the validity of formulas $\mathfrak{G} \not\models \phi$. Now we apply Lemma 4.4.9(4) and (7) to obtain a $t \geq m$ such that $\mathfrak{G}' \oplus \mathfrak{L}_{g_t} \not\models \phi$, for some $t \geq m$. To finish the proof it is sufficient to show that $\mathfrak{G}' \oplus \mathfrak{L}_{g_t}$ is an L-frame. To see this, observe that $\mathfrak{G}' \oplus \mathfrak{L}_{g_t}$ is a generated subframe of $\mathfrak{G}' \oplus \mathfrak{L}$, which is a generated subframe of $\mathfrak{F}$. Therefore, $\mathfrak{G}' \oplus \mathfrak{L}_{g_t}$ is an L-frame and thus L has the finite model property.

(2) The result follows immediately from (1). □

4.4.14. Remark. In fact, Theorem 4.4.13 can be strengthened. It is proved in [8] that every extension of **RN** has the poly-size model property. This means that every non-theorem of L can be refuted in a frame which has the size polynomial in the length of ϕ. It is also shown in [8] that for every function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, there exists an extension of **KG** that has the fmp but does not have the f-size model property.

4.5 The finite model property in extensions of KG

In this section we show that in extensions of **KG** the situation is completely different. We prove that there are continuum many extensions of **KG** without the finite model property. We also show that there is exactly one extension of **KG** that has the pre-finite model property.

4.5.1 Extensions of KG without the finite model property

First we discuss a systematic method of constructing logics without the fmp. Let $\mathfrak{G}$ be a finite rooted **KG**-frame that is not isomorphic to an **RN**-frame. The simplest such frame is $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$. Let $\mathfrak{H}$ be isomorphic to $\mathfrak{L} \oplus \mathfrak{G}$ and suppose $L = Log(\mathfrak{H})$. If $\mathfrak{G}$ is isomorphic to $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$, then the frame $\mathfrak{H}$ is isomorphic to the frame shown in Figure 4.4. We will prove that L lacks the finite model property. First we characterize the finite rooted L-frames.

4.5.1. THEOREM. *Let $\mathfrak{G}$ be a finite rooted **KG**-frame that is not isomorphic to an **RN**-frame. Let $\mathfrak{H}$ be isomorphic to $\mathfrak{L} \oplus \mathfrak{G}$ and suppose $L = Log(\mathfrak{H})$. A finite rooted **KG**-frame $\mathfrak{F}$ is an L-frame iff either of the following two conditions is satisfied.*

1. *$\mathfrak{F}$ is an **RN**-frame.*
2. *$\mathfrak{F}$ is isomorphic to a p-morphic image of a generated subframe of $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathbf{2}_* \oplus \mathfrak{G}$, where each $\mathfrak{F}_i$ is either empty or isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$.*

Proof. First we show that if a finite rooted frame satisfies the conditions of the theorem, then it is an L-frame. Since $\mathfrak{L}$ is a generated subframe of $\mathfrak{H}$ we have that every **RN**-frame is an L-frame. By Theorem 4.2.7, every frame of the form $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathbf{2}_*$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$ or $\mathbf{4}_*$, is a p-morphic image of $\mathfrak{L}$. Therefore, $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathbf{2}_* \oplus \mathfrak{G}$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{G}$. Thus if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathbf{2}_* \oplus \mathfrak{G}$, then $\mathfrak{F}$ is an L-frame.

Conversely, let $\mathfrak{F}$ be a finite rooted L-frame. Then by Lemma 3.4.26, $\mathfrak{F}$ is a p-morphic image of a generated subframe $\mathfrak{H}'$ of $\mathfrak{H}$. If $\mathfrak{H}'$ is a generated subframe of $\mathfrak{L}$, then $\mathfrak{F}$ is an **RN**-frame. Now suppose that $\mathfrak{H}'$ is isomorphic to $\mathfrak{L} \oplus \mathfrak{H}''$, where $\mathfrak{H}''$ is a generated subframe of $\mathfrak{G}$. By Theorem 4.2.7, every finite p-morphic image of $\mathfrak{L}$ has the form $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathbf{2}_*$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$

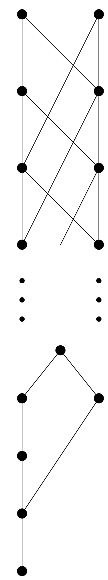

Figure 4.4: The frame $\mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$

or $\mathbf{4}_*$. Thus, if $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}'$, then $\mathfrak{F}$ is a p-morphic image of $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{H}''$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$ or $\mathbf{4}_*$. It is easy to see that $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{H}''$ is a generated subframe of $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{G}$.

□

4.5.2. Theorem. *Let $\mathfrak{G}$ be a finite rooted* **KG***-frame that is not isomorphic to an* **RN***-frame. Let $\mathfrak{H}$ be isomorphic to $\mathfrak{L} \oplus \mathfrak{G}$ and suppose $L = Log(\mathfrak{H})$. Then L does not have the finite model property.*

Proof. Consider the Jankov-de Jongh formulas $\chi_1 = \chi(\mathbf{2}_* \oplus \mathfrak{G})$ and $\chi_2 = \chi(\mathfrak{L}_{g_4})$ with separated variables. Let $\phi = \chi_1 \vee \chi_2$. It is easy to see that $\mathbf{2}_* \oplus \mathfrak{G}$ is a p-morphic image of $\mathfrak{H}$ (we just map all the points in $\mathfrak{L}$ to the top node of $\mathbf{2}_* \oplus \mathfrak{G}$). Hence, $\mathfrak{H} \not\models \chi_1$. Obviously, $\mathfrak{L}_{g_4}$ is a generated subframe of $\mathfrak{H}$. This means that $\mathfrak{H} \not\models \chi_2$. Therefore, $\mathfrak{H} \not\models \phi$. Now suppose there is a finite rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \phi$, whence $\mathfrak{F} \not\models \chi_1$ and $\mathfrak{F} \not\models \chi_2$. Then $\mathfrak{F} \not\models \chi_1$ implies that $\mathbf{2}_* \oplus \mathfrak{G}$ is a p-morphic image of a generated subframe of $\mathfrak{F}$. Hence, if $\mathfrak{F}$ is an **RN**-frame, then $\mathbf{2}_* \oplus \mathfrak{G}$ is also an **RN**-frame, which is a contradiction, by Theorem 4.2.10. Thus, $\mathfrak{F} \not\models \chi_1$ implies $\mathfrak{F}$ is not an **RN**-frame. By (2) of Theorem 4.5.1, this means that $\mathfrak{F}$ is a p-morphic image of some $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{H}''$, where $\mathfrak{H}''$ is a generated subframe of $\mathfrak{G}$ and each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$ or $\mathbf{4}_*$.

Next we show that $\mathfrak{L}_{g_4}$ cannot be a p-morphic image of a generated subframe of $\mathfrak{F}$. Let $\mathfrak{F}'$ be a generated subframe of $\mathfrak{F}$ and $f : \mathfrak{F}' \to \mathfrak{L}_{g_4}$ be a p-morphism. If $|max(\mathfrak{F}')| = 1$, then clearly $\mathfrak{L}_{g_4}$ cannot be a p-morphic image of $\mathfrak{F}'$. Now suppose $\mathfrak{F}'$ has two maximal points u_1 and u_2. Then $f(u_1) \neq f(u_2)$ and $f(u_1)$ and $f(u_2)$

are the maximal points of $\mathfrak{L}_{g_4}$. Let u be a point of the second layer of $\mathfrak{F}'$. Then since the top layers of $\mathfrak{F}'$ are sums of $\mathbf{2}_*$'s and $\mathbf{4}_*$'s we have uRu_1 and uRu_2. Therefore, $f(u) \neq f(u_1)$ and $f(u) \neq f(u_2)$. But then u should be mapped to a point of the second layer of $\mathfrak{L}_{g_4}$, which consists of only one point. This implies that this point of the second layer of $\mathfrak{L}_{g_4}$ sees both maximal points of $\mathfrak{L}_{g_4}$, which is a contradiction. Hence, no generated subframe of $\mathfrak{F}$ can be p-morphically mapped onto $\mathfrak{L}_{g_4}$, and $\mathfrak{F} \models \chi_2$. This contradicts our earlier assumption that $\mathfrak{F} \not\models \chi_2$. Thus, there is no finite L-frame that refutes both χ_1 and χ_2. This means that L does not have the finite model property. □

Next we show that there are continuum many extensions of **KG** without the finite model property. To construct these extensions we will need to construct infinite antichains of finite **KG**-frames.

Recall from the previous chapter that for every frame $\mathfrak{F}$ and $\mathfrak{G}$:

$$\mathfrak{F} \leq \mathfrak{G} \text{ iff } \mathfrak{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathfrak{G}.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are Heyting algebras. Then

$$\mathfrak{A} \leq \mathfrak{B} \text{ iff } \mathfrak{A} \text{ is a subalgebra of a homomorphic image of } \mathfrak{B}.$$

In the next lemma we show how to construct antichains of finite **KG** and **RN**-frames. These antichains will allow us to show that both **KG** and **RN** have continuum many extensions. The antichain in Lemma 4.5.3(3) was constructed by Kracht [73].

4.5.3. Lemma.

1. *The sequence* $\Gamma = \{\mathfrak{L}_{g_k} \oplus \mathbf{2}_* : k \geq 4\}$ *of rooted* **KG***-frames forms an* $\leq$*-antichain.*
2. *The sequences* $\Delta_1 = \{\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_* : k \geq 4$ *and* k *is even*$\}$ *and* $\Delta_2 = \{\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_* : k \geq 5$ *and* k *is odd*$\}$ *of rooted* **KG***-frames form* $\leq$*-antichains.*
3. $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_* \not\leq (\bigoplus_{i=1}^{n} \mathfrak{F}_i)\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ *where each* $\mathfrak{F}_i$ *is isomorphic to* $\mathbf{2}_*$ *or* $\mathbf{4}_*$ *and* $k \neq m$.
4. *The sequences* $\Upsilon_1 = \{\bigoplus_{i=1}^{k} \mathbf{4}_* \oplus \mathfrak{L}_{g_4} : k \in \omega\}$ *and* $\Upsilon_2 = \{\bigoplus_{i=1}^{k} \mathbf{4}_* \oplus \mathfrak{L}_{g_5} : k \in \omega\}$ *of rooted* **RN***-frames form* $\leq$*-antichains.*

Proof. (1) Consider any two frames $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ and $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ in Γ and suppose $m > k$. Then $|\mathfrak{L}_{g_k} \oplus \mathbf{2}_*| < |\mathfrak{L}_{g_m} \oplus \mathbf{2}_*|$ and clearly $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ cannot be a p-morphic image of a generated subframe of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$. Now suppose there exists a generated subframe $\mathfrak{H}$ of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ such that $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{H}$. If $\mathfrak{H}$ is a proper generated subframe of $\mathfrak{L}_m \oplus \mathbf{2}_*$, then $\mathfrak{H}$ is an **RN**-frame. On the other

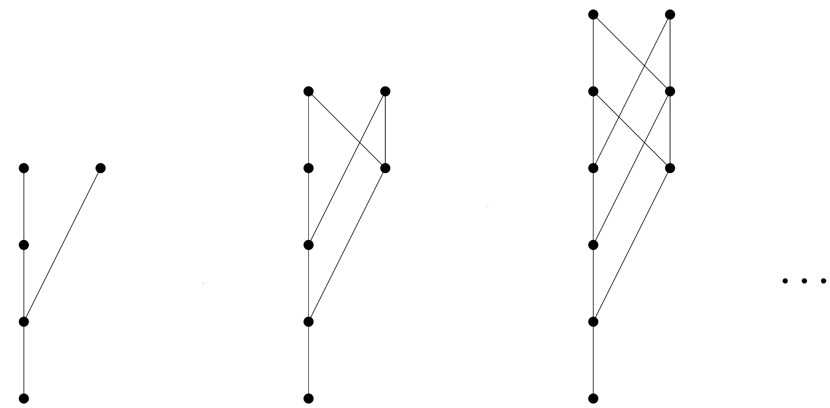

Figure 4.5: The frames $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$, $\mathfrak{L}_{g_6} \oplus \mathbf{2}_*$ and $\mathfrak{L}_{g_8} \oplus \mathbf{2}_*$

hand, by Theorem 4.2.10, $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ is not an **RN**-frame and therefore cannot be a p-morphic image of $\mathfrak{H}$. Thus, $\mathfrak{H}$ is isomorphic to $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ and $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$. Then the least point of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ is mapped to the least point of $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ and no other point of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ is mapped to the least point of $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$; otherwise $\mathfrak{L}_k \oplus \mathbf{2}_*$ would be a p-morphic image of $\mathfrak{L}_{g_m}$ which is a contradiction, since $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is not an **RN**-frame. This gives us that $\mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_m}$, which is a contradiction by Lemma 4.2.13.

(2) We prove that Δ_1 is a $\leq$-antichain. Suppose for $m > k$ we have that $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphic image of a generated subframe of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$. Then there exist $\mathfrak{H}$ and f such that $\mathfrak{H}$ is a generated subframe of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ and $f : \mathfrak{H} \to \mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphism. Obviously $\mathfrak{H}$ contains the first three layers of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$. Moreover, the restriction of f to the first three layers of $\mathfrak{H}$ is an isomorphism. To see this, observe that if not then the first three layers of $\mathfrak{H}$ should be mapped to the top point of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$. Then $\mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ would be a p-morphic image of $\mathbf{2}_* \oplus \mathfrak{L}_{g_m}$. This implies that $\mathfrak{L}_{g_k}$ is a p-morphic image $\mathfrak{L}_{g_m}$, which contradicts Theorem 4.2.12. Hence, the restriction of f to the first three layers of $\mathfrak{H}$ is an isomorphism. Then there exists a point x in $\mathfrak{H}$ of depth $d(x) > 3$ such that $d(f(x)) \leq 3$. Otherwise, $\mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphic image of a generated subframe of $\mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ which, by (1), is a contradiction. For every point y of $\mathfrak{H}$ of depth $d(y) \leq 3$ we have that xRy and hence $f(x)Rf(y)$. This is a contradiction because for every point u of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ of depth ≤ 3 there exists a point z of depth ≤ 3 such that $\neg(uRz)$. Therefore, there is no generated subframe of $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$ that can be p-morphically mapped onto $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$.

(3) The proof is a routine adaptation of the proof of (2).

(4) The proof is similar to (1) and (2) and is based on the fact that there is no p-morphism from $\bigoplus_{i=1}^{n} \mathbf{4}$ onto $\bigoplus_{i=1}^{m} \mathbf{4}$ for $m, n \in \omega$ and $m \neq n$.

$\square$

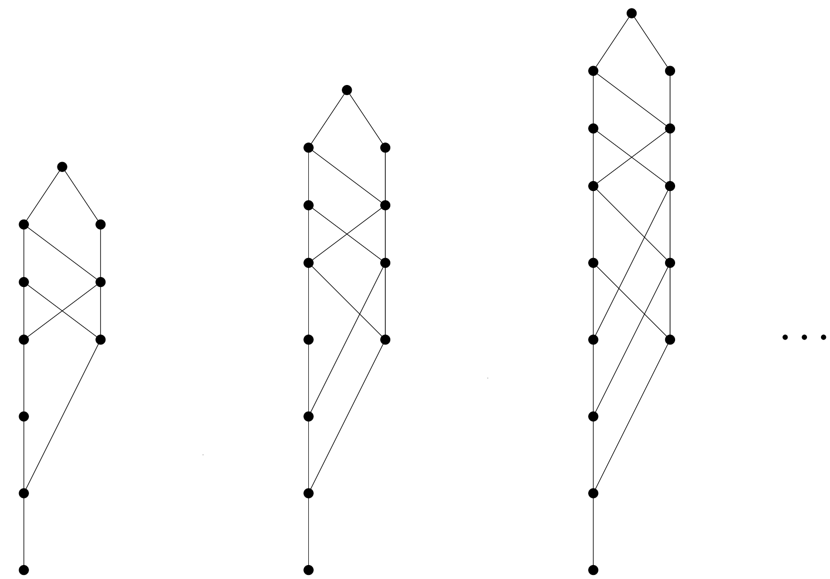

Figure 4.6: The frames in Δ_1

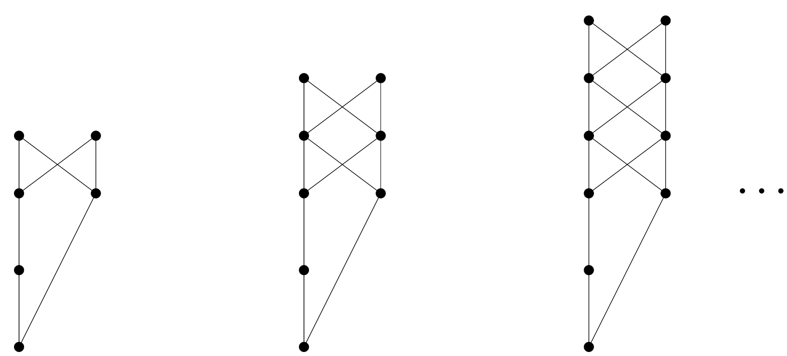

Figure 4.7: The frames in Υ_1

4.5.4. THEOREM.

1. *There are continuum many extensions of* **RN**.

2. *There are continuum many subvarieties of* $\mathcal{RN}$.

3. *There are continuum many extensions of* **KG** *with the finite model property.*

4. *There are continuum many finitely approximable subvarieties of* $\mathcal{KG}$.

Proof. The theorem is proved in the same way as Lemma 3.4.19 and Theorem 3.4.20. $\square$

Denote by $\mathfrak{H}_k$ the frame $\mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{L}$, where $k \geq 4$ is even. Let $\Theta = \{\mathfrak{H}_k : k$ is even $\geq 4\}$. For every subset $\Theta' \subseteq \Theta$, let $Log(\Theta') = \bigcap\{Log(\mathfrak{H}_k) : \mathfrak{H}_k \in \Theta'\}$.

4.5.5. THEOREM.

1. *$Log(\mathfrak{H}_k)$ lacks the finite model property, for every $k \geq 4$.*

2. *For every $\Theta' \subseteq \Theta$, the logic $Log(\Theta')$ lacks the finite model property.*

3. *For every $\Theta_1, \Theta_2 \subseteq \Theta$, such that $\Theta_1 \neq \Theta_2$ we have $Log(\Theta_1) \neq Log(\Theta_2)$.*

Proof. (1) The result follows immediately from Theorem 4.5.2, since $\mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is not an **RN**-frame.

(2) First we show that a finite rooted frame $\mathfrak{F}$ is a $Log(\Theta')$-frame iff $\mathfrak{F}$ is a $Log(\mathfrak{H}_k)$-frame for some $\mathfrak{H}_k \in \Theta'$. Let $\mathfrak{F}$ be a finite rooted $Log(\Theta')$-frame. Let $Log(\mathfrak{F})$ be the logic of $\mathfrak{F}$. Then $Log(\mathfrak{F}) \supseteq Log(\Theta')$. Since in the lattice of intermediate logics the logic of a finite rooted frame is a splitting [24, Theorem 10.49] and every splitting is completely meet irreducible[4] [24, Proposition 10.46] we have that there is $\mathfrak{H}_k \in \Theta'$ such that $Log(\mathfrak{F}) \supseteq Log(\mathfrak{H}_k)$. This means that $\mathfrak{F}$ is a $Log(\mathfrak{H}_k)$-frame. Now the same technique as in the proof of Theorem 4.5.2 and (1) shows that $Log(\Theta')$ lacks the fmp for every $\Theta' \subseteq \Theta$.

(3) Suppose $\Theta_1, \Theta_2 \subseteq \Theta$, such that $\Theta_1 \neq \Theta_2$. Without loss of generality assume that there is $\mathfrak{H}_k \in \Theta_1$ and $\mathfrak{H}_k \notin \Theta_2$. Then it is easy to see that $\mathfrak{G}_k := \mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{H}_k$ and hence $\mathfrak{G}_k$ is a $Log(\Theta_1)$-frame. Suppose $\mathfrak{G}_k$ is a $Log(\Theta_2)$-frame. Then, as was shown in (2), there exists $\mathfrak{H}_m \in \Theta_2$ such that $m \neq k$ and $\mathfrak{G}_k$ is a $Log(\mathfrak{H}_m)$-frame. Similarly to Theorem 4.5.1 we can show that all finite rooted frames of $Log(\mathfrak{H}_m)$ are finite rooted **RN**-frames or p-morphic images of generated subframes of $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathbf{2}_*$ or $\mathbf{4}_*$. Then $\mathfrak{G}_k$ is a p-morphic image of a generated subframe of $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathbf{2}_* \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathbf{2}_*$, which is a contradiction by Lemma 4.5.3(3). Therefore, $\mathfrak{G}_k$ is not a $Log(\Theta_2)$-frame. Then the Jankov-de Jongh formula of $\mathfrak{G}_k$ belongs to $Log(\Theta_2)$ but does not belong to $Log(\Theta_1)$. Thus, $Log(\Theta_1) \neq Log(\Theta_2)$. $\square$

[4]Recall that an element a of a lattice λ is called *completely meet irreducible* if $\bigwedge_{i \in I} b_i \leq a$ implies that there is $i_0 \in I$ such that $b_{i_0} \leq a$.

4.5.6. COROLLARY.

1. *There are continuum many extensions of* **KG** *without the finite model property.*

2. *There are continuum many subvarieties of* $\mathcal{KG}$ *that are not finitely approximable.*

Proof. The proof follows immediately from Theorem 4.5.5. □

4.5.2 The pre-finite model property

We will now characterize the logics that bound the finite model property in extensions of **KG**.

4.5.7. DEFINITION. A logic L is said to have the *pre-finite model property* if L does not have the fmp, but all proper extensions of L do have the fmp.

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ denote the frames $\mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{L} \oplus \mathbf{2}_*$ and $\mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{L} \oplus \mathbf{2}_*$, respectively. The frames $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are shown in Figure 4.8.

4.5.8. LEMMA.

1. $\mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ *is a p-morphic image of* $\mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$.

2. $\mathfrak{T}_1$ *is a p-morphic image of* $\mathfrak{T}_2$.

Proof. (1) Let $\mathbf{2}_*\oplus\mathfrak{L}_{g_4}\oplus\mathbf{2}_*$ and $\mathbf{2}_*\oplus\mathfrak{L}_{g_5}\oplus\mathbf{2}_*$ be labeled as it is shown in Figure 4.9. Define a map $f : \mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_* \to \mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ by putting: $f(y_i) = x_i$, for every $i = 1, \ldots, 5$ and $f(y_6) = x_5$. It is now easy to check that f is a p-morphism.

(2) The proof is a simple adaptation of the proof of (1). □

The next theorem was first established by Gerciu [48]. However, his proof was very sketchy. Here we give a full proof of this result skipping just some technical details.

4.5.9. THEOREM. *Let* $L \supseteq$ **KG**.

1. *If* L *does not have the fmp, then* $L \subseteq Log(\mathfrak{T}_1)$.

2. $Log(\mathfrak{T}_1)$ *is the only extension of* **KG** *with the pre finite model property.*

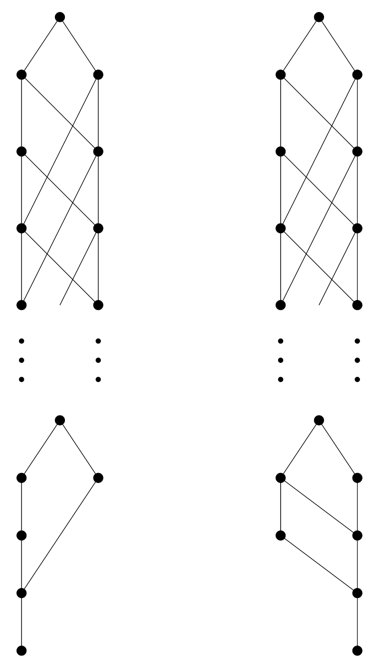

Figure 4.8: The frames $\mathfrak{T}_1$ and $\mathfrak{T}_2$

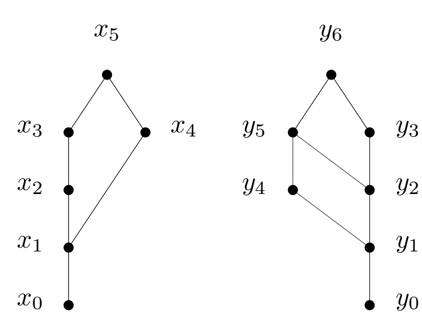

Figure 4.9: The frames $\mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ and $\mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$ with the labels

Proof. (1) Suppose $L \supseteq \mathbf{KG}$ and L does not have the finite model property. Then there is a formula ϕ such that $L \not\vdash \phi$ and for every finite L-frame $\mathfrak{G}$ we have $\mathfrak{G} \models \phi$. By Corollary 3.4.3, there is a finitely generated, rooted descriptive L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \phi$. By our assumption, $\mathfrak{F}$ is infinite. This implies that $Log(\mathfrak{F})$ also lacks the fmp. Obviously, we have $L \subseteq Log(\mathfrak{F})$. Hence, to prove that $L \subseteq Log(\mathfrak{T}_1)$, it is sufficient to show that $Log(\mathfrak{F}) \subseteq Log(\mathfrak{T}_1)$. We will prove this by showing that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$.

By Theorem 4.3.10, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where $k, n \in \omega$ and each $\mathfrak{F}_i$ is a cyclic frame. Since $\mathfrak{F}$ is infinite, there is $j \leq n$ such that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$. Let j be the least such. First suppose $j > 1$. This means that $\mathfrak{F}$ is isomorphic to $\mathfrak{G} \oplus \mathfrak{F}_j \oplus \mathfrak{F}_{j-1} \oplus \ldots \oplus \mathfrak{F}_n \oplus \mathfrak{L}_{g_k}$, where $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{G}$ is a finite frame. If there is no i with $n \geq i \geq j-1$ such that $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ or $\mathfrak{L}_{f_l}$ for some $m \geq 4$ and $l \geq 2$, then the same argument as in the proof Theorem 4.4.13 shows that $Log(\mathfrak{F})$ has the fmp, which is a contradiction. So, there is such i and we take the least. Then two cases are possible: $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$, for $m \geq 4$ or $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{f_l}$, for $l \geq 2$. We only consider the case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$, for $m \geq 4$. The case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{f_l}$, for $l \geq 2$ is similar. Next we define an equivalence relation on $\mathfrak{F}$ which leaves $\mathfrak{F}_i$ and $\mathfrak{F}_j$ untouched and identifies all the points above $\mathfrak{F}_j$, all the points below $\mathfrak{F}_i$ and all the points between $\mathfrak{F}_i$ and $\mathfrak{F}_j$. Now we will define this relation more precisely. Let E be an equivalence relation on $\mathfrak{F}$ such that for every $w, v \in \mathfrak{F}$:

- wEv if $w, v \in \mathfrak{G}$,
- wEv if $w = v$, for $w, v \in \mathfrak{F}_j$,
- wEv if $w, v \in \mathfrak{F}_{j-1} \oplus \ldots \oplus \mathfrak{F}_{i-1}$,
- wEv if $w = v$, for $w, v \in \mathfrak{F}_i$,
- wEv if $w, v \in \mathfrak{F}_{i+1} \oplus \ldots \oplus \mathfrak{L}_{g_k}$.

Then E is a bisimulation equivalence and $\mathfrak{F}/E$ is isomorphic to $\mathbf{2}_* \oplus \mathfrak{F}_j \oplus \mathbf{2}_* \oplus \mathfrak{F}_i \oplus \mathbf{2}_*$, where $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for $m \geq 4$. Suppose $m \geq 4$. Looking at the structure of $\mathfrak{L}_{g_m}$ we see that if m is even, then the subframe of $\mathfrak{L}_{g_m}$ consisting of the last three layers of $\mathfrak{L}_{g_m}$ is isomorphic to $\mathfrak{L}_{g_4}$, and if m is odd, then the subframe of $\mathfrak{L}_{g_m}$ consisting of the last three layers of $\mathfrak{L}_{g_m}$ is isomorphic to $\mathfrak{L}_{g_5}$. Therefore, if m is even and $m > 4$ then by identifying all but the points of the last three layers of $\mathfrak{L}_{g_m}$ we obtain a p-morphic image of $\mathfrak{L}_{g_m}$ that is isomorphic to $\mathbf{2}_* \oplus \mathfrak{L}_{g_4}$ and if m is odd and $m > 5$ then by identifying all but the points of the last three layers of $\mathfrak{L}_{g_m}$ we obtain a p-morphic image of $\mathfrak{L}_{g_m}$ that is isomorphic to $\mathbf{2}_* \oplus \mathfrak{L}_{g_5}$. Thus, if $m > 4$ and m is even the frame $\mathfrak{H}_1 := \mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_* \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}/E$ and if $m > 5$ and

m is odd the frame $\mathfrak{H}_2 := \mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_* \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}/E$. Clearly, if $m = 4$, then $\mathfrak{F}/E$ is isomorphic to $\mathfrak{H}_1' := \mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ and if $m = 5$, then $\mathfrak{F}/E$ is isomorphic to $\mathfrak{H}_2 := \mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$. It is easy to see that $\mathfrak{H}_1'$ is a p-morphic image of $\mathfrak{H}_1$ and that $\mathfrak{H}_2'$ is a p-morphic image of $\mathfrak{H}_2$. Now by identifying the greatest element of $\mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ with the least point of $\mathfrak{L} \oplus \mathbf{2}_*$ we obtain that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{H}_1'$. Exactly the same argument shows that $\mathfrak{T}_2$ is a p-morphic image of $\mathfrak{H}_2'$. Finally, Lemma 4.5.8(2) ensures that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{T}_2$, which means that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$.

The proof in case $j = 1$ is analogous, with the only difference that we also use that, by Theorem 4.2.7, $\mathbf{2}_* \oplus \mathfrak{L}$ is a p-morphic image of $\mathfrak{L}$, and hence $\mathbf{2}_* \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ and $\mathbf{2}_* \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$. Therefore, $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$ and $Log(\mathfrak{T}_1) \supseteq Log(\mathfrak{F})$.

(2) Suppose L has the pre-fmp. Then by (1) $L \supseteq Log(\mathfrak{T}_1)$. If $L \supsetneq Log(\mathfrak{T}_1)$, then L does not have the pre-fmp. Therefore, $L = Log(\mathfrak{T}_1)$. □

4.5.3 The axiomatization of RN

First we show that **RN** is not a subframe logic and hence by Theorem 3.4.16, cannot be axiomatized by subframe formulas. Denote by $\mathfrak{K}_4$, $\mathfrak{K}_5$ and $\mathfrak{K}_6$ the frames $\mathfrak{L}_{g_4} \oplus \mathbf{2}_*$, $\mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$, and $\mathfrak{L}_{g_5} \oplus \mathbf{2}_*$, respectively (see Figure 4.11).

4.5.10. THEOREM. *The following holds.*

1. **RN** *is not a subframe logic.*
2. **RN** *is not a cofinal subframe logic.*

Proof. By Theorem 4.2.10, neither $\mathfrak{K}_4$ nor $\mathfrak{K}_6$ is an **RN**-frame. However, as follows from Figure 4.10, both $\mathfrak{K}_4$ and $\mathfrak{K}_6$ are subframes of $\mathfrak{L}$. Moreover, they are cofinal subframes. Therefore, **RN** is neither a subframe logic nor a cofinal subframe logic.[5]

□

Next we show that **RN** is finitely axiomatizable by subframe formulas and Jankov-de Jongh formulas. That **RN** is finitely axiomatizable was first shown by Kuznetsov and Gerciu [83] without using these formulas. Kracht [73] gave an axiomatization of **RN** by subframe and Jankov-de Jongh formulas. However, the formula $\chi(\mathfrak{K}_6)$, see below, is missing in his axiomatization. Consider the frames $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$ shown in Figure 4.11 and let $\mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6$ be the corresponding Heyting algebras shown in Figure 4.12. Recall that the frames $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$ are shown in Figure 4.3.

[5] For proving the theorem it is of course sufficient to find one (cofinal) subframe of $\mathfrak{L}$ that is not an **RN**-frame. However, both frames $\mathfrak{K}_4$ and $\mathfrak{K}_6$ play an important role in our investigations and it is useful to know that, in fact, both of them are subframes of $\mathfrak{L}$.

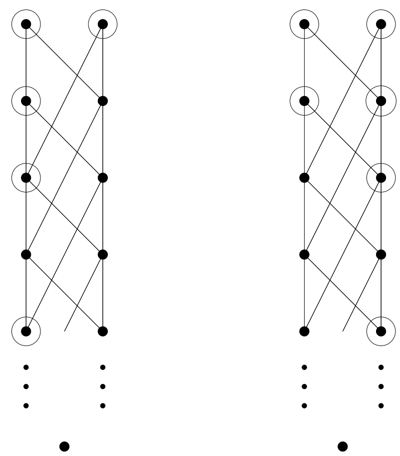

Figure 4.10: Subframes of $\mathfrak{L}$

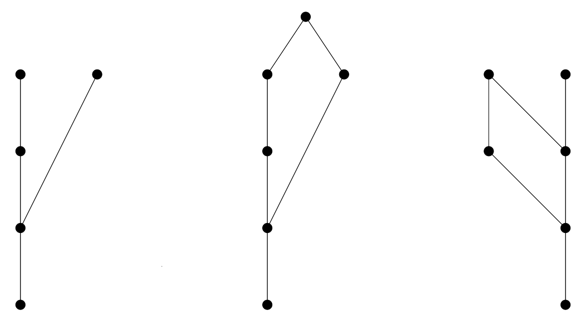

Figure 4.11: The frames $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$

4.5.11. THEOREM.

1. (a) $\mathbf{RN} = \mathbf{IPC} + \bigwedge_{i=1}^{3} \beta(\mathfrak{K}_i) + \bigwedge_{i=4}^{6} \chi(\mathfrak{K}_i)$.
   (b) $\mathcal{RN} = \mathcal{HA} + [\bigwedge_{i=1}^{3} \beta(\mathfrak{K}_i) = 1] + [\bigwedge_{i=4}^{6} \chi(\mathfrak{A}_i) = 1])$.
2. (a) $\mathbf{RN} = \mathbf{IPC} + \phi_{KG} + \bigwedge_{i=4}^{6} \chi(\mathfrak{K}_i)$.
   (b) $\mathcal{RN} = \mathcal{HA} + [\phi_{KG} = 1] + [\bigwedge_{i=4}^{6} \chi(\mathfrak{A}_i) = 1]$.

Proof. (1) As was mentioned above $\mathbf{RN} \supseteq \mathbf{KG}$. Moreover, by Theorem 4.2.10 none of $\mathfrak{K}_i$ for $i = 4, 5, 6$ is an **RN**-frame. We first prove the following claim.

4.5.12. CLAIM. *A finitely generated rooted* **KG**-*frame* $\mathfrak{F}$ *is an* **RN**-*frame iff* $\mathfrak{K}_i \not\leq \mathfrak{F}$, *for each* $i = 4, 5, 6$. [6]

[6] In terms of the previous chapter, this means that for every $\mathfrak{F} \in \mathbb{FG}(\mathbf{KG}) \setminus \mathbb{FG}(\mathbf{RN})$ there exists some $i = 4, 5, 6$ such that $\mathfrak{K}_i \leq \mathfrak{F}$.

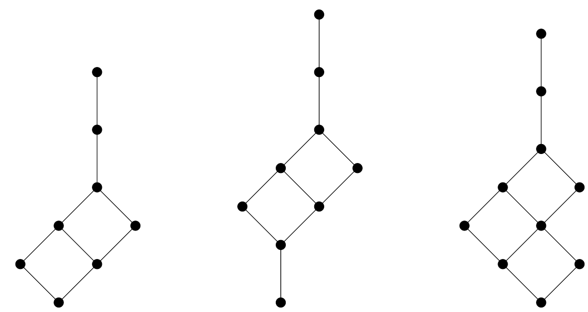

Figure 4.12: The algebras $\mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6$

Proof. Suppose $\mathfrak{F}$ is an **RN**-frame and $\mathfrak{K}_i \leq \mathfrak{F}$, for each $i = 4, 5, 6$. Then the $\mathfrak{K}_i$s are also **RN**-frames, for every $i = 4, 5, 6$, which is a contradiction by Theorem 4.2.10.

Conversely, since $\mathfrak{F}$ is a **KG**-frame, by Theorem 4.3.9, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where all $\mathfrak{F}_i$'s are cyclic frames. As $\mathfrak{F}$ is not an **RN**-frame, by Theorem 4.4.12, there exists $i \leq n$ such that $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ or $\mathfrak{L}_{f_l}$, for some $m \geq 4$ and $l \geq 2$. We take the least such i. We again consider the case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for some $m \geq 4$. The proof for the other case is similar. We define a bisimulation equivalence that identifies all the points above $\mathfrak{F}_i$, identifies all the points below $\mathfrak{F}_i$ and leaves the points of $\mathfrak{F}_i$ untouched. Now we define this relation more precisely. Two cases are possible.

Case 1. $i > 1$. Define an equivalence relation E on $\mathfrak{F}$ by putting for every $w, v \in \mathfrak{F}$:

- wEv for $w, v \in \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_{i-1}$,
- wEv if $w = v$, for $w, v \in \mathfrak{F}_i$,
- wEv for $w, v \in \mathfrak{F}_{i+1} \oplus \ldots \oplus \mathfrak{F}_n \oplus \mathfrak{L}_{g_k}$.

Then E is a bisimulation equivalence and $\mathfrak{F}/E$ is isomorphic to $\mathbf{2}_* \oplus \mathfrak{F}_i \oplus \mathbf{2}_*$. Next we apply exactly the same argument as in the proof of Theorem 4.5.9. If $m > 4$ is even then $\mathbf{2}_* \oplus \mathfrak{L}_{g_4}$ is a p-morphic image of $\mathfrak{L}_{g_m}$ and if $m > 4$ is odd, then $\mathbf{2}_* \oplus \mathfrak{L}_{g_5}$ is a p-morphic image of $\mathfrak{L}_{g_m}$. Therefore, if $m > 4$ and m is even then $\mathfrak{G}_1 := \mathbf{2}_* \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}/E$ and if $m > 4$ is odd then $\mathfrak{G}_2 := \mathbf{2}_* \oplus \mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}/E$. Clearly, if $m = 4$, $\mathfrak{F}/E$ is isomorphic to $\mathfrak{K}_5 = \mathbf{2}_* \oplus \mathfrak{L}_{g_4} \oplus \mathbf{2}_*$ and if $m = 5$ then $\mathfrak{F}/E$ is isomorphic to $\mathfrak{K}'_5 = \mathbf{2}_* \oplus \mathfrak{L}_{g_5} \oplus \mathbf{2}_*$ is a p-morphic image of $\mathfrak{F}/E$. It is easy to see that $\mathfrak{K}_5$ is a p-morphic image of $\mathfrak{G}_1$ and $\mathfrak{K}'_5$ is a p-morphic image of $\mathfrak{G}_2$. Finally, by Lemma 4.5.8, $\mathfrak{K}_5$ is a p-morphic image of $\mathfrak{K}'_5$, which gives us that $\mathfrak{K}_5$ is a p-morphic image of $\mathfrak{F}$.

Case 2. $i = 1$. This case is similar to Case 1, except that if $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_4}$, then $\mathfrak{K}_4$ is a p-morphic image of $\mathfrak{F}$ and if $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_5}$ then $\mathfrak{K}_6$ is a p-morphic image of $\mathfrak{F}$.

$\square$

The result now follows from Corollary 3.4.14 by replacing $\mathbb{FG}(\mathbf{IPC})$ with $\mathbb{FG}(\mathbf{KG})$.

(2) is an immediate consequence of (1), Theorem 4.3.4 and Corollary 4.3.5.

$\square$

4.6 Locally tabular extensions of RN and KG

In this section we give criteria of local tabularity in extensions of **KG** and **RN**. For the definition of locally tabular intermediate logics and locally finite varieties of Heyting algebras consult Sections 2.1.2 and 2.3.5.

4.6.1. Definition.

1. A logic L is called *pre-locally tabular* if L is not locally tabular but every proper extension of L is locally tabular.

2. A variety $\mathbf{V}$ is called *pre-locally finite* if $\mathbf{V}$ is not locally finite but every proper subvariety of $\mathbf{V}$ is locally finite.

Pre-local tabularity and pre-local finiteness are dual notions. That is, an intermediate logic is pre-locally tabular iff the corresponding variety of Heyting algebras is pre-locally finite. Now we prove that there is only one pre-locally tabular extension of **KG**. This fact will immediately provide us with a criterion of local tabularity in extensions of **KG**.

Let $\mathfrak{K}$ denote the frame $\mathbf{2}_* \oplus \mathfrak{L}$. $\mathfrak{K}$ is shown in Figure 4.13. It is easy to see that $\mathfrak{K}$ is obtained from $\mathfrak{L}$ by identifying the two maximal nodes of $\mathfrak{L}$.

4.6.2. Theorem. *$Log(\mathfrak{K})$ is complete with respect to $\{\mathbf{2}_* \oplus \mathfrak{L}_{g_k} : k \in \omega\}$.*

Proof. Suppose $\mathfrak{K} \not\models \phi$, for some formula ϕ. Then by Lemma 4.4.9, there exists a descriptive valuation V and a point x of $\mathfrak{K}$ of finite depth such that $(\mathfrak{K}, V), x \not\models \phi$. We consider the generated subframe $\mathfrak{F}_x$ of $\mathfrak{K}$ generated by the point x. Then it is easy to see that $\mathfrak{F}_x$ is isomorphic to $\mathbf{2}_* \oplus \mathfrak{L}_{g_k}$ for some $k \in \omega$ and that $\mathfrak{F}_x \not\models \phi$. Therefore, $Log(\mathfrak{K})$ is complete with respect to $\{\mathbf{2}_* \oplus \mathfrak{L}_{g_k} : k \in \omega\}$. $\square$

4.6.3. Definition. Let $\mathbf{RN.KC} = \mathbf{RN} + (\neg p \vee \neg\neg p)$.

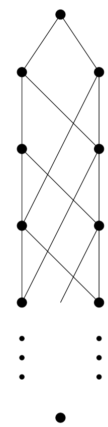

Figure 4.13: The frame $\mathfrak{K}$

4.6.4. THEOREM. $Log(\mathfrak{K}) = \mathbf{RN.KC}$.

Proof. It is well known see, e.g., [24, Proposition 2.37] that a descriptive frame $\mathfrak{F}$ validates $\neg p \vee \neg\neg p$ iff $max(\mathfrak{F})$ is a singleton set. $\mathfrak{K}$ is a p-morphic image of $\mathfrak{L}$ therefore it is an **RN**-frame. $\mathfrak{K}$ has a greatest element, thus $\mathfrak{K}$ is an **RN.KC**-frame and $Log(\mathfrak{K}) \supseteq \mathbf{RN.KC}$.

Conversely, **RN.KC** is an extension of **RN**. By Theorem 4.4.13, **RN.KC** has the finite model property. Finite rooted **RN.KC**-frames, then are finite rooted **RN**-frames with a greatest element. Similar arguments as in Theorem 4.2.8 show that every finite rooted **RN.KC**-frame is a p-morphic image of a generated subframe of $\mathfrak{K}$. Therefore, $\mathbf{RN.KC} \supseteq Log(\mathfrak{K})$.

□

Now we are ready to prove a criterion of local tabularity for extensions of **RN**. We will again use the criterion formulated in Theorem 3.4.23.

4.6.5. THEOREM. *For every $L \supseteq \mathbf{KG}$:*

$$L \text{ is not locally tabular iff } L \subseteq Log(\mathfrak{K}).$$

Proof. We first show that $Log(\mathfrak{K})$ is not locally tabular. Observe that for every point x of $\mathfrak{K}$ the point-generated subframe $\mathfrak{F}_x$ is 2-generated and $sup(\{|\mathfrak{F}_x| : x \in \mathfrak{K}\}) = \omega$. Therefore, by Theorem 3.4.23, $Log(\mathfrak{K})$ is not locally tabular. Thus, if there are infinitely many pairwise non-equivalent formulas in n variables in $Log(\mathfrak{K})$, then there are infinitely many pairwise non-equivalent formulas in n variables in every $L \subseteq Log(\mathfrak{K})$. Therefore, if $L \subseteq Log(\mathfrak{K})$, then L is not locally tabular.

Now suppose L is not locally tabular. Then by Theorem 3.4.23, there are two cases:

Case 1. There exists $n \in \omega$ such that there is an n-generated infinite rooted L-frame $\mathfrak{F}$. By Theorem 4.3.10, $\mathfrak{F}$ is isomorphic to $\bigoplus_{i=1}^{m} \mathfrak{G}_i$, where each $\mathfrak{G}_i$ is a cyclic frame. Since $\mathfrak{F}$ is infinite there is $j \leq m$ such that $\mathfrak{G}_j$ is isomorphic to $\mathfrak{L}$. Again two cases are possible:

Case 1.1. $j > 1$. Similarly to the other cases we define a bisimulation equivalence that identifies all the points above $\mathfrak{G}_j$, all the points below $\mathfrak{G}_j$ and leaves the points in $\mathfrak{G}_j$ untouched. More precisely, let E be an equivalence relation on $\mathfrak{F}$ such that for every $w, v \in \mathfrak{F}$:

- wEv if $w, v \in \mathfrak{G}_{j+1} \oplus \ldots \oplus \mathfrak{G}_n$,
- wEv if $w = v$, for $w, v \in \mathfrak{G}_j$,
- wEv if $w, v \in \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_{j-1}$.

Then E is a bisimulation equivalence and $\mathfrak{F}/E$ is isomorphic to $\mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_*$. Finally, by identifying the least two points of $\mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_*$ we obtain a p-morphic image of $\mathbf{2}_* \oplus \mathfrak{L} \oplus \mathbf{2}_*$ isomorphic to $\mathfrak{K}$. Therefore, $\mathfrak{K}$ is a p-morphic image of $\mathfrak{F}$ and $L \subseteq Log(\mathfrak{F})$.

Case 1.2. $j = 1$. In the same way as in Case 1 we obtain that $\mathfrak{L}$ is a p-morphic image of $\mathfrak{F}$. As we mentioned above the p-morphism that identifies the two maximal points of $\mathfrak{L}$ give us a frame isomorphic to $\mathfrak{K}$.

Case 2. There exists $n \in \omega$ such that the cardinality of $sup(\{|\mathfrak{H}| : \mathfrak{H}$ is a n-generated finite rooted L-frame $\}) = \omega$. This means that for every $m \in \omega$ there is a finite rooted n-generated frame $\mathfrak{H}$ such that $|\mathfrak{H}| > m$. Since every $\mathfrak{H}$ is a **KG**-frame, every $\mathfrak{H}$ is isomorphic to $\bigoplus_{i=1}^{s} \mathfrak{H}_i$, where every $\mathfrak{H}_i$ is finite and cyclic. Now consider these $\mathfrak{H}_i$'s. We again have two cases: the cardinality of the family $\mathfrak{H}_i$'s is bounded or it is not bounded.

Case 2.1. For every $m \in \omega$ there exists an n-generated finite rooted frame $\mathfrak{H} = \bigoplus_{i=1}^{s} \mathfrak{H}_i$ and a cyclic frame $\mathfrak{H}_i$, for $i \leq s$ such that $|\mathfrak{H}_i| > m$. Then the same technique as in Case 1 shows that for every $k \in \omega$ the frame $\mathbf{2}_* \oplus \mathfrak{L}_{g_k}$ is an L-frame. By Theorem 4.6.2 this implies that $L \subseteq Log(\mathfrak{K})$.

Case 2.2. There is $m \in \omega$ such that for every n-generated finite rooted L-frame $\mathfrak{H} = \bigoplus_{i=1}^{s} \mathfrak{H}_i$, we have $|\mathfrak{H}_i| \leq m$, for $i = 1, \ldots, s$. By Claim 4.5.12, $s \leq 2n$. Therefore, $|\mathfrak{H}| \leq m \cdot 2n$ and by Theorem 3.4.23, L is locally tabular, which contradicts our assumptions.

□

4.6.6. COROLLARY. *If $L \supseteq$ **KG** is decidable, then it is decidable whether L is locally tabular.*

Proof. By Theorem 4.6.5, L is not locally tabular iff $L \vdash \phi$ for every axiom ϕ of **RN.KC**. This problem is clearly decidable if L is decidable. $\square$

Next we give another criterion of local tabularity in extensions of **RN**. By Theorem 4.2.10 every finite rooted L-frame is isomorphic to $\mathfrak{L}_{g_k} \oplus \bigoplus_{i=1}^{n} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$, and $k, n \in \omega$.

4.6.7. DEFINITION.

1. The *initial segment* of a frame $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$, is the frame $\mathfrak{L}_{g_k}$.

2. The *internal depth* of a finite rooted **RN**-frame $\mathfrak{F}$ is the depth of its initial segment. Denote by $d_I(\mathfrak{F})$ the internal depth of a frame $\mathfrak{F}$.

3. Define the *internal depth* of a logic $L \supseteq$ **RN** as $sup\{d_I(\mathfrak{F}) : \mathfrak{F}$ is a finite rooted L-frame$\}$. We denote by $d_I(L)$ the internal depth of L.

4.6.8. THEOREM. *A logic $L \supseteq$ **RN** is locally tabular iff $d_I(L) < \omega$.*

Proof. First suppose $d_I(L) = \omega$. Then for every $m \in \omega$ there exists $k > m$ such that $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$ is an L-frame, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$. Then $\mathbf{2}_* \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$. We map all the points in $\bigoplus_{i=1}^{n} \mathfrak{F}_i$ onto the top node of $\mathbf{2}_* \oplus \mathfrak{L}_{g_k}$. Then by Theorem 4.1.23, $Log(\mathfrak{K}) \subseteq L$ and by Theorem 4.6.5, L is not locally tabular.

Now suppose $d_I(L) = m < \omega$. Let $\mathfrak{F}$ be an n-generated rooted L-frame. Then, by Lemma 4.6.2, $\mathfrak{F}$ isomorphic to a finite sum of cyclic frames, therefore $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{s} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathbf{2}_*$ or $\mathbf{4}_*$. Then since $d_I(L) = m$ we have $|\mathfrak{L}_{g_k}| \leq m$, and by Lemma 4.6.2, $s \leq 2n$. Therefore, $|\mathfrak{H}| \leq (m + 2n) \cdot 2$. Thus, the cardinality of every n-generated rooted L-frame is bounded by $|\mathfrak{H}|$. Therefore, by Theorem 3.4.23, L is locally tabular.

$\square$

Part II

Lattices of cylindric modal logics

Chapter 5

Cylindric modal logic and cylindric algebras

In the second part of this thesis we investigate lattices of two-dimensional cylindric modal logics. Cylindric modal logics can be seen as finite variable fragments of the classical first-order logic **FOL** and also arise naturally as multi-dimensional products of the well-known modal logic **S5**.

The idea of "approximating" **FOL** by its finite variable fragments goes back to Tarski. Tarski and his collaborators developed the theory of cylindric algebras—the algebraic models of **FOL** [60]. In particular, cylindric algebras of dimension n are Boolean algebras with n additional operators. They are algebraic models of the n-variable fragment of **FOL**. Therefore, finite dimensional cylindric algebras provide an algebraic semantics for finite variable fragments of **FOL**, and so give their algebraic "approximation".

Because of the close connection between Boolean algebras with additional operators and modal logic, which we will discuss in this chapter, this approach can be formulated purely in modal logic terms. Venema [125] defined cylindric modal logic—the modal logic counterpart of cylindric algebras—which gives a modal approximation of **FOL**. Cylindric modal logic can be also approached from the point of view of products of modal logics of Gabbay and Shehtman [44]. In the framework of products of modal logics, cylindric modal logics constitute a special case, namely products of the well-known modal logic **S5**.

The one variable fragment of **FOL** is **S5**. This logic has a lot of "nice" properties: **S5** is finitely axiomatizable, has the finite model property and is decidable. Moreover, the lattice of normal extensions of **S5** is rather simple: it is an $(\omega + 1)$-chain. Every proper normal extension of **S5** is tabular, is finitely axiomatizable and is decidable (see Scroggs [111]). In contrast to this, the three variable fragment of **FOL**—the three dimensional cylindric modal logic is much more complicated and no longer has "nice" properties. It has been shown by Maddux [88] that three-dimensional cylindric modal logic is undecidable and has continuum many undecidable extensions. Kurucz [79] strengthened this by showing that the fmp also fails for all these logics [43, Theorem 8.12]. It follows from Monk [99] and Johnson [67] that three dimensional cylindric modal logics are not finitely axiomatizable.

In this thesis we investigate in detail two-dimensional cylindric modal logic. We will show that the two-dimensional case is not as complicated as the three-dimensional, but is not as simple as the one-dimensional case. We consider two different formalisms: cylindric modal logic without diagonal and cylindric modal logic with diagonal. As we will see below, the former corresponds to the two-dimensional substitution-free fragment of **FOL**, whereas the latter corresponds to the full two-dimensional fragment of **FOL**.

The chapter is organized as follows. In the first section we recall some basic facts from modal logic. In section two we discuss many-dimensional modal logics. Section three introduces two-dimensional cylindric modal logic. In the final section we discuss two-dimensional cylindric algebras and their topological representation.

5.1 Modal Logic

In this section we recall the basic facts about modal logic. Most of these were already discussed in Chapter 2 for intermediate logics. Let $\mathcal{ML}$ be an extension of the propositional language $\mathcal{L}$ with the modal operator $\Diamond$ and let $\textsc{Form}(\mathcal{ML})$ be the set of all formulas of $\mathcal{ML}$.

5.1.1. Definition. The *basic modal logic* **K** is the smallest set of formulas that contains **CPC** and the axioms:

1. $\Box(p \to q) \to (\Box p \to \Box q)$.
2. $\Box p \leftrightarrow \neg\Diamond\neg p$.

and is closed under the rules (MP), (Subst) and

Necessitation (N) : from ϕ infer $\Box\phi$.

A *normal modal logic* is a set of formulas $L \subseteq \textsc{Form}(\mathcal{ML})$ that contains **K** and is closed under (MP), (Subst) and (N).

Next we recall the Kripke semantics for normal modal logics; see, e.g., [18, Definitions 1.19 and 1.20] and [24, §3.2].

5.1.2. Definition.

1. A *modal Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and R is a binary relation on W.

2. A *modal Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a Kripke frame and V is an arbitrary map $V : \textsc{Prop} \to \mathcal{P}(W)$, called a *valuation*.

Let $\mathfrak{M} = (W, R, V)$ be a modal Kripke model and consider an element w of W. For a formula $\phi \in \textsc{Form}(\mathcal{ML})$ the following provides an inductive definition of $\mathfrak{M}, w \models \phi$.

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$,
2. $\mathfrak{M}, w \models \phi \wedge \psi$ iff $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$,
3. $\mathfrak{M}, w \models \phi \vee \psi$ iff $\mathfrak{M}, w \models \phi$ or $\mathfrak{M}, w \models \psi$,
4. $\mathfrak{M}, w \models \phi \to \psi$ iff $\mathfrak{M}, w \not\models \phi$ or $\mathfrak{M}, w \models \psi$,
5. $\mathfrak{M}, w \not\models \bot$,
6. $\mathfrak{M}, w \models \Diamond\phi$ iff there exists v such that wRv and $\mathfrak{M}, v \models \phi$,
7. $\mathfrak{M}, w \models \Box\phi$ iff for all v such that wRv we have $\mathfrak{M}, v \models \phi$.

Since in Part II of this thesis we will only be concerned with modal logics, we call "modal Kripke frames" simply "Kripke frames".

5.1.3. Remark. The definitions of truth, validity, completeness, and the fmp remain the same as in the intuitionistic case. The same holds for all the definitions, constructions and theorems in Section 2.1.1. We will refer to these theorems as the modal analogues of the corresponding theorems for intermediate logics.

5.1.4. Definition. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. $\mathfrak{F}$ is called *rooted* if there exists $w \in W$ such that for every $v \in W$ we have wR^*v, where R^* is the reflexive and transitive closure of R.

We have the following analogue of Corollary 2.1.15; see e.g., [24, Proposition 1.11].

5.1.5. Theorem. *If a modal logic L is Kripke complete, then L is Kripke complete with respect to the class of its rooted frames.*

Next we recall the axiomatizations of some well-known modal logics; see, e.g., [18, §4.1] and [24, §3.8].

5.1.6. Definition. Let

1. $\mathbf{K4} = \mathbf{K} + (\Diamond\Diamond p \to \Diamond p)$,
2. $\mathbf{S4} = \mathbf{K4} + (p \to \Diamond p)$,

3. $\mathbf{S5} = \mathbf{S4} + (p \to \Box\Diamond p)$.

We also recall the completeness results for these logics; see, e.g., [18, §4.2 and §4.3] and [24, §5.2].

5.1.7. THEOREM.

1. $\mathbf{K}$ *is complete with respect to the class of all finite rooted frames.*

2. $\mathbf{K4}$ *is complete with respect to the class of all finite transitive rooted frames.*

3. $\mathbf{S4}$ *is complete with respect to the class of all finite transitive and reflexive rooted frames.*

4. $\mathbf{S5}$ *is complete with respect to the class of all finite transitive, reflexive and symmetric rooted frames.*

5.1.1 Modal algebras

In this section we discuss the algebraic semantics for modal logic. In the same way as Boolean algebras provide an algebraic semantics for the classical propositional calculus modal algebras provide an algebraic semantics for modal logic.

5.1.8. DEFINITION. A *modal algebra* is a pair $\mathfrak{B} = (B, \Diamond)$, where B is a Boolean algebra and $\Diamond : B \to B$ is a map satisfying the following two conditions for every $a, b \in B$:

1. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,

2. $\Diamond 0 = 0$.

We also assume that $\Box : B \to B$ is defined by $\Box a = -\Diamond - a$, for every $a \in B$.

The interpretation of a modal formula in a modal algebra is defined in the same way as in Section 2.2.2; the interpretation of the modal operators is as follows:

- $v(\Diamond\phi) = \Diamond v(\phi)$,
- $v(\Box\phi) = \Box v(\phi)$.

As in Section 2.2.2 with every normal modal logic L we associate a variety $\mathbf{V}_L$ of modal algebras that validate all the theorems of L. Using the standard Lindenbaum-Tarski construction we can show that every normal modal logic is complete with respect to its algebraic semantics; see, e.g., [18, Theorem 5.27] and [24, Theorem 7.52].

5.1.9. THEOREM. *Every normal modal logic L is complete with respect to $\mathbf{V}_L$.*

Moreover, we have that the lattice of all normal modal logics is dually isomorphic to the lattice of all varieties of modal algebras; see, e.g., [24, Theorem 7.54].

5.1.10. THEOREM. *There exists a lattice anti-isomorphism between the lattice of normal extensions of a normal modal logic L and the lattice of subvarieties of $\mathbf{V}_L$.*

The notion of a filter was defined in Section 2.2.3. Next we recall the definitions of ultrafilters and modal filters; see, e.g., [24, §7.4 and §7.7].

5.1.11. DEFINITION. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra and $F \subseteq B$ be a filter. Then

1. F is called an *ultrafilter* if for every $a \in B$ we have

$$a \in B \text{ or } -a \in B.$$

2. F is called a *modal filter* if for every $a \in B$ we have

$$a \in B \text{ implies } \Box a \in B.$$

The next theorem is an analogue of Theorem 2.3.11 (1)–(2); see, e.g., [24, Proposition 7.69].

5.1.12. THEOREM. *Let $\mathfrak{B}$ be a modal algebra. Then there is a lattice anti-isomorphism between the lattice of congruences on $\mathfrak{B}$ and the lattice of modal filters of $\mathfrak{B}$.*

We will use this correspondence in the subsequent chapters.

5.1.2 Jónsson-Tarski representation

The dual frames of modal algebras are similar to the descriptive frames of intuitionistic logic. This duality was explicitly formulated by Goldblatt [51, 52]. However, the idea of this duality goes back to Jónsson and Tarski [71].

5.1.13. DEFINITION. A *modal general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$, where (W, R) is a modal Kripke frame and $\mathcal{P}$ is a set of subsets of W, i.e. $\mathcal{P} \subseteq \mathcal{P}(W)$ such that

1. $\emptyset, W \in \mathcal{P}$,

2. If $U_1, U_2 \in \mathcal{P}$, then $U_1 \cap U_2 \in \mathcal{P}$,

3. If $U \in \mathcal{P}$, then $(W \setminus U) \in \mathcal{P}$,[1]

4. If $U \in \mathcal{P}$, then $R^{-1}(U) \in \mathcal{P}$.

Next we introduce descriptive frames for modal logic; see, e.g., [18, Definition 5.65] and [24, §8.4].

5.1.14. DEFINITION. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a modal general frame.

1. $\mathfrak{F}$ is called *differentiated* if for each $w, v \in W$,

$$w \neq v \text{ implies there is } U \in \mathcal{P} \text{ such that } w \in U \text{ and } v \notin U.$$

2. $\mathfrak{F}$ is called *tight* if for every $w, v \in W$,

$$\neg(wRv) \text{ implies that there is } U \in \mathcal{P} \text{ such that } v \in U \text{ and } w \notin R^{-1}(U).$$

3. $\mathfrak{F}$ is called *refined* if it is differentiated and tight.

4. $\mathfrak{F}$ is called *compact* if for every $\Gamma \subseteq \mathcal{P}$ with the finite intersection property we have $\bigcap \Gamma \neq \emptyset$.

5. $\mathfrak{F}$ is called *descriptive* if it is refined and compact.

Note that for every descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ the algebra $(\mathcal{P}, \cup, \cap, \setminus, \emptyset, R^{-1})$ is a modal algebra. In fact, every modal algebra can be represented in such a way; see, e.g., [18, Theorem 5.43] and [24, Theorem 8.51].

5.1.15. THEOREM. *For every modal algebra $\mathfrak{B}$ there exists a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ such that $\mathfrak{B}$ is isomorphic to $(\mathcal{P}, \cup, \cap, \setminus, \emptyset, R^{-1})$.*

We quickly sketch the main idea of the proof. Let $W_{\mathfrak{B}}$ be the set of all ultrafilters of $\mathfrak{B}$, and let $\mathcal{P}_{\mathfrak{B}} = \{\widehat{a} : a \in B\}$, where $\widehat{a} = \{w \in W_{\mathfrak{B}} : a \in w\}$. We define $R_{\mathfrak{B}}$ on $W_{\mathfrak{B}}$ by

$$wR_{\mathfrak{B}}v \text{ iff } a \in v \text{ implies } \Diamond a \in w \text{ for each } a \in B,$$

which is equivalent to

$$wR_{\mathfrak{B}}v \text{ iff } \Box a \in w \text{ implies } a \in v.$$

Then $(W_{\mathfrak{B}}, R_{\mathfrak{B}}, \mathcal{P}_{\mathfrak{B}})$ is a descriptive frame and $\mathfrak{B}$ is isomorphic to the modal algebra $(\mathcal{P}_{\mathfrak{B}}, \cup, \cap, \setminus, \emptyset, R_{\mathfrak{B}}^{-1})$.

[1] Therefore, $\mathcal{P}$ is a Boolean subalgebra of the powerset algebra $\mathcal{P}(W)$.

5.1.16. Remark. The notions of generated subframes, generated submodels, p-morphisms and disjoint unions of modal descriptive frames and the preservation results are exactly the same as in Section 2.3.1.

We will finish this section by reformulating the representation theorem for modal algebras in topological terms.

5.1.17. Definition. A triple $\mathcal{X} = (X, \mathcal{O}, R)$ is called a *modal space* if $(X, \mathcal{O})$ is a Stone space and R is a point-closed and clopen relation; that is, for every $x \in X$, the set $R(x)$ is closed and for every clopen $U \subseteq X$, the set $R^{-1}(U)$ is clopen.

Similar to Esakia spaces, a triple $\mathcal{X} = (X, \mathcal{O}, R)$ is a modal space iff R is a clopen relation on X satisfying the following condition:

$$\neg(xRy) \text{ implies there is a clopen } U \text{ such that } y \in U \text{ and } x \notin R^{-1}(U).$$

Note that in the case R is a partial order, this condition becomes equivalent to the Priestley separation axiom. We also note that for every clopen relation R we have that $R^{-1}(U)$ is closed for every closed set U. Then the representation theorem of modal algebras can be formulated as follows.

5.1.18. Theorem. *For every modal algebra $\mathfrak{B}$ there exists a modal space $\mathcal{X} = (X, \mathcal{O}, R)$ such that $\mathfrak{B}$ is isomorphic to $(\mathcal{CP}(X), \cup, \cap, \setminus, \emptyset, R^{-1})$, where $\mathcal{CP}(X)$ is the Boolean algebra of all clopens of $\mathcal{X}$.*

The correspondence between modal descriptive frames and modal spaces is even more straightforward than in the intuitionistic case. For every modal space $\mathcal{X} = (X, \mathcal{O}, R)$, the triple $(X, R, \mathcal{CP}(X))$ is a modal descriptive frame. Conversely, if $\mathfrak{F} = (W, R, \mathcal{P})$ is a modal descriptive frame, then define topology on W by letting $\mathcal{P}$ be a basis for the topology. Then the triple $(W, \mathcal{O}_{\mathcal{P}}, R)$ is a modal space.

In Part I we defined bisimulation equivalences for intuitionistic descriptive frames. Now we will give an analogous definition of bisimulation equivalence for modal descriptive frames.

5.1.19. Definition. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. An equivalence relation Q on W is called a *bisimulation equivalence* if the following two conditions are satisfied:

1. For every $w, v, u \in W$, wQv and vRu imply there is $u' \in W$ such that wRu' and $u'Qu$. In other words, $RQ(w) \subseteq QR(w)$.
2. For every $w, v \in W$, if $\neg(wQv)$ then w and v are *separated* by a Q-saturated admissible set; that is, there exists $U \in \mathcal{P}$ such that $Q(U) = U$, $w \in U$ and $v \notin U$.

We reformulate the definition of bisimulation equivalence in topological terms.

5.1.20. DEFINITION. Let $\mathcal{X} = (X, \mathcal{O}, R)$ be a modal space. An equivalence relation Q on W is called a *bisimulation equivalence* if the following two conditions are satisfied:

1. For every $x, y, z \in X$, xQy and yRz imply there is $z' \in X$ such that xRz' and $z'Qz$. In other words, $RQ(x) \subseteq QR(x)$.
2. For every $x, y \in X$, if $\neg(xQy)$ then x and y are *separated* by a Q-saturated clopen; that is, there exists a clopen $U \subseteq X$ such that $Q(U) = U$, $x \in U$ and $y \notin U$.

We order the set of all bisimulation equivalences of $\mathcal{X}$ by set-theoretic inclusion. Then we have the following analogue of Theorem 2.3.10.

5.1.21. THEOREM. *The lattice of subalgebras of a modal algebra $\mathfrak{B}$ is dually isomorphic to the lattice of bisimulation equivalences of its dual $\mathcal{X}$.*

As in the case of Heyting algebras, the category of modal descriptive frames is isomorphic to the category of modal spaces, and is dual to the category of all modal algebras. In this part of the thesis we mostly use the topological duality between modal algebras and modal spaces.

5.2 Many-dimensional modal logics

In this section we extend the notions defined in the previous section for modal logics to their multi–dimensional analogues.

5.2.1 Basic definitions

Let $\mathcal{ML}_n$ be an extension of the propositional language $\mathcal{L}$ with n modal operators $\Diamond_1, \ldots, \Diamond_n$. Let FORM($\mathcal{ML}_n$) be the set of all formulas of $\mathcal{ML}_n$. Many-dimensional normal modal logics are obtained as straightforward generalizations of normal modal logics; see, e.g., [43, §1.4].

5.2.1. DEFINITION. The *minimal n-modal logic* $\mathbf{K}_n$ is the smallest set of formulas that contains **CPC**, the following axioms for $i \leq n$:

1. $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$,
2. $\Box_i p \leftrightarrow \neg\Diamond_i\neg p$.

and is closed under the rules (MP), (Subst) and

Necessitation (N)$_i$: from ϕ infer $\Box_i\phi$.

An *n-normal modal logic* is a set $L \subseteq \textsc{Form}(\mathcal{ML}_n)$ that contains $\mathbf{K}_n$ and is closed under (MP), (Subst) and (N)$_i$, for each $i \leq n$.

5.2.2. Remark. The Kripke semantics for many-dimensional modal logics is obtained by a straightforward generalization of the uni-modal case.

Throughout, we will skip the prefix n in "n-normal modal logics" if it is clear from the context. As we saw in Part I, an important class of frames is the class of rooted frames. Next we discuss rooted Kripke frames for many-dimensional modal logics; see, e.g, [43, §1.4].

5.2.3. Definition. Let $\mathcal{F} = (W, R_1, \dots, R_n)$ be a many-dimensional Kripke frame. Then $\mathcal{F}$ is called *rooted* if there is a point $w \in W$ that is related to every point $v \in W$ by the reflexive and transitive closure of the relation $\bigcup_{i=1}^n R_i$. The point w is called a *root* of $\mathcal{F}$.

We have the following analogue of Theorem 5.1.5; see, e.g., [43, Proposition 1.11]

5.2.4. Theorem. *If a many-dimensional modal logic L is Kripke complete, then L is Kripke complete with respect to the class of its rooted frames.*

5.2.2 Products of modal logics

In this section we recall the fusion and product of modal logics. For an extensive study of many-dimensional modal logics we refer to [43] and [95].

5.2.5. Definition. Let L_1 and L_2 be normal modal logics with the modal operators $\Diamond_1$ and $\Diamond_2$, respectively. The *fusion* $L_1 \otimes L_2$ of L_1 and L_2 is the smallest normal modal logic, in the language $\mathcal{ML}_2$, containing $L_1 \cup L_2$.

Consider the following formulas called *right* and *left commutativity formulas*, and the *Church-Rosser formula.*

1. $\mathbf{com^r} := \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$
2. $\mathbf{com^l} := \Diamond_2\Diamond_1 p \to \Diamond_1\Diamond_2 p$
3. $\mathbf{chr} := \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p.$

The next theorem gives a semantic characterization of $\mathbf{com^r}$, $\mathbf{com^l}$ and $\mathbf{chr}$; see, e.g., [43, §5.1].

5.2.6. Theorem. *For every frame $\mathcal{F} = (W, R_1, R_2)$ we have*

1. $\mathcal{F} \models \mathbf{com^r}$ *iff* $(\forall w, v, u \in W)(wR_1v \wedge vR_2u \ \rightarrow (\exists z)(wR_2z \wedge zR_1u))$,
2. $\mathcal{F} \models \mathbf{com^l}$ *iff* $(\forall w, v, u \in W)(wR_2v \wedge vR_2u \ \rightarrow (\exists z)(wR_1z \wedge zR_2u))$,
3. $\mathcal{F} \models \mathbf{chr}$ *iff* $(\forall w, v, u \in W)(wR_1v \wedge wR_2u \ \rightarrow (\exists z)(vR_1z \wedge uR_2z))$.

Proof. The proof follows directly from the Sahlqvist correspondence, because $\mathbf{com^r}$, $\mathbf{com^l}$ and **chr** are Sahlqvist formulas; see, e.g., [18, §3.6]. □

Next we define the product of Kripke frames and the product of modal logics. These constructions were introduced in [115] and [44], (see also [43, §5.1]).

5.2.7. DEFINITION.

1. Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be Kripke frames. The *product* of $\mathcal{F}$ and $\mathcal{F}'$ is the frame $\mathcal{F} \times \mathcal{F}' := (W \times W', R_1, R_2)$, where

$$(w, w')R_1(v, v') \ \text{ iff } \ wRv \text{ and } w' = v',$$

and

$$(w, w')R_2(v, v') \ \text{ iff } \ w'R'v' \text{ and } w = v.$$

The frame $\mathcal{F} \times \mathcal{F}'$ is called a *product frame*.

2. Let L_1 and L_2 be Kripke complete normal modal logics. The *product* $L_1 \times L_2$ of L_1 and L_2 is defined as

$$L_1 \times L_2 := Log(\{\mathcal{F} \times \mathcal{F}' : \mathcal{F} \text{ is an } L_1\text{-frame and } \mathcal{F}' \text{ is an } L_2\text{-frame}\})$$

Product logics can be axiomatized by the commutativity and the Church-Rosser formulas. Let

$$\mathbf{com} = \mathbf{com}^r \wedge \mathbf{com}^l.$$

The next theorem, gives a sufficient condition when a product logic is axiomatized by **com** and **chr**; see, e.g., [43, Theorem 5.9].

5.2.8. THEOREM. *If L_1 and L_2 are normal uni-modal logics axiomatized by Sahlqvist formulas, then*

$$L_1 \times L_2 = L_1 \otimes L_2 + \mathbf{com} + \mathbf{chr}.$$

The rest of this thesis is devoted to the two-dimensional products of the modal logic **S5**.

5.3 Cylindric modal logics

In this section we introduce cylindric modal logics, investigate their Kripke semantics, and discuss the connection with **FOL**. We start with the logic $\mathbf{S5}^2$, which is the substitution-free fragment of **FOL**.

5.3.1 $\mathbf{S5} \times \mathbf{S5}$

We consider a very special case of products of modal logics. In particular, we look at the product $L_1 \times L_2$, for $L_1 = L_2 = \mathbf{S5}$. We first simplify the axiomatization of $\mathbf{S5} \times \mathbf{S5}$.

5.3.1. LEMMA. *Let $\mathcal{F} = (W, R_1, R_2)$ be a frame such that R_1 and R_2 are symmetric relations. Then the following three conditions are equivalent:*

1. $\mathcal{F} \models \mathbf{com^r}$,
2. $\mathcal{F} \models \mathbf{com^l}$,
3. $\mathcal{F} \models \mathbf{chr}$.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{F} \models \mathbf{com^r}$. We will show that $\mathcal{F} \models \mathbf{com^l}$. Suppose $w, v, u \in W$, wR_2v and vR_1u. Then since R_1 is symmetric, we have uR_1v and vR_2w. From $\mathcal{F} \models \mathbf{com^r}$ and Theorem 5.2.6(1) it follows that there exists $z \in W$ such that uR_2z and zR_1w. By the symmetry of R_1, we get wR_1z and zR_2u. By Theorem 5.2.6(2), this means that $\mathcal{F} \models \mathbf{com^l}$. The proof of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) is similar. $\square$

It is well known that the axioms of $\mathbf{S5}$ are Sahlqvist formulas; see, e.g., [18, §3.6]. Thus we have the following corollary of Theorem 5.2.8; see, e.g., [43, Corollary 5.11 and Theorem 5.12].

5.3.2. COROLLARY.

$$\mathbf{S5} \times \mathbf{S5} = \mathbf{S5} \otimes \mathbf{S5} + \mathbf{com}^r = \mathbf{S5} \otimes \mathbf{S5} + \mathbf{com}^l = \mathbf{S5} \otimes \mathbf{S5} + \mathbf{chr}.$$

Proof. Apply Lemma 5.3.1 and Theorem 5.2.8. $\square$

WARNING. From now on we use the abbreviation $\mathbf{S5}^2$ for $\mathbf{S5} \times \mathbf{S5}$. We denote by $\mathcal{F}, \mathcal{G}, \ldots$, the frames of $\mathbf{S5}^2$. We also denote by $\mathfrak{F}, \mathfrak{G}, \ldots$, the frames in a similarity type with an additional constant d (see Section 5.3.2). Since the relations in $\mathbf{S5}^2$-frames are equivalence relations we denote them by E_1 and E_2.

In Definition 5.2.3 we defined rooted frames for many-dimensional modal logics. The next lemma characterizes the rooted $\mathbf{S5}^2$-frames.

5.3.3. LEMMA. *Let $\mathcal{F} = (W, E_1, E_2)$ be an $\mathbf{S5}^2$-frame. Then $\mathcal{F}$ is rooted iff for every $w, v \in W$, there exists $u \in W$ such that wE_1u and uE_2v.*

Proof. It is easy to see that if $\mathcal{F}$ satisfies the above condition, then every point of W is a root of $\mathcal{F}$. Conversely, suppose $\mathcal{F}$ is rooted. Let $w, v \in W$, and let r be a root of $\mathcal{F}$. Then there are two finite sequences $r_0, \ldots, r_k$ and $r'_0, \ldots, r'_m$ such that $r_0 = r'_0 = r$, $r_k = w$, $r'_m = v$ and $r_i(E_1 \cup E_2)r_{i+1}$ for $i < k$ and $r'_i(E_1 \cup E_2)r'_{i+1}$ for $i < m$. It follows that there is a sequence $w_0, \ldots, w_n$ for $n = k + m$ such that $w_0 = w$, $w_n = v$ and $w_i(E_1 \cup E_2)w_{i+1}$ for $i < n$. We will prove the lemma by induction on the length of this sequence. If $n = 1$, then $w(E_1 \cup E_2)v$. Without loss of generality we may assume that wE_1v. So, wE_1v and vE_2v. Now suppose that $n > 1$. Then, by the induction hypothesis, there is a u such that wE_1u and uE_2w_{n-1}. If $w_{n-1}E_2v$, then uE_2v and the statement of the lemma is satisfied. If $w_{n-1}E_1v$, then there exists u' such that uE_1u' and $u'E_2v$. This means that wE_1u' and $u'E_2v$, and so, the condition of the lemma is satisfied. □

Next we introduce general definitions that will be used in subsequent chapters. Note that for every $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$ the intersection $E_1 \cap E_2$ of E_1 and E_2 is also an equivalence relation.

5.3.4. DEFINITION. Let $\mathcal{F} = (W, E_1, E_2)$ be an $\mathbf{S5}^2$-frame.

1. Let E_0 denote the equivalence relation $E_1 \cap E_2$.
2. For $i = 1, 2, 3$ we call the E_i-equivalence classes the *E_i-clusters*.
3. For $w \in W$ and $i = 1, 2, 3$ let $E_i(w)$ denote the E_i-cluster containing w.
4. For $X \subseteq W$ and $i = 1, 2, 3$ we let $E_i(X)$ denote $\bigcup_{x \in X} E_i(x)$.

5.3.5. LEMMA. *Let $\mathcal{F} = (W, E_1, E_2)$ be an $\mathbf{S5}^2$-frame. Then $\mathcal{F}$ is isomorphic to a product frame iff $E_0(w) = \{w\}$ for every $w \in W$.*

Proof. It is easy to see that if $\mathcal{F}$ is (isomorphic to) a product $\mathbf{S5}^2$-frame, then $E_0(w) = \{w\}$ for every $w \in W$. Conversely, let $\mathcal{F}$ be such that $E_0(w) = \{w\}$ for every $w \in W$. Fix $w \in W$ and let $\mathcal{F}' := (E_1(w), E_1 \restriction E_1(w))$ and $\mathcal{F}'' := (E_2(w), E_2 \restriction E_2(w))$. Obviously $\mathcal{F}'$ and $\mathcal{F}''$ are $\mathbf{S5}$-frames. It is now routine to check that $\mathcal{F}' \times \mathcal{F}''$ is isomorphic to $\mathcal{F}$. □

The next lemma gives a characterization of rooted $\mathbf{S5}^2$-frames.

5.3.6. LEMMA. *Let $\mathcal{F} = (W, E_1, E_2)$ be an $\mathbf{S5}^2$-frame. Let $\{C_i\}_{i \in I}$ and $\{C^j\}_{j \in J}$ be the families of all E_1 and E_2-clusters of $\mathcal{F}$, respectively. Then $\mathcal{F}$ is rooted iff $C_i \cap C^j \neq \emptyset$ for every $i \in I$ and $j \in J$.*

Proof. It is easy to see that if the condition of the lemma is satisfied, then for every $w, v \in W$ we have $E_1(w) \cap E_2(v) \neq \emptyset$. Therefore, by Lemma 5.3.3, $\mathcal{F}$ is rooted. Conversely, let C_i and C^j be an E_1 and E_2-cluster of $\mathcal{F}$, respectively. Suppose $w \in C_i$ and $v \in C^j$. Then, by Lemma 5.3.3, there exists $z \in W$ such that wE_1z and zE_2v, which means that $z \in C_i \cap C^j$, and so $C_i \cap C_j \neq \emptyset$. □

We use the terms rectangles, squares, and quasi-squares to denote the following rooted $\mathbf{S5}^2$-frames.

5.3.7. Definition.

1. We call rooted product frames *rectangles*. Let **Rect** denote the class of all rectangles. We denote by $\mathbf{n} \times \mathbf{m}$ the finite rectangle consisting of n E_1-clusters and m E_2-clusters.

2. A rectangle that is isomorphic to $\mathcal{G} \times \mathcal{G}$, for some **S5**-frame $\mathcal{G}$, is called a *square*. We denote by **Sq** the class of all squares. Let $\mathbf{n} \times \mathbf{n}$ denote the finite square consisting of n E_1 and E_2-clusters.

3. Call a rooted $\mathbf{S5}^2$-frame $\mathcal{F}$ a *quasi-square* if the cardinality of E_1-clusters of $\mathcal{F}$ is the same as the cardinality of E_2-clusters of $\mathcal{F}$.

It is clear that $\mathbf{Sq} \subseteq \mathbf{Rect}$. We will see in the next chapter that $\mathbf{S5}^2$ is complete with respect to the classes of finite rectangles and finite squares.

5.3.2 Cylindric modal logic with the diagonal

Let $\mathcal{ML}_2^d$ be the extension of $\mathcal{ML}_2$ with a constant d. We call this constant the *diagonal*. Let $\textsc{Form}(\mathcal{ML}_2^d)$ be the set of all formulas of $\mathcal{ML}_2^d$.

5.3.8. Definition. The *two-dimensional cylindric modal logic* $\mathbf{CML}_2$ is the smallest set of formulas of $\textsc{Form}(\mathcal{ML}_2^d)$ that contains $\mathbf{S5}^2$, the axioms:

1. $\Diamond_i(d)$,

2. $\Diamond_i(d \wedge p) \to \neg\Diamond_i(d \wedge \neg p)$,

and is closed under (MP), (Subst) and $(\mathrm{N})_i$, for $i = 1, 2$.

We now define the Kripke semantics for this new similarity type.

5.3.9. Definition.

1. A frame of the language $\mathcal{ML}_2^d$ is a quadruple (W, R_1, R_2, D) such that (W, R_1, R_2) is a two-dimensional Kripke frame and $D \subseteq W$.

2. A model of the language $\mathcal{ML}_2^d$ is a tuple (W, R_1, R_2, D, V), where (W, R_1, R_2, D) is a frame of $\mathcal{ML}_2^d$ and $V : \textsc{Prop} \cup \{d\} \to W$ is a valuation such that $V(d) = D$. If $w \in V(d)$ we write $w \models d$.

The next proposition characterizes the frames of $\mathbf{CML}_2$; see, e.g., [60] and [125].

5.3.10. PROPOSITION. *A frame $\mathfrak{F} = (W, E_1, E_2, D)$ is a* $\mathbf{CML}_2$*-frame iff the following three conditions are satisfied:*

1. *(W, E_1, E_2) is an* $\mathbf{S5}^2$*-frame,*

2. *For each $i = 1, 2$, every E_i-cluster of $\mathfrak{F}$ contains a unique point from D.*

Proof. The right to left direction is straightforward. Now assume $\mathfrak{F}$ is a $\mathbf{CML}_2$-frame. Since $\mathbf{CML}_2$ contains $\mathbf{S5}^2$ we have that (1) is satisfied. To show (2) suppose for $i = 1, 2$ there exists an E_i-cluster C such that $D \cap C = \emptyset$. Then for every $w \in C$ we have that $w \not\models \Diamond_i d$, which contradicts Definition 5.3.8(1). Now suppose that for $i = 1, 2$ there exists an E_i-cluster C such that $D \cap C = \{w, v\}$ and $w \neq v$. Let V be a valuation such that $V(p) = \{w\}$. Then $w \models \Diamond_i(d \wedge p)$. On the other hand, $v \models d \wedge \neg p$. Hence, $w \models \Diamond_i(d \wedge \neg p)$, which contradicts Definition 5.3.8(2). □

5.3.11. COROLLARY. *For every* $\mathbf{CML}_2$*-frame $\mathfrak{F} = (W, E_1, E_2, D)$, the cardinality of the set of all E_1-clusters of $\mathfrak{F}$ is the same as the cardinality of the set of all E_2-clusters of $\mathfrak{F}$.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the sets of all E_1 and E_2-clusters of $\mathfrak{F}$, respectively. Define $f : \mathcal{E}_1 \to \mathcal{E}_2$ by putting $f(C) = E_2(C \cap D)$. Suppose $C_1, C_2 \in \mathcal{E}_1$, $C_1 \neq C_2$, $C_1 \cap D = \{x\}$, and $C_2 \cap D = \{y\}$. Since every E_i-cluster of $\mathcal{X}$ contains a unique point from D, it follows that $f(C_1) = E_2(x) \neq E_2(y) = f(C_2)$. Therefore, f is an injection. Now suppose $C' \in \mathcal{E}_2$ and $C' \cap D = \{x\}$. If we let $C = E_1(x)$, then $f(C) = E_2(x) = C'$. Thus, f is a surjection. Hence, we obtain that f is a bijection. □

The next theorem shows the completeness of $\mathbf{CML}_2$ with respect to its Kripke semantics; see, e.g., [124, §3.2.2].

5.3.12. THEOREM. $\mathbf{CML}_2$ *is Kripke complete.*

Proof. The result follows immediately from the Sahlqvist correspondence, because (1) and (2) are Sahlqvist formulas. □

The definition of rooted frames in this similarity type is the same as for $\mathcal{ML}_2$.

5.3.13. PROPOSITION. *Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a* $\mathbf{CML}_2$*-frame. $\mathfrak{F}$ is rooted iff (W, E_1, E_2) is a rooted* $\mathbf{S5}^2$*-frame.*

Proof. Apply Lemma 5.3.3. □

5.3.3 Product cylindric modal logic

Similar to the diagonal-free case we can define product frames in the signature with the diagonal.

5.3.14. DEFINITION.

1. A rooted $\mathbf{CML_2}$-frame $\mathfrak{F} = (W \times W, E_1, E_2, D)$ is called a *cylindric square* or *square* $\mathbf{CML_2}$*-frame*, if $(W \times W, E_1, E_2)$ is a square and $D = \{(w, w) : w \in W\}$. Let $\mathbf{CSq}$ denote the class of all cylindric squares.

2. Let $\mathbf{PCML_2}$ denote the logic of $\mathbf{CSq}$; that is, $\mathbf{PCML_2} = Log(\mathbf{CSq})$. We call $\mathbf{PCML_2}$ the *product cylindric modal logic*.

We note that $\mathbf{CML_2} \neq \mathbf{PCML_2}$. In fact, $\mathbf{PCML_2}$ can be obtained by adding to $\mathbf{CML_2}$ the *Henkin axiom*:

$$(\mathrm{H}) = \Diamond_i(p \wedge \neg q \wedge \Diamond_j(p \wedge q)) \to \Diamond_j(\neg d \wedge \ \Diamond_i p), \quad i \neq j, \quad i, j = 1, 2.$$

or the *Venema axiom*

$$(\mathrm{V}) = d \wedge \Diamond_i(\neg p \wedge \Diamond_j p) \to \Diamond_j(\neg d \wedge \Diamond_i p), \quad i \neq j, \quad i, j = 1, 2.$$

For the next theorem see [60, Theorem 3.2.65(ii)] and [124, Proposition 3.5.8].

5.3.15. THEOREM. $\mathbf{PCML_2} = \mathbf{CML_2}$+(H)= $\mathbf{CML_2}$+(V).

Since both (H) and (V) are Sahlqvist formulas we have the following theorem; see, e.g., [124, Theorem 3.5.4].

5.3.16. THEOREM. $\mathbf{PCML_2}$ *is Kripke complete.*

Now we give a useful characterization of $\mathbf{PCML_2}$-frames, which will allow us to construct rather simple finite $\mathbf{CML_2}$-frames that are not $\mathbf{PCML_2}$-frames. Suppose (W, E_1, E_2, D) is a $\mathbf{CML_2}$-frame. We call $w \in D$ a *diagonal point*, and $w \in W \setminus D$ a *non-diagonal point*. Also, call an E_0-cluster C a *diagonal* E_0*-cluster* if it contains a diagonal point. Otherwise we call C a *non-diagonal* E_0*-cluster.*

5.3.17. DEFINITION. Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a $\mathbf{CML_2}$-frame. $\mathfrak{F}$ is said to satisfy $(*)$ if there exists a diagonal point $x_0 \in D$ such that $E_0(x_0) = \{x_0\}$ and there exists a non-singleton E_0-cluster C which is either E_1 or E_2-related to x_0.

The next theorem characterizes $\mathbf{PCML_2}$-frames. A similar characterization can be found in [60, Lemma 3.2.59, Theorem 3.2.65]. However, our characterization uses Venema's axiom, while the one in [60] uses Henkin's axiom. Moreover, our proof below appears to be simpler than the original one in [60].

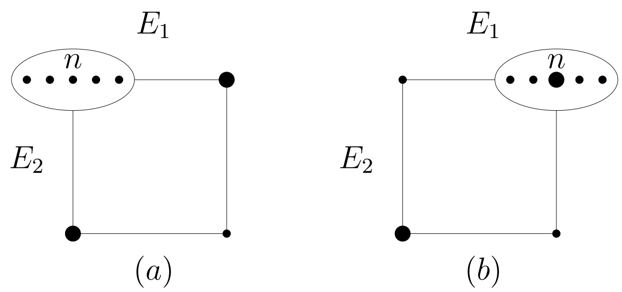

Figure 5.1: $\mathbf{CML}_2$ and $\mathbf{PCML}_2$-frames

5.3.18. THEOREM. *Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a $\mathbf{CML}_2$-frame. Then $\mathfrak{F}$ is an $\mathbf{PCML}_2$-frame iff $\mathfrak{F}$ does not satisfy* $(*)$.

Proof. Suppose $\mathfrak{F}$ satisfies $(*)$. We show that (V) does not hold in $\mathfrak{F}$, implying that $\mathfrak{F}$ is not a $\mathbf{PCML}_2$-frame. Let x_0 be a diagonal point with $E_0(x_0) = \{x_0\}$ and C be a non-singleton E_0-cluster, say E_1-related to x_0 (the case when C is E_2-related to x_0 is proved similarly). Choose two different points y and z from C. Then $y \in (W \setminus \{z\}) \cap E_2(z)$, and so $x_0 \in D \cap E_1((W \setminus \{z\}) \cap E_2(z))$. On the other hand, $E_1(z) = E_1(x_0)$. If $x_0 \in E_2((W \setminus D) \cap E_1(z))$, then there exists $u \in W \setminus D$ that is E_1 and E_2 related to x_0, which contradicts the fact that $E_0(x_0) = \{x_0\}$. Finally, if we define a valuation V on $\mathfrak{F}$ by $V(p) = \{z\}$, then $(\mathfrak{F}, V), x_0 \not\models (V)$. Thus, by Theorem 5.3.15, $\mathfrak{F}$ is not a $\mathbf{PCML}_2$.

Conversely, suppose $\mathfrak{F}$ is not a $\mathbf{PCML}_2$-frame. We show that $(*)$ holds in $\mathfrak{F}$. We know that (V) does not hold in $\mathfrak{F}$. Therefore, there exist a point $x \in W$ and a set $A \subseteq X$ such that $x \in D \cap E_i((W \setminus A) \cap E_j(A))$ but $x \notin E_j((W \setminus D) \cap E_i(A))$ for $i, j = 1, 2$ and $i \neq j$. Since $x \in D \cap E_i((W \setminus A) \cap E_j(A))$, we have $x \in D$ and there exist points $y, z \in W$ such that xE_iy, yE_jz, $y \notin A$ and $z \in A$. From $y \notin A$ and $z \in A$ it follows that y and z are different. Also xE_iy and yE_jz imply that there exists a point $u \in W$ such that xE_ju and uE_iz. If $u \neq x$, then, by Proposition 5.3.10, u is a non-diagonal point, and so $u \in (W \setminus D) \cap E_i(A)$. But then $x \in E_j((W \setminus D) \cap E_i(A))$, which contradicts our assumption. Thus, $u = x$ and xE_iz. Therefore, yE_0z and both y and z are E_i-related to x. Moreover, if $E_0(x) \neq \{x\}$, then by choosing a point $u \in E_0(x)$ different from x we again obtain that $u \in (W \setminus D) \cap E_i(A)$, and so $x \in E_j((W \setminus D) \cap E_i(A))$, which is impossible. Therefore, $E_0(x) = \{x\}$ and $E_0(y)$ is a non-singleton E_0-cluster E_i-related to x_0. Thus, $(*)$ holds in $\mathfrak{F}$. $\square$

Using this criterion it is easy to see that the $\mathbf{CML}_2$-frames shown in Figure 5.1(b) are $\mathbf{PCML}_2$-frames, while the $\mathbf{CML}_2$-frames shown in Figure 5.1(a) are not. Moreover, the smallest $\mathbf{CML}_2$-frame that is not a $\mathbf{PCML}_2$-frame is the frame shown in Figure 5.1(a), where the non-singleton E_0-cluster contains only two points.

5.3.4 Connection with FOL

As we mentioned in the introduction to this chapter, one of the main reasons for studying $\mathbf{S5}^2$ and $\mathbf{CML}_2$ ($\mathbf{PCML}_2$) is that they axiomatize the two-variable fragments of **FOL**. $\mathbf{S5}^2$ corresponds to a “clean”, substitution-free fragment of **FOL**, whereas $\mathbf{PCML}_2$ is the full fragment of **FOL**. To see this, consider the following translation of the formulas of the language of $\mathbf{S5}^2$ and $\mathbf{CML}_2$ to the first order language:

- $p^t = P(x_1, x_2)$,
- $(\cdot)^t$ is a homomorphism for the Booleans,
- $(\Diamond_1\varphi)^t = \exists x_1 \varphi^t$,
- $(\Diamond_2\varphi)^t = \exists x_2 \varphi^t$,
- $d^t = (x_1 = x_2)$.

This translation preserves the validity of formulas; see, e.g., [43, §3.5] and [124, Proposition 4.1.7].

5.3.19. Theorem. *Let $\phi \in \mathrm{Form}(\mathcal{ML}_2)$ and $\psi \in \mathrm{Form}(\mathcal{ML}_2^d)$. Then*

1. $\mathbf{S5}^2 \vdash \phi$ *iff* $\mathbf{FOL} \vdash \phi^t$.
2. $\mathbf{PCML}_2 \vdash \psi$ *iff* $\mathbf{FOL} \vdash \psi^t$.

Note that the analogue of this theorem for the one-variable fragment of **FOL** was first proved by Wajsberg [128], who showed that **S5** axiomatizes the one-variable fragment of **FOL**. Similarly one can show that the logics $\mathbf{S5}^n$ and $\mathbf{PCML}_n$ of n-ary products of **S5**-frames are the substitution-free and full n-variable subfragments of **FOL**. However, for $n \geq 3$ the logics $\mathbf{S5}^n$ and $\mathbf{PCML}_n$ no longer have “good” properties. That $\mathbf{S5}^n$ is not finitely axiomatizable for $n \geq 3$ follows from Johnson [67], and that $\mathbf{PCML}_n$ is not finitely axiomatizable for $n \geq 3$ follows from Monk [99] (see also [43, Theorems 8.1 and 8.2]).

5.4 Cylindric algebras

Two-dimensional cylindric algebras are algebraic models of two-dimensional cylindric modal logics. Note that historically cylindric algebras were introduced by Tarski much earlier than cylindric modal logics.

5.4.1 $\mathbf{Df}_2$-algebras

5.4.1. Definition. [60, Definition 1.1.2] An algebra $\mathcal{B} = (B, \Diamond_1, \Diamond_2)$ is said to be a *two-dimensional diagonal-free cylindric algebra*, or a $\mathbf{Df}_2$-algebra for short, if B is a Boolean algebra and each $\Diamond_i : B \to B$, $i = 1, 2$, satisfies the following axioms for every $a, b \in B$:

1. $\Diamond_i 0 = 0$,
2. $a \leq \Diamond_i a$,
3. $\Diamond_i(\Diamond_i a \wedge b) = \Diamond_i a \wedge \Diamond_i b$,
4. $\Diamond_1 \Diamond_2 a = \Diamond_2 \Diamond_1 a$.

Since $\mathbf{Df}_2$-algebras are equationally defined, the class of all $\mathbf{Df}_2$-algebras forms a variety.

5.4.2. Definition. Let $\mathbf{Df}_2$ denote the variety of all two-dimensional diagonal-free cylindric algebras.

Using the standard Lindenbaum-Tarski construction we can show that $\mathbf{S5}^2$ is complete with respect to $\mathbf{Df}_2$; see, e.g., [124, §4.2].

5.4.3. Theorem. $\mathbf{S5}^2 \vdash \phi$ *iff* ϕ *is valid in every* $\mathbf{Df}_2$*-algebra.*

Let $\Lambda(\mathbf{S5}^2)$ denote the lattice of all normal extensions of $\mathbf{S5}^2$ and let $\Lambda(\mathbf{Df}_2)$ denote the lattice of subvarieties of $\mathbf{Df}_2$. Then we have the following corollary of Theorem 5.4.3 and the modal logic analogue of Theorem 2.2.19.

5.4.4. Corollary. $\Lambda(\mathbf{S5}^2)$ *is dually isomorphic to* $\Lambda(\mathbf{Df}_2)$.

By adapting Definition 5.1.11 to the case of $\mathbf{Df}_2$ we obtain that a filter F of a $\mathbf{Df}_2$-algebra $\mathcal{B} = (B, \Diamond_1, \Diamond_2)$ is a $\mathbf{Df}_2$-filter provided for each $a \in B$, if $a \in F$, then $\Box_i a \in F$. Therefore, we have the following corollary; see, e.g., [60, Theorem 2.3.4 and Remark 2.3.6].

5.4.5. Corollary. *There exists a lattice isomorphism between the lattice of congruences of* $(B, \Diamond_1, \Diamond_2)$ *and the lattice of* $\mathbf{Df}_2$*-filters of* $(B, \Diamond_1, \Diamond_2)$.

Recall from [23] that every algebra $\mathfrak{A}$ has at least two congruence relations, the diagonal $\Delta = \{(a, a) : a \in \mathfrak{A}\}$ and $\mathfrak{A}^2$. Recall also that an algebra $\mathfrak{A}$ is *simple* if Δ and $\mathfrak{A}^2$ are the only congruence relations of $\mathfrak{A}$. It is well known that every simple algebra is subdirectly irreducible. In the case of $\mathbf{Df}_2$, the converse is also true; see [60, Theorems 2.4.43, 2.4.14].

5.4.6. THEOREM. *Let $\mathcal{B} = (B, \Diamond_1, \Diamond_2)$ be a* $\mathbf{Df}_2$*-algebra. Then $\mathcal{B}$ is subdirectly irreducible iff $\mathcal{B}$ is simple.*

5.4.7. REMARK. We mention that a $\mathbf{Df}_1$-algebra or *Halmos' monadic algebra* is a pair $(B, \Diamond)$ such that B is a Boolean algebra and $\Diamond$ is an unary operator on B satisfying conditions 1–3 of Definition 5.4.1; see e.g., [58, p.40]. The unary operator $\Diamond$ is called a *monadic operator*, and $\mathbf{Df}_1$-algebras are widely known as *monadic algebras*. They provide algebraic completeness for **S5**. Some of the most important proprieties of $\mathbf{Df}_1$-algebras are:

- Every finitely generated $\mathbf{Df}_1$-algebra is finite. Therefore, $\mathbf{Df}_1$ is *locally finite* and $\mathbf{Df}_1$ is generated by its finite algebras.
- The lattice of all subvarieties of $\mathbf{Df}_1$ is a countable increasing chain $\mathcal{V}_1 \subsetneq \mathcal{V}_2 \subsetneq \ldots$ that converges to $\mathbf{Df}_1$.

For a proof of these and other related results we refer to Halmos [58], Bass [3], Monk [100], and Kagan and Quackenbush [72].

5.4.2 Topological representation

The dual spaces of $\mathbf{Df}_2$-algebras can be obtained by adjusting the general duality between modal algebras and modal spaces.

5.4.8. DEFINITION. A triple $\mathcal{X} = (X, E_1, E_2)$ is said to be a $\mathbf{Df}_2$*-space*, if (X, E_1) and (X, E_2) are modal spaces and (X, E_1, E_2) is an $\mathbf{S5}^2$-frame.

We have the following representation theorem for $\mathbf{Df}_2$-algebras.

5.4.9. THEOREM. *Every* $\mathbf{Df}_2$*-algebra can be represented as $(\mathcal{CP}(X), E_1, E_2)$ for the corresponding* $\mathbf{Df}_2$*-space (X, E_1, E_2).*

Proof. (Sketch). By Theorem 5.1.18 we need to verify that in the dual space $\mathcal{X} = (X, E_1, E_2)$ of $(B, \Diamond_1, \Diamond_2)$, the relations E_1 and E_2 commute; that is $(\forall x, y, z \in X)(xE_1y \wedge yE_2z \rightarrow (\exists u)(xE_2u \wedge uE_1z))$; and conversely, that in every $\mathbf{Df}_2$-space we have $E_1E_2(A) = E_2E_1(A)$ for every $A \in \mathcal{CP}(X)$. The former follows immediately from the Sahlqvist correspondence (see [18, Theorems 3.54 and 5.91]), and the latter is obvious, since $E_i(A) = \bigcup_{x \in A} E_i(x)$ and E_i commutes with $\bigcup$ for $i = 1, 2$. $\square$

Consequently, every finite $\mathbf{Df}_2$-algebra is represented as an algebra $(\mathcal{P}(X), E_1, E_2)$, where $\mathcal{P}(X)$ denotes the power set algebra of X, for the corresponding finite $\mathbf{S5}^2$-frame (X, E_1, E_2).

Now we can obtain dual descriptions of algebraic concepts of $\mathbf{Df}_2$-algebras. To obtain the dual description of $\mathbf{Df}_2$-filters we need the following definition. Let $\mathcal{X} = (X, E_1, E_2)$ be $\mathbf{Df}_2$-space. A subset U of X is said to be *saturated* if $E_1(U) = E_2(U) = U$.

5.4.10. THEOREM. *Let $\mathcal{B} = (B, \Diamond_1, \Diamond_2)$ be a $\mathbf{Df}_2$-algebra and $\mathcal{X} = (X, E_1, E_2)$ be its dual $\mathbf{Df}_2$-space.*

1. *The lattice of $\mathbf{Df}_2$-filters of $\mathcal{B}$ is isomorphic to the lattice of closed saturated subsets of $\mathcal{X}$.*

2. *Congruences of $\mathcal{B}$ correspond to closed saturated subsets of $\mathcal{X}$.*

Proof. The proof is an easy adaptation of Theorem 2.3.11. □

Bisimulation equivalences for modal descriptive frames and modal spaces were defined in Section 5.1 (see Definitions 5.1.19 and 5.1.20). For convenience we formulate the definition for $\mathbf{Df}_2$-spaces.

5.4.11. DEFINITION. Let $\mathcal{X} = (X, E_1, E_2)$ be a $\mathbf{Df}_2$-space. An equivalence relation Q on W is called a *bisimulation equivalence* if:

1. For every $x, y, z \in X$ and $i = 1, 2$, xQy and yE_iz imply that there is $u \in X$ such that xE_iu and uQz. In other words, $E_iQ(x) \subseteq QE_i(x)$.

2. For every $x, y \in X$ and $i = 1, 2$, if $\neg(xQy)$ then x and y are *separated* by a Q-saturated clopen; that is, there exists a clopen $U \subseteq X$ such that $Q(U) = U$, $x \in U$ and $y \notin U$.

Note that since E_1, E_2 and Q are equivalence relations, Q is a bisimulation equivalence iff it is separated and $QE_i(x) = E_iQ(x)$ for every $x \in X$ and $i = 1, 2$. To obtain the dual description of subalgebras of $\mathbf{Df}_2$-algebras, we order the set of all bisimulation equivalences of a $\mathbf{Df}_2$-space $\mathcal{X}$ by set-theoretical inclusion.

5.4.12. THEOREM. *The lattice of subalgebras of $\mathcal{B} \in \mathbf{Df}_2$ is dually isomorphic to the lattice of bisimulation equivalences of its dual $\mathcal{X}$.*

Proof. The proof is a routine adaptation of the proof of Theorem 2.3.10. □

For any $\mathbf{Df}_2$-space $\mathcal{X} = (X, E_1, E_2)$ and a bisimulation equivalence Q, let $\mathcal{X}/Q$ denote the quotient space of $\mathcal{X}$ by Q. That is, $\mathcal{X}/Q$ is a $\mathbf{Df}_2$-space $\mathcal{X}/Q = (X/Q, (E_1)_Q, (E_2)_Q)$, where $X/Q = \{Q(x) : x \in X\}$, the topology on X/Q is the quotient topology (i.e., the opens of $\mathcal{X}/Q$ are, up to homeomorphism, the Q-saturated opens of $\mathcal{X}$) and $Q(x)(E_i)_QQ(y)$ iff there are $x' \in Q(x)$ and $y' \in Q(y)$ such that $x'E_iy'$ for $i = 1, 2$. The next lemma can be found in [60, Theorem 2.7.17].

5.4.13. LEMMA. *Let $\mathcal{B} = (B, \Diamond_1, \Diamond_2)$ be a $\mathbf{Df}_2$-algebra and $\mathcal{X} = (X, E_1, E_2)$ be its dual $\mathbf{Df}_2$-space. Then $\mathcal{B}$ is simple iff X and $\emptyset$ are the only saturated subsets of $\mathcal{X}$.*

Proof. Suppose there is a closed saturated subset of X distinct from X and $\emptyset$. Then by Theorem 5.4.10 the corresponding congruence relation of $\mathcal{B}$ will be proper and non-trivial. This is a contradiction since $\mathcal{B}$ is simple. The proof for the other direction is similar. $\square$

The next theorem connects the simple $\mathbf{Df}_2$-algebras and rooted $\mathbf{Df}_2$-spaces; see e.g., [60, Theorem 2.7.17].

5.4.14. THEOREM. *Let $\mathcal{B} = (B, \lozenge_1, \lozenge_2)$ be a $\mathbf{Df}_2$-algebra and $\mathcal{X} = (X, E_1, E_2)$ its dual $\mathbf{Df}_2$-space. Then $\mathcal{B}$ is simple iff $\mathcal{X}$ is rooted.*

Proof. By Lemma 5.4.13, all we need to show is that $\mathcal{X}$ is rooted iff X and $\emptyset$ are the only closed saturated subsets of $\mathcal{X}$. If $\mathcal{X}$ is rooted, then there exists $x \in X$ such that $E_1E_2(x) = X$. Thus, X and $\emptyset$ are the only saturated subsets of $\mathcal{X}$. Now we show that if $\mathcal{X}$ is not rooted, then there exists a closed saturated subset U different from X and $\emptyset$. Suppose $\mathcal{X}$ is not rooted. Then there are two distinct points x and y such that there is no u with xE_1u and uE_2y. Let $U = E_1E_2(x)$. By the commutativity of E_1 and E_2 we have $U = E_2E_1(x)$. Therefore, U is saturated. Since E_1 and E_2 are closed relations, U is a closed set. Moreover, $x \in U$ and $y \notin U$ imply that U is different from X and $\emptyset$. Therefore, if $\mathcal{X}$ is not rooted there exists a closed saturated subset of X that is different from X and $\emptyset$. $\square$

5.4.3 $\mathbf{CA}_2$-algebras

In this section we define cylindric algebras with the diagonal. They represent algebraic models of cylindric modal logic with the diagonal.

5.4.15. DEFINITION. [60, Definition 1.1.1] A quadruple $\mathfrak{B} = (B, \lozenge_1, \lozenge_2, d)$ is said to be a *two-dimensional cylindric algebra*, or a $\mathbf{CA}_2$-*algebra* for short, if $(B, \lozenge_1, \lozenge_2)$ is a $\mathbf{Df}_2$-algebra and $d \in B$ is a constant satisfying the following conditions for all $a \in B$ and $i = 1, 2$.

1. $\lozenge_i(d) = 1$;
2. $\lozenge_i(d \wedge a) \leq -\lozenge_i(d \wedge -a)$.

Let $\mathbf{CA}_2$ denote the variety of all two-dimensional cylindric algebras.

Again the standard Lindenbaum-Tarski argument shows that $\mathbf{CML}_2$ is complete with respect to $\mathbf{CA}_2$; see, e.g., [125, §4.2].

5.4.16. THEOREM. $\mathbf{CML}_2 \vdash \phi$ *iff* ϕ *is valid in every* $\mathbf{CA}_2$-*algebra.*

Let $\Lambda(\mathbf{CML}_2)$ denote the lattice of normal extensions of $\mathbf{CML}_2$ and let $\Lambda(\mathbf{CA}_2)$ denote the lattice of subvarieties of $\mathbf{CA}_2$. Then we have the following corollary of Theorem 5.4.16 and the modal logic analogue of Theorem 2.2.19.

5.4.17. COROLLARY. *$\Lambda(\mathbf{CML}_2)$ is dually isomorphic to $\Lambda(\mathbf{CA}_2)$.*

Since we will only deal with two-dimensional cylindric algebras, we simply refer to them as cylindric algebras. Below we will generalize the duality for $\mathbf{Df}_2$-algebras to $\mathbf{CA}_2$-algebras.

5.4.18. DEFINITION. A quadruple (X, E_1, E_2, D) is said to be a *cylindric space* if the triple (X, E_1, E_2) is a $\mathbf{Df}_2$-space and D is a clopen subset of X such that every E_i-cluster $i = 1, 2)$ of X contains a unique point from D.

The following is an immediate consequence of this definition. For an algebraic analogue see [60, Theorem 1.5.3].

5.4.19. PROPOSITION. *Suppose $\mathcal{X}$ is a cylindric space. Then the cardinality of the set of all E_1-clusters of $\mathcal{X}$ is equal to the cardinality of the set of all E_2-clusters of $\mathcal{X}$.*

Proof. The proof is identical to the proof of Corollary 5.3.11. □

We have the following topological representation of cylindric algebras.

5.4.20. THEOREM. *Every cylindric algebra $\mathfrak{B} = (B, \Diamond_1, \Diamond_2, d)$ can be represented as $(\mathcal{CP}(X), E_1, E_2, D)$ for the corresponding cylindric space $\mathcal{X} = (X, E_1, E_2, D)$.*

Proof. The proof is a routine adaptation of Theorem 5.4.9 to cylindric algebras. □

Consequently, every finite cylindric algebra is represented as the algebra $(\mathcal{P}(X), E_1, E_2, D)$ for the corresponding finite $\mathbf{CML}_2$-frame (X, E_1, E_2, D) (see also [60, Theorem 2.7.34]).

In order to obtain the dual description of homomorphic images and subalgebras of cylindric algebras, as well as subdirectly irreducible and simple cylindric algebras, we need the following two definitions. Suppose $\mathcal{X}$ is a cylindric space. A bisimulation equivalence Q of X is called a *cylindric bisimulation equivalence* if $Q(D) = D$. A cylindric space $\mathcal{X}$ is called a *cylindric quasi-square* if its D-free reduct is a quasi-square $\mathbf{Df}_2$-space (see Definition 5.3.7). In other words, a cylindric space is a quasi-square if it is rooted; that is $E_1E_2(x) = X$ for every $x \in X$.

5.4.21. Theorem.

1. *The lattice of congruences of a cylindric algebra $\mathfrak{B}$ is isomorphic to the lattice of closed saturated subsets of its dual $\mathcal{X}$.*

2. *The lattice of subalgebras of a cylindric algebra $\mathfrak{B}$ is dually isomorphic to the lattice of cylindric bisimulation equivalences of its dual $\mathcal{X}$.*

3. *A cylindric algebra $\mathfrak{B}$ is subdirectly irreducible iff it is simple iff its dual $\mathcal{X}$ is a cylindric quasi-square.*

Proof. A routine adaptation of Theorems 5.4.6, 5.4.10, 5.4.12 and 5.4.14 to cylindric algebras. For (3) also see [60, Theorems 2.4.43, 2.4.14]. □

5.4.4 Representable cylindric algebras

In this final section we discuss the representable cylindric algebras, that is, the cylindric algebras corresponding to $\mathbf{PCML}_2$.

5.4.22. Definition.

1. For a rectangle $\mathcal{F} = (W \times W', E_1, E_2)$ let $\mathcal{F}^+ = (\mathcal{P}(W \times W'), E_1, E_2)$ denote the complex algebra of $\mathcal{F}$. We call $\mathcal{F}^+$ a *rectangular algebra.*[2] Let $\mathbb{RECT}$ denote the class of all rectangular $\mathbf{Df}_2$-algebras.

2. Call a rectangular algebra $\mathcal{F}^+$ a *square algebra* if $\mathcal{F}$ is isomorphic to a square. Let $\mathbb{SQ}$ denote the class of all square algebras.

3. For a $\mathbf{CML}_2$-square $\mathfrak{F} = (W \times W, E_1, E_2, D)$ let $\mathfrak{F}^+ = (P(W \times W), E_1, E_2, D)$ denote the complex algebra of $\mathfrak{F}$. We call $\mathfrak{F}^+$ the *cylindric square algebra.* Let $\mathbb{CSQ}$ denote the class of all cylindric square algebras.[3]

4. Also let Fin$\mathbb{RECT}$, Fin$\mathbb{SQ}$ and Fin$\mathbb{CSQ}$ denote the classes of all finite rectangular, square and cylindric square algebras, respectively.

5.4.23. Definition. [60, Definitions 5.1.33(v), 3.1.1(vii) and Remark 1.1.13]

- A $\mathbf{Df}_2$-algebra $\mathcal{B}$ is said to be *rectangularly* (*square*) *representable* if $\mathcal{B} \in \mathbf{SP}(\mathbb{RECT})$ ($\mathcal{B} \in \mathbf{SP}(\mathbb{SQ})$).

---

[2] Note that the concept of a "rectangular algebra" is different from the one of a "rectangular element" defined in [60, Definition 1.10.6].

[3] The rectangular and square algebras are defined in [60, Definitions 3.1.1(v) and 5.1.33(iii)], where they are called "two-dimensional (diagonal-free) cylindric set and uniform cylindric set algebras". However, since we only work with two-dimensional cylindric algebras, we find the terms "rectangular algebra" and "square algebra" more convenient.

- A cylindric algebra $\mathfrak{B}$ is called *representable* if $\mathfrak{B} \in \mathbf{SP}(\mathbb{CSQ})$.[4]

It is known that a $\mathbf{Df}_2$-algebra is rectangularly representable iff it is square representable. We simply call such algebras *representable* [60, Definition 3.1.1]. The classes of representable $\mathbf{Df}_2$ and $\mathbf{CA}_2$-algebras are also closed under homomorphic images, and so form varieties which are usually denoted by $\mathbf{RDf}_2$ and $\mathbf{RCA}_2$, respectively. For the proof of the next theorem consult [60, Theorem 5.1.47] and [43, Corollary 5.10].

5.4.24. Theorem.

1. $\mathbf{RDf}_2 = \mathbf{Df}_2 = \mathbf{HSP}(\mathbb{RECT}) = \mathbf{HSP}(\mathbb{SQ}) = \mathbf{SP}(\mathbb{RECT}) = \mathbf{SP}(\mathbb{SQ})$.
2. $\mathbf{RCA}_2 = \mathbf{HSP}(\mathbb{CSQ}) = \mathbf{SP}(\mathbb{CSQ})$.
3. $\mathbf{RCA}_2 \subsetneq \mathbf{CA}_2$.

Let

$$\text{(H)} := \Diamond_i(a \wedge -b \wedge \Diamond_j(a \wedge b)) \leq \Diamond_j(-d \wedge \Diamond_i a), \quad i \neq j, \quad i,j = 1,2.$$

and

$$\text{(V)} := d \wedge \Diamond_i(-a \wedge \Diamond_j a) \leq \Diamond_j(-d \wedge \Diamond_i a), \quad i \neq j, \quad i,j = 1,2.$$

We call (H) and (V) the *Henkin* and *Venema inequalities*, respectively. Then $\mathbf{RCA}_2$ is axiomatized by adding either of these inequalities to the axiomatization of $\mathbf{CA}_2$; see, e.g., [60, Theorem 3.2.65(ii)] or [124, Proposition 3.5.8]).

5.4.25. Theorem. $\mathbf{RCA}_2 = \mathbf{CA}_2$+(H)$= \mathbf{CA}_2$+(V).

Note that Theorem 5.4.24(1) is an algebraic formulation of Theorem 5.3.2(3). Therefore, we have that $\mathbf{PCML}_2$ is complete with respect to $\mathbf{RCA}_2$ [124, §4.2].

5.4.26. Theorem. $\mathbf{PCML}_2 \vdash \phi$ *iff* ϕ *is valid in every* $\mathbf{RCA}_2$*-algebra.*

Let $\Lambda(\mathbf{PCML}_2)$ denote the lattice of normal extensions of $\mathbf{CML}_2$ and let $\Lambda(\mathbf{RCA}_2)$ denote the lattice of subvarieties of $\mathbf{CA}_2$. We again have the following corollary of Theorem 5.4.16 and the modal logic analogue of Theorem 2.2.19.

5.4.27. Corollary. $\Lambda(\mathbf{PCML}_2)$ *is dually isomorphic to* $\Lambda(\mathbf{RCA}_2)$.

As in Section 5.3.3 (see Theorem 5.3.18), we can give the dual characterization of representable cylindric algebras, and construct rather simple finite non-representable cylindric algebras. We say that a cylindric space $\mathcal{X}$ satisfies $(*)$ if its underlying $\mathbf{CML}_2$-frame satisfies $(*)$ (see Definition 5.3.17). In the terminology of [60], a cylindric space satisfies $(*)$ iff the corresponding cylindric algebra has at least one defective atom (see [60, Lemma 3.2.59]).

[4]The definition of representability is not quite the same as the original one from [60] but is equivalent to it.

5.4.28. Theorem. *A cylindric algebra $\mathfrak{B}$ is representable iff its dual cylindric space $\mathcal{X}$ does not satisfy* $(*)$.

Proof. The proof is almost identical to the one of Theorem 5.3.18. For the details we refer to [14, Theorem 3.4]. $\square$

For an algebraic analogue of Theorem 5.4.28 see [60, Lemma 3.2.59, Theorem 3.2.65]. Using this criterion it is easy to see that the cylindric algebras corresponding to the cylindric spaces shown in Figure 5.1(b) are representable, while the cylindric algebras corresponding to the cylindric spaces shown in Figure 5.1(a) are not. Moreover, the smallest non-representable cylindric algebra is the algebra corresponding to the cylindric space shown in Figure 5.1(a), where the non-singleton E_0-cluster contains only two points.

Chapter 6

Normal extensions of $\mathbf{S5}^2$

In this chapter, which is based on [12], we study the lattice of normal extensions of $\mathbf{S5}^2$. It is known that $\mathbf{S5}^2$ has the finite model property and is decidable [110]; in fact, it has a NEXPTIME-complete satisfiability problem [93]. It is neither tabular nor locally tabular [60] and it lacks the interpolation property [27]. In addition, we show that every proper normal extension of $\mathbf{S5}^2$ is locally tabular, i.e., $\mathbf{S5}^2$ is pre-locally tabular. As a corollary we obtain that every normal extension of $\mathbf{S5}^2$ has the finite model property. We also characterize all tabular extensions of $\mathbf{S5}^2$ by showing that there are exactly six pre-tabular extensions of $\mathbf{S5}^2$. A classification of normal extensions of $\mathbf{S5}^2$ will also be provided.

The lattice of normal extensions of $\mathbf{S5}$ has been well-investigated. It is known that it forms an $(\omega + 1)$-chain, and that every normal extension of $\mathbf{S5}$ is finitely axiomatizable, has the finite model property and is decidable (see [111]). Moreover, every proper normal extension of $\mathbf{S5}$ is tabular. On the other hand, the lattice of normal extensions of $\mathbf{S5}^3$ is much more complicated. It has been shown that $\mathbf{S5}^3$ is not finitely axiomatizable (see Monk [99] and Johnson [67]), that there are continuum many undecidable extensions of $\mathbf{S5}^3$ (see Maddux [88] and Gabbay et al. [43, Theorem 8.5]), and that each of these extensions lacks the finite model property (see Kurucz [79] and Gabbay et al. [43, Theorem 8.12]). We show that the lattice of all normal extensions of $\mathbf{S5}^2$, although complex, is still manageable to some extend.

6.1 The finite model property of $\mathbf{S5}^2$

In this section we prove that $\mathbf{S5}^2$ has the finite model property. Moreover, we show that it is complete with respect to the classes of finite rectangles and finite squares. We also state algebraic analogues of these results.

There is a wide variety of proofs available for the decidability of the classical first-order logic with two variables. Equivalent results were stated and proved using quite different methods in first-order, modal and algebraic logic. We present a short historic overview.

Decidability of the validity of equality-free first-order sentences in two variables was proved by Scott [110]. The proof uses a reduction to the set of prenex formulas of the form $\exists^2\forall^n\varphi$, whose validity is decidable by Gödel [50]. Scott's result was extended by Mortimer [101], who included equality in the language and showed that such sentences cannot enforce infinite models, obtaining decidability as a corollary. A simpler proof was provided in Grädel et al. [55]. They showed that any satisfiable formula can actually be satisfied in a model whose size is single exponential in the length of the formula. Segerberg [113] proved the fmp and decidability for the so-called "two-dimensional modal logic", which is a cylindric modal logic enriched with the operation of involution. For an algebraic proof see [60, Lemma 5.1.24 and Theorem 5.1.64]. A mosaic type proof can be found in Marx and Mikulás [94]. A proof using quasimodels is provided in [43, Theorem 5.22]. Here we give a proof via the filtration method.

6.1.1. THEOREM. $\mathbf{S5}^2$ *has the finite model property.*

Proof. Suppose $\mathbf{S5}^2 \not\vdash \phi$. Then, by Theorems 5.3.2 and 5.2.4, there exists a rooted $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$ and a valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \phi$. Let $Sub(\phi)$ be the set of all subformulas of ϕ. Define an equivalence relation $\equiv$ on W by

$$w \equiv v \text{ iff } w \models \psi \Leftrightarrow v \models \psi \text{ for all } \psi \in Sub(\phi).$$

Let $[w]$ denote the $\equiv$-equivalence class containing the point w, let $W/_{\equiv} = \{[w] : w \in W\}$. We define E_i^f on $W/_{\equiv}$ by

$$[w]E_i^f[v] \text{ iff } w \models \Diamond_i\psi \Leftrightarrow v \models \Diamond_i\psi \text{ for all } \Diamond_i\psi \in Sub(\phi).$$

Let $\mathcal{F}^f = (W/_{\equiv}, E_1^f, E_2^f)$ and define V^f on $\mathcal{F}^f$ by $[w] \in V^f(p)$ iff $w \in V(p)$. The standard filtration argument (see e.g. [18, Theorem 2.39]) shows that for every $\psi \in Sub(\phi)$:

$$(\mathcal{F}, V), w \models \psi \text{ iff } (\mathcal{F}^f, V^f), [w] \models \psi.$$

Therefore, $(\mathcal{F}^f, V^f) \not\models \phi$. We show that $\mathcal{F}^f$ is an $\mathbf{S5}^2$-frame. It follows immediately from the definition of E_i^f that E_i^f is an equivalence relation. Note that for $w, v \in W$ we have:

$$wE_iv \text{ implies } [w]E_i^f[v].$$

We prove that E_1^f and E_2^f commute. Suppose $[w]E_1^f[v]$ and $[v]E_2^f[u]$. Then since $\mathcal{F}$ is rooted, by Lemma 5.3.3, there exists $z \in W$ such that wE_2z and zE_1u. Therefore, $[w]E_2^f[z]$ and $[z]E_1^f[u]$, which means that E_1^f and E_2^f commute and thus $\mathcal{F}^f$ is an $\mathbf{S5}^2$-frame.

$\square$

6.1.2. COROLLARY.

1. *$\mathbf{S5}^2$ is decidable.*

2. *$\mathbf{Df}_2$ is finitely approximable.*

3. *The equational theory of $\mathbf{Df}_2$ is decidable.*

Proof. (1) The result follows immediately from Theorem 6.1.1 since every finitely axiomatizable logic with the fmp is decidable (see Section 2.1.2).

(2) The result follows immediately from Theorem 6.1.1 and the modal logic analogue of Theorem 2.3.27(1).

(3) Apply (1) and the modal logic analogue of Theorem 2.3.27(6). □

Questions concerning the computational complexity of $\mathbf{S5}^2$ and its normal extensions will be addressed in Chapter 8. Here we show that $\mathbf{S5}^2$ is not only complete, for finite $\mathbf{S5}^2$-frames, but also complete for finite rectangles and finite squares.

6.1.3. DEFINITION. For a class K of algebras or frames let $\mathrm{Fin}(\mathsf{K})$ denote the class of finite members of K.

We now show that $\mathbf{S5}^2$ is complete with respect to $\mathrm{Fin}(\mathbf{Rect})$ and $\mathrm{Fin}(\mathbf{Sq})$. The next lemma is an analogue of Theorem 2.3.9. It shows a connection between p-morphisms and bisimulation equivalences.

6.1.4. LEMMA. *Let $\mathcal{F} = (W, E_1, E_2)$ be a finite $\mathbf{S5}^2$-frame and Q an equivalence relation on W. Let $\mathcal{F}/Q = (W/Q, E_1', E_2')$, where for $i = 1, 2$:*

$$Q(w)E_i'Q(v) \textit{ iff there exist } w' \in Q(w) \textit{ and } v' \in Q(v) \textit{ with } w'E_iv'.$$

Let the function $f_Q : W \to W/Q$ be defined by $f_Q(w) = Q(w)$ for any $w \in W$. Then the following two conditions are equivalent:

1. *Q is a bisimulation equivalence,*

2. *f_Q is a p-morphism.*

Proof. The proof is similar to the proof of Theorem 2.3.9. □

Next we prove a number of auxiliary lemmas.

6.1.5. LEMMA. *For every finite rectangle $\mathcal{F}$, there exists a finite square $\mathcal{G}$ such that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

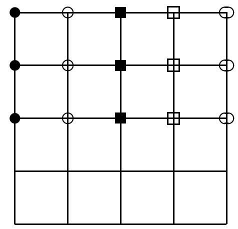

Figure 6.1: A bisimulation equivalence of a finite square

Proof. Suppose $\mathcal{F}$ is isomorphic to $\mathbf{m} \times \mathbf{n}$ with $n > m$. Let $\mathcal{G}$ be the square $\mathbf{n} \times \mathbf{n}$. Define Q on $\mathbf{n} \times \mathbf{n}$ by identifying all points (k, i), (k, j) such that $k \in n$ and $m - 1 \leq i, j < n$ (see Figure 6.1, where the points of the same color are identified). It is routine to check that Q is a bisimulation equivalence of $\mathbf{n} \times \mathbf{n}$, and that the quotient of $\mathbf{n} \times \mathbf{n}$ by Q is a rectangle isomorphic to $\mathbf{m} \times \mathbf{n}$. Thus, by Lemma 6.1.4, $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. □

6.1.6. Definition. Let $\mathcal{F} = (W, E_1, E_2)$ be a rooted $\mathbf{S5}^2$-frame.

1. $\mathcal{F}$ is said to be a *bicluster* if $E_1(w) = E_2(w) = W$ for each $w \in W$.

2. $\mathcal{F}$ is said to be *regular* if every E_0-cluster of $\mathcal{F}$ has the same cardinality.

6.1.7. Lemma. *For every finite bicluster $\mathcal{F}$, there exists a finite square $\mathcal{G}$ such that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

Proof. Suppose $\mathcal{F}$ consists of n points. Consider the square $\mathbf{n} \times \mathbf{n}$. Define an equivalence relation Q on $\mathbf{n} \times \mathbf{n}$ by

$$(k, m)R(k', m') \text{ iff } k - m \equiv k' - m' \pmod{n}.$$

This means that every Q-equivalence class contains a unique point from every E_i-cluster (see Figure 6.2, where points of the same color are identified). Since $QE_i(k, m) = n \times n = E_iQ(k, m)$ for each $k, m \in n$ and $i = 1, 2$, we have that Q is a bisimulation equivalence of $\mathbf{n} \times \mathbf{n}$. It should be clear now that the quotient of $\mathbf{n} \times \mathbf{n}$ by Q is isomorphic to $\mathcal{F}$, thus by Lemma 6.1.4, $\mathcal{F}$ is a p-morphic image of $\mathbf{n} \times \mathbf{n}$.

□

6.1.8. Lemma. *For every finite regular rooted $\mathcal{F}$, there exists a finite rectangle $\mathcal{G}$ such that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

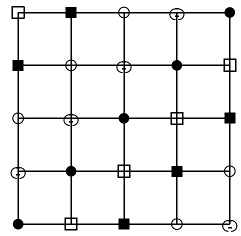

Figure 6.2: The partition of a finite square

Proof. Let $\{C_i\}_{i=1}^n$ and $\{C^j\}_{j=1}^m$ be the sets of all E_1 and E_2-clusters of $\mathcal{F}$, respectively. Also let C_i^j denote the E_0-cluster $C_i^j = C_i \cap C^j$. Since $\mathcal{F}$ is regular, the cardinality of every C_i^j is the same. Let $|C_i^j| = k > 0$ for every $i \leq n$ and $j \leq m$. Now consider the rectangle $\mathbf{nk} \times \mathbf{mk}$. Let Δ^j be $(jk \times nk) \backslash ((j-1)k \times nk)$ and Δ_i be $(mk \times ik) \backslash (mk \times (i-1)k)$. Therefore, if we think of $\mathbf{nk} \times \mathbf{mk}$ as the rectangle shown in Figure 6.3, then Δ_i is the rectangle consisting of all the rows of $\mathbf{nk} \times \mathbf{mk}$ between the $(i-1)k$-th and ik-th rows and Δ_j is the rectangle consisting of all the columns of $\mathbf{nk} \times \mathbf{mk}$ between the $(j-1)k$-th and jk-th columns. Also let $\Delta_i^j = \Delta_i \cap \Delta^j$. Then Δ_i^j is the square with k E_1 and E_2-clusters. Define a partition Q on $\mathbf{nk} \times \mathbf{mk}$ by sewing each square Δ_i^j into a bicluster as in the proof of Lemma 6.1.7. It follows that Q is a bisimulation equivalence, and that the quotient of $\mathbf{nk} \times \mathbf{mk}$ is isomorphic to $\mathcal{F}$. Hence, $\mathcal{F}$ is a p-morphic image of $\mathbf{nk} \times \mathbf{mk}$.

□

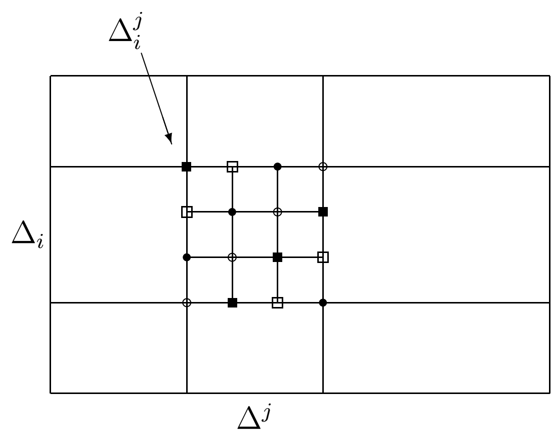

Figure 6.3: The rectangle $\mathbf{nk} \times \mathbf{mk}$.

6.1.9. Lemma. *For every finite rooted $\mathbf{S5}^2$-frame $\mathcal{F}$, there exists a finite regular frame $\mathcal{G}$ such that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

Proof. Let C_i, C^j and C_i^j be the same as in the proof of Lemma 6.1.8. Also let $k = max\{|C_i^j| : i \leq n, j \leq m\}$. Obviously all $|C_i^j| > 0$ and therefore $k > 0$. Consider the regular frame $\mathcal{G}$ which is obtained from $\mathcal{F}$ by replacing every E_0-cluster of $\mathcal{F}$ by an E_0-cluster containing k points. Let Q be an equivalence relation on $\mathcal{G}$ identifying $k-(|C_i^j|+1)$ points in each E_0-cluster of $\mathcal{G}$, (see Figure 6.4, where filled circles represent the identified points). Note that in the E_0-clusters with k points we do not identify any points. It should be clear that Q is a bisimulation equivalence of $\mathcal{G}$, and that the quotient of $\mathcal{G}$ by Q is isomorphic to $\mathcal{F}$. Therefore, $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. $\square$

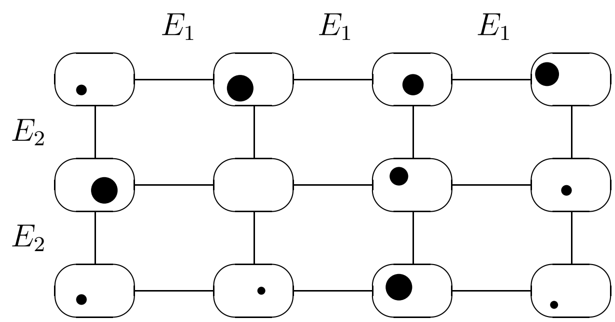

Figure 6.4: The identifications in a regular frame

Now we are ready to prove that $\mathbf{S5}^2$ is complete with respect to finite rectangles and squares. This result was first proved by Segerberg [113]. His technique is very similar to ours with the exception that he considers the similarity type with one additional unary operation, the involution. The result also follows from Mortimer [101]. A short algebraic proof can be found in Andréka and Nemeti [1]. For a different frame theoretic proof using quasi-models see [43, Theorem 5.25].

6.1.10. Theorem. $\mathbf{S5}^2$ *is complete with respect to* Fin(**Rect**) *and* Fin(**Sq**).

Proof. By Theorem 6.1.1, if $\mathbf{S5}^2 \not\vdash \phi$, then ϕ is refuted in a finite rooted $\mathbf{S5}^2$-frame. By Lemmas 6.1.7–6.1.9, every finite rooted $\mathbf{S5}^2$-frame is a p-morphic image of a finite rectangle and by Lemma 6.1.5, it is a p-morphic of a finite square. Since p-morphic images preserve validity of formulas, the result follows. $\square$

6.1.11. Theorem. $\mathbf{Df}_2$ *is generated by finite rectangular* (*square*) *algebras, that is* $\mathbf{Df}_2 = \mathbf{HSP}(\text{Fin}\mathbb{RECT}) = \mathbf{HSP}(\text{Fin}\mathbb{SQ})$.

Proof. Follows immediately from Theorems 6.1.10 and 5.4.3. $\square$

6.1.12. Remark. More algebraic properties of $\mathbf{Df}_2$ are discussed in [12]. In particular, a characterization of finitely approximable $\mathbf{Df}_2$-algebras, projective and injective $\mathbf{Df}_2$-algebras, and absolute retracts of $\mathbf{Df}_2$ is given in [12, §3.1 and §3.2].

6.2 Locally tabular extensions of $\mathbf{S5}^2$

In this section we investigate locally tabular extensions of $\mathbf{S5}^2$ and locally finite subvarieties of $\mathbf{Df}_2$. We recall that a logic L is locally tabular if for every $n \in \omega$ there are only finitely many pairwise non-L-equivalent formulas in n variables, and that a variety $\mathbf{V}$ is locally finite if every finitely generated $\mathbf{V}$-algebra is finite. As follows from Theorem 2.3.27, a logic is locally tabular iff its corresponding variety of algebras is locally finite. It is well known that $\mathbf{S5}$ is locally tabular; see, e.g, [58]. Now we show that $\mathbf{S5}^2$ is not locally tabular. We will approach the problem from an algebraic perspective. It was Tarski who first noticed that $\mathbf{Df}_2$ is not locally finite. Below we sketch Tarski's example. It can also be found in any of these references: Henkin, Monk and Tarski [60, Theorem 2.1.11], Halmos [58, p.92], Erdős, Faber and Larson [33].

6.2.1. Example. Consider the infinite square $\omega \times \omega$. Let $g = \{(n, m) : n \leq m\}$. Then the $\mathbf{Df}_2$-algebra $\mathbb{G} \subseteq P(\omega \times \omega)$ generated by g is infinite. Indeed, let $g_1 = (\omega \times \omega) \setminus E_2((\omega \times \omega) \setminus g)$ and $g_2 = (\omega \times \omega) \setminus E_1(g \setminus g_1)$. Then it is easy to check that $g_1 = \{(0, n) : n \in \omega\}$ and $g_2 = \{(n, 0) : n \in \omega\}$. Then clearly $g_1 \cap g_2 = \{(0, 0)\}$. Now let $S := (\omega \times \omega) \setminus (g_1 \cup g_2)$ and $g' := g \cap S_1$. We define g'_1 and g'_2 for the infinite square S in the same way we defined g_1 and g_2 for $\omega \times \omega$. That is, $g'_1 = S \setminus E_2(S \setminus g')$ and $g'_2 = S \setminus E_1(g' \setminus g'_1)$ Then $g'_1 = \{(1, n) : n > 0\}$ and $g'_2 = \{(n, 1) : n \in \omega\}$. Therefore, $g'_1 \cap g'_2 = \{(1, 1)\}$. Continuing this process we obtain that every element of the diagonal $\Delta = \{(n, n)\}_{n \in \omega}$ is an element of $\mathbb{G}$. Hence $\mathbb{G}$ is infinite. In fact, every singleton $\{(n, m)\}$ of $\omega \times \omega$ belongs to $\mathbb{G}$, since $\{(n, m)\} = E_2(n, n) \cap E_1(m, m)$.

In contrast to this, we will prove that every proper subvariety of $\mathbf{Df}_2$ is locally finite. First we prove an auxiliary lemma.

6.2.2. Lemma.

(i) *The rectangle* $\mathbf{k} \times \mathbf{m}$ *is a p-morphic image of every rooted* $\mathbf{S5}^2$*-frame* $\mathcal{F} = (W, E_1, E_2)$ *containing* k E_1*-clusters and* m E_2*-clusters.*

(ii) *The rectangle* $\mathbf{k}' \times \mathbf{m}'$ *is a p-morphic image of* $\mathbf{k} \times \mathbf{m}$ *for every* $k \geq k'$ *and* $m \geq m'$.

Proof. (i) Consider $\mathcal{F}/E_0 = (W/E_0, E'_1, E'_2)$. It is easy to see that E_0 is a bisimulation equivalence and that $\mathcal{F}/E_0$ is isomorphic to $\mathbf{k} \times \mathbf{m}$. Hence, $\mathbf{k} \times \mathbf{m}$ is a p-morphic image of $\mathcal{F}$.

(ii) A similar argument to Lemma 6.1.5 shows that $\mathbf{k}' \times \mathbf{m}'$ is a p-morphic image of $\mathbf{k} \times \mathbf{m}$ for $k' \leq k$ and $m' \leq m$. $\square$

6.2.3. Definition. Let a simple $\mathbf{Df}_2$-algebra $\mathcal{B}$ and its dual $\mathcal{X}$ be given, $i = 1, 2$ and $n > 0$.

1. $\mathcal{X}$ is said to be of E_i-*depth* n if the number of E_i-clusters of $\mathcal{X}$ is exactly n.

2. The E_i-depth of $\mathcal{X}$ is said to be *infinite* if $\mathcal{X}$ has infinitely many E_i-clusters.

3. $\mathcal{B}$ is said to be of E_i-*depth* $n < \omega$ if the E_i-depth of $\mathcal{X}$ is n.

4. The E_i-depth of $\mathcal{B}$ is said to be of *infinite* if $\mathcal{X}$ is of infinite E_i-depth.

5. $\mathbf{V} \subseteq \mathbf{Df}_2$ is said to be of E_i-*depth* $n < \omega$ if n is the maximal E_i-depth of the simple members of $\mathbf{V}$, and $\mathbf{V}$ is of E_i-*depth* ω if there is no bound on the E_i-depth of simple members of $\mathbf{V}$.

For a simple $\mathbf{Df}_2$-algebra $\mathcal{B}$ and its dual $\mathcal{X}$, let $d_i(\mathcal{B})$ and $d_i(\mathcal{X})$ denote the E_i-depth of $\mathcal{B}$ and $\mathcal{X}$, respectively. Similarly, let $d_i(\mathbf{V})$ denote the E_i-depth of a variety $\mathbf{V} \subseteq \mathbf{Df}_2$.

Consider the following formulas:

$$D_i^n := \bigwedge_{k=1}^{n} \Diamond p_k \to \bigvee_{k \neq l,\ 1 \leq k,l \leq n} \Diamond(p_k \wedge \Diamond_i p_l)$$

where $n \in \omega$, $i = 1, 2$ and $\Diamond\phi := \Diamond_1\Diamond_2\phi$, for every formula ϕ.

We have the following characterization of varieties of E_i-depth n, where $i = 1, 2$ and $0 < n < \omega$:

6.2.4. THEOREM. *Let $\mathcal{B}$ a simple $\mathbf{Df}_2$-algebra, and $\mathbf{V}$ be a variety of $\mathbf{Df}_2$-algebras.*

1. *D_i^n is valid in a simple $\mathcal{B}$ iff the E_i-depth of $\mathcal{B}$ is less than or equal to n.*

2. *$\mathbf{V}$ is of E_i-depth n iff $\mathbf{V} \subseteq \mathbf{Df}_2 + D_i^n$ and $\mathbf{V} \not\subseteq \mathbf{Df}_2 + D_i^{n-1}$.*

Proof. It is easy to see that D_i^n is a Sahlqvist formula, for every $n \in \omega$. Now apply the standard Sahlqvist algorithm (see [18, §3.6] for the details). $\square$

6.2.5. DEFINITION. For a variety $\mathbf{V}$, let $\mathrm{SI}(\mathbf{V})$ and $\mathrm{S}(\mathbf{V})$ denote the classes of all subdirectly irreducible and simple $\mathbf{V}$-algebras, respectively. Let also $\mathrm{FinSI}(\mathbf{V})$ and $\mathrm{FinS}(\mathbf{V})$ denote the classes of all finite subdirectly irreducible and simple $\mathbf{V}$-algebras, respectively.

Now we reformulate Theorem 3.4.23 in algebraic terms; see [7].

6.2.6. THEOREM. *A variety $\mathbf{V}$ of a finite signature is locally finite iff the class $\mathrm{SI}(\mathbf{V})$ is uniformly locally finite; that is, for each natural number n there is a natural number $M(n)$ such that $|A| \leq M(n)$ for each n-generated $\mathcal{A} \in \mathrm{SI}(\mathbf{V})$.*

6.2.7. LEMMA. $\mathbf{Df}_2 + D_i^m$ *is locally finite for any* $0 < m < \omega$ *and* $i = 1, 2$.

Proof. Since $\mathrm{SI}(\mathbf{Df}_2) = \mathrm{S}(\mathbf{Df}_2)$ and $\mathbf{Df}_2$ has a finite signature, it is sufficient to show that $\mathrm{S}(\mathbf{Df}_2 + D_i^m)$ is uniformly locally finite for each $i = 1, 2$. We will prove that $\mathrm{S}(\mathbf{Df}_2 + D_1^m)$ is uniformly locally finite. The case of $\mathrm{S}(\mathbf{Df}_2 + D_2^m)$ is completely analogous. Suppose $\mathcal{B} = (B[g_1, ..., g_n], \Diamond_1, \Diamond_2)$ is an n-generated simple algebra from the variety $\mathbf{Df}_2 + D_1^m$, where $g_1, ..., g_n$ denote the generators of $\mathcal{B}$. Then for each $a \in B[g_1, ..., g_n]$, there is a polynomial $P(g_1, ..., g_n)$, including Boolean operations as well as $\Diamond_1$ and $\Diamond_2$, such that $a = P(g_1, ..., g_n)$. Let $B_1 = \{\Diamond_1 b : b \in B\}$, and let $\mathcal{X}$ be the dual of $\mathcal{B}$. Every E_1-saturated subset of $\mathcal{X}$ is a union of E_1-clusters. Since there are at most m E_1-clusters of $\mathcal{X}$, there are at most 2^m distinct E_1-saturated sets. Since elements of $\mathcal{B}$ of the form $\Diamond_1 b$ correspond to E_1-saturated clopens, we obtain that $|B_1| \leq 2^m$. Suppose $B_1 = \{a_1, ..., a_k\}$, $k \leq 2^m$. Then for every element b of $\mathcal{B}$, there exists $a_j \in B_1$ such that $\Diamond_1 b = a_j$. Therefore, by substituting every subformula of $P(g_1, \ldots, g_n)$ of the form $\Diamond_1 b$ by a_j, we obtain that $a = P'(g_1, ..., g_n, a_1, \ldots, a_k)$, where P' is a new $\Diamond_1$-free polynomial. Thus, $B[g_1, ..., g_n]$ is generated by $g_1, ..., g_n, a_1, ..., a_k$ as a $\mathbf{Df}_1$-algebra. Since $\mathbf{Df}_1$ is locally finite, there exists $M(n)$ such that $|B[g_1, ..., g_n]| \leq M(n)$. Therefore, $\mathrm{S}(\mathbf{Df}_2 + D_i^m)$ is uniformly locally finite. $\square$

We proceed by showing that the join of two locally finite varieties is locally finite.

6.2.8. LEMMA. *The join of two locally finite varieties is locally finite.*

Proof. Suppose $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are locally finite varieties. In order to arrive at a contradiction, suppose that $A \in \mathbf{V} = \mathbf{HSP}(\mathbf{V}_1 \cup \mathbf{V}_2)$ is a finitely generated infinite algebra. $A \in \mathbf{V}$ implies there exists a family $\{A_i\}_{i \in I}$ with $A_i \in \mathbf{V}_1 \cup \mathbf{V}_2$ such that $A \in \mathbf{HS}(\prod_{i \in I} A_i)$. For each $i \in I$ we have $A_i \in \mathbf{V}_1$ or $A_i \in \mathbf{V}_2$. Let $I_1 = \{\, i \in I \mid A_i \in \mathbf{V}_1 \,\}$ and $I_2 = \{\, i \in I \mid A_i \in \mathbf{V}_2 \backslash \mathbf{V}_1 \,\}$. Obviously $\prod_{i \in I} A_i$ is isomorphic to $\prod_{i \in I_1} A_i \times \prod_{i \in I_2} A_i$. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ are varieties, $\prod_{i \in I_1} A_i \in \mathbf{V}_1$ and $\prod_{i \in I_2} A_i \in \mathbf{V}_2$. Hence, there exist algebras $A_1 = \prod_{i \in I_1} A_i$ in $\mathbf{V}_1$ and $A_2 = \prod_{i \in I_2} A_i$ in $\mathbf{V}_2$ such that $A \in \mathbf{HS}(A_1 \times A_2)$. Therefore there is an algebra $A' \in \mathbf{V}$ such that A is a homomorphic image of A' and there is an embedding ι of A' into $A_1 \times A_2$. Without loss of generality we may assume that A' is finitely generated.[1] Since A is infinite, A' is infinite as well. Let π_i be the natural projection of $A_1 \times A_2$ onto A_i ($i = 1, 2$). Then A' is (isomorphic to) a subalgebra of $\pi_1\iota(A') \times \pi_2\iota(A')$. Therefore at least one of $\pi_i\iota(A')$ is infinite. On the other hand, the latter two algebras, being homomorphic images of A', are finitely generated. Hence, at least one of $\mathbf{V}_i$ is not locally finite, which is a contradiction. $\square$

[1] If A' is not finitely generated then we consider a finitely generated subalgebra of A' generated by the elements of A' that are mapped to the generators of A.

Now we are in a position to prove that every proper subvariety of $\mathbf{Df}_2$ is locally finite.

6.2.9. LEMMA. *If a variety* $\mathbf{V} \subseteq \mathbf{Df}_2$ *is not locally finite, then* $\mathbf{V} = \mathbf{Df}_2$.

Proof. Suppose $\mathbf{V}$ is not locally finite. Then there exists a finitely generated infinite $\mathbf{V}$-algebra $\mathcal{B}$. Let $\mathcal{X}$ be the dual of $\mathcal{B}$. Then either there exists an infinite rooted generated subframe of $\mathcal{X}$, or $\mathcal{X}$ consists of infinitely many finite rooted generated subframes.

First suppose that $\mathcal{X}$ contains an infinite rooted generated subframe $\mathcal{X}_0$. If either the E_1 or E_2-depth of $\mathcal{X}_0$ is finite, then the $\mathbf{Df}_2$-algebra corresponding to $\mathcal{X}_0$ belongs to $\mathbf{Df}_2 + D_i^n$ for some $n \in \omega$. Let $\mathcal{B}_0$ be the $\mathbf{Df}_2$-algebra dual to $\mathcal{X}_0$. Then $\mathcal{B}_0$ is a homomorphic image of $\mathcal{B}$ and is finitely generated. Moreover, by our assumption this algebra is infinite. This is a contradiction by Lemma 6.2.7. Thus, both the E_1 and E_2-depths of $\mathcal{X}_0$ are infinite. Consider $\mathcal{X}_0/E_0$ and denote it by $\mathcal{Y}$. Since both depths of $\mathcal{X}_0$ are infinite, $\mathcal{Y}$ is an infinite rectangle of infinite E_1 and E_2-depths. We will show that the complex algebra $(\mathcal{P}(n \times n), E_1, E_2)$ is a subalgebra of $(\mathcal{CP}(\mathcal{Y}), E_1, E_2)$ for any $n < \omega$.

6.2.10. CLAIM. *There exists a bisimulation equivalence* Q *of* $\mathcal{Y}$ *such that* $\mathcal{Y}/Q$ *is isomorphic to the square* $\mathbf{n} \times \mathbf{n}$.

Proof. Pick $n-1$ points $x_1, \ldots, x_{n-1} \in Y$ such that $\neg(x_p E_i x_q)$, $p \neq q$, $1 \leq p, q \leq n-1$ and $i = 1, 2$. Obviously $\bigcup_{k=1}^{n-1} E_1(x_k)$ is a closed E_1-saturated set and $U_1 = Y \setminus \bigcup_{k=1}^{n-1} E_1(x_k)$ is an open E_1-saturated set. Hence, there exists a non-empty E_1-saturated clopen $C_1 \subseteq U_1$. Clearly, $Y = C_1 \cup (Y \setminus C_1)$. Now consider $U_2 = Y \setminus (C_1 \cup \bigcup_{k=2}^{n-1} E_1(x_k))$. Since $x_1 \in U_2$, U_2 is non-empty and obviously is an E_1-saturated open set. Hence, there exists an E_1-saturated clopen C_2 such that $x_2 \in C_2$ and $C_2 \subseteq U_2$. Then $(Y \setminus C_1) = C_2 \cup ((Y \setminus C_1) \setminus C_2)$. Now let $U_3 = Y \setminus (C_1 \cup C_2 \cup \bigcup_{k=3}^{n-1} E_1(x_k))$. Since $x_2 \in U_3$, U_3 is a non-empty E_1-saturated open set, and there exists an E_1-saturated clopen C_3 such that $x_3 \in C_3$ and $C_3 \subseteq U_3$. Therefore, $(Y \setminus (C_1 \cup C_2) = C_3 \cup (((Y \setminus C_1) \setminus C_2) \setminus C_3)$. We continue this process $(n-1)$ times. At each stage U_k is non-empty, since $x_{k-1} \in U_k$. As a result we get a partition of Y into n E_1-saturated clopens $C_1, C_2, \ldots C_{n-1}, C_n = Y \setminus \bigcup_{j=1}^{n-1} C_j$. Now in exactly the same way we select n E_2-saturated clopens $D_1, D_2, \ldots D_n$ such that $\bigcup_{i=1}^{n} D_i = Y$ and $D_i \cap D_j = \emptyset$. Consider the partition $Q = \{C_j \cap D_k\}_{1 \leq j,k \leq n}$ (see Figure 6.5).

Since every C_j is E_1-saturated and every D_k is E_2-saturated, by Lemma 5.3.6, we have that $C_j \cap D_k \neq \emptyset$, for every $1 \leq j, k \leq n$. Moreover, as C_j and D_k are clopens, $C_j \cap D_k$ is also clopen for every $1 \leq j, k \leq n$. Thus, Q is a partition of Y into n^2 clopens. Now we show that $QE_i(x) \subseteq E_iQ(x)$ for each $x \in Y$ and $i = 1, 2$. If $y \in QE_1(x)$, then there exists $z \in E_1(x)$ such that yQz. Also suppose that $x \in C_j \cap D_k$. Then $z, y \in C_j \cap D_l$ for some l. As $\mathcal{Y}$ is rooted,

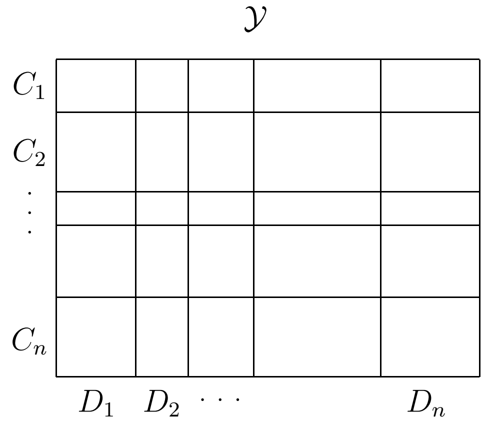

Figure 6.5: The partition of $\mathcal{Y}$

by Lemma 5.3.6, $E_1(y) \cap D_k \neq \emptyset$. Moreover, since C_j is E_1-saturated, we have that $E_1(y) \subseteq C_j$. Therefore, $E_1(y) \cap (C_j \cap D_k) = E_1(y) \cap D_k \neq \emptyset$. Hence, there exists $u \in E_1(y) \cap (C_j \cap D_k)$, which means that $u \in E_1(y)$ and $u \in Q(x)$. Thus, $y \in E_1Q(x)$ and $QE_1(x) \subseteq E_1Q(x)$. That $QE_2(x) \subseteq E_2Q(x)$ is proved similarly. Thus, Q is a bisimulation equivalence of $\mathcal{Y}$. Moreover, it follows from the construction of Q that $\mathcal{Y}/Q$ is isomorphic to $\mathbf{n} \times \mathbf{n}$. $\square$

Therefore, $(\mathbf{n} \times \mathbf{n})^+ = (\mathcal{P}(n \times n), E_1, E_2)$ is a subalgebra of $(\mathcal{CP}(\mathcal{Y}), E_1, E_2)$. Since $\mathbf{Df}_2$ is generated by finite square algebras (see Theorem 5.4.24) it follows that $\mathbf{V} = \mathbf{Df}_2$.

Now suppose that $\mathcal{X}$ consists of infinitely many finite rooted frames which we denote by $\{\mathcal{X}_j\}_{j \in J}$. If both the E_1 and E_2-depths of the members of $\{\mathcal{X}_j\}_{j \in J}$ are bounded by some integer n, then their corresponding algebras belong to $\mathbf{Df}_2 + D_i^n$ for some $i = 1, 2$. This means that there is an infinite finitely generated algebra in $\mathbf{Df}_2 + D_i^n$, which, by Lemma 6.2.7, is a contradiction. Therefore, we can assume that either the E_1 or E_2-depth of $\{\mathcal{X}_j\}_{j \in J}$ are not bounded by any integer. We distinguish the following two cases:

Case 1. $\mathcal{X}$ consists of two families $\{\mathcal{X}'_j\}_{j \in J'}$ and $\{\mathcal{X}''_j\}_{j \in J''}$ such that the E_2-depth of the members of the first family is bounded by some integer n, but the E_1-depth of them is not bounded by any integer; and conversely, the E_1-depth of the members of the second family is bounded by some integer m, but the E_2-depth of them is unbounded.

Consider the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$, where $\mathbf{V}_1$ denotes the variety generated by the algebras corresponding to the members of the first family, while $\mathbf{V}_2$ denotes the variety generated by the algebras corresponding to the members of the second family. Observe that $\mathcal{B} \in \mathbf{V}_1 \vee \mathbf{V}_2 = \mathbf{HSP}(\mathbf{V}_1 \cup \mathbf{V}_2)$.

Now from Lemma 6.2.7 it follows that both $\mathbf{V}_1$ and $\mathbf{V}_2$ are locally finite. By Lemma 6.2.8, $\mathbf{V}_1 \vee \mathbf{V}_2$ is locally finite. Therefore, $\mathcal{B}$ is finite, which contradicts our assumption.

Case 2. Both the E_1 and E_2-depths of $\mathcal{X}_j$ are not bounded by any integer. Therefore, for every $n \in \omega$, there exists $\mathcal{X}_j$ such that $d_1(\mathcal{X}_j), d_2(\mathcal{X}_j) > n$. By Lemma 6.2.2, $\mathbf{n} \times \mathbf{n}$ is a p-morphic image of $\mathcal{X}_j$, which by Theorem 6.1.11 means that $\mathbf{V} = \mathbf{Df}_2$.

Thus, if $\mathbf{V}$ is not locally finite, then $\mathbf{V} = \mathbf{Df}_2$, which completes the proof of the lemma. □

Recall from Chapter 4 that a variety $\mathbf{V}$ is called pre-locally finite if $\mathbf{V}$ is not locally finite but every proper subvariety of $\mathbf{V}$ is locally finite, and that a logic L is called pre-locally tabular if L is not locally tabular but every proper extension of L is locally tabular.

6.2.11. COROLLARY.

1. *$\mathbf{V} \in \Lambda(\mathbf{Df}_2)$ is locally finite iff $\mathbf{V}$ is a proper subvariety of $\mathbf{Df}_2$.*
2. *$\mathbf{Df}_2$ is the only pre-locally finite subvariety of $\mathbf{Df}_2$.*
3. *Every variety $\mathbf{V} \subseteq \mathbf{Df}_2$ is finitely approximable.*

Proof. The result follows immediately from Lemma 6.2.9 and Corollary 6.1.2(2). □

6.2.12. COROLLARY.

1. *$L \in \Lambda(\mathbf{S5}^2)$ is locally tabular iff L is a proper normal extension of $\mathbf{S5}^2$.*
2. *$\mathbf{S5}^2$ is the only pre-locally tabular extension of $\mathbf{S5}^2$.*
3. *Every normal extension of $\mathbf{S5}^2$ has the finite model property.*

Proof. The result is an immediate consequence of Corollary 6.2.11 and Theorem 5.4.3. □

6.3 Classification of normal extensions of $\mathbf{S5}^2$

In this section we prove a more specific version of Theorem 6.2.12. In particular, for each proper normal extension of $\mathbf{S5}^2$, we describe the class of its finite rooted frames in terms of their E_1 and E_2-depths. The E_1 and E_2-depth of an $\mathbf{S5}^2$-frame $\mathcal{F}$ is defined in exactly the same way as in Definition 6.2.3. For every logic $L \supseteq \mathbf{S5}^2$ let $\mathbf{F}_L$ denote the set of all finite L-frames modulo isomorphism. It follows from Theorem 6.2.12 that every extension of $\mathbf{S5}^2$ is complete with respect to $\mathbf{F}_L$.

6.3.1. Definition. For every logic $L \supseteq \mathbf{S5}^2$ let $d_i(L) = d_i(\mathbf{F}_L)$.

6.3.2. Theorem. *For every proper normal extension L of $\mathbf{S5}^2$ there exists a natural number n such that $\mathbf{F}_L$ can be divided into three disjoint classes $\mathbf{F}_L = \mathbf{F}_1 \uplus \mathbf{F}_2 \uplus \mathbf{F}_3$, where $d_2(\mathbf{F}_1),\ d_1(\mathbf{F}_2) \leq n$ and $d_1(\mathbf{F}_3),\ d_2(\mathbf{F}_3) \leq n$. (Note that any two of the classes $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ may be empty.)*

Proof. Suppose L is a proper normal extension of $\mathbf{S5}^2$. By Theorem 6.1.10, $\mathbf{S5}^2$ is complete with respect to the class of all finite squares. Therefore, there exists a square $\mathbf{n} \times \mathbf{n}$ such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Let n be the minimal number such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Consider three subclasses of $\mathbf{F}_L$: $\mathbf{F}_1 = \{\mathcal{F} \in \mathbf{F}_L : d_1(\mathcal{F}) > n\}$, $\mathbf{F}_2 = \{\mathcal{F} \in \mathbf{F}_L : d_2(\mathcal{F}) > n\}$ and $\mathbf{F}_3 = \{\mathcal{F} \in \mathbf{F}_L : d_1(\mathcal{F}), d_2(\mathcal{F}) \leq n\}$. It is obvious that $\mathbf{F}_L = \mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3$. We prove that $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are disjoint.

Let us show that if $\mathcal{F} \in \mathbf{F}_1$, then $d_2(\mathcal{F}) \leq n$ and if $\mathcal{F} \in \mathbf{F}_2$, then $d_1(\mathcal{F}) \leq n$. Suppose $\mathcal{F} \in \mathbf{F}_1 \cup \mathbf{F}_2$, $d_1(\mathcal{F}) = k$, $d_2(\mathcal{F}) = m$ and both $k, m > n$. By Lemma 6.2.2(i), $\mathbf{k} \times \mathbf{m}$ is a p-morphic image of $\mathcal{F}$, and by Lemma 6.2.2(ii), $\mathbf{n} \times \mathbf{n}$ is a p-morphic image of $\mathbf{k} \times \mathbf{m}$. So, $\mathbf{n} \times \mathbf{n}$ is a p-morphic image of $\mathcal{F}$, and hence $\mathbf{n} \times \mathbf{n}$ belongs to $\mathbf{F}_L$, which is a contradiction. Thus, $\mathcal{F} \in \mathbf{F}_1$ implies $d_1(\mathcal{F}) > n$ and $d_2(\mathcal{F}) \leq n$, and $\mathcal{F} \in \mathbf{F}_2$ implies $d_1(\mathcal{F}) \leq n$ and $d_2(\mathcal{F}) > n$. Also, if $\mathcal{F} \in \mathbf{F}_3$, then $d_1(\mathcal{F}), d_2(\mathcal{F}) \leq n$. This shows that all the three classes are disjoint. $\square$

From this theorem we obtain the following classification of normal extensions of $\mathbf{S5}^2$.

6.3.3. Theorem. *For every $L \in \Lambda(\mathbf{S5}^2)$, either $L = \mathbf{S5}^2$, or $L = \bigcap_{i \in S} L_i$ for some $S \subseteq \{1, 2, 3\}$, where $d_1(L_1), d_2(L_2), d_1(L_3), d_2(L_3) < \omega$.*

Proof. The theorem follows immediately from Theorem 6.3.2 by taking $L_i = Log(\mathbf{F}_i)$ for $i = 1, 2, 3$. $\square$

6.3.4. Theorem. *For every $\mathbf{V} \in \Lambda(\mathbf{Df}_2)$, either $\mathbf{V} = \mathbf{Df}_2$, or $\mathbf{V} = \bigvee_{i \in S} \mathbf{V}_i$ for some $S \subseteq \{1, 2, 3\}$, where $d_1(\mathbf{V}_1), d_2(\mathbf{V}_2), d_1(\mathbf{V}_3), d_2(\mathbf{V}_3) < \omega$.*

Proof. The result is an immediate consequence of Theorem 6.3.3. $\square$

6.4 Tabular and pre tabular extension of $\mathbf{S5}^2$

Recall that a logic is tabular if it is the logic of a single finite frame, and that a logic is pre-tabular if it is not tabular, but all its proper normal extensions are tabular. Also recall that a variety is finitely generated if it is generated by a single finite algebra, and that a variety is pre-finitely generated if it is not finitely generated, but all its proper subvarieties are finitely generated. In this section we characterize the tabular logics over $\mathbf{S5}^2$ by showing that there exist exactly six pre-tabular logics in $\Lambda(\mathbf{S5}^2)$. The characterization of finitely generated and pre-finitely generated subvarieties of $\mathbf{Df}_2$ will follow from the characterization of tabular and pre-tabular logics.

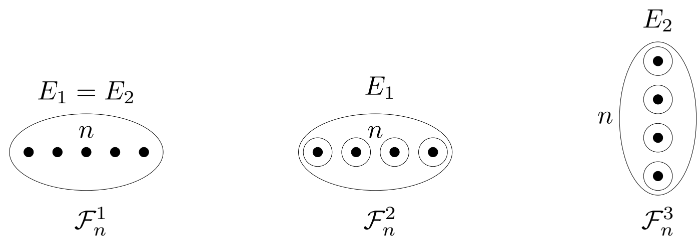

Figure 6.6: The frames $\mathcal{F}_n^1 - \mathcal{F}_n^3$.

6.4.1. Definition. Let $\mathcal{F} = (W, E_1, E_2)$ be a finite rooted $\mathbf{S5}^2$-frame.

1. $\mathcal{F}$ is said to be E_i-*discrete* if for every $w \in W$, $i, j = 1, 2$ and $i \neq j$, $E_i(w) = \{w\}$ and $E_j(w) = W$.
2. $\mathcal{F}$ is said to be an E_i-*quasi-bicluster* if $\mathcal{F}$ consists of two E_i-clusters, one of these clusters is a singleton set, and $E_j(w) = W$ for every $w \in W$, $i, j = 1, 2$ and $i \neq j$.
3. $\mathcal{F}$ is said to be a *quasi-rectangle* of type (n, m) if it is obtained from $\mathbf{n} \times \mathbf{m}$ by replacing a point of $\mathbf{n} \times \mathbf{m}$ by a finite E_0-cluster.
4. $\mathcal{F}$ is said to be a *quasi-square* of type (n, n) if it is obtained from $\mathbf{n} \times \mathbf{n}$ by replacing a point of $\mathbf{n} \times \mathbf{n}$ by a finite E_0-cluster.

We also recall that $\mathcal{F}$ is a bicluster if $E_1(w) = E_2(w) = W$ for every $w \in W$. We will use the following notation (see Figures 6.6 and 6.7):

1. Let $\mathcal{F}_n^1$ be a bicluster consisting of n points,
2. Let $\mathcal{F}_n^2$ be a E_2-discrete frame consisting of n points,
3. Let $\mathcal{F}_n^3$ be a E_1-discrete frame consisting of n points,
4. Let $\mathcal{F}_n^4$ be a E_2-quasi-bicluster frame, whose non-singleton E_2-cluster consists of n points,
5. Let $\mathcal{F}_n^5$ be a E_1-quasi-bicluster $\mathcal{F}$, whose non-singleton E_1-cluster consists of n points,
6. Let $\mathcal{F}_n^6$ be a quasi-square frame of type $(2, 2)$, whose non-singleton E_0-cluster consists of n points.

6.4.2. Definition. For every $i = 1, \ldots, 6$, let $L_i := Log(\{\mathcal{F}_n^i\}_{n \in \omega})$.

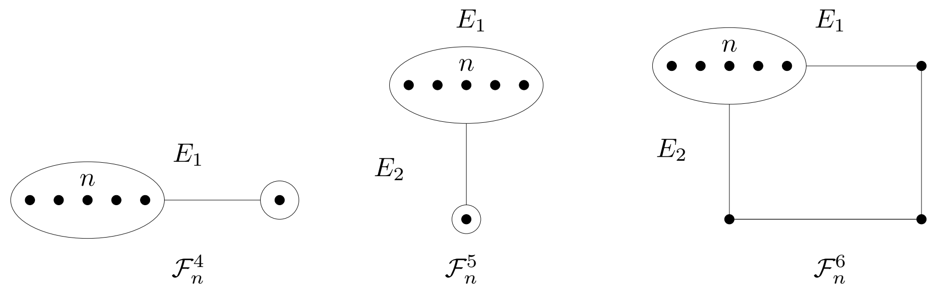

Figure 6.7: The frames $\mathcal{F}_n^4 - \mathcal{F}_n^6$.

We will prove that $L_1, \ldots, L_6$ are the only pre-tabular logics in $\Lambda(\mathbf{S5}^2)$. For this we need to show that every non-tabular logic is contained in one of the six logics described above. In the previous section we defined the E_1 and E_2-depths of a logic $L \supseteq \mathbf{S5}^2$. Now we define the girth of L.

6.4.3. Definition. For a finite rooted $\mathbf{S5}^2$-frame $\mathcal{F}$ and $w \in W$:

1. We call the number of elements of $E_0(w)$ the *girth* of w and denote it by $g(w)$.
2. We define the *girth* of $\mathcal{F}$ as $sup\{g(w) : w \in W\}$, and denote it by $g(\mathcal{F})$.
3. The *girth* of $L \supseteq \mathbf{S5}^2$, is $n > 0$, if there is $\mathcal{F} \in \mathbf{F}_L$ whose girth is n, and the girths of all the other members of $\mathbf{F}_L$ are less than or equal to n.
4. The *girth* of $L \supseteq \mathbf{S5}^2$ is said to be ω if the girths of the members of $\mathbf{F}_L$ are not bounded by any integer. Let $g(L)$ denote the girth of L.

6.4.4. Lemma. *Let $L \in \Lambda(\mathbf{S5}^2)$. Then L is tabular iff the E_1-depth, the E_2-depth and the girth of L are bounded by some integer.*

Proof. There exist only finitely many finite non-isomorphic rooted frames whose E_1-depth, E_2-depth and girth are all bounded by some integer. Thus, if the E_1-depth, the E_2-depth and the girth of L are bounded by some n, then L is tabular. Conversely, suppose L is tabular. Then $L = Log(\mathcal{F})$ for some finite frame $\mathcal{F}$. It follows from Jónsson's Lemma that every finite rooted L-frame is a p-morphic image of a generated subframe of $\mathcal{F}$, and therefore, has the cardinality $\leq |\mathcal{F}|$. Thus, the E_1-depth, the E_2-depth and the girth of $\mathbf{F}_L$ are bounded by some integer n, and so the E_1-depth, the E_2-depth and the girth of L are bounded by n. $\square$

It follows that if L is not tabular, then either the E_1-depth, the E_2-depth or the girth of L is not bounded. Consequently, no L_i is tabular for $i = 1, \ldots, 6$. Now we show that these logics are the only pre-tabular normal extensions of $\mathbf{S5}^2$.

6.4.5. THEOREM.

1. *If L is not tabular, then $L \subseteq L_i$ for some $i = 1, \ldots, 6$.*

2. *$L_1, \ldots, L_6$ are the only pre-tabular logics in $\Lambda(\mathbf{S5}^2)$.*

Proof. (1) Suppose L is not tabular. Then by Lemma 6.4.4, the E_1-depth, the E_2-depth or the girth of L is not bounded. We distinguish the following cases:

Case 1. If the E_1-depth of L is not bounded, then for any $n \in \omega$, there is a finite rooted L-frame $\mathcal{F} = (W, E_1, E_2)$ whose E_1-depth is $\geq n$. It is easy to see that E_1 is a bisimulation equivalence of $\mathcal{F}$. Consider the quotient of $\mathcal{F}$ by E_1. Then $\mathcal{F}/E_1$ is isomorphic to $\mathcal{F}^3_{d_1(\mathcal{F})}$. Hence, $\mathcal{F}^3_{d_1(\mathcal{F})}$ is a p-morphic image of $\mathcal{F}$, and so $\mathcal{F}^3_{d_1(\mathcal{F})} \in \mathbf{F}_L$. Therefore, for every $n \in \omega$, there exists $m > n$ such that $\mathcal{F}^3_m$ is an L-frame. Thus, $L_3 \supseteq L$.

Case 2. If the E_2-depth of L is not bounded, then similar to Case 1 we have $L_2 \supseteq L$.

Case 3. If the girth of L is not bounded, then for any n there is a finite rooted $\mathcal{F} = (W, E_1, E_2)$ such that $\mathcal{F} \in \mathbf{F}_L$ and $g(\mathcal{F}) \geq n$. But then, at least one of the following four cases holds:

Case 3.1. For every $w \in W$, $E_1(w) = E_2(w) = W$. This means that $\mathcal{F}$ consists of one E_1-cluster and one E_2-cluster. In this case $\mathcal{F}$ is isomorphic to $\mathcal{F}^1_{g(\mathcal{F})}$ and therefore, $\mathcal{F}^1_{g(\mathcal{F})}$ is an L-frame.

Case 3.2 For every $w \in W$, we have $E_1(w) = W$ but $E_2(w) \neq W$. This means that $\mathcal{F}$ consists of one E_1-cluster and at least two E_2-clusters. Let C denote an E_0-cluster of $\mathcal{F}$ containing $g(\mathcal{F}) \geq n$ points. We define an equivalence relation Q on W that leaves all the points in C untouched and identifies all the other points of $\mathcal{F}$:

- vQu for any $v, u \in W \setminus C$,
- vQu iff $v = u$, for any $v, u \in C$.

It is routine to check that Q is a bisimulation equivalence of $\mathcal{F}$, and that the quotient of $\mathcal{F}$ by Q is isomorphic to $\mathcal{F}^4_{g(\mathcal{F})}$. Thus, $\mathcal{F}^4_{g(\mathcal{F})}$ is a p-morphic image of $\mathcal{F}$, and so $\mathcal{F}^4_{g(\mathcal{F})}$ is an L-frame.

Case 3.3. For every $w \in W$ we have $E_2(w) = W$ but $E_2(w) \neq W$. Then the same argument as in Case 3.2 shows that $\mathcal{F}^5_{g(\mathcal{F})}$ is an L-frame.

Case 3.4. For every $w \in W$ we have $E_1(w) \neq W$ and $E_2(w) \neq W$. This means that $\mathcal{F}$ consists of at least two E_1 and at least two E_2-clusters. Let C denote an E_0-cluster containing $g(\mathcal{F}) \geq n$ points. First we define an equivalence relation Q on W that leaves points in C untouched and identifies all the points in other E_0-clusters of $\mathcal{F}$:

- vQu iff vE_0u, for every $v, u \in W \setminus C$,
- vQu iff $w = v$, for every $v, u \in C$.

It is routine to check that Q is a bisimulation equivalence. Let $\mathcal{G} = \mathcal{F}/Q$. Then $\mathcal{G}$ is isomorphic to a quasi-rectangle with just one non-singleton E_0-cluster C (i.e., a frame obtained from a finite rectangle by replacing one point by the E_0-cluster C). Let $\mathcal{G} = (W', E'_1, E'_2)$. Next we define an equivalence relation Q' on W' that leaves points of C untouched, identifies all the other points in the E_1-cluster containing C, identifies all the other points in the E_2-cluster containing C, and identifies all the remaining points:

- $wQ'v$, for any $w, v \in W' \setminus (E'_1(C) \cup E'_2(C))$,
- $wQ'v$, for any $w, v \in E'_2(C) \setminus C$,
- $wQ'v$, for any $w, v \in E'_1(C) \setminus C$,
- $wQ'v$ if $w = v$, for any $w, v \in C$.

Then it is again routine to check that Q' is a bisimulation equivalence and that $\mathcal{G}/Q'$ is isomorphic to a quasi-square of type $(2, 2)$ with the non-singleton E'_0-cluster C. Therefore, $\mathcal{F}^6_{g(\mathcal{F})}$ is a p-morphic image of $\mathcal{F}$. Thus, $\mathcal{F}^6_{g(\mathcal{F})}$ is an L-frame.

Consequently, for every $n \in \omega$ there exists $m \geq n$ such that one of $\mathcal{F}^1_m$, $\mathcal{F}^4_m$, $\mathcal{F}^5_m$, $\mathcal{F}^6_m$ is an L-frame. This implies that one of L_1, L_4, L_5, L_6 contains L. This concludes the proof that if L is not tabular, then L is contained in one of the six logics $L_1, \ldots, L_6$.

(2) First we show that every L_i, for $i = 1, \ldots, 6$ is a pre-tabular logic. As we mentioned above, by Lemma 6.4.4, no L_i is tabular. It is also easy to see that all these logics are incomparable. Now suppose L is a proper normal extension of L_i for some $i = 1, \ldots, 6$. If L is not tabular, then $L \subseteq L_j$ for some $j = 1, \ldots, 6$ and $j \neq i$. This implies that L_j is a proper extension of L_i, which is a contradiction because these logics are incomparable. Therefore, all the L_i are pre-tabular. Now suppose L is a pre-tabular logic. Then L is not tabular and by (1), $L \subseteq L_i$ for some $i = 1, \ldots, 6$. If $L \subsetneq L_i$, then L is not pre-tabular, because L_i is a non-tabular extension of L. Therefore, $L = L_i$. This means that $L_1, \ldots, L_6$ are the only pre tabular logics in $\Lambda(\mathbf{S5}^2)$. □

6.4.6. COROLLARY. *A logic $L \supseteq \mathbf{S5}^2$ is tabular iff none of $L_1 - L_6$ contains L.*

Proof. Suppose L is not tabular. Then, by the proof of Theorem 6.4.5, L_i contains L for some $i = 1, \ldots, 6$. On the other hand, if L is tabular, by Lemma 6.4.4, it cannot have a non-tabular normal extension. Therefore, $L_i \not\supseteq L$ for every $i = 1, \ldots, 6$. □

For $i = 1, \ldots, 6$ let $\mathbf{V}_i$ be the subvariety of $\mathbf{Df}_2$ corresponding to L_i. It is easy to see that $\mathbf{V}_i$ is generated by the complex algebras of the frames $\mathcal{F}_n^i$, $n \in \omega$. Then we have the following algebraic analogue of Theorem 6.4.5.

6.4.7. COROLLARY.

1. *$\mathbf{V}_1, \ldots, \mathbf{V}_6$ are the only pre-finitely generated varieties in $\Lambda(\mathbf{Df}_2)$.*
2. *A variety $\mathbf{V} \subseteq \mathbf{Df}_2$ is finitely generated iff none of $\mathbf{V}_1 - \mathbf{V}_6$ is a subvariety of $\mathbf{V}$.*

Proof. Follows from Theorem 6.4.5 and Corollary 6.4.6. □

Another characterization of tabular logics in $\Lambda(\mathbf{S5}^2)$ can be found in [12, §5] and [14, §7].

Chapter 7

Normal extensions of $\mathbf{CML}_2$

In Chapter 6, we investigated the lattice $\Lambda(\mathbf{S5}^2)$ of normal extensions of $\mathbf{S5}^2$ and its dual lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of $\mathbf{Df}_2$. In this chapter, which is based on [14], we investigate the lattice $\Lambda(\mathbf{CML}_2)$ of all normal extensions of $\mathbf{CML}_2$ and its dual lattice $\Lambda(\mathbf{CA}_2)$ of all subvarieties of $\mathbf{CA}_2$. We show that there exists a continuum of normal extensions of $\mathbf{PCML}_2$ and continuum many subvarieties of $\mathbf{RCA}_2$. We also show that there exists a continuum of normal logics in between $\mathbf{CML}_2$ and $\mathbf{PCML}_2$ and a continuum of varieties in between $\mathbf{RCA}_2$ and $\mathbf{CA}_2$. In Section 7.2 we describe the only pre-locally tabular extension of $\mathbf{CML}_2$ and the only pre-locally finite subvariety of $\mathbf{CA}_2$. We also characterize locally tabular extensions of $\mathbf{CML}_2$ and locally finite subvarieties of $\mathbf{CA}_2$. In Section 7.3 a characterization of the tabular and pre-tabular logics in $\Lambda(\mathbf{CML}_2)$ will be given together with a characterization of the finitely generated and pre-finitely generated subvarieties of $\mathbf{CA}_2$. Finally, we give a rough classification of the lattice structure of $\Lambda(\mathbf{CML}_2)$ and $\Lambda(\mathbf{CA}_2)$.

7.1 Finite CML_2-frames

In this section we discuss the finite model property of $\mathbf{CML}_2$ and $\mathbf{PCML}_2$. We show that $\mathbf{PCML}_2$ is complete with respect to finite cylindric squares, construct an infinite antichain of finite cylindric squares, and prove that the cardinality of both $\Lambda(\mathbf{CML}_2)$ and of $\Lambda(\mathbf{PCML}_2)$ is that of the continuum.

7.1.1 The finite model property

We start by showing that $\mathbf{CML}_2$ has the finite model property. This result was first proved in [60, Theorem 4.2.7] using algebraic technique. Here we present a proof using the filtration method.

7.1.1. Theorem.

1. $\mathbf{CML}_2$ *has the finite model property.*

2. $\mathbf{CML}_2$ *is decidable.*

Proof. (1) The proof is similar to the proof of the fmp of $\mathbf{S5}^2$ (see Theorem 6.1.1). It is based on the filtration method. Suppose $\mathbf{CML}_2 \nvdash \phi$. Then by Theorems 5.3.12 and 5.2.4, there exists a rooted $\mathbf{CML}_2$-frame $\mathfrak{F} = (W, E_1, E_2, D)$ and a valuation V such that $(\mathfrak{F}, V) \not\models \phi$. Let

- $\Sigma := Sub(\phi) \cup \{d\} \cup \{\Diamond_1(d \wedge \psi), \Diamond_2(d \wedge \psi) : \psi \in Sub(\phi)\}$.

We filter out $(\mathfrak{F}, V)$ through Σ in the same way as in the proof of Theorem 6.1.1. Let $\mathfrak{F}^f := (W^f, E_1^f, E_2^f, [D], V^f)$ denote the resulting model, where (W^f, E_1^f, E_2^f, V^f) is defined as in the proof of Theorem 6.1.1 and $[D] = \bigcup_{y \in D}[y]$. Similar to the proof of Theorem 6.1.1, we can show that (W^f, E_1^f, E_2^f) is a finite rooted $\mathbf{S5}^2$-frame and $(\mathfrak{F}^f, V^f) \not\models \phi$. By Proposition 5.3.10, in order to show that $\mathfrak{F}^f$ is a $\mathbf{CML}_2$-frame, we need to prove that for each $i = 1, 2$, every E_i^f-cluster of $\mathfrak{F}^f$ contains a unique point of $[D]$. Suppose $C \subseteq W^f$ is an E_i^f-cluster and $[x] \in C$. Since $\mathfrak{F}$ is a $\mathbf{CML}_2$-frame, by Proposition 5.3.10, there exists $y \in D$ such that xE_iy. Therefore, $[x]E_i^f[y]$ and $[y] \in [D]$, implying that every E_i^f-cluster contains a point from $[D]$. Now we show that such a point is unique. Assume there are $[y], [z] \in W^f$ such that $[y], [z] \in [D]$, $[y] \neq [z]$ and $[y]E_i^f[z]$. (Note that, $[y], [z] \in [D]$ and $[y] \neq [z]$ imply $\neg(yE_iz)$ for each $i = 1, 2$.) Then $y \models d$ and $z \models d$, and there is $\psi \in \Sigma$ such that w.l.o.g. $y \models \psi$ and $z \not\models \psi$. There are two cases:

Case 1. $\psi \in Sub(\phi)$. Then $y \models \Diamond_i(d \wedge \psi)$ and $z \not\models \Diamond_i(d \wedge \psi)$, which is a contradiction since $[y]E_i^f[z]$ and $\Diamond_i(d \wedge \psi) \in \Sigma$.

Case 2. $\psi = \Diamond_j(d \wedge \chi)$ (for some $j = 1, 2$) and $\chi \in Sub(\phi)$. This implies that $y \models \chi$ and $z \not\models \chi$ and we are back to Case 1.

Therefore, $[y] = [z]$ and every E_i^f-cluster contains a unique point from $[D]$. Thus, $\mathfrak{F}^f$ is a $\mathbf{CML}_2$-frame.

(2) The result follows immediately from (1) and the fact that $\mathbf{CML}_2$ is finitely axiomatizable. $\square$

7.1.2. Corollary.

1. $\mathbf{CA}_2$ *is finitely approximable.*

2. *The equational theory of* $\mathbf{CA}_2$ *is decidable.*

Proof. The result follows from Theorems 7.1.1, 5.4.16 and an analogue of Theorem 2.3.27 for modal logics. □

The product cylindric modal logic $\mathbf{PCML}_2$ also has the finite model property. This result follows from Mortimer [101], see also [60, Theorem 4.2.9], [95, Theorem 2.3.5] and [92]. However, the proof of this result is much more complicated than the proof of the fmp of $\mathbf{CML}_2$. We will skip it.

7.1.3. THEOREM. $\mathbf{PCML}_2$ *has the finite model property and is decidable.*

7.1.4. COROLLARY. $\mathbf{RCA}_2$ *is finitely approximable and its equational theory is decidable.*

Proof. The result follows immediately from Theorems 7.1.3 and 5.4.26. □

Next we define cylindric p-morphisms and cylindric bisimulation equivalences for $\mathbf{CML}_2$-frames.

7.1.5. DEFINITION.

1. Let $\mathfrak{F} = (W, E_1, E_2, D)$ and $\mathfrak{G} = (W', E_1', E_2', D')$ be $\mathbf{CML}_2$-frames. A map $f : W \to W'$ is called a *cylindric p-morphism* if f is a p-morphism between (W, E_1, E_2) and (W', E_1', E_2'), and $f^{-1}(D') = D$.
2. Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a $\mathbf{CML}_2$-frame. An equivalence relation Q on W is called a *cylindric bisimulation equivalence* if Q is a bisimulation equivalence of (W, E_1, E_2) and $Q(D) = D$.

Now similar to the diagonal-free case we will spell out the connection between cylindric p-morphisms and cylindric bisimulation equivalences.

7.1.6. LEMMA.

1. *Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a $\mathbf{CML}_2$-frame and Q an equivalence relation on W. Let the map $f_Q : W \to W/Q$ be defined by $f_Q(w) = Q(w)$ for every $w \in W$. Then the following two conditions are equivalent:*
   - (*a*) *Q is a cylindric bisimulation equivalence,*
   - (*b*) *f_Q is a cylindric p-morphism.*
2. *Let $\mathfrak{F} = (W, E_1, E_2, D)$ and $\mathfrak{F}' = (W', E_1', E_2', D')$ be $\mathbf{CML}_2$-frames and $f : W \to W'$ a map. Let the relation $Q_f \subseteq W \times W$ be defined by wQ_fv iff $f(w) = f(v)$ for any $w, v \in W$. Then the following two conditions are equivalent:*

(*a*) *f is a cylindric p-morphism,*

(*b*) *Q_f is a cylindric bisimulation equivalence.*

Proof. The proof is an easy generalization of the proof for the case of $\mathbf{S5}^2$. $\square$

In order to prove that $\mathbf{PCML}_2$ is complete with respect to the class of finite cylindric squares $\mathbf{CSq}$ we will show that every finite $\mathbf{PCML}_2$-frame is a cylindric p-morphic image of a finite cylindric square.

7.1.7. Lemma. *For every finite rooted $\mathbf{PCML}_2$-frame $\mathfrak{F}$ there is a finite cylindric square $\mathfrak{G}$ such that $\mathfrak{F}$ is a cylindric p-morphic image of $\mathfrak{G}$.*

Proof. (Sketch) By Theorem 5.3.18, we have to show that every $\mathbf{CML}_2$-frame $\mathfrak{F}$ that does not satisfy $(*)$ is a cylindric p-morphic image of some finite cylindric square $\mathfrak{G}$. All we need to do is to check that Lemmas 6.1.7–6.1.9 hold for $\mathbf{CML}_2$-frames not satisfying the $(*)$-condition. We will skip the technical details. $\square$

Now we are ready to prove that $\mathbf{PCML}_2$ is complete with respect to the class of all finite cylindric squares (see [101], [60, Theorem 4.2.9] and [95, Theorem 2.3.10]).

7.1.8. Theorem. $\mathbf{PCML}_2$ *is complete with respect to* $\mathrm{Fin}(\mathbf{CSq})$.

Proof. Let $\mathbf{PCML}_2 \not\vdash \phi$. By Theorem 7.1.3, there is a finite rooted $\mathbf{PCML}_2$-frame $\mathfrak{F}$ that refutes ϕ. By Lemma 7.1.7, $\mathfrak{F}$ is a p-morphic image of some finite cylindric square $\mathfrak{G}$. Therefore, $\mathfrak{G}$ also refutes ϕ. $\square$

The next corollary is an algebraic version of Theorem 7.1.8.

7.1.9. Corollary. $\mathbf{RCA}_2$ *is generated by* $\mathrm{Fin}(\mathbb{CSQ})$.

7.1.2 The Jankov-Fine formulas

The modal logic analogues of the Jankov-de Jongh formulas are known in the literature as *Jankov-Fine formulas* and were first defined by Fine [41] (see also Rautenberg [105]) for an algebraic version. We consider the Jankov-Fine formulas for $\mathbf{S5}^2$ and $\mathbf{CML}_2$ (see [18, §3.4] and [43, §8.4 p. 399]). Let $\mathcal{F} = (W, E_1, E_2)$ be a finite $\mathbf{S5}^2$-frame. For each point $w \in W$ we introduce a propositional variable p_w, and consider the formulas

$$\begin{aligned}
\delta(\mathcal{F}) &:= \Box_1\Box_2\Big(\bigvee_{w\in W}(p_w \wedge \neg \bigvee_{v\in W\setminus\{w\}} p_v) \\
&\qquad \wedge \bigwedge_{\substack{i=1,2\\ w,v\in W, wE_iv}} (p_w \to \Diamond_i p_v) \quad \wedge \bigwedge_{\substack{i=1,2\\ w,v\in W, \neg(wE_iv)}} (p_w \to \neg\Diamond_i p_v)\Big), \\
\chi(\mathcal{F}) &:= \neg\delta(\mathcal{F}).
\end{aligned}$$

7.1.10. THEOREM. *Let $\mathcal{F} = (W, E_1, E_2)$ be a finite rooted $\mathbf{S5}^2$-frame and let $\mathcal{G} = (U, S_1, S_2)$ be a rooted (descriptive) $\mathbf{S5}^2$-frame. Then*

$$\mathcal{G} \not\models \chi(\mathcal{F}) \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$

Proof. (Sketch) Suppose $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. Define a valuation V on $\mathcal{F}$ by $V(p_w) = \{w\}$ for any $w \in W$. Then $(\mathcal{F}, V) \not\models \chi(\mathcal{F})$ by the definition of $\chi(\mathcal{F})$. Now if $\mathcal{G} \models \chi(\mathcal{F})$, then since p-morphic images preserve validity of formulas, we would also have $\mathcal{F} \models \chi(\mathcal{F})$, a contradiction. Therefore, $\mathcal{G} \not\models \chi(\mathcal{F})$.

For the converse, we use the argument of [43, Claim 8.36]. Suppose that $\mathcal{G} \not\models \chi(\mathcal{F})$. Then there is a valuation V' on $\mathcal{G}$ and a point $u \in W'$ such that $(\mathcal{G}, V'), u \not\models \chi(\mathcal{F})$. Therefore, $(\mathcal{G}, V'), u \models \delta(\mathcal{F})$. Define a map $f : U \to W$ by

$$f(t) = w \iff (\mathcal{G}, V'), t \models p_w,$$

for every $t \in U$ and $w \in W$. From $\mathcal{G}$ being rooted and the truth of the first conjunct of $\delta(\mathcal{F})$ it follows that f is well defined. The truth of the first two conjuncts of $\delta(\mathcal{F})$ together with $\mathcal{F}$ being rooted implies that f is surjective. Finally, the truth of the second and third conjuncts of $\delta(\mathcal{F})$ guarantee that f is a p-morphism. (If $\mathcal{G}$ is a descriptive frame, then it immediately follows from the definition of f that the inverse image of every point of $\mathcal{F}$ is an admissible subset of $\mathcal{G}$.) Therefore, $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. □

7.1.11. REMARK. Theorem 7.1.10 is an analogue of Theorem 3.3.3 for $\mathbf{S5}^2$-frames. In this case we do not require that $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{G}$ ($\mathcal{F}$ is simply a p-morphic image of $\mathcal{G}$), because $\mathcal{G}$ is a rooted $\mathbf{S5}^2$-frame.

Suppose $\mathfrak{F} = (W, E_1, E_2, D)$ is a finite rooted $\mathbf{CML}_2$-frame, $\mathcal{F} = (W, E_1, E_2)$ diagonal-free reduct of $\mathfrak{F}$, and $\delta(\mathcal{F})$—the Jankov-Fine formula of $\mathcal{F}$.

Let

$$\begin{aligned}
\delta_d(\mathfrak{F}) \quad &:= \quad \delta(\mathcal{F}) \wedge \Box_1\Box_2\Big(\bigwedge_{w\in D}(p_w \to d) \wedge \bigwedge_{w\notin D}(p_w \to \neg d)\Big),\\
\chi_d(\mathfrak{F}) \quad &:= \quad \neg\delta_d(\mathfrak{F}).
\end{aligned}$$

7.1.12. THEOREM. *Let $\mathfrak{F} = (W, E_1, E_2, D)$ be a finite rooted $\mathbf{CML}_2$-frame and $\mathfrak{G} = (W', E_1', E_2', D')$ be a rooted (descriptive) $\mathbf{CML}_2$-frames. Then*

$$\mathfrak{G} \not\models \chi_d(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a cylindric } p\text{-morphic image of } \mathfrak{G}.$$

Proof. The proof is similar to the proof of Theorem 7.1.10. The additional conjunct of $\delta_d(\mathfrak{F})$ guarantees that the map f constructed in the proof of Theorem 7.1.10 is a cylindric p-morphism, i.e., $f^{-1}(D') = D$. □

7.1.3 The cardinality of $\Lambda(\mathbf{CML}_2)$

Let $\mathcal{F}$ and $\mathcal{G}$ be rooted $\mathbf{S5}^2$-frames, and let $\mathfrak{F}$ and $\mathfrak{G}$ be rooted $\mathbf{CML}_2$-frames. We write

$$\mathcal{F} \leq \mathcal{G} \quad \text{iff} \quad \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$
$$\mathfrak{F} \leq \mathfrak{G} \quad \text{iff} \quad \mathfrak{F} \text{ is a cylindric } p\text{-morphic image of } \mathfrak{G}.$$

It follows from Theorems 7.1.10 and 7.1.12 that if $\mathcal{F}$ and $\mathfrak{F}$ are finite then

1. $\mathcal{G} \not\models \chi(\mathcal{F})$ iff $\mathcal{F} \leq \mathcal{G}$,
2. $\mathfrak{G} \not\models \chi_d(\mathfrak{F})$ iff $\mathfrak{F} \leq \mathfrak{G}$.

Now we show that the cardinality of $\Lambda(\mathbf{PCML}_2)$ as well as the cardinality of $\Lambda(\mathbf{CML}_2) \setminus \Lambda(\mathbf{PCML}_2)$ is that of the continuum. In the next chapter we prove that the cardinality of $\Lambda(\mathbf{S5}^2)$ is countable. First we construct $\leq$-antichains of finite $\mathbf{CML}_2$-frames.

7.1.13. LEMMA. *Every two non-isomorphic finite squares are $\leq$-incomparable.*

Proof. Let $\mathfrak{F}$ and $\mathfrak{G}$ be two non-isomorphic finite squares. Then $\mathfrak{F}$ is isomorphic to $(n \times n, E_1, E_2, D)$ and $\mathfrak{G}$ is isomorphic to $(m \times m, E_1', E_2', D')$ where $n \neq m$. Without loss of generality we may assume that $n > m$. Then obviously $\mathfrak{F}$ can not be a cylindric p-morphic image of $\mathfrak{G}$. Suppose $\mathfrak{G}$ is a proper cylindric p-morphic image of $\mathfrak{F}$. Then by Lemma 7.1.6(2), there exists a cylindric bisimulation equivalence Q of $\mathfrak{F}$ such that $\mathfrak{F}/Q = (W/Q, E_1', E_2', D')$ is isomorphic to $\mathfrak{G}$. Therefore, Q must identify points from different E_1 or E_2-clusters of $\mathfrak{F}$. Without loss of generality we may assume that Q identifies points from different E_1-clusters C_1 and C_2. Let $x_1 \in C_1$ be the diagonal point of C_1 and $x_2 \in C_2$ be the diagonal point of C_2. Since $Q(D) = D$, we have that $x_1 Q x_2$. Let $E_1(x_1) \cap E_2(x_2) = \{y_1\}$. Since $x_2 Q x_1$ and $x_1 E_1 y_1$, there exists y_2 in $\mathfrak{F}$ such that $y_1 Q y_2$ and $y_2 E_1 x_2$. Consider $Q(x_1)$ and $Q(y_1)$. It is obvious that $Q(x_1) E_1' Q(y_1)$. Also since $Q(x_1) = Q(x_2)$ and $Q(y_1) = Q(y_2)$ it follows that $Q(x_1) E_2' Q(y_1)$. Therefore, $Q(x_1) E_0' Q(y_1)$. Also $Q(x_1) \neq Q(y_1)$ since $x_1 \in D$, $y_1 \notin D$ and $Q(D) = D$. Therefore, there exists a non-singleton E_0-cluster of $\mathfrak{F}/Q$, which is impossible since $\mathfrak{F}/Q$ is isomorphic to $\mathfrak{G}$ and $\mathfrak{G}$ is a square. Thus, $\mathfrak{G}$ is not a proper p-morphic image of $\mathfrak{F}$, and so every two non-isomorphic finite squares are $\leq$-incomparable. $\square$

As an immediate consequence of Lemma 7.1.13 we obtain the following theorem.

7.1.14. THEOREM.

1. *The cardinality of $\Lambda(\mathbf{PCML}_2)$ is that of the continuum.*
2. *The cardinality of $\Lambda(\mathbf{RCA}_2)$ is that of the continuum.*

Proof. (1) Let $\mathfrak{F}_n$ be the square $(n \times n, E_1, E_2, D)$. Consider the family $\Delta = \{\mathfrak{F}_n\}_{n\in\omega}$. From Lemma 7.1.13 it follows that Δ forms a $\leq$-antichain. Now the result follows from Theorem 7.1.12 and the modal logic analogues of Theorems 3.4.18 and 3.4.20.

(2) follows from (1). □

For $n > 1$ let $\mathfrak{G}_n$ denote the finite cylindric space obtained from $\mathfrak{F}_n$ by replacing a singleton non-diagonal E_0-cluster by a two-element E_0-cluster. For example, $\mathfrak{G}_2$ is shown in Figure 5.1(a) on page 138, where the non-singleton E_0-cluster contains two points. Obviously $\mathfrak{G}_n$ satisfies $(*)$, and so is not a $\mathbf{PCML}_2$-frame. Similar to Lemma 7.1.13, we can prove the following lemma.

7.1.15. LEMMA. *The family $\{\mathfrak{G}_n\}_{n\in\omega}$ forms a $\leq$-antichain.*

As an immediate consequence of Lemma 7.1.15 and the fact that every $\mathfrak{G}_n$ is a $\mathbf{CML}_2$-frame but not a $\mathbf{PCML}_2$-frame, we obtain the following theorem.

7.1.16. THEOREM.

1. *The cardinality of $\Lambda(\mathbf{CML}_2) \setminus \Lambda(\mathbf{PCML}_2)$ is that of the continuum.*

2. *The cardinality of $\Lambda(\mathbf{CA}_2) \setminus \Lambda(\mathbf{RCA}_2)$ is that of the continuum.*

Finally, note that because $\mathfrak{G}_n$ is not a $\mathbf{PCML}_2$-frame, the Fine-Jankov formula $\chi_d(\mathfrak{G}_n)$ of $\mathfrak{G}_n$ belongs to $\mathbf{PCML}_2$. Then the same argument as in Theorem 7.1.14 shows that $\Gamma, \Gamma' \subseteq \{\mathfrak{G}_n\}_{n\in\omega}$ and $\Gamma \neq \Gamma'$ imply $\mathbf{PCML}_2 \cap Log(\Gamma) \neq \mathbf{PCML}_2 \cap Log(\Gamma')$. Therefore, we obtain the following corollary.

7.1.17. COROLLARY.

1. *There exists a continuum of logics in between $\mathbf{CML}_2$ and $\mathbf{PCML}_2$.*

2. *There exists a continuum of varieties in between $\mathbf{RCA}_2$ and $\mathbf{CA}_2$.*

Note that there are only countably many finitely axiomatizable logics. Therefore, Theorem 7.1.14 also implies that there exist continuum many non-finitely axiomatizable extensions of $\mathbf{PCML}_2$ and of $\mathbf{CML}_2$.

7.2 Locally tabular extensions of $\mathbf{CML}_2$

In the previous chapter we proved that $\mathbf{Df}_2$ is pre-locally finite. It is known (see, e.g., [60, Theorem 2.1.11]) that $\mathbf{RCA}_2$, and hence every variety in the interval $[\mathbf{RCA}_2, \mathbf{CA}_2]$, is not locally finite (the result could be obtained by using the Example 6.2.1). In this section, we present a criterion for a variety of cylindric algebras to be locally finite, and show that there exists exactly one pre-locally finite subvariety of $\mathbf{CA}_2$. The corresponding results for cylindric modal logics will also be stated.

Let $\mathfrak{B}$ be a cylindric algebra and $\mathcal{X}$ be its dual cylindric space. Recall that a cylindric space is a quasi-square if it is rooted and the number of the E_1 and E_2-clusters of $\mathcal{X}$ is the same. We have that $\mathfrak{B}$ is simple iff $\mathcal{X}$ is a quasi-square (see Theorem 5.4.21(3)). Therefore, we have that the cardinalities of the sets E_1 and E_2-clusters of $\mathcal{X}$ coincide.

7.2.1. Definition.

1. A quasi-square $\mathcal{X}$ is said to be of *depth* n $(0 < n < \omega)$ if the number of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is equal to n.
2. A quasi-square $\mathcal{X}$ is said to be of an *infinite depth* if the cardinality of the set of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is infinite.
3. A simple cylindric algebra $\mathfrak{B}$ is said to be of *depth* n if its dual $\mathcal{X}$ is of depth n.
4. A simple cylindric algebra $\mathfrak{B}$ is said to be of an *infinite depth* if its dual $\mathcal{X}$ is of an infinite depth.
5. A variety $\mathbf{V}$ of cylindric algebras is said to be of *depth* n if there is a simple $\mathbf{V}$-algebra of depth n and the depth of every other simple $\mathbf{V}$-algebra is less than or equal to n.
6. A variety $\mathbf{V}$ is said to be of *depth* ω if the depth of simple members of $\mathbf{V}$ is not bounded by any natural number.

Recall that there exists a formula measuring the depth of a variety of cylindric algebras (see Theorem 6.2.4). Let $d(\mathbf{V})$ denote the depth of the variety $\mathbf{V}$. Our goal is to show that a variety $\mathbf{V}$ of cylindric algebras is locally finite iff $d(\mathbf{V}) < \omega$. For this we need the following definition.

7.2.2. Definition.

1. Call a quasi-square $\mathcal{X}$ *uniform* if every non-diagonal E_0-cluster of $\mathcal{X}$ is a singleton set, and every diagonal E_0-cluster of $\mathcal{X}$ contains only two points.
2. Call a simple cylindric algebra $\mathfrak{B}$ *uniform* if its dual quasi-square $\mathcal{X}$ is uniform.

Finite uniform quasi-squares are shown in Figure 7.1, where big dots denote the diagonal points. Let $\mathcal{X}_n$ denote the uniform quasi-square of depth n. Also let $\mathfrak{B}_n$ denote the uniform cylindric algebra of depth n. It is obvious that $\mathcal{X}_n$ is (isomorphic to) the dual cylindric space of $\mathfrak{B}_n$. Let $\mathbf{U}$ denote the variety generated by all finite uniform cylindric algebras; that is $\mathbf{U} = \mathbf{HSP}(\{\mathfrak{B}_n\}_{n\in\omega})$.

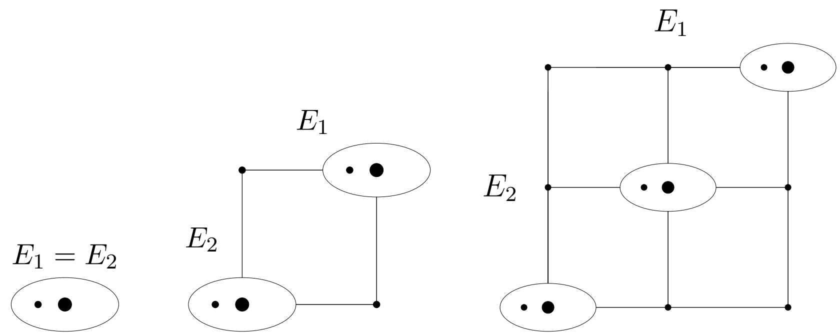

Figure 7.1: Uniform quasi-squares

7.2.3. PROPOSITION. $\mathbf{U} \subseteq \mathbf{RCA}_2$.

Proof. Since none of the diagonal E_0-clusters of $\mathcal{X}_n$ is a singleton set, $\mathcal{X}_n$ does not satisfy ($*$). Therefore, each $\mathfrak{B}_n$ is representable by Theorem 5.4.28. Thus, $\{\mathfrak{B}_n\}_{n\in\omega} \subseteq \mathbf{RCA}_2$, implying that $\mathbf{U} \subseteq \mathbf{RCA}_2$. □

7.2.4. LEMMA.

1. *If $\mathfrak{B}$ is a simple cylindric algebra of an infinite depth, then each $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.*

2. *If $\mathfrak{B}$ is a simple cylindric algebra of depth $2n$, then $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.*

Proof. (1) Suppose that $\mathfrak{B}$ is a simple cylindric algebra of an infinite depth. Let $\mathcal{X}$ be the dual cylindric space of $\mathfrak{B}$. Then $\mathcal{X}$ is a quasi-square with infinitely many E_1 and E_2-clusters. As in the proof of Claim 6.2.10, for each n, we can divide $\mathcal{X}$ into n-many E_1-saturated disjoint clopen sets $G_1, \ldots, G_n$. We let $D_i = D \cap G_i$ and $F_i = E_2(D_i)$ for $i = 1, \ldots, n$. Obviously each of the D_i's and F_i's is clopen. Define an equivalence relation Q of $\mathcal{X}$ by

- xQy if $x, y \in D$ and there exists $i = 1, \ldots, n$ such that $x, y \in D_i$,
- xQy if $x, y \in X \setminus D$ and there exist $1 \leq j, k \leq n$ such that $x, y \in G_j \cap F_k$.

It is easy to check, or transform the proof of Claim 6.2.10, that Q is a cylindric bisimulation equivalence of $\mathcal{X}$, and that $\mathcal{X}/Q$ is isomorphic to $\mathcal{X}_n$. Therefore, by Theorem 5.4.21(2), each $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.

(2) Suppose that $\mathfrak{B}$ is a simple cylindric algebra of depth $2n$. Let $\mathcal{X}$ be the dual cylindric space of $\mathfrak{B}$. Then $\mathcal{X}$ is a quasi-square. Moreover, there are exactly $2n$ E_1-clusters and exactly $2n$ E_2-clusters of $\mathcal{X}$. Obviously all of them are clopens. Let $C_1, \ldots, C_{2n}$ be the E_1-clusters of $\mathcal{X}$ and let $G_i = C_{2i-1} \cup C_{2i}$ for $i = 1, \ldots, n$. Obviously every G_i is an E_1-saturated clopen. Now applying the same technique as in (1), we obtain that $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$. □

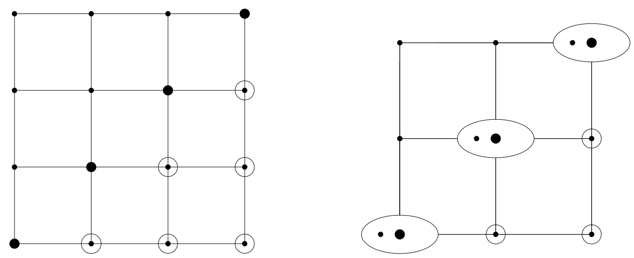

Figure 7.2: Generators of square and uniform quasi-square algebras

7.2.5. THEOREM. *For a variety* $\mathbf{V}$ *of cylindric algebras,* $d(\mathbf{V}) = \omega$ *iff* $\mathbf{U} \subseteq \mathbf{V}$.

Proof. It is obvious that $d(\mathbf{U}) = \omega$. So, if $\mathbf{U} \subseteq \mathbf{V}$, then obviously $d(\mathbf{V}) = \omega$. Conversely, suppose $d(\mathbf{V}) = \omega$. We want to show that every finite uniform cylindric algebra belongs to $\mathbf{V}$. Since $d(\mathbf{V}) = \omega$, the depth of the simple members of $\mathbf{V}$ is not bounded by any integer. So, either there exists a family of simple $\mathbf{V}$-algebras of increasing finite depth, or there exists a simple $\mathbf{V}$-algebra of an infinite depth. In either case, it follows from Lemma 7.2.4 that $\{\mathfrak{B}_n\}_{n\in\omega} \subseteq \mathbf{V}$. Therefore, $\mathbf{U} \subseteq \mathbf{V}$ since $\{\mathfrak{B}_n\}_{n\in\omega}$ generates $\mathbf{U}$. □

Our next task is to show that $\mathbf{U}$ is not locally finite. For this we will need the following lemma.

7.2.6. LEMMA.

1. *Every finite square algebra is* 1*-generated.*

2. *Every finite uniform algebra is* 1*-generated.*

Proof. (1) For a finite cylindric square $\mathfrak{F}_n = (n \times n, E_1, E_2, D)$, consider the set $g = \{(k, m) : k < m\}$. It follows from Example 6.2.1 that the cylindric algebra generated by g contains all singleton subsets of $n \times n$. Hence, $(P(n \times n), E_1, E_2, D)$ is generated by g.

(2) is proved similar to (1). Let $\mathfrak{B}$ be a finite uniform algebra, and let $\mathcal{X}$ be its dual cylindric quasi-square. Then the same argument as above shows that every E_0-cluster of $\mathcal{X}$ belongs to the algebra generated by the lower triangle g' (see Figure 7.2, where big dots represent the diagonal points and points in circles represent the points that belong to the sets g and g', respectively). Hence it is left to show that for every diagonal E_0-cluster C and $x \in C$, the singleton set $\{x\}$ belongs to the algebra generated by g'. But for any $x \in C$, either $x \in D$ and hence $\{x\} = C \cap D$ or $x \notin D$ and $\{x\} = C \setminus D$. Thus, every singleton set belongs to the cylindric algebra generated by g', and so g' generates $\mathfrak{B}$. □

7.2.7. Remark. Note that the $\mathbf{Df}_2$-reducts of finite uniform algebras are not generated by g'. Indeed, the $\mathbf{Df}_2$-algebra generated by g' does not contain the singleton sets from non-singleton E_0-clusters.

7.2.8. Corollary. $\mathbf{U}$ *is not locally finite.*

Proof. Follows from Lemma 7.2.6 and Theorem 6.2.6. □

Next we show that if a variety of cylindric algebras is of finite depth, then it is locally finite.

7.2.9. Theorem. *If* $d(\mathbf{V}) < \omega$*, then* $\mathbf{V}$ *is locally finite.*

Proof. The proof is similar to the proof of Lemma 6.2.7 for the diagonal-free case: To show $\mathbf{V}$ is locally finite, by Theorem 6.2.6, it is sufficient to prove that the cardinality of every n-generated simple $\mathbf{V}$-algebra is bounded by some natural number $M(n)$. Let $\mathfrak{B}$ be an n-generated simple $\mathbf{V}$-algebra. Let also $B_i = \{\Diamond_i b : b \in B\}$, for $i = 1, 2$. Since $d(\mathbf{V}) < \omega$, we have $|B_1| = |B_2| < \omega$. Suppose $\mathfrak{B}$ is generated by $G = \{g_1, \dots, g_k\}$. Then as a Boolean algebra $\mathfrak{B}$ is generated by $G \cup B_1 \cup B_2 \cup \{d\}$. Since the variety of Boolean algebras is locally finite, there exists $M(n) < \omega$ such that $|\mathfrak{B}| \leq M(n)$ (in fact, $|\mathfrak{B}| \leq 2^{2^{n+2|B_1|+1}}$). Thus, $\mathbf{V}$ is locally finite. □

Finally, combining Theorem 7.2.5 with Corollary 7.2.8 and Theorem 7.2.9, we obtain the following characterization of locally finite varieties of cylindric algebras.

7.2.10. Theorem.

1. *For* $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$ *the following conditions are equivalent:*

   (a) $\mathbf{V}$ *is locally finite,*

   (b) $d(\mathbf{V}) < \omega$,

   (c) $\mathbf{U} \not\subseteq \mathbf{V}$.

2. $\mathbf{U}$ *is the only pre-locally finite subvariety of* $\mathbf{CA}_2$.

Proof. (1) The equivalence $(b) \Leftrightarrow (c)$ is shown in Theorem 7.2.5. The implication $(b) \Rightarrow (a)$ is proved in Theorem 7.2.9. Finally, by Corollary 7.2.8, $\mathbf{U}$ is not locally finite. Therefore, if $\mathbf{V} \supseteq \mathbf{U}$, then $\mathbf{V}$ is not locally finite either. Thus, $(a) \Rightarrow (c)$.

(2) First we show that $\mathbf{U}$ is pre-locally finite. Suppose $\mathbf{V} \subsetneq \mathbf{U}$. Then by Theorem 7.2.5, $d(\mathbf{V}) < \omega$, and so by Theorem 7.2.9, $\mathbf{V}$ is locally finite. Now suppose $\mathbf{V}$ is pre-locally finite. Then again by Theorem 7.2.9, $d(\mathbf{V}) = \omega$. By Theorem 7.2.5, $\mathbf{U} \subseteq \mathbf{V}$ and since $\mathbf{V}$ is pre-locally finite, $\mathbf{V} = \mathbf{U}$. □

In order to formulate Theorem 7.2.10 in terms of logics, we need the following terminology. We define the *depth* of a logic $L \supseteq \mathbf{CML_2}$ as the depth of its corresponding variety of cylindric algebras. We denote by $d(L)$ the depth of L. Let $L_{\mathbf{U}}$ denote the logic of all finite uniform quasi-squares (rooted $\mathbf{CML}_2$-frames); that is $L_{\mathbf{U}} = Log(\{\mathcal{X}_n\}_{n\in\omega})$.

7.2.11. COROLLARY.

1. *For $L \in \Lambda(\mathbf{CML}_2)$ the following conditions are equivalent:*
   - (a) *L is locally tabular,*
   - (b) *$d(L) < \omega$,*
   - (c) *$L_{\mathbf{U}} \not\supseteq L$.*
2. *$L_{\mathbf{U}}$ is the only pre-locally tabular extension of* $\mathbf{CML}_2$.

Proof. The result follows from Theorem 7.2.10. □

Therefore, in contrast to the diagonal-free case, there exist uncountably many normal extensions of $\mathbf{CML}_2$ ($\mathbf{PCML}_2$) which are not locally tabular. Since every locally tabular logic has the finite model property we obtain from Theorem 7.2.10 that every normal extension of $\mathbf{CML}_2$ of finite depth has the finite model property. We leave it as an open problem whether every normal extension of $\mathbf{CML}_2$ has the finite model property.

7.2.12. OPEN QUESTION.

1. *Does every normal extension of* $\mathbf{CML}_2$ ($\mathbf{PCML}_2$) *have the finite model property?*
2. *Is every subvariety of* $\mathbf{CA}_2$ ($\mathbf{RCA}_2$) *finitely approximable?*

7.3 Tabular and pre-tabular extensions of $\mathbf{CML}_2$

In Chapter 6 we showed that there are exactly six pre tabular logics in $\Lambda(\mathbf{S5}^2)$ (Theorem 6.4.5). The situation is more complex in $\Lambda(\mathbf{CML}_2)$. In this section we show that there exist exactly fifteen pre-tabular logics in $\Lambda(\mathbf{CML}_2)$, and that six of them belong to $\Lambda(\mathbf{PCML}_2)$. It trivially implies a characterization of tabular logics of $\Lambda(\mathbf{CML}_2)$.

Consider the finite quasi-squares $\mathfrak{F}_n^i$ shown in Figures 7.3 and 7.4, where $i = 1, \ldots, 15$ and $n \geq 2$. Again big dots represent the diagonal points. The pattern according to which the quasi-squares are depicted is the following: First come the frames of depth 1, then the frames of depth 2, and finally the frames of depth 3;

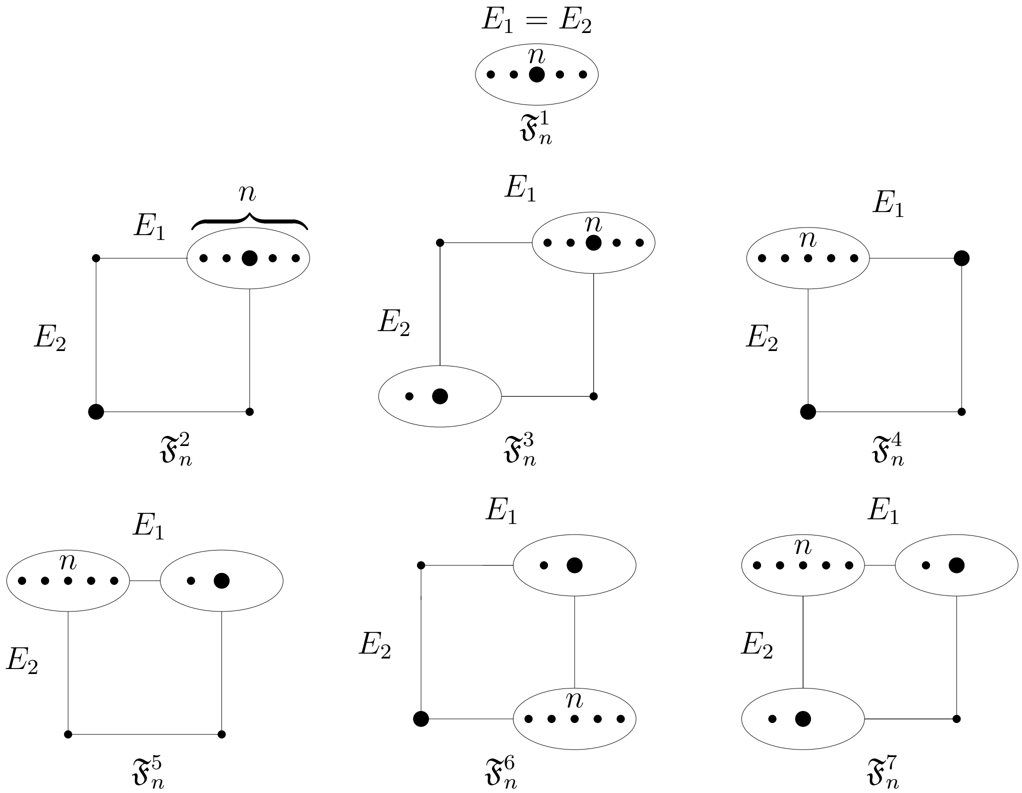

Figure 7.3: Quasi-squares $\mathfrak{F}_n^1 - \mathfrak{F}_n^7$

quasi-squares with more clusters come later in the list; the first and last quasi-squares (of the same depth) do not satisfy $(*)$. As can be seen from the figure, each E_0-cluster of $\mathfrak{F}_n^i$ consists of one, two or n points. For each $i = 1, \ldots, 15$ let $L_i := Log(\{\mathfrak{F}_n^i : n \geq 2\})$. From Theorem 5.3.18 it follows that only $\mathfrak{F}_n^1$ $\mathfrak{F}_n^2$, $\mathfrak{F}_n^3$, $\mathfrak{F}_n^7$, $\mathfrak{F}_n^{14}$ and $\mathfrak{F}_n^{15}$ are $\mathbf{PCML}_2$-frames, and so only L_1, L_2, L_3, L_7, L_{14} and L_{15} belong to $\Lambda(\mathbf{PCML}_2)$.

Now we are in a position to prove that $L_1 - L_{15}$ are the only pre-tabular normal extensions of $\mathbf{CML}_2$. As we saw in the proof of Theorem 6.4.5, for this it is sufficient to show that $L_1 - L_{15}$ are incomparable, they are not tabular, and that every normal extensions of $\mathbf{CML}_2$ that is not tabular is contained in exactly one of $L_1 - L_{15}$.

7.3.1. LEMMA. $L_3 \supseteq L_{\mathbf{U}}$.

Proof. Suppose $\mathcal{X}_n$ is the finite uniform square of depth n. We show that $\mathfrak{F}_n^3$ is a cylindric p-morphic image of $\mathcal{X}_n$. Fix a diagonal E_0-cluster, say C of $\mathcal{X}_n$, and let $D \cap C = \{x_0\}$. Define an equivalence relation Q on $\mathcal{X}_n$ by

- xQy if $x = y$ for all $x, y \in C$,

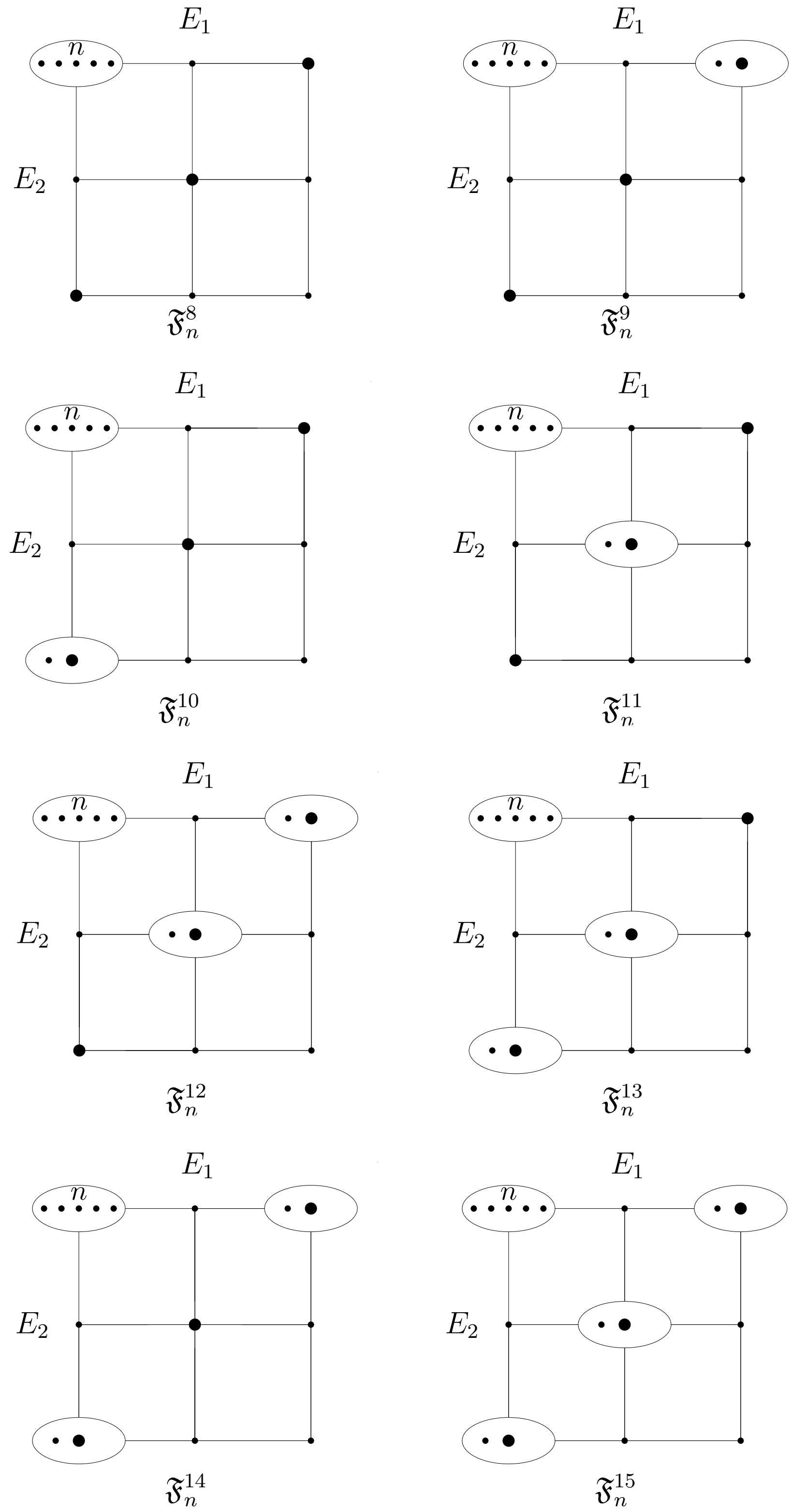

Figure 7.4: Quasi-squares $\mathfrak{F}_n^8 - \mathfrak{F}_n^{15}$

- xQy for all $x, y \in E_1(C) \setminus C$,
- xQy for all $x, y \in E_2(C) \setminus C$,
- xQy for all $x, y \in D \setminus \{x_0\}$,
- Finally, looking at the subframe of $\mathcal{X}_n$ based on the set $Y = X \setminus (E_1(C) \cup E_2(C) \cup D)$ we see that it is isomorphic to the $(n-1) \times (n-1)$-square. Then Q is defined on Y in the same way as in Lemma 6.1.7; that is, we let each of the remaining $n-1$ Q-equivalence classes consist of $n-1$ points chosen so that each Q-equivalence class contains exactly one point from each E_i-cluster of Y for $i = 1, 2$.

It is a matter of routine verification that Q is a cylindric bisimulation equivalence of $\mathcal{X}_n$, and that $\mathcal{X}_n/Q$ is isomorphic to $\mathfrak{F}_n^3$. Therefore, by Lemma 7.1.6(1), $\mathfrak{F}_n^3$ is a cylindric p-morphic image of $\mathcal{X}_n$ for every n, implying that $L_3 \supseteq L_{\mathbf{U}}$. □

Consequently, if $d(L) = \omega$, then $L_3 \supseteq L$. Suppose $d(L) < \omega$. Then L is locally tabular by Corollary 7.2.11. Recall that $\mathbf{F}_L$ denotes the class of finite rooted L-frames modulo isomorphism. Since L is locally tabular, L is complete with respect to $\mathbf{F}_L$.

7.3.2. Definition. Let $\mathcal{X} = (X, E_1, E_2, D)$ be a finite quasi-square. Fix $x \in X$.

1. The *girth* of x is the number of elements of $E_0(x)$.
2. The *diagonal girth* of $\mathcal{X}$ is the maximum of the girths of all $x \in E_0(D)$.
3. The *non-diagonal girth* of $\mathcal{X}$ is the maximum of the girths of all $x \in X \setminus E_0(D)$.
4. The diagonal (resp. non-diagonal) girth of L is n if there is $\mathcal{X} \in \mathbf{F}_L$ whose diagonal (resp. non-diagonal) girth is n, and the diagonal (resp. non-diagonal) girth of every other member of $\mathbf{F}_L$ is less than or equal to n.
5. The diagonal (resp. non-diagonal) girth of L is ω if the diagonal (resp. non-diagonal) girths of the members of $\mathbf{F}_L$ are not bounded by any integer.

7.3.3. Lemma. *Let $L \in \Lambda(\mathbf{CML}_2)$. Then L is tabular iff the depth, the diagonal girth and the non-diagonal girths of L are bounded by some integer.*

Proof. The proof is the same as the proof of Lemma 6.4.4. □

It follows that if a a normal extension L of $\mathbf{CML}_2$ has a finite depth and is not tabular, then either the diagonal girth or non-diagonal girth of L is ω.

7.3.4. LEMMA. *If $L \in \Lambda(\mathbf{CML}_2)$ has a finite depth and an infinite diagonal girth, then one of $L_1 - L_3$ contains L.*

Proof. Since the diagonal girth of L is ω, for each n there is $\mathcal{X} \in \mathbf{F}_L$ whose diagonal girth is $m \geq n$. Let C denote a diagonal E_0-cluster of $\mathcal{X}$ containing m points. Then two cases are possible:

Case 1. $d(\mathcal{X}) = 1$. Then $\mathcal{X}$ is isomorphic to $\mathfrak{F}^1_m$.

Case 2. $d(\mathcal{X}) \geq 2$. Then we define an equivalence relation Q on X such that Q leaves points of C untouched, identifies all the points in every non-diagonal E_0-cluster of X, and identifies all the non-diagonal points in every diagonal E_0-cluster of X different from C:

- xQy if $x = y$ for any $x, y \in C \cup D$,
- xQy if xE_0y for any $x, y \in X \setminus (C \cup D)$.

It is easy to see that Q is a cylindric bisimulation equivalence. Let $\mathcal{Y}$ denote $\mathcal{X}/Q$. Then by the definition of Q, every non-diagonal E_0-cluster of $\mathcal{Y}$ is a singleton set and every diagonal E_0-cluster different from C contains either one or two points. Again two cases are possible:

Case 2.1. $d(\mathcal{Y}) = 2$. Then $\mathcal{Y}$ is isomorphic to $\mathfrak{F}^2_m$ or $\mathfrak{F}^3_m$.

Case 2.2. $d(\mathcal{Y}) > 2$. Then we define an equivalence relation Q' on Y such that Q' leaves the points of C untouched, identifies all the other points in the E_1-cluster containing C, identifies all the other points in the E_2-cluster containing C, identifies all the other diagonal points, and identifies all the other remaining points:

- $xQ'y$ if $x = y$ for any $x, y \in C$,
- $xQ'y$ for any $x, y \in E_1(C) \setminus C$,
- $xQ'y$ for any $x, y \in E_2(C) \setminus C$,
- $xQ'y$ for any $x, y \in D \setminus C$,
- $xQ'y$ for any $x, y \in Y \setminus (E_1(C) \cup E_2(C) \cup D)$.

It is routine to check that Q' is a cylindric bisimulation equivalence, and that $\mathcal{Y}/Q'$ is isomorphic to $\mathfrak{F}^3_m$.

Therefore, by Lemma 7.1.6(1), for every $n \in \omega$, there exists $m > n$ such that either $\mathfrak{F}^1_m$, $\mathfrak{F}^2_m$ or $\mathfrak{F}^3_m$ is a cylindric p-morphic image of $\mathcal{X}$. Thus, $L_1 \supseteq L$, $L_2 \supseteq L$ or $L_3 \supseteq L$.

$\square$

7.3.5. LEMMA. *If $L \in \Lambda(\mathbf{CML}_2)$ has a finite depth and an infinite non-diagonal girth, then one of $L_4 - L_{15}$ contains L.*

Proof. Since the non-diagonal girth of L is ω, for each n there is $\mathcal{X} \in \mathbf{F}_L$ whose non-diagonal girth is $m \geq n$. Let C denote a non-diagonal E_0-cluster of $\mathcal{X}$ containing m points. Since non-diagonal E_0-clusters exist only in quasi-squares of depth > 1, we have $d(\mathcal{X}) > 1$. As in the previous lemma, define an equivalence relation Q on X by

- xQy if $x = y$ for any $x, y \in C \cup D$,
- xQy if xE_0y for any $x, y \in X \setminus (C \cup D)$.

It is easy to see that Q is a cylindric bisimulation equivalence. By the definition of Q, every non-diagonal E_0-cluster of $\mathcal{X}/Q$ is a singleton set and every diagonal E_0-cluster different from C contains either one or two points. Since $d(\mathcal{X}) > 1$, three cases are possible:

Case 1. $d(\mathcal{X}) = 2$. Then $\mathcal{X}/Q$ is isomorphic to one of $\mathfrak{F}^4_m - \mathfrak{F}^7_m$.

Case 2. $d(\mathcal{X}) = 3$. Then $\mathcal{X}/Q$ is isomorphic to one of $\mathfrak{F}^8_m - \mathfrak{F}^{15}_m$.

Case 3. $d(\mathcal{X}) > 3$. Let $\mathcal{Y} = \mathcal{X}/Q$. Let C' denote the diagonal E_0-cluster E_1-related to C, and let C'' denote the diagonal E_0-cluster E_2-related to C. Let also C''' be the non-diagonal E_0-cluster $E_1(C'') \cap E_2(C')$. Next we define an equivalence relation Q' on Y such that Q' leaves the points of C untouched, identifies all the non-diagonal points in C' (if such points exists), identifies all the non-diagonal points in C'' (if such points exist), identifies all the remaining points in the E_1-cluster containing C and C'', identifies all the remaining points in the E_2-cluster containing C and C'', identifies all the points in C''', identifies all the remaining points in the E_1-cluster containing C'' and C''', identifies all the remaining points in the E_2-cluster containing C' and C''', identifies all the remaining diagonal points, and identifies all the remaining non-diagonal points:

- $xQ'y$ if $x = y$ for any $x, y \in C \cup ((C' \cup C'') \cap D)$,
- $xQ'y$ for any $x, y \in D \setminus (C' \cup C'')$,
- $xQ'y$ for any $x, y \in X \setminus (D \cup E_1(C') \cup E_2(C') \cup E_1(C'') \cup E_2(C'')$,
- $xQ'y$ if xE_0y for any $x, y \in (C' \cup C'' \cup C''') \setminus D)$,
- $xQ'y$ for any $x, y \in E_2(C) \setminus (C \cup C'')$,
- $xQ'y$ for any $x, y \in E_1(C) \setminus (C \cup C')$,
- $xQ'y$ for any $x, y \in E_2(C') \setminus (C''' \cup C')$,

- $xQ'y$ for any $x, y \in E_1(C'') \setminus (C''' \cup C'')$.

It is a matter of routine verification that Q' is a cylindric bisimulation equivalence. Moreover, there are four cases possible. Either both C' and C'' are singleton sets, C' is a singleton set and C'' is not, C'' is a singleton set and C' is not, or neither C' nor C'' are singleton sets. In the first case $\mathcal{Y}/Q'$ is isomorphic to $\mathfrak{F}_m^{11}$, in the second case $\mathcal{Y}/Q'$ is isomorphic to $\mathfrak{F}_m^{13}$, in the third case $\mathcal{Y}/Q'$ is isomorphic to $\mathfrak{F}_m^{12}$, and finally in the fourth case $\mathcal{Y}/Q'$ is isomorphic to $\mathfrak{F}_m^{15}$.

Consequently, going through all these cases for every $n \in \omega$ there exists $m \geq n$ such that at least one of $\mathfrak{F}_m^4 - \mathfrak{F}_m^{15}$ is a cylindric p-morphic image of $\mathcal{X}$. Therefore, at least one of $L_4 - L_{15}$ contains L. □

7.3.6. Corollary.

1. *$L_1 - L_{15}$ are the only pre tabular logics in $\Lambda(\mathbf{CML}_2)$.*
2. *L_1, L_2, L_3, L_7, L_{14} and L_{15} are the only pre tabular logics in $\Lambda(\mathbf{PCML}_2)$.*

Proof. (1) The proof is similar to the proof of Theorem 6.4.5. It is easy to see that all L_i's are incomparable. By Lemma 7.3.3, none of L_1–L_{15} is tabular. If $L \supsetneq L_i$ and L is not tabular, then by Lemmas 7.3.1, 7.3.4 and 7.3.5, there is $j \neq i$ such that $L_j \supseteq L \supsetneq L_i$. This is a contradiction since all L_i's are incomparable. Therefore, every L_i is pre-tabular. Finally, if L is pre-tabular, then again by Lemmas 7.3.1, 7.3.4 and 7.3.5, $L_i \supseteq L$ for some $i = 1, \dots, 15$. Since L is pre-tabular, L_i cannot be a proper extension of L. Therefore, $L = L_i$.

(2) The result is an immediate consequence of (1) since, as we mentioned above, out of $L_1 - L_{15}$, only L_1, L_2, L_3, L_7, L_{14}, L_{15} belong to $\Lambda(\mathbf{PCML}_2)$.

□

For every $i = 1, \dots 15$ let $\mathbf{V}_i$ be the subvariety of $\mathbf{CA}_2$ corresponding to L_i. Then we have the following analogue of Theorem 7.3.6.

7.3.7. Corollary.

1. *$\mathbf{V}_1 - \mathbf{V}_{15}$ are the only pre-finitely generated varieties in $\Lambda(\mathbf{CA}_2)$.*
2. *$\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$, $\mathbf{V}_7$, $\mathbf{V}_{14}$ and $\mathbf{V}_{15}$ are the only pre-finitely generated varieties in $\Lambda(\mathbf{RCA}_2)$.*

It follows from Corollary 7.3.6 that a logic $L \supseteq \mathbf{CML}_2$ (resp. $L \supseteq \mathbf{PCML}_2$) is tabular iff L is not contained in one of the fifteen (resp. six) pre-tabular logics. Another characterization of tabular logics in $\Lambda(\mathbf{CML}_2)$ can be found [14, §7].

We close this section with a very rough description of the lattice structure of normal extensions of $\mathbf{CML}_2$. We need the following notation:

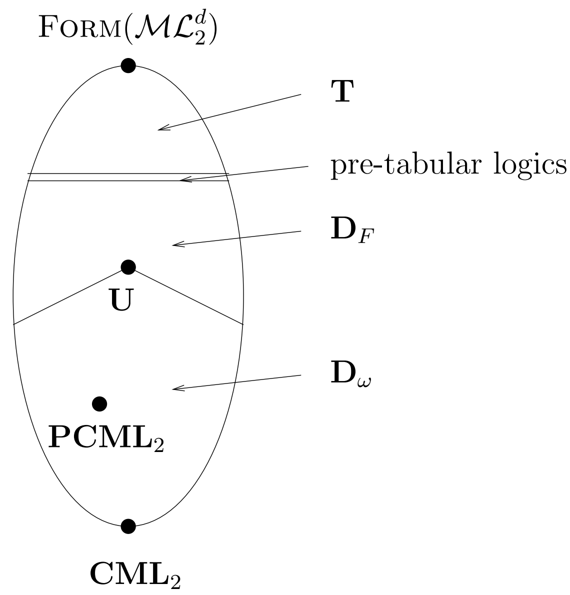

Figure 7.5: Rough picture of $\Lambda(\mathbf{CML}_2)$

$$\mathbf{T} = \{L \in \Lambda(\mathbf{CML}_2) : L \text{ is tabular}\},$$

$$\mathbf{D}_F = \{L \in \Lambda(\mathbf{CML}_2) : d(L) < \omega \text{ and } L \notin \mathbf{T}\},$$

$$\mathbf{D}_\omega = \{L \in \Lambda(\mathbf{CML}_2) : d(L) = \omega\}.$$

Let also $\text{Form}(\mathcal{ML}_2^d)$ denote the inconsistent logic.

7.3.8. Theorem.

1. $\{\mathbf{T}, \mathbf{D}_F, \mathbf{D}_\omega\}$ *is a partition of* $\Lambda(\mathbf{CML}_2)$.
2. $\text{Form}(\mathcal{ML}_2^d)$ *is the greatest element of* $\mathbf{T}$.
3. $\mathbf{T}$ *does not have minimal elements.*
4. $\mathbf{D}_F$ *has precisely fifteen maximal elements.*
5. $\mathbf{D}_F$ *does not have minimal elements.*
6. $L_{\mathbf{U}}$ *and* $\mathbf{CML}_2$ *are the greatest and least elements of* $\mathbf{D}_\omega$*, respectively.*

Proof. (1) and (2) hold by definition. For (3) observe that for every tabular extension of $\mathbf{CML}_2$, there is a tabular extension of $\mathbf{CML}_2$ properly contained in it. Therefore, $\mathbf{T}$ cannot have a minimum. (4) follows from Corollary 7.3.6. For (5) observe that for every extension of $\mathbf{CML}_2$ of finite depth there is an extension of $\mathbf{CML}_2$ of finite depth properly contained in it. (6) follows from Theorem 7.2.5. $\square$

The lattice $\Lambda(\mathbf{CML}_2)$ can be roughly depicted as shown in Figure 7.5. The detailed investigation of the upper part of $\Lambda(\mathbf{CML}_2)$ can be found in [14, §7]. In particular, a complete characterization of the lattice structure of the extensions of $\mathbf{CML}_2$ of depth one is given in [14, 7.1]. Obviously, there is a close connection between $\mathbf{S5}^2$ and $\mathbf{CML}_2$. $\mathbf{S5}^2$ can be seen as a diagonal-free reduct of $\mathbf{CML}_2$. Moreover, we can define a reduct functor from the lattice $\Lambda(\mathbf{CML}_2)$ into the lattice $\Lambda(\mathbf{S5}^2)$. This reduct functor and the properties that are preserved and reflected by it are investigated in [14, §7].

Chapter 8

Axiomatization and computational complexity

In this chapter, based on [17] and [16], we show that every normal extension of $\mathbf{S5}^2$ is finitely axiomatizable, and that every proper normal extension of $\mathbf{S5}^2$ has the polynomial size model property and an NP-complete satisfiability problem.

WARNING. In this chapter, by the complexity of a logic we mean the complexity of its satisfiability problem.

It is well known that the logic $\mathbf{S5}$ is NP-complete (see [84] and [43, Theorem 16(i)]) and that $\mathbf{S5}^2$ is NEXPTIME-complete (see [93] and [43, Theorem 5.26]). Explicit bounds on the size of finite models are known. Every $\mathbf{S5}$-consistent formula ϕ is satisfiable in a model of size $|\phi| + 1$ [84]. For $\mathbf{S5}^2$ the models need to be much larger. Every $\mathbf{S5}^2$-consistent formula ϕ can be satisfied in a product model of size $2^{f(|\phi|)}$, where f is a linear function (see [55] and [43, Theorem 5.25]). Both bounds are optimal. Here we recall that $|\phi|$ is the length of ϕ.

In Corollary 6.2.12 we proved that every normal extension of $\mathbf{S5}^2$ has the finite model property. Using this result we show that every proper normal extension L of $\mathbf{S5}^2$ has the poly-size model property. That is, there is a polynomial $P(n)$ such that every L-consistent formula ϕ is satisfied in an L-frame consisting of at most $P(|\phi|)$ points. We recall that ϕ is L-consistent if $\neg\phi \notin L$.

With every proper normal extension L of $\mathbf{S5}^2$ we associate a natural number $b(L)$—the bound of L. We show that for every L, there exists a polynomial $P(\cdot)$ of degree $b(L) + 1$ such that every L–consistent formula ϕ is satisfiable on an L-frame whose universe is bounded by $P(|\phi|)$. We also show that this bound is optimal.

In addition, we show that every proper normal extension L of $\mathbf{S5}^2$ is axiomatizable by Jankov-Fine formulas. In fact, for every proper normal extension L of $\mathbf{S5}^2$, we find a finite set $\mathbf{M}_L$ of finite rooted $\mathbf{S5}^2$-frames such that an arbitrary finite rooted $\mathbf{S5}^2$-frame is a frame for L iff it does not have any frame in $\mathbf{M}_L$

as a p-morphic image. This condition yields a finite axiomatization of L. Furthermore, we show that whether $\mathcal{F}$ is an L-frame is decidable in deterministic polynomial time. This, together with the poly-size model property of L, implies NP-completeness of (satisfiability for) L.

Finally, we note that general complexity results for (uni)modal logics were investigated before. Bull and Fine proved that every normal extension of **S4.3** has the finite model property, is finitely axiomatizable and therefore is decidable (see [18, Theorems 4.96, 4.101]). Hemaspaandra strengthened the second result by showing that every normal extension of **S4.3** is NP-complete (see, e.g., [18, Theorem 6.41]). The proof of finite axiomatizability uses Kruskal's theorem on well-quasi-orderings [18, Theorem 4.99]. Kracht uses the same technique for showing that every extension of the intermediate logic of leptonic strings is finitely axiomatizable [73, Theorem 14, Proposition 15]. We take the same line of research beyond unimodal logics. However, as we will see below, the theory of well-quasi-orderings does not suffice for our purposes; instead, we will use better-quasi-orderings.

8.1 Finite axiomatization

In this section we prove that every normal extension of $\mathbf{S5}^2$ is finitely axiomatizable. Let $\mathbf{F}_{\mathbf{S5}^2}$ be the class of finite rooted $\mathbf{S5}^2$-frames modulo isomorphism Recall that for $\mathcal{F}, \mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$ we put

$$\mathcal{F} \leq \mathcal{G} \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$

It is routine to check that $\leq$ is a partial order on $\mathbf{F}_{\mathbf{S5}^2}$. We write $\mathcal{F} < \mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \not\leq \mathcal{F}$. Then $\mathcal{F} < \mathcal{G}$ implies $|\mathcal{F}| < |\mathcal{G}|$ and we see that there are no infinite descending chains in $(\mathbf{F}_{\mathbf{S5}^2}, <)$. Thus, for any non-empty $A \subseteq \mathbf{F}_{\mathbf{S5}^2}$, the set $\min(A)$ of $<$-minimal elements of A is non-empty, and indeed for any $\mathcal{G} \in A$ there is an $\mathcal{F} \in \min(A)$ such that $\mathcal{F} \leq \mathcal{G}$.

Now we again apply the technique of frame-based formulas to show that every normal extension of $\mathbf{S5}^2$ is axiomatizable by Jankov-Fine formulas. Since every normal extension of $\mathbf{S5}^2$ has the finite model property, instead of considering the finitely generated rooted descriptive frames, as in the case of intermediate logics (see Chapter 3), we restrict ourselves to finite rooted $\mathbf{S5}^2$-frames. In order to make this chapter more self-contained we supply proofs for the next results, even though they can be easily derived from the results of Section 3.4.

Let L be a proper normal extension of $\mathbf{S5}^2$. By completeness of $\mathbf{S5}^2$ with respect to $\mathbf{F}_{\mathbf{S5}^2}$, the set $\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$ is non-empty. Let $\mathbf{M}_L = \min(\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L)$. Note that $\mathbf{M}_L$ is a shorthand of $\mathbf{M}(L, \leq)$ used in Section 3.4.

8.1.1. THEOREM. *For any proper normal extension L of $\mathbf{S5}^2$ and $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$, $\mathcal{G} \in \mathbf{F}_L$ iff no $\mathcal{F} \in \mathbf{M}_L$ is a p-morphic image of $\mathcal{G}$.*

Proof. Let $\mathcal{G} \in \mathbf{F}_L$. Then since p-morphisms preserve validity of formulas, every p-morphic image of $\mathcal{G}$ belongs to $\mathbf{F}_L$ and hence can not be in $\mathbf{M}_L$. Conversely, if $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$ then there is $\mathcal{F} \in \mathbf{M}_L$ such that $\mathcal{F} \leq \mathcal{G}$; that is, $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. □

8.1.2. THEOREM. *Every proper normal extension L of $\mathbf{S5}^2$ is axiomatizable by the axioms of $\mathbf{S5}^2$ plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$.*

Proof. Let $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$. Then by Theorem 8.1.1, $\mathcal{G} \in \mathbf{F}_L$ iff there is no $\mathcal{F} \in \mathbf{M}_L$ with $\mathcal{F} \leq \mathcal{G}$, iff (by Theorem 7.1.10) there is no $\mathcal{F} \in \mathbf{M}_L$ with $\mathcal{G} \not\models \chi(\mathcal{F})$, iff $\mathcal{G} \models \chi(\mathcal{F})$ for all $\mathcal{F} \in \mathbf{M}_L$. Thus, $\mathcal{G} \models \{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$ iff $\mathcal{G} \in \mathbf{F}_L$.

Let L' be the logic axiomatized by the axioms of $\mathbf{S5}^2$ plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$. From the above it is clear that $\mathbf{F}_{L'} = \mathbf{F}_L$. But L (resp. L') is sound and complete with respect to $\mathbf{F}_L$ (resp. $\mathbf{F}_{L'}$). So, $L' = L$. □

It follows that a proper normal extension L of $\mathbf{S5}^2$ is finitely axiomatizable whenever $\mathbf{M}_L$ is finite. We now proceed to show that $\mathbf{M}_L$ is indeed finite for every proper normal extension L of $\mathbf{S5}^2$.

Fix a proper normal extension L of $\mathbf{S5}^2$. Since $\mathbf{S5}^2$ is complete with respect to $\{\mathbf{n} \times \mathbf{n} : n \geq 1\}$ (see Theorem 6.1.10), there is $n \geq 1$ such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Let $n(L)$ be the least such.

8.1.3. LEMMA. *Let L be as above, and write n for $n(L)$.*

1. *If $\mathcal{G} \in \mathbf{F}_L$, then $d_1(\mathcal{G}) < n$ or $d_2(\mathcal{G}) < n$.*

2. *If $\mathcal{G} \in \mathbf{M}_L$, then $d_1(\mathcal{G}) \leq n$ or $d_2(\mathcal{G}) \leq n$.*

Proof.

1. If $\mathcal{G} \in \mathbf{F}_L$ and $d_1(\mathcal{G}) \geq n$ and $d_2(\mathcal{G}) \geq n$, then by Lemma 6.2.2, $\mathbf{n} \times \mathbf{n}$ is a p-morphic image of $\mathcal{G}$. So, $\mathbf{n} \times \mathbf{n} \in \mathbf{F}_L$, a contradiction.[1]

2. If $\mathcal{G} \in \mathbf{M}_L$ and both depths of $\mathcal{G}$ are greater than n, then again $\mathbf{n} \times \mathbf{n}$ is a p-morphic image of $\mathcal{G}$. Therefore, $\mathbf{n} \times \mathbf{n} < \mathcal{G}$. However, $\mathcal{G}$ is a minimal element of $\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$, implying that $\mathbf{n} \times \mathbf{n}$ belongs to $\mathbf{F}_L$, which is false.

□

8.1.4. COROLLARY. *$\mathbf{M}_L$ is finite iff $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$.*

Proof. By Lemma 8.1.3, $\mathbf{M}_L = \bigcup_{k \leq n(L)} \{\mathcal{F} \in \mathbf{M}_L : d_1(\mathcal{F}) = k\} \cup \bigcup_{k \leq n(L)} \{\mathcal{F} \in \mathbf{M}_L : d_2(\mathcal{F}) = k\}$. Thus, $\mathbf{M}_L$ is finite if and only if $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. □

[1] This result also follows immediately from Theorem 6.3.2.

Since $\mathbf{M}_L$ is a $\leq$-antichain in $\mathbf{F}_{\mathbf{S5}^2}$, to show that $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$, it is enough to prove that for any k, the set $\{\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2} : d_i(\mathcal{F}) = k\}$ does not contain an infinite $\leq$-antichain. Without loss of generality we can consider the case when $i = 2$.

Fix $k \in \omega$. For every $n \in \omega$ let $\mathcal{M}_n$ denote the set of all $n \times k$ matrices[2] (m_{ij}) with coefficients in ω ($i < n$, $j < k$). Let $\mathcal{M} = \bigcup_{n\in\omega} \mathcal{M}_n$. Define $\preccurlyeq$ on $\mathcal{M}$ by putting $(m_{ij}) \preccurlyeq (m'_{ij})$ if we have $(m_{ij}) \in \mathcal{M}_n$, $(m'_{ij}) \in \mathcal{M}_{n'}$, $n \leq n'$, and there is a surjection $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. It is easy to see that $(\mathcal{M}, \preccurlyeq)$ is a quasi-ordered set (i.e., $\preccurlyeq$ is reflexive and transitive).

Let $\mathbf{F}^k_{\mathbf{S5}^2} = \{\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2} : d_2(\mathcal{F}) = k\}$. For each $\mathcal{F} \in \mathbf{F}^k_{\mathbf{S5}^2}$ we fix enumerations $F_0, \ldots, F_{n-1}$ of the E_1-clusters of $\mathcal{F}$ (where $n = d_1(\mathcal{F})$) and $F^0, \ldots, F^{k-1}$ of the E_2-clusters of $\mathcal{F}$. Define a map $H : \mathbf{F}^k_{\mathbf{S5}^2} \to \mathcal{M}$ by putting $H(\mathcal{F}) = (m_{ij})$ if $|F_i \cap F^j| = m_{ij}$ for $i < d_1(\mathcal{F})$ and $j < k$. As $\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2}$, it follows that $m_{ij} > 0$ for each such i, j. Recall that a map $f : P \to P'$ between ordered sets $(P, \leq)$ and $(P' \leq')$ is *order reflecting* if $f(w) \leq' f(v)$ implies $w \leq v$ for any $w, v \in P$.

8.1.5. LEMMA. *$H : (\mathbf{F}^k_{\mathbf{S5}^2}, \leq) \to (\mathcal{M}, \preccurlyeq)$ is an order-reflecting injection.*

Proof. Since $\mathbf{F}_{\mathbf{S5}^2}$ consists of non-isomorphic frames, H is one-one. Now let $\mathcal{F} = (W, E_1, E_2)$, $\mathcal{G} = (U, S_1, S_2)$, $\mathcal{F}, \mathcal{G} \in \mathbf{F}^k_{\mathbf{S5}^2}$, and $(m_{ij}), (m'_{ij}) \in \mathcal{M}$ be such that $H(\mathcal{F}) = (m_{ij})$, $H(\mathcal{G}) = (m'_{ij})$, and $(m_{ij}) \preccurlyeq (m'_{ij})$. We need to show that $\mathcal{F} \leq \mathcal{G}$. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. Then there is surjective $f : n' \to n$ such that $m_{f(i)j} \leq m'_{ij}$ for $i < n'$ and $j < k$. Then $|G_i \cap G^j| \geq |F_{f(i)} \cap F^j| > 0$ for any $i < n'$ and $j < k$. Hence there exists a surjection $h^j_i : G_i \cap G^j \to F_{f(i)} \cap F^j$. Define $h : U \to W$ by putting $h(u) = h^j_i(u)$, where $i < n', j < k$, and $u \in G_i \cap G^j$. It is obvious that h is well defined and onto.

Now we show that h is a p-morphism. If uS_1v, then $u, v \in G_i$ for some $i < n'$. Therefore, $h(u), h(v) \in F_{f(i)}$, and so $h(u)E_1h(v)$. Analogously, if uS_2v, then $u, v \in G^j$ for some $j < k$, $h(u), h(v) \in F^j$, and so $h(u)E_2h(v)$. Now suppose $u \in G_i \cap G^j$ for some $i < n'$ and $j < k$. If $h(u)E_2h(v)$, then $h(u), h(v) \in F^j$ and $v \in G^j$. As both u and v belong to G^j it follows that uS_2v. Finally, if $h(u)E_1h(v)$, then $h(u) \in F_{f(i)} \cap F^j$ and $h(v) \in F_{f(i)} \cap F^{j'}$, for some $j' < k$. Therefore, there exists $z \in G_i \cap G^{j'}$ (since $z \in G_i$ we have uS_1z) such that $h(z) = h(v)$. Thus, h is an onto p-morphism, implying that $\mathcal{F} \leq \mathcal{G}$. Thus, H is order reflecting. □

8.1.6. COROLLARY. *If $\Delta \subseteq \mathbf{F}^k_{\mathbf{S5}^2}$ is a $\leq$-antichain, then $H(\Delta) \subseteq \mathcal{M}$ is a $\preccurlyeq$-antichain.*

Proof. Immediate. □

Now we will show that there are no infinite $\preccurlyeq$-antichains in $\mathcal{M}$. For this we define a quasi-order $\sqsubseteq$ on $\mathcal{M}$ included in $\preccurlyeq$ and show that there are no infinite

[2]By an $n \times k$ matrix we mean a matrix with n rows and k columns.

$\sqsubseteq$-antichains in $\mathcal{M}$. To do so we first introduce two quasi-orders $\sqsubseteq_1$ and $\sqsubseteq_2$ on $\mathcal{M}$ and then define $\sqsubseteq$ as the intersection of these quasi-orders. For $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$, we say that:

- $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ if there is a one-one order-preserving map $\varphi : n \to n'$ (i.e., $i < i' < n$ implies $\varphi(i) < \varphi(i')$) such that $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$;

- $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ if there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$.

Let $\sqsubseteq$ be the intersection of $\sqsubseteq_1$ and $\sqsubseteq_2$.

8.1.7. Lemma. *For any $(m_{ij}), (m'_{ij}) \in \mathcal{M}$, if $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \preccurlyeq (m'_{ij})$.*

Proof. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. If $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ and $(m_{ij}) \sqsubseteq_2 (m'_{ij})$. By $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ there is a one-one order-preserving map $\varphi : n \to n'$ with $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$; and by $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ there is a map $\psi : n' \to n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Let $\mathrm{rng}(\varphi) = \{\varphi(i) : i < n\}$. Define $f : n' \to n$ by putting

$$f(i) = \begin{cases} \varphi^{-1}(i), & \text{if } i \in \mathrm{rng}(\varphi), \\ \psi(i), & \text{otherwise.} \end{cases}$$

Then f is a surjection. Moreover, for $i < n'$ and $j < k$, if $i \in \mathrm{rng}(\varphi)$, then $m_{f(i)j} = m_{\varphi^{-1}(i)j} \leq m'_{ij}$ by the definition of $\sqsubseteq_1$; and if $i \notin \mathrm{rng}(\varphi)$, then $m_{f(i)j} = m_{\psi(i)j} \leq m'_{ij}$ by the definition of $\sqsubseteq_2$. Therefore, $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Thus, $(m_{ij}) \preccurlyeq (m'_{ij})$. $\square$

Thus, it is left to show that there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. For this we use the theory of better-quasi-orderings (bqos). Our main source of reference is Laver [85].

For any set $X \subseteq \omega$ let $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\}$, and for $n < \omega$ let $[X]^n = \{Y \subseteq X : |Y| = n\}$. We say that Y is an initial segment of X if there is $n \in \omega$ such that $Y = \{x \in X : x \leq n\}$.

8.1.8. Definition. Let X be an infinite subset of ω. We say that $\mathcal{B} \subseteq [X]^{<\omega}$ is a *barrier on* X if $\emptyset \notin \mathcal{B}$ and:

- for every infinite $Y \subseteq X$, there is an initial segment of Y in $\mathcal{B}$;

- $\mathcal{B}$ is an antichain with respect to $\subseteq$.

A *barrier* is a barrier on some infinite $X \subseteq \omega$.

Note that for any $n \geq 1$, $[\omega]^n$ is a barrier on ω.

8.1.9. DEFINITION.

1. If s, t are finite subsets of ω, we write $s \lhd t$ to mean that there are $i_1 < \ldots < i_k$ and j $(1 \leq j < k)$ such that $s = \{i_1, \ldots, i_j\}$ and $t = \{i_2, \ldots, i_k\}$.

2. Given a barrier $\mathcal{B}$ and a quasi-ordered set $(Q, \leq)$, we say that a map $f : \mathcal{B} \to Q$ is *good* if there are $s, t \in \mathcal{B}$ such that $s \lhd t$ and $f(s) \leq f(t)$.

3. Let $(Q, \leq)$ be a quasi-order. We call $\leq$ a *better-quasi-ordering* (*bqo*) if for every barrier $\mathcal{B}$, every map $f : \mathcal{B} \to Q$ is good.

Now we recall basic constructions and properties of bqos.

8.1.10. PROPOSITION. *If $(Q, \leq)$ is a bqo, there are no infinite $\leq$-antichains in Q.*

Proof. Let $(\xi_n)_{n\in\omega}$ be an infinite sequence of distinct elements of Q. As we pointed out, $\mathcal{B} = [\omega]^1 = \{\{n\} : n < \omega\}$ is a barrier. Define a map $\theta : \mathcal{B} \to Q$ by $\theta(\{n\}) = \xi_n$. Since $(Q, \leq)$ is a bqo, θ is good. Therefore, there are $\{n\}, \{m\} \in \mathcal{B}$ such that $\{n\} \lhd \{m\}$ (i.e., $n < m$) and $\xi_n \leq \xi_m$. So, no infinite subset of Q forms a $\leq$-antichain. $\square$

We write On for the class of all ordinals. Let $(Q, \leq)$ be a quasi-order. Define $\leq^*$ on the class $\bigcup_{\alpha\in On} Q^\alpha$, and on any set contained in it, by $(x_i)_{i<\alpha} \leq^* (y_i)_{i<\beta}$ if there is a one-one order-preserving map $\varphi : \alpha \to \beta$ such that $x_i \leq y_{\varphi(i)}$ for all $i < \alpha$.

Let $\mathcal{P}(Q)$ be the power set of Q. The order $\leq$ can be extended to $\mathcal{P}(Q)$ as follows: For $\Gamma, \Delta \in \mathcal{P}(Q)$, we say that $\Gamma \leq \Delta$ if for all $\delta \in \Delta$ there is $\gamma \in \Gamma$ with $\gamma \leq \delta$. Recall that $(P, \leq')$ is a *suborder* of $(Q, \leq)$ if $P \subseteq Q$ and $\leq' = \leq \cap P^2$.

8.1.11. THEOREM.

1. *$(\omega, \leq)$ is a bqo.*

2. *Any suborder of a bqo is a bqo.*

3. *If $\leq$ and $\leq'$ are bqos on Q, then $\leq \cap \leq'$ is also a bqo on Q.*

4. *If $(Q, \leq)$ is a bqo, then $(\bigcup_{\alpha\in On} Q^\alpha, \leq^*)$ is also a (proper class) bqo. Hence, by (2), its suborders $(Q^k, \leq^*)$ and $(\bigcup_{n<\omega} Q^n, \leq^*)$ are bqos.*

5. *If $(Q, \leq)$ is a bqo, then $(\mathcal{P}(Q), \leq)$ is a bqo.*

Proof. (1) follows from Lemma 1.2 of [85]. (2) is trivial.

(3): By [85, Lemma 1.8], $(Q \times Q, \leq \otimes \leq')$ is a bqo, where we define $(x, x') \leq \otimes \leq' (y, y')$ iff $x \leq y$ and $x' \leq' y'$. By (2), its suborder $(\{(q, q) : q \in Q\}, \leq \otimes \leq')$ is also a bqo, and this is isomorphic to $(Q, \leq \cap \leq')$.

(4) See [85, Theorem 1.10].

(5) Finally to show $(\mathcal{P}(Q), \leq)$ is a bqo we adapt the proof of Lemma 1.3 of [85]. Let $\mathcal{B}$ be a barrier and consider $f : \mathcal{B} \to \mathcal{P}(Q)$. Suppose f is not good. Then for each $s, t \in \mathcal{B}$ with $s \lhd t$ we have $f(s) \not\leq f(t)$. Let $\mathcal{B}(2) = \{s \cup t : s, t \in \mathcal{B}$ and $s \lhd t\}$. Thus for every element $s \cup t \in \mathcal{B}(2)$ there is an element $\delta_{st} \in f(t)$ such that for every $\gamma \in f(s)$ we have $\gamma \not\leq \delta_{st}$.

Define a map $h : \mathcal{B}(2) \to Q$ by putting $h(s \cup t) = \delta_{st}$ for every $s \cup t \in \mathcal{B}(2)$. It is easy to see that h is well defined. It is known (see, e.g., [85, p. 35]) that $\mathcal{B}(2)$ is a barrier. Since $(Q, \leq)$ is a bqo, h is good, so there exist $s \cup t, s' \cup t' \in \mathcal{B}(2)$ with $s \cup t \lhd s' \cup t'$ and $h(s \cup t) \leq h(s' \cup t')$. It is easy to check (see [85, p. 35]) that $t = s'$. But now $\delta_{s't'} = h(s' \cup t') \geq h(s \cup t) \in f(t) = f(s')$. This contradicts the definition of $\delta_{s't'}$, hence f is good and therefore $(\mathcal{P}(Q), \leq)$ is a bqo. □

8.1.12. Remark. A quasi-order $\leq$ on a set Q is called a *well-quasi-ordering* (*wqo*) if for any sequence $(x_i)_{i<\omega}$ in Q there exist $i < j < \omega$ with $x_i \leq x_j$. As we said in the introduction to this chapter, wqos have been used to prove finite axiomatizability results in modal logic on many previous occasions. The following facts are known about them (see e.g. [85]):

1. Any bqo is a wqo.

2. If $\leq$ and $\leq'$ are wqos on Q, then $\leq \cap \leq'$ is also a wqo on Q.

3. (Higman's Lemma, proved in [63]) If $(Q, \leq)$ is a wqo then $(\bigcup_{n \in \omega} Q^n, \leq^*)$ is also a wqo.

An example of a wqo $(Q, \leq)$ for which $(\bigcup_{\alpha \in On} Q^\alpha, \leq^*)$ is not a wqo, was constructed by Rado [104]: let $Q = \{(i, j) : i < j < \omega\}$, ordered by $(i, j) \leq (k, l)$ iff either $i = k$ and $j \leq l$, or else $i, j < k$. This is a wqo on Q. Now for $i < \omega$ let ξ_i be the sequence $((i, i+1), (i, i+2), \ldots)$. Then $\xi_i \not\leq^* \xi_j$ for all $i < j < \omega$. This example can be used to show that for a wqo $(Q, \leq)$, in general $(\mathcal{P}(Q), \leq)$ fails to be a wqo, even if we restrict to finite subsets of Q (see also the discussion on p. 33 of [85]). This failure is why we use bqos and not wqos here.

By Proposition 8.1.10, to show that there are no $\sqsubseteq$-antichains in $\mathcal{M}$ it suffices to show that $(\mathcal{M}, \sqsubseteq)$ is a bqo. It follows from Theorem 8.1.11(3) that the intersection of two bqos is again a bqo. Hence, it is enough to prove that $(\mathcal{M}, \sqsubseteq_1)$ and $(\mathcal{M}, \sqsubseteq_2)$ are bqos.

8.1.13. Lemma. *$(\mathcal{M}, \sqsubseteq_1)$ is a bqo.*

Proof. By Theorem 8.1.11(1), $(\omega, \leq)$ is a bqo. By Theorem 8.1.11(4), $(\omega^k, \leq^*)$ is also a bqo. Note that ω^k is the set of all k-tuples of natural numbers. It follows from the definition of $\leq^*$ that $(m_1, \dots, m_k) \leq^* (m'_1, \dots, m'_k)$ iff $m_i \leq m'_i$, for every $i \leq k$. Then $(\omega^k)^n$ is the set of all matrices with coefficients in ω that have n rows and k columns. Now, by spelling out the definition of $\leq^{**}$ we obtain that for $(m)_{ij} \in (\omega^k)^n$ and $(m)'_{ij} \in (\omega^k)^{n'}$, we have $(m)_{ij} \leq^{**} (m)'_{ij}$ iff there is a one-one order-preserving map $\varphi : n \to n'$ such that $(m_{i1}, \dots, _{ik}) \leq^* (m_{\varphi(i)1}, \dots, m_{\varphi(i)k})$, which means that for each $j \leq k$, we have $m_{ij} \leq m_{\varphi(i)j}$. Therefore, $(m)_{ij} \sqsubseteq_1 (m)'_{ij}$ and $(\mathcal{M}, \sqsubseteq_1) \cong (\bigcup_{n<\omega}(\omega^k)^n, \leq^{**})$. By Theorem 8.1.11(4), $(\bigcup_{n<\omega}(\omega^k)^n, \leq^{**})$ is a bqo, implying that $(\mathcal{M}, \sqsubseteq_1)$ is a bqo as well.[3] $\square$

It remains to show that $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

8.1.14. Lemma. *$(\mathcal{M}, \sqsubseteq_2)$ is a bqo.*

Proof. For a matrix $(m_{ij}) \in \mathcal{M}_n$ let $m_i = (m_{i0}, \dots, m_{ik-1})$ denote the i-th row of (m_{ij}). Note that each row of (m_{ij}) is a $1 \times k$ matrix, and so $m_i \in \mathcal{M}_1$ for any $i < n$. We write $\mathrm{row}(m_{ij})$ for the set $\{m_i : i < n\}$. Obviously, $\mathrm{row}(m_{ij}) \in \mathcal{P}(\mathcal{M}_1) \subseteq \mathcal{P}(\mathcal{M})$. Consider an arbitrary barrier $\mathcal{B}$ and a map $f : \mathcal{B} \to \mathcal{M}$. We need to show that f is good with respect to $\sqsubseteq_2$. Define $g : \mathcal{B} \to \mathcal{P}(\mathcal{M})$ by $g(s) = \mathrm{row}(f(s))$. Since $(\mathcal{M}, \sqsubseteq_1)$ is a bqo, by Theorem 8.1.11(5), $(\mathcal{P}(\mathcal{M}), \sqsubseteq_1)$ is also a bqo. Hence, there are $s, t \in \mathcal{B}$ such that $s \lhd t$ and $g(s) \sqsubseteq_1 g(t)$. Therefore, for each $\delta \in g(t)$ there is $\gamma \in g(s)$ with $\gamma \sqsubseteq_1 \delta$.

Now we show that $f(s) \sqsubseteq_2 f(t)$. Write (m_{ij}) for $f(s)$ and (m'_{ij}) for $f(t)$. Suppose that $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. We define $\psi : n' \to n$ as follows. Let $i < n'$. Then $m'_i \in g(t)$. By the above, we may choose $\psi(i) < n$ such that $m_{\psi(i)} \sqsubseteq_1 m'_i$. This defines ψ, and we have $m_{\psi(i)j} \leq m'_{ij}$ for any $i < n'$ and $j < k$. Thus, $f(s) \sqsubseteq_2 f(t)$, f is a good map, and so $(\mathcal{M}, \sqsubseteq_2)$ is a bqo. $\square$

It follows that $(\mathcal{M}, \sqsubseteq)$ is a bqo. Therefore, there are no infinite $\sqsubseteq$-antichains in $\mathcal{M}$. Thus, by Lemma 8.1.7, there are no infinite $\preccurlyeq$-antichains in $\mathcal{M}$.

Now we are in a position to prove the first main result of this chapter, which was obtained jointly with I. Hodkinson, see [16, Theorem 3.16].

8.1.15. Theorem. *Every normal extension of* $\mathbf{S5}^2$ *is finitely axiomatizable.*

Proof. Clearly, $\mathbf{S5}^2$ is finitely axiomatizable. Suppose L is a proper normal extension of $\mathbf{S5}^2$. Then, by Theorem 8.1.2, L is axiomatizable by the $\mathbf{S5}^2$ axioms plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$. Since there are no infinite $\preccurlyeq$-antichains in $\mathcal{M}$, by Corollary 8.1.6, there are no infinite antichains in $\mathbf{F}^k_{\mathbf{S5}^2}$, for each $k \in \omega$. Therefore, $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. Thus, $\mathbf{M}_L$ is finite by Corollary 8.1.4. It follows that L is finitely axiomatizable. $\square$

[3] To apply this theorem, we needed to require in the definition of $\sqsubseteq_1$ on $\mathcal{M}$ that φ is order preserving.

8.1.16. Corollary. *The lattice of normal extensions of* $\mathbf{S5}^2$ *is countable.*

Proof. This immediately follows from Theorem 8.1.15 since there are only countably many finitely axiomatizable normal extensions of $\mathbf{S5}^2$. $\square$

8.1.17. Corollary.

1. *Every subvariety of* $\mathbf{Df}_2$ *is finitely axiomatizable.*
2. *The lattice of subvarieties of* $\mathbf{Df}_2$ *is countable.*

8.1.18. Remark. Note that Theorem 8.1.15 and Corollaries 8.1.16 and 8.1.17 show one more difference between the diagonal free case and the case with the diagonal. As follows from Theorem 7.1.14 $\mathbf{CML}_2$ and $\mathbf{PCML}_2$, (resp. $\mathbf{CA}_2$ and $\mathbf{RCA}_2$) have continuum many normal extensions (resp. subvarieties) and continuum many of them are not finitely axiomatizable.

8.2 The poly-size model property

In this section we prove that every proper normal extension of $\mathbf{S5}^2$ has the poly-size model property. First we introduce some terminology.

Recall from Theorem 6.3.2 that for every proper normal extension L of $\mathbf{S5}^2$, we have that $\mathbf{F}_L = \mathbf{F}_1 \uplus \mathbf{F}_2 \uplus \mathbf{F}_3$, where $d_1(\mathbf{F}_2)$, $d_2(\mathbf{F}_1)$, $d_1(\mathbf{F}_3)$ and $d_2(\mathbf{F}_3)$ are bounded by some natural number n. Now we introduce the following parameters: for $i \in \{1,2\}$, $k \in \{1,2,3\}$, let $p_i^k = d_i(\mathbf{F}_k)$. The parameter p_i^k gives the E_i-depth of the class $\mathbf{F}_k$. We call a parameter *finite*, if it is not ω. Note that the only parameters which may be infinite are p_1^1 and p_2^2. Let $b(L)$ denote the maximum between all the finite parameters of L, and call it the *bound* of L. Note that if p_1^1 and p_2^2 are ω, then $b(L) = n$, where n is the minimal natural number such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$.

Let $|\phi|$ denote the *modal size* of the formula ϕ, that is the number of subformulas of ϕ of the form $\Diamond_1\psi$ and $\Diamond_2\chi$. Recall that a polynomial $P(n)$ is said to be of *degree* k if n^k occurs in $P(n)$ and n^m does not occur in $P(n)$ for any $m > k$.

8.2.1. Theorem. *Let L be a proper normal extension of* $\mathbf{S5}^2$ *with bound $b(L)$. Then every L–satisfiable formula ϕ is satisfiable in an L-frame of size $P(|\phi|)$, for $P(|\phi|)$ a polynomial of degree $b(L)+1$. Moreover, if all the parameters of L are finite, then $P(|\phi|)$ is just linear in $|\phi|$.*

In the proof we create small models from large ones taking care that 1) the frame of the small model is still a frame of the logic, and 2) certain formulas are still satisfied in the small model. For this we will need two lemmas proved below. For the first part we use Lemma 8.2.2, for the latter part we use Lemma 8.2.3.

8.2.2. LEMMA. *Let $\mathcal{F} = (W, E_1, E_2)$ be a finite $\mathbf{S5}^2$-frame and Q an equivalence relation on W. If either of the following three cases $(1), (2a), (2b)$ holds, then Q is a bisimulation equivalence and $f_Q : W \to W/B$ is a p-morphism from $\mathcal{F}$ onto $\mathcal{F}/Q$.*

1. *$Q \subseteq E_0$ (that is, Q identifies only points from E_0-clusters).*

(2a). *$Q \subseteq E_2$ and uQv implies that for every $u' \in E_1(u)$ there exists some $v' \in E_1(v)$ with $u'Qv'$.*

(2b). *$Q \subseteq E_1$ and uQv implies that for every $u' \in E_2(u)$ there exists some $v' \in E_2(v)$ with $u'Qv'$.*

Proof. The proof is a routine verification. $\square$

8.2.3. LEMMA. *For any proper normal extension L of $\mathbf{S5}^2$, if ϕ is L-satisfiable, then it is satisfiable in an L-frame $\mathcal{F} = (W, E_1, E_2)$ such that*

$$|W| \leq d_1(\mathcal{F})|\phi| + d_2(\mathcal{F})|\phi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1.$$

Moreover, the size of any E_0-cluster in $\mathcal{F}$ is at most $|\phi|$.

Proof. Let $\mathcal{F} = (W, E_1, E_2)$ be an L-frame satisfying formula ϕ. Then there exists a valuation V on $\mathcal{F}$ and a point $w \in W$ such that $(\mathcal{F}, V), w \models \phi$. The next claim is the analogue of what is known as Tarski's test in first-order logic; see, e.g., Chang and Kiesler [25, Proposition 3.1.2]

8.2.4. CLAIM. *Let $\mathfrak{M} = (\mathcal{F}, V)$ be a model based on some $\mathbf{S5}^2$-frame $\mathcal{F} = (W, E_1, E_2)$. Let $W' \subseteq W$ and let $\mathfrak{M}' = (W', E_1', E_2', V')$ be a submodel of $\mathfrak{M}$ obtained by restricting E_1 and E_2 and V to W'. Suppose W' satisfies the next two conditions:*

(*i*) *For every $\Diamond_1\psi \in Sub(\phi)$ and E_1-cluster C_i of $\mathcal{F}$, if there exists $x \in C_i$ such that $\mathfrak{M}, x \models \psi$, then there exists $y \in C_i \cap W'$ such that $\mathfrak{M}, y \models \psi$.*

(*ii*) *For every $\Diamond_2\psi \in Sub(\phi)$ and E_2-cluster C^j of $\mathcal{F}$, if there exists $x \in C^j$ such that $\mathfrak{M}, x \models \psi$, then there exists $y \in C^j \cap W'$ such that $\mathfrak{M}, y \models \psi$.*

Then for every $v \in W'$ and $\psi \in Sub(\phi)$, we have

$$\mathfrak{M}, v \models \psi \text{ iff } \mathfrak{M}', v \models \psi.$$

Proof. We prove the claim by induction on the size of $\psi \in Sub(\phi)$. The Boolean clauses are trivial. Let $\psi = \Diamond_i\chi$, $i = 1, 2$. Then $\mathfrak{M}', v \models \Diamond_i\chi$ implies that there exists $v' \in W'$ such that vE_iv' and $\mathfrak{M}', v' \models \chi$. But then by the induction hypothesis $\mathfrak{M}, v' \models \chi$, and hence $\mathfrak{M}, v \models \Diamond_i\chi$. Conversely, $\mathfrak{M}, v \models \Diamond_i\chi$ implies that χ is satisfied in $E_i(v)$. From (i) and (ii) it follows that there exists $y \in W'$ such that vE_iy and $y \models \chi$. But then by the induction hypothesis $\mathfrak{M}', y \models \chi$, and hence $\mathfrak{M}', v \models \Diamond_i\chi$. $\square$

Now we will create a small satisfying model from $\mathcal{F}$. For every E_1-cluster C_i ($1 \leq i \leq d_1(\mathcal{F})$) and every $\Diamond_1\psi \in Sub(\phi)$, we choose a point $x \in C_i$ such that $x \models \psi$ (if such a point exists at all). We do the same for E_2-clusters and $\Diamond_2\psi \in Sub(\phi)$. Moreover, if there are E_0-clusters of W which do not contain any selected points, we choose one point from each of them. Let W' denote the set of all selected points plus w. (Note that if $\mathcal{F}$ is a product-frame, then $W = W'$.) Define the relation Q on W as follows: By the definition of W', for each E_0-cluster C_i^j of $\mathcal{F}$ we have chosen at least one point $witness(C_i^j) \in C_i^j$ to be in W'. Let $\mathcal{F}' = (W', E_1', E_2')$ be the frame obtained by restricting E_1 and E_2 to W'. Now let Q be the smallest equivalence relation which identifies the points from $C_i^j \setminus W'$ with $witness(C_i^j)$ and define $f_Q : \mathcal{F} \to \mathcal{F}/Q$ by putting $f_Q(w) = Q(w)$ for any $w \in W$. Then it is easy to see that $\mathcal{F}_Q$ is isomorphic to $\mathcal{F}'$, and $Q \subseteq E_0$. Therefore, by Lemma 8.2.2(1), $\mathcal{F}'$ is (isomorphic to) a p-morphic image of $\mathcal{F}$. Thus, $\mathcal{F}'$ is also an L-frame.

Finally, consider the model $\mathfrak{M}' = (\mathcal{F}', V')$, where V' is the restriction of V to W', i.e., $V(p) = V'(p) \cap W'$, for every $p \in$ Prop. Then W' satisfies the conditions of Claim 8.2.4, and so $\mathfrak{M}', w \models \phi$. Note, that $|W'| \leq d_1(\mathcal{F})|\phi| + d_2(\mathcal{F})|\phi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1$. Indeed, there exist $d_1(\mathcal{F})$-many E_1-clusters and $d_2(\mathcal{F})$-many E_2-clusters of W. From every E_i-cluster, $i = 1, 2$, we select at most $|\phi|$ points. So, we select $(d_1(\mathcal{F})|\phi| + d_2(\mathcal{F})|\phi|)$-many points, and then from every E_0-cluster which does not contain any selected point, we choose an additional point. Obviously there are $d_1(\mathcal{F}) \cdot d_2(\mathcal{F})$-many E_0-clusters in W, hence $|W'| \leq d_1(\mathcal{F})|\phi| + d_2(\mathcal{F})|\phi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1$. ☐

Now we can prove Theorem 8.2.1.

Proof of Theorem 8.2.1. Let L be as in the theorem with bound $b(L)$. Let ϕ be L-satisfiable. Then there exist an L-frame $\mathcal{F} = (W, E_1, E_2)$, a valuation V on $\mathcal{F}$ and $w \in W$ such that $(\mathcal{F}, V), w \models \phi$. By Lemma 8.2.3, we may assume that

$$|W| \leq d_1(\mathcal{F})|\phi| + d_2(\mathcal{F})|\phi| + d_1(\mathcal{F}) \cdot d_2(\mathcal{F}) + 1.$$

Moreover, the size of any E_0-cluster in $\mathcal{F}$ is at most $|\phi|$. Hence, every E_1-cluster of $\mathcal{F}$ contains at most $d_2(\mathcal{F})|\phi|$ points and every E_2-cluster contains at most $d_1(\mathcal{F})|\phi|$ points. We split the proof in three cases.

Case 1: [All parameters are finite or $\mathcal{F} \in \mathbf{F}_3$]. In this case, $d_1(\mathcal{F})$ and $d_2(\mathcal{F})$ are both smaller than $b(L)$, whence ϕ is satisfied in a frame with at most $2b(L)|\phi| + b(L)^2 + 1$ points, which is a linear function in $|\phi|$.

Case 2: [$\mathcal{F} \in \mathbf{F}_2$ and $d_2(\mathcal{F})$ is unbounded]. Because $\mathcal{F} \in \mathbf{F}_2$, $d_1(\mathcal{F}) \leq b(L)$, but $d_2(\mathcal{F})$ is unbounded, whence the frame might be too large. We make it smaller by defining an equivalence relation Q on W, and factoring $\mathcal{F}$ through it. To this end we say that two E_2-clusters C^p and C^q are *equivalent* if

$$|C_i \cap C^p| = |C_i \cap C^q| \text{ for all } i \text{ between 1 and } d_1(\mathcal{F}).$$

Because the size of the E_0-clusters $C_i \cap C^j$ is bounded by $|\phi|$, the number of non-equivalent E_2-clusters is bounded by $|\phi|^{d_1(\mathcal{F})}$. Indeed, to every E_2-cluster C^p of $\mathcal{F}$ corresponds the sequence of natural numbers $\overline{n} = (n_1, \ldots, n_{d_1(\mathcal{F})})$, where $n_1 = |C_1^p|, \ldots, n_{d_1(\mathcal{F})} = |C_{d_1(\mathcal{F})}^p|$. Obviously, $n_j \leq |\phi|$ for $1 \leq j \leq d_1(\mathcal{F})$, and to equivalent E_2-clusters correspond the same sequences. Now since there exist only $|\phi|^{d_1(\mathcal{F})}$-many different sequences $\overline{n} = (n_1, \ldots, n_{d_1(\mathcal{F})})$, there exist only $|\phi|^{d_1(\mathcal{F})}$-many non-equivalent E_2-clusters.

Next we define a submodel of $\mathfrak{M} = (\mathcal{F}, V)$ which still satisfies ϕ, its underlying frame is a p-morphic image of $\mathcal{F}$ and it is of the right (small) size.

For every E_1-cluster C_i of $\mathcal{F}$ $(1 \leq i \leq d_1(\mathcal{F}))$ and every $\Diamond_1\psi \in Sub(\phi)$, we choose a point $x \in C_i$ such that $\mathfrak{M}, x \models \psi$ (if such a point exists at all). Denote by S the set of selected points plus w. It is easy to see that

$$|E_2(S)| \leq (d_1(\mathcal{F})|\phi| + 1)d_1(\mathcal{F})|\phi|$$

Indeed, from every E_1-cluster we select at most $|\phi|$ points. There are $d_1(\mathcal{F})$ E_1-clusters in $\mathcal{F}$. So, we select points from at most $d_1(\mathcal{F})|\phi| + 1$ different E_2-clusters and every E_2-cluster of $\mathcal{F}$ contains at most $d_1(\mathcal{F})|\phi|$ points.

Now from each equivalence class of E_2-clusters (see above) let us choose one representative C^p and let W' be $E_2(S)$ plus this set of representatives. For $i = 1, 2$, let E_i' and V' be the restrictions of E_i and V to W'. Consider $\mathcal{F}' = (W', E_1', E_2')$ and $\mathfrak{M}' = (\mathcal{F}', V')$. Then W' again satisfies the conditions of Claim 8.2.4. Therefore, $\mathfrak{M}', w \models \phi$. The number of points in W' is bounded by

$$|E_2(S)| + (|\phi|^{d_1(\mathcal{F})} \cdot d_1(\mathcal{F})|\phi|) \leq b(L)^2|\phi|^2 + b(L)|\phi| + b(L)|\phi|^{b(L)+1}.$$

Finally, almost the same construction as in Lemma 8.2.3 will provide us with a p-morphism from $\mathcal{F}$ to $\mathcal{F}'$. For every E_2-cluster $C^q \subseteq W \setminus W'$, let $C^p \subseteq W'$ be a E_2-cluster which is equivalent to C^q. Then the E_0-clusters C_i^p and C_i^q contain the same number of points for every $i = 1, \ldots, d_1(\mathcal{F})$. Suppose $C_i^p = \{w_{i_1}, \ldots, w_{i_{n_i}}\}$ and $C_i^q = \{v_{i_1}, \ldots, v_{i_{n_i}}\}$. Let Q be the smallest equivalence relation such that $w_{i_r} Q v_{i_r}$ holds for all $r = 1, \ldots, n_i$ and $i = 1, \ldots, d_1(\mathcal{F})$. Then Q satisfies condition (2b) of Lemma 8.2.2. Thus by Lemma 8.2.2, f_Q is a p-morphism from $\mathcal{F}$ onto $\mathcal{F}/Q$. But $\mathcal{F}/Q$ is isomorphic to $\mathcal{F}'$, so the latter is in $\mathbf{F}_L$.

Therefore, ϕ is satisfiable in an L-frame containing at most $P(|\phi|)$-many points, for $P(\cdot)$ a polynomial of degree $b(L) + 1$.

Case 3: [$\mathcal{F} \in \mathbf{F}_1$ and $d_1(\mathcal{F})$ is unbounded]. This case is symmetric to Case 2. This finishes the proof of the theorem. □

The next corollary is a joint result with M. Marx [17, Corollary 9].

8.2.5. COROLLARY. *Every proper normal extension of* $\mathbf{S5}^2$ *has the poly-size model property.*

Proof. Let L be a proper normal extension of $\mathbf{S5}^2$ and ϕ an L-consistent formula. Then $\neg\phi \notin L$ and by Corollary 6.2.12, there is a finite L-frame $\mathcal{F}$ refuting $\neg\phi$. Thus, $\mathcal{F}$ satisfies ϕ, and by Theorem 8.2.1, there exists an L-frame $\mathcal{F}'$ which satisfies ϕ and whose universe is bounded by a polynomial of degree $b(L)+1$ in $|\phi|$. Therefore, L has the poly-size model property. $\Box$

8.3 Logics without the linear-size model property

In the previous section we showed that all proper normal extensions of $\mathbf{S5}^2$ have the poly-size model property. In this section we show that our bound is indeed optimal by constructing proper normal extensions L_k of $\mathbf{S5}^2$ and formulas ϕ_k^n such that the size of the smallest L_k-frame satisfying ϕ_k^n is a polynomial of degree $b(L_k)+1$ in $|\phi_k^n|$. (Of course, the logics L_k will have an infinite parameter, namely $p_2^2(L_k)$ will be ω.)

Let a finite $\mathbf{S5}^2$-frame $\mathcal{F}$ be given and let $\{C_i\}_{i=1}^n$ and $\{C^j\}_{j=1}^m$ be the sets of E_1 and E_2-clusters of $\mathcal{F}$, respectively. Recall from the previous section that two distinct E_2-clusters C^p and C^q are *equivalent* if

$$|C_i \cap C^p| = |C_i \cap C^q| \text{ for all } i \text{ between 1 and } n.$$

Fix any natural number $k \geq 2$. For any natural number n, let $\mathcal{G}_k^n$ be an $\mathbf{S5}^2$-frame of E_1-depth k such that every E_2-cluster of $\mathcal{G}_k^n$ contains exactly $k+n$ points and no two distinct E_2-clusters of $\mathcal{G}_k^n$ are equivalent to each other. Note that $\mathcal{G}_k^n$ is not unique, since there are several (though finitely many) frames with this property. Let $\mathcal{F}_k^n$ be the maximal one with this property, that is $|\mathcal{G}_k^n| \leq |\mathcal{F}_k^n|$, for any $\mathcal{G}_k^n$. The cases for $k=2$ and $k=3$ are shown in Figure 8.1.

Let $L_k = \bigcap_{n\in\omega} Log(\mathcal{F}_k^n)$, where $Log(\mathcal{F}_k^n)$ is the logic of the frame $\mathcal{F}_k^n$ for $n \in \omega$. Obviously, $p_2^2(L_k) = \omega$ and $b(L_k) = k$.
Now for $n > k$, let $\phi_k^n = Q_k \wedge \psi^n$, where

$$\begin{aligned} Q_k &= \textstyle\bigwedge_{i=1}^k \Diamond_1\Diamond_2 p_i \wedge \Box_1\Box_2[\bigwedge_{i=1}^k(\Diamond_1 p_i \leftrightarrow p_i) \wedge \bigwedge_{1\leq i\neq j\leq k} \neg(p_i \wedge p_j)], \\ \psi^n &= \Box_1[\textstyle\bigwedge_{i=1}^n \Diamond_2 q_i \wedge \Box_2(\bigwedge_{1\leq i\neq j\leq n} \neg(q_i \wedge q_j))]. \end{aligned}$$

It is not difficult to show that

$$Q_k \text{ is satisfiable in } \mathcal{F} \text{ iff } \mathcal{F} \text{ contains at least } k\text{-many } E_1\text{-clusters} \tag{8.1}$$
$$\psi^n \text{ is satisfiable in } \mathcal{F} \text{ iff all } E_2\text{-clusters of } \mathcal{F} \text{ contain at least } n \text{ points} \tag{8.2}$$

Thus, the formula ϕ_k^n is satisfiable in the frame $\mathcal{F}_k^{n-k}$. The next claim states that in the logic L_k we cannot do better.

$$\mathcal{F}_k^{n-k} \text{ is the smallest } L_k\text{-frame satisfying } \phi_k^n. \tag{8.3}$$

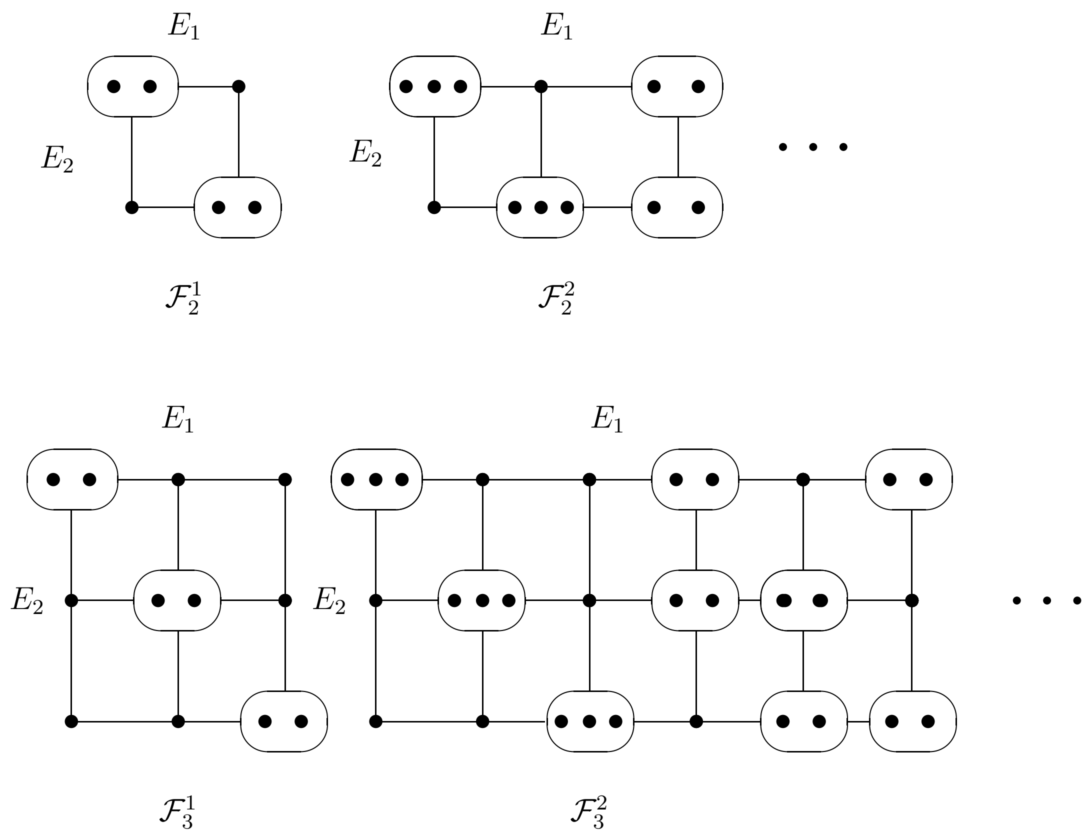

Figure 8.1: $\mathcal{F}_k^n$ frames for $k = 2$ and $k = 3$.

In order to prove (8.3), suppose ϕ_k^n is satisfiable in a finite L_k-frame $\mathcal{F}$. Then $\mathcal{F}$ is a p-morphic image of some $\mathcal{F}_k^i$, $i \in \omega$; that is, there is an onto p-morphism $f : \mathcal{F}_k^i \to \mathcal{F}$. As ψ^n is satisfied in $\mathcal{F}$, by (8.2), $i \geq n-k$.

Let $i = n-k$. The argument when $i > n-k$ is similar. Since Q_k is satisfiable in $\mathcal{F}$, (8.1) implies that $\mathcal{F}$ contains k-many E_1-clusters. Thus, f cannot identify points from different E_1-clusters of $\mathcal{F}_k^{n-k}$. Also note that since ψ^n is satisfiable in $\mathcal{F}$ and every E_2-cluster of $\mathcal{F}_k^{n-k}$ contains n points, f cannot identify points from the same E_2-cluster. Let us show that f cannot identify points from different E_2-clusters either. To see this, suppose there exist $w \in C_i^p$ and $v \in C_i^q$ such that $f(w) = f(v)$. Since f is a p-morphism, for any $j = 1, \ldots, k$ and $w' \in C_j^p$ there exists $v' \in C_j^q$ such that $f(w') = f(v')$. Now since C^p is not equivalent to C^q, at least two points from some C_j^p will be identified by f. Hence the number of points of the E_2-cluster $f(C^p)$ of $\mathcal{F}$ is strictly less than n, which again contradicts the satisfiability of ψ^n in $\mathcal{F}$. Therefore, f should be the identity map, and so $\mathcal{F} = \mathcal{F}_k^{n-k}$.

Now we compute the size of $\mathcal{F}_k^{n-k}$. As in Theorem 8.2.1, to every E_2-cluster C^p of $\mathcal{F}_k^{n-k}$ we correspond the sequence of natural numbers $(m_1, \ldots, m_k)$, where $m_1 = |C_1^p|, \ldots, m_k = |C_k^p|$. From the definition of $\mathcal{F}_k^{n-k}$ it follows that $m_1 + \ldots + m_k = n$. But then the number of different sequences $(m_1, \ldots, m_k)$ will be

$$\binom{n-1}{k} = \frac{(n-1)!}{k!(n-(k+1))!} = \frac{(n-1)\ldots(n-k)}{k!} \geq \frac{(n-k)^k}{k!}.$$

Furthermore, every E_2-cluster of $\mathcal{F}_k^{n-k}$ contains precisely n points. So the size of $\mathcal{F}_k^{n-k}$ is at least $\frac{n(n-k)^k}{k!}$, hence

$$\text{The size of } \mathcal{F}_k^{n-k} \text{ is a polynomial of degree } k+1 \text{ in } n. \tag{8.4}$$

Putting (8.3) and (8.4) together we obtain the following theorem, which is a joint result with M. Marx, see [17, Theorem 10].

8.3.1. THEOREM. *There exist infinitely many proper normal extensions L_k of $\mathbf{S5}^2$ and formulas ϕ_k^n such that the size of the smallest L_k-frame satisfying ϕ_k^n is a polynomial of degree $b(L_k)+1$ in $|\phi_k^n|$.*

8.4 NP-completeness

Note that Theorem 8.1.15 and the fact that every normal extension L of $\mathbf{S5}^2$ is complete with respect to the class of finite frames $\mathbf{F}_L$, for which the membership is decidable (up to isomorphism), imply that L is decidable. This section will be devoted to showing that if L is a proper normal extension of $\mathbf{S5}^2$, then the satisfiability problem for L is NP-complete. Fix such an L. We will see in Corollary 8.4.3 below that NP-completeness follows from the poly-size model property if we can decide in time polynomial in $|W|$ whether a finite structure $\mathcal{S} = (W, R_1, R_2)$ is in $\mathbf{F}_L$ (up to isomorphism). It suffices to decide in polynomial time (1) whether $\mathcal{S}$ is a (rooted $\mathbf{S5}^2$-) frame; (2) whether a given frame is in $\mathbf{F}_L$. The first is easy. We concentrate on the second.

By Lemma 8.1.3(1), there is $n(L) \in \omega$ such that for each frame $\mathcal{G} = (U, S_1, S_2)$ in $\mathbf{F}_L$ we have $d_1(\mathcal{G}) < n(L)$ or $d_2(\mathcal{G}) < n(L)$. So, if both depths of a given frame $\mathcal{G}$ are greater than or equal to $n(L)$ (which obviously can be checked in polynomial time in the size of $\mathcal{G}$), then $\mathcal{G} \notin \mathbf{F}_L$. So, without loss of generality we may assume that $d_1(\mathcal{G}) < n(L)$.

By Theorem 8.1.1, $\mathcal{G}$ is in $\mathbf{F}_L$ iff it has no p-morphic image in $\mathbf{M}_L$. Because $\mathbf{M}_L$ is a fixed finite set, it suffices to provide, for an arbitrary fixed frame $\mathcal{F} = (W, E_1, E_2)$, an algorithm that decides in time polynomial in the size of $\mathcal{G}$ whether there is a p-morphism from $\mathcal{G}$ onto $\mathcal{F}$. If we considered every map $f : U \to W$ and checked whether it is a p-morphism, it would take exponential time in the size of $\mathcal{G}$ (since there are $|W|^{|U|}$ different maps from U to W). Now we will give a different algorithm to check in polynomial time in $|U|$ whether the fixed frame $\mathcal{F}$ is a p-morphic image of a given frame $\mathcal{G} = (U, S_1, S_2)$ with $d_1(\mathcal{G}) < n(L)$. We show that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$ iff there exists a partial map g from $\mathcal{G}$ to $\mathcal{F}$ satisfying conditions that can be checked in polynomial time.

8.4.1. LEMMA. *$\mathcal{F}$ is a p-morphic image of $\mathcal{G}$ iff there is a partial surjective map $g : U \to W$ with the following properties:*

1. *For each $u \in U$, there is $v \in \mathrm{dom}(g)$ such that uS_1v.*
2. *For each $v \in \mathrm{dom}(g)$, the restriction $g \restriction (\mathrm{dom}(g) \cap S_1(v))$ is one-one and has range $E_1(g(v))$.*
3. *For each $u \in U$ there is $w \in W$ such that*
   (a) *$g(v)E_2w$ for all $v \in \mathrm{dom}(g) \cap S_2(u)$,*
   (b) *for each E_0-cluster $Y \subseteq E_2(w)$,*

$$\text{if } X_Y = S_1(g^{-1}(Y)) \cap S_2(u), \text{ then } |Y \setminus g(X_Y)| \leq |X_Y \setminus \mathrm{dom}(g)|.$$

Proof. It is easy to see that a map $f : U \to W$ is a p-morphism iff the f-image of every S_i-cluster of $\mathcal{G}$ is an E_i-cluster of $\mathcal{F}$, for $i = 1, 2$.

Suppose there is a surjective p-morphism $f : U \to W$. Then for each S_1-cluster $C \subseteq U$, the map $f \restriction C$ is a surjection from C onto $E_1(f(u))$ for any $u \in C$, so we may choose $C' \subseteq C$ such that $f \restriction C'$ is a bijection from C' onto $E_1(f(u))$. Let $U' = \bigcup\{C' : C$ is an S_1-cluster of $\mathcal{G}\}$. Then it is easy to check that $g = f \restriction U'$ satisfies Conditions 1–2 of the lemma. To check Condition 3, take any $u \in U$, and put $w = f(u)$. Fix any E_0-cluster $Y \subseteq E_2(w)$. Pick any $x \in S_2(u)$. Note that $f(x) \in E_2(w)$. Define X_Y as in the lemma. Then $x \in X_Y$

iff $x \in S_1(g^{-1}(Y))$, iff there is $z \in U'$ such that xS_1z and $g(z) \in Y$, iff $f(x)E_1f(z)$ and $f(z) \in Y$, iff $f(x) \in Y$. Now f maps $S_2(u)$ onto $E_2(w)$. Therefore, f maps X_Y onto Y. Thus, f must map a subset of $X_Y \setminus U'$ onto $Y \setminus g(X_Y \cap U')$, so we have $|X_Y \setminus U'| \geq |Y \setminus g(X_Y \cap U')|$ as required.

Conversely, let g be as stated. By Condition 2 of the lemma, g is surjective. We will extend g to a p-morphism $f : U \to W$. Since U is a disjoint union of S_2-clusters, it is enough to define f on an arbitrary S_2-cluster of $\mathcal{G}$. Pick $u \in U$. We will extend $g \restriction S_2(u)$ to the whole of $S_2(u)$. Pick $w \in W$ according to Condition 3 of the lemma. By Condition 3a, $g(S_2(u)) \subseteq E_2(w)$. Now we extend g to f such that $f(S_2(u)) = E_2(w)$ and $f(x)E_1g(v)$ whenever $v \in \mathrm{dom}(g)$ and $x \in S_2(u) \cap S_1(v)$.

For each E_0-cluster $Y \in E_2(w)$, define X_Y as in the lemma. By Conditions 1 and 2, $S_2(u) = \bigcup\{X_Y : E_0(Y) = Y \text{ and } Y \subseteq E_2(w)\}$, and $X_Y \cap X_{Y'} = \emptyset$ whenever $E_0(Y) = Y$, $E_0(Y') = Y'$ and $Y \cap Y' = \emptyset$. For each E_0-cluster $Y \subseteq E_2(w)$, we consider the restriction of g to X_Y (this restriction may be empty), observe that its image is a subset of Y. We extend $g \restriction X_Y$ to a surjection from X_Y onto Y. By Condition 3, $|X_Y \setminus \mathrm{dom}(g)| \geq |Y \setminus g(X_Y)|$. So, there exists a surjection $f_{X_Y} : X_Y \to Y$ extending g. Repeating this for every $Y \subseteq E_2(w)$ in turn yields an extension of g to $S_2(u)$. Repeating for a representative u of each S_2-cluster in turn yields an extension of g to U as required.

It is left to show that f is a p-morphism. But it follows immediately from the construction of f that $f \restriction S_i(u) : S_i(u) \to E_i(f(u))$ is surjective for each $u \in U$ and $i = 1, 2$. As we pointed out above, this implies that f is a p-morphism.

$\square$

8.4.2. Corollary. *It is decidable in polynomial time in the size of $\mathcal{G}$ whether $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

Proof. By Lemma 8.4.1, it is enough to check whether there exists a partial map $g : U \to W$ satisfying Conditions 1–3 of the lemma. There are at most $n(L)$ S_1-clusters in $\mathcal{G}$, and the restriction of g to each S_1-cluster is one-one; hence, $d = |\mathrm{dom}(g)| \leq n(L) \cdot |W|$, and this is independent of $\mathcal{G}$. There are at most $d^{|W|}$ maps from a set of size at most d into W. Obviously, there are $\binom{|U|}{d} \leq |U|^d$ subsets of U of size d. Hence there are at most $d^{|W|}|U|^d$ partial maps which may satisfy Conditions 1 and 2 of the lemma. Our algorithm enumerates all partial maps from U to W with domain of size at most d, and for each one, checks whether it satisfies Conditions 1–3 or not. It is not hard to see that this check can be done in P-time; indeed, it is clear that Conditions 1 and 2 can be checked in time polynomial in $|U|$ and there is a first-order sentence $\sigma_{\mathcal{F}}$ such that $\mathcal{G} \models \sigma_{\mathcal{F}}$ iff $\mathcal{G}$ satisfies Condition 3. The algorithm states that $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$ if and only if it finds a map satisfying the conditions. Therefore, this is a P-time algorithm checking whether $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$. $\square$

The next corollary is a joint result with I. Hodkinson, see [16, Corollary 4.3].

8.4.3. Corollary. *Let L be a proper normal extension of* $\mathbf{S5}^2$.

1. *It can be checked in polynomial time in $|U|$ whether a finite $\mathbf{S5}^2$-frame $\mathcal{G} = (U, S_1, S_2)$ is an L-frame.*
2. *The satisfiability problem for L is NP-complete.*
3. *The validity problem for L is co-NP-complete.*

Proof.

1. Follows directly from Theorem 8.1.1, Corollary 8.4.2, and the fact (shown in the proof of Theorem 8.1.15) that $\mathbf{M}_L$ is finite.
2. It is a well-known result of modal logic (see, e.g., [18, Lemma 6.35]) that if L is a consistent normal modal logic having the poly-size model property, and the problem of whether a finite structure $\mathcal{A}$ is an L-frame is decidable in time polynomial in the size of $\mathcal{A}$, then the satisfiability problem of L is NP-complete. The poly-size model property of every $L \supsetneq \mathbf{S5}^2$ is proved in Corollary 8.2.5. (1) implies that the problem $\mathcal{G} \in \mathbf{F}_L$ can be decided in polynomial time in the size of $\mathcal{G}$. The result follows.
3. Follows directly from (2).

$\square$

Summary

In this thesis we study classes of intermediate and cylindric modal logics. Intermediate logics are the logics that contain the intuitionistic propositional calculus **IPC** and are contained in the classical propositional calculus **CPC**. Cylindric modal logics are finite variable fragments of the classical first-order logic **FOL**. They are also closely related to n-dimensional products of the well-known modal logic **S5**. In this thesis we investigate:

1. The lattice of extensions of the intermediate logic **RN** of the Rieger-Nishimura ladder.
2. Lattices of two-dimensional cylindric modal logics. In particular, we study:
   (a) The lattice of normal extensions of the two-dimensional cylindric modal logic $\mathbf{S5}^2$ (without the diagonal).
   (b) The lattice of normal extensions of the two-dimensional cylindric modal logic $\mathbf{CML}_2$ (with the diagonal).

Our methods are a mixture of algebraic, frame-theoretic and order-topological techniques. In Part I of the thesis we give an overview of Kripke, algebraic and general-frame semantics for intuitionistic logic and we study in detail the structure of finitely generated Heyting algebras and their dual descriptive frames. We also discuss what we call frame-based formulas. In particular, we look at the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas and we construct a unified framework for these formulas.

After that we investigate the logic **RN** of the Rieger-Nishimura ladder. The Rieger-Nishimura ladder is the dual frame of the one-generated free Heyting algebra described by Rieger [106] and Nishimura [102]. Its logic is the greatest 1-conservative extension of **IPC**. It was studied earlier by Kuznetsov and Gerciu [83], Gerciu [48] and Kracht [73]. We describe the finitely generated and finite

descriptive frames of **RN** and provide a systematic analysis of its extensions. We also study a slightly weaker intermediate logic **KG**, introduced by Kuznetsov and Gerciu. **KG** is closely related to **RN** and plays an important role in our investigations. While studying extensions of **KG** and **RN** we introduce some general techniques. For example, we give a systematic method for constructing intermediate logics without the finite model property, we give a method for constructing infinite antichains of finite Kripke frames that implies the existence of a continuum of logics with and without the finite model property. We also introduce a gluing technique for proving the finite model property for large classes of logics. In particular, we show that every extension of **RN** has the finite model property. Finally, we give a criterion of local tabularity in extensions of **RN** and **KG**.

In Part II of the thesis we investigate in detail lattices of two-dimensional cylindric modal logics. The lattice of extensions of one-dimensional cylindric modal logic, is very simple: it is an $(\omega+1)$-chain, Scroggs [111]. In contrast to this, the lattice of extensions of the three-dimensional cylindric modal logic is too complicated to describe. In this thesis we concentrate on two-dimensional cylindric modal logics. We consider two similarity types: two-dimensional cylindric modal logics with and without diagonal. Cylindric modal logic with the diagonal corresponds to the full two-variable fragment of **FOL** and the cylindric modal logic without the diagonal corresponds to the two-variable substitution-free fragment of **FOL**.

Cylindric modal logic without the diagonal is the two-dimensional product of **S5**, which we denote by $\mathbf{S5}^2$. It had been shown that the logic $\mathbf{S5}^2$ is finitely axiomatizable, has the finite model property, is decidable Henkin et al. [60], Segerberg [113], Scott [110] and has a NEXPTIME-complete satisfiability problem Marx [93]. We show that every proper normal extension of $\mathbf{S5}^2$ is also finitely axiomatizable, has the finite model property, and is decidable. Moreover, we prove that in contrast to $\mathbf{S5}^2$ itself, each of its proper normal extensions has an NP-complete satisfiability problem. We also show that the situation for cylindric modal logics with the diagonal is different. There are continuum many non-finitely axiomatizable extensions of the cylindric modal logic $\mathbf{CML}_2$. We leave it as an open problem whether all of them have the finite model property. Finally, we give a criterion of local tabularity for two-dimensional cylindric modal logics with and without diagonal and characterize the pre-tabular cylindric modal logics.

Samenvatting

In dit proefschrift bestuderen we klassen van intermediaire en cylindrische modale logica's. Intermediaire logica's zijn die logica's die de intuitionistische propositielogica **IPC** omvatten en bevat zijn in de klassieke propostielogica **CPC**. Cylindrische modale logica's zijn eindige-variabele fragmenten van de klassieke eersteorde logica **FOL**. Ze zijn ook sterk gerelateerd aan de n-dimensionale producten van de bekende modale logica **S5**. In dit proefschrift bestuderen we :

1. De tralie van uitbreidingen van de intermediaire logica **RN** van de Rieger-Nishimuraladder.
2. Tralies van twee-dimensionale cylindrische modale logica's. In het bijzonder bestuderen we:
   (a) De tralie van de normale uitbreidingen van de twee-dimensionale cylindrische modale logica $\mathbf{S5}^2$ (zonder de diagonaal).
   (b) De tralie van de normale uitbreidingen van de twee-dimensionale cylindrische modale logica $\mathbf{CML}^2$ (met de diagonaal).

Onze methoden zijn een mengsel van algebraische, orde-topologische en gegeneraliseerde-frametechnieken. In Deel I van het proefschrift geven we een overzicht van de algebraische, Kripke- and gegeneraliseerde-framesemantiek voor de intuitionistische logica en bestuderen we in detail de structuur van de eindig gegenereerde Heytingalgebra's en hun duale descriptieve frames. We bediscussieren ook wat we frame-gebaseerde formules zullen noemen. In het bijzonder bekijken we de Jankov-deJongh-formules, subframeformules en cofinale-subframeformules en construeren we een algemeen kader voor dergelijke formules.

Hierna onderzoeken we de logica **RN** van de Rieger-Nishimuraladder. De Rieger-Nishimuraladder is het duale frame van de vrije Heytingalgebra op 1 generator zoals beschreven door Rieger [106] en Nishimura [102]. De logica van dit tralie is de sterkste 1-conservatieve uitbreiding van **IPC**. **RN** is eerder bestudeerd door Kuznetsov en Gerciu [83], Gerciu [48] en Kracht [73]. We geven een systematische analyse van dit systeem en zijn uitbreidingen. We bestuderen ook een iets zwakkere intermediaire logica **KG**, geintroduceerd door Kuznetsov and Gerciu. **KG** is sterk gerelateerd aan **RN** en speelt een belangrijke rol in ons onderzoek. Bij het bestuderen van de uitbreidingen van **KG** en **RN** introduceren we enkele algemene technieken. Bijvoorbeeld geven we een systematische methode voor de constructie van intermediaire logica's zonder de eindige modeleigenschap, en verder een methode voor de constructie van oneindige antiketens van eindige Kripkeframes die het bestaan impliceert van een continuum van logica's met en zonder de eindige modeleigenschap. We introduceren ook een lijmtechniek voor het bewijzen van de eindige modeleigenschap voor grote klassen van logica's. In het bijzonder laten we zien dat iedere uitbreiding van **RN** de eindige modeleigenschap heeft. Tenslotte geven we een criterium voor locale tabulariteit in uitbreidingen van **RN** en **KG**.

In Deel II van het proefschrift onderzoeken we in detail tralies van de tweedimensionale cylindrische modale logica's. De tralie van de uitbreidingen van de één-dimensionale cylindrische modale logica is erg eenvoudig: het is een $(\omega+1)$-keten; [111]. Daarentegen is de tralie van uitbreidingen van de drie-dimensionale cylindrische modale logica te gecompliceerd om te beschrijven. In dit proefschrift concentreren we ons op twee-dimensionale cylindrische modale logica's. We beschouwen twee similariteitstypen: twee-dimensionale cylindrische modale logica's met en zonder diagonaal. Cylindrische modale logica met diagonaal correspondeert met het volledige twee-variabele fragment van **FOL** en de cylindrische modale logica zonder diagonaal correspondeert met het substitutievrije twee-variabele fragment van **FOL**.

Cylindrische modale logica zonder diagonaal is het twee-dimensionale product van **S5**, dat we aanduiden met $\mathbf{S5}^2$. Het was al bewezen dat de logica $\mathbf{S5}^2$ eindig axiomatiseerbaar is, de eindige modeleigenschap heeft, beslisbaar is Henkin et al. [60], Segerberg [113], Scott [110] en een NEXPTIME-volledig satisfactieprobleem heeft Marx [93]. We laten zien dat iedere echte normale uitbreiding van $\mathbf{S5}^2$ ook eindig axiomatiseerbaar is, de eindige modeleigenschap heeft en beslisbaar is. Bovendien bewijzen we dat, in tegenstelling tot $\mathbf{S5}^2$, iedere echte normale uitbreiding van $\mathbf{S5}^2$ een NP-volledig satisfactieprobleem heeft. We tonen tevens aan dat de situatie bij cylindrische modale logica's met diagonaal anders is. Er zijn continuum veel niet eindig axiomatiseerbare uitbreidingen van de cylindrische modale logica $\mathbf{CML}_2$. We laten het probleem open of al deze uitbreidingen de eindige modeleigenschap hebben. Tenslotte geven we een criterium voor locale tabulariteit van twee-dimensionale cylindrische modale logica's met en zonder diagonaal en karakteriseren we de pretabulaire cylindrische modale logica's.

Bibliography

[1] H. Andréka and I. Nemeti. Simple proof of decidability of the universal theory of cylindric set algebras of dimension 2. In *Algebraic Logic and the Methodology of Applying it*, TEMPUS Summer School. Budapest, Hungary, 1994.

[2] R. Balbes and P. Dwinger. *Distributive Lattices.* University of Missouri Press, 1974.

[3] H. Bass. Finite monadic algebras. *Proceedings of the American Mathematical Society*, 9:258–268, 1958.

[4] F. Bellissima. Finitely generated free Heyting algebras. *Journal of Symbolic Logic*, 51:152–165, 1986.

[5] J. van Benthem. Two simple incomplete modal logics. *Theoria*, 44:25–37, 1978.

[6] G. Bezhanishvili. Varieties of monadic Heyting algebras. Part II: Duality theory. *Studia Logica*, 62:21–48, 1999.

[7] G. Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46:531–548, 2001.

[8] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The logic of the Rieger-Nishimura ladder. *manuscript*, 2005.

[9] G. Bezhanishvili and S. Ghilardi. An algebraic approach to subframe logics. Part I. *manuscript*, 2005.

[10] G. Bezhanishvili and R. Grigolia. Subalgebras and homomorphic images of the Rieger-Nishimura lattice. In *Proceedings of the Institute of Cybernetics*, volume 1, pages 9–16. Georgian Academy of Sciences, Tbilisi, 2000.

[11] G. Bezhanishvili and R. Grigolia. Locally finite varieties of Heyting algebras. *Algebra Universalis*, 2005. to appear.

[12] N. Bezhanishvili. Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case. *Algebra Universalis*, 48:11–42, 2002.

[13] N. Bezhanishvili. De Jongh's characterization of intuitionistic propositional calculus. In J. van Benthem, A. Troelstra, F. Veltman, and A. Visser, editors, *Liber Amicorum Dick de Jongh*. University of Amsterdam, 2004.

[14] N. Bezhanishvili. Varieties of two-dimensional cylindric algebras II. *Algebra Universalis*, 51:177–206, 2004.

[15] N. Bezhanishvili and D. de Jongh. Intuitionistic logic. ESSLLI course notes, 2005.

[16] N. Bezhanishvili and I. Hodkinson. All normal extensions of S5-squared are finitely axiomatizable. *Studia Logica*, 78:443–457, 2004.

[17] N. Bezhanishvili and M. Marx. All proper normal extensions of S5-square have the polynomial size model property. *Studia Logica*, 73:367–382, 2003.

[18] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[19] W. Blok. *Varieties of interior algebras*. PhD thesis, University of Amsterdam, 1976.

[20] W. Blok. On the degree of incompleteness in modal logics and the covering relation on the lattice of modal logics. Technical Report 78-07, Department of Mathematics, University of Amsterdam, 1978.

[21] W. Blok. The lattice of modal logics: an algebraic investigation. *Journal of Symbolic Logic*, 45:221–236, 1980.

[22] R. Bull. That all normal extensions of S4.3 have the finite model property. *Zeitschrift für mathematische Logic und Grundlagen der Mathematik*, 12:341–344, 1966.

[23] R. Burris and H. Sankappanavar. *A Course in Universal Algebra*. Springer, 1981.

[24] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[25] C.C. Chang and H.J. Kiesler. *Model Theory*. North-Holland, Amsterdam, 1990. 3rd edition.

[26] A. Citkin. On admissible rules of intuitionistic propositional calculus. *Math. USSR Sbornik*, 31:279–288, 1977.

[27] S. Comer. Classes without the amalgamation property. *Pacific Journal of Mathematics*, 28:309–318, 1969.

[28] D. van Dalen. Intuitionistic Logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 3, pages 225–339. Kluwer, Reidel, Dordrecht, 1986.

[29] B. Davey. On the lattice of subvarieties. *Houston Journal of Mathematics*, 5:183–192, 1979.

[30] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[31] M. Dummett and E. Lemmon. Modal logics between S4 and S5. *Zeitschrift für mathematische Logic und Grundlagen der Mathematik*, 5:250–264, 1959.

[32] R. Engelking. *General Topology*. Heldermann Verlag, 1989.

[33] P. Erdős, V. Faber, and J. Larson. Sets of natural numbers of positive density and cylindric set algebras of dimension 2. *Algebra Universalis*, 12:81–92, 1981.

[34] L. Esakia. To the theory of modal and superintuitionistic systems. In *Proceedings of the USSR symposium on the theory of the logical inferences*, pages 147–172. Nauka, Moscow, 1979.

[35] L. Esakia. *Heyting Algebras I, Duality Theory*. Metsniereba Press, Tbilisi, 1985. (in Russian).

[36] L. Esakia. Gödel-Löb modal system – addendum. In *Proceedings of the third International Conference, Smirnov's Readings*, pages 77–79, Moscow, 2001. Russian Academy of Sciences.

[37] L. Esakia and R. Grigolia. The criterion of Brouwerian and closure algebras to be finitely generated. *Bull. of Sect. of Log.*, 6:46–52, 1977.

[38] L. L. Esakia. Topological Kripke models. *Soviet Mathematics Doklady*, 15:147–151, 1974.

[39] K. Fine. The logics containing S4.3. *Zeitschrift für mathematische Logic und Grundlagen der Mathematik*, 17:371–376, 1971.

[40] K. Fine. An incomplete logic containing S4. *Theoria*, 40:110–116, 1974.

[41] K. Fine. Logics containing K4, Part I. *Journal of Symbolic Logic*, 39:229–237, 1974.

[42] K. Fine. Logics containing K4, Part II. *Journal of Symbolic Logic*, 50:619–651, 1985.

[43] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications.* North-Holland Publishing Company, 2004.

[44] D.M. Gabbay and V.B. Shehtman. Products of modal logics, Part I. *Logic Journal of the IGPL*, 6:73–146, 1998.

[45] M. Gehrke, J. Harding, and Y. Venema. MacNeille completions and canonical extensions. *Transactions of the AMS*, 358(2):573–590, 2006.

[46] M. Gehrke and B. Jónsson. Bounded distributive lattices with operators. *Mathematica Japonica*, 40:207–215, 1994.

[47] M. Gehrke and B. Jónsson. Bounded distributive lattice expansions. *Mathematica Scandibavica*, 94:13–45, 2004.

[48] V. Gerciu. The finite approximability of superintuitionistic logics. *Mat. Issled.*, 7(1(23)):186–192, 1972. (Russian).

[49] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Annals of Pure and Applied Logic*, 86:1–32, 1997.

[50] K. Gödel. Zum Entscheidungsproblem des logischen Funktionenkalküls. Monatshefte für Mathematik und Physik, 40:433–443, 1933.

[51] R. Goldblatt. Metamathematics of modal logic, Part I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[52] R. Goldblatt. Metamathematics of modal logic, Part II. *Reports on Mathematical Logic*, 7:21–52, 1976.

[53] R. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 38:173–241, 1989.

[54] R. Goldblatt, I. Hodkinson, and Y. Venema. Erdős graphs resolve Fine's canonicity problem. *Bull. Symbolic Logic*, 10:186–208, 2004.

[55] E. Grädel, P. Kolaitis, and M. Vardi. On the decision problem for two–variable first order logic. *Bulletin of Symbolic Logic*, 3:53–69, 1997.

[56] G. Grätzer. *Universal Algebra.* Springer-Verlag, 1978. Second Edition.

[57] R. Grigolia. *Free Algebras of Non-Classical Logics.* "Metsniereba", Tbilisi, 1987. (Russian).

[58] P. Halmos. *Algebraic Logic.* Chelsea Publishing Company, 1962.

[59] A. Hendriks. *Computations in Propositional Logic.* PhD thesis, ILLC, University of Amsterdam, 1996.

[60] L. Henkin, D. Monk, and A. Tarski. *Cylindric Algebras. Parts I & II.* North-Holland, 1971 & 1985.

[61] A. Heyting. Die formalen Regeln der intuitionistischen Logik. *Sitzungberichte der preussischen Akademie der Wissenschaften*, 31:42–56, 1930.

[62] A. Heyting. *Intuitionism, an Introduction.* North-Holland, Amsterdam, 1956. 3rd rev. ed. (1971).

[63] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.*, 2:326–336, 1952.

[64] V.A. Jankov. The relationship between deducibility in the intuitionistic propositional calculus and finite implicational structures. *Soviet Mathematics Doklady*, 4:1203–1204, 1963.

[65] V.A. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Mathematics Doklady*, 9:806–807, 1968.

[66] S. Jaśkowski. Recherches sur lesystème de lalogique intuitioniste. In *Actes Du Congrès Intern. De Phil. Scientifique. VI. Phil. Des Mathèmatiques, Act. Sc. Et Ind 393*, pages 58–61. Paris, 1936.

[67] J. Johnson. Nonfinitizability of classes of representable polyadic algebras. *Journal of Symbolic Logic*, 34:344–352, 1969.

[68] P. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[69] D. de Jongh. *Investigations on the Intuitionistic Propositional Calculus.* PhD thesis, University of Wisconsin, 1968.

[70] D. de Jongh and A. Troelstra. On the connection of partially ordered sets with some pseudo-Boolean algebras. *Indagationes Mathematicae*, 28:317–329, 1966.

[71] B. Jónsson and A. Tarski. Boolean algebras with operators, Part I. *American Journal of Mathematics*, 73:891–939, 1951.

[72] J. Kagan and R. Quackenbush. Monadic algebras. *Reports on Mathematical Logic*, 7:53–62, 1976.

[73] M. Kracht. Prefinitely axiomatizable modal and intermediate logics. *Mathematical Logic Quarterly*, 39:301–322, 1993.

[74] M. Kracht. Splittings and the finite model property. *Journal of Symbolic Logic*, 58:139–157, 1993.

[75] M. Kracht. *Tools and Techniques in Modal Logic*. North-Holland, 1999.

[76] S.A. Kripke. Semantical analysis of modal logic, Part I. *Zeitschrift für mathematische Logic und Grundlagen der Mathematik*, 9:67–96, 1963.

[77] S.A. Kripke. Semantical considerations on modal logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

[78] S.A. Kripke. Semantical analysis of intuitionistic logic. I. In *Formal Systems and Recursive Functions, Proceeedings of the 8th Logic Colloquium*, pages 92–130. North-Holland, 1965.

[79] A. Kurucz. $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ lacks the finite model property. In F. Wolter, H. Wansing, M. de Rijke, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 3 of *CSLI Lecture Notes*, pages 321–328. CSLI Publications, Stanford, 2002.

[80] A. Kuznetsov. On finitely generated pseudo-Boolean algebras and finitely approximable varieties. In *Proceedings of the 12nd USSR Algebraic Colloquium*, page 281, Sverdlovsk, 1973. in Russian.

[81] A. Kuznetsov. On superintuitionistic logics. In *Proceedings of the International Congress of Mathematicians*, volume 1, pages 243–249, Vancouver, 1974.

[82] A. Kuznetsov. Some classification problems for superintuitionistic logics. In *Proceedings of the 3rd USSR Conference in Mathematical Logic*, pages 119–122, Novosibirsk, 1974. in Russian.

[83] A. Kuznetsov and V. Gerciu. Superintuitionistic logics and finite approximability. *Soviet Mathematics Doklady*, 11(6):1614–1619, 1970.

[84] R. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM journal of computing*, 6(3):467–480, 1977.

[85] R. Laver. Better-quasi-orderings and a class of trees. In Gian-Carlo Rota, editor, *Studies in Foundations and Combinatorics*, volume 1 of *Advances in Mathematics Supplementary Studies*, pages 31–48. Academic Press, 1978.

[86] C.I. Lewis. *A Survey in Symbolic Logic*. University of California Press, Berkeley, 1918.

[87] S. MacLane. *Category Theory for the Working Mathematician*. Springer, Berlin, 1971.

[88] R. Maddux. The equational theory of CA_3 is undecidable. *Journal of Symbolic Logic*, 45:311–317, 1980.

[89] L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras. *Algebra and Logic*, 16:427–455, 1977.

[90] L. Maksimova. Interpolation properties of superintuitionistic logics. *Studia Logica*, 38:419–428, 1979.

[91] L. Maksimova. Interpolation theorems in modal logic and amalgamable varieties of topological Boolean algebras. *Algebra and Logic*, 18:348–370, 1979.

[92] M. Marx. Mosaics and cylindric modal logic of dimension two. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, number 87 in CSLI Lecture Notes, pages 141–157. CSLI Publications, Stanford, 1997.

[93] M. Marx. Complexity of products of modal logics. *Journal of Logic and Computation*, 92:221–238, 1999.

[94] M. Marx and Sz. Mikulás. Decidability of cylindric set algebras of dimension two and first-order logic with two variables. *Journal of Symbolic Logic*, 64:1563–1572, 1999.

[95] M. Marx and Y. Venema. *Multi-dimensional Modal Logic*. Applied Logic Series. Kluwer Academic Publisher, 1997.

[96] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[97] J.C.C. McKinsey and A. Tarski. On closed elements of closure algebras. *Annals of Mathematics*, 47:122–162, 1946.

[98] J.C.C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, pages 1–15, 1948.

[99] D. Monk. Nonfinitizability of classes of representable cylindric algebras. *Journal of Symbolic Logic*, 34:331–343, 1969.

[100] D. Monk. On equational classes of algebraic versions of logic I. *Mathematica Scandinavica*, 27:53–71, 1970.

[101] M. Mortimer. On languages with two variables. *Zeitchrift für mathematische Logik und Grundlagen der Mathematik*, 21:135–140, 1975.

[102] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *Journal of Symbolic Logic*, 25:327–331, 1960.

[103] H. Priestley. Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society*, 24:507–530, 1972.

[104] R. Rado. Partial well ordering of sets of vectors. *Mathematica*, 1:89–95, 1954.

[105] W. Rautenberg. Splitting lattices of logics. *Archiv für Mathematische Logik*, 20:155–159, 1980.

[106] L. Rieger. On the lattice theory of Brouwerian propositional logic. *Acta fac. rerum nat. Univ. Car.*, 189:1–40, 1949.

[107] V. V. Rybakov. *Admissibility of Logical Inference Rules*. Elsevier, 1997.

[108] V.V. Rybakov. Rules of inference with parameters for intuitionistic logic. *Journal of Symbolic Logic*, 57:33–52, 1992.

[109] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In S. Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 110–143, Amsterdam, 1975. North-Holland.

[110] D. Scott. A decision method for validity of sentences in two variables. *Journal of Symbolic Logic*, 27:477, 1962.

[111] S. G. Scroggs. Extensions of the Lewis system S5. *Journal of Symbolic Logic*, 16:111–120, 1951.

[112] K. Segerberg. *An essay in classical modal logic*, volume 13 of *Philosophical Studies*. Uppsala, 1971.

[113] S. Segerberg. Two-dimensional modal logic. *Journal of Philosophical logic*, 2:77–96, 1973.

[114] V. Shehtman. On incomplete propositional logic. *Soviet Mathematics Doklady*, 18:985–989, 1977.

[115] V. Shehtman. Two-dimensional modal logics. *Mathematical Notes*, 5:759–772, 1978. (in Russian).

[116] V.B. Shehtman. Rieger-Nishimura lattices. *Soviet Mathematics Doklady*, 19:1014–1018, 1978.

[117] S.K. Sobolev. On the finite approximability of superintuitionistic logics. *Mathematics of the USSR, Sbornik*, 31:257–268, 1977.

[118] E. Spaan. *Complexity of Modal Logics*. PhD thesis, ILLC, University of Amsterdam, 1993.

[119] A Tarski. Der Aussagenkalkül und die Topologie. *Fund. Math.*, 31:103–134, 1938.

[120] S. K. Thomason. An incompleteness theorem in modal logic. *Theoria*, 40:30–34, 1974.

[121] E. Tomaszewski. *On Sufficiently Rich Sets of Formulas*. PhD thesis, Institute of Philosophy, Jagiellonian University, Kraków, 2003.

[122] A. Troelstra. On intermediate propositional logics. *Indagationes Mathematicae*, 27:141–152, 1965.

[123] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics, an Introduction*. North-Holland, Amsterdam, 1988. two volumes.

[124] Y. Venema. *Many-Dimensional Modal Logic*. PhD thesis, ILLC, University of Amsterdam, 1992.

[125] Y. Venema. Cylindric modal logic. *Journal of Symbolic Logic*, 60:591–663, 1995.

[126] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. Elsevier, 2006. To appear.

[127] A. Visser, D. de Jongh, J. van Benthem, and G. Renardel de Lavalette. NNIL a study in intuitionistic logic. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal logics and Process Algebra: a bisimulation perspective*, pages 289–326, 1995.

[128] M. Wajsberg. Ein erweiterter Klassenkalkül. *Monatshefte für Mathematik und Physik*, 40:113–126, 1933.

[129] F. Wolter. The finite model property of tense logics. *Journal of Symbolic Logic*, 60:757–774, 1995.

[130] F. Wolter. The structure of lattices of subframe logics. *Annals of Pure and Applied Logic*, 86:545–551, 1997.

[131] F. Wolter and M. Zakharyaschev. Modal decision problems. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. Elsevier, 2006. To appear.

[132] M. Zakharyaschev. Syntax and semantics of modal logics containing S4. *Algebra and Logic*, 27:408–428, 1988.

[133] M. Zakharyaschev. Syntax and semantics of intermediate logics. *Algebra and Logic*, 28:262–282, 1989.

[134] M. Zakharyaschev. Canonical formulas for K4. Part I : Basic results. *Journal of Symbolic Logic*, 57:1377–1402, 1992.

[135] M. Zakharyaschev. Canonical formulas for K4. Part II : Cofinal subframe logics. *Journal of Symbolic Logic*, 61:421–449, 1996.

Index

Titles in the ILLC Dissertation Series:

ILLC DS-2001-01: **Maria Aloni**
Quantification under Conceptual Covers

ILLC DS-2001-02: **Alexander van den Bosch**
Rationality in Discovery - a study of Logic, Cognition, Computation and Neuropharmacology

ILLC DS-2001-03: **Erik de Haas**
Logics For OO Information Systems: a Semantic Study of Object Orientation from a Categorial Substructural Perspective

ILLC DS-2001-04: **Rosalie Iemhoff**
Provability Logic and Admissible Rules

ILLC DS-2001-05: **Eva Hoogland**
Definability and Interpolation: Model-theoretic investigations

ILLC DS-2001-06: **Ronald de Wolf**
Quantum Computing and Communication Complexity

ILLC DS-2001-07: **Katsumi Sasaki**
Logics and Provability

ILLC DS-2001-08: **Allard Tamminga**
Belief Dynamics. (Epistemo)logical Investigations

ILLC DS-2001-09: **Gwen Kerdiles**
Saying It with Pictures: a Logical Landscape of Conceptual Graphs

ILLC DS-2001-10: **Marc Pauly**
Logic for Social Software

ILLC DS-2002-01: **Nikos Massios**
Decision-Theoretic Robotic Surveillance

ILLC DS-2002-02: **Marco Aiello**
Spatial Reasoning: Theory and Practice

ILLC DS-2002-03: **Yuri Engelhardt**
The Language of Graphics

ILLC DS-2002-04: **Willem Klaas van Dam**
On Quantum Computation Theory

ILLC DS-2002-05: **Rosella Gennari**
Mapping Inferences: Constraint Propagation and Diamond Satisfaction

ILLC DS-2002-06: **Ivar Vermeulen**
A Logical Approach to Competition in Industries

ILLC DS-2003-01: **Barteld Kooi**
Knowledge, chance, and change

ILLC DS-2003-02: **Elisabeth Catherine Brouwer**
Imagining Metaphors: Cognitive Representation in Interpretation and Understanding

ILLC DS-2003-03: **Juan Heguiabehere**
Building Logic Toolboxes

ILLC DS-2003-04: **Christof Monz**
From Document Retrieval to Question Answering

ILLC DS-2004-01: **Hein Philipp Röhrig**
Quantum Query Complexity and Distributed Computing

ILLC DS-2004-02: **Sebastian Brand**
Rule-based Constraint Propagation: Theory and Applications

ILLC DS-2004-03: **Boudewijn de Bruin**
Explaining Games. On the Logic of Game Theoretic Explanations

ILLC DS-2005-01: **Balder David ten Cate**
Model theory for extended modal languages

ILLC DS-2005-02: **Willem-Jan van Hoeve**
Operations Research Techniques in Constraint Programming

ILLC DS-2005-03: **Rosja Mastop**
What can you do? Imperative mood in Semantic Theory

ILLC DS-2005-04: **Anna Pilatova**
A User's Guide to Proper names: Their Pragmatics and Semanics

ILLC DS-2006-01: **Troy Lee**
Kolmogorov complexity and formula size lower bounds

ILLC DS-2006-02: **Nick Bezhanishvili**
Lattices of intermediate and cylindric modal logics

ILLC DS-2006-03: **Clemens Kupke**
Finitary coalgebraic logics