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LAW OF THE ABSORPTION TIME OF SOME POSITIVE SELF-SIMILAR MARKOV PROCESSES[1]

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Let X be a spectrally negative self-similar Markov process with 0 as an absorbing state. In this paper, we show that the distribution of the absorption time is absolutely continuous with an infinitely continuously differentiable density. We provide a power series and a contour integral representation of this density. Then, by means of probabilistic arguments, we deduce some interesting analytical properties satisfied by these functions, which include, for instance, several types of hypergeometric functions. We also give several characterizations of the Kesten's constant appearing in the study of the asymptotic tail distribution of the absorbtion time. We end the paper by detailing some known and new examples. In particular, we offer an alternative proof of the recent result obtained by Bernyk, Dalang and Peskir [*Ann. Probab.* **36** (2008) 1777–1789] regarding the law of the maximum of spectrally positive Lévy stable processes.

1. Introduction. Let $X = ((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x>0})$ be a self-similar Hunt process with values in $[0, \infty)$. It means that X is a right-continuous strong Markov process with quasi-left continuous trajectories and there exists $\alpha > 0$ such that X enjoys the following self-similarity property: for each $c > 0$ and $x \geq 0$,

$$\text{the law of the process } (c^{-1}X_{c^\alpha t})_{t\geq 0}, \text{ under } \mathbb{Q}_x, \text{ is } \mathbb{Q}_{x/c}.$$

$1/\alpha$ is called the index of self-similarity. The purpose of the paper is to describe the law of the stopping time

$$T_0 = \inf\{s > 0; X_s = 0\}$$

with the usual convention that $\inf\{\varnothing\} = \infty$. The class of positive self-similar Markov processes (for short pssMp) has been introduced and studied by Lamperti [14]. In particular, he showed that for each fixed $\alpha > 0$, there is a bijective correspondence between pssMp with index α and (possibly killed) real-valued Lévy processes, that is, processes with stationary and independent increments. More specifically, by introducing the additive functional

$$\Sigma_t = \inf\left\{s > 0; A_s = \int_0^s X_r^{-\alpha}\,dr > t\right\},$$

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Lamperti [14] showed that the process $\xi = (\xi_t)_{t\geq 0}$, defined by

$$\xi_t = \log(X_{\Sigma_t}), \qquad 0 \leq t < T_0, \tag{1.1}$$

is a (possibly killed) Lévy process. We denote the law of the process ξ when starting at 0 by P. It is plain that

$$\Sigma_t = \int_0^t e^{\alpha \xi_s}\, ds$$

and writing $q \geq 0$ for the killing rate of the Lévy process, one gets the identity in distribution

$$(T_0, \mathbb{Q}_x) \stackrel{(d)}{=} (x^\alpha \Sigma_{\mathbf{e}_q}, \mathrm{P}),$$

where $\mathbf{e}_q$ is an independent exponential random variable of parameter q (we have $\mathbf{e}_0 = \infty$). Lamperti [14] explained that, either $q > 0$ and X reaches 0 by a jump, that is,

$$\mathbb{Q}_x(X_{T_{0-}} > 0, T_0 < \infty) = 1 \qquad \forall x > 0,$$

or ξ drifts to $-\infty$ and X reaches 0, that is,

$$\mathbb{Q}_x(X_{T_{0-}} = 0, T_0 < \infty) = 1 \qquad \forall x > 0.$$

We gather these two possibilities in the following hypothesis.

H: *Either* $q > 0$ *or* $\lim_{t\to\infty} \xi_t = -\infty$ *a.s. and* $q = 0$.

The law of T_0 or equivalently of $\Sigma_{\mathbf{e}_q}$ turns out to be a key object in various settings. It appears, for instance, in the study of coagulation-fragmentation processes [4] and continuous state branching processes with immigration [23]. We also mention that, recently, in the SLE context, Alberts and Sheffield [1] describe a measure-valued function supported on the intersection of a chordal SLE(κ) curve with $\mathbb{R}$, $4 < \kappa < 8$, in terms of the law of the absorption time T_0 of some Bessel processes which form the class of pssMp having continuous trajectories. The law of $\Sigma_{\mathbf{e}_q}$ is also critical for the pricing of Asian options in mathematical finance (see, e.g., [26]), but also for computing perpetuities in insurance mathematics (see, e.g., [10]).

Unfortunately, beside some isolated cases the distribution of T_0 is not attainable. We mention the papers [8, 12] and [23] where such examples can be found and refer to the survey paper [6] for a description of these cases. Besides, two notable exceptions might be worth mentioning: when X is a Bessel process of negative index and when X is a regular spectrally negative stable Lévy process killed upon entering the negative half-line. In the former case, several proofs can be found in the literature, see, for instance, the excellent monograph of Yor [31] and the more recent survey papers of Matsumoto and Yor [16] and [17]. However, most of the proofs rely on the knowledge of the semigroup of Bessel processes. For the second case, Bernyk, Dalang and Peskir [2] derive a representation of the distribution of T_0 by inverting, in a nontrivial way, the known expression of the Wiener–Hopf factorization of stable one-sided Lévy processes. Our approach will differ from these two cases since we do not have, in general, access neither to the semigroup of X nor to the Laplace transform of T_0.

The remaining part of the paper is organized as follows. In the next section, we state our main results including the smoothness and the representation as an absolutely convergent power series of the distribution of T_0. The proof of these results is presented in Section 3. Finally, in the last section, we present a few consequences of the main result and we detail some known and new examples. We also mention that some of the results stated in Theorem 2.3 below were announced without proofs in the note [25].

2. Main results. Henceforth, we assume that X is a pssMp of index $1/\alpha > 0$ and of the spectrally negative type. It means that it is associated via the Lamperti mapping to a possibly killed Lévy process ξ which is spectrally negative. We exclude the cases when ξ is degenerate, that is, when ξ is the negative of a subordinator or a pure drift process. We recall that P (resp., E) stands for the law (resp., the expectation operator) of ξ with $\xi_0 = 0$. The law of ξ is determined by its Laplace exponent $\bar{\psi}(u) = \psi(u) - q$, where $q \geq 0$ is the killing rate and ψ admits the following Lévy–Khintchine representation: for any $u \geq 0$,

$$\psi(u) = \bar{b}u + \frac{\sigma}{2}u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur\mathbb{I}_{\{|r|<1\}})\nu(dr),$$

where $\bar{b} \in \mathbb{R}, \sigma \geq 0$ and the measure ν is such that $\int_{-\infty}^{0}(1 \wedge r^2)\nu(dr) < +\infty$. We shall refer to ξ (resp., $\bar{\psi}$) as the underlying Lévy process (resp., Laplace exponent) of X. Let us now proceed by recalling some basic properties of the Laplace exponent ψ, which can be found, for instance, in Bertoin [3]. First, it is plain that $\lim_{u\to\infty} \psi(u) = +\infty$ and by monotone convergence, one gets $\mathrm{E}[\xi_1] = \bar{b} + \int_{-\infty}^{-1} r\nu(dr) \in [-\infty, \infty)$. We shall also need the value of the constant $\Lambda = \lim_{u\to\infty} \frac{\psi(\alpha u)}{u}$ which is given (see [3], Corollary VII.5) by

$$\Lambda = \begin{cases} \alpha b = \alpha\Big(\bar{b} - \displaystyle\int_{-1}^{0} r\nu(dr)\Big), & \text{if } \sigma = 0 \text{ and } \displaystyle\int_{-\infty}^{0}(1 \wedge r)\nu(dr) < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since we have excluded the degenerate cases, we easily check that $b > 0$. Next, we recall that the mapping $u \mapsto \psi(u)$ is continuous and increasing on $[\phi(0), \infty)$, where $\phi(0)$ stands for the largest solution to the equation $\psi(u) = 0$. Thus, ψ has a well-defined inverse function $\phi : [0, \infty) \to [\phi(0), \infty)$ which is also continuous and increasing. In order to simplify the notation we write, for any $q \geq 0$, $\gamma = \phi(q) > 0$. Then, it is easily seen that

$$\mathrm{E}[e^{\gamma\xi_1}] = 1.$$

We also note that the condition H is equivalent to the requirement $\phi(q) > 0$. Next, we set $\psi_\gamma(u) = \psi(u+\gamma) - \psi(\gamma)$ and observing that $\psi_\gamma(0) = 0$, we deduce that ψ_γ is the Laplace exponent of a conservative spectrally negative Lévy process. We also point out that $\psi'_\gamma(0^+) = \psi'(\gamma) > 0$ and $\lim_{u\to\infty} \frac{\psi_\gamma(u)}{u} = \lim_{u\to\infty} \frac{\psi(u)}{u}$.

We proceed by introducing more notation taken from Patie [21] and [27]. First, for a function f and for any $\alpha > 0$, we write

$$a_s(f;\alpha) = \prod_{k=1}^{\infty} \frac{f(\alpha(k+s))}{f(\alpha k)}, \qquad s \in \mathbb{C},$$

whenever the infinite product exists. Note that, for instance, $a_0(\psi;\alpha) = 1$ and for any $n = 1, 2, \dots,$

$$a_n(\psi;\alpha) = \left(\prod_{k=1}^{n} \psi(\alpha k)\right)^{-1}. \tag{2.1}$$

Next, we introduce, for any $\rho \in \mathbb{C}$ such that $\mathfrak{Re}(\rho) > 0$, the power series

$$\mathcal{I}_\psi(\rho;z) = \frac{1}{\Gamma(\rho)} \sum_{n=0}^{\infty} a_n(\psi;\alpha)\Gamma(\rho+n)z^n, \tag{2.2}$$

where Γ stands for the Gamma function. By means of classical criteria, it is easily seen that the function $z \mapsto \mathcal{I}_\psi(\rho;z)$ is analytic in the disc $\{z \in \mathbb{C}; |z| < \Lambda\}$. In particular, in the case $\Lambda = +\infty$, that is, when the process ξ has paths of unbounded variations, $\mathcal{I}_\psi(\rho;z)$ is an entire function in z. Moreover, for any $|z| < \Lambda$, the mapping $\rho \mapsto \mathcal{I}_\psi(\rho;z)$ is a meromorphic function defined for all complex numbers ρ except at the poles of the Gamma function, which are the points $\rho = 0, -1, \dots$ However, they are removable singularities. Indeed, for any $|z| < \Lambda$ and any integer $N \in \mathbb{N}$, one has, by means of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$,

$$\mathcal{I}_\psi(0;z) = 1$$

and

$$\mathcal{I}_\psi(-N;z) = \sum_{n=0}^{N} (-1)^n \frac{\Gamma(N+1)}{\Gamma(N+1-n)} a_n(\psi;\alpha) z^n.$$

Thus, by uniqueness of the analytic continuation, for any $|z| < \Lambda$, $\mathcal{I}_\psi(\rho;z)$ is an entire function in ρ. Before stating our main result, we show that in the case $\Lambda = \alpha b$, the power series (2.2) can be represented, in the left half-plane, as another convergent power series which corresponds to an analytic continuation in this domain. To this end, we aim to use the co-called Euler transformation; see, for example, [19], page 294. However, this transformation can be performed if and only if the singularity of the function $\mathcal{I}_{\psi_\gamma}(\rho;z)$ on the circle $|z| = \Lambda$ is located at the point $z = \Lambda$. In order to show that our family of functions satisfies this property, we first provide a contour integral representation of $\mathcal{I}_{\psi_\gamma}(\rho;z)$ which turns out to be an analytic continuation in the entire complex plane cut along the positive real axis. Then, we are able to apply the Euler transformation to derive a series representation.

PROPOSITION 2.1. *Let $\Lambda = \alpha b$, then $\mathcal{I}_{\psi_\gamma}(\rho; z)$ is analytic in the disc $|z| < \alpha b$ and for any fixed $\rho = 0, -1, \dots,$ the mapping $z \mapsto \mathcal{I}_{\psi_\gamma}(\rho; z)$, as a polynomial, is an entire function.*

Moreover, for any $\rho \neq 0, -1, \dots, \mathcal{I}_{\psi_\gamma}(\rho; z)$ admits an analytic continuation in the entire complex plane cut along the positive real axis given by

$$\mathcal{I}_{\psi_\gamma}(\rho; z) = \frac{1}{2i\pi\Gamma(\rho)} \int_{-i\infty}^{i\infty} a_s(\varphi_\gamma; \alpha)\Gamma(s+\rho)\Gamma(-s)\left(-\frac{z}{\alpha}\right)^s ds, \quad |\arg(-z)| < \pi, \tag{2.3}$$

where the contour is indented to ensure that all poles (resp., nonnegative poles) of $\Gamma(\rho+s)$ [resp., $\Gamma(-s)$] lie to the left (resp., right) of the intended imaginary axis.

Consequently, for any $\rho \in \mathbb{C}$, $\mathcal{I}_{\psi_\gamma}(\rho; z)$ admits, in the half-plane $\mathfrak{Re}(z) < \frac{\alpha b}{2}$, the following power series representation

$$\mathcal{I}_{\psi_\gamma}(\rho; z) = \left(1 - \frac{z}{\alpha b}\right)^{-\rho} \sum_{n=0}^{\infty} \mathcal{I}_{\psi_\gamma}(-n; \alpha b)\frac{\Gamma(\rho+n)}{n!\Gamma(\rho)}\left(\frac{z}{z-\alpha b}\right)^n. \tag{2.4}$$

Finally, for any fixed $\mathfrak{Re}(z) < \frac{\alpha b}{2}$, $\mathcal{I}_{\psi_\gamma}(\rho; z)$ is an entire function in the argument ρ.

REMARK 2.2. A specific instance of the mapping $\mathcal{I}_{\psi_\gamma}(\rho; x)$ when $\Lambda = \alpha b$, is the hypergeometric function ${}_2F_1$. In this case, the representation (2.4) is known as the Euler transformation which has the remarkable feature that the power series on the right-hand side of (2.4) is still an hypergeometric function ${}_2F_1$. We refer to the Section 4.3 below for more details on this example.

We are now ready to state our main result.

THEOREM 2.3. *Let $q \geq 0$, assume that $\phi(q) > 0$ and set $\gamma = \phi(q)$ and $\gamma_\alpha = \gamma/\alpha$. Then, there exists a constant $C_\gamma > 0$ such that*

$$\mathcal{I}_{\psi_\gamma}(\gamma_\alpha; -t) \sim \frac{t^{-\gamma_\alpha}}{C_\gamma} \qquad \text{as } t \to \infty \tag{2.5}$$

($f(t) \sim g(t)$ as $t \to a$ means that $\lim_{t\to a} \frac{f(t)}{g(t)} = 1$ for any $a \in [0, \infty]$) and

$$S(t) = C_\gamma t^{-\gamma_\alpha} \mathcal{I}_{\psi_\gamma}(\gamma_\alpha; -t^{-1}), \qquad t > 0, \tag{2.6}$$

where, by self-similarity, we have set $S(tx^{-\alpha}) = \mathbb{Q}_x(T_0 \geq t)$, $x, t > 0$. *Finally, the law of* T_0 *under* $\mathbb{Q}_1$ *is absolutely continuous with an infinitely continuously differentiable density denoted by* s *and given by*

$$s(t) = \gamma_\alpha C_\gamma t^{-\gamma_\alpha - 1} \mathcal{I}_{\psi_\gamma}(1+\gamma_\alpha; -t^{-1}), \qquad t > 0.$$

REMARK 2.4. In the case $\Lambda = \infty$, we easily check that, for any $\Re\mathfrak{e}(\rho) > 0$, the mapping $x \mapsto \mathcal{I}_{\psi_\gamma}(\rho; x)$ is increasing on $[0, \infty)$. Hence, we deduce from the above theorem that the entire function $z \mapsto \mathcal{I}_{\psi_\gamma}(\gamma_\alpha; z)$ has no real zeros.

In the above theorem, the constant C_γ is characterized by the behavior of the function $\mathcal{I}_{\psi_\gamma}(\gamma_\alpha; -t)$ for large values of t. In what follows, we provide some representations of this constant in terms of the Laplace exponent ψ_γ.

PROPOSITION 2.5.

(1) *If* $\Lambda = \alpha b$, *then*

$$C_\gamma = \alpha^{\gamma_\alpha} a_{-\gamma_\alpha}(\varphi_\gamma; \alpha),$$

where $\varphi_\gamma(u) = b - \int_0^\infty e^{-ur} \int_{-\infty}^{-r} e^{\gamma v} \nu(dv)\, dr$.

(2) *Otherwise, we have*

$$C_\gamma = \begin{cases} \psi'_\gamma(0^+), & \text{if } \gamma_\alpha = 1, \\ \alpha^n \psi'_\gamma(0^+) \left(\displaystyle\prod_{k=1}^{n} \varphi_\gamma(\alpha k) \right)^{-1}, & \text{if } \gamma_\alpha = n+1, n = 1, 2 \ldots, \\ \dfrac{\alpha^{2\gamma_\alpha}}{\Gamma(1-\gamma_\alpha)} a_{-\gamma_\alpha}(\bar{\varphi}_\gamma; \alpha), & \text{otherwise}, \end{cases}$$

where $\varphi_\gamma(\alpha u) = \psi_\gamma(\alpha u)/\alpha u$ *and*

$$\bar{\varphi}_\gamma(u) = \frac{\hat{b}}{u} + \frac{\sigma}{2} + \int_0^\infty e^{-ur} \int_{-\infty}^{-r} \int_{-\infty}^{-s} e^{\gamma v} \nu(dv)\, ds\, dr$$

with $\hat{b} = \bar{b} + \sigma\gamma + \int_{-\infty}^{0} (e^{\gamma r} - \mathbb{I}_{\{|r|<1\}}) r \nu(dr)$.

(3) *Finally, if* $q = 0$ *and* $0 < \gamma_\alpha < 1$, *then*

$$\mathcal{I}_\psi(r) \sim C_\gamma \Gamma(1-\gamma_\alpha) r^{\gamma_\alpha} \mathcal{I}_{\psi_\gamma}(r) \qquad \text{as } r \to \infty,$$

where $\mathcal{I}_\psi(r) = \sum_{n=0}^\infty a_n(\psi; \alpha) r^n$ *is an entire function.*

3. Proofs.

3.1. *A useful analytic continuation.* The first claim of Proposition 2.1 follows from the discussion preceding the proposition. Thus, let us assume that

$\rho \neq 0, -1, \ldots$ Since $\psi'_\gamma(0^+) > 0$, ψ_γ is well defined and analytic in the positive right half-plane and $\psi_\gamma(u) > 0$ for any $u > 0$. Our next aim is to extend the coefficients $a_n(\psi_\gamma, \alpha)$ to a function of the complex variable. Since the paths of the Lévy process ξ are of bounded variation, its Laplace exponent ψ_γ admits the following representation (see [3], Section VII.3):

$$\psi_\gamma(u) = u\big(b - \hat{v}_\gamma(u)\big),$$

where $\hat{v}_\gamma(u) = \int_0^\infty e^{-ur} \int_{-\infty}^{-r} e^{\gamma v} \nu(dv)\, dr$. Thus, for any $n \geq 0$, we have

$$a_n(\psi_\gamma, \alpha) = \frac{1}{\Gamma(n+1)\alpha^n} a_n(\varphi_\gamma; \alpha)$$

with $a_n(\varphi_\gamma; \alpha)^{-1} = \prod_{k=1}^n \varphi_\gamma(\alpha k)$ and $a_0(\varphi_\gamma; \alpha) = 1$. It is plain that the mapping $\hat{v}_\gamma$ is analytic in $F_{-\gamma} = \{s \in \mathbb{C}; \mathfrak{Re}(s) > -\gamma\}$ and $\hat{v}_\gamma(u)$ is decreasing on $\mathbb{R}^+$ with $0 < \hat{v}_\gamma(0) < b$ since $\psi_\gamma(0^+) > 0$. Then, we may write

$$\begin{aligned} a_s(\psi_\gamma; \alpha) &= \frac{1}{\Gamma(s+1)\alpha^s} a_s(\varphi_\gamma; \alpha) \\ &= \frac{1}{\Gamma(s+1)\alpha^s} \prod_{k=1}^\infty \frac{\varphi_\gamma(\alpha(k+s))}{\varphi_\gamma(\alpha k)}, \end{aligned}$$

where the infinite product is easily seen to be absolutely convergent for any $\mathfrak{Re}(s) > 0$ by taking the logarithm and noting that $|\hat{v}_\gamma(s)| \leq \hat{v}_\gamma(\mathfrak{Re}(s))$; see, for example, [29], Section 1.41. Moreover, $a_s(\varphi_\gamma; \alpha)$ satisfies the functional equation

$$a_{s+1}(\varphi_\gamma; \alpha) = \frac{1}{\varphi_\gamma(\alpha(s+1))} a_s(\varphi_\gamma; \alpha),$$

which shows that $a_s(\varphi_\gamma; \alpha)$ is analytic in the half-plane $F_{-\gamma-1} = \{s \in \mathbb{C}; \mathfrak{Re}(s) > -1 - \gamma\}$. Consequently, $a_s(\varphi_\gamma; \alpha)$ is bounded on any closed subset of $F_{-\gamma-1}$. Then, we set $G(s) = \Gamma(s+\rho)\Gamma(-s)a_s(\varphi_\gamma; \alpha)$ and define

$$\mathfrak{I}_{\mathfrak{L}_R} = -\frac{1}{2i\pi\Gamma(\rho)} \int_{\mathfrak{L}_R} G(s)\left(-\frac{z}{\alpha}\right)^s ds,$$

where the integral is taken in a clockwise direction round the contour $\mathfrak{L}_R$, consisting of a large semi-circle, of center the origin and radius R, lying to the right of the imaginary axis. This contour is intended to ensure that all poles (resp., nonnegative poles) of $\Gamma(\rho + s)$ [resp., $\Gamma(-s)$] lie to the left (resp., right) of the intended imaginary axis. This contour is always possible since we have assumed that $\rho \neq 0, -1, \ldots$. We can split $\mathfrak{I}_{\mathfrak{L}_R}$ up into two integrals, $\mathfrak{I}_{\mathfrak{A}_{iR}}$ along the imaginary axis and, writing $s = Re^{i\theta}$,

$$\mathfrak{I}_{\mathfrak{C}_R} = -\frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} G(Re^{i\theta})\left(-\frac{z}{\alpha}\right)^{Re^{i\theta}} Re^{i\theta}\, d\theta.$$

Recalling the following well-known asymptotic formulae (see, e.g., [20], Section 2.4), as $|s| \to \infty$,

$$\Gamma(s+\rho) \sim \sqrt{2\pi} e^{-Re^{i\theta}} R^{Re^{i\theta}+\rho-1/2} e^{i\theta(Re^{i\theta}+\rho-1/2)}, \qquad |\theta| < \pi,$$

$$\Gamma(-s) \sim e^{-\pi R|\sin\theta|} e^{Re^{i\theta}} R^{-Re^{i\theta}-1/2} e^{i\theta(-Re^{i\theta}-1/2)}, \qquad |\theta| < \pi,$$

and

$$\left|\left(-\frac{z}{\alpha}\right)^s\right| \sim |\alpha z|^{R\cos\theta} e^{-R\sin\theta \arg(-z)},$$

we deduce that as $|s| \to \infty$

$$\left|G(s)\left(-\frac{z}{\alpha}\right)^s\right| \sim aR^{\Re\mathfrak{e}(\rho)-1}|\alpha z|^{R\cos\theta} \times \begin{cases} e^{-R|\sin\theta|(\pi+\arg(-z))}, & 0 < \theta \le \pi/2, \\ e^{-R|\sin\theta|(\pi-\arg(-z))}, & -\pi/2 \le \theta < 0, \end{cases} \tag{3.1}$$

where a is a positive constant. On the one hand, along the path $\mathfrak{A}_{iR}$ we have $\theta = \pm\frac{\pi}{2}$ and thus as $|z| \to \infty$

$$\left|G(s)\left(-\frac{z}{\alpha}\right)^s\right| \sim aR^{\Re\mathfrak{e}(\rho)-1} e^{\pm(\pi/2)\Im(\rho)} \begin{cases} e^{-R(\pi+\arg(-z))}, & \theta = \pi/2, \\ e^{-R(\pi-\arg(-z))}, & \theta = -\pi/2. \end{cases}$$

For the integral (2.3) to converge absolutely, it is therefore required that $|\arg(-z)| < \pi$. On the other hand, the asymptotic estimate (3.1) gives, as $R \to \infty$,

$$\mathfrak{I}_{\mathfrak{C}_R} \to 0 \qquad \text{if } |z| < 1 \text{ and } |\arg(-z)| < \pi.$$

Thus, as $R \to \infty$,

$$\mathfrak{I}_{\mathfrak{L}_R} \to -\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} G(s)\left(-\frac{z}{\alpha}\right)^s ds.$$

Finally, evaluating $\mathfrak{I}_{\mathfrak{L}_R}$ by the Cauchy integral theorem and letting $R \to \infty$, we get

$$\frac{1}{2i\pi}\int_{-i\infty}^{i\infty} G(s)\left(-\frac{z}{\alpha}\right)^s ds = \frac{1}{\Gamma(\rho)}\sum_{n=0}^{\infty} a_n(\psi_\gamma,\alpha)\Gamma(\rho+n)z^n,$$

$$|z| < 1 \text{ and } |\arg(-z)| < \pi.$$

Therefore, the integral (2.3) offers an analytic continuation of the mapping $z \mapsto \mathcal{I}_{\psi_\gamma}(\rho; z)$ in the entire complex plane cut along the positive real axis. Moreover, we deduce from such an analytic continuation that the power series (2.2) has an unique singularity on the circle $|z| = \alpha b$ located at the point $z = \alpha b > 0$. Now, following a device developed for hypergeometric series (see Nørlund [19], pages 294 and 295), we introduce the function $\mathcal{H}$ defined for some $a \in \mathbb{C}$ by

$$\mathcal{H}_{\psi_\gamma,a}(\rho; z) = (1-z)^{-\rho}\mathcal{I}_{\psi_\gamma}\left(\rho; \frac{a\alpha bz}{z-1}\right).$$

Note that

$$\mathcal{I}_{\psi_\gamma}(\rho; az) = \left(1 - \frac{z}{\alpha b}\right)^{-\rho} \mathcal{H}_{\psi_\gamma,a}\left(\rho; \frac{z}{z - \alpha b}\right). \tag{3.2}$$

Thus, denoting by $(b_n)_{n\geq 0}$ the coefficients of the power series $\mathcal{H}_{\psi_\gamma,a}(\rho; z)$, we have $b_0 = a_0$ and by means of residues calculus, with $\mathfrak{C}$ a circle around 0 of small radius and with positive orientation, we have for $n \geq 1$,

$$\begin{aligned} b_n &= \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{\mathcal{H}_{\psi_\gamma,a}(\rho; z)}{z^{n+1}} \, dz \\ &= (-1)^n \frac{1}{2\pi i} \int_{\mathfrak{C}} (1 - z)^{-\rho} \mathcal{I}_{\psi_\gamma}\left(\rho; \frac{a\alpha b z}{z - 1}\right) \frac{dz}{z^{n+1}} \, dv \\ &= \frac{1}{\Gamma(\rho)} \sum_{k=0}^{n} (-a\alpha b)^k a_k(\psi_\gamma; \alpha) \frac{\Gamma(\rho + n)}{\Gamma(n - k + 1)}. \end{aligned}$$

Thus, one gets

$$(1 - z)^{-\rho} \mathcal{I}_{\psi_\gamma}\left(\rho; \frac{a\alpha b z}{z - 1}\right) = \sum_{n=0}^{\infty} \mathcal{I}_{\psi_\gamma}(-n; a\alpha b) \frac{\Gamma(\rho + n)}{\Gamma(\rho) n!} z^n.$$

From Weierstrass's double series theorem, the above identity is true if $|z| < \frac{1}{1+|a|}$. Moreover, the function on the left-hand side has a singularity at $z = 1$ and $z = \frac{1}{1-a}$. Thus, the series on the right-hand side is convergent if $|z| < 1$ and $|z(1 - a)| < 1$. By choosing $a = 1$, we conclude by observing that the series on the right-hand side of (3.2) is convergent for $\mathfrak{Re}(z) < \frac{\alpha b}{2}$.

3.2. *The distribution of* T_0. We proceed by introducing the Ornstein–Uhlenbeck process $U = (U_t)_{t\geq 0}$ defined by

$$U_t = e^{\tilde{\alpha} t} X_{\tau(t)}, \qquad t \geq 0,$$

where $\tilde{\alpha} = \alpha^{-1}$ and $\tau(t) = 1 - e^{-t}$. Next, we put

$$H_0 = \inf\{s > 0; U_s = 0\}$$

and set

$$1 - K(x) = \mathbb{Q}_x(H_0 < \infty), \qquad x > 0.$$

We are now ready to state the following.

PROPOSITION 3.1. *Assume that the condition* H *holds. Then, for any* $x > 0$ *and* $t > 0$, *we have*

$$K(xt^{-\tilde{\alpha}}) = \mathbb{Q}_x(T_0 \geq t) \tag{3.3}$$

and P *is increasing on* $\mathbb{R}^+$ *with* $\lim_{x\to\infty} K(x) = 1$ *and* $K(0) = 0$.

PROOF. First, a simple time change yields the following identity in distribution:

$$H_0 \overset{(d)}{=} -\log(1 - T_0 \wedge 1).$$

Thus, we deduce that

$$\begin{aligned} 1 - K(x) &= \mathbb{Q}_x(H_0 < \infty) \\ &= \mathbb{Q}_x(T_0 < 1). \end{aligned}$$

Then, invoking the self-similarity property of X we obtain the identity

$$\mathbb{Q}_x(T_0 \geq t) = \mathbb{Q}_{xt^{-\tilde{\alpha}}}(T_0 \geq 1)$$

from which we deduce the identity (3.3) and the properties stated on P. □

According to Proposition 3.1, our goal now is to derive an expression of the function $K(x) = 1 - \mathbb{Q}_x(H_0 < \infty)$, $x > 0$. Relying on the following identity:

$$K(x) = \lim_{a\to\infty} \lim_{q\to 0} \mathbb{E}_x\left[e^{-qH_a} \mathbb{I}_{\{H_a < H_0\}}\right],$$

where $H_a = \inf\{s > 0; U_s \geq a\}$, the problem reduces to the computation of the functional $\mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<H_0\}}]$. Actually, for technical reasons, we must deal first with the functional $\mathbb{E}_x^{(\gamma)}[e^{-qH_a}]$ which is the Laplace transform of the first passage time above for the Ornstein–Uhlenbeck process associated to the pssMp X with underlying Laplace exponent ψ_γ. Finally, by means of Doob h-transform arguments, we will be able to relate the latter functional to the former one.

We use the notation introduced in Theorem 2.3 and take first X with underlying Laplace exponent ψ_γ. We denote its law (resp., its expectation operator) by $\mathbb{Q}^{(\gamma)}$ (resp., $\mathbb{E}^{(\gamma)}$). In order to simplify the notation we set, without loss of generality, $\alpha = 1$. We recall that $\psi_\gamma(0) = 0$ and $\psi_\gamma(0^+) > 0$ and hence the condition H does not hold. Next, we simply write $Q^{(\gamma)} = (Q_t^{(\gamma)})_{t\geq 0}$ for the semigroup of X, that is, for any bounded Borelian function g and $t, x > 0$, one has

$$Q_t^{(\gamma)} g(x) = \mathbb{E}_x^{(\gamma)}[g(X_t)].$$

From [5], we have that $Q^{(\gamma)}$ is a Feller semigroup on $[0, \infty)$. Next, we say, for any $r \in \mathbb{R}$, that a function I is r-invariant for $Q^{(\gamma)}$ if

$$e^{-rt} Q_t^{(\gamma)} I(x) = I(x), \qquad x > 0.$$

We start with the following lemma which is obtained readily from [27], Theorem 1.

LEMMA 3.2. *For any $r > 0$, the mapping $x \mapsto \mathcal{I}_{\psi_\gamma}(-rx)$ is $-r$-invariant for $Q^{(\gamma)}$.*

Following a device developed by the author in [22], we show how to construct some specific time–space invariant functions for the semigroup $Q^{(\gamma)}$ in terms of its r-invariant functions. We now state the following result which is a slight generalization of [22], Theorem 1 and Corollary 3.2.

LEMMA 3.3. *For any* $\mathfrak{Re}(\rho) > 0$, *the mapping* $x \mapsto \mathcal{I}_{\psi_\gamma}(\rho; -x)$ *satisfies the identity, for any* $0 \leq t < 1$,

$$(1-t)^{-\rho} Q_t^{(\gamma)}\big(d_{(1-t)^{-1}}\mathcal{I}_{\psi_\gamma}\big)(\rho; -x) = \mathcal{I}_{\psi_\gamma}(\rho; -x), \qquad x > 0, \tag{3.4}$$

where $d_c f(x) = f(cx)$, $c > 0$.

Next, we introduce the stopping time D_a defined, for any $a > 0$, by

$$D_a = \inf\{0 < s \leq 1; X_s = a(1-s)\}.$$

Writing $(a)_+ = \max(a, 0)$, we have

$$e^{-H_a} \overset{(d)}{=} (1 - D_a)_+ \tag{3.5}$$

and, in particular, for $a = 0$, since $D_0 \overset{(d)}{=} T_0 \wedge 1$, we obtain

$$e^{-H_0} \overset{(d)}{=} (1 - T_0)_+.$$

For any $a > 0$, we set

$$\kappa(a) = \inf\{\kappa \in \mathbb{R}^+; \mathcal{I}_{\psi_\gamma}(\kappa; -a) = 0\}$$

with the usual convention that $\inf\{\varnothing\} = \infty$, for the smallest positive real zero of the function $\mathcal{I}_{\psi_\gamma}(\cdot; -a)$. We are now ready to state the following.

COROLLARY 3.4. *Let* $0 \leq x \leq a$. *Then, for any* $\rho \in \mathbb{C}$ *with* $\mathfrak{Re}(\rho) < \kappa(a)$, *we have*

$$\mathbb{E}_x^{(\gamma)}[(1 - D_a)_+^{-\rho}] = \frac{\mathcal{I}_{\psi_\gamma}(\rho; -x)}{\mathcal{I}_{\psi_\gamma}(\rho; -a)}.$$

Consequently, for any real κ *such that* $\kappa < \kappa(a)$, *the mapping* $x \mapsto \mathcal{I}_{\psi_\gamma}(\kappa; -x)$ *is positive on* $\mathbb{R}^+$.

PROOF. Since X under $\mathbb{Q}^{(\gamma)}$ is a Feller process on $[0, \infty)$, we can start by fixing $x = 0$ and $a > 0$. Then, recalling that $\mathcal{I}_{\psi_\gamma}(0, -a) = 1$, we observe that $\mathcal{I}_{\psi_\gamma}(\kappa; -a)$ is positive for any $0 \leq \kappa < \kappa(a)$ reals. The existence of such an interval follows from the fact that the zeros of a nonconstant holomorphic function are isolated. Thus, by combining the identity (3.4) with the Dynkin formula (see, e.g., [11], Theorem 12.4), applied to the bounded stopping time D_a, we deduce, for any $0 \le \kappa < \kappa(a)$, that

$$\mathbb{E}_0^{(\gamma)}[(1-D_a)_+^{-\kappa}] = \frac{1}{\mathcal{I}_{\psi_\gamma}(\kappa; -a)}. \tag{3.6}$$

Next, we recall, from identity (3.5), that

$$e^{\kappa H_a} \stackrel{(d)}{=} (1-D_a)_+^{-\kappa}.$$

Since H_a is a positive random variable, as a Laplace transform, the left-hand side on identity (3.6) is analytic in the half-plane $\{\rho \in \mathbb{C}; \mathfrak{Re}(\rho) < \kappa(a)\}$ and positive on $\mathbb{R}^+$; see, for example, [30], Chapter II. Then, let us assume that there exists a complex number $\rho(a)$ in the strip $0 \le \mathfrak{Re}(\rho(a)) < \kappa(a)$ such that $\mathcal{I}_{\psi_\gamma}(\rho(a); -a) = 0$. However, as the left-hand side of (3.6) is analytic with respect to the argument κ in this strip, we deduce, by the principle of analytic continuation, that this is not possible. Moreover, we get that $\mathcal{I}_{\psi_\gamma}(\rho; -a)$ has no zeros on $\{\rho \in \mathbb{C}; \mathfrak{Re}(\rho) < \kappa(a)\}$ and is positive on $\{\kappa \in \mathbb{R}; \kappa < \kappa(a)\}$. Finally, let us consider a real number a_1 such that $0 < a_1 \le a$. Clearly, $\mathbb{Q}_0^{(\gamma)}$-a.s. $(1-D_{a_1})_+^{-\kappa} \le (1-D_a)_+^{-\kappa}$, for any $0 \le \kappa < \kappa(a) \wedge \kappa(a_1)$. Then we deduce from (3.6), for any $0 \le \kappa < \kappa(a) \wedge \kappa(a_1)$, that

$$0 < \frac{1}{\mathcal{I}_{\psi_\gamma}(\kappa; -a_1)} \le \frac{1}{\mathcal{I}_{\psi_\gamma}(\kappa; -a)}.$$

Thus, it is not difficult to see that $\kappa(a_1) \ge \kappa(a)$. Therefore, since $\kappa(x) \ge \kappa(a)$, for any $0 \le x \le a$, the strong Markov property and the absence of positive jumps of X complete the proof. $\square$

The choice of starting our computation under the law $\mathbb{Q}^{(\gamma)}$ was motivated by the previous proof where it was necessary to start X at 0 in order to get some information about the sign of the function $\mathcal{I}_{\psi_\gamma}(\kappa, -a)$. This device would not have been possible under $\mathbb{Q}$. We proceed to the proof of Theorem 2.3 which we now split into two parts: the case when X reaches 0 continuously, that is, $q = 0$ and $\mathrm{E}[\xi_1] < 0$ and the case when X reaches 0 by a jump, that is, $q > 0$.

3.2.1. *Continuous killing*. Here, we assume that $q = 0$ and $\mathrm{E}[\xi_1] < 0$. Thus, in this case, $\gamma = \phi(0)$ and $\psi_\gamma(u) = \psi(\gamma + u)$ with $\psi'_\gamma(0^+) > 0$.

LEMMA 3.5. *Writing $\kappa'(a) = \kappa(a) - \gamma > 0$, we have, for any $\kappa < \kappa'(a)$ and $0 < x \le a$,*

$$\mathbb{E}_x\big[(1-D_a)^{-\kappa}\mathbb{I}_{\{D_a < T_0 \wedge 1\}}\big] = \frac{x^\gamma}{a^\gamma}\frac{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma; -x)}{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma; -a)}.$$

In particular, for any $0 < x \leq a$, *we have*

$$\mathbb{Q}_x[D_a < T_0 \wedge 1] = \frac{x^\gamma}{a^\gamma} \frac{\mathcal{I}_{\psi_\gamma}(\gamma; -x)}{\mathcal{I}_{\psi_\gamma}(\gamma; -a)}.$$

PROOF. We start by using the fact that the function $x \mapsto x^{-\gamma}$ is excessive for $Q_t^{(\gamma)}$; see, for example, [28]. In particular, one has, for any $t > 0$ and for any F a $\mathcal{F}_t$-measurable and bounded random variable,

$$\mathbb{E}_x^{(\gamma)}[F] = \mathbb{E}_x[X_t^\gamma F, t < T_0], \qquad x > 0.$$

Note that this relation also holds for any $\mathcal{F}_\infty$-stopping time. Moreover, proceeding as in the proof of Corollary 3.4, one gets that the Mellin transform of the positive random variable $(1 - D_a)_+$ is well defined for any real κ such that $\kappa \leq 0$. Thus, since X has no positive jumps, one obtains by means of both Corollary 3.4 and the optional stopping theorem, for any $\kappa \leq 0$,

$$\begin{aligned} \mathbb{E}_x\big[(1 - D_a)_+^{-\kappa} \mathbb{I}_{\{D_a < T_0\}}\big] &= \frac{x^\gamma}{a^\gamma} \mathbb{E}_x^{(\gamma)}\big[(1 - D_a)_+^{-(\kappa+\gamma)}\big] \\ &= \frac{x^\gamma}{a^\gamma} \frac{\mathcal{I}_{\psi_\gamma}(\kappa + \gamma; -x)}{\mathcal{I}_{\psi_\gamma}(\kappa + \gamma; -a)}. \end{aligned}$$

We deduce that $\kappa'(a) > 0$ and the proof is completed by letting $\kappa \to 0$. $\square$

We are now ready to complete the proof of Theorem 2.3 in the case $\gamma = \phi(0)$. One gets that

$$\begin{aligned} \mathbb{Q}_x[D_a < T_0 \wedge 1] &= \mathbb{Q}_x[\tau(H_a) < \tau(H_0) \wedge 1] \\ &= \mathbb{Q}_x[H_a < H_0] \end{aligned}$$

since τ is increasing and $\tau^{-1}(1) = \infty$. Thus, as X has no positive jumps, one deduces that

$$\begin{aligned} \lim_{a \to \infty} \mathbb{Q}_x[D_a < T_0 \wedge 1] &= \mathbb{Q}_x[H_0 = \infty] \\ &= K(x). \end{aligned}$$

As we have learnt from Corollary 3.4 and Lemma 3.5 that the mapping $x \mapsto \mathcal{I}_{\psi_\gamma}(\gamma; -x)$ is positive on $\mathbb{R}^+$, it means that there exists a constant $C_\gamma > 0$ such that

$$\mathcal{I}_{\psi_\gamma}(\gamma; -x) \sim C_\gamma^{-1} x^{-\gamma} \qquad \text{as } x \to \infty.$$

Then, recalling that $\lim_{x\to\infty} K(x) = 1$, we obtain

$$K(x) = C_\gamma x^\gamma \mathcal{I}_{\psi_\gamma}(\gamma; -x).$$

Hence, we deduce the expression of S from the identity $S(t) = K(t^{-1})$. Finally, the series $\mathcal{I}_{\psi_\gamma}(\gamma; -x)$ being absolutely continuous, the expression of the density s is obtained by differentiating terms by terms. Indeed, one has

$$\begin{aligned} s(t) &= -\frac{d}{dt}S(t) \\ &= C_\gamma t^{-\gamma-1}\frac{1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}(-1)^n a_n(\psi)(\gamma+n)\Gamma(\gamma+n)t^{-n} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}C_\gamma t^{-\gamma-1}\mathcal{I}_{\psi_\gamma}(1+\gamma; -t). \end{aligned}$$

The expression of the successive derivatives are obtained by means of an induction argument.

3.2.2. *X reaches 0 by a jump*. Throughout this part, we assume that ξ is a spectrally negative Lévy process killed at some independent exponential time of parameter $q > 0$. Recall that, for any $u \geq 0$, $\bar{\psi}(u) = \psi(u) - q$, ϕ is such that $\psi \circ \phi(u) = u$ and with $\gamma = \phi(q)$, we easily see that $\psi_\gamma(u) = \bar{\psi}(u+\gamma)$ and $\psi'_\gamma(0^+) > 0$.

LEMMA 3.6. *Writing $\kappa'(a) = \kappa(a) - \gamma > 0$, we have, for any $\kappa < \kappa'(a)$ and $0 < x \leq a$,*

$$\mathbb{E}_x\big[(1 - D_a)_+^{-\kappa}\mathbb{I}_{\{D_a < T_0\}}\big] = \frac{x^\gamma}{a^\gamma}\frac{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma; -x)}{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma; -a)}.$$

In particular,

$$\mathbb{Q}_x[D_a < T_0 \wedge 1] = \frac{x^\gamma}{a^\gamma}\frac{\mathcal{I}_{\psi_\gamma}(\gamma; -x)}{\mathcal{I}_{\psi_\gamma}(\gamma; -a)}.$$

REMARK 3.7. Writing $D_a^+ = \inf\{s > 0; X_s = a(1+s)\}$ and $K(x; a) = \mathbb{Q}_x[D_a < T_0 \wedge 1]$, we deduce from [22], Corollary 3.2, the following identity:

$$\mathbb{Q}_x[D_a^+ < T_0] = K(-x, -a), \qquad 0 < x \leq a < \Lambda.$$

It would be interesting to prove such a formula directly from the definition of D_a and D_a^+.

PROOF OF LEMMA 3.6. Let us observe from the Lamperti mapping (1.1) that the semigroup $(Q_t)_{t\geq 0}$ of X is given for a function f positive and measurable on $\mathbb{R}^+$ by

$$Q_t f(x) = \mathbb{E}_x^q[e^{-qA_t}f(X_t)], \qquad t \geq 0, x > 0,$$

where $\mathbb{E}^q$ stands for the expectation operator associated to the law of X with underlying Laplace exponent ψ. Thus, for any $\mathcal{F}_\infty$-stopping time T, one has

$$\mathbb{E}_x[f(X_T)] = \mathbb{E}_x^q[e^{-qA_T} f(X_T)].$$

Moreover, as ξ has independent increments, it is plain that the process $(e^{-qt+\gamma\xi_t})_{t\geq 0}$ is a P^q-martingale, where P^q stands for the law of the Lévy process with Laplace exponent ψ. By time change, one deduces that the process $(X_t^\gamma e^{-qA_t})_{t\geq 0}$ is a $\mathbb{Q}_1$-martingale. Thus, one can define a new probability measure, which we denote by $\mathbb{Q}^{(\gamma)}$, as follows, for any $t>0$ and for any F a $\mathcal{F}_t$-measurable and bounded random variable,

$$\mathbb{E}_x^{(\gamma)}[F] = \mathbb{E}_x^q[X_t^\gamma e^{-qA_t} F], \qquad x>0.$$

It is easily seen that the underlying Laplace exponent of X, under $\mathbb{Q}^{(\gamma)}$, is ψ_γ. Hence, one gets by the absence of positive jumps for X and an application of the optional stopping theorem, that, for any $0<x\leq a$ and $\kappa\leq 0$,

$$\begin{aligned}
\mathbb{E}_x\big[(1-D_a)_+^{-\kappa}\mathbb{I}_{\{D_a<T_0\}}\big] &= \mathbb{E}_x^q\big[e^{-qA_{D_a}}(1-D_a)_+^{-\kappa}\mathbb{I}_{\{D_a<T_0\}}\big]\\
&= \left(\frac{x}{a}\right)^\gamma \mathbb{E}_x^{(\gamma)}\big[(1-D_a)_+^{-(\kappa+\gamma)}\big]\\
&= \left(\frac{x}{a}\right)^\gamma \frac{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma;-x)}{\mathcal{I}_{\psi_\gamma}(\kappa+\gamma;-a)},
\end{aligned}$$

where the last line follows from Corollary 3.4 since $\psi_\gamma'(0^+)>0$. The proof of the lemma is complete. $\square$

The proof of the theorem is completed by following a line of reasoning similar to the previous case.

3.3. *Proof of Proposition* 2.5. Let us start by pointing out that it is not difficult to check that we have, in all cases, $C_\gamma>0$. Moreover, let us first assume that $\lim_{u\to\infty}\frac{\psi(u)}{u}=b$. From Proposition 2.1, we have

$$\mathcal{I}_{\psi_\gamma}(\gamma_\alpha;-z) = \frac{1}{2i\pi\Gamma(\gamma_\alpha)}\int_{-i\infty}^{i\infty} a_s(\varphi_\gamma;\alpha)\Gamma(s+\gamma_\alpha)\Gamma(-s)\left(\frac{z}{\alpha}\right)^s ds, \qquad |\arg(z)|<\pi.$$

Hence, upon displacement of the path to the left in order to include the first pole of $\Gamma(s+\gamma_\alpha)$ we obtain, from Theorem 2.3 and a residue computation, that

$$\mathcal{I}_{\psi_\gamma}(\gamma_\alpha;-z) = \alpha^{\gamma_\alpha} a_{-\gamma_\alpha}(\varphi_\gamma;\alpha) z^{-\gamma_\alpha} + o(z^{-\gamma_\alpha}),$$

which gives the characterization of C_γ in this case.

For the other case, that is, when $\Lambda = +\infty$, one may follow a line of reasoning similar to the proof of Proposition 2.1. Indeed, as $0 < \psi'_\gamma(0^+) < \infty$, we have, for any $u > 0$,

$$\begin{aligned}\psi_\gamma(\alpha u) &= \hat{b}\alpha u + \frac{\sigma}{2}(\alpha u)^2 + \int_{-\infty}^{0}(e^{\alpha u r} - 1 - \alpha u r)e^{\gamma r}\nu(dr)\\ &= (\alpha u)^2\bar{\varphi}_\gamma(\alpha u),\end{aligned}$$

where $\hat{b} = \bar{b} + \sigma\gamma + \int_{-\infty}^{0}(e^{\gamma r} - \mathbb{I}_{\{|r|<1\}})r\nu(dr)$ and

$$\bar{\varphi}_\gamma(\alpha u) = \frac{\hat{b}}{\alpha u} + \frac{\sigma}{2} + \int_0^\infty e^{-\alpha u r}\int_{-\infty}^{-r}\int_{-\infty}^{-s} e^{\gamma v}\nu(dv)\,ds\,dr.$$

Thus, as above, one may define the function

$$\begin{aligned}a_s(\psi_\gamma;\alpha) &= \frac{1}{\alpha^2\Gamma^2(s+1)}a_s(\bar{\varphi}_\gamma;\alpha)\\ &= \frac{1}{\alpha^2\Gamma^2(s+1)}\prod_{k=1}^{\infty}\frac{\bar{\varphi}_\gamma(\alpha(k+s+1))}{\bar{\varphi}_\gamma(\alpha k)}\end{aligned}$$

and observe the identity

$$a_{s+1}(\bar{\varphi}_\gamma;\alpha) = \frac{1}{\bar{\varphi}_\gamma(\alpha(s+1))}a_s(\bar{\varphi}_\gamma;\alpha)$$

with $a_0(\bar{\varphi}_\gamma;\alpha) = 1$. Hence, $a_s(\bar{\varphi}_\gamma;\alpha)$ is a meromorphic function in $F_{-\gamma} = \{s \in \mathbb{C}; \mathfrak{Re}(s) > -\gamma - 1\}$ with simple poles at the points $s_k = -k-1$ for $k = 0, 1, \dots$ and $s_k > -\gamma - 1$. We obtain, writing $\bar{G}(s) = \frac{a_s(\bar{\varphi}_\gamma;\alpha)}{\Gamma(s+1)}\Gamma(s+\gamma_\alpha)\Gamma(-s)$, the following identity:

$$\mathcal{I}_{\psi_\gamma}(\gamma_\alpha; -z) = \frac{1}{2i\pi\Gamma(\gamma_\alpha)}\int_{-i\infty}^{i\infty}\bar{G}(s)\left(\frac{z}{\alpha^2}\right)^s ds,$$

which is now valid in the sector $|\arg(z)| < \pi/2$. As above, after a displacement of the path to the left in order to include the first pole of $\Gamma(s+\gamma_\alpha)$ we obtain, from Theorem 2.3 and a residue computation, that

$$\begin{aligned}\mathcal{I}_{\psi_\gamma}(\gamma_\alpha; -z) = \frac{1}{\Gamma(\gamma_\alpha)}\Bigg(&\sum_{k=1}^{[\gamma_\alpha]}\frac{\Gamma(k)\operatorname{Res}_{j=-k}a_j(\bar{\varphi}_\gamma;\alpha)}{\Gamma(1-k)}\left(\frac{z}{\alpha^2}\right)^{-k}\\ &+ \operatorname{Res}_{s=-\gamma_\alpha}\bar{G}(s)\left(\frac{z}{\alpha^2}\right)^{-s}\Bigg)\\ &+ o(z^{-\gamma_\alpha}),\end{aligned}$$

where the sum is 0 if $[\gamma_\alpha]$, the integer part of γ_α, is lower than 1. Since $a_s(\bar{\varphi}_\gamma,\alpha)$ has a simple pole at $j = -1, \dots, -[\gamma_\alpha]$, the terms in the sum vanish. Hence, if

γ_α is not an integer $\bar{G}(s)$ has a simple pole at $-\gamma_\alpha$ and the expression of C_γ follows readily in this case. If $\gamma_\alpha = n+1$, then $\bar{G}(s)$ has a double pole at $-(n+1)$ and using the recurrence relations of both the gamma function and $a_s(\bar{\varphi}_\gamma;\alpha)$, we deduce that

$$\begin{aligned}
&\mathrm{Res}_{s=-(n+1)}\,\bar{G}(s)\\
&\quad= \lim_{s\to -n-1}\frac{d}{ds}\big((s+n+1)^2\bar{G}(s)\big)\\
&\quad= \lim_{s\to -n-1}\frac{d}{ds}\left(\alpha^{-n-2}\prod_{k=1}^{n}\varphi_\gamma\big(\alpha(s+k)\big)\psi_\gamma\big(\alpha(s+n+1)\big)a_{s+n+1}(\bar{\varphi}_\gamma;\alpha)\right)\\
&\quad= \alpha^{-n-2}\Gamma(n+1)\psi_\gamma'(0^+)\prod_{k=1}^{n}\varphi(\alpha k)
\end{aligned}$$

and the result follows. The second part of the proposition is proved as follows. Let us recall that in [27], the expression of the Laplace transform of T_0, in the case $\mathrm{E}[\xi_1]<0$, $q=0$ and $\gamma<\alpha$ is given for any $r, x \geq 0$ as follows:

$$\mathbb{E}_x[e^{-rT_0}] = \mathcal{N}_{\psi,\gamma}(rx), \tag{3.7}$$

where

$$\mathcal{N}_{\psi,\gamma}(r) = \mathcal{I}_\psi(r) - C(\gamma)r^{\gamma_\alpha}\mathcal{I}_{\psi_\theta}(r)$$

and the positive constant $C(\gamma)$ is characterized by

$$\mathcal{I}_\psi(r) \sim C(\gamma)r^{\gamma_\alpha}\mathcal{I}_{\psi_\gamma}(r) \qquad \text{as } r\to\infty.$$

Next, let us write $\hat{F}(r) = \mathbb{E}_1[1-e^{-rT_0}]$. Then, from (3.7), one deduces easily that

$$\hat{F}(r) \sim C(\gamma)r^{\gamma_\alpha} \qquad \text{as } r\to 0,$$

which is equivalent, according to Bingham, Goldie and Teugels [7], Corollary 8.1.7, to

$$S(t) \sim \frac{C(\gamma)}{\Gamma(1-\gamma_\alpha)}t^{-\gamma_\alpha} \qquad \text{as } t\to\infty,$$

which completes the proof.

4. Some final remarks and illustrative examples.

We start by offering a few consequences of Theorem 2.3.

COROLLARY 4.1. *With the notation used and introduced in Theorem 2.3, we have, writing* $s^{(m)} = \frac{d^m}{dt^m}s$,

$$s^{(m)}(t) = (-1)^m\frac{\Gamma(m+1+\gamma_\alpha)}{\Gamma(\gamma_\alpha)}C_\gamma t^{-\gamma_\alpha-1-m}\mathcal{I}_{\psi_\gamma}(m+1+\gamma_\alpha;-t^{-1}), \qquad t>0.$$

Moreover,

$$S(t) \sim C_\gamma t^{-\gamma_\alpha} \qquad \text{as } t \to \infty$$

and, for any $m = 0, 1 \ldots,$

$$s^{(m)}(t) \sim (-1)^m C_\gamma \frac{\Gamma(m+1+\gamma_\alpha)}{\Gamma(\gamma_\alpha)} t^{-\gamma_\alpha - 1 - m} \qquad \text{as } t \to \infty.$$

As pointed out by several authors (see Carmona, Petit and Yor [8], Rivero [28] and Maulik and Zwart [18]) the study of the exponential functional is also motivated by its connection to some interesting random affine equations which have been deeply studied by Kesten [13]. Relying on a result of Kesten, Rivero ([28], Lemma 4) shows that there exists a constant $C > 0$ such that one has the following asymptotic behavior

$$S(t) \sim C t^{-\tilde{\alpha}\gamma} \qquad \text{as } t \to \infty,$$

whenever the Lévy process satisfies a set of conditions. As we have excluded the case when $-\xi$ is a subordinator, it is not difficult to verify that the Lévy processes we consider in this paper satisfy Rivero's conditions. Hence, Theorem 2.3 and Proposition 2.5 offers several characterizations of the Kesten's constant. We also point out that the asymptotic behavior of the density in Corollary 4.1 could not be deduced directly from Rivero's result since we do not know whether or not the density is ultimately monotone.

4.1. *The Bessel processes.* We consider ξ to be a 2-scaled Brownian motion with drift $2b \in \mathbb{R}$ and killed at some independent exponential time of parameter $q > 0$, that is, $\bar{\psi}(u) = 2u^2 + 2bu - q$ and $2\phi(q) = \sqrt{2q + b^2} - b$. Note that $\psi_{\phi(q)}(u) = 2u^2 + (2b + \phi(q))u$. Its associated self-similar process X is well known to be a Bessel process of index b killed at a rate $q \int_0^t X_s^{-2}\, ds$. Moreover, we obtain, setting $\varrho = b + 2\phi(q)$,

$$\begin{aligned}
\mathcal{I}_{\psi_{\phi(q)}}(\rho; -x) &= \frac{\Gamma(\varrho+1)}{\Gamma(\rho)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\rho+n)}{n!\Gamma(n+\varrho+1)} (x/2)^n \\
&= \Phi(\rho, \varrho+1; -x/2),
\end{aligned}$$

where Φ stands for the confluent hypergeometric function. We refer to Lebedev ([15], Section 9) for useful properties of this function. Next, using the following asymptotic:

$$\Phi(\rho, \varrho+1; -x) \sim \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1-\rho)} x^{-\rho} \qquad \text{as } x \to \infty,$$

we get that $C_{\phi(q)} = \frac{\Gamma(\varrho+1-\phi(q))}{2^{\phi(q)}\Gamma(\varrho+1)}$. Thus, we obtain, recalling that, for any $q > 0$, $\varrho - \phi(q) = b + \phi(q) > 0$,

$$\begin{aligned}
s_{\phi(q)}(t) &= \phi(q)\frac{\Gamma(\varrho+1-\phi(q))}{2^{\phi(q)}\Gamma(\varrho+1)} t^{-\phi(q)-1}\Phi\bigl(1+\phi(q), \varrho+1; -(2t)^{-1}\bigr) \\
&= \frac{b+\phi(q)}{2^{\phi(q)}\Gamma(\phi(q))} t^{-\phi(q)-1} \int_0^1 e^{-u/(2t)}(1-u)^{\varrho-\phi(q)-1}u^{\phi(q)}\,du,
\end{aligned}$$

which is expression (5.a) in [31], page 105. Considering now the case $q = 0$ and $b < 0$, we obtain readily that $\phi(0) = -b$ and

$$\begin{aligned}
s_{\phi(0)}(t) &= \frac{2^b}{\Gamma(-b)} t^{b-1}\Phi\bigl(1-b, 1-b; -(2t)^{-1}\bigr) \\
&= \frac{2^b}{\Gamma(-b)} t^{b-1} e^{-1/(2t)}.
\end{aligned}$$

Hence, we deduce the well-known identity $(T_0, \mathbb{Q}_1) \stackrel{(d)}{=} \frac{1}{2G_{-b}}$ where we recall that G_{-b} stands for a Gamma random variable of parameter $-b > 0$.

4.2. *Law of the maximum of spectrally positive stable Lévy processes.* Let Z be an α-stable spectrally negative Lévy process, with $1 < \alpha < 2$. Let us denote by X the process Z killed upon entering into the negative half-line. X is then a pssMp. Next, we denote by $\hat{Z}$ the dual of Z, that is, $\hat{Z} = -Z$ which is a α-stable spectrally positive Lévy process. Then, by means of the translation invariance of Lévy processes, we deduce readily the following identities:

$$\begin{aligned}
\mathbb{Q}_x(T_0 \le t) &= \mathbb{P}_x\Bigl(\inf_{0<s\le t} Z_s \le 0\Bigr) \\
&= \mathbb{P}\Bigl(\max_{0<s\le t} \hat{Z}_s \ge x\Bigr),
\end{aligned}$$

which can be written as follows:

$$\mathbb{P}\Bigl(\max_{0\le s\le t} \hat{Z}_s \ge x\Bigr) = K(xt^{-\alpha}), \qquad x, t > 0. \tag{4.1}$$

The Laplace exponent of the underlying Lévy process of X has been computed Patie [23] in terms of the Pochhammer symbol. Instead of using this expression, we follow an alternative route. Indeed, in [21], the author computed the unique increasing invariant function, say P_+, of the Ornstein–Uhlenbeck process defined by

$$\tilde{U}_t = e^{-t/\alpha} X_{e^t-1}, \qquad t \ge 0.$$

The function P_+, is given, with C a constant to be determined and writing $\tilde{\alpha} = 1/\alpha$, by

$$P_+(x) = Cx^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-\tilde{\alpha})}{\Gamma(\alpha n+\alpha)} \alpha^n x^{\alpha n}$$
$$= Cx^{\alpha-1} {}_2\Psi_1 \left(\begin{matrix} (1,1), (1, 1-\tilde{\alpha}) \\ (\alpha, \alpha) \end{matrix} \middle| \alpha x^{\alpha} \right), \qquad x \geq 0,$$

where ${}_2\Psi_1$ stands for the Wright hypergeometric function. From Remark 3.7, we have $K(x) = K_+(e^{i\pi/\alpha}x)$. Note that $K(0) = 0$ and using the large asymptotic of the function ${}_2\Psi_1$ (details can be found in [24]), we get as $x \to \infty$,

$${}_2\Psi_1 \left(\begin{matrix} (1,1), (1, 1-\tilde{\alpha}) \\ (\alpha, \alpha) \end{matrix} \middle| -x^{\alpha} \right) \sim \left(\frac{\sin(\tilde{\alpha}\pi)}{\pi} \right)^{-1} x^{1-\alpha}.$$

Hence, by setting $C = \frac{\sin(\tilde{\alpha}\pi)}{\pi}$, we obtain the required condition $\lim_{x\to\infty} K(\infty) = 1$ and

$$K(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-1} {}_2\Psi_1 \left(\begin{matrix} (1,1), (1, 1-\tilde{\alpha}) \\ (\alpha, \alpha) \end{matrix} \middle| -x^{\alpha} \right).$$

Next, from identity (4.1), we find that

$$\mathbb{P}\Big(\max_{0 \leq s \leq 1} \hat{Z}_s \geq x \Big) = P(x),$$

where $\hat{Z}$ is a spectrally positive stable process of index α. Thus, by differentiating, one gets the following expression for the density:

$$k(x) = \frac{\sin(\tilde{\alpha}\pi)}{\pi} x^{\alpha-2} {}_2\Psi_1 \left(\begin{matrix} (1,1), (1, 1-\tilde{\alpha}) \\ (\alpha, \alpha-1) \end{matrix} \middle| -x^{\alpha} \right),$$

which is the expression found by Bernyk, Dalang and Peskir [2], Theorem 1.

4.3. *The self-similar saw-tooth processes.* Finally, we consider the so-called saw-tooth process introduced and deeply studied by Carmona, Petit and Yor [9]. It is a self-similar positive Markov process of index $\alpha = 1$ with underlying Lévy process the sum of a drift of parameter $b = 1$ and the negative of a compound Poisson process of parameter $\beta > 0$ whose jumps are exponentially distributed with parameter $\delta + \beta - 1 > 0$, that is,

$$\psi(u) = u \frac{u + \delta - 1}{u + \delta + \beta - 1}, \qquad u \geq 0.$$

Moreover, in [9], the authors show that

$$\phi(q) = \tfrac{1}{2}\big(q - (\delta - 1) + \bar{\phi}(q)\big), \qquad q \geq 0,$$

where $\bar{\phi}(q) = \sqrt{(q-(\delta-1))^2 + 4(\delta+\beta-1)q}$. Let us proceed with the case $q = 0$. Note, for $1-\beta < \delta < 1$, that $\gamma = 1-\delta$ and

$$\psi_{1-\delta}(u) = u\frac{u+1-\delta}{u+\beta}.$$

Thus,

$$a_n(\psi_{1-\delta}, 1) = \frac{\Gamma(n+1+\beta)\Gamma(2-\delta)}{\Gamma(1+\beta)\Gamma(n+1)\Gamma(n+2-\delta)}, \qquad a_0 = 1,$$

and for $|z| < 1$

$$\begin{aligned}\mathcal{I}_{\psi_{1-\delta}}(\rho; -z) &= \frac{\Gamma(2-\delta)}{\Gamma(\rho)\Gamma(1+\beta)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\rho+n)\Gamma(n+1+\beta)}{\Gamma(n+2-\delta)n!} z^n \\ &= {}_2F_1(\rho, 1+\beta, 2-\delta; -z),\end{aligned}$$

where ${}_2F_1(a, b; x)$ stands for the hypergeometric function; see Lebedev [15], Section 9, for a detailed account on this function. Next, recalling the identity

$${}_2F_1(-n, 1+\beta, \delta; 1) = \frac{\Gamma(2-\delta)\Gamma(n+1-\delta-\beta)}{\Gamma(2-\delta+n)\Gamma(1-\delta-\beta)},$$

we recover from (2.4) the well-known identity

$${}_2F_1(\rho, 1+\beta, 2-\delta; z) = (1-z)^{-\rho}{}_2F_1\left(\rho, 1-\delta-\beta, \delta; \frac{z}{z-1}\right),$$

which provides an analytic continuation of the hypergeometric function into the half-plane $\mathfrak{Re}(z) < \frac{1}{2}$. Finally, using the asymptotic

$${}_2F_1(\rho, 1+\beta, 2-\delta; -x) \sim \frac{\Gamma(2-\delta)\Gamma(1+\beta-\rho)}{\Gamma(2-\delta-\rho)\Gamma(1+\beta)} x^{-\rho} \qquad \text{as } x \to \infty,$$

one obtains

$$S(t) = \frac{\Gamma(1+\beta)}{\Gamma(2-\delta)\Gamma(\beta+\delta)} t^{\delta-1} {}_2F_1(1-\delta, 1+\beta, \delta; -t^{-1}).$$

Moreover, after some easy computations, one gets for $\gamma = \phi(q)$, $q > 0$,

$$\psi_{\phi(q)}(u) = u\frac{u+\phi(q)}{u+\beta+\delta+\phi(q)-1}.$$

Thus, proceeding as above, we obtain

$$\mathcal{I}_{\psi_{\phi(q)}}(\rho; -z) = {}_2F_1\big(\rho, \beta+\delta+\phi(q), 1+\phi(q); -z\big)$$

and

$$\begin{aligned}S(t) = &\frac{\Gamma(\beta+\delta+\phi(q))\Gamma(1+\bar{\phi}(q)-\phi(q))}{\Gamma(1+\bar{\phi}(q))\Gamma(\beta+\delta)} \\ &\times t^{-\phi(q)} {}_2F_1\big(\phi(q), \beta+\delta+\phi(q), 1+\bar{\phi}(q); -t^{-1}\big).\end{aligned}$$

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