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LAWS OF LARGE NUMBERS FOR A CELLULAR AUTOMATON

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We prove laws of large numbers for a cellular automaton in the space $\{0, 1, \dots, p-1\}^Z$ with p being a prime number. The dynamics τ of the system are defined by $\tau\eta(x) = \eta(x-1) + \eta(x+1) \bmod p$ for $\eta \in X$.

1. Introduction. The cellular automaton considered in this paper is a dynamical system in the configuration space $X = \{0, 1, \dots, p-1\}^Z$, where Z is the one dimensional integer lattice and $p \geq 2$ is a prime number. The evolution rule of the system is specified by a mapping τ: $X \to X$ such that for any $\eta \in X$,

$$\tau\eta(x) = \eta(x-1) + \eta(x+1) \bmod p. \tag{1.1}$$

Let σ^+ and σ^- denote the right-shift and the left-shift on X, respectively: $\sigma^+\eta(x) = \eta(x+1)$, $\sigma^-\eta(x) = \eta(x-1)$. Then $\tau\eta = (\sigma^+ + \sigma^-)\eta \bmod p$. Let τ^n be the nth iterate of the map τ. Then $\tau^n\eta = (\sigma^+ + \sigma^-)^n\eta \bmod p$. Therefore,

$$\tau^n\eta(x) = \sum_{i=0}^{n} \binom{n}{i} \eta(2i - n + x) \bmod p \tag{1.2}$$

for $x \in Z$. We will sometimes denote the system by $\{\eta_n: n \geq 0\}$, where $\eta_n = \tau\eta_{n-1}$ for $n \geq 1$. If the system starts from the configuration $\eta_0 = \delta_0$ with $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for all $x \neq 0$, then η_n gives the nth row of Pascal's triangle mod p.

This dynamical system has been studied for the case $p = 2$ by several people and it was proved by Miyamoto (1979) and Lind (1984) independently [see Durrett (1988)] that if μ is a shift invariant product probability measure on X such that $\mu\{\eta(x) = 1\} \neq 0, 1$, then as $N \to \infty$,

$$\frac{1}{N} \sum_{n=0}^{N-1} \tau^n\mu \Rightarrow \mu_{1/2}, \tag{1.3}$$

where $\tau^n\mu$ is the distribution of the nth iteration when the system starts from μ, $\mu_{1/2}$ is the product probability measure such that $\mu_{1/2}(\eta(x) = 1) = 1/2$ for all $x \in X$ and "$\Rightarrow$" means weak convergence.

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In this paper we will generalize the above result by showing that a similar convergence theorem holds for all prime numbers p. More precisely, we have the following:

THEOREM 1. *Let* $\theta \equiv (\theta_0, \theta_1, \dots, \theta_{p-1})$ *be such that* $\theta_i \ge 0$, $\Sigma_i \theta_i = 1$ *and* μ_θ *be the product measure on* X *with density* θ, *that is*, $\mu_\theta(\eta(x) = i) = \theta_i$, $i \in Z/(p)$. *Then*:

(i) $\tau^n \mu_\theta$ *does not converge unless* $\theta_0 = 1$ *or* $\theta_i = 1/p$, $\forall\ i$;
(ii) *if* $\theta_k < 1$, $\forall\ k$ *then*

$$\frac{1}{N} \sum_{n=0}^{N-1} \tau^n \mu_\theta \Rightarrow \mu_{1/p} \quad \text{as } N \to \infty, \tag{1.4}$$

where $\mu_{1/p}$ *is the shift invariant product measure on* X *such that* $\mu_{1/p}\{\eta(x) = k\} = 1/p$ *for all* $0 \le k \le p - 1$ *and* $x \in Z$.

We will also show in this paper a result which contains a strong law of large numbers for the system:

THEOREM 2. *Let* μ_θ *be a measure on* X *satisfying the conditions of Theorem 1. For any cylinder set* B *in* X, *there exist two constants* c_1 *and* c_2 *depending on* μ_θ *and* B *such that*

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} I_B \circ \tau^n(\eta) = c_1 \qquad \mu_\theta\text{-}a.s.$$

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} I_B \circ \tau^n(\eta) = c_2 \qquad \mu_\theta\text{-}a.s.$$

and $c_1 \le \mu_{1/p}(B) \le c_2$. *If* $\mu_\theta = \mu_{1/p}$, *then* $c_1 = c_2 = \mu_{1/p}(B)$.

Because of the technical difficulties arising from some irregularities of nonprime integers, we are unable to prove the above results for general integers at this point. Some discussion about this matter will be given in the last section of this paper. In Section 2 we discuss some useful tools which are convenient to the analysis of this kind of system. The proofs of the theorems will be given in Section 3. In the proof of the first theorem we adopt an approach similar to that of Lind which uses the ideas discussed in Section 2. Two properties of Pascal's triangle mod p are the keys to our results, whose proofs along with other results about the triangle will be given in Section 4.

2. Characters and the dual system. In this section, we will discuss the characteristic functions of $\{0, 1, \dots, p - 1\}^Z$-valued random variables and then show a self duality of our system in terms of the characteristic functions. The results in this section do not require p to be a prime number.

We consider $\{0, 1, \dots, p - 1\} = Z/(p)$ as an Abelian group with the usual mod p addition and $X \equiv \{0, 1, \dots, p - 1\}^Z$ as the direct sum of the copies of

$Z/(p)$. Let $Z/(p)$ have the discrete topology and X have the product topology. Then X is a locally compact Abelian group. A continuous function $\gamma\colon X \to \mathscr{C}$ is called a continuous character if $|\gamma(x)| = 1$ for all $x \in X$ and $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $x, y \in X$. Let $\hat{X}$ be the collection of all such continuous characters. Our first step is to identify $\hat{X}$.

For any map $\phi\colon Z \to Z/(p)$ with finite support, that is, $\phi(k) = 0$ for all but finitely many $k \in Z$, we denote $\phi_k = \phi(k)$, $k \in Z$ and let $\gamma_\phi \in \hat{X}$ be such that

$$\gamma_\phi(\eta) = \exp\left\{ i\frac{2\pi}{p} \sum_{k \in Z} \phi(k)\eta(k) \right\}, \qquad \forall\, \eta \in X.$$

We have the following:

LEMMA 2.1. *$\hat{X} = \{\gamma_\phi$: ϕ has finite or empty support$\}$.*

PROOF. Obviously $\hat{X} \supset \{\gamma_\phi\colon \phi$ has finite or empty support$\}$. Let $\delta_k \in X$ be such that $\delta_k(k) = 1$ and $\delta_k(z) = 0$, for $z \neq k$. For any $\gamma \in \hat{X}$, we can write $\gamma(\delta_k) = e^{i(2\pi/p)\gamma_k}$ for a unique real number γ_k in $[0, p)$. Since $e^{i(2\pi\gamma)_k} = \gamma(p\delta_k) = 1$, we know that γ_k is in $Z/(p)$. Let $\phi_\gamma\colon Z \to Z/(p)$ be such that $\phi_\gamma(k) = \gamma_k$, $\forall\, k \in Z$. We claim that ϕ_γ has finite support.

Indeed, if not we can choose infinitely many points $\{z_n\colon n \geq 1\} \subset Z$ and some $g \neq 0$ in $Z/(p)$ such that $\phi_\gamma(z_n) = g$ or $\gamma(\delta_{z_n}) = e^{i(2\pi/p)g}$. Let $x_n \in X$ be such that

$$x_n(z) = \begin{cases} 1, & z = z_j,\ 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, as $n \to \infty$, $x_n \to x_\infty$ where

$$x_\infty(z) = \begin{cases} 1, & z = z_j \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\gamma(x_n) \to \gamma(x_\infty)$. On the other hand, it is clear that $\gamma(x_n) = e^{i(2\pi/p)ng}$ and hence $\gamma(x_n)$ does not converge to anything. This contradiction shows that ϕ_γ has finite support. Finally, we can easily see $\gamma = \gamma_{\phi_\gamma}$. This proves the lemma. □

Note that from the proof of Lemma 2.1 we have for any $\gamma \in \hat{X}$ a unique $\phi_\gamma \in X$ with finite support such that

$$\gamma(\eta) = \exp i\frac{2\pi}{p} \sum_k \phi_\gamma(k)\eta(k). \tag{2.1}$$

Suppose μ is a probability measure on X. Define its characteristic function $\hat{\mu}$ as

$$\hat{\mu}(\gamma) = \int_X \gamma(x)\mu(dx), \qquad \forall\, \gamma \in \hat{X}.$$

Then, we have the following:

LEMMA 2.2. *Let μ_i, $i = 1, 2$, be two probability measures on X. Then $\mu_1 = \mu_2$ if and only if $\hat{\mu}_1(\gamma) = \hat{\mu}_2(\gamma)$ for all $\gamma \in \hat{X}$.*

PROOF. See Rudin [(1962), page 29, 1.7.3(b)]. □

LEMMA 2.3. *Suppose $\{\mu_n: 1 \le n \le \infty\}$ is a family of probability measures on X. Then $\mu_n \Rightarrow \mu_\infty$ as $n \to \infty$ if and only if $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = \hat{\mu}_\infty(\gamma)$, $\forall\ \gamma \in \hat{X}$.*

PROOF. Since X is compact, every subsequence of $\{\mu_n: 1 \le n \le \infty\}$ has a subsequence converging to a limit point. From Lemma 2.2, all these limits are the same if and only if $\lim_{n\to\infty} \hat{\mu}_n(\gamma) = \hat{\mu}_\infty(\gamma)$, $\forall\ \gamma \in \hat{X}$. Therefore, Lemma 2.3 is true. □

Next we show a self duality of the system in terms of the characteristic functions. For $\gamma \in \hat{X}$ define $\tau\gamma \in \hat{X}$ as

$$\tau\gamma(\eta) = \exp i \frac{2\pi}{p} \sum_k \left[\phi_\gamma(k-1) + \phi_\gamma(k+1)\right]\eta(k). \tag{2.2}$$

Let $H: \hat{X} \times X \to \mathscr{C}$ be such that $H(\gamma, \eta) = \exp i(2\pi/p)\Sigma_k \phi_\gamma(k)\eta(k)$. Then from (2.2) and (1.1), we have $H(\gamma, \tau\eta) = H(\tau\gamma, \eta)$. Let $\{\gamma_n: n \ge 0\}$ be the system in $\hat{X}$ with $\gamma_0 = \gamma$ and $\gamma_n = \tau\gamma_{n-1}$. Then $\{\gamma_n: n \ge 0\}$ is the dual system of $\{\eta_n\}$ in the sense that

$$H(\gamma_0, \eta_n) = H(\gamma_n, \eta_0). \tag{2.3}$$

From (1.1) and (2.2), we know those two systems have the same dynamics. Suppose η_0 has the distribution μ_θ and $\gamma_0 = \gamma$ for some $\gamma \in \hat{X}$. From (2.3) and the definition of the characteristic function, we have

$$\left(\widehat{\tau^n \mu_\theta}\right)(\gamma) = \hat{\mu}_\theta(\tau^n\gamma). \tag{2.4}$$

We will apply this duality in Section 3.

3. Proof of the theorems. It is clear from the definition that the dynamics of the cellular automaton considered here are completely determined by the properties of Pascal's triangle mod p. Before proving the theorems we list below two lemmas whose proofs, along with a study of some other properties of the triangle, will be given in the next section. From now on we will always assume that p is a prime number.

Suppose $\eta \in X$ is such that $\eta(x) = 0$, if $|x| > m$ for some integer m, but $\eta \ne 0$. Then the system $\{\eta_n\}$ with $\eta_0 = \eta$ is a Pascal trapezoid mod p. For $0 < i < p$ and $\xi \in X$, let $N_i(\xi)$ denote the number of i's in the configuration ξ:

$$N_i(\xi) = \text{the number of } j\text{'s such that } \xi(j) = i.$$

The following lemma says that in most rows of Pascal's trapezoid mod p, the number of the terms which equal some $i \in \{1, \dots, p-1\}$ can be very large.

LEMMA 3.1. *Suppose $\eta \neq 0$ but has finite support. Then, for any i, $0 < i < p$ and $M > 0$,*

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N_i(\eta_n)\le M)}\right] = 0.$$

The second lemma we need says that in each column l of Pascal's triangle mod p, *the number of nonzero terms is relatively small.*

LEMMA 3.2. *For any integer l,*

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=0}^{N-1} I\left\{n : n \ge |l|,\ n + l \text{ is even and } \binom{n}{(n+l)/2} \not\equiv 0 \bmod p\right\}\right] = 0.$$

PROOF OF THEOREM 1.

(i) From (2.4) we have, for any $m \ge 1$,

$$\left(\widehat{\tau^{p^m}\mu_\theta}\right)(\delta_0) = \left(\sum_{k=0}^{p-1}\theta_k e^{i(2\pi/p)k}\right)^2 \quad\text{and}\quad \left(\widehat{\tau^{p^m+1}\mu_\theta}\right)(\delta_0) = \left(\sum_{k=0}^{p-1}\theta_k e^{i(2\pi/p)k}\right)^4,$$

where $\sum_{k=0}^{p-1}\theta_k e^{i(2\pi/p)k} \neq 0, 1$ unless $\theta_0 = 1$ or $\theta_k = 1/p$. Thus, we can conclude that $\widehat{\tau^n\mu_\theta}(\delta_0)$ does not converge and hence by Lemma 2.3, $\tau^n\mu_\theta$ cannot converge.

(ii) By (2.4) and Lemma 2.3, (1.4) is equivalent to

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\hat{\mu}_\theta(\tau^n\gamma) = \hat{\mu}_{1/p}(\gamma) \tag{3.1}$$

for all $\gamma \in \hat{X}$, where $\hat{\mu}_{1/p}(\gamma)$ is the characteristic function of the measure $\mu_{1/p}$. It is easy to see that

$$\hat{\mu}_{1/p}(\gamma) = \begin{cases} 1, & \text{if } \gamma \equiv 1, \\ 0, & \text{otherwise.}\end{cases}$$

When $\gamma \equiv 1$, (3.1) is trivial.

When $\gamma \not\equiv 1$, γ can be identified by a unique element ϕ_γ in X with finite support [see (2.1)]. Let

$$A_k^n = \{x \in Z : \phi_{\tau^n\gamma}(x) = k\}$$

for $k = 1, \dots, p-1$. Then $|A_k^n| < \infty$ and

$$\begin{aligned}\hat{\mu}_\theta(\tau^n\gamma) &= E_{\mu_\theta}\prod_{k\neq 0}\exp i\frac{2\pi}{p}k\sum_{x\in A_k^n}\eta(x)\\ &= \prod_{k\neq 0}E_{\mu_\theta}\exp i\frac{2\pi}{p}k\sum_{x\in A_k^n}\eta(x) \qquad (3.2)\\ &= \prod_{k\neq 0}\left(E_{\mu_\theta}e^{i(2\pi/p)k\eta(0)}\right)^{|A_k^n|}.\end{aligned}$$

It follows from the conditions of the theorem on the measure μ_θ that there is a $\delta > 0$ such that for any nonzero integer k,

$$\left|E_{\mu_\theta} e^{i(2\pi/p)k\eta(0)}\right| = \left|\sum_{j=0}^{p-1} \theta_j e^{i(2\pi/p)kj}\right| < 1 - \delta.$$

Therefore, for any $\varepsilon > 0$, from (3.2), there is an $M > 0$ such that if $|A_k^n| \geq M$, $\forall\, k \neq 0$, then

$$|\hat{\mu}_\theta(\tau^n \gamma)| < \varepsilon. \tag{3.3}$$

For this fixed M, from Lemma 3.1, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, |A_k^n| < M,\, \exists\, k \neq 0)} = 0. \tag{3.4}$$

Therefore, from (3.3) and (3.4),

$$\limsup_{N\to\infty} \left|\frac{1}{N} \sum_{n=0}^{N-1} \hat{\mu}_\theta(\tau^n \gamma)\right| \leq \varepsilon.$$

Since ε is arbitrary,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \hat{\mu}_\theta(\tau^n \gamma) = 0.$$

Thus, (3.1) is true for the case $\gamma \not\equiv 1$, which completes the proof of the theorem. □

Proof of Theorem 2. Let f_1 and f_2 denote the lower limit and the upper limit of $(1/N)\sum_{n=0}^{N-1} I_B \circ \tau^n(\eta)$, respectively. Let η have the distribution μ_θ and $\mathscr{F}_\infty = \bigcap_{n=0}^{\infty} \sigma\{\eta(x)\colon |x| \geq n\}$. Since μ_θ is a shift invariant product measure, $\mathscr{F}_\infty$ is trivial with respect to μ_θ. Therefore to prove that f_1 and f_2 are constants μ_θ-a.s., it suffices to show that they are in $\mathscr{F}_\infty$, or, equivalently, to show that for any finite cylinder set A, these limits are independent of A.

Let $M_1 > 0$ and $M_2 > 0$ be two arbitrary integers and

$$A = \{\eta\colon \eta(i) = a_i,\, i = -M_1, \dots, 0, \dots, M_1\},$$
$$B = \{\eta\colon \eta(i) = b_i,\, i = -M_2, \dots, 0, \dots, M_2\}.$$

Rewrite the formula (1.2) in the following way. Let

$$c(n,j) = \begin{cases} \binom{n}{(n+j)/2}, & \text{if } n+j \text{ is even and } |j| \leq n, \\ 0, & \text{if } n+j \text{ is odd and } |j| \leq n. \end{cases}$$

Then

$$\tau^n \eta(k) = \sum_{j=-n}^{n} c(n,j)\eta(j+k). \tag{3.5}$$

Now write

$$\frac{1}{N}\sum_{n=0}^{N-1} I_B \circ \tau^n(\eta) = \frac{1}{N}\sum_{n'} I_B \circ \tau^{n'}(\eta) + \frac{1}{N}\sum_{n''} I_B \circ \tau^{n''}(\eta),$$

where the first summation is over all n' such that $n' < N$ and $c(n', j) = 0 \bmod p$ for $-(M_1 + M_2) \le j \le M_1 + M_2$ and the second summation is over all the remaining terms. It is easy to see from (3.5) that the first sum is independent of A. The second sum can be written as

$$\begin{aligned}
&\frac{1}{N}\sum_{n''} I_B \circ \tau^{n''}(\eta)\\
&= \sum_{j=-M_1-M_2}^{M_1+M_2} (\text{number of } n''\colon n'' < N \quad \text{and} \quad c(n'', j) \ne 0 \bmod p)/N\\
&= \sum_{j=-M_1-M_2}^{M_1+M_2} \left\{ \frac{1}{N} \sum_{n''=0}^{N-1} I\left\{ n''\colon n'' \ge |j|,\ n'' + j \text{ is even and} \right.\right.\\
&\qquad\qquad \left.\left. \binom{n''}{(n''+j)/2} \ne 0 \bmod p \right\}\right\}
\end{aligned}$$

and it follows from Lemma 3.2 that, for each j,

$$\frac{1}{N}\sum_{n''=0}^{N-1} I\left\{ n''\colon n'' \ge |j|,\ n'' + j \text{ is even and } \binom{n''}{(n''+j)/2} \ne 0 \bmod p \right\} \to 0$$

as $N \to \infty$. Therefore

$$f_1 = \liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} I_B \circ \tau^n(\eta) = \liminf_{N\to\infty} \frac{1}{N}\sum_{n'} I_B \circ \tau^{n'}(\eta)$$

is independent of A. This is obviously true for f_2 also. Therefore, by Theorem 1, $c_1 \equiv f_1 \le \mu_{1/p}(B) \le f_2 \equiv c_2$ μ_θ-a.s. The last statement of the theorem follows from the fact that $\tau\mu_{1/p} = \mu_{1/p}$, Birkhoff's ergodic theorem, and what we have just proved. □

4. Pascal's triangle mod p. In this section, we will prove Lemmas 3.1 and 3.2 and highlight some insights into Pascal's triangle or Pascal's trapezoid. As mentioned above, by Pascal's triangle, we refer to the system $\{\eta_n\colon n \ge 0\}$ with $\eta_0 = \delta_0$ and, by Pascal's trapezoid, we refer to the system with η_0 having finite support. From a rescaling argument (Lemma 4.4), we will see that there is a Pascal's triangle embedded in a Pascal's trapezoid. Thus some properties of the triangle can be transformed into the properties of the trapezoids.

First, let us look at Pascal's triangle. Let F be the field $Z/(p) = \{0, 1, \dots, p-1\}$ with mod p addition and $F[x]$ the polynomials on F. The nth row of Pascal's triangle mod p consists of the coefficients of the polynomial $(1 + x)^n$

$\in F[x]$. Since F is a field, it is convenient to express the integer n as a polynomial in p:

$$n = a_m p^m + a_{m-1} p^{m-1} + \cdots + a_1 p + a_0,$$

where $0 \le a_i < p$, $0 \le i \le m$ and $a_m \ne 0$. This expression is unique. Thus, using the fact that if p is a prime number then $(1+x)^p \equiv 1 + x^p$ in $F[x]$, we have, in $F(x)$,

$$\begin{aligned}(1+x)^n &= \prod_{i=0}^{m} (1+x)^{a_i p^i} \\ &= \prod_{i=0}^{m} \left(1 + x^{p^i}\right)^{a_i} \\ &= \prod_{i=0}^{m} \left[\sum_{j=0}^{a_i} \binom{a_i}{j} x^{j p^i} \right] \qquad (4.1) \\ &= \sum_{\substack{0 \le k_0 \le a_0 \\ 0 \le k_1 \le a_1 \\ \vdots \\ 0 \le k_m \le a_m}} \binom{a_0}{k_0}\binom{a_1}{k_1} \cdots \binom{a_m}{k_m} x^{k_0 + k_1 p + \cdots + k_m p^m}.\end{aligned}$$

Note that in this formula, $\binom{a_i}{k_i} \ne 0 \bmod p$ for $i = 0, 1, \ldots, m$.

In Lemmas 4.1 and 4.2 we will show that most rows of the triangle contain a large number of representatives of any integer from 1 to $p-1$. This is a special case of Lemma 3.1. Later we will use a rescaling argument to generalize this result to obtain Lemma 3.1, which is basically Lemma 4.2 with the triangle being replaced by trapezoids.

For $i = 1, \ldots, p-1$ and $n = a_m p^m + a_{m-1} p^{m-1} + \cdots + a_1 p + a_0$, let

$$N^i(n) = |\{j : a_j = i\}|.$$

LEMMA 4.1. *For $M > 0$, we have*

$$\lim_{N \to \infty} \left[\frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, N^i(n) \le M)} \right] = 0.$$

PROOF. First, we consider $N = (p-1)p^m$. Any integer $n < N$ can be written as $n = a_m p^m + a_{m-1} p^{m-1} + \cdots + a_1 p + a_0$ with $0 \le a_m < (p-1)$ and $0 \le a_i \le (p-1)$, $0 \le i \le m-1$. Out of $(p-1)p^m$ choices of these n, there are $\sum_{j=0}^{[M]} \binom{m}{j} (p-1)^{m-j}$ of them satisfying $N^i(n) \le M$. So,

$$\frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, N^i(n) \le M)} \le \frac{1}{(p-1)p^m} \sum_{j=0}^{[M]} \binom{m}{j} (p-1)^{m-j} \to 0$$

as $m \to \infty$.

In general, let m_N be such that $(p-1)p^{m_N} \le N < (p-1)p^{m_N+1}$. Then,

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N^i(n)\le M)} \\ &\quad \le \frac{(p-1)p^{m_N+1}}{N}\,\frac{1}{(p-1)p^{m_N+1}} \sum_{n=0}^{(p-1)p^{m_N+1}} I_{(n:\, N^i(n)\le M)} \\ &\quad \le p\left[\frac{1}{(p-1)p^{m_N+1}} \sum_{n=0}^{(p-1)p^{m_N+1}} I_{(n:\, N^i(n)\le M)}\right] \to 0 \end{aligned}$$

as $N \to \infty$ from the above special case. □

For $0 < i < p$, let $N_i(n)$ denote the number of i in the nth row of Pascal's triangle mod p:

$$N_i(n) = \text{the number of } j\text{'s such that } 0 \le j \le n \quad \text{and} \quad \binom{n}{j} \equiv i \bmod p.$$

Lemma 4.2. *For any* i, $0 < i < p$ *and* $M > 0$,

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N_i(n)\le M)}\right] = 0.$$

Proof. Consider the field $F = Z/(p)$. We know that $G = F \setminus \{0\}$ is a group with order $p-1$. So, $i^{l(p-1)+1} = i$ in G or F for any $l \ge 0$. For each $n = a_m p^m + a_{m-1}p^{m-1} + \cdots + a_1 p + a_0$, by carefully taking k_j's as 0 or 1 in (4.1), we know there are at least $\sum_{l:\, l(p-1)+1\le N^i(n)} \binom{N^i(n)}{l(p-1)+1}$ i's appearing in the nth row of Pascal's triangle mod p. But $\sum_{l:\, l(p-1)+1\le K} \binom{K}{l(p-1)+1} \nearrow \infty$ as $K \to \infty$. So,

$$M^* \equiv \min\left\{K\colon \sum_{l:\, l(p-1)+1\ge K} \binom{K}{l(p-1)+1} > M\right\} < \infty.$$

Thus,

$$\frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N_i(n)\le M)} \le \frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N^i(n)\le M^*)} \to 0$$

as $N \to \infty$, from Lemma 4.1. □

The possibility of rescaling comes from the following fact.

Lemma 4.3. *If the system* $\{\eta_n\}$ *is Pascal's triangle* mod p *in the space* $\{0,1,\ldots,p-1\}^Z$, *then the system* $\{\xi_n\}$ *with* $\xi_n(x) = \eta_{np^K}(p^K x)$ *is Pascal's triangle* mod p *in the space* $\{0,1,\ldots,p-1\}^{p^K Z}$.

PROOF. Note that $(1+x)^{kp^m} = (1+x^{p^m})^k$ in $F[x]$. We have

$$\sum_{j=0}^{kp^m} \binom{kp^m}{j} x^j = \sum_{i=0}^{k} \binom{k}{i} x^{ip^m}.$$

Comparing the coefficients in $F[x]$, we see that

(a) $\binom{np^K}{j} = 0$ in F if $j \le np^K$ and $j \ne ip^K$, $i \le n$;

(b) $\binom{np^K}{ip^K} = \binom{n}{i}$ in F if $i \le n$.

Hence, by (1.2) we have, in F,

$$\begin{aligned}\xi_n(x) &= \sum_{i=0}^{n} \binom{np^K}{ip^K} \eta(2ip^K - np^K + p^K x)\\ &= \sum_{i=0}^{n} \binom{n}{i} \xi(2i - n + x).\end{aligned}$$

Thus, in the space $\{0, 1, \dots, p-1\}^{p^K Z}$, the system $\{\xi_n\}$ has dynamics defined by (1.2). $\square$

Suppose $\eta \in X$ has finite support, that is, $\eta(x) = 0$, if $|x| > m$ for some m, but $\eta \ne 0$. Without loss of generality, we assume $\eta(0) \ne 0$. Call m a support range of η. Then, the system $\{\eta_n\}$ with $\eta_0 = \eta$ is Pascal's trapezoid mod p. We rescale time and space by a factor p^K, where K is any integer such that $p^K > m$. Define a system $\{\xi_n\}$ as

$$\xi_n(x) = \eta_{np^K}(p^K x).$$

From Lemma 4.3 we see that the system $\{\xi_n : n \ge 0\}$ is Pascal's triangle mod p starting with $\xi_0 = \eta_0 \delta_0$. This leads to the following result:

LEMMA 4.4. *Suppose a support range m of η satisfies $m < p^K$. Then, for any i, $0 < i < p$ and $M > 0$,*

$$\lim_{N\to\infty} \left[\frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, N_i(\eta_{np^K}) \le M)} \right] = 0.$$

PROOF. Since F is a field, there is a j such that $j\xi(0) = i$ in F. Note that $N_i(\xi_n) = N_j(n) > M$ implies $N_i(\eta_{np^K}) > M$, where $N_j(n)$ is as in Lemma 4.2. We have

$$\frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, N_i(\eta_{np^K}) \le M)} \le \frac{1}{N} \sum_{n=0}^{N-1} I_{(n:\, N_j(n) \le M)} \to 0$$

as $N \to \infty$, from Lemma 4.2. $\square$

PROOF OF LEMMA 3.1. Let m be a support range of η, that is, $\eta(x) = 0$, $|x| > m$. For any $\varepsilon > 0$, choose a $K > 0$ such that $m/p^K < \varepsilon$. Regrouping the sum according to the remainder of n divided by p^K, we have

$$\frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N_\iota(\eta_n) \le M)}$$

$$= \frac{1}{N}\sum_{r=0}^{[p^K-m]-1} \sum_{n:\, np^K+r<N} I_{(n:\, N_\iota(\eta_{np^K+r}) \le M)}$$

$$+ \frac{1}{N}\sum_{r=[p^K-m]}^{p^K-1} \sum_{n:\, np^K+r<N} I_{(n:\, N_\iota(\eta_{np^K+r}) \le M)}$$

$$(4.2) = \sum_{r=0}^{[p^K-m]-1} \frac{[(N-r)/p^K]+1}{N} \frac{1}{[(N-r)/p^K]+1} \sum_{n=0}^{[(N-r)/p^K]} I_{(n:\, N_\iota(\eta_{np^K+r}) \le M)}$$

$$+ \sum_{r=[p^K-m]}^{p^K-1} \frac{[(N-r)/p^K]+1}{N} \frac{1}{[(N-r)/p^K]+1} \sum_{n=0}^{[(N-r)/p^K]} I_{(n:\, N_\iota(\eta_{np^K+r}) \le M)}$$

$$\le \sum_{r=0}^{[p^K-m]-1} \frac{[(N-r)/p^K]+1}{N} \frac{1}{[(N-r)/p^K]+1} \sum_{n=0}^{[(N-r)/p^K]} I_{(n:\, N_\iota(\eta_{np^K+r}) \le M)}$$

$$+ \frac{[(N-r)/p^K]+1}{N}(p^K - [p^K - m]).$$

Note that $\eta_{np^K+r} = \tau^{np^K}\eta_r$. When $r < p^K - m$, the support range of η_r is at most $m + r < p^K$. From Lemma 4.4, each term in the first sum tends to 0 as $N \to \infty$. From (4.2), we have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} I_{(n:\, N_i(\eta_n) \le M)} \le 0 + \frac{1}{p^K}\left(p^K - [p^K - m]\right)$$

$$\le \frac{m}{p^K} < \varepsilon.$$

Since ε is arbitrary, the lemma follows. □

For the proof of Lemma 3.2, we need:

LEMMA 4.5. *Suppose the integers* $k > l$ *can be written as*

$$k = a_m p^m + \cdots + a_1 p + a_0,$$

$$l = b_m p^m + \cdots + b_1 p + b_0,$$

where $0 \le a_i < p$, $0 \le b_i < p$. *Then,* $\binom{k}{l} = 0 \bmod p$ *if and only if there is some* i, $0 \le i \le m - 1$, *such that* $a_i < b_i$.

PROOF. Note that $\binom{k}{l}$ is the coefficient of x^l in $(1+x)^k$ in $F[x]$. From (4.1), we know the nonzero coefficients come from those powers l with $a_i \geq b_i$, $\forall\, i$. Thus the lemma is true. □

PROOF OF LEMMA 3.2. Observe that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} I\left\{n: n+l \text{ is even and } \binom{n}{(n+l)/2} \neq 0 \bmod p\right\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k:\, 0\leq 2k+l<N} I\left\{k: \binom{2k+l}{k+l} \neq 0 \bmod p\right\} \\
&= \lim_{N\to\infty} \frac{1}{N} \left|\left\{k: 0 \leq 2k+l < N \quad \text{and} \quad \binom{2k+l}{k+l} \neq 0 \bmod p\right\}\right|.
\end{aligned}$$

Suppose N has the form

$$N = a_m p^m + \cdots + a_1 p + a_0$$

with $a^m > 0$. We claim that

$$\left|\left\{k: 0 \leq 2k+l < N \quad \text{and} \quad \binom{2k+l}{k+l} \neq 0 \bmod p\right\}\right| < (p+1)^{m+1}/2^{m+1}.$$

This, combined with the above observation, will prove the lemma.

Let $k = b_m p^m + \cdots + b_1 p + b_0$ and $l = c_m p^m + \cdots + c_1 p + c_0$. Then

$$\begin{aligned}
2k + l &= (2b_m + c_m)p^m + \cdots + (2b_1 + c_1)p + (2b_0 + c_0), \\
k + l &= (b_m + c_m)p^m + \cdots + (b_1 + c_1)p + (b_0 + c_0).
\end{aligned}$$

Write

$$2b_0 + c_0 = \begin{cases} 2b_0 + c_0, & \text{if } 2b_0 + c_0 < p, \\ p + (2b_0 + c_0 - p), & \text{if } p \leq 2b_0 + c_0 < 2p, \\ 2p + (2b_0 + c_0 - 2p), & \text{if } 2b_0 + c_0 \geq 2p \end{cases}$$

and

$$b_0 + c_0 = \begin{cases} b_0 + c_0, & \text{if } b_0 + c_0 < p, \\ p + (b_0 + c_0 - p), & \text{if } b_0 + c_0 \geq p. \end{cases}$$

From this and Lemma 4.5, it is clear that in order to have

$$\binom{2k+l}{l} \neq 0 \bmod p, \tag{4.3}$$

b_0 must satisfy either

$$2b_0 + c_0 < p \tag{4.4}$$

or

$$\begin{aligned} b_0 + c_0 &\geq p, \\ 2b_0 + c_0 &< 2p. \end{aligned} \tag{4.5}$$

Assume b_0 satisfies (4.4) or (4.5). Then $2k + l$ and $k + l$ can be rewritten as

$$2k + l = (2b_m + c_m)p^m + \cdots + (2b_2 + c_2)p^2 + (2b_1 + c_1')p + d_0,$$
$$k + l = (b_m + c_m)p^m + \cdots + (b_2 + c_2)p^2 + (b_1 + c_1')p + d_0,$$

where $c_1' = c_1$ and $d_0 = b_0 + c_0$ if b_0 satisfies (4.4), or $c_1' = c_1 + 1$ and $d_0 = b_0 + c_0 - p$ if b_0 satisfies (4.5). Apply the previous argument to $(2b_1 + c_1')$ and $(b_1 + c_1')$. We see that b_1 must satisfy conditions similar to (4.4) or (4.5) to have (4.3). Repeat this procedure m times. We conclude that if (4.3) holds then b_i, $i = 0, 1, \ldots, m$, must satisfy either

$$2b_i + c_i' < p \tag{4.6}$$

or

$$\begin{aligned} b_i + c_i' &\geq p, \\ 2b_i + c_i' &< 2p, \end{aligned} \tag{4.7}$$

where c_i' are some integers. Out of p possible choices of b_i from 0 to $p - 1$, there are no more than $(p + 1)/2$ of them that satisfy conditions (4.6) or (4.7). Therefore

$$\begin{aligned} &\left| \left\{ k : 0 \leq 2k + l < N \quad \text{and} \quad \binom{2k + l}{k + l} \neq 0 \bmod p \right\} \right| \\ &\quad \leq \left| \{ (b_m, \ldots, b_0) : b_i \text{ satisfies } (2.12) \text{ or } (2.13) \} \right| \\ &\quad < (p + 1)^{m+1} / 2^{m+1}. \end{aligned}$$

□

5. Discussion. In this section, we discuss some other features of the above system.

The first problem is what will happen if the condition of Theorem 1(ii) fails, that is, $\theta_k = 1$ for some k. In case $p = 2$, η_n will be 0 after at most two steps. In case $p > 2$, if a is the smallest positive integer with $p^a = 1$ in $Z/(p)$, then the system $\{\eta_n\}$ will appear periodically with period a.

Next, we consider the situation when p is not a prime number. One simple case is that the modulus is the product of two prime numbers p and q. In this case, we consider $Z/(pq)$ as the direct product of the groups $G_p = \{0, q, 2q, \ldots, (p - 1)q\}$ and $G_q = \{0, p, 2p, \ldots, (q - 1)p\}$, where each of them uses the addition mod pq and hence is $Z/(p)$ and $Z/(q)$, respectively. Suppose $\{\eta_n^p\}$ as a G_p^Z-valued system and $\{\eta_n^q\}$ as a G_q^Z-valued system are defined as in (1.1). Then, Theorem 1 holds for each of them. Now we can see that $\{\eta_n^p + \eta_n^q\}$ is the system mod pq and (1.4) holds with the limiting distribution $\mu_{1/pq}$. Therefore the theorem holds for modulus pq with those initial distributions which are the sum of two simple cases.

Based on the above observation for the case of nonprime numbers, we see that we should consider the case when the modulus is a power of a prime number. Even though little is known so far, we can see it differs from the prime number case in the following example. Let the modulus be 4 and $\theta_0 + \theta_4 = 1$. Then it is clear that 1 and 3 will never appear in the system and hence (1.4) will not be true. However, we are still quite convinced, from some computer simulations, that (1.4) is true when $\theta_1 > 0$ or $\theta_3 > 0$.

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