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LAW OF LARGE NUMBERS AND CENTRAL LIMIT THEOREM FOR LINEAR CHEMICAL REACTIONS WITH DIFFUSION

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Two mathematical models of chemical reactions with diffusion for a single reactant in a one-dimensional volume are compared, namely, the deterministic and the stochastic models. The deterministic model is given by a partial differential equation, the stochastic one by a space-time jump Markov process. By the law of large numbers the consistency of the two models is proved. The deviation of the stochastic model from the deterministic model is estimated by a central limit theorem. This limit is a distribution-valued Gauss–Markov process and can be represented as the mild solution of a certain stochastic partial differential equation.

0. Introduction.

0.1. Mathematical models of chemical reactions. Mathematical models of chemical reactions have been described by Gardiner, McNeil, Walls, and Matheson [16], Haken [18], Nicolis and Prigogine [31], and Arnold [3]. Following Arnold [3] there are two main principles according to which reactions in a spatial domain are modeled:

(1) global description (i.e., without diffusion, spatially homogeneous, or "well-stirred" case) versus local description (i.e., including diffusion, spatially inhomogeneous case);
(2) deterministic description (macroscopic, phenomenological, in terms of concentrations) versus stochastic description (on the level of particles, taking into account internal fluctuations).

The combination of these two principles gives rise to four mathematical models, namely,

G.1 global deterministic model—ordinary differential equation;
G.2 global stochastic model—jump Markov process;
L.1 local deterministic model—partial differential equation;
L.2 local stochastic model—space-time jump Markov process.

The relation between the two global models G.1 and G.2 has been thoroughly investigated by Kurtz in a number of papers (cf. [27], [28] and also references therein), and the consistency of G.1 and L.1 as well as of G.2 and L.2 was proved by Arnold [3]. Since the global models will not concern us here we shall only review the mathematical details of the local models as given by Arnold in [3].

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Following Arnold [3] and Arnold and Theodosopulu [6], we want to be conceptual rather than computational and, consequently, shall only deal with the case of one chemical reactant in a one-dimensional volume. Moreover, we shall assume reflection at the boundary (zero flux boundary conditions) in both models and that the reaction is linear.

L.*1.* *Local deterministic model.* Let $C(r) := b(r) - d(r) = c_1 r + c_0$, $r \in \mathbb{R}$, $c_0 \geq 0$, $b(r) = br + c_0$, and $d(r) = dr$, $b, d \geq 0$. Δ denotes the Laplacian and $D > 0$ the diffusion coefficient. Then the concentration $X(t,q)$, $q \in [0,1]$, is given by the following PDE:

$$\begin{aligned} &\frac{\partial}{\partial t} X(t,q) = D\Delta X(t,q) + C(X(t,q)), \\ (0.1) \qquad &\frac{\partial}{\partial q} X(t,0) = \frac{\partial}{\partial q} X(t,1) = 0, \\ &X_0 \in H_2 := \{ y \in H^2 \colon \dot{y}(0) = \dot{y}(1) = 0 \}, \end{aligned}$$

where H^2 is the space of real valued functions on $[0,1]$, twice differentiable in the generalized sense with square integrable second derivative. Then to any T, $\rho_0 \in \mathbb{R}_+$ there is a ρ_T such that if $0 \leq X_0(q) \leq \rho_0$ then there exists a unique global solution $X \in \mathscr{C}^1([0,\infty), L_2(0,1) \cap \mathscr{C}([0,\infty), H_2)$ such that for all $t \in [0,T]$, $T > 0$, $0 \leq X(t,q) \leq \rho_T$ (Kuiper [25], Arnold and Theodosopulu [6]), where $L_2(0,1)$ is the space of real valued square integrable functions on $[0,1]$.

We shall consider $D\Delta$ here (and throughout the paper) as a closed operator with domain H_2 from (0.1), and let $U(t)$ be the semigroup generated by $A := D\Delta + c_1$. Then, by "variation of constants"

$$X(t) = U(t)X_0 + \int_{[0,t]} U(t-s) c_0 \, ds. \tag{0.2}$$

L.*2.* *Local stochastic model.* Divide the volume V into N cells of equal size $v = V/N$, where neighboring cells are linked by diffusion and in each cell reaction is going on. We set:

$$X_{\#,t}^{v,N,j} := \text{number of particles of the reactant } X \text{ in the } j\text{th cell at time } t,$$

where the superscript indicates that for each cell the process depends essentially on the two parameters v (cell size) and N (number of cells), since the third parameter, the volume V, is equal to vN. Hence,

$$X_{\#,t}^{v,N} := \left(X_{\#,t}^{v,N,1}, \ldots, X_{\#,t}^{v,N,N} \right)$$

is modeled as a jump Markov process with state space $\mathbb{N}^N$ and the following transition intensities:

$$
(0.3)\qquad
\begin{aligned}
&\left.\begin{aligned} p_{k,k+e_j} &= vb\left(\frac{k_j}{v}\right) \\ p_{k,k-e_j} &= vd\left(\frac{k_j}{v}\right)\end{aligned}\right\} && j = 1,\dots,N,\\
&p_{k,k+e_{j+1}-e_j} = DN^2k_j, && j = 1,\dots,N-1,\\
&p_{k,k+e_{j-1}-e_j} = DN^2k_j, && j = 2,\dots,N,\\
&p_{k,m} = 0, && \text{otherwise}.
\end{aligned}
$$

$k = (k_1,\dots,k_j,\dots,k_N)$, $k \pm e_j, k + e_{j\pm 1} - e_j, k + m \in \mathbb{N}^N$, e_j the jth unit vector in $\mathbb{R}^N$, and the birth and death rates are the functions from L.1. Note that particles reflect at the boundary.

We shall assume that $X^{v,N}_{\#,t}$ is defined on some probability space $(\Omega, \mathscr{F}, \mathscr{F}^{v,N}_t, P)$, where $\mathscr{F}^{v,N}_t := \sigma(X^{v,N}_{\#,s},\ s \le t)$. Under the above assumptions the probabilities

$$
P_k(t) := P\{X^{v,N}_{\#,t} = k\}, \qquad k \in \mathbb{N},
$$

are the unique solution of Kolmogorov's backward equation (which is called in the application-oriented literature the "multivariate master equation") (cf. Arnold [3]).

In order to compare L.1 and L.2 we map the volume onto the interval $[0,1]$, i.e., we divide by V, and the jth cell onto $((j-1)/N, j/N]$. To get the density in each cell we divide $X^{v,N}_{\#,t}$ by v and, consequently, the description of the local stochastic model can be given by

$$
(0.4)\qquad X^{v,N}(t,q) := \frac{X^{v,N,j}_{\#,t}}{v}, \qquad q \in \left(\frac{j-1}{N}, \frac{j}{N}\right], \ j = 1,\dots,N.
$$

Hence, $X^{v,N}$ is a process with values in the space of real valued cadlag step functions on $[0,1]$ with constancy intervals $((j-1)/N, j/N]$, $j = 1,\dots,N$. This space will be denoted by $\mathbb{H}^N$. We abbreviate

$$
(H_0, \langle\cdot,\cdot\rangle_0) := (L_2(0,1), \langle\cdot,\cdot\rangle_0),
$$

where $\langle\cdot,\cdot\rangle_0$ is the standard scalar product on $L_2(0,1)$, and we will denote the corresponding norm by $|\cdot|_0$. Then we have

$$
\begin{aligned}
X^{v,N}(t) &\in \mathbb{H}^N \subset H_0,\\
X(t) &\in H_2 \subset H_0,
\end{aligned}
$$

with $X(t)$ from (0.1) or (0.2). This means that we can compare $X^{v,N}$ and X as processes with values in H_0.

0.2. Statement of the problem. Following Arnold and Theodosopulu [6] and Kurtz [27] we represent both $X^{v,N}$ and $X^{v,N} - X$ as solutions to stochastic evolution equations driven by a martingale:

(i) Define

$$\Delta_N\colon H_0 \to \mathbb{H}^N \subset H_0$$

$$(0.5)\qquad (\Delta_N y)(q) := N^3\int_{j/N}^{j+1/N} y(p)\,dp - 2\int_{j-1/N}^{j/N} y(p)\,dp + \int_{j-2/N}^{j-1/N} y(p)\,dp,$$

$$q \in \left(\frac{j-1}{N}, \frac{j}{N}\right],\ j = 1,\dots,N$$

$$A_N(t) := D\Delta_N + c_1, \qquad U_N(t) := \exp(tA_N),$$

where we set

$$y(q) := y(-q), \qquad q \in \left[-\frac{1}{N}, 0\right],$$

$$y(q) := y(2-q), \qquad q \in \left[1, 1 + \frac{1}{N}\right].$$

This last definition reflects the zero flux boundary condition in the approximation scheme Δ_N, and we obtain Δ_N is self-adjoint, dissipative and $|(\Delta_N - \Delta)\varphi|_0 \to 0$ for all $\varphi \in H_2^c := H_2 \cap \mathscr{C}^2[0,1]$ (Kotelenez [24]). Since both $U(t)$ and $U_N(t)$ are bounded by e^{tc_1} this entails the strong convergence of $U_N(t)$ to $U(t)$ uniformly on bounded intervals by the Trotter–Kato theorem (Davies [12], Corollary 3.18 and Kato [22]).

(ii) Set

$$(0.6)\qquad Z^{v,N}(t) := X^{v,N}(t) - X_0^{v,N} - \int_0^t A_N X^{v,N}(s)\,ds - tc_0.$$

By the subsequent Lemma 0.1 $Z^{v,N}$ is a square integrable cadlag martingale provided $X_0^{v,N}$ is a.s. uniformly bounded on $\mathbb{H}^N$.

Variation of constants yields:

$$(0.7)\quad X^{v,N}(t) = U_N(t)X_0^{v,N} + \int_0^t U^N(t-s)\,dZ^{v,N}(s) + \int_0^t U^N(t-s)c_0\,ds$$

and

$$(0.8)\quad X^{v,N}(t) - X(t) = U_N(t)\left(X_0^{v,N} - X_0\right) + \int_0^t U_N(t-s)\,dZ^{v,N}(s) + \varepsilon_N(t)$$

with

$$\varepsilon_N(t) := [U_N(t) - U(t)]X_0 + \int_0^t [U_N(t-s) - U(t-s)]c_0\,ds$$

or

$$(0.9)\quad X^{v,N}(t) - X(t) = U(t)\left(X_0^{v,N} - X_0\right) + \int_0^t U(t-s)\,dZ^{v,N}(s) + \delta_N(t)$$

with

$$\delta_N(t) := \int_0^t (A_N - A)X^{v,N}(s)\,ds.$$

At this stage (0.9) is just a formal expression since, in the definition of δ_N, $X^{v,N}$

is not in the domain of A. Therefore, we construct an extension of A to $\tilde{A}$ on Hilbert distribution spaces. Set

$$H_\alpha := \mathscr{D}\big((-D\Delta)^{\alpha/2}\big), \qquad \alpha \in \mathbb{R}_+,$$

where $(-D\Delta)^{\alpha/2}$ (the $\alpha/2$th power of $(-D\Delta)$) is defined through the spectral resolution of $(-D\Delta)$ (cf. Yosida [33]), and $\mathscr{D}(\cdot)$ denotes the domain of the operator. Setting

$$\langle \varphi, \psi \rangle_\alpha := \Big\langle (I - D\Delta)^{\alpha/2}\varphi, (I - D\Delta)^{\alpha/2}\psi \Big\rangle_0, \qquad \varphi, \psi \in H_\alpha,$$

I being the identity operator, then $(H_\alpha, \langle \cdot, \cdot \rangle_\alpha)$ becomes a real separable Hilbert space. Since

$$\begin{Bmatrix} \phi_n := \sqrt{2}\cos(n\pi(\cdot)), & n \ge 1 \\ \phi_0 := 1 & \end{Bmatrix}$$

is a complete orthonormal system (CONS) of eigenvectors of Δ we have

$$H_\alpha := \left\{ f \in H_0 = L_2(0,1) \colon \sum_{n=0}^{\infty} \langle f, \phi_n \rangle_0^2 (1 + n^2)^\alpha < \infty \right\}$$

and

$$\langle \varphi, \psi \rangle_\alpha = \sum_n \langle \varphi, \phi_n \rangle_0 \langle \varphi, \phi_n \rangle_0 (1 + Dn^2\pi^2)^\alpha.$$

Identify H_0 with its strong dual H_0' and extend the scalar product $\langle \cdot, \cdot \rangle_0$ to the dual pairing $(\cdot, \cdot)$ between H_α and its strong dual $H_\alpha' =: H_{-\alpha}$. For $\alpha \in \mathbb{R}$ let

$$l_{2,\alpha} := \left\{ (a_n) \in R^\infty \colon \sum_n a_n^2 (1 + n^2)^\alpha < \infty \right\}.$$

Then we have

$$l_{2,\alpha} \cong H_\alpha, \qquad \alpha \in \mathbb{R}.$$

Thus, we obtain

$$H_\alpha \subset H_0 = H_0' \subset H_{-\alpha}, \qquad \alpha \in \mathbb{R}_+ \tag{0.10}$$

is isomorphic to

$$l_{2,\alpha} \subset l_{2,0} \subset l_{2,-\alpha},$$

and we easily see that the imbedding

$$H_\alpha \hookrightarrow H_\beta$$

is Hilbert–Schmidt iff $\alpha > \beta + \frac{1}{2}$. (Note that this is just Maurin's theorem for the standard Hilbert–Sobolev spaces $H^\alpha(0,1)$; cf. Adams [1]).

Moreover, for $\alpha \ge 0$ and $\varphi, \psi \in H_0$,

$$\langle \varphi, \psi \rangle_{-\alpha} = \Big\langle (I - D\Delta)^{-\alpha/2}\varphi, (I - D\Delta)^{-\alpha/2}\psi \Big\rangle_0$$

by definition of H_α and the dual norm, and $U(t) = e^{D\Delta t}e^{c_1 t}$ and $(I - D\Delta)^{-\alpha/2}$

commute. Therefore $U(t)$ is extendible to a strongly continuous semigroup $U_{-\alpha}(t)$ on $H_{-\alpha}$ s.t.

$$|U_{-\alpha}(t)|_{\mathscr{L}(H_{-\alpha})} = |U(t)|_{\mathscr{L}(H_0)} \le e^{c_1 t} \tag{0.11}$$

(cf. Kantorovič and Akilov [21]).

In (0.11) $\mathscr{L}(H_{-\alpha})$ denotes the usual operator norm on $H_{-\alpha}$, $\alpha \ge 0$, and we used the fact that $D\Delta$ is dissipative. Let $A_{-\alpha}$ denote the generator of $U_{-\alpha}(t)$. Since $|\cdot|_{-\alpha+2}$ is equivalent to the graph norm of $A_{-\alpha}$ we have

$$D(A_{-\alpha}) = H_{-\alpha+2}. \tag{0.12}$$

Consequently, (0.9) has meaning as an equation on $H_{-\alpha}$ for $\alpha \ge 2$ if we substitute $A_{-\alpha}$ for A in the definition of δ_N.

In what follows we will just write $\tilde{A}$ and $\tilde{U}(t)$ for any of the extended operators $A_{-\alpha}, U_{-\alpha}(t)$, $\alpha > 0$.

Thus, returning to (0.8) and (0.9), we see that the limit behaviour of $X^{v,N} - X$ and $Y^{v,N} := \gamma(v, N)(X^{v,N} - X)$ (for a suitable renormalization constant $\gamma(v, N)$) is essentially a consequence of the limit behaviour of the (locally s.i.c.) martingales $Z^{v,N}$ and $M^{v,N} := \gamma(v, N)Z^{v,N}$ and of the properties of the stochastic convolution integral in (0.8) (resp. (0.9)) provided that ε_N and δ_N suitably tend to 0. The limit theorem for $Y^{v,N}$ (Theorem 3.1) will be called the central limit theorem (CLT).

0.3. Main lemmas. Let $\delta(\cdot)$ denote the Fréchet derivative, $\mathscr{A}^{v,N}$ the weak infinitesimal operator of the pair process $(X^{v,N}, Z^{v,N})$ with state space $\mathbb{H}^{2N}$ and $Z^{v,N}$ from (0.6), $\lambda^{v,N}(x)$ the waiting time parameter, and $\sigma^{v,N}(x, dw)$ the jump distribution function of $X^{v,N}$.

LEMMA 0.1 (Kurtz [27]). *Let h be a bounded continuously differentiable function of $z \in \mathbb{H}^N$. Then h is in the domain of the weak infinitesimal operator $\mathscr{A}^{v,N}$ and*

$$\begin{aligned} &E\left(h(Z^{v,N}(t+s)) - h(Z^{v,N}(t)) \middle| \mathscr{F}_t^{v,N}\right) \\ &\qquad \left(\mathscr{F}_t^{v,N} := \sigma\left(X^{v,N}(u),\, u \in [0,t]\right)\right) \\ &= \int_{[t,t+s]} E\Bigg(\lambda^{v,N}(X^{v,N}(u)) \\ &\qquad \cdot \int_{\mathbb{H}_N} \Big[h(w - X^{v,N}(u) + Z^{v,N}(u)) - h(Z^{v,N}(u)) \\ &\qquad\qquad - \left\langle w - X^{v,N}(u), \delta h(Z^{v,N}(u))\right\rangle_0\Big] \\ &\qquad \cdot \sigma^{v,N}(X^{v,N}(u), dw) \Big| \mathscr{F}_t^{v,N}\Bigg)\, du \end{aligned} \tag{0.13}$$

(cf. *also Arnold and Theodosopulu* [6]).

Let H be a real separable Hilbert space, M an H-valued square integrable cadlag martingale and $V(t)$ a strongly continuous (s.c.) semigroup of operators on H (cf. Curtain and Pritchard [10]).

DEFINITION 0.1. $\int_{[0,t]} V(t-s)\,dM(s)$ is called a stochastic convolution-type integral, which we will denote by stochastic $*$-integral.

DEFINITION 0.2. A s.c. semigroup of operators $V(t)$ is of contraction-type on $[0,T]$ if there is a $\mu \in [0,\infty)$ such that

$$|V(t)|_{L(H)} \le e^{\mu t} \quad \text{for all } t \in [0,T].$$

REMARK 0.1. From Theorem 2 in Kotelenez [23] (also see this reference for a more detailed description of the stochastic $*$-integral) we have for the contraction-type case, abbreviating $\int V\,dM := \int_{[0,\cdot]} V(\cdot - s)\,dM(s)$:

If M is cadlag, then $\int V\,dM$ has a cadlag version;

If M is continuous, then $\int V\,dM$ has a continuous version.

Note that $U(t)$ and $\tilde{U}(t)$ are of contraction-type by (0.11).

The following lemma is a special case of Theorem 1 in Kotelenez [23].

LEMMA 0.2. *If $V(t)$ is of contraction-type, then for all $\delta > 0$:*

$$P\left\{\sup_{0\le t\le T}\left|\int_{[0,t]} V(t-s)\,dM(s)\right|_H \ge 2\delta\right\} < \frac{e^{4\mu T}}{\delta^2} E|M(T)|_H^2. \tag{0.14}$$

Finally, we give an easily verifiable estimate on the speed of convergence of $\Delta_N \to \Delta$. Let $|||\cdot|||$ denote the sup-norm *in $r \in [0,1]$ and $\varphi^{(i)}$ the ith derivative of $\varphi \in H_0$ (if it exists). Then there is constant $c < \infty$* s.t. *for all sufficiently smooth $\varphi \in D(\Delta)$.*

$$|||(\Delta_N - \Delta)\varphi||| \le c\min\left\{|||\varphi^{(2)}|||, \frac{1}{N}|||\varphi^{(3)}|||\right\}. \tag{0.15}$$

Concluding remarks. 1. The choice of the $H_{-\alpha}$ rather than $H^{-\alpha} := (H_0^\alpha)^*$ is determined by the need to extend $U(t)$ and A, in order to arrive at an SPDE for the CLT in the Hilbert distribution space in which the limit of the martingale sequence $M^{v,N}(t)$ defines a (σ-additive) measure $M(t)$. ($H_0^\alpha = H_0^\alpha(0,1)$) is the subspace of $H^\alpha(0,1)$ generated by the C^∞-functions with compact support in $(0,1)$ (cf. Adams [1].) A representation of the elements from $H_{-\alpha}$ can be found in DaPrato and Grisvard [11]; e.g., H_{-2} can be represented as the quotient space $(H_0)^2/G_{D\Delta}$, where $G_{D\Delta}$ is the graph of $D\Delta$.

2. The LLN in H_0 can also be proved by our method in the case of the nonlinear (polynomial) reaction provided the leading coefficient in the reaction polynomial $f(r)$ is negative (Kotelenez [24]).

3. $X^{v,N}$ can be looked at as a branching random walk with immigration. Our CLT is related in spirit to results obtained by Martin-Löf [30] and, especially, by Holley and Stroock [19]. Holley and Stroock (loc. cit.) have investigated scaling limit theorems for a sequence of critical branching Brownian motions on the Schwartz space $\mathscr{S}'$. The limit in their CLT is an infinite-dimensional Ornstein–Uhlenbeck process (OUP) which is characterized by $\frac{1}{2}\Delta$, the unbounded "drift," generating a contraction semigroup $T(t)$, and the identity operator as bounded "diffusion coefficient." In our case the limit is characterized by the unbounded "drift" $D\tilde{\Delta} + c_1$, generating $\tilde{U}(t)$, and the unbounded "diffusion coefficient" $\hat{F}^{(1/2)*}(t)$, which measures the intensity of the fluctuations around the deterministic limit X in terms of the Laplacian plus chemical reaction. Apart from the fact that Holley and Stroock look only at the critical case ($c_1 = 0$) without immigration ($c_0 = 0$), they simultaneously scale space and time and arrive at a bounded "diffusion coefficient." Moreover, they have a spatially continuous model (Brownian motion and just one operator $A = \frac{1}{2}\Delta$), whereas we have a sequence of discrete models (branching random walks linked with a sequence of operators $A_N \to A$) approximating the continuous model. Accordingly, our sequence $X^{v,N}$ depends on two parameters, the cell size v and the number of cells N, and we derive under the general hypothesis $N \to \infty$ the LLN (thermodynamic limit—cf. Nicolis and Prigogine [31]) in H_0 if $N^2/v \to \infty$, in $H_{-\alpha}$, $\alpha \in [2, \frac{5}{2}]$, if $v \to \infty$, and in $H_{-\alpha}$, $\alpha > \frac{5}{2}$, if $vN \to \infty$, i.e., the cell size v can be kept constant. Intuitively, this means that for (fast) growing cells reaction in the model becomes dominant and smoothes $X^{v,N}$ so that it becomes close to X even in the stronger norms. Moreover, we obtain the CLT in $H_{-\alpha}$, $\alpha > \frac{7}{2}$ if $v/N \to 0$. Consequently, the two parameters in our setup give us more freedom concerning the limit behaviour of our discrete models.

4. The mathematical framework in our approach is the calculus of evolution equations on Hilbert space which on the one hand allows us to make direct use of weak compactness criteria for processes with values in complete separable metric spaces (cf. Billingsley [7], Kurtz [28]) and on the other hand yields an easy and straightforward representation of the OUP as the mild solution of an SPDE, which allows us to describe the smallest Hilbert distribution spaces on which the OUP lives.

5. Finally, the central limit theorem can be considered as an example of how to arrive in a natural way, namely through the internal noise of the system, at an SPDE within the framework of semigroup theory. Moreover, we prove weak convergence of solutions of evolution equations on $D_{H_{-\alpha}}[0, \infty)$ by using the submartingale-type inequality for stochastic *-integrals (Lemma 0.2). For weaker convergence usually used in SPDE see Viot [32].

6. The relation between evolution equations and branching diffusions as measure processes has been investigated by Dawson [14] where a diffusion approximation of the branching diffusion is given.

0.4. Notation. Whenever possible we shall suppress the superindex v and write just X^N instead of $X^{v,N}$, etc.

1. The law of large numbers (LLN). The following assumption

$$|||X_0^{v,N}||| \le K(v, N) \quad \text{a.s.} \tag{1.1}$$

for some finite constant $K(v, N)$ depending on v and N implies that $Z^{v,N}$ is a square integrable cadlag martingale. (1.1) will be assumed throughout the paper.

Simple estimates allow us to extend the validity of Lemma 0.1 to

$$h(z) = \langle z, \varphi\rangle_0^2, \qquad Z \in \mathbb{H}^N, \qquad \varphi \in H_0.$$

Let $\varphi \in H_0$ and define $\varphi_N, \dot{\varphi}_N \in \mathbb{H}^N$ by

$$\varphi_N\left(\frac{j}{N}\right) = N\int_{(j-1)/N}^{j/N} \varphi,$$

$$\cdot\dot{\varphi}_N\left(\frac{j}{N}\right) = \frac{N^2}{\sqrt{2}}\left\{\left(\int_{j/N}^{(j+1)/N}\varphi - \int_{(j-1)/N}^{j/N}\varphi\right)^2 + \left(\int_{(j-2)/N}^{(j-1)/N}\varphi - \int_{(j-1)/N}^{j/N}\varphi\right)^2\right\}^{1/2},$$

where $\int_a^b\varphi := \int_a^b\varphi(q)\,dq$, $[a, b] \subset [0,1]$, and we set $|C|(r) := (b + d)r + c_0$.

LEMMA 1.1

$$E\langle Z^N(t), \varphi\rangle_0^2 = \frac{1}{vN}\int_{[0,t]} E\{\langle \varphi_N^2, |C|(X^N(s))\rangle_0 + \langle \dot{\varphi}_N^2, 2DX^N(s)\rangle_0\}\,ds \tag{1.2}$$

PROOF. From Lemma 0.1 we obtain

$$\begin{aligned}
E\langle Z^N(t), \varphi\rangle_0^2 &= \int_{[0,t]} E\bigg\{\lambda^N(X^N(s)) \\
&\qquad\cdot\int_{\mathbb{H}^N}\langle w - X^N(s) + Z^N(s), \varphi\rangle_0^2 - \langle Z^N(s), \varphi\rangle_0^2 \\
&\qquad -2\langle Z^N(s), \varphi\rangle_0\langle w - X^N(s), \varphi\rangle_0\sigma^N(X^N(s), dw)\bigg\}\,ds \\
&= \int_{[0,t]} E\left\{\lambda^N(X^N(s))\int_{\mathbb{H}^N}\langle w - X^N(s), \varphi\rangle_0^2\sigma^N(X^N(s), dw)\right\}ds \\
&= \int_{[0,t]} E\Bigg\{\sum_j \frac{1}{v}\left[\int_{(j-1)/N}^{j/N}\varphi\right]^2 |C|\left(X^N\left(s, \frac{j}{N}\right)\right) \\
&\qquad + \sum_j \frac{DN^2}{v}\left[\int_{j/N}^{(j+1)/N}\varphi - \int_{(j-1)/N}^{j/N}\varphi\right]^2 X^N\left(s, \frac{j}{N}\right) \\
&\qquad + \sum_j \frac{DN^2}{v}\left[\int_{(j-2)/N}^{(j-1)/N}\varphi - \int_{(j-1)/N}^{j/N}\varphi\right]^2 X^N\left(s, \frac{j}{N}\right)\Bigg\}\,ds \\
&= \frac{1}{vN}\int_{[0,t]} E\{\langle \varphi_N^2, |C|(X^N(s))\rangle_0 + \langle \dot{\varphi}_N^2, 2DX^N(s)\rangle_0\}\,ds. \qquad \square
\end{aligned}$$

COROLLARY 1.1. *There is a constant $K(t) < \infty$, depending only on t, such that for arbitrary $\varphi \in H_0$:*

$$(1.3) \qquad E\langle Z^N(t), \varphi\rangle_0^2 \le \frac{K(t)}{vN}\big(|||EX_0^N||| + c_0\big)\langle (I - D\Delta_N)\varphi, \varphi\rangle_0.$$

PROOF. The proof with

$$K(t) \ge \max\{t^2 \exp(c_1 t)c_0, t\cdot \exp(c_1 t)\}(b + d + 2D)$$

is an easy consequence of (1.2), (0.7), and the fact that $U_N(t)$ is positivity preserving, i.e., maps positive functions onto positive ones (Davies [12], Theorem 7.16 and Kotelenez [24]). □

COROLLARY 1.2. *Let $\alpha > \frac{3}{2}$. Then, there is a constant $L(t) < \infty$, depending only on t and α such that*

$$(1.4) \qquad E|Z^N(t)|_{-\alpha}^2 \le \frac{L(t)}{vN}\big(|||EX_0^N||| + c_0\big).$$

PROOF. Let $\{\varphi_n\}$ be a CONS in H_α. Then

$$\begin{aligned} E|Z^N(t)|_{-\alpha}^2 &= \sum_{n=0}^{\infty} E\langle Z^N(t), \varphi_n\rangle_0^2 \\ &\le \frac{K(t)\big(|||EX_0^N||| + c_0\big)}{vN} 4\sum_{n=0}^{\infty}\langle (I - D\Delta)\varphi_n, \varphi_n\rangle_0 \\ &= \frac{\big(|||EX_0^N||| + c_0\big)}{vN} 4K(t)\sum_{n=0}^{\infty}\big(|\varphi_n|_0^2 + D|\dot\varphi_n|_0^2\big) \end{aligned}$$

by

$$(1.5) \qquad \langle -D\Delta_N\varphi, \varphi\rangle_0 \le 4\langle -D\Delta\varphi, \varphi\rangle_0, \qquad \varphi \in H_0$$

and partial integration. Set

$$L(t) := 4K(t)\left(\sum_{n=0}^{\infty}|\varphi_n|_0^2 + D|\dot\varphi_n|_0^2\right).$$

$L(t) < \infty$, since $H_\alpha \hookrightarrow H_1$ is Hilbert–Schmidt for $\alpha > \frac{3}{2}$. □

THEOREM 1.1(LLN). *Let $\alpha \in \{0\} \cup [2, \infty)$ and assume*

(I) $N \to \infty$,
$(\mathrm{II})_\alpha$ $N^2/v = V^2/v^3 \to 0$, *if* $\alpha = 0$,
$v \to \infty$, *if* $\alpha \in [2, \frac{5}{2}]$,
$vN \to \infty$, *if* $\alpha > \frac{5}{2}$,
(III) $|X_0^N - X_0|_{-\alpha} \to 0$ *in probability*, $X^N = X^{v,N}$,
(IV) $\sup_N |||EX_0^N||| < \infty$.

Then for all $T > 0$, $\delta > 0$

$$(1.6) \qquad \lim P\left\{\sup_{0\le t\le T}|X^N(t) - X(t)|_{-\alpha} > \delta\right\} = 0.$$

PROOF. (i) In the case $\alpha = 0$, (1.6) follows directly from (0.8), (1.3), Lemma 0.2, and $\varepsilon_N \to 0$ in (0.8) (by the Trotter–Kato theorem), thus improving upon the LLN obtained by Arnold and Theodosopulu [6].

(ii) In the case $\alpha > \frac{5}{2}$, (1.6) follows from (0.9), (1.4), Lemma 0.2, and $\delta_N \to 0$ on $H_{-\alpha}$, $\alpha > \frac{5}{2}$ by (0.15) and by

$$(1.7) \qquad H_\gamma \hookrightarrow (H^\gamma(0,1)) \hookrightarrow) \mathscr{C}^m[0,1], \qquad \gamma > m + \tfrac{1}{2}$$

(continuously imbedded into the m times continuously differentiable functions; cf. Lions and Magenes [29]).

(iii) For $\alpha = 2$ we cannot show that $\delta_N \to 0$ on $H_{-\alpha}$. Therefore, we use (0.8). Since $U_N(t)$ does not commute with $(I - D\Delta)^{-1}$ (which defines the $|\cdot|_{-2}$ norm) we introduce

$$(\varphi, \psi)_N := \langle (I - D\Delta_N)\varphi, (I - D\Delta_N)\psi\rangle_0, \qquad \varphi, \psi \in H_0$$

as a norm approximating $|\cdot|_{-2}$, which satisfies the easily verifiable inequality

$$(1.8) \qquad |(I - D\Delta)^{-1}\varphi|_0 \le 10|(I - D\Delta_N)^{-1}\varphi|_0, \qquad \varphi \in H_0.$$

Then (1.6) follows using (1.3), Lemma 0.2 wrt H_0, endowed with $(\cdot,\cdot)_N$, (1.7), and

$$|(I - D\Delta_N)^{1/2}|_{\mathscr{L}(H_0)} \to |(I - D\Delta)^{1/2}|_{\mathscr{L}(H_0)}$$

(Kato [22] and for details Kotelenez [24]). □

REMARK 1.1. Condition $(\mathrm{II})_\alpha$ shows that for weaker distances between X^N and X we obtain more freedom concerning the limit behaviour of the cell size v. This was first observed by Arnold [4].

2. Weak convergence of the accompanying Martingale. Let $D_{H_{-\gamma}}[0,\infty)$ denote the complete separable metric space of $H_{-\gamma}$-valued cadlag functions (cf. Billingsley [7] and Kurtz [28]), $\gamma \ge 0$. (1.2) shows that the sequence of variances

$$E\langle M^N(t), \varphi\rangle_0^2$$

of the normalized martingales

$$M^N := (vN)^{1/2} Z^N, \qquad Z^N \text{ from } (0.6),$$

tends to

$$\int_{[0,t]} \left[\langle (\dot\varphi)^2, 2DX(s)\rangle_0 + \langle \varphi^2, |C|(X(s))\rangle_0\right] ds.$$

Integrating by parts we see that this expression is equal to (setting $\partial_q := \partial/\partial q$)

$$\int_{[0,t]} \left[\langle -\partial_q(2DX(s)\partial_q\varphi), \varphi\rangle_0 + \langle \varphi, |C|(X(s))\varphi\rangle_0\right] ds.$$

But this quadratic form on H_1 determines a unique (in distribution) process on $D_{H_{-\alpha}}[0,\infty)$, $\alpha > \frac{3}{2}$. Indeed, let $J\colon H_{-\alpha} \hookrightarrow H_\alpha$ denote the Riesz representation, i.e., for $\varphi^* \in H_{-\alpha}$, $\psi \in H_\alpha$, we have $(\varphi^*, \psi) = \langle \psi, J(\varphi^*)\rangle_\alpha$ (cf. Yosida [33]). Fix

$\alpha > \frac{3}{2}$ and define a bounded operator $B(t)$ on $H_{-\alpha}$ by

$$\langle B(t)\varphi^*, \psi^*\rangle_{-\alpha} := \int_{[0,t]} \langle F(s)J(\varphi^*), J(\psi^*)\rangle_0 \, ds, \tag{2.1}$$

$$F(t) := -\partial_q\big(2DX(t)\partial_q\big) + |C|(X(t)).$$

(i.1) $B(t)$ is a nuclear operator on $H_{-\alpha}$ because $H_\alpha \hookrightarrow H_1$ is Hilbert–Schmidt.
(i.2) $\langle B(t)\varphi^*, \varphi^*\rangle_{-\alpha}$ is increasing and continuous in t.

Hence, (cf. Itô [20])

(ii) there is a unique (in distribution) $H_{-\alpha}$-valued Gaussian process M on some probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ with independent increments, continuous sample paths, and characteristic functional

$$\tilde{E}\exp\big(i\langle M(t), \varphi^*\rangle_{-\alpha} = \exp\big(-\tfrac{1}{2}\langle B(t)\varphi^*, \varphi^*\rangle_{-\alpha}\big), \tag{2.2}$$

where $\tilde{E}(\cdot) = \int(\cdot)\, d\tilde{P}$.

The following two lemmas show that the assumptions of Kurtz [28], Theorem 2.7, are satisfied for M^N, and from the uniqueness of the limit point M we may conclude that $M^N \Rightarrow M$, where $\Rightarrow$ denotes weak convergence.

LEMMA 2.1. *Let $\alpha \in [2, \infty)$ and assume*

(I) $N \to \infty$,
$(\mathrm{II})_\alpha$ $v \to \infty$, *if* $\alpha \in [2, \frac{5}{2}]$,
$vN \to \infty$, *if* $\alpha > \frac{5}{2}$,
(III) $|X_0^N - X_0|_{-\alpha} \to 0$ *in probability*, $X^N = X^{v,N}$,
(IV) $\sup_N |||EX_0^N||| < \infty$.

Then for any $t > 0$

$$M^N(t) \Rightarrow M(t) \quad \text{on } H_{-\alpha}. \tag{2.3}$$

PROOF. (A) Take $\varphi \in H_\alpha$. Then

$$\mathbb{X}^N(t, \varphi) := E\exp\big(i\langle M^N(t), \varphi\rangle\big)_0 \to \exp\left(-\tfrac{1}{2}\int_{[0,t]}\langle F(s)\varphi, \varphi\rangle_0\right).$$

The proof follows that of Kurtz [27], Theorem 3.1; cf. Kotelenez [24].

(B) The relative compactness of $P\{M^N(t) \in \cdot\}$ we obtain from Araujo and Giné [2], Theorem 4.17, as follows:

(B.i) Let $\mathrm{Cov}(N, t)$ denote the covariance operator of $M^N(t) \in H_{-\alpha}$ (cf. Kuo [26], Chapter 1, Theorem 2.1). Then

$$B(N, t) := [\mathrm{Cov}(N, t)]^{1/2}$$

is Hilbert–Schmidt.

Let $\{\varphi_n^*\}$ be a CONS in $H_{-\alpha}$. Then

$$\sup_N \sum_{n=m}^{\infty} |B(N,t)\varphi_n^*|_{-\alpha}^2 = \sup_N \sum_{n=m}^{\infty} \langle \mathrm{Cov}(N,t)\varphi_n^*, \varphi_n^*\rangle_{-\alpha}$$

$$= \sup_N \sum_{n=m}^{\infty} E\langle M^N(t), \varphi_n^*\rangle_{-\alpha} \quad \text{(Kuo, [26], p. 15)}$$

$$= \sup_N \sum_{n=m}^{\infty} E\langle M^N(t), J(\varphi_n^*)\rangle_0^2.$$

$$\leq 4K(t)\Big(\sup_N |||EX_0^N||| + c_0\Big) \sum_{n=m}^{\infty} |J(\varphi_n^*)|_1^2 \qquad (< \infty \ \forall\, m)$$

by (1.3) and the proof of Corollary 1.2

$$\to 0, \quad \text{if } m \to \infty.$$

(B.ii) $$|1 - \mathbb{X}^N(t,\varphi)|^2 = 2(1 - \mathrm{Re}\,\mathbb{X}^N(t,\varphi))$$
$$\leq \langle \mathrm{Cov}(N,t)\varphi^*, \varphi^*\rangle_{-\alpha}, \qquad \varphi^* = J^{-1}(\varphi)$$
$$= |B(N,t)\varphi^*|_{-\alpha}^2 \quad \text{(cf. Kuo [26], p. 21)}.$$

Hence, $P\{M^N(t) \in \cdot\}$ is relatively compact.

(C) (A) and (B) together imply (2.3) (Buldygin [8]). □

LEMMA 2.2. *Let $\alpha > 2$. Then for any $T > 0$, $s > 0$ there exist random variables $\gamma_N^T(s) \geq 0$ s.t. for all $t \in [0,T]$*

$$E\big(|M^N(t+s) - M^N(t)|_{-\alpha}^2 | \mathscr{F}_t^N\big) \leq E\big(\gamma_N^T(s) | \mathscr{F}_t^N\big) \tag{2.4}$$

and $\lim_{s\to 0} \limsup_{N\to\infty} E(\gamma_N^T(s)) = 0$.

PROOF. (i) Let $\{\varphi_N\}$ be a CONS in H_α such that all $\varphi_n \in \mathscr{C}^\infty[0,1]$. By Lemma 0.1 (cf. Corollary 1.2):

$$E\big(|M^N(t+s) - M^N(t)|_{-\alpha}^2 | \mathscr{F}_t^N\big)$$

$$= vN \sum_n \int_{[t,t+s]} E\Big(\lambda^N(X^N(u)) \int \langle w - X^N(u), \varphi_n\rangle_n^2 \sigma^N(X^N(u), dw) | \mathscr{F}_t^N\Big)\, du$$

$$(**) \quad \leq 2\sum_n \int_{[t,t+s]} E\big(\langle \varphi_n, \varphi_n |C|(X^N(u))\rangle_0 + \langle \dot\varphi_n, \dot\varphi_n 2DX^N(u)\rangle_0 | \mathscr{F}_t^N\big)\, du$$

(cf. (1.2))

$$\leq 2\sum_n \int_{[t,t+s]} E\big(\langle \varphi_n, \varphi_n(b+d)X^N(u)\rangle_0 + \langle \dot\varphi_n, \dot\varphi_n 2DX^N(u)\rangle_0 | \mathscr{F}_t^N\big)\, du$$
$$+ 2c_0 \sum_n |\varphi_n|_0^2 s.$$

From (1.7) with $\beta = \alpha - \frac{1}{2} - \varepsilon > \frac{3}{2}$ for some $\varepsilon > 0$

$$\langle \varphi_n, \varphi_n X^N(u)\rangle_0 + \langle \dot\varphi_n, \dot\varphi_n X^N(u)\rangle_0 \leq C(D,\beta)|\varphi_n|_\beta^2 \sup_{0\leq t\leq T} \langle 1, X^N(t)\rangle_0$$

for some constant $C(D,\beta)$ depending only on D and β. Thus, $(**)$ can be estimated from above by

$$[2C(D,\beta)(b+d+2D)+c_0]\sum_n |\varphi_n|_\beta^2 \sup_{t\le T}\langle 1, X^N(t)\rangle_0 s =: \gamma_N^T(s).$$

Hence (2.4), since $H_\alpha \hookrightarrow H_\beta$ is Hilbert–Schmidt, and by Lemma 1.1

$$E\Big(\sup_{t\le T}\langle 1, X^N(t)\rangle_0\Big) \le 2e^{T(b+1)} \times\left[\max\big(E\langle 1, X_0^N\rangle_0 + c_0, 1\big)\left(1+\left(\frac{b+d}{vN}\right)^{1/2}\right)\right]. \qquad (2.5)$$

□

Theorem 2.1. *Let $\alpha \in (2,\infty)$ and assume*

(I) $N\to\infty$,
$(\mathrm{II})_\alpha$ $v\to\infty$, *if* $\alpha\in(2,\frac{5}{2}]$,
$vN\to\infty$, *if* $\alpha>\frac{5}{2}$,
(III) $|X_0^N - X_0|_{-\alpha}\to 0$ *in probability*, $X^N = X^{v,N}$,
(IV) $\sup_N |||EX_0^N||| < \infty$.

Then

$$M^N \Rightarrow M \text{ on } D_{H_{-\alpha}}[0,\infty),$$

where M is the Gaussian independent increment process defined by (2.2).

Proof. (i) From Lemma 2.2 condition (b) and from Lemma 2.1 condition (a) of Kurtz [28], Theorem 2.7 follows, i.e., M^N is relatively compact.

(ii) Weak convergence follows now from the uniqueness (in distribution) of M. □

We will now represent M by a Wiener integral wrt a cylindrical Brownian motion (cf. Dawson [13], [14], Kuo [26]). Therefore, we analyze the covariance of $M(t)$.

Lemma 2.3. *If $c_0 = 0$ and $X_0(q)\equiv 0$ then $X(t,q)\equiv 0$ for all t, q. If $c_0>0$ or $X_0(q)\not\equiv 0$ then $X(t,q)>0$ for all q and all $t>0$.*

Proof. The proof is an application of the Feynman–Kac formula. First we show that

$$X(t,q) = EX_0(\xi(T,q,T-t))\exp(c_1 t) + c_0\int_{[T-t,T]} E\big(\exp(c_1(v-T+t)\big)\,dv, \qquad (2.6)$$

where $\xi(t,q,s)$, $t\in[s,T]$, $q\in[0,1]$ is a Brownian motion in $[0,1]$ with reflection at the boundary $\{0,1\}$ and intensity $\sigma^2 = 2D$. Consequently,

$$X(t,q) \ge \int_{[0,1]} X_0(y)p(q,t,y)\,dy\cdot\exp(-|c_1|T) + c_0 t\cdot\exp(-|c_1|T), \qquad (2.7)$$

with $p(q, t, y)$ the transition probability density of the Brownian motion $\xi(t, q, 0)$, which is strictly positive for all q, y, and $t > 0$ (cf. Dynkin [15], Chapter 10, Section 6, (10.68′)). From this the lemma follows since $X_0 \in H_2$ implies that X_0 is continuous by Sobolev's imbedding theorem. For more details see Kotelenez [24] as well as Gihman and Skorohod [17] Part 1, Section 23. □

By Lemma 2.3 we may without loss of generality assume that $\dot{X}(t, q)$ is strictly positive for all $t > 0$. This implies that the symmetric positive unbounded operator $F(t)$ from (2.1) has H_2 as its maximal domain in H_0 (which can be verified by partial integration). Consequently, $F(t)$ is self-adjoint (Davies [12]) and $\mathscr{D}(F^{1/2}(t)) = H_1$. This implies by interpolation (Lions and Magenes [29], Chapter 1, Theorem 5.1)

$$\hat{F}^{1/2}(t) \in \mathscr{L}(H_\alpha, H_{\alpha-1}) \quad \text{for all } \alpha \in [1,2],\ t > 0, \tag{2.8}$$

where $\hat{F}^{1/2}(t)$ is equal to $F^{1/2}(t)|_{H_\alpha}$ and considered as a bounded operator with values in $H_{\alpha-1}$.

Fix $\alpha > \frac{1}{2}$. Since the imbedding i_0^α: $H_\alpha \hookrightarrow H_0$ is Hilbert–Schmidt we have $(i_0^\alpha, H_\alpha, H_0)$ as an abstract Wiener space (Kuo [26]), and with the isometry from H_0 to H_α $j_\alpha^0 := (I - D\Delta)^{-\alpha/2}$ we have

$$Q^{1/2} := j_\alpha^0 \ i_0^\alpha \in \mathscr{L}_2(H_\alpha)$$

(Hilbert–Schmidt operators on H_α), $Q^{1/2} \geq 0$, and $Q^{1/2}$ is self-adjoint. Let $Q^{(1/2)*}$ be the dual operator of $Q^{1/2}$. Then, $Q^{(1/2)*} \in \mathscr{L}_2(H_{-\alpha})$, and

$$Q^* := \left(Q^{(1/2)*}\right)^2 \in \mathscr{L}_1(H_{-\alpha}),$$

(nuclear operators on $H_{-\alpha}$).

Take an $H_{-\alpha}$-valued Wiener process W on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ (we may assume that this is possible on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$) with covariance operator Q^* and from (2.8) we obtain for the dual operator of $\hat{F}^{(1/2)}(t)$: $\hat{F}^{(1/2)*}(t) \in \mathscr{L}(H_{-\alpha}, H_{-\alpha-1})$, whence

$$\int \hat{F}^{(1/2)*}\, dW \tag{2.9}$$

is a well defined $H_{-\alpha-1}$-valued Gaussian martingale with continuous sample paths (cf. Curtain and Pritchard [10]).

Lemma 2.4. *W is a cylindrical Brownian motion on H_0 and*

$$M \overset{\mathscr{D}}{=} \int \hat{F}^{(1/2)*}\, dW \quad (\textit{equal in distribution}), \tag{2.10}$$

where M is defined by (2.2).

Proof. (i) Let $\{\psi_n^*\}_{n \in \mathbb{N}}$ be a CONS in $H_{-\alpha}$ consisting of eigenvectors of $Q^{(1/2)*}$ and $\lambda_n \geq 0$ the corresponding eigenvalues. Then,

$$dW(t) = \sum \lambda_n \psi_n^*\, d\beta_n(t),$$

where β_n are one-dimensional mutually independent standard Wiener processes

(Curtain and Pritchard [10]). For $\varphi^* \in H_{-\alpha-1}$ we obtain:

$$\begin{aligned}\tilde{E}\left\langle \int_{[0,t]} \hat{F}^{(1/2)*}(s)\, dW(s), \varphi^* \right\rangle_{-\alpha-1}^2 &= \tilde{E}\left\langle \sum_n \int_{[0,t]} \hat{F}^{(1/2)*}(s)\lambda_n \psi_n^*\, d\beta_n(s), \varphi^* \right\rangle_{-\alpha-1}^2 \\ &= \sum_n \int_{[0,t]} \langle \hat{F}^{(1/2)*}(s) Q^{(1/2)*}\psi_n^*, \varphi^* \rangle_{-\alpha-1}^2\, ds \\ &= \sum_n \int_{[0,t]} \langle J(\psi_n^*), Q^{1/2}\hat{F}^{1/2}(s) J(\varphi^*)\rangle_\alpha^2\, ds \\ &= \int_{[0,t]} |Q^{1/2}\hat{F}^{1/2}(s) J(\varphi^*)|_\alpha^2\, ds \\ &= \int_{[0,t]} |i_0^\alpha \hat{F}^{1/2}(s) J(\varphi^*)|_0^2\, ds \\ &= \int_{[0,t]} \langle F(s) J(\varphi^*), J(\varphi^*)\rangle_0\, ds \\ &= \tilde{E}\langle M(t), \varphi^*\rangle_{-\alpha-1}^2.\end{aligned}$$

Since both M and $\int \hat{F}^{(1/2)*}\, dW$ are Gaussian with independent increments, (2.10) follows.

(ii) If we take I (identity) instead of $F(t)$ in step (i) we obtain that W is a cylindrical Brownian motion on H_0. □

3. The central limit theorem (CLT)

LEMMA 3.1. *Under the assumptions of Theorem* 2.1

$$\int \tilde{U}\, dM^N \Rightarrow \int \tilde{U}\, dM \quad \text{on } D_{H_{-\alpha}}[0,\infty). \tag{3.1}$$

PROOF. (i) Since $D_{H_{-\alpha}}[0,\infty)$ is complete and separable (Kurtz [28]), weak convergence and convergence wrt the Prohorov metric, denoted by d_p, are equivalent (cf. Billingsley [7], Appendix III, Theorem 5).

(ii) Take a smooth CONS$\{\varphi_n^*\}$ for $H_{-\alpha}$. Denote by π_k the projection in $H_{-\alpha}$ onto $\mathscr{L}(\varphi_1^*, \dots, \varphi_k^*)$, i.e., onto the finite-dimensional subspace spanned by $\varphi_1^*, \dots, \varphi_k^*$, and $\pi_k^\perp := I - \pi_k$. Set

$$P^N(\cdot) := P\left\{ \int \tilde{U}\, dM^N \in \cdot \right\},$$

$$P^{N,k}(\cdot) := P\left\{ \int \tilde{U}\pi_k\, dM^N \in \cdot \right\},$$

$$\tilde{P}^k(\cdot) := \tilde{P}\left\{ \int \tilde{U}\pi_k\, dM \in \cdot \right\},$$

$$\tilde{P}(\cdot) := \tilde{P}\left\{ \int \tilde{U}\, dM \in \cdot \right\}.$$

(iii) Note that $\{\varphi_n\}$ is a CONS in H_α, where $\varphi_n = \mathscr{J}(\varphi_n^*)$. Hence, Lemma 0.2 (cf. step (B) in the proof of Lemma 2.1) yields:

$$P\left\{\sup_{0\le t\le T}\left|\int_{[0,t]}\tilde{U}\pi_k^\perp\,dM^N\right|_{-\alpha}\ge 2\delta\right\}$$
$$\le \frac{4K(T)}{\delta^2}\left(\sup_N|||EX_0^N|||+c_0\right)\left(\sum_{n=k+1}^\infty|\varphi_n|_0^2+D|\dot{\varphi}_n|_0^2\right),$$

since $\pi_k^\perp\,dM^N = d\pi_k^\perp\,M^N$. For fixed δ and T the r.h.s. can be made arbitrarily small independent of N by choosing k large enough. This ensues by the definition of the metric $\mathscr{d}$ on $D_{H_{-\alpha}}[0,\infty)$ (cf. Kurtz [28]): For any $\varepsilon > 0$ there is a $k_1(\varepsilon)$ such that for all N and all $k \ge k_1(\varepsilon)$:

$$P\left\{\mathscr{d}\left(\int\tilde{U}\pi_k^\perp\,dM^N,0\right)>\frac{\varepsilon}{3}\right\}\le\frac{\varepsilon}{3},$$

whence for all N and all $k \ge k_1(\varepsilon)$

$$d_P(P^N, P^{N,k}) \le \frac{\varepsilon}{3}.$$

(iv) Similarly, we obtain a $k_2(\varepsilon)$ such that for all $k \ge k_2(\varepsilon)$:

$$d_P(\tilde{P}, \tilde{P}^k) \le \frac{\varepsilon}{3}.$$

(v) Set $k := \max(k_1(\varepsilon), k_2(\varepsilon))$. Note that by the choice of $\{\varphi_1^*,\dots,\varphi_k^*\}$ $\pi_k M^N(t) \in \mathscr{D}(\tilde{A})$. Hence, by partial integration (Kotelenez [24]):

$$\int_{[0,t]}\tilde{U}(t-s)\pi_k\,dM^N(s) = \pi_k M^N(t) + \int_{[0,t]}\tilde{U}(t-s)\tilde{A}\pi_k M^N(s)\,ds.$$

Since

$$\varphi^*(\cdot)\to(\pi_k\varphi^*)(\cdot)+\int_{[0,\cdot]}\tilde{U}(\cdot-s)\tilde{A}\pi_k\varphi^*)(s)\,ds$$

is a continuous map from $D_{H_{-\alpha}}[0,\infty)$ into itself (cf. Kurtz [28]), $M^N \Rightarrow M$ entails

$$\int\tilde{U}\pi_k\,dM^N \Rightarrow \int\tilde{U}\pi_k\,dM$$

(Billingsley [7], Chapter 1, Section 5, Theorem 5.1). Consequently, we may choose an $N(\varepsilon)$ such that for $k = \max(k_1(\varepsilon), k_2(\varepsilon))$ and all $N \ge N(\varepsilon)$

$$d_P(P^{N,k}, \tilde{P}^k) \le \frac{\varepsilon}{3},$$

whence from (iii) and (iv)

$$d_P(P^N, \tilde{P}) \le \varepsilon \quad \text{for all } N \ge N(\varepsilon).$$ □

Let $\alpha > \frac{1}{2}$ and $\mathbf{Y}_0$ be a square integrable $H_{-\alpha}$-valued random variable on $(\Omega, \mathscr{F}, P)$ such that there is an $H_{-\alpha}$-valued random variable Y_0 on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, $\mathbf{Y}_0 \overset{\mathscr{D}}{=} Y_0$, and Y_0 independent of $M(t)$ for all $t > 0$. Further, let $\mathscr{C}$ denote an arbitrary $\int\tilde{U}\,dM$-continuity set of $D_{H_{-\alpha-3}}[0,\infty)$ (cf. Billingsley [7]) and F an arbitrary element from $\sigma(Y_0)$. We shall make the following asymptotic independence assumption:

$$(3.2)\qquad \begin{gathered} Y_0^N \to \mathbf{Y}_0 \quad \text{in probability on } H_{-\alpha-3}, \\ P\left\{\left(\int \tilde{U}\, dM^N \in \mathscr{C}\right) \cap F\right\} \to \tilde{P}\left\{\left(\int \tilde{U}\, dM \in \mathscr{C}\right)\right\} P\{F\}. \end{gathered}$$

$Y_0^N \to \mathbf{Y}_0$ in probability implies

$$\tilde{U}(\cdot)Y_0^N \to \tilde{U}(\cdot)\mathbf{Y}_0$$

on $(\mathscr{C}_{H_{-\alpha-3}}[0,\infty), \mathscr{d}|_C)$, where $\mathscr{d}|_C$ is the metric $\mathscr{d}$, restricted to $\mathscr{C}_{H_{-\alpha-3}}[0,\infty)$, which is equivalent to the metric obtained from the uniform convergence on bounded intervals (cf. Billingsley [7], Chapter 3, Section 14), and $\mathscr{C}_{H_{-\gamma}}[0,\infty)$ denotes the continuous functions from $[0,\infty)$ into $H_{-\gamma}$, $\gamma \geq 0$.

Moreover, (0.15) and (1.6) imply

$$(3.3)\qquad \sup_{0 \leq t \leq T} |(A_N - \tilde{A})X^N(t)|_{-\alpha-3} \leq \frac{K(D,\alpha)}{N} \sup_{0 \leq t \leq T} \langle 1, X^N(t)\rangle_0$$

for some constant $K(D,\alpha) < \infty$ depending on D and α. This implies by (2.5)

$$E\left(\int_{[0,T]} |(vN)^{1/2}(A_N - \tilde{A})X^N(s)|_{-\alpha-3}^2\, ds\right) \to 0 \quad \text{for any } T > 0,$$

if $v/N \to 0$.

THEOREM 3.1 (CLT). *Let $\alpha > \frac{1}{2}$ be an arbitrary number and assume*

(I) $v/N \to 0$ }
(II) $vN \to \infty$ } *entailing $N \to \infty$,*
(III) $\sup_N |||EX_0^N||| < \infty$,
(IV) $Y_0^N \Rightarrow Y_0$ *in the sense of* (3.2), *where Y_0 is independent of M and $\tilde{E}|Y_0|_{-\alpha}^2 < \infty$.*

Then for $Y^N := Y^{v,N} := (vN)^{1/2}(X^{v,N} - X)$:

(i) $$Y^N \Rightarrow Y \quad \text{on } D_{H_{-\alpha-3}}[0,\infty) \text{ (converges weakly)},$$

where

$$(3.4)\qquad Y(t) = \tilde{U}(t)Y_0 + \int_{[0,t]} \tilde{U}(t-s)\hat{F}^{(1/2)*}(s)\, dW(s)$$

is the mild solution of the stochastic partial differential equation

$$(3.5)\qquad \begin{gathered} dY(t) = (D\tilde{\Delta} + c_1)Y(t)\, dt + \hat{F}^{(1/2)*}(t)\, dW(t), \\ Y(0) = Y_0. \end{gathered}$$

(ii)

$$(3.6)\qquad Y \in \mathscr{C}_{H_{-\alpha}}[0,T] \quad \text{a.s. } \textit{for all } T > 0,$$

and (if $c_0 > 0$ or $X_0(q) \not\equiv 0$ then) $Y(t), t > 0$, does not define a σ-additive measure on $H_{-\gamma}$ for $\gamma \leq \frac{1}{2}$, i.e., (3.6) *is the maximal regularity of Y on the Hilbert scale* (0.10).

(iii) *Y is a Markov process, and its weak generator is given by*

$$\mathscr{A}(t)g(t,\varphi^*) = \frac{\partial}{\partial t}g(t,\varphi^*) + \langle (D\tilde{\Delta} + c_1)\delta g(t,\varphi^*), \varphi^* \rangle_{-\alpha-1} \tag{3.7}$$
$$+ \tfrac{1}{2}\mathrm{Tr}\{Q^{(1/2)*}\hat{F}^{(1/2)0}(t)\delta^2 g(t,\varphi^*)\hat{F}^{(1/2)*}(t)Q^{(1/2)*}\},$$

where $g \in B([0,T] \times H_{-\alpha-1})$ (real valued measurable functions g with domain $[0,T] \times H_{-\alpha-1}$) s.t. $\partial g/\partial t$, δg, $\delta^2 g$, and $D\tilde{\Delta}\,\delta g$ exist, are continuous in x and t, and uniformly bounded in norm on $[0,T] \times H_{-\alpha-1}$, and $\hat{F}^{(1/2)0}(t)$ is the dual operator of $\hat{F}^{(1/2)*}(t)$ after identifying the duals of $H_{-\alpha}$ and $H_{-\alpha-1}$ with $H_{-\alpha}$ and $H_{-\alpha-1}$, respectively.

PROOF. (i) The weak convergence follows from Lemma 3.1, our assumption (3.2), (3.3), and the fact that the addition between continuous and arbitrary elements from $D_{H_{-\alpha-3}}[0,\infty)$ yields a continuous map from $\mathscr{C}_{H_{-\alpha-3}}[0,\infty), \mathscr{d})$ $\times D_{H_{-\alpha-3}}([0,\infty), \mathscr{d}) \to D_{H_{-\alpha-3}}([0,\infty), \mathscr{d})$ (cf. Billingsley [7], Chapter 1, Section 4, Theorem 4.5, problem 7 and Section 5, Theorem 5.1 and Kotelenez [24]). The representation (3.4) for Y follows from Lemmas 2.4 and 3.1.

(ii) The proof of (3.6) is an easy generalization of the proof of Proposition 5 in Dawson [13], and we will just sketch the main steps: Let $\lambda > 0$ and $b(s)$ be a one-dimensional standard Brownian motion. Then there is a constant $K < \infty$ s.t. for all sufficiently large $a\lambda$ and for any $T > 0$

$$P\left\{\sup_{0\le t\le T}\left(\int_0^t e^{-\lambda(t-s)}\,db(s)\right)^2 \ge a\right\}$$
$$\le K\left\{(a\lambda)^{1/2}\exp(-a\lambda)\left[\lambda T + \log\left((a\lambda)^{1/2}\right)\right]\right. \tag{3.8}$$
$$\left. + (a\lambda)^{1/4}\exp\left(-\frac{a\lambda}{2}\right)\left(\log\left((a\lambda)^{1/2}\right)\right)^{1/2}\exp(-\lambda T)\right\}.$$

Take the eigenvectors $\{\phi_n\}$ of $(I - D\Delta)$ with eigenvalues $\lambda_n := (1 + Dn^2\pi^2)$ and note that $\phi_n^\gamma := \lambda_n^{-\gamma/2}\phi_n$ is a CONS for H_γ, $\gamma \in \mathbb{R}$. Moreover, $U(t)\phi_n^{-\gamma} = e^{\lambda_n - 1}\cdot e^{tc_1}\phi_n^{-\gamma}$, $\gamma \ge 0$.

Obviously, we may w.l.o.g. assume $c_1 = 1$. Then

$$\int_0^t \tilde{U}(t-s)\,dM(s) = \sum_n \int_0^t e^{-\lambda_n(t-s)}\,dm_n^\gamma(s)\phi_u^{-\gamma}$$

with

$$m_n^\gamma(t) = \langle M(t), \phi_n^{-\gamma}\rangle_{-\gamma}, \qquad \gamma \ge 0,$$

by Fourier expansion in $H_{-\gamma}$, where we will determine γ in what follows. If $[m_n^\gamma](t)$ denotes the quadratic variation of $m_n^\gamma(t)$ then we have

$$[m_n^\gamma](t) = \int_0^t \langle F(s)\phi_n^\gamma, \phi_n^\gamma\rangle\,ds \qquad (F(s) \text{ from } (2.1))$$
$$\le C(T)t|\phi_n^\gamma|_1^2 = C(T)t\lambda_n^{1-\gamma}$$

for some constant $C(T)$ by $|||X(t)||| \le \rho_T < \infty$ for $t \le T$. Note that $m_n^\gamma(t)$ can be written as $b([m_n^\gamma](t))$. Now take $\gamma > \frac{1}{2}$ and $\varepsilon > 0$ s.t. $\gamma - \varepsilon > \frac{1}{2}$, which implies $\sum \lambda_n^{-\gamma+\varepsilon} < \infty$. Then, as in Dawson (loc. cit.)

$$\tilde{P}\left\{\sup_{0 \le t \le T}\left(\int_0^t e^{-\lambda_n(t-s)}\,dm_n(s)\right)^2 \ge \lambda_n^{-\gamma+\varepsilon}\right\}$$

$$\le \tilde{P}\left\{\sup_{0 \le t \le T}\left(\int_0^t e^{-\lambda_n(t-s)}\,db(s)\right)^2 \ge C(T)^{-1}\lambda_n^{\varepsilon-1}\right\}.$$

Since $\lambda_n \lambda_n^{\varepsilon-1} = \lambda_n^\varepsilon \to \infty$ we obtain (3.6) by Fourier expansion, (3.8), the Borel–Cantelli lemma, and the obvious fact that $\tilde{U}(\cdot)y_0 \in \mathscr{C}_{H_{-\alpha}}[0,\infty)$.

Since the convolution integral in (3.4) is Gaussian it must have a second moment in order to define a σ-additive measure on $H_{-\gamma}$. For $0 \le s < t$

$$\tilde{E}\left|\int_s^t \tilde{U}(t-p)\,dM(p)\right|_{-\gamma}^2 = \sum_n \int_s^t e^{-2\lambda_n(t-p)}\,d[m_n^\gamma](p)$$

$$\ge C(t,s)\sum_n \int_s^t e^{-2\lambda_n(t-p)}\,ds\,\lambda_n^{-\gamma+1},$$

with $C(t,s) > 0$ determined by $\inf_{u \in [s,t],\, q \in [0,1]} X(u,q) > 0$ (unless $X(u,q) \equiv 0$, contradicting our assumption, cf. (2.6))

$$\sim \sum_n n^{-2\gamma} = \infty \quad \text{if } \gamma \le \tfrac{1}{2}.$$

(iii) The Markov property follows from Arnold, Curtain, and Kotelenez [5]; (3.7) follows from Curtain [9], (6.1). □

REMARK. (i) For $\alpha > \frac{3}{2}$, (3.6) follows from Kotelenez [23] since $\tilde{U}(t)$ is of contraction-type (cf. Remark 0.1.).

(ii) The maximal regularity shows in particular that our Gaussian approximation to the reaction and diffusion system does not live on the function space H_0. Consequently, (nonlinear) operations which may have a meaning on H_0 do not (necessarily) have a meaning in the limit.

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