Mathematics and Financial Economics (2022) 16:447–480
https://doi.org/10.1007/s11579-022-00313-9

Law-Invariant Functionals that Collapse to the Mean: Beyond Convexity

Felix-Benedikt Liebrich[1] · **Cosimo Munari**[2]

Received: 15 July 2021 / Accepted: 8 February 2022 / Published online: 28 March 2022

Abstract

We establish general "collapse to the mean" principles that provide conditions under which a law-invariant functional reduces to an expectation. In the convex setting, we retrieve and sharpen known results from the literature. However, our results also apply beyond the convex setting. We illustrate this by providing a complete account of the "collapse to the mean" for quasiconvex functionals. In the special cases of consistent risk measures and Choquet integrals, we can even dispense with quasiconvexity. In addition, we relate the "collapse to the mean" to the study of solutions of a broad class of optimisation problems with law-invariant objectives that appear in mathematical finance, insurance, and economics. We show that the corresponding quantile formulations studied in the literature are sometimes illegitimate and require further analysis.

Keywords Law invariance · Quasiconvex functionals · Consistent risk measures · Nonconvex Choquet integrals · Optimisation problems

JEL Classification C61 · D81

1 Introduction

The expression "collapse to the mean" refers to a variety of results about law-invariant functionals defined on spaces of random variables. Their red thread is the fundamental tension existing between law invariance and "linearity" properties. In mathematical finance, insurance, or economics, a random variable typically models the future unknown value of a financial or economic variable of interest, e.g., the payoff of an asset; the return on a portfolio of assets; the capital level of a financial institution; the net worth of an agent. The functional

✉ Felix-Benedikt Liebrich
felix.liebrich@insurance.uni-hannover.de

Cosimo Munari
cosimo.munari@bf.uzh.ch

1 Institute of Actuarial and Financial Mathematics & House of Insurance, Leibniz University Hannover, Hannover, Germany

2 Center for Finance and Insurance and Swiss Finance Institute, University of Zurich, Zurich, Switzerland

under consideration models the “value” of said variable, e.g., a price; a risk measure; a capital requirement; or a preference index. In this context, the assumption of law invariance posits that “value” is only sensitive to the distribution of the underlying variables with respect to a reference probability measure. This has an important practical implication in that it allows to compute the “value” of a random variable by means of statistical estimation. Law invariance has been thoroughly studied in insurance pricing (see, e.g., [9, 10, 64–66]), risk management (see, e.g., [7, 28, 29, 36, 38, 39, 42, 44, 60, 62]), and decision theory (sometimes under the name of symmetry or probabilistic sophistication; see, e.g., [2, 3, 26, 37, 46, 48, 49, 56, 63]). The assumption of “linearity” covers a spectrum of local linearity properties such as affinity along a one-dimensional space or translation invariance and captures the presence of a frictionless determinant of “value”, e.g., a riskless investment opportunity; a liquidly traded asset without transaction costs; a desirable prospect; an unambiguous event. Their formal description is postponed to Sect. 3. As the term suggests, the “collapse to the mean” is concerned with properties under which the only functionals that are simultaneously law invariant and “linear” in this weak sense are expectations or, more generally, functions of the expectation with respect to the reference probability measure. Apart from their intrinsic mathematical interest, these results are an important litmus test because functionals that are fully determined by expectation typically fail to capture “value” in an adequate risk-sensitive way. As a result, to avoid an inadequate representation of “value” one would be forced to choose between law invariance and other properties that are often desirable on their own merits.

To our knowledge, the earliest “collapse to the mean” is recorded in [16], which proves that the expectation is the only law-invariant Choquet integral defined on the space of bounded random variables that is convex and linear along a nonconstant random variable. In the setting of Choquet pricing, the result shows that the combination of law invariance, a common postulate in insurance pricing, and the existence of a frictionless risky traded asset is only compatible with frictionless markets where prices are determined by the expectation with respect to the physical probability measure and where, as a consequence, obvious arbitrage opportunities arise. In the setting of Choquet expected utility models, the result shows that a capacity that is submodular and law invariant with respect to the physical probability measure must coincide with it whenever it admits nontrivial unambiguous events. The collapse for Choquet integrals was later extended, again in a bounded setting, to convex cash-additive risk measures in [28]. A general picture for convex functionals beyond the bounded setting has recently been discussed in [8]. We also refer to [21] and [23] for versions of the collapse for linear maps satisfying weak semicontinuity properties and for conditionally convex maps, respectively. The “collapse to the mean” can also be reinterpreted from the recent perspective of [67]. There, it is shown that a law-invariant functional on bounded random variables is “risk neutral”, i.e., a function of the expectation, precisely when it is “dependence neutral”, i.e., the functional applied to a sum of random variables only depends on their marginal distributions. Notably, [67] does not impose convexity assumptions.

The goal of this paper is to present general formulations of the “collapse to the mean” that both extend the known results from the literature and can be applied, in the spirit of [67], beyond the world of convex functionals. This is important to capture situations where the presence of market frictions or other imperfections makes convexity too strong a property and calls to replace it with weaker properties, e.g., quasiconvexity. The general “collapse to the mean” principle is stated in Theorem 4.1, which in turn is derived from a sharp version of the Fréchet-Hoeffding bounds recorded in Lemma A.2. A complementary geometric version of the general principle is stated in Proposition 4.3. We illustrate the versatility of these tools in five case studies.

Collapse for convex functionals. In Sect. 5.1 we revisit the known "collapse to the mean" for convex functionals. We provide two versions under the assumption that the underlying functional is translation invariant along a nonconstant random variable, see Theorem 5.1 and Theorem 5.2. If the random variable has zero expectation, the functional collapses to a function of the expectation. Otherwise, it collapses to a specific function, namely an affine function, of the expectation. In addition, we provide new dual characterizations of the collapse in terms of weaker translation invariance properties and conjugate functions. This enriches the results in [8, 16, 28]. Compared to this literature, our proofs are new and more self contained.

Collapse for quasiconvex functionals. In Sect. 5.2 we take up the study of quasiconvex functionals. This is an important extension in view of the economic interpretation of quasiconvexity, which is a more elementary mathematical formulation of the diversification principle; see, e.g., [18, 24, 29, 30, 42, 52]. We extend both convex versions of the collapse, see Theorem 5.3 and Theorem 5.5, by means of the aforementioned sharp Fréchet-Hoeffding bounds. Moreover, we demonstrate sharpness of our results.

Collapse for consistent risk measures. In Sect. 5.3 we focus on cash-additive functionals that are monotonic with respect to second-order stochastic dominance. This class of risk measures is named "consistent" in [47] and contains the family of law-invariant convex risk measures, but also functionals that are neither convex nor quasiconvex. The literature on the connection between risk measures and stochastic dominance is rich; see, e.g., [6, 22, 40, 54, 55]. Monotonicity with respect to stochastic dominance is also investigated in decision theory; see, e.g., [2, 17, 26, 56]. The collapse for consistent risk measures in Theorem 5.7 is an exhaustive characterisation of the collapse for consistent risk measures and is again based on the sharp version of the Fréchet-Hoeffding bounds.

Collapse for Choquet integrals. In Sect. 5.4 we take one further step beyond convexity and consider Choquet integrals associated with a variety of different law-invariant capacities. In the case of submodular capacities, the Choquet integral is convex and a related collapse to the mean has been obtained in [16] and confirmed in [3]. Here, we go beyond submodular capacities and consider the case of coherent as well as Jaffray-Philippe capacities. The corresponding Choquet integrals are neither convex nor quasiconvex and play a natural role in decision theory under ambiguity; see, e.g., [19, 35]. In Theorem 5.11 we use the sharp Fréchet-Hoeffding bounds to derive a collapse result for this general class of Choquet integrals. In the spirit of [3, 48, 49], we highlight the economic implications by reformulating the collapse in terms of existence of unambiguous events. We also include a version of the collapse for α-maxmin expected utilities, which are related to Choquet capacities but cannot be expressed as Choquet expected utilities. The corresponding collapse is related to the results in [50]. Our strategy based on Fréchet-Hoeffding bounds allows us to dispense with the additional regularity condition imposed in that paper.

Collapse in optimisation problems. In Sect. 5.5 we focus on a general optimisation problem that encompasses a variety of important problems in economics, finance, and insurance, including the maximisation of expected investment returns or expected utility from terminal wealth (von Neumann-Morgenstern utility, rank-dependent utility, Yaari utility, S-shaped utility from prospect theory). More precisely, we study the maximisation of a general law-invariant objective subject to a general law-invariant constraint and a "budget" constraint expressed in terms of a "pricing density". A common intuition for such optimisation problems is that, *if* a solution exists, then all or some of these solutions have to be *antimonotone* with the pricing density. This allows to reduce the original problem to an optimisation problem involving quantile functions, which is substantially simpler and for which solution techniques are available; see, e.g., [12, 14, 34, 59, 60, 68, 69]. We provide a slight improvement over the existing results — see in particular [68] — by establishing more general sufficient conditions for the existence of antimonotone solutions. In particular, we highlight some conditions that are often omitted in the literature. In addition, we conduct a careful analysis showing that our result is sharp in the sense that, if any of the conditions is removed, the validity of the result forces the budget constraint to "collapse to the mean": The pricing density is necessarily constant, and the corresponding pricing rule reduces to the expectation with respect to the physical probability measure. This points to an issue in the literature, where the reduction to a quantile formulation is sometimes invoked even though some of the aforementioned conditions are not satisfied. In this situation, the reduction might be illegitimate unless extra analysis of the *specific* structure of the problem is carried over.

The paper is organised as follows. In Sect. 2 we describe the underlying setting and introduce the necessary notation. Section 3 provides guidelines for the interpretation of the "collapse to the mean" in the context of pricing theory, risk management, and decision theory. In Sect. 4 we state the general "collapse to the mean" principle and establish a useful geometric counterpart for convex sets. In Sect. 5 we provide a range of applications to convex and quasiconvex functionals, consistent risk measures, and Choquet integrals. In addition, we discuss a general optimisation problem involving law invariance, provide a result about optimal solutions, and show what can go wrong when passing to its quantile formulation. All mathematical details, proofs, and auxiliary results are relegated to Appendices A–G.

2 Setting and notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. A Borel measurable function $X : \Omega \to \mathbb{R}$ is called a random variable. By L^0 we denote the set of equivalence classes of random variables with respect to almost-sure equality under $\mathbb{P}$. As is customary, we do not explicitly distinguish between an element of L^0 and any of its representatives. In particular, the elements of $\mathbb{R}$ are naturally identified with random variables that are almost-surely constant under $\mathbb{P}$. For two random variables $X, Y \in L^0$ we write $X \sim Y$ whenever X and Y have the same law under $\mathbb{P}$, i.e., the probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ on the Borel sets of the real line agree. The expectation operator under $\mathbb{P}$ is denoted by $\mathbb{E}[\cdot]$, the conditional expectation with respect to a σ-field $\mathcal{G} \subset \mathcal{F}$ by $\mathbb{E}[\cdot|\mathcal{G}]$. The standard Lebesgue spaces are denoted by L^p for $p \in [1, \infty]$. We say that a set $\mathcal{X} \subset L^0$ is *law invariant* if $X \in \mathcal{X}$ for every $X \in L^0$ such that $X \sim Y$ for some $Y \in \mathcal{X}$.

Assumption 2.1 We denote by $(\mathcal{X}, \mathcal{X}^*)$ a pair of law-invariant vector subspaces of L^1 containing L^∞. We assume that $XY \in L^1$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{X}^*$ and denote by $\sigma(\mathcal{X}, \mathcal{X}^*)$ the weakest linear topology on $\mathcal{X}$ with respect to which, for every $Y \in \mathcal{X}^*$, the linear functional on $\mathcal{X}$ given by $X \mapsto \mathbb{E}[XY]$ is continuous.

As $\mathcal{X}$ and $\mathcal{X}^*$ contain L^∞ by assumption, the pairing on $\mathcal{X} \times \mathcal{X}^*$ given by $(X, Y) \mapsto \mathbb{E}[XY]$ is separating. In particular, when equipped with the topology $\sigma(\mathcal{X}, \mathcal{X}^*)$, the space $\mathcal{X}$ is a locally convex Hausdorff topological vector space. We say that a (nonempty) set $\mathcal{C} \subset \mathcal{X}$ is *convex* if it contains the convex combination of any of its elements, and $\sigma(\mathcal{X}, \mathcal{X}^*)$-*closed* if it contains the limit of any $\sigma(\mathcal{X}, \mathcal{X}^*)$-convergent net of its elements. The *(upper) support functional* of $\mathcal{C}$ is the map $\sigma_{\mathcal{C}} : \mathcal{X}^* \to [-\infty, \infty]$ given by

$$\sigma_{\mathcal{C}}(Y) := \sup_{X \in \mathcal{C}} \mathbb{E}[XY].$$

Throughout the paper we focus on functionals $\varphi\colon \mathcal{X} \to [-\infty, \infty]$. The *(effective) domain* of φ is

$$\mathrm{dom}(\varphi) := \{X \in \mathcal{X}\,;\ \varphi(X) \in \mathbb{R}\}.$$

We say that φ is *proper* if $\mathrm{dom}(\varphi)$ is nonempty. Moreover, the functional φ is called:

(1) *convex* if for all $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda X + (1-\lambda)Y) \le \lambda\varphi(X) + (1-\lambda)\varphi(Y).$$

(2) *quasiconvex* if for all $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda X + (1-\lambda)Y) \le \max\{\varphi(X), \varphi(Y)\}.$$

(3) $\sigma(\mathcal{X}, \mathcal{X}^*)$-*lower semicontinuous* if for all nets $(X_\alpha) \subset \mathcal{X}$ and $X \in \mathcal{X}$,

$$X_\alpha \xrightarrow{\sigma(\mathcal{X},\mathcal{X}^*)} X \implies \varphi(X) \le \liminf_\alpha \varphi(X_\alpha).$$

(4) *law invariant* if for all $X, Y \in \mathcal{X}$,

$$X \sim Y \implies \varphi(X) = \varphi(Y).$$

(5) *expectation invariant* if for all $X, Y \in \mathcal{X}$,

$$\mathbb{E}[X] = \mathbb{E}[Y] \implies \varphi(X) = \varphi(Y).$$

(6) an *affine function of the expectation* if there exist $a, b \in \mathbb{R}$ such that, for every $X \in \mathcal{X}$,

$$\varphi(X) = a\mathbb{E}[X] + b.$$

In the sequel we use that quasiconvexity and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuity are equivalent to every lower level set $\{X \in \mathcal{X}\,;\ \varphi(X) \le m\}$, $m \in \mathbb{R}$, being convex and $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed, respectively. As $\mathcal{X}$ contains all constant random variables, expectation invariance is equivalent to having for every $X \in \mathcal{X}$

$$\varphi(X) = \varphi(\mathbb{E}[X]).$$

The *conjugate* of (a not necessarily convex) φ is the functional $\varphi^*\colon \mathcal{X}^* \to [-\infty, \infty]$ given by

$$\varphi^*(Y) := \sup_{X \in \mathcal{X}} \{\mathbb{E}[XY] - \varphi(X)\}.$$

The next lemma records the well-known dual representations of convex closed sets and convex lower-semicontinuous functionals, which are direct consequences of the Hahn-Banach theorem; see, e.g., [70,Theorem 1.1.9, Theorem 2.3.3].

Proposition 2.2 *Let $\mathcal{C} \subset \mathcal{X}$ be convex and $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed. Then,*

$$\mathcal{C} = \bigcap_{Y \in \mathcal{X}^*} \{X \in \mathcal{X}\,;\ \mathbb{E}[XY] \le \sigma_{\mathcal{C}}(Y)\}.$$

Let $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ be proper, convex, and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous. Then,

$$\varphi(X) = \sup_{Y \in \mathcal{X}^*} \{\mathbb{E}[XY] - \varphi^*(Y)\}, \quad X \in \mathcal{X}.$$

For quasiconvex functionals on $\mathcal{X}$, the property of law invariance is equivalent to other well-known properties such as dilatation monotonicity and Schur convexity (also known as monotonicity with respect to the convex order), to which our corresponding results therefore naturally apply. We refer to [7,Theorem 3.6, Proposition 5.6] for a proof in our general setting.

Proposition 2.3 *Let $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ be proper, quasiconvex, and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous. Then, the following statements are equivalent:*

(i) *φ is law invariant.*
(ii) *φ is dilatation monotone, i.e., for every $X \in \mathcal{X}$ and every σ-field $\mathcal{G} \subset \mathcal{F}$,*

$$\mathbb{E}[X|\mathcal{G}] \in \mathcal{X} \implies \varphi(X) \geq \varphi(\mathbb{E}[X|\mathcal{G}]).$$

(iii) *φ is Schur convex, i.e., for all $X, Y \in \mathcal{X}$,*

$$\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)] \textit{ for every convex } f\colon \mathbb{R} \to \mathbb{R} \implies \varphi(X) \geq \varphi(Y).$$

3 Interpretation

Our results admit a range of interpretations depending on the interpretation of the functional φ. We highlight the following relevant situations:

(a) φ is a pricing rule.
(b) φ is a risk or deviation measure.
(c) φ is a preference index (up to a sign).[1]

As mentioned in the introduction, in each of these three cases the properties of (quasi)convexity, lower semicontinuity, and law invariance are thoroughly investigated in the literature. Our versions of the "collapse to the mean" will involve local "linearity" properties of the following type for given $Z \in \mathcal{X}$ and $a \in \mathbb{R}$:[2]

(1) $\varphi(X + tZ) = \varphi(X) + at$ for all (respectively, for some) $X \in \mathcal{X}$ and all $t \in \mathbb{R}$.
(2) $\varphi(X + tZ) \leq \varphi(X)$ for all (respectively, for some) $X \in \mathcal{X}$ and all $t \geq 0$.

If $a \leq 0$, each statement in (1) implies the corresponding statement in (2). The first property stipulates affinity of φ along direction Z whereas the second property stipulates that Z be a direction of recession for φ. Both the "for all" and the "for some" formulation of properties (1)–(2) above are verified to trigger a collapse in the extant literature. One of the specific features of our approach is to systematically pursue both versions. Where possible, we additionally try to find the minimal set of $X \in \mathcal{X}$ at which a direction Z as above needs to be anchored to entail a collapse.

Properties (1)–(2) are encountered in the literature in each of the three areas of application mentioned above:

(a) If φ is a pricing rule, then the first property holds whenever Z is a frictionless payoff (in particular, if $\varphi(0) = 0$, then $\varphi(tZ) = t\varphi(Z)$ for every $t \in \mathbb{R}$, showing that any multiple of Z can be transacted with zero bid-ask spread) and the second property holds whenever, e.g., Z is negative and φ is nondecreasing (adding Z decreases prices).

[1] That is, $-\varphi$ numerically represents a preference relation $\succeq$ on the set $\mathcal{X}$: For all $X, Y \in \mathcal{X}$ we have $X \succeq Y$ if and only if $-\varphi(X) \geq -\varphi(Y)$. This convention allows to relate (quasi)convexity of φ to convexity of $\succeq$.

[2] As these properties are known in the literature under a variety of different names, we decided to avoid assigning to them a specific name and shall always state them explicitly in the corresponding statements.

(b) If φ is a risk measure, then the first property is the standard translation invariance introduced in [5], where Z represents the payoff of the eligible asset, and the second property holds whenever, e.g., Z is positive and φ is nonincreasing (adding Z decreases risk). If φ is a deviation measure and $a = 0$, then the first property stipulates that Z be a zero deviation element.
(c) If φ is a preference index, then the first property corresponds to the cash-additivity property of risk orders axiomatised in [24]. The second property stipulates that Z be a desirable prospect in the spirit, e.g., of [1,Definition 8.2], or at least a neutral prospect, i.e., adding Z makes the aggregate element at least as preferable as the original one.

The "collapse to the mean" states that, under suitable regularity properties, the combination of law invariance together with the local "linearity" properties (1)–(2) forces the functional φ to be expectation invariant. In special cases, φ is even reduced to an affine function of the expectation. The interpretation of the collapse will therefore depend on the interpretation of φ. In each of the three areas of application mentioned above the collapse is a particularly restrictive result for the following reasons:

(a) If a pricing rule depends only on the expectation with respect to the reference probability measure, then prices are likely to be inconsistent with market prices and will typically engender arbitrage opportunities.
(b) If a risk measure depends only on the expectation with respect to the reference probability measure, then it arguably captures risk in an unsatisfactory manner: large losses can be compensated by (equally likely) large gains.
(c) If a preference index depends only on the expectation with respect to the reference probability measure, then it can only model the preferences of a risk-neutral agent.

We refer to this section to attach concrete interpretations of our results in the aforementioned settings.

4 The general "collapse to the mean" principle

This section contains our prototype version of the "collapse to the mean", which will later be exploited to obtain a variety of results for specific classes of functionals. This general result shows that the expectation is, up to an affine transformation, the only linear and $\sigma(\mathcal{X}, \mathcal{X}^*)$-continuous functional that is dominated above by a law-invariant functional which fulfills a suitable local translation invariance property. It should be noted that the result holds for a general law-invariant functional without any additional property.

Theorem 4.1 *Let* $\varphi \colon \mathcal{X} \to (-\infty, \infty]$ *be law invariant and contain a constant random variable in its domain, i.e.,* $\mathrm{dom}(\varphi) \cap \mathbb{R} \neq \emptyset$. *Let* $x \in \mathrm{dom}(\varphi) \cap \mathbb{R}$ *admit* $a \in \mathbb{R}$ *and a nonconstant* $Z \in \mathcal{X}$ *such that*

$$\varphi(x + tZ) = \varphi(x) + at, \quad t \in \mathbb{R}.$$

Then, $\mathrm{dom}(\varphi^*) \subset \mathbb{R}$. *In particular, if there exist* $c \in \mathbb{R}$ *and* $Y \in \mathcal{X}^*$ *such that*

$$\varphi(X) \geq \mathbb{E}[XY] + c, \quad X \in \mathcal{X},$$

then Y *must be constant.*

We complement the previous theorem with a geometrical counterpart about convex sets. Recall that the *recession cone* of a convex set $\mathcal{C} \subset \mathcal{X}$ is defined by

$$\mathcal{C}^\infty := \{X \in \mathcal{X}\, ;\; \{X\} + \mathcal{C} \subset \mathcal{C}\},$$

the set of all directions of recession of $\mathcal{C}$. Before stating the announced result, it is useful to highlight the following dual representation of the recession cone of a law-invariant set.

Lemma 4.2 *Let $\mathcal{C} \subset \mathcal{X}$ be convex and $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed. Then,*

$$\mathcal{C}^\infty = \bigcap_{Y \in \mathrm{dom}(\sigma_{\mathcal{C}})} \{X \in \mathcal{X}\,;\ \mathbb{E}[XY] \le 0\}. \tag{4.1}$$

If $\mathcal{C}$ is law invariant, then

$$\mathcal{C}^\infty = \bigcap_{Y \in \mathrm{dom}(\sigma_{\mathcal{C}})} \left\{X \in \mathcal{X}\,;\ \int_0^1 q_X(s) q_Y(s) ds \le 0\right\}. \tag{4.2}$$

In particular, $\mathcal{C}^\infty$ is law invariant itself.

We now turn to the announced geometrical version of the "collapse to the mean", which generalises an earlier result formulated in [43,Proposition 5.10] and provides a simpler proof. It shows that a convex and $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed set that is law invariant and admits a nonzero direction of recession with zero expectation must be determined by expectation: Whether or not a random variable belongs to the set depends exclusively on its mean. In particular, the set must contain infinitely many affine spaces.

Proposition 4.3 *Let $\mathcal{C} \subset \mathcal{X}$ be convex, $\sigma(\mathcal{X}, \mathcal{X}^*)$-closed, and law invariant. If there exists a nonzero $Z \in \mathcal{C}^\infty$ such that $\mathbb{E}[Z] = 0$, then* $\mathrm{dom}(\sigma_{\mathcal{C}}) \subset \mathbb{R}$ *and*

$$\mathcal{C} = \{X \in \mathcal{X}\,;\ -\sigma_{\mathcal{C}}(-1) \le \mathbb{E}[X] \le \sigma_{\mathcal{C}}(1)\}. \tag{4.3}$$

5 Applications

5.1 Collapse to the mean: The convex case

As stated in the introduction, a variety of "collapse to the mean" results have been established in the literature for convex functionals. Early versions of the collapse to the mean were obtained in [16] for convex Choquet integrals and in [28] for convex monetary risk measures. The focus of both papers was on bounded random variables. A general version of the collapse to the mean for convex functionals beyond the bounded setting has recently been established in [8]. To best appreciate the differences with the quasiconvex case, we devote this section to revisiting the most general results from the literature and complementing them with additional conditions.

We start by revisiting [8,Theorem 4.7]. This result states that, under convexity and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuity, a functional that is law invariant and affine (in particular, linear) along a nonconstant random variable with zero expectation must be, in our terminology, expectation invariant. We provide a self-contained proof of this result and complement it by a number of weak translation invariance conditions and by a dual condition expressed in terms of the conjugate functional.

Theorem 5.1 *Let $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ be proper, convex, $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

(i) φ is expectation invariant.
(ii) φ is the supremum of a family of affine functions of the expectation.

(iii) *There exists a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] = 0$ *such that*

$$\varphi(X + tZ) = \varphi(X), \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

(iv) *There exist* $a \in \mathbb{R}$ *and a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] = 0$ *such that*

$$\varphi(X + tZ) = \varphi(X) + at, \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

(v) *For every* $X \in \mathcal{X}$ *there exists a nonconstant* $Z_X \in \mathcal{X}$ *with* $\mathbb{E}[Z_X] = 0$ *such that*

$$\varphi(X + tZ_X) \leq \varphi(X), \quad t \geq 0.$$

(vi) *There exist* $X \in \mathrm{dom}(\varphi)$ *and a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] = 0$ *such that*

$$\varphi(X + tZ) \leq \varphi(X), \quad t \geq 0.$$

(vii) $\mathrm{dom}(\varphi^*) \subset \mathbb{R}$.

We turn to revisiting [8,Theorem 4.5]. This result states that, under convexity and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuity, a functional that is law invariant and translation invariant along a nonconstant random variable with nonzero expectation must collapse to the mean up to an affine transformation. We provide a compact proof of this result and complement it by a dual condition expressed in terms of the conjugate functional.

Theorem 5.2 *Let* $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ *be proper, convex,* $\sigma(\mathcal{X}, \mathcal{X}^*)$*-lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

(i) φ *is an affine function of the expectation.*
(ii) *There exist* $a \in \mathbb{R}$ *and a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] \neq 0$ *such that*

$$\varphi(X + tZ) = \varphi(X) + at, \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

(iii) *There exist* $a \in \mathbb{R}$*, a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] \neq 0$*, and* $x \in \mathrm{dom}(\varphi) \cap \mathbb{R}$ *such that*

$$\varphi(x + tZ) = \varphi(x) + at, \quad t \in \mathbb{R}.$$

(iv) $\mathrm{dom}(\varphi^*) \subset \mathbb{R}$ *and* $|\mathrm{dom}(\varphi^*)| = 1$.

5.2 Collapse to the mean: The quasiconvex case

In this section we investigate to which extent the collapse to the mean documented above generalises to quasiconvex functionals. It should be noted that, being heavily based on conjugate duality, the proofs in the convex case do not admit a direct adaptation to the quasiconvex case. In fact, we tackle the collapse to the mean in our more general setting by pursuing a different strategy based on the analysis of recession directions and their interaction with law invariance discussed in Sect. 4.

Our first result establishes that Theorem 5.1 continues to hold if we replace convexity with quasiconvexity provided the condition involving conjugate functions is appropriately adapted to a condition involving sublevel sets. In the accompanying remark we show the link between these two conditions.

Theorem 5.3 *Let* $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ *be proper, quasiconvex,* $\sigma(\mathcal{X}, \mathcal{X}^*)$*-lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

(i) φ *is expectation invariant.*

(ii) There exists a nonconstant $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] = 0$ *such that*

$$\varphi(X + tZ) = \varphi(X), \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

(iii) For every $X \in \mathcal{X}$ *there exists a nonconstant* $Z_X \in \mathcal{X}$ *with* $\mathbb{E}[Z_X] = 0$ *such that*

$$\varphi(X + tZ_X) \leq \varphi(X), \quad t \geq 0.$$

(iv) For every $m \in \mathbb{R}$ *we have* $\mathrm{dom}(\sigma_{\{\varphi \leq m\}}) \subset \mathbb{R}$.

Remark 5.4 Let $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ be proper, convex, and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous. Moreover, take $m \in \mathbb{R}$ such that $\{\varphi \leq m\} \neq \emptyset$. The proof of Theorem 5.1 demonstrates that $\mathrm{dom}(\varphi^*) \subset \mathbb{R}$ (point (vii) in Theorem 5.1) is a direct consequence of $\mathrm{dom}(\sigma_{\{\varphi \leq m\}}) \subset \mathbb{R}$ (point (iv) in Theorem 5.3).

Example C.1 shows that point (vi) in Theorem 5.1 is specific to the convex case and cannot be added to the equivalent conditions in Theorem 5.3.

We turn to the collapse to the mean established in Theorem 5.2. The next result shows that, if convexity is relaxed to quasiconvexity, then the collapse to the mean continues to hold in the presence of translation invariance (point (ii) in Theorem 5.2).

Theorem 5.5 *Let* $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ *be proper, quasiconvex,* $\sigma(\mathcal{X}, \mathcal{X}^*)$*-lower semicontinuous, and law invariant. Then, the following statements are equivalent:*

(i) φ *is an affine function of the expectation.*
(ii) There exist $a \in \mathbb{R}$ *and a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] \neq 0$ *such that*

$$\varphi(X + tZ) = \varphi(X) + at, \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

(iii) There exist $a \in \mathbb{R}$ *and a nonconstant* $Z \in \mathcal{X}$ *with* $\mathbb{E}[Z] \neq 0$ *such that*

$$\varphi(x + tZ) = \varphi(x) + at, \quad x \in \mathbb{R}, \ t \in \mathbb{R}.$$

Example D.1 shows that point (iii) in Theorem 5.2 fails to produce a collapse to the mean under mere quasiconvexity. In particular, this observation holds regardless of the expectation of the nonconstant random variable along which local translation invariance in the sense of point (iii) in Theorem 5.2 holds. Moreover, the example demonstrates that Theorem 5.5 cannot be improved. Also, point (iii) in Theorem 5.5 does not imply expectation invariance of φ without quasiconvexity of the latter; cf. Example D.2.

We close this section by discussing a corresponding version of the collapse for *deviation measures*. These functionals are designed to measure the degree of variability within a given financial position and are studied, for instance, in [11, 33, 53, 58]. More precisely, given a functional $\mathcal{D}\colon \mathcal{X} \to [0, \infty]$ and a random variable $Z \in \mathcal{X}$, we say that $\mathcal{D}$ is Z*-translation insensitive* if

$$\mathcal{D}(X + tZ) = \mathcal{D}(X), \quad X \in \mathcal{X}, \ t \in \mathbb{R}.$$

In a financial context, the previous property mean that Z is a financial variable that does not affect the deviation of a financial position from a benchmark. 1-translation insensitivity is a common minimal assumption in the definition of deviation measures.

We first discuss Theorem 5.3 in the present context. Suppose that $\mathcal{D}\colon \mathcal{X} \to [0, \infty]$ is proper, quasiconvex, $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous, and law invariant. Also, suppose $\mathcal{D}$ is Z-translation insensitive for some $Z \in \mathcal{X}$ with $\mathbb{E}[Z] = 0$. Then, $\mathcal{D}$ is expectation invariant by Theorem 5.3. In view of the fact that this means $\mathcal{D}(X) = \mathcal{D}(tX)$ for all $X \in \mathcal{X}$ with

$\mathbb{E}[X] = 0$ and all $t > 0$, $\mathcal{D}$ is therefore unsuited to capture the variability or spread in a given financial position. If otherwise $\mathbb{E}[Z] \neq 0$, then 1-translation insensitivity is without alternative in the law-invariant case as shown by the next corollary. The statement follows directly from Theorem 5.5.

Corollary 5.6 *Let $\mathcal{D}\colon \mathcal{X} \to [0, \infty]$ be proper, quasiconvex, $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous, and law invariant. Moreover, suppose $\mathcal{D}$ is Z-translation insensitive for some $Z \in \mathcal{X}$ with* $\mathbb{E}[Z] \neq 0$. *Then, one of the following two alternatives holds:*

(i) *$\mathcal{D}$ is constant.*
(ii) *Z is constant.*

5.3 Collapse to the mean: The case of consistent risk measures

In this and the following section, we leave convexity further behind and establish a collapse to the mean for classes of law-invariant functionals beyond the quasiconvex family. Here we focus on functionals that are translation invariant along constants and monotonic with respect to second-order stochastic dominance. Following the terminology in [47], we refer to them as *consistent risk measures*. This class covers the family of law-invariant convex risk measures but also includes nonconvex functionals, e.g., minima of law-invariant convex risk measures. As translation invariance along constants implies that convexity and quasiconvexity are equivalent, the class of consistent risk measures contains functionals that are not quasiconvex. Consequently, we cannot resort to the quasiconvex results in Sect. 5.2.

First, a *consistent risk measure* is a proper functional $\varphi : \mathcal{X} \to (-\infty, \infty]$ that is:

(1) *cash-additive*, i.e., $\varphi(X + m) = \varphi(X) + m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.
(2) *consistent with second-order stochastic dominance*, i.e., for all $X, Y \in \mathcal{X}$,

$$\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)] \text{ for every nondecreasing convex } f : \mathbb{R} \to \mathbb{R} \implies \varphi(X) \geq \varphi(Y).$$

(3) *normalised*, i.e., $\varphi(0) = 0$.

Given its defining properties, a consistent risk measure takes only finite values on L^∞. Moreover, every consistent risk measure is automatically dilatation monotone and law invariant by property (2). In case $\mathcal{X} = L^\infty$, every normalised, law-invariant, and convex risk measure is a consistent risk measure. The same holds for normalised, law-invariant, $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous convex risk measures by Proposition 2.3.

Our main result, Theorem 5.7, establishes a collapse to the mean for consistent risk measures. We show that linearity along a nonconstant random variable is sufficient to reduce the functional to a mere expectation. In line with our previous result, we also provide an equivalent condition for the collapse in terms of directions of recession and conjugate functions. The theorem assumes that the consistent risk measure in question is $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous. This assumption is easy to satisfy. Every consistent risk measure on L^∞ is $\sigma(L^\infty, \mathcal{X}^*)$-lower semicontinuous, no matter the choice of the space $L^\infty \subset \mathcal{X}^* \subset L^1$. Moreover, Proposition E.2 shows that every consistent risk measure on L^∞ extends uniquely to a $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous consistent risk measure.

Theorem 5.7 *Let $\varphi\colon \mathcal{X} \to (-\infty, \infty]$ be a $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous consistent risk measure. Then, the following are equivalent:*

(i) *φ coincides with the expectation under $\mathbb{P}$.*

(ii) There exists a nonconstant $Z \in \mathcal{X}$ *such that*

$$\varphi(tZ) = t\varphi(Z), \quad t \in \mathbb{R}.$$

(iii) There exists a nonconstant $U \in \mathcal{X}$ *such that* $\mathbb{E}[U] = 0$ *and*

$$\sup_{t \geq 0} \varphi(tU) \leq 0.$$

Any of the previous statements implies:

(iv) $\mathrm{dom}(\varphi^*) = \{1\}$.

Statements (i)–(iv) are equivalent if, additionally,

$$\varphi(\lambda X) \leq \lambda\varphi(X), \quad X \in \mathcal{X},\ \lambda \in [0, 1]. \tag{5.1}$$

Theorem 5.7 is sharp: Item (iv) does not imply items (i)–(iii) without the additional assumption (5.1). This is illustrated by Example E.3, which can also be found in [41].

Remark 5.8 Condition (5.1) means that the risk measure φ is *star shaped* in the sense of [15]. By [15,Proposition 2], the latter is equivalently characterised by the fact that the *acceptance set* $\mathcal{A}_\varphi := \{X \in \mathcal{X}\,;\ \varphi(X) \leq 0\}$ is star shaped about 0. Consistent risk measures satisfying (5.1) are characterised in [15,Theorem 11]. We would like to stress here that (5.1) is a mild requirement. By [47,Theorem 3.3] or Lemma E.1 below, a consistent risk measure $\varphi\colon L^\infty \to \mathbb{R}$ is represented by a family $\mathcal{T}$ of convex law-invariant risk measures τ in that

$$\varphi(X) = \inf_{\tau \in \mathcal{T}} \tau(X), \quad X \in L^\infty.$$

If each $\tau \in \mathcal{T}$ is normalised, i.e., $\tau(0) = 0$, then φ has property (5.1).

Remark 5.9 Theorem 5.7 provides a key technical tool in [41]. This paper considers a finite set $\mathcal{I}$ of agents, each measuring risk with a consistent risk measures φ_i. For an aggregate loss $X \in L^\infty$ one tries to minimise the aggregated risk

$$\sum_{i \in \mathcal{I}} \varphi_i(X_i)$$

subject to the constraint $\sum_{i \in \mathcal{I}} X_i = X$. However, the reference measures $\mathbb{P}_i$ determining consistency of φ_i are allowed to be heterogeneous. As elaborated there, existence of minimisers can generally only be guaranteed if these heterogeneous beliefs are suitably compatible. The employed notion of compatibility is based on Theorem 5.7.

5.4 Collapse to the mean: the case of choquet integrals

For a review of capacities and Choquet integrals, we refer to [51] and the references therein. As mentioned in the introduction, the research on the collapse to the mean of law-invariant functionals was triggered by [16] whose focus lies on Choquet integrals associated with special submodular law-invariant capacities. As the property of submodularity is equivalent to convexity of the Choquet integral, the "collapse to the mean" established there can be seen as a special case of the results in Sect. 5.1. In this section, we extend the "collapse to the mean" to Choquet integrals and utility functionals that are generally not convex or concave. It should be noted that we cannot resort to the quasiconvex results in Sect. 5.2; in view of translation invariance along constants, a Choquet integral is quasiconvex if and only if it is convex.

A *capacity* is a function $\mu\colon \mathcal{F} \to [0,1]$ such that $\mu(\emptyset)=0$, $\mu(\Omega)=1$, and $\mu(A)\le\mu(B)$ for all $A, B \in \mathcal{F}$ with $A \subset B$. In this section shall consider the space B_b of all bounded $\mathcal{F}$-measurable random variables (not equivalence classes) and its dual space **ba** consisting of all bounded signed charges on $\mathcal{F}$. Δ denotes the set of nonnegative and normalised elements $\xi \in$ **ba**, i.e. $\xi(\Omega)=1$. These are also called "finitely additive probabilities" in the literature. The *anticore* of μ is the set

$$\operatorname{acore}(\mu) := \{\xi \in \Delta\,;\ \forall\, A \in \mathcal{F},\ \xi(A) \le \mu(A)\}$$

and is always weak* compact and convex. The *dual capacity* $\overline{\mu}\colon \mathcal{F} \to [0,1]$ is defined by

$$\overline{\mu}(A) := 1 - \mu(A^c).$$

The *Choquet integral* associated with μ is the functional $\mathbb{E}_\mu\colon B_b \to \mathbb{R}$ defined by

$$\mathbb{E}_\mu[X] := \int_{-\infty}^{0} (\mu(X > s) - 1)\, ds + \int_0^\infty \mu(X > s) ds.$$

If μ is countably additive, i.e., a probability measure, then the Choquet integral reduces to a standard expectation. Moreover, for every $X \in B_b$, $t \ge 0$, and $c \in \mathbb{R}$, the Choquet integral satisfies

$$\mathbb{E}_\mu[X] = -\mathbb{E}_{\overline{\mu}}[-X] \quad \text{and} \quad \mathbb{E}_\mu[tX + c] = t\mathbb{E}_\mu[X] + c. \tag{5.2}$$

We say that μ is:

(1) *exact* if $\mu = \max_{\xi \in \operatorname{acore}(\mu)} \xi(\cdot)$.

(2) *coherent* or an *upper envelope* if there exists a set $\mathcal{Q}$ of probability measures on $\mathcal{F}$ such that

$$\mu(A) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{Q}(A), \quad A \in \mathcal{F}.$$

(3) *submodular* or *2-alternating* if, for all $A, B \in \mathcal{F}$,

$$\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B).$$

(4) *law invariant* or *symmetric* if, for all $A, B \in \mathcal{F}$,

$$\mathbb{P}(A) = \mathbb{P}(B) \implies \mu(A) = \mu(B).$$

By [61], each submodular capacity is exact. One can also show that each coherent capacity is exact. However, exactness or coherence do not imply submodularity, cf. [37]. The Choquet integral $\mathbb{E}_\mu$ is convex if and only if μ is submodular, and law invariant if and only if μ is law invariant. In that case, we can unambiguously define the Choquet integral $\mathbb{E}_\mu$ on the space L^∞ as will be tacitly done below.

Our first target is the extension of Theorem 5.2 to a wide class of nonconvex Choquet integrals. To this end, we focus on so-called Jaffray-Philippe (JP) capacities introduced in [35]. A capacity μ is a *JP capacity* if there is a pair (ν, α) of an exact capacity ν and $\alpha \in [0,1]$ such that

$$\mu(A) = \alpha\nu(A) + (1-\alpha)\overline{\nu}(A), \quad A \in \mathcal{F}.$$

As special cases, JP capacities encompass both submodular and coherent capacities, as well as *neo-additive capacities* [19]. It has already been observed, e.g., in [35] that the case $\alpha = \frac{1}{2}$ is peculiar, hence we exclude it in our investigation. Under this exclusion, the following key result holds. The argument underlying its proof is that, for law-invariant capacities, exactness and coherence are equivalent properties.

Proposition 5.10 *Let $\alpha \neq \frac{1}{2}$. For a JP capacity μ represented by the pair (ν, α), the following are equivalent:*

(i) *μ is law invariant.*
(ii) *ν is law invariant.*
(iii) *ν is a law-invariant upper envelope, i.e., there is a law-invariant set of probability densities $\mathcal{D} \subset L^1$ such that*

$$\nu(A) = \sup_{D \in \mathcal{D}} \mathbb{E}[D\mathbf{1}_A], \quad A \in \mathcal{F}.$$

Theorem 5.11, our "collapse to the mean" for nonconvex Choquet integrals, encompasses [16,Theorem 3.1], which has been established by means of convex duality under the assumption of submodularity and an additional continuity assumption, and [3,Proposition 3.3], which retains the submodularity assumption of [16] but dispenses with continuity. Our proof is direct and solely based on Theorem 4.1. Note that the collapse for a Choquet integral is equivalent to the underlying capacity μ (and its representing exact capacity ν in the JP case) reducing to the reference probability measure. This is verified to occur whenever μ admits a *nontrivial unambiguous event*, i.e., an event $A \in \mathcal{F}$ such that $\mathbb{P}(A) \in (0, 1)$ and $\mu(A) + \mu(A^c) = 1$. This notion of (un)ambiguity and its interaction with law invariance has been investigated in the literature on submodular and coherent capacities; see, e.g., [3, 48, 49]. Our result extends the corresponding "collapse to the reference probability" to JP capacities. Our proof deviates from the ones encountered in the literature, which are based on the convex range of the reference probability and Lyapunov's Convexity Theorem. Once again, we only need to rely on the sharp version of the Fréchet-Hoeffding bounds.

Theorem 5.11 *Let μ be a law-invariant JP capacity represented by a pair (ν, α). Moreover, assume $\alpha \neq \frac{1}{2}$. Then, the following statements are equivalent:*

(i) *$\mathbb{E}_\mu$ coincides with the expectation under $\mathbb{P}$, or equivalently, $\mu = \nu = \mathbb{P}$.*
(ii) *There exist $a \in \mathbb{R}$ and a nonconstant $Z \in L^\infty$ such that*

$$\mathbb{E}_\mu[X + tZ] = \mathbb{E}_\mu[X] + at, \quad X \in L^\infty, \ t \in \mathbb{R}.$$

(iii) *There exist $a \in \mathbb{R}$ and a nonconstant $Z \in L^\infty$ such that*

$$\mathbb{E}_\mu[tZ] = at, \quad t \in \mathbb{R}.$$

(iv) *There exists a nonconstant $Z \in L^\infty$ such that*

$$\mathbb{E}_\mu[-Z] = -\mathbb{E}_\mu[Z].$$

(v) *There exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) \in (0, 1)$ and*

$$\mathbb{E}_\mu[-\mathbf{1}_A] = -\mathbb{E}_\mu[\mathbf{1}_A].$$

(vi) *There exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) \in (0, 1)$ and*

$$\mu(A^c) = 1 - \mu(A).$$

Theorem 5.11 fails both if one drops the assumption of law invariance and if μ is replaced by a general law-invariant capacity; cf. Example F.1.

Remark 5.12 The key feature of Theorem 5.11 is stating the wide range of conditions under which the Choquet integral collapses to the mean. The latter is equivalent to the collapse of the corresponding capacity to the reference probability. In the decision theory literature the focus is typically on capacities and unambiguous events, i.e., the equivalence between points (i) and (vi) in Theorem 5.11. This particular equivalence alternatively follows with Marinacci's Uniqueness Theorem [49,Theorem 1] in lieu of sharp Fréchet-Hoeffding bounds.

We now turn our attention from Choquet integrals to *α-maxmin expected utility* (α*-MEU) functionals* axiomatised, for instance, in [32]. More precisely, we consider a weak* compact and convex set $\mathcal{Z} \subset \Delta$, a convex combination parameter $\alpha \in [0, 1]$, and the functional $\varphi_{\mathcal{Z},\alpha} \colon B_b \to \mathbb{R}$ defined by

$$\varphi_{\mathcal{Z},\alpha}(X) := \alpha \cdot \max_{\xi \in \mathcal{Z}} \int X \, d\xi + (1-\alpha) \cdot \min_{\xi \in \mathcal{Z}} \int X \, d\xi. \tag{5.3}$$

In the α-MEU framework, utility is computed applying $\varphi_{\mathcal{Z},\alpha}$ to random variables U that are utility evaluations of (state-dependent) acts. The utility computation thereby interpolates between an optimistic view $\max_{\xi \in \mathcal{Z}} \int U \, d\xi$ and a pessimistic view expressed by $\min_{\xi \in \mathcal{Z}} \int U \, d\xi$ according to the weight α.

We shall prove a variant of [50,Theorem 1 & Proposition 1].[3] There, the ranking of events provided by the JP capacity $\mu_{\mathcal{Z},\alpha}(A) = \varphi_{\mathcal{Z},\alpha}(\mathbf{1}_A)$, $A \in \mathcal{F}$, is considered.[4]

Definition 5.13 $\varphi_{\mathcal{Z},\alpha}$ encodes *weak probabilistic beliefs* if there is a $\widehat{\xi} \in \Delta$ with *convex range*, i.e.,

$$\{\widehat{\xi}(B) ;\ B \in \mathcal{F},\ B \subset A\} = [0, \widehat{\xi}(A)], \quad A \in \mathcal{F}, \tag{5.4}$$

such that $\mu_{\mathcal{Z},\alpha}(A) = \mu_{\mathcal{Z},\alpha}(B)$ whenever $A, B \in \mathcal{F}$ satisfy $\widehat{\xi}(A) = \widehat{\xi}(B)$.

In our Theorem 5.14 below we will focus on the special case in which $\widehat{\xi}$ in Definition 5.13 agrees with $\mathbb{P}$. This is only seemingly less general than [50]. There, the assumption of *monotone continuous* preferences is made, which corresponds to the Lebesgue property of risk measures and has the strong implication that $\mathcal{Z}$ only contains countably additive measures. Lemma F.2 below shows that then $\widehat{\xi}$ is countably additive as well.

While we thus dispense with monotone continuity, we assume $\mathcal{Z}$ be the anticore of a capacity. Consequently, we can additionally show that the distinction between "(weak) probabilistic beliefs" and law invariance of (the preferences represented by) $\varphi_{\mathcal{Z},\alpha}$ becomes superfluous; all these notions agree.

Theorem 5.14 *Assume $\alpha \neq \frac{1}{2}$ and that $\mathcal{Z}$ is the anticore of a capacity. Then, the following statements are equivalent:*

(i) $\varphi_{\mathcal{Z},\alpha}$ is law invariant.
(ii) $\mu_{\mathcal{Z},\alpha}$ is law invariant.

In that case, statements (i)–(vi) in Theorem 5.11 remain equivalent if $\mathbb{E}_{\mu_{\mathcal{Z},\alpha}}$ is replaced by $\varphi_{\mathcal{Z},\alpha}$.

Remark 5.15 (i) Another result which proves the collapse of α-MEU preferences to subjective expected utility ("to the mean") is [31,Proposition 3]. There, this collapse is proved under the existence of an essential, unambiguous, and complement symmetric event. Essentiality is akin to nontriviality above, unambiguity is the same notion that we use here. The key difference is that we focus on law-invariant functionals and preferences and do not have to resort to the (behavioural) concept of complement symmetric events (cf. [31,Definition 4]).

[3] While [50] defines measurability in terms of a λ-system on Ω, we consider the standard case of an underlying σ-algebra.

[4] Note that $\varphi_{\mathcal{Z},\alpha} \neq \mathbb{E}_{\mu_{\mathcal{Z},\alpha}}$ holds in general because the representing ν may not be submodular.

(ii) While the focus of [50] is on α-MEU preferences, it is stated at the end of that paper that similar arguments would deliver a collapse result for Choquet integrals associated with JP capacities. Here, we have pursued the opposite path starting from Choquet integrals, and our collapse result recorded in Theorem 5.11 is proved without relying on "monotone continuity" assumptions.

5.5 Collapse to the mean in optimisation problems

In this section we focus on a class of optimisation problems involving law invariance at the level of both the objective function and the optimisation domain. We investigate the existence of optimal solutions that are *antimonotone* with respect to a "pricing density" appearing in the budget constraint under a list of suitable assumptions. Anti- and comonotonicity of pairs of random variables are recalled in Appendix A. We prove sharpness of our existence result in the sense that, if any of the listed assumptions is removed, then the result continues to hold only in the trivial situation where the budget constraint "collapses to the mean". This is relevant in applications because a key monotonicity assumption on the optimisation domain is sometimes omitted in the literature, in which case, contrary to what is sometimes stated, the general result cannot be invoked and one has to proceed case by case.

Throughout the entire section we focus on the optimisation problem

$$\begin{cases} \varphi(X) = \max \\ X \in \mathcal{C} \\ \mathbb{E}[DX] = p \end{cases}$$

under the following basic assumptions:

(1) $\varphi\colon \mathcal{X} \to [-\infty, \infty]$ is law invariant,
(2) $\mathcal{C} \subset \mathcal{X}$ is law invariant,
(3) $D \in \mathcal{X}^*$ satisfies $\mathbb{E}[D] > 0$ and $p \in \mathbb{R}$.

The last constraint is typically interpreted as a budget constraint where D plays the role of a "pricing density". We say that the quadruple $(\varphi, \mathcal{C}, D, p)$ is *feasible* if the optimisation problem admits an optimal solution. In this case, we denote by $\mathbf{Max}(\varphi, \mathcal{C}, D, p)$ the corresponding optimal value. This problem has been extensively studied in the literature, see, e.g., [12, 14, 34, 60, 68, 69], and the recent overview in [59]. In this literature, one encounters the following two types of statements about optimal solutions:

- There exists an optimal solution that is antimonotone with D.
- All optimal solutions are antimonotone with D.

As mentioned in the introduction, these statements are very useful because they allow to reduce the original problem to a deterministic optimisation problem involving quantile functions; see, e.g., [59].

We start by providing a slight extension to the extant results about existence of optimal solutions that are antimonotone with the "pricing density". To this effect, it is convenient to define the following notions:

(1) $\mathcal{C}$ is *increasing* if $X + m \in \mathcal{C}$ for all $X \in \mathcal{C}$ and $m \geq 0$.
(2) φ is *weakly increasing* if $\varphi(X + m) \geq \varphi(X)$ for all $X \in \mathcal{X}$ and $m \geq 0$.
(3) φ is *increasing* if $\varphi(X + m) > \varphi(X)$ for all $X \in \mathcal{X}$ with $\varphi(X) \in \mathbb{R}$ and $m > 0$.

The next result shows that antimonotone optimal solutions always exist provided that both $\mathcal{C}$ is increasing and φ is weakly increasing. If φ is also increasing, then every optimal solution must be antimonotone with the "pricing density".

Theorem 5.16 *Let $(\varphi, \mathcal{C}, D, p)$ be a feasible quadruple.*

(i) If $\mathcal{C}$ is increasing and φ is weakly increasing, then there exists an optimal solution that is antimonotone with D.
(ii) If $\mathcal{C}$ is increasing, φ is increasing, and $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$, then all optimal solutions are antimonotone with D.

The previous result is sometimes stated without the monotonicity assumption on the domain $\mathcal{C}$ (see, e.g., [59]) or it is said that the monotonicity assumption on $\mathcal{C}$ is made without loss of generality (see, e.g., [68]).[5] The remainder of the section is devoted to showing that all the assumptions in Theorem 5.16, including the monotonicity assumption on $\mathcal{C}$, are necessary for the result to hold. More precisely, we show that, if any of the assumptions is removed, then for every choice of a nonconstant "pricing density" one can find a concrete formulation of the optimisation problem for which the result does not hold. Equivalently, one can preserve the result after discarding any of the preceding assumptions only under a "collapse to the mean": The "pricing density" must be constant, and the "pricing rule" in the budget constraint can be expressed by a standard expectation.

Proposition 5.17 *For every nonconstant $D \in \mathcal{X}^*$ with $\mathbb{E}[D] > 0$ there exists a feasible quadruple $(\varphi, \mathcal{C}, D, p)$ such that:*

(i) φ is weakly increasing but no optimal solution is antimonotone with D.
(ii) $\mathcal{C}$ is increasing but no optimal solution is antimonotone with D.
(iii) φ is increasing and $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$ but there exist optimal solutions that are not antimonotone with D.
(iv) $\mathcal{C}$ is increasing and $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$ but there exist optimal solutions that are not antimonotone with D.

We strengthen the previous result in two ways. In a first step, we show that imposing no condition on the domain $\mathcal{C}$ besides law invariance leads to counterexamples independently of the choice of both the "pricing density" D and the objective function φ.

Proposition 5.18 *(i) For every law-invariant $\varphi \colon \mathcal{X} \to [-\infty, \infty]$ and for every nonconstant $D \in \mathcal{X}^*$ with $\mathbb{E}[D] > 0$ there exists a feasible quadruple $(\varphi, \mathcal{C}, D, p)$ such that no optimal solution is antimonotone with D.*
(ii) For every law-invariant $\varphi \colon \mathcal{X} \to [-\infty, \infty]$ such that $\varphi(X) \in \mathbb{R}$ for some nonconstant $X \in \mathcal{X}$, and for every nonconstant $D \in \mathcal{X}^*$ with $\mathbb{E}[D] > 0$, there exists a feasible quadruple $(\varphi, \mathcal{C}, D, p)$ such that $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$, but there exist optimal solutions that are not antimonotone with D.

We reinforce the same message by showing that the monotonicity assumption on $\mathcal{C}$ remains critical even if we impose more structure on the set $\mathcal{C}$ itself. We illustrate this by focusing on two common choices in the literature, starting from an "interval-like" set.

[5] We highlight that the result is also typically stated without the finiteness assumption of the optimal value. This is often justified because the special choice of φ and $\mathcal{C}$ ensures finiteness.

Proposition 5.19 *Let $\mathcal{C} \subset \mathcal{X}$ be law invariant and such that*

$$\mathcal{C} = \{X \in \mathcal{X}\,;\ a \leq X \leq b\}$$

for suitable constants $a < b$. For every nonconstant $D \in \mathcal{X}^*$ with $\mathbb{E}[D] > 0$ there exists a feasible quadruple $(\varphi, \mathcal{C}, D, p)$ such that:

(i) φ is weakly increasing but no optimal solution is antimonotone with D.
(ii) φ is increasing and $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$ but there exist optimal solutions that are not antimonotone with D.

We conclude by focusing on the situation where $\mathcal{C}$ admits a maximum with respect to a suitable preference relation $\preceq$ which we assume to be *compatible with the expectation*: for all $X, Y \in \mathcal{X}$,

$$X \succeq Y \implies \mathbb{E}[X] \geq \mathbb{E}[Y].$$

This weak compatibility property is satisfied by many preference relations encountered in the literature, including the convex order and second-order stochastic dominance.

Proposition 5.20 *Let $\mathcal{C} \subset \mathcal{X}$ be law invariant and such that, for a nonconstant $B \in \mathcal{C}$ and a preference $\succeq$ compatible with the expectation,*

$$\mathcal{C} \subset \{Y \in \mathcal{X}\,;\ Y \preceq B\}.$$

For every nonconstant $D \in \mathcal{X}^*$ with $\mathbb{E}[D] > 0$ there exists a feasible quadruple $(\varphi, \mathcal{C}, D, p)$ such that:

(i) φ is weakly increasing but no optimal solution is antimonotone with D.
(ii) φ is increasing and $\mathbf{Max}(\varphi, \mathcal{C}, D, p) \in \mathbb{R}$ but there exist optimal solutions that are not antimonotone with D.

Acknowledgements We would like to thank two anonymous referees, the Editor, the participants of the Weekly Seminars on Risk Management and Actuarial Science at the University of Waterloo and of the Mathematical Finance Seminar at the University of Bielefeld, and Ruodu Wang for their valuable input which helped improving earlier versions of this manuscript. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Funding Open Access funding enabled and organized by Projekt DEAL.

A The key tool: Sharp Fréchet–Hoeffding bounds

In this brief section we record the main tool that is needed to establish our "collapse to the mean" results, which consists of a sharp formulation of the well-known Fréchet-Hoeffding bounds. For any random variable $X \in L^0$ we denote by q_X a *fixed* quantile function of X, i.e., a function $q_X \colon (0, 1) \to \mathbb{R}$ satisfying for every $s \in (0, 1)$

$$\inf\{x \in \mathbb{R}\,;\ \mathbb{P}(X \leq x) \geq s\} \leq q_X(s) \leq \inf\{x \in \mathbb{R}\,;\ \mathbb{P}(X \leq x) > s\}.$$

As the distribution function of X has at most countably many plateaus, any two quantile functions of X coincide almost surely with respect to the Lebesgue measure on $(0, 1)$. For $X, Y \in L^0$ we say that X and Y are *comonotone* if for all $x, y \in \mathbb{R}$,

$$\mathbb{P}(X \leq x, Y \leq y) = \min\{\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)\};$$

or, equivalently, there are nondecreasing functions $f, g: \mathbb{R} \to \mathbb{R}$ and $Z \in L^0$ with $X = f(Z)$ and $Y = g(Z)$ (cf. [27, Lemma 4.89]). Similarly, we say that X and Y are *antimonotone* if for all $x, y \in \mathbb{R}$,

$$\mathbb{P}(X \leq x, Y \leq y) = \max\{\mathbb{P}(X \leq x) + \mathbb{P}(Y \leq y) - 1, 0\};$$

or, equivalently, X and $-Y$ or $-X$ and Y are comonotone.

In the proof of the sharp version of the Fréchet-Hoeffding bounds and in the sequel, we will repeatedly use the fact that, by nonatomicity, for all $X, Y \in L^0$ we can always find $X' \sim X$ and $Y' \sim Y$ such that X' and Y' are comonotone. The analogue for anticomonotonicity holds as well. In fact, the following stronger result is well known; see, e.g., [27,Lemmas 4.89 & A.32].

Lemma A.1 *For all $X \in \mathcal{X}$ and $Y \in \mathcal{X}^*$ there exist $X', X'' \sim X$ such that X' and Y are comonotone and X'' and Y are antimonotone.*

The next result connecting the range of special integrals and quantile functions builds on early work by Fréchet and Hoeffding on joint distribution functions (see [12]) and Chebyshev, Hardy, and Littlewood on rearrangement inequalities (see [45]). Its general formulation in our setting is essentially due to Luxemburg; see [45,Theorem 9.1]. However, as the statements found in the literature contain only portions of the statement we need, we provide a complete proof in our general framework.

Lemma A.2 *For all $X \in \mathcal{X}$ and $Y \in \mathcal{X}^*$ the functions*

$$(0, 1) \ni s \mapsto q_X(s)q_Y(s) \text{ and } (0, 1) \ni s \mapsto q_X(s)q_Y(1-s)$$

are both Lebesgue integrable on $(0, 1)$ and

$$\begin{aligned} \min_{X' \sim X} \mathbb{E}[X'Y] &= \textstyle\int_0^1 q_X(1-s)q_Y(s)ds, \\ \max_{X' \sim X} \mathbb{E}[X'Y] &= \textstyle\int_0^1 q_X(s)q_Y(s)ds. \end{aligned} \tag{A.1}$$

The minimum (respectively maximum) is attained by $X' \sim X$ if and only if X' and Y are antimonotone (respectively comonotone). Moreover, if both X and Y are nonconstant,

$$\int_0^1 q_X(1-s)q_Y(s)ds < \mathbb{E}[X]\mathbb{E}[Y] < \int_0^1 q_X(s)q_Y(s)ds. \tag{A.2}$$

Proof First, let X and Y be positive. For every $X' \sim X$, Fubini's theorem yields

$$\begin{aligned} \mathbb{E}[X'Y] &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbf{1}_{[0,X')}(x)\mathbf{1}_{[0,Y)}(y)dxdy\right] = \int_0^\infty \int_0^\infty \mathbb{E}[\mathbf{1}_{\{X'>x\}}\mathbf{1}_{\{Y>y\}}]dxdy \\ &= \int_0^\infty \int_0^\infty \mathbb{P}(X' > x, Y > y)dxdy \leq \int_0^\infty \int_0^\infty \min\{\mathbb{P}(X' > x), \mathbb{P}(Y > y)\}dxdy \\ &= \int_0^\infty \int_0^\infty \int_0^1 \mathbf{1}_{[F_X(x),1]}(s)\mathbf{1}_{[F_Y(y),1]}(s)dsdxdy \\ &= \int_0^1 \int_0^{q_Y(s)} \int_0^{q_X(s)} dxdyds = \int_0^1 q_X(s)q_Y(s)ds. \end{aligned}$$

We have equality if and only if $\mathbb{P}(X' > x, Y > y) = \min\{\mathbb{P}(X' > x), \mathbb{P}(Y > y)\}$, or equivalently $\mathbb{P}(X' \leq x, Y \leq y) = \min\{\mathbb{P}(X' \leq x), \mathbb{P}(Y \leq y)\}$, for almost all $x, y \in \mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{R}$. By right continuity of distribution functions, this holds if and only if X' and Y are comonotone. Note that, by Lemma A.1, we do find $X' \sim X$ such that X' and Y are comonotone. This proves the integrability of $q_X q_Y$, the right-hand side equality in (A.1), and the corresponding attainability assertion. In a similar way, we obtain

$$\begin{aligned}\mathbb{E}[X'Y] &= \int_0^\infty \int_0^\infty \mathbb{P}(X' > x, Y > y)dxdy \\ &\geq \int_0^\infty \int_0^\infty \max\{\mathbb{P}(X' > x) - \mathbb{P}(Y \leq y), 0\}dxdy \\ &= \int_0^\infty \int_0^\infty \int_0^1 \mathbf{1}_{[0,1-F_X(x)]}(s)\mathbf{1}_{[F_Y(y),1]}(s)dsdxdy \\ &= \int_0^1 \int_0^{q_Y(s)} \int_0^{q_X(1-s)} dxdyds = \int_0^1 q_X(1-s)q_Y(s)ds.\end{aligned}$$

We have equality if and only if $\mathbb{P}(X' > x, Y > y) = \max\{\mathbb{P}(X' > x) - \mathbb{P}(Y \leq y), 0\}$, or equivalently $\mathbb{P}(X' \leq x, Y \leq y) = \max\{\mathbb{P}(X' \leq x) + \mathbb{P}(Y \leq y) - 1, 0\}$, for almost all $x, y \in \mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{R}$. By right continuity of distribution functions, this holds if and only if X' and Y are antimonotone. Note that, by Lemma A.1, we do find $X' \sim X$ such that X' and Y are antimonotone. This proves the integrability of $q_X(1-\cdot)q_Y$, the left-hand side equality in (A.1), and the corresponding attainability assertion. The statement for general X and Y follows by applying (A.1) and the attainability result to the positive and negative parts of X and Y exploiting the fact that $q_{\max\{X,0\}} = \max\{q_X, 0\}$ and $q_{\max\{-X,0\}} = \max\{-q_X(1-\cdot), 0\}$ almost surely with respect to the Lebesgue measure on $(0, 1)$, and similarly for Y. For the attainability assertion, one observes that X and Y are comonotone if and only if $\max\{X, 0\}$ and $\max\{Y, 0\}$ as well as $\max\{-X, 0\}$ and $\max\{-Y, 0\}$ are comonotone and $\max\{X, 0\}$ and $\max\{-Y, 0\}$ as well as $\max\{-X, 0\}$ and $\max\{Y, 0\}$ are antimonotone, and similarly for antimonotonicity.

Now, take general nonconstant X and Y. Observe that

$$\begin{aligned}&2\left(\int_0^1 q_X(s)q_Y(s)ds - \mathbb{E}[X]\mathbb{E}[Y]\right) \\ &= \int_0^1 \int_0^1 q_X(s)q_Y(s)dtds + \int_0^1 \int_0^1 q_X(t)q_Y(t)dtds - 2\int_0^1 \int_0^1 q_X(s)q_Y(t)dtds \\ &= \int_0^1 \int_0^1 \big(q_X(s) - q_X(t)\big)\big(q_Y(s) - q_Y(t)\big)dtds.\end{aligned}$$

The integrand in the last expression is nonnegative. Moreover, we can invoke nonconstancy of X and Y to find some $\alpha \in (0, \frac{1}{2})$ such that $q_X(t) - q_X(s) > 0$ and $q_Y(t) - q_Y(s) > 0$ for all $s < \alpha$ and $t > 1 - \alpha$. This shows the right-hand side inequality in (A.2). Repeating the argument by replacing X with $-X$ delivers the left-hand side inequality in (A.2) and concludes the proof. $\square$

Remark A.3 The strict inequality in (A.2) is seldom stated in the literature and is related to a rearrangement inequality by Chebyshev; see, e.g., [25]. An alternative proof can be obtained from [68,Lemma 8]. Indeed, by nonatomicity of $(\Omega, \mathcal{F}, \mathbb{P})$ we find two independent random variables U_1 and U_2 with uniform distribution over $(0, 1)$. Hence, $X' := q_X(U_1) \sim X$ and $Y' := q_Y(U_2) \sim Y$ are independent as well. Let $\alpha \in (0, \frac{1}{2})$ be such that $q_X(s) < q_X(t)$ and

$q_Y(s) < q_Y(t)$ for all $s \le \alpha$ and $t \ge 1-\alpha$, which is possible as X and Y are not constant. Set $R = (\{U_1 \ge 1-\alpha\} \cap \{U_2 \le \alpha\}) \times (\{U_1 \le \alpha\} \cap \{U_2 \ge 1-\alpha\})$ and note that $(\mathbb{P} \otimes \mathbb{P})(R) > 0$ and that

$$\Omega \times \Omega \ni (\omega, \omega') \mapsto \left(X'(\omega) - X'(\omega')\right) \cdot \left(Y'(\omega) - Y'(\omega')\right)$$

is negative $\mathbb{P} \otimes \mathbb{P}$-almost surely on R. As the random variables X' and Y' can therefore not be comonotone, we obtain

$$\mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X'Y'] < \mathbb{E}[q_X(U_1)q_Y(U_1)] = \int_0^1 q_X(s)q_Y(s)ds.$$

The other inequality follows by exchanging X with $-X$.

B Proofs accompanying Section 4

Proof of Theorem 4.1 If $\mathrm{dom}(\varphi^*) = \emptyset$, the assertion trivially holds. Hence, suppose we can select $Y \in \mathrm{dom}(\varphi^*)$. By an affine transformation of φ, we can assume without loss of generality that $\varphi^*(Y) = 0$. For all $k \in \mathbb{N}$ and $Z' \sim Z$, we observe that

$$\varphi(x + kZ) - \varphi(x) = \varphi(x + kZ') - \varphi(x) \ge k\mathbb{E}[Z'Y] + x\mathbb{E}[Y] - \varphi(x).$$

In the same vein,

$$\begin{aligned}\varphi(x + kZ) - \varphi(x) &= \varphi(x) - \varphi(x - kZ)\\ &= \varphi(x) - \varphi(x - kZ') \le \varphi(x) + k\mathbb{E}[Z'Y] - x\mathbb{E}[Y].\end{aligned}$$

As a result, for every $k \in \mathbb{N}$,

$$\sup_{Z'\sim Z} \mathbb{E}[Z'Y] \le \frac{2\left(\varphi(x) - x\mathbb{E}[Y]\right)}{k} + \inf_{Z'\sim Z} \mathbb{E}[Z'Y].$$

Letting $k \to \infty$, we infer that

$$\sup_{Z'\sim Z} \mathbb{E}[Z'Y] = \inf_{Z'\sim Z} \mathbb{E}[Z'Y].$$

As Z is nonconstant, Lemma A.2 implies that Y has to be constant. □

Proof of Lemma 4.2 To show (4.1), fix an arbitrary $U \in \mathcal{C}$. It follows from Proposition 2.2 that

$$\begin{aligned}\mathcal{C}^\infty &= \{X \in \mathcal{X}\,;\ \forall k \in \mathbb{N},\ U + kX \in \mathcal{C}\}\\ &= \bigcap_{Y\in\mathrm{dom}(\sigma_\mathcal{C})} \{X \in \mathcal{X}\,;\ \forall k \in \mathbb{N},\ \mathbb{E}[(U + kX)Y] \le \sigma_\mathcal{C}(Y)\}\\ &= \bigcap_{Y\in\mathrm{dom}(\sigma_\mathcal{C})} \{X \in \mathcal{X}\,;\ \forall k \in \mathbb{N},\ \mathbb{E}[XY] \le \tfrac{1}{k}(\sigma_\mathcal{C}(Y) - \mathbb{E}[UY])\}\\ &= \bigcap_{Y\in\mathrm{dom}(\sigma_\mathcal{C})} \{X \in \mathcal{X}\,;\ \mathbb{E}[XY] \le 0\}.\end{aligned}$$

To show (4.2), note that law invariance of $\mathcal{C}$ together with Lemma A.2 imply for every $Y \in \mathcal{X}^*$

$$\sigma_\mathcal{C}(Y) = \sup_{X\in\mathcal{C}} \mathbb{E}[XY] = \sup_{X\in\mathcal{C}} \sup_{X'\sim X} \mathbb{E}[X'Y] = \sup_{X\in\mathcal{C}} \int_0^1 q_X(s)q_Y(s)ds.$$

This shows that $\sigma_{\mathcal{C}}$ is a law-invariant functional and, thus, $\mathrm{dom}(\sigma_{\mathcal{C}})$ is a law-invariant set. As a result, we infer from (4.1) together with Lemma A.2 that

$$\begin{aligned}
\mathcal{C}^{\infty} &= \bigcap_{Y \in \mathrm{dom}(\sigma_{\mathcal{C}})} \{X \in \mathcal{X};\ \mathbb{E}[XY] \le 0\} = \bigcap_{Y \in \mathrm{dom}(\sigma_{\mathcal{C}})} \bigcap_{Y' \sim Y} \{X \in \mathcal{X};\ \mathbb{E}[XY'] \le 0\} \\
&= \Big\{X \in \mathcal{X};\ \forall\, Y \in \mathrm{dom}(\sigma_{\mathcal{C}}),\ \sup_{Y' \sim Y} \mathbb{E}[XY'] \le 0\Big\} \\
&= \Big\{X \in \mathcal{X};\ \forall\, Y \in \mathrm{dom}(\sigma_{\mathcal{C}}),\ \int_0^1 q_X(s) q_Y(s) ds \le 0\Big\}.
\end{aligned}$$

This representation clearly shows that $\mathcal{C}^{\infty}$ is law invariant. □

Proof of Proposition 4.3 Since $Z \in \mathcal{C}^{\infty}$ by assumption, Lemma 4.2 implies that, for every $Y \in \mathrm{dom}(\sigma_{\mathcal{C}})$,

$$\int_0^1 q_Z(s) q_Y(s) ds \le 0.$$

Note that Z is nonconstant by assumption. If there existed a nonconstant $Y \in \mathrm{dom}(\sigma_{\mathcal{C}})$, then Lemma A.2 would entail the impossible chain of inequalities

$$0 = \mathbb{E}[Z]\mathbb{E}[Y] < \int_0^1 q_Z(s) q_Y(s) ds \le 0.$$

This yields $\mathrm{dom}(\sigma_{\mathcal{C}}) \subset \mathbb{R}$. By positive homogeneity of $\sigma_{\mathcal{C}}$, Proposition 2.2 implies

$$\mathcal{C} = \bigcap_{Y \in \mathrm{dom}(\sigma_{\mathcal{C}})} \{X \in \mathcal{X};\ \mathbb{E}[XY] \le \sigma_{\mathcal{C}}(Y)\} = \{X \in \mathcal{X};\ -\sigma_{\mathcal{C}}(-1) \le \mathbb{E}[X] \le \sigma_{\mathcal{C}}(1)\}.$$

This delivers the desired claims and concludes the proof. □

C Mathematical details of Section 5.1

Proof of Theorem 5.1 It is straightforward to verify that (ii) implies (iii), which in turn implies (iv), and that (v) implies (vi). Also note that $\mathrm{dom}(\varphi) \cap \mathbb{R} \neq \emptyset$ by dilatation monotonicity recorded in Proposition 2.3.

(i) implies (ii): If (i) holds, then Proposition 2.2 yields for every $X \in \mathcal{X}$

$$\varphi(X) = \varphi(\mathbb{E}[X]) = \sup_{Y \in \mathcal{X}^*} \{\mathbb{E}[\mathbb{E}[X]Y] - \varphi^*(Y)\} = \sup_{Y \in \mathrm{dom}(\varphi^*)} \{\mathbb{E}[Y]\mathbb{E}[X] - \varphi^*(Y)\}.$$

(iv) implies (vii): This is a direct consequence of Proposition 2.2 and Theorem 4.1.
(vii) implies (v): This is a direct consequence of Proposition 2.2.
(vi) implies (i): Let X and Z be as in the assertion of (vi) and consider the nonempty convex set $\mathcal{C} := \{V \in \mathcal{X};\ \varphi(V) \le \varphi(X)\}$. As $Z \in \mathcal{C}^{\infty}$, it follows from Proposition 4.3 that $\mathrm{dom}(\sigma_{\mathcal{C}}) \subset \mathbb{R}$. Note that, for every $Y \in \mathrm{dom}(\varphi^*)$,

$$\sigma_{\mathcal{C}}(Y) = \sup_{V \in \mathcal{C}} \{\mathbb{E}[VY] - \varphi(V) + \varphi(V)\} \le \varphi^*(Y) + \varphi(X) < \infty.$$

Hence, $\mathrm{dom}(\varphi^*) \subset \mathbb{R}$. Together with Proposition 2.2, for every $V \in \mathcal{X}$

$$\varphi(V) = \sup_{Y \in \mathrm{dom}(\varphi^*)} \{\mathbb{E}[VY] - \varphi^*(Y)\} = \sup_{Y \in \mathrm{dom}(\varphi^*)} \{\mathbb{E}[Y]\mathbb{E}[V] - \varphi^*(Y)\} = \varphi(\mathbb{E}[V]).$$

This concludes the proof of the equivalence. □

Proof of Theorem 5.2 It is clear that (i) implies (ii), which in turn implies (iii). Now, assume that (iii) holds. By Proposition 2.2 and Theorem 4.1, $\emptyset \neq \mathrm{dom}(\varphi^*) \subset \mathbb{R}$. Moreover, each $y \in \mathrm{dom}(\varphi^*)$ must satisfy

$$\sup_{t\in\mathbb{R}}\{(y\mathbb{E}[Z]-a)t\}+yx-\varphi(x)=\sup_{t\in\mathbb{R}}\{\mathbb{E}[(x+tZ)y]-\varphi(x+tZ)\}\leq\varphi^*(y)<\infty,$$

showing that $\mathrm{dom}(\varphi^*)=\{\frac{a}{\mathbb{E}[Z]}\}$. The proof that (iii) implies (iv) is complete. Finally, assume that (iv) holds and let $y \in \mathbb{R}$ be the unique scalar such that $\varphi^*(y) < \infty$. It immediately follows from Proposition 2.2 that

$$\varphi(X)=\mathbb{E}[Xy]-\varphi^*(y)=y\mathbb{E}[X]-\varphi^*(y).$$

This shows that (iv) implies (i) and concludes the proof of the equivalence. □

Example C.1 Let the functional $\rho\colon \mathcal{X} \to \mathbb{R}$ be defined by

$$\rho(X)=\frac{1}{2}\mathbb{E}[X]+\int_{1/2}^{1}q_X(s)ds.$$

Note that ρ is convex, $\sigma(\mathcal{X}, L^\infty)$-lower semicontinuous, and law invariant. Set

$$\mathcal{C}_m=\begin{cases}\{X\in\mathcal{X}\,;\ \rho(X)\leq m\} & \text{if } m<0,\\ \{X\in\mathcal{X}\,;\ \mathbb{E}[X]\leq 2m\} & \text{if } m\geq 0.\end{cases}$$

Define the functional $\varphi\colon \mathcal{X} \to \mathbb{R}$ by setting

$$\varphi(X)=\inf\{m\in\mathbb{R}\,;\ X\in\mathcal{C}_m\}=\begin{cases}\rho(X) & \text{if } \rho(X)<0,\\ \frac{1}{2}\max\{\mathbb{E}[X],0\} & \text{if } \rho(X)\geq 0.\end{cases}$$

For all $X \in \mathcal{X}$ and $m \in \mathbb{R}$ we have $\varphi(X) \leq m$ if and only if $X \in \mathcal{C}_m$, showing that φ is quasiconvex and $\sigma(\mathcal{X}, L^\infty)$-lower semicontinuous. Moreover, φ is clearly law invariant and satisfies $\varphi(0) = 0$. Now, use nonatomicity to find a random variable Z such that

$$\mathbb{P}(Z=2)=1-\mathbb{P}(Z=-1)=\tfrac{1}{3}.$$

A direct calculation shows that $\mathbb{E}[Z] = 0$ and $\rho(Z) = \frac{1}{2}$. As a result, we obtain for every $m \geq 0$ that $\varphi(0+mZ) = m\varphi(Z) = 0 = \varphi(0)$, showing that φ satisfies point (vi) in Theorem 5.1. However, φ is not expectation invariant. To see this, compare a random variable X with $\mathbb{P}(X = 4) = \mathbb{P}(X = -6) = \frac{1}{2}$ to the constant random variable $Y = -1$. Then, we have $\mathbb{E}[X] = \mathbb{E}[Y] = -1$, but $\varphi(X) = 0$, while $\rho(Y) = -1 = \varphi(Y)$.

D Mathematical details of Section 5.2

Proof of Theorem 5.3 It is clear that (i) implies (ii), which in turn implies (iii). Now, assume that (iii) holds. Take $m \in \mathbb{R}$ and set $\mathcal{C}_m = \{\varphi \leq m\}$. If $\mathcal{C}_m = \emptyset$, then we have $\mathrm{dom}(\sigma_{\mathcal{C}_m}) = \emptyset$. Hence, suppose that $\mathcal{C}_m \neq \emptyset$ and take any $X \in \mathcal{C}_m$. By assumption, for every $t \geq 0$ we have $X + tZ_X \in \mathcal{C}_m$. This implies that $Z_X \in \mathcal{C}_m^\infty$. It follows from Proposition 4.3 that $\mathrm{dom}(\sigma_{\mathcal{C}_m}) \subset \mathbb{R}$, showing that (iii) implies (iv). Finally, assume that (iv) holds. For every $m \in \mathbb{R}$ set again $\mathcal{C}_m = \{\varphi \leq m\}$. As $\mathrm{dom}(\sigma_{\mathcal{C}_m}) \subset \mathbb{R}$ and $\sigma_{\mathcal{C}_m}$ is positively homogeneous, it follows from Proposition 2.2 that

$$\mathcal{C}_m=\{X\in\mathcal{X}\,;\ -\sigma_{\mathcal{C}_m}(-1)\leq\mathbb{E}[X]\leq\sigma_{\mathcal{C}_m}(1)\}.$$

As a consequence, we obtain for every $X \in \mathcal{X}$

$$\varphi(X) = \inf\{m \in \mathbb{R};\ X \in \mathcal{C}_m\} = \inf\{m \in \mathbb{R};\ -\sigma_{\mathcal{C}_m}(-1) \leq \mathbb{E}[X] \leq \sigma_{\mathcal{C}_m}(1)\}.$$

In particular, $\varphi(X) = \varphi(\mathbb{E}[X])$ for every $X \in \mathcal{X}$. This shows that (iv) implies (i). □

Proof of Theorem 5.5 It is easy to see that (i) implies (ii) and that (ii) implies (iii). Assume now that (iii) holds. Suppose $m \in \mathbb{R}$ is such that $\{\varphi \leq m\} \neq \emptyset$. By dilatation monotonicity of φ recorded in Proposition 2.3, we find $x \in \mathbb{R}$ such that $\varphi(x) \leq m$. Making use of dilatation monotonicity once more, we infer for all $t \geq 0$ that

$$\varphi\left(x + t(\mathbb{E}[Z] - Z)\right) = \varphi(x + t\mathbb{E}[Z]) - t\varphi(Z) \leq \varphi(x + tZ) - t\varphi(Z) = \varphi(x) \leq m.$$

As $U := \mathbb{E}[Z] - Z$ belongs to the recession cone of $\{\varphi \leq m\}$ and $\mathbb{E}[U] = 0$, Proposition 4.3 implies that $\mathrm{dom}(\sigma_{\{\varphi \leq m\}}) \subset \mathbb{R}$. By Theorem 5.3, φ is expectation invariant. In particular,

$$\begin{aligned}\varphi(X) &= \varphi\left(X - \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}Z + \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}Z\right) = \varphi\left(X - \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}Z\right) + \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}\varphi(Z)\\ &= \varphi\left(\mathbb{E}\left[X - \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}Z\right]\right) + \tfrac{\mathbb{E}[X]}{\mathbb{E}[Z]}\varphi(Z) = \varphi(0) + \tfrac{\varphi(Z)}{\mathbb{E}[Z]}\mathbb{E}[X]\end{aligned}$$

for every $X \in \mathcal{X}$. That is, φ is an affine function of the expectation as stated in (i). □

Example D.1 Let $\mathcal{X}$ and $\mathcal{X}^*$ be arbitrary, but conforming to Assumption 2.1. Define $\varphi\colon \mathcal{X} \to [0, 1]$ by

$$\varphi(X) := \begin{cases} 0 & X = 0, \\ 1 & \text{otherwise}. \end{cases}$$

Clearly, φ is proper, quasiconvex, law invariant, and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous. Moreover, for all $0 \neq x \in \mathbb{R}$, all nonconstant random variables Z and all $t \in \mathbb{R}$, $\varphi(x + tZ) = \varphi(x) = 1$. However, φ is clearly not expectation invariant.

Example D.2 Consider the space $\mathcal{X} = L^\infty$ and let $Z \in L^\infty$ satisfy $\mathbb{P}(Z = 1) = 1 - \mathbb{P}(Z = 0) = \frac{1}{2}$. The set

$$\mathcal{C} := \{x + tZ';\ x, t \in \mathbb{R},\ Z \sim Z'\}$$

is not convex, but law invariant. At last, we define $\varphi\colon L^\infty \to [0, \infty]$ by

$$\varphi(X) := \begin{cases} 0 & X \in \mathcal{C}, \\ \infty & \text{otherwise}. \end{cases}$$

Then, for all $x \in \mathbb{R}$ and $t \geq 0$,

$$\varphi(x + tZ) = 0 = \varphi(x),$$

while φ is simultaneously not expectation invariant; for a random variable U uniformly distributed over $[-1, 1]$, $\varphi(U) = \infty \neq 0 = \varphi(\mathbb{E}[U])$.

E Mathematical details of Section 5.3

The following representation result from [47] will play a crucial role in the proof of Theorem 5.7. In the terminology of [13], it shows that any consistent risk measure on L^∞ can be expressed as a minimum of adjusted Expected Shortfalls.

Lemma E.1 *([47,Theorem 3.1]) The* Expected Shortfall *of $X \in \mathcal{X}$ at level $p \in [0,1]$ is*

$$\mathrm{ES}_p(X) := \begin{cases} \frac{1}{1-p}\int_p^1 q_X(s)ds & \text{if } p < 1, \\ \inf\{x \in \mathbb{R};\ \mathbb{P}(X \le x) = 1\} & \text{if } p = 1. \end{cases}$$

Let $\varphi\colon L^\infty \to \mathbb{R}$ be a consistent risk measure. Then, for every $X \in L^\infty$,

$$\varphi(X) = \min_{Y \in \mathcal{A}_\varphi} \sup_{p \in [0,1]} \{\mathrm{ES}_p(X) - \mathrm{ES}_p(Y)\}.$$

where $\mathcal{A}_\varphi := \{Y \in L^\infty;\ \varphi(Y) \le 0\}$ denotes the acceptance set of φ.

Proposition E.2 *Let $\varphi\colon L^\infty \to \mathbb{R}$ be a consistent risk measure. Then, there is a unique, $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous, consistent risk measure $\overline{\varphi}\colon \mathcal{X} \to (-\infty, \infty]$ that extends φ.*

Proof Note that φ is dilatation monotone in the sense of [57]. In addition, by [47,Theorem 3.5], φ has the Fatou property, i.e., for every uniformly bounded sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ converging to $X \in L^\infty$ almost surely, $\varphi(X) \le \liminf_{n\to\infty} \varphi(X_n)$. Let Π denote the set of finite measurable partitions of Ω. For $X \in L^1$ and $\pi \in \Pi$ we write $\mathbb{E}[X|\pi] := \mathbb{E}[X|\sigma(\pi)]$, where $\sigma(\pi)$ is the σ-field generated by π. [57,Theorem 4] proves that the functional $\varphi^\sharp\colon L^1 \to (-\infty, \infty]$ defined by

$$\varphi^\sharp(X) := \sup_{\pi \in \Pi} \varphi(\mathbb{E}[X|\pi]),$$

is a $\sigma(L^1, L^\infty)$-lower semicontinuous, dilatation monotone in the sense of [57], cash-additive extension of φ. *A fortiori*, the restriction of $\varphi^\sharp$ to $\mathcal{X}$, denoted by $\overline{\varphi}$, is a $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuous, dilatation monotone in the sense of [57], cash-additive extension of φ. It remains to verify consistency of $\varphi^\sharp$, which implies that of $\overline{\varphi}$. By [47,Theorem B.3], it suffices to check for dilatation monotonicity in the sense of [47]. To this end, suppose $X, Y \in L^1$ satisfy $\mathbb{E}[Y|X] = X$. Let $(\pi_n)_{n\in\mathbb{N}} \subset \sigma(\mathcal{X})$ be an increasing sequence of finite measurable partitions such that $X_n = \mathbb{E}[X|\pi_n] \to X$ in L^1. For all $n \in \mathbb{N}$, $\mathbb{E}[Y|X_n] = \mathbb{E}[X|\pi_n]$ holds, which entails

$$\varphi^\sharp(Y) \ge \limsup_{n\to\infty} \varphi^\sharp(X_n) \ge \liminf_{n\to\infty} \varphi^\sharp(\mathbb{E}[X|\pi_n]) \ge \varphi^\sharp(X) = \varphi^\sharp(\mathbb{E}[Y|X]).$$

This is the desired dilatation monotonicity of $\varphi^\sharp$. Uniqueness of $\overline{\varphi}$ can be seen to be a consequence of the uniqueness statement in [57,Theorem 4]. □

Proof of Theorem 5.7 It is trivial to see that (i) implies (ii). In order to see that (ii) implies (iii), recall first that φ is dilatation monotone as observed above. Hence, we may estimate

$$a = \varphi(Z) \ge \varphi(\mathbb{E}[Z]) = \mathbb{E}[Z] = -\mathbb{E}[-Z] = -\varphi(\mathbb{E}[-Z]) \ge -\varphi(-Z) = a.$$

This means that $a = \mathbb{E}[Z]$. Set $U = Z - \mathbb{E}[Z]$ and use cash-additivity of φ to infer for every $t \ge 0$ that

$$\varphi(tU) = \varphi(tZ - t\mathbb{E}[Z]) = \varphi(tZ) - t\mathbb{E}[Z] = ta - t\mathbb{E}[Z] = 0.$$

This yields the desired implication.

Now, we claim that (iii) implies (i). We first consider the case $\mathcal{X} = L^\infty$ and fix an arbitrary $X \in L^\infty$. Using Lemma E.1, we have

$$\varphi(X) \le \inf_{t>0} \sup_{p\in[0,1]} \{\mathrm{ES}_p(X) - t\mathrm{ES}_p(U)\}. \tag{E.1}$$

As $\mathbb{E}[U] = 0$ by assumption, Lemma A.2 implies that $\mathrm{ES}_p(U) > 0$ for every $p \in (0, 1)$. Let $q \in (0, 1)$ be arbitrary and choose $t_0 > 0$ such that $\mathrm{ES}_1(X) - t_0\mathrm{ES}_q(U) \leq \mathbb{E}[X]$. Note that

$$\inf_{t>0} \sup_{p\in[0,1]} \{\mathrm{ES}_p(X) - t\mathrm{ES}_p(U)\} = \inf_{t>t_0} \sup_{p\in[0,1]} \{\mathrm{ES}_p(X) - t\mathrm{ES}_p(U)\}.$$

Moreover, for all $p \in [q, 1]$ and $t > t_0$,

$$\mathrm{ES}_p(X) - t\mathrm{ES}_p(U) \leq \mathrm{ES}_1(X) - t_0\mathrm{ES}_q(U) \leq \mathbb{E}[X] = \mathrm{ES}_0(X) - t\mathrm{ES}_0(U).$$

As a result, we get

$$\inf_{t>0} \sup_{p\in[0,1]} \mathrm{ES}_p(X) - t\mathrm{ES}_p(U) = \inf_{t>t_0} \sup_{p\in[0,q]} \mathrm{ES}_p(X) - t\mathrm{ES}_p(U). \tag{E.2}$$

Now, for all $p \in [0, q]$,

$$|\mathrm{ES}_p(X) - \mathbb{E}[X]| = -\tfrac{1}{1-p} \int_0^p q_X(s)ds + \tfrac{p}{1-p} \int_0^1 q_X(s)ds \leq \tfrac{2q}{1-q}\|X\|_\infty.$$

Combining this inequality with (E.1) and (E.2) yields

$$\varphi(X) \leq \inf_{t>t_0} \left\{\mathbb{E}[X] + \tfrac{2q}{1-q}\|X\|_\infty - \inf_{p\in[0,q]} t\mathrm{ES}_p(U)\right\} = \mathbb{E}[X] + \tfrac{2q}{1-q}\|X\|_\infty.$$

We conclude by noting that, by dilatation monotonicity,

$$\mathbb{E}[X] = \varphi(\mathbb{E}[X]) \leq \varphi(X) \leq \lim_{q\downarrow 0}\{\mathbb{E}[X] + \tfrac{2q}{1-q}\|X\|_\infty\} = \mathbb{E}[X].$$

This shows that $\varphi(X) = \mathbb{E}[X]$ whenever $X \in L^\infty$. To conclude the proof of the implication, we consider the case of a general space $\mathcal{X}$. Note that for an arbitrary finite sub-σ-field such that $\mathbb{E}[U|\mathcal{G}] \in L^\infty$ is nonconstant, dilatation monotonicity implies

$$\sup_{t\geq 0} \varphi\left(t\mathbb{E}[U|\mathcal{G}]\right) = \sup_{t\geq 0} \varphi(tU) \leq 0.$$

The preceding argument shows that φ coincides with the expectation under $\mathbb{P}$ when restricted to L^∞. By, e.g., [7,Lemma 4.1], L^∞ is dense in $\mathcal{X}$ with respect to $\sigma(\mathcal{X}, \mathcal{X}^*)$. Take a net $(X_\alpha) \subset L^\infty$ satisfying $X_\alpha \to X$ with respect to $\sigma(\mathcal{X}, \mathcal{X}^*)$. By dilatation monotonicity and $\sigma(\mathcal{X}, \mathcal{X}^*)$-lower semicontinuity,

$$\mathbb{E}[X] = \varphi(\mathbb{E}[X]) \leq \varphi(X) \leq \liminf_\alpha \varphi(X_\alpha) = \liminf_\alpha \mathbb{E}[X_\alpha] = \mathbb{E}[X].$$

This delivers (i). Clearly, (i) implies (iv). We conclude by proving that (iv) implies (iii) under the additional assumption (5.1) that $\varphi(\lambda X) \leq \lambda\varphi(X)$ for all $\lambda \in [0, 1]$ and $X \in \mathcal{X}$. To this end, let $A \in \mathcal{F}$ satisfy $\mathbb{P}(A) = \frac{1}{2}$ and set $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$. For every $\mathcal{G}$-measurable, positive, nonconstant $Y \in L^\infty$ with $\mathbb{E}[Y] = 1$ and for every $n \in \mathbb{N}$ we claim that

$$\sup\{\mathbb{E}[XY];\ X \in \mathcal{A}_\varphi,\ X \text{ is } \mathcal{G}\text{-measurable},\ \|X\|_\infty > n\} = \infty. \tag{E.3}$$

To see this, observe that

$$\sup\{\mathbb{E}[XY];\ X \in \mathcal{A}_\varphi,\ X \text{ is } \mathcal{G}\text{-measurable},\ \|X\|_\infty \leq n\} \leq n\mathbb{E}[Y] < \infty.$$

At the same time,

$$\sup\{\mathbb{E}[XY];\ X \in \mathcal{A}_\varphi,\ X \text{ is } \mathcal{G}\text{-measurable}\} = \sup_{X\in\mathcal{A}_\varphi} \mathbb{E}[\mathbb{E}[X|\mathcal{G}]Y],$$

where we used that $\mathbb{E}[X|\mathcal{G}] \in \mathcal{A}_\varphi$ holds for every $X \in \mathcal{A}_\varphi$ by dilatation monotonicity. As a consequence, by $\mathcal{G}$-measurability of Y,

$$\sup\{\mathbb{E}[XY];\ X \in \mathcal{A}_\varphi,\ X \text{ is } \mathcal{G}\text{-measurable}\} = \sup_{X \in \mathcal{A}_\varphi} \mathbb{E}[XY] = \varphi^*(Y) = \infty.$$

This delivers (E.3). Now, for $n \in \mathbb{N}$ define $Y_n = \frac{n-1}{n}\mathbf{1}_A + \frac{n+1}{n}\mathbf{1}_{A^c} \in L^\infty$ and note that Y_n is $\mathcal{G}$-measurable, positive, nonconstant, and satisfies $\mathbb{E}[Y_n] = 1$. It follows from (E.3) that we find a $\mathcal{G}$-measurable $X_n \in \mathcal{A}_\varphi \leq 0\}$ with $\|X_n\|_\infty > n$ and $\mathbb{E}[X_nY_n] \geq 1$. As $\mathbb{E}[X_n]\mathbb{E}[Y_n] = \mathbb{E}[X_n] \leq \varphi(X_n) \leq 0$ by dilatation monotonicity and cash-additivity, X_n cannot be constant by Lemma A.2. Using compactness of the appropriate unit sphere in $\mathbb{R}^2$, we can assume without loss of generality that there is a suitable $\mathcal{G}$-measurable $U \in L^\infty$ such that $U \neq 0$ and

$$\frac{X_n}{\|X_n\|_\infty} \to U.$$

By our additional assumption, for every $t > 0$ we eventually have $t\frac{X_n}{\|X_n\|_\infty} \in \mathcal{A}_\varphi$ and, thus, $tU \in \mathcal{A}_\varphi$ or, equivalently, $\varphi(tU) \leq 0$. To prove (iii), it remains to show that $\mathbb{E}[U] = 0$. To this effect, note that $\frac{X_nY_n}{\|X_n\|_\infty} \to U$. As a result, applying dilatation monotonicity again,

$$0 \geq \varphi(U) \geq \mathbb{E}[U] = \lim_{n\to\infty} \frac{\mathbb{E}[X_nY_n]}{\|X_n\|_\infty} \geq \lim_{n\to\infty} \frac{1}{\|X_n\|_\infty} = 0.$$

This concludes the proof. □

Example E.3 Suppose $\mathcal{X} = L^\infty$ and define convex law-invariant risk measures $\tau_1, \tau_2 \colon L^\infty \to \mathbb{R}$ by

$$\begin{aligned} &\tau_1(X) := \mathbb{E}[X] + 1 \\ &\tau_2(X) = \operatorname{ess\,sup}(X) = \inf\{m \in \mathbb{R} \mid X \leq m\}, \end{aligned} \quad X \in L^\infty.$$

Then $\varphi := \min\{\tau_1, \tau_2\}$ is a nonconvex consistent risk measure. Clearly, $\varphi^* \geq \tau_1^*$ and $\operatorname{dom}(\tau_1^*) = \{1\}$. In particular, $\operatorname{dom}(\varphi^*) = \{1\}$ and statement (iv) in Theorem 5.7 holds true. Nevertheless, none of the equivalent statements (i)–(iii) hold true. As an example, let $X \in L^\infty$ satisfy $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$ and note that $\varphi(X) = 1 > 0 = \mathbb{E}[X]$.

F Mathematical details of Section 5.4

Proof of Proposition 5.10 Assume that (i) holds, i.e., μ is law invariant. Its dual capacity is given by $\overline{\mu} = \alpha\overline{\nu} + (1-\alpha)\nu$. As $\alpha \neq \frac{1}{2}$, we may recover ν as

$$\nu = \tfrac{\alpha}{2\alpha-1}\mu - \tfrac{1-\alpha}{2\alpha-1}\overline{\mu}. \tag{F.1}$$

As the dual capacity $\overline{\mu}$ is also law invariant, the value of the right-hand side in (F.1) only depends on the $\mathbb{P}$-probability of its argument. This implies law invariance of ν.

Assuming (ii), we may apply Lemma 3.1, Remark 3.2, and Proposition 3.3 of [4] to the law-invariant exact capacity ν to find a family $\mathfrak{C}$ of concave functions $g \colon [0,1] \to [0,1]$ satisfying $g(0) = 0$, $g(1) = 1$, and $\nu(A) = \sup_{g\in\mathfrak{C}} g(\mathbb{P}(A))$, $A \in \mathcal{F}$. The Choquet integrals $\mathbb{E}_{g\circ\mathbb{P}}[\cdot]$ are law-invariant coherent risk measures on L^∞. By [62,Proposition 1.1] and Proposition 2.2 there is a law-invariant set $\mathcal{D}_g \subset L^1$ of probability densities such that

$$\mathbb{E}_{g\circ\mathbb{P}}[X] = \sup_{D\in\mathcal{D}_g} \mathbb{E}[DX], \quad X \in L^\infty.$$

At last, $\mathcal{D} := \bigcup_{g\in\mathfrak{C}} \mathcal{D}_g$ satisfies

$$\nu(A) = \sup_{g\in\mathfrak{C}} \mathbb{E}_{g\circ\mathbb{P}}[\mathbf{1}_A] = \sup_{D\in\mathcal{D}} \mathbb{E}[D\mathbf{1}_A], \quad A \in \mathcal{F}.$$

Suppose now that (iii) holds. Let $A \in \mathcal{F}$, $p := \mathbb{P}(A)$, and observe that $\overline{\nu}(A) = \inf_{D\in\mathcal{D}} \mathbb{E}[D\mathbf{1}_A]$. By Lemma A.2,

$$\begin{aligned}\mu(A) &= \alpha \sup_{D\in\mathcal{D}} \sup_{D'\sim D} \mathbb{E}[D'\mathbf{1}_A] + (1-\alpha) \inf_{D\in\mathcal{D}} \inf_{D'\sim D} \mathbb{E}[D'\mathbf{1}_A]\\ &= \alpha \sup_{D\in\mathcal{D}} \int_{1-p}^{1} q_D(s)ds + (1-\alpha) \inf_{D\in\mathcal{D}} \int_0^p q_D(s)\, ds.\end{aligned}$$

This proves law invariance of μ. □

Proof of Theorem 5.11 Take any $A \in \mathcal{F}$. By (5.2), we conclude that $\mathbb{E}_\mu[-\mathbf{1}_A] = \mathbb{E}_\mu[\mathbf{1}_{A^c} - 1] = \mu(A^c) - 1$, which suffices to verify the chain of implications (i) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii). Moreover, as $\mathbb{E}_\mu[0] = 0$, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) holds. Thus, it remains to prove that (iii) implies (i). By the polarisation identity in (F.1),

$$\begin{aligned}\mathbb{E}_\nu[-Z] &= \tfrac{\alpha}{2\alpha-1}\mathbb{E}_\mu[-Z] - \tfrac{1-\alpha}{2\alpha-1}\mathbb{E}_{\overline{\mu}}[-Z] = \mathbb{E}_\mu[-Z]\\ &= -\mathbb{E}_\mu[Z] = \tfrac{1-\alpha}{2\alpha-1}\mathbb{E}_{\overline{\mu}}[Z] - \tfrac{\alpha}{2\alpha-1}\mathbb{E}_\mu[Z] = -\mathbb{E}_\nu[Z].\end{aligned}$$

Using (5.2) once more, we verify $\mathbb{E}_\nu[tZ] = t\mathbb{E}_\nu[Z]$ for every $t \in \mathbb{R}$. Now, by Proposition 5.10, there exists a family $\mathcal{D} \subset L^1$ of probability densities such that, for every $A \in \mathcal{F}$,

$$\nu(A) = \sup_{D\in\mathcal{D}} \mathbb{E}[D\mathbf{1}_A].$$

Note furthermore that each $X \in L^\infty$ and each $D \in \mathcal{D}$ satisfy $\mathbb{E}_\nu[X] \geq \mathbb{E}[DX]$. By Theorem 4.1, D must be constant. This forces $\nu = \overline{\nu} = \mathbb{P}$, and consequently $\mu = \mathbb{P}$, that is, (i) holds. □

Example F.1 (1) Fix $A_0 \in \mathcal{F}$ with $\mathbb{P}(A_0) = \frac{1}{2}$ and let $\mathcal{Q}$ denote the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$ such that $\mathbb{Q}(A_0) = \frac{1}{2}$. One observes that $\{\mathbb{P}\} \subsetneq \mathcal{Q}$. Now consider the JP capacity μ represented by $\nu := \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{Q}(\cdot)$ and $\alpha = \frac{1}{3}$. μ is not law invariant and satisfies

$$\mathbb{E}_\mu[t\mathbf{1}_{A_0}] = \tfrac{1}{2}t = t\mathbb{E}_\mu[\mathbf{1}_{A_0}], \quad t \in \mathbb{R}.$$

Nevertheless, μ does not collapse to any probability measure because it lacks additivity. To see this, fix any event $B \in \mathcal{F}$ with $B \subset A_0$ and $\mathbb{P}(B) \in (0, \mathbb{P}(A))$. One verifies that

$$\nu(B) = \tfrac{1}{2},\ \nu(B^c) = 1,\ \overline{\nu}(B) = 0,\ \overline{\nu}(B^c) = \tfrac{1}{2},$$

whence $\mu(B) = \frac{1}{6}$ and $\mu(B^c) = \frac{2}{3}$ follows. Hence, $\mu(B) + \mu(B^c) < \mu(\Omega) = 1$.

(2) Consider the law-invariant capacity $\mu := T \circ \mathbb{P}$, where $T\colon [0,1] \to [0,1]$ is defined by

$$T(p) = \tfrac{1}{2}\mathbf{1}_{[\frac{1}{2},1)}(p) + \mathbf{1}_{\{1\}}(p).$$

μ is not a JP capacity and satisfies $\mathbb{E}_\mu[t\mathbf{1}_A] = \frac{1}{2}t = t\mathbb{E}_\mu[\mathbf{1}_A]$, $t \in \mathbb{R}$, for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = \frac{1}{2}$.

The key to the following lemma is to adapt the proof of [4,Proposition 3.1].

Lemma F.2 *Suppose $\alpha \neq \frac{1}{2}$ and that the preferences encoded by the functional $\varphi_{\mathcal{Z},\alpha}$ in (5.3) are weak probabilistic beliefs with respect to a $\widehat{\xi} \in \Delta$ with convex range. Moreover, assume they satisfy the monotone continuity axiom. Then $\widehat{\xi}$ is a countably additive atomless probability measure.*

Proof By [50,Lemma 1], $\mathcal{Z}$ only contains countably additive measures. Applying (F.1) with $\mu = \mu_{\mathcal{Z},\alpha}$, we see that the capacity $\nu := \max_{\mathbb{Q}\in\mathcal{Z}} \mathbb{Q}(\cdot)$ satisfies

$$\forall A, B \in \mathcal{F}: \quad \widehat{\xi}(A) = \widehat{\xi}(B) \quad \Longrightarrow \quad \nu(A) = \nu(B). \tag{F.2}$$

The compactness of $\mathcal{Z}$ and Dini's Theorem imply that $\nu(A_n) \downarrow 0$ whenever $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ satisfies $A_n \downarrow \emptyset$. Combining (5.4) and (F.2), there is a unique nondecreasing function $T : [0,1] \to [0,1]$ such that $\nu = T \circ \widehat{\xi}$. If we can prove $T \geq id_{[0,1]}$, $\widehat{\xi}$ has to be countably additive. Let $0 < m \leq n$ be integers. Invoke (5.4) repeatedly to find a finite measurable partition $\pi \subset \mathcal{F}$ of Ω such that $\widehat{\xi}(B) = \frac{1}{n}$, $B \in \pi$. Let $\mathcal{M}$ be the set of all $\pi' \subset \pi$ with cardinality m. For $\mathbb{Q} \in \mathcal{Z}$ we observe

$$\begin{aligned}\binom{n}{m} T\left(\tfrac{m}{n}\right) &= \sum_{\pi'\in\mathcal{M}} \nu\Big(\bigcup_{B\in\pi'} B\Big) \geq \sum_{\pi'\in\mathcal{M}} \sum_{B\in\pi'} \mathbb{Q}(B)\\ &= \sum_{B\in\pi} \Big(\sum_{\pi'\in\mathcal{M}:\ B\in\pi'} \mathbb{Q}(B)\Big) = \binom{n-1}{m-1}.\end{aligned}$$

Hence, $T(\frac{m}{n}) \geq \frac{m}{n}$. Next, given an arbitrary $p \in (0,1]$ and a sequence $(p_n)_{n\in\mathbb{N}}$ of rationals such that $p_n \uparrow p$, $T(p) \geq \sup_{n\in\mathbb{N}} T(p_n) \geq \sup_{n\in\mathbb{N}} p_n = p$ follows. $\square$

Lemma F.3 *Suppose ν is a law-invariant capacity. Then the functional $\psi : L^\infty \to \mathbb{R}$ defined by*

$$\psi(X) = \sup_{\xi\in\mathrm{acore}(\nu)} \int X\, d\xi \tag{F.3}$$

is law invariant as well.

Proof For the Choquet integral $\mathbb{E}_\nu$, $\xi \in \mathrm{acore}(\nu)$ fixed, and $X \in L^\infty$, $\int X\, d\xi \leq \mathbb{E}_\nu[X]$ holds. Considering the (bi)conjugate functions

$$\begin{aligned}&\mathbb{E}_\nu^*(\xi) := \sup\nolimits_{X\in L^\infty} \textstyle\int X\, d\xi - \mathbb{E}_\nu[X], \quad \xi \in \mathbf{ba},\\ &\mathbb{E}_\nu^{**}(X) := \sup\nolimits_{\xi\in\mathbf{ba}} \textstyle\int X\, d\xi - \mathbb{E}_\nu^*(\xi), \quad X \in L^\infty.\end{aligned}$$

one can therefore show that $\psi = \mathbb{E}_\nu^{**}$, i.e., ψ is the maximal convex and lower semicontinuous function $g : L^\infty \to \mathbb{R}$ with $g \leq \mathbb{E}_\nu$. Now define the functional $\Psi := \sup_{\xi\in\mathrm{acore}(\nu)} f_\xi$, where

$$f_\xi(X) := \sup_{X'\sim X} \int X'\, d\xi, \quad X \in L^\infty. \tag{F.4}$$

Clearly, $\psi \leq \Psi$. In [20,p. 16f.] it is verified that functionals of shape (F.4) are continuous, law invariant, and subadditive. Therefore Ψ is lower semicontinuous, law invariant, and subadditive. Law invariance of the Choquet integral $\mathbb{E}_\nu$ and $\psi \leq \mathbb{E}_\nu$ also shows $\Psi \leq \mathbb{E}_\nu$. The aforementioned universal property of ψ therefore implies $\Psi = \psi$. In particular, ψ is law invariant. $\square$

Proof of Theorem 5.14 Clearly, law invariance of $\varphi_{\mathcal{Z},\alpha}$ implies law invariance of $\mu_{\mathcal{Z},\alpha}$. Conversely, $\mu_{\mathcal{Z},\alpha}$ is law invariant if and only if the capacity $\nu := \max_{\xi\in\mathcal{Z}} \xi(\cdot)$ is law invariant

(Proposition 5.10). As $\mathcal{Z} = \mathrm{acore}(\tau)$ for a capacity τ, one shows that $\mathcal{Z} = \mathrm{acore}(\nu)$ as well. By Lemma F.3, the map ψ defined in (F.3) is law invariant. As for $X \in L^\infty$ we can write $\varphi_{\mathcal{Z},\alpha}(X) = \alpha\psi(X) - (1-\alpha)\psi(-X)$, the functional $\varphi_{\mathcal{Z},\alpha}$ is also law invariant. Equivalence between statements (i)–(vi) is established with slight modifications to the proof of Theorem 5.11. □

G Proofs accompanying Section 5.5

Proof of Theorem 5.16 Let $X \in \mathcal{X}$ be an optimal solution. To prove (i), let $X' \sim X$ be antimonotone with D. Note that $\mathbb{E}[DX'] \leq \mathbb{E}[DX]$ by Lemma A.2 and set

$$m = \frac{\mathbb{E}[DX] - \mathbb{E}[DX']}{\mathbb{E}[D]} \geq 0.$$

As $X \in \mathcal{C}$, we have $X' \in \mathcal{C}$ by law invariance of $\mathcal{C}$. As $\mathcal{C}$ is increasing, $X' + m \in \mathcal{C}$. Note that $\mathbb{E}[D(X'+m)] = \mathbb{E}[DX] = p$. In addition, $\varphi(X'+m) \geq \varphi(X') = \varphi(X)$ because the function φ is weakly increasing and law invariant. We conclude that $X' + m$ is an optimal solution. It remains to observe that $X' + m$ is antimonotone with D by construction.

To establish (ii), assume towards a contradiction that X is not antimonotone with D — which entails in particular that D and X are nonconstant — and take X' and m as above. The same argument as above shows that $X' + m$ is an optimal solution. From Lemma A.2 we derive

$$\mathbb{E}[DX] > \mathbb{E}[D'X],$$

which means in particular that $m > 0$. This yields $\varphi(X'+m) > \varphi(X') = \varphi(X)$ because φ is increasing and law invariant, and because $\varphi(X) \in \mathbb{R}$. However, this contradicts the optimality of X. In conclusion, X and D have to be antimonotone. □

Proof of Proposition 5.17 Let $Z \in \mathcal{X}$ be nonconstant and comonotone with D. Note that Z is not antimonotone with D due to Lemma A.2. Up to an appropriate translation, we can assume that $\mathbb{E}[Z] = 0$. Set $p = \mathbb{E}[DZ]$ and observe that $p > \mathbb{E}[D]\mathbb{E}[Z] = 0$ again by Lemma A.2. We claim that there always exist a law-invariant functional φ and a law-invariant set $\mathcal{C}$ such that $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple with the required properties and with respect to which Z is an optimal solution.

First, consider the law-invariant set $\mathcal{C} = \{X \in \mathcal{X}\,;\ \mathbb{E}[X] \leq 0\}$ and set for every $X \in \mathcal{X}$

$$\varphi(X) = \mathbb{E}[X].$$

Clearly, φ is both weakly increasing and increasing. Note that $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple and Z is an optimal solution with $\varphi(Z) \in \mathbb{R}$. This shows (iii). In addition, by Lemma A.2, any optimal solution $X \in \mathcal{X}$ that is antimonotone with D would need to satisfy

$$0 < p = \mathbb{E}[DX] \leq \mathbb{E}[D]\mathbb{E}[X] = \mathbb{E}[D]\mathbb{E}[Z] = 0,$$

which is clearly impossible. This shows that (i) holds.

Next, consider the law-invariant set $\mathcal{C} = \{Z' + m\,;\ Z' \sim Z,\ m \in \mathbb{R}\}$ and set for every $X \in \mathcal{X}$

$$\varphi(X) = \begin{cases} -|\mathbb{E}[X]| & \text{if } X \in \mathcal{C}, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{C}$ is increasing. Note that $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple and Z is an optimal solution with $\varphi(Z) \in \mathbb{R}$. This shows that (iv) holds. In addition, by Lemma A.2, any optimal solution $X \in \mathcal{X}$ that is antimonotone with D would have to satisfy

$$0 < p = \mathbb{E}[DX] \leq \mathbb{E}[D]\mathbb{E}[X] = \mathbb{E}[D]\mathbb{E}[Z] = 0,$$

which is clearly impossible. This shows that (ii) holds. □

Proof of Proposition 5.18 To show (i), take any nonconstant $Z \in \mathcal{X}$ that is comonotone with D and set $p = \mathbb{E}[DZ]$. In addition, set $\mathcal{C} = \{Z' \in \mathcal{X};\ Z' \sim Z\}$. It is clear that $\mathcal{C}$ is law invariant and that $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple with respect to which Z is optimal. If $X \in \mathcal{X}$ is another optimal solution, then we must have $X \sim Z$ as well as $\mathbb{E}[DX] = \mathbb{E}[DZ]$. As Z is nonconstant, it follows from Lemma A.2 that X cannot be antimonotone with D. To show (ii), it suffices to repeat the same argument under the additional condition that $\varphi(Z) \in \mathbb{R}$, which is possible by assumption. □

Proof of Proposition 5.19 By assumption on D, we find $k \in \mathbb{R}$ such that $\mathbb{P}(D \leq k) \in (0, 1)$ and $\mathbb{E}[D\mathbf{1}_{\{D\leq k\}}] \neq 0$. Define for every $X \in \mathcal{X}$

$$\varphi(X) = \frac{1}{\mathbb{P}(D > k)} \int_{\mathbb{P}(D\leq k)}^{1} q_X(s)ds.$$

Note that φ is both weakly increasing and increasing. Indeed, for all $X \in \mathcal{X}$ and $m > 0$ we have $\varphi(X) \in \mathbb{R}$ and $\varphi(X + m) = \varphi(X) + m > \varphi(X)$. Now, set

$$Z = a\mathbf{1}_{\{D\leq k\}} + b\mathbf{1}_{\{D>k\}} \in \mathcal{C}$$

as well as $p = \mathbb{E}[DZ]$. Note that Z is not constant and satisfies $\varphi(X) \leq b = \varphi(Z)$ for every $X \in \mathcal{C}$. As a result, $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple and Z is an optimal solution. Since, by construction, Z is not antimonotone with D, we infer that (ii) holds. In addition, take any optimal solution $X \in \mathcal{X}$ that is antimonotone with D. From $X \leq b$ and

$$\varphi(X) = \varphi(Z) = b,$$

we infer that $q_X(s) = b$ for almost every $s \in [\mathbb{P}(D \leq k), 1)$. Consequently, $q_X(s) = b$ holds for almost every $s \in (0, \mathbb{P}(D \leq k)]$ as well by antimonotonicity. As a result, we must have $X = b$, from which we deduce

$$a\mathbb{E}[D\mathbf{1}_{\{D\leq k\}}] + b\mathbb{E}[D\mathbf{1}_{\{D>k\}}] = \mathbb{E}[DZ] = \mathbb{E}[DX] = b\mathbb{E}[D].$$

Hence, $\mathbb{E}[D\mathbf{1}_{\{D\leq k\}}] = 0$, a contradiction to the choice of k. To avoid this contradiction, D has to be constant. This shows that (i) holds. □

Proof of Proposition 5.20 Let $Z \sim B$ be comonotone with D. Set $p = \mathbb{E}[DZ]$ and define for every $X \in \mathcal{X}$

$$\varphi(X) = \mathbb{E}[X].$$

Clearly, φ is both weakly increasing and increasing. Note that $(\varphi, \mathcal{C}, D, p)$ is a feasible quadruple with respect to which Z is an optimal solution with $\varphi(Z) \in \mathbb{R}$. As Z is nonconstant and comonotone with D, it follows from Lemma A.2 that Z is not antimonotone with D, showing (ii). In addition, take any optimal solution $X \in \mathcal{X}$ that is antimonotone with D. If X were nonconstant, then we would derive from Lemma A.2 that

$$p = \mathbb{E}[DX] < \mathbb{E}[D]\mathbb{E}[X] = \mathbb{E}[D]\mathbb{E}[Z] < \mathbb{E}[DZ] = p,$$

which is absurd. Hence, X must be constant and equal to $\frac{p}{\mathbb{E}[D]}$ or equivalently $\frac{\mathbb{E}[DZ]}{\mathbb{E}[D]}$. By optimality and compatibility with the expectation, $X \in \mathcal{C}$ yields

$$\frac{\mathbb{E}[DZ]}{\mathbb{E}[D]} = \mathbb{E}[X] \leq \mathbb{E}[B] = \mathbb{E}[Z].$$

This implies $\mathbb{E}[DZ] \leq \mathbb{E}[D]\mathbb{E}[Z]$, which is, however, in contrast to the comonotonicity between Z and D by Lemma A.2. This shows that (i) holds. □

References

1. Aliprantis, C.D., Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics. American Mathematical Society, Providence (2003)
2. Allais, M.: Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. Econometrica **21**(4), 503–546 (1953)
3. Amarante, M.: Bipolar behavior of submodular, law-invariant capacities. STRM (2021). https://doi.org/10.1515/strm-2020-0025
4. Aouani, Z., Chateauneuf, A.: Exact capacities and star-shaped distorted probabilities. Math. Soc. Sci. (2008). https://doi.org/10.1016/j.mathsocsci.2008.01.006
5. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. Math. Finance (1999). https://doi.org/10.1111/1467-9965.00068
6. Bäuerle, N., Müller, A.: Stochastic orders and risk measures: consistency and bounds. Insur. Math. Econ. (2006). https://doi.org/10.1016/j.insmatheco.2005.08.003
7. Bellini, F., Koch-Medina, P., Munari, C., Svindland, G.: Law-invariant functionals on general spaces of random variables. SIAM J. Financ. Math. (2021). https://doi.org/10.1137/20M1341258
8. Bellini, F., Koch-Medina, P., Munari, C., Svindland, G.: Law-invariant functionals that collapse to the mean. Insur. Math. Econ. (2021). https://doi.org/10.1016/j.insmatheco.2021.03.002
9. Borch, K.H.: Economics of Insurance. North Holland, Amsterdam (1974)
10. Bühlmann, H.: Mathematical Methods in Risk Theory. Springer, Berlin (1970)
11. Brutti Righi, M.: A composition between risk and deviation measures. Ann. Oper. Res. (2019). https://doi.org/10.1007/s10479-018-2913-0
12. Burgert, C., Rüschendorf, L.: On the optimal risk allocation problem. Stat. Decis. (2006). https://doi.org/10.1524/stnd.2006.24.1.153
13. Burzoni, M., Munari, C., Wang, R.: Adjusted expected shortfall. J. Bank. Financ. (2022). https://doi.org/10.1016/j.jbankfin.2021.106297
14. Carlier, G., Dana, R.-A.: Law invariant concave utility functions and optimization problems with monotonicity and comonotonicity constraints. Stat. Decis. (2006). https://doi.org/10.1524/stnd.2006.24.1.127
15. Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C., Wang, R.: Star-shaped risk measures. Preprint (2021). arXiv:2103.15790v1
16. Castagnoli, E., Maccheroni, F., Marinacci, M.: Choquet insurance pricing: A caveat. Math. Financ. (2004). https://doi.org/10.1111/j.0960-1627.2004.00201.x
17. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M.: Stochastic dominance analysis without the independence axiom. Manag. Sci. (2017). https://doi.org/10.1287/mnsc.2015.2388
18. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L.: Risk measures: Rationality and diversification. Math. Financ. (2011). https://doi.org/10.1111/j.1467-9965.2010.00450.x
19. Chateauneuf, A., Eichberger, J., Grant, S.: Choice under uncertainty with the best and worst in mind: Neo-additive capacities. J. Econ. Theory (2007). https://doi.org/10.1016/j.jet.2007.01.017
20. Chen, S., Gao, N., Leung, D. H., Li, L.: Automatic Fatou property of law-invariant risk measures. Preprint (2021). arXiv:2107.08109v1
21. Chen, S., Gao, N., Leung, D. H., Li, L.: Do law-invariant linear functionals collapse to the mean?. Preprint (2021). arXiv:2107.11239v2
22. De Giorgi, E.: Reward-risk portfolio selection and stochastic dominance. J. Bank. Financ. (2005). https://doi.org/10.1016/j.jbankfin.2004.05.027
23. Delbaen, F.: Commonotonicity and time-consistency for Lebesgue-continuous monetary utility functions. Financ. Stoch. (2021). https://doi.org/10.1007/s00780-021-00459-2
24. Drapeau, S., Kupper, M.: Risk preferences and their robust representation. Math. Oper. Res. (2013). https://doi.org/10.1287/moor.1120.0560

25. Fink, A.M., Jodeit, M., Jr.: On Chebyshev's other inequality. Inst. Math. Stat. Lecture Notes Monograph Ser. (1984). https://doi.org/10.1214/lnms/1215465637
26. Fishburn, P.C.: Decision and Value Theory. Wiley, Hoboken (1964)
27. Föllmer, H., Schied, A.: Stochastic Finance: An Introduction in Discrete Time, 4th edn. De Gruyter, Berlin (2016)
28. Frittelli, M., Rosazza Gianin, E.: Law invariant convex risk measures. Adv. Math. Econ. (2005). https://doi.org/10.1007/4-431-27233-X_2
29. Gao, N., Leung, D., Munari, C., Xanthos, F.: Fatou property, representations, and extensions of law-invariant risk measures on general Orlicz spaces. Financ. Stoch. (2018). https://doi.org/10.1007/s00780-018-0357-7
30. Gao, N., Munari, C.: Surplus-invariant risk measures. Math. Oper. Res. (2020). https://doi.org/10.1287/moor.2019.1035
31. Ghirardato, P., Marinacci, M.: Range convexity and ambiguity averse preferences. Econ. Theory (2001). https://doi.org/10.1007/PL00004118
32. Ghirardato, P., Maccheroni, F., Marinacci, M.: Differentiating ambiguity and ambiguity attitude. J. Econ. Theory (2004). https://doi.org/10.1016/j.jet.2003.12.004
33. Grechuk, B., Molyboha, A., Zabarankin, M.: Maximum entropy principle with general deviation measures. Math. Oper. Res. (2009). https://doi.org/10.1287/moor.1090.0377
34. He, H., Zhou, X.Y.: Portfolio choice via quantiles. Math. Financ. (2011). https://doi.org/10.1111/j.1467-9965.2010.00432.x
35. Jaffray, J.-Y., Philippe, F.: On the existence of subjective upper and lower probabilities. Math. Oper. Res. (1997). https://doi.org/10.1287/moor.22.1.165
36. Jouini, E., Schachermayer, W., Touzi, N.: Law invariant risk measures have the Fatou property. Adv. Math. Econ. (2006). https://doi.org/10.1007/4-431-34342-3_4
37. Kadane, J.B., Wasserman, L.: Symmetric, coherent, Choquet capacities. Ann. Stat. (1996). https://doi.org/10.1214/aos/1032526967
38. Krätschmer, V., Schied, A., Zähle, H.: Comparative and qualitative robustness for law-invariant risk measures. Financ. Stoch. (2014). https://doi.org/10.1007/s00780-013-0225-4
39. Kusuoka, S.: On law invariant coherent risk measures. Adv. Math. Econ. (2001). https://doi.org/10.1007/978-4-431-67891-5_4
40. Leitner, J.: A short note on second-order stochastic dominance preserving coherent risk measures. Math. Financ. (2005). https://doi.org/10.1111/j.1467-9965.2005.00255.x
41. Liebrich, F.-B.: Risk sharing under heterogeneous beliefs without convexity. Preprint (2021). arXiv:2108.05791v2
42. Liebrich, F.-B., Svindland, G.: Efficient allocations under law-invariance: a unifying approach. J. Math. Econ. (2019). https://doi.org/10.1016/j.jmateco.2019.05.002
43. Liebrich, F.-B., Svindland, G.: Risk sharing for capital requirements with multidimensional security markets. Financ. Stoch. (2019). https://doi.org/10.1007/s00780-019-00402-6
44. Liu, F., Cai, J., Lemieux, C., Wang, R.: Convex risk functionals: representation and applications. Insur. Math. Econ. (2020). https://doi.org/10.1016/j.insmatheco.2019.10.007
45. Luxemburg, W.A.J.: Rearrangement-invariant Banach function spaces. Queen's Pap. Pure Appl. Math. **10**, 83–144 (1967)
46. Machina, M.J., Schmeidler, D.: A more robust definition of subjective probability. Econometrica (1992). https://doi.org/10.2307/2951565
47. Mao, T., Wang, R.: Risk aversion in regulatory capital principles. SIAM J. Financ. Math. (2020). https://doi.org/10.1137/18M121842X
48. Marinacci, M.: Upper probabilities and additivity. Sankhyā Indian J. Stat. **61**(3), 358–361 (1999)
49. Marinacci, M.: A uniqueness theorem for convex-ranged probabilities. Decis. Econ. Finance (2000). https://doi.org/10.1007/s102030070003
50. Marinacci, M.: Probabilistic sophistication and multiple priors. Econometrica (2002). https://doi.org/10.1111/1468-0262.00303
51. Marinacci, M., Montrucchio, L.: Introduction to the mathematics of ambiguity. In: *Uncertainty in Economic Theory: a Collection of Essays in Honor of David Schmeidler's 65th Birthday*. Routledge (2004)
52. Mastrogiacomo, E., Rosazza Gianin, E.: Portfolio optimization with quasiconvex risk measures. Math. Oper. Res. (2015). https://doi.org/10.1287/moor.2015.0711
53. Nendel, M., Riedel, F., Schmeck, M.D.: A decomposition of general premium principles into risk and deviation. Insur. Math. Econ. (2021). https://doi.org/10.1016/j.insmatheco.2021.05.006
54. Ogryczak, W., Ruszczyński, A.: On consistency of stochastic dominance and mean-semideviation models. Math. Program. **89**(2), 217–232 (2001)

55. Ogryczak, W., Ruszczyński, A.: Dual stochastic dominance and related mean-risk models. SIAM J. Optim. (2002). https://doi.org/10.1137/S1052623400375075
56. Quirk, J.P., Saposnik, R.: Admissibility and measurable utility functions. Rev. Econ. Stud. (1962). https://doi.org/10.2307/2295819
57. Rahsepar, M., Xanthos, F.: On the extension property of dilatation monotone risk measures. STRM (2020). https://doi.org/10.1515/strm-2020-0006
58. Rockafellar, R., Uryasev, S., Zabarankin, M.: Generalized deviations in risk analysis. Financ. Stoch. (2006). https://doi.org/10.1007/s00780-005-0165-8
59. Rüschendorf, L., Vanduffel, S.: On the construction of optimal payoffs. Decis. Econ. Finance (2020). https://doi.org/10.1007/s10203-019-00272-9
60. Schied, A.: On the Neyman–Pearson problem for law-invariant risk measures and robust utility functionals. Ann. Appl. Probab. (2004). https://doi.org/10.1214/105051604000000341
61. Schmeidler, D.: Integral representation without additivity. Proc. Am. Math. Soc. (1986). https://doi.org/10.2307/2046508
62. Svindland, G.: Continuity properties of law-invariant (quasi-)convex risk functions on L^∞. Math. Fin. Econ. (2010). https://doi.org/10.1007/s11579-010-0026-x
63. Wakker, P.P., Yang, J.: Concave/convex weighting and utility functions for risk: a new light on classical theorems. Insur. Math. Econ. (2021). https://doi.org/10.1016/j.insmatheco.2021.07.002
64. Wang, S.S.: A class of distortion operators for pricing financial and insurance risks. J. Risk Insur. (2000). https://doi.org/10.2307/253675
65. Wang, S.S.: A universal framework for pricing financial and insurance risks. ASTIN Bull. (2002). https://doi.org/10.2143/AST.32.2.1027
66. Wang, S.S., Young, V.R., Panjer, H.H.: Axiomatic characterization of insurance prices. Insur. Math. Econ. (1997). https://doi.org/10.1016/S0167-6687(97)00031-0
67. Wang, R., Wu, Q.: Dependence and risk attitudes: an equivalence. Preprint (2020). https://doi.org/10.2139/ssrn.3707709
68. Xu, Z.Q.: A new characterization of comonotonicity and its application in behavioral finance. J. Math. Anal. Appl. (2014). https://doi.org/10.1016/j.jmaa.2014.03.053
69. Xu, Z.Q.: A note on the quantile formulation. Math. Financ. (2016). https://doi.org/10.1111/mafi.12072
70. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2002)