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LAWS OF LARGE NUMBERS FOR SUMS OF EXTREME VALUES

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Let $X_1, X_2, \cdots$, be a sequence of nonnegative i.i.d. random variables with common distribution F, and for each $n \ge 1$ let $X_{1n} \le \cdots \le X_{nn}$ denote the order statistics based on $X_1, \cdots, X_n$. Necessary and sufficient conditions are obtained for averages of the extreme values $X_{n+1-i,n} i = 1, \cdots, k_n + 1$ of the form: $k_n^{-1} \sum_{i=1}^{k_n} (X_{n+1-i,n} - X_{n-k_n,n})$, where $k_n \to \infty$ and $n^{-1}k_n \to 0$, to converge in probability or almost surely to a finite positive constant. In the process, characterizations are given of the classes of distributions with regularly varying upper tails and of distributions with "exponential-like" upper tails.

1. Introduction. Let $X_1, X_2, \cdots$, be a sequence of i.i.d. nonnegative unbounded random variables having common distribution F. For each $n \ge 1$, let $X_{1n} \le \cdots \le X_{nn}$ denote the order statistics based on $X_1, \cdots, X_n$. We will find necessary and sufficient conditions under which averages of the $k_n + 1$ extreme order statistics $X_{n-k_n,n}, \cdots, X_{nn}$, of the form

$$T_n = k_n^{-1} \sum_{i=1}^{k_n} (X_{n+1-i,n} - X_{n-k_n,n})$$

converge in probability or almost surely to a finite positive constant c, whenever k_n is a sequence of positive integers satisfying

$$\text{(K)} \qquad 1 \le k_n < n, k_n \to \infty \quad \text{and} \quad n^{-1}k_n \to 0.$$

Sarhan (1955) has shown that T_n is the best linear unbiased estimate of c based on the extreme order statistics, $X_{n-k_n,n}, \cdots, X_{nn}$, when $X_1, \cdots, X_n$ are independent identically distributed (i.i.d.) exponential random variables with mean $0 < c < \infty$. This fact is the key to the characterization of laws of large numbers for T_n. For this purpose we introduce a family of distributions that behave like an exponential distribution in their upper tails.

DEFINITION 1. For any finite constant $c > 0$, let

$$\mathscr{E}_c = \left\{ F: \lim_{x\to\infty} \int_x^\infty \frac{(1 - F(y))}{1 - F(x)} \, dy = c \right\}.$$

$\mathscr{E}_c$ will be called the class of distributions which have an *exponential-like upper tail* with asymptotic mean c.

We will show that $T_n \to_p c$ for all sequences k_n satisfying (K) if and only if $F \in \mathscr{E}_c$. In the process, we will also derive a similar characterization of the class distributions with regularly varying upper tails with exponent $-c^{-1}$.

DEFINITION 2. For any finite constant c > 0, let

$$\mathscr{R}_c = \left\{ G: \quad \text{for each} \quad t > 0 \lim_{x\to\infty} \frac{(1 - G(xt))}{1 - G(x)} = t^{-c^{-1}} \right\}.$$

$\mathscr{R}_c$ is the class of distributions with regularly varying upper tails of exponent $-c^{-1}$.

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Let $Y_1, Y_2, \cdots$, be a sequence of i.i.d. nonnegative unbounded random variables with common distribution G; and for each $n \geq 1$, let $Y_{1n} \leq \ldots \leq Y_{nn}$ denote the order statistics based on $Y_1, \ldots, Y_n$. Consider the statistic

$$S_n = k_n^{-1} \sum_{i=1}^{k_n} (\ln Y_{n+1-i,n} - \ln Y_{n-k_n,n}).$$

It will be shown that there exists a finite positive constant c such that $S_n \rightarrow_p c$ for all sequences k_n satisfying (K) if and only if $G \in \mathscr{R}_c$. (The statistic S_n appears in Hill (1975).)

De Haan (1979), de Haan and Resnick (1980), and Teugels (1981) have recently obtained some results closely related to the above characterization of the class $\mathscr{R}_c$.

De Haan (1979), and de Haan and Resnick (1980) have shown that if $G \in \mathscr{R}_c$ and k_n satisfies (K) then

$$(\ln Y_{nn} - \ln Y_{n-k_n,n})/\ln k_n \rightarrow_p c.$$

Theorem 2 of Teugels (1981) implies that if $G \in \mathscr{R}_c$ then for all sequences k_n satisfying (K)

$$k_n^{-1}(\textstyle\sum_{i=1}^{n-k_n} Y_{in} - (n - k_n)\nu\, Y_{n+1-k_n,n})/Y_{n+1-k_n,n} \rightarrow_p (c - 1)^{-1},$$

where $\nu = 0$ if $1 < c < \infty$ and $\nu = EY_1$ if $1/2 < c < 1$.

Another related problem concerns the possible limiting distributions of the extreme order statistics $X_{n-k_n,n}$ and $X_{k_n,n}$, when suitably normalized. For solutions to this problem refer to Smirnov (1949), Chibisov (1964), Smirnov (1967), Mejzler (1978), and Balkema and de Haan (1978a, 1978b). Also see Polfeldt (1970) and Nagaraja (1980) for an investigation into the limiting distribution of finite averages of extreme order statistics.

REMARK 1. If a sequence of positive integers k_n is chosen so that $(n - k_n)n^{-1} \rightarrow p \in [0, 1)$, then the following is true:

$$T_n \rightarrow (1-p)^{-1} \int_p^1 F^{-1}(u)\, du - F^{-1}(p) \quad \text{(finite)} \quad \text{a.s.}$$

if and only if $EX < \infty$ and F^{-1} is continuous at p. (When $p = 0$, replace $F^{-1}(0)$ by $F^{-1}(0+)$.)

This statement follows from the strong law of large numbers for linear functions of order statistics of either van Zwet (1980) or Mason (1982), combined with the fact that $X_{n-k_n,n} \rightarrow F^{-1}(p)$ a.s. if and only if F^{-1} is continuous at p. This fact will be used later on in the appendix.

The statistics T_n and S_n have a potential application as parameter estimates.

EXAMPLE 1. If it is known that X has a distribution of the form:

$$1 - F(x) = \exp(-(x - b)c^{-1}) + o(\exp(-xc^{-1})),$$

where $0 \leq b < \infty$ and $0 < c < \infty$, then since $F \in \mathscr{E}_c$, T_n is a consistent estimate of c for any sequence k_n satisfying (K). (If more smoothness assumptions are added to F a consistent estimate for b also exists.) Such distributions occur as the distribution of the maximum waiting time in the $GI/G/1$ queue. See Iglehart (1972) for details. The same family of distributions occur as the stationary waiting time distribution in the $GI/PH/c$ queue. Refer to Neuts and Takahashi (1980).

EXAMPLE 2. It is well known that there exist normalizing constants α_n and β_n such that $\alpha_n Y_{nn} - \beta_n \rightarrow_d \Phi_c$; where Φ_c is the extreme value distribution $\Phi_c(x) = \exp(-x^{-c^{-1}})$ for $x > 0$, if and only if the distribution G of Y is in $\mathscr{R}_c$. The statistic S_n is a consistent estimate of the parameter c for all sequences k_n satisfying (K) if and only if $G \in \mathscr{R}_c$.

2. The Main Results. For any $0 < \alpha < 1$, set $S_n(\alpha) \equiv S_n$ and $T_n(\alpha) \equiv T_n$, when $k_n = [n^\alpha]$. (As usual $[x]$ denotes the greatest integer $\leq x$.) The following two theorems characterize laws of large numbers for T_n and S_n.

THEOREM 1. *Let $X_1, X_2, \cdots$, be a sequence of* i.i.d. *nonnegative random variables with common distribution F. There exists a finite positive constant c such that*

(*A*) *for some* $0 < \alpha < 1$, $T_n(\alpha) \to_p c$ *if and only if*
(B) $T_n(\alpha) \to c$ a.s. *if and only if*
(C) $T_n \to_p c$ *for every sequence* k_n *satisfying* (K) *if and only if*
(D) $F \in \mathscr{E}_c$.

PROOF. Postponed until Section 3.

REMARK 2. It would seem reasonable that $T_n \to_p c$ for some sequence k_n satisfying (K) would be enough to insure that $F \in \mathscr{E}_c$. The example given in the appendix shows that this conjecture is not true. There exists a finite positive constant c, a sequence k_n satisfying (K) and a distribution F such that $T_n \to_p c$ but $F \notin \mathscr{E}_c$. Hence the word *every* cannot be replaced by the word *some* in statement (C). On the other hand, statement (A) says that if k_n is of the form $k_n = [n^\alpha]$ for some $0 < \alpha < 1$ and $T_n \to_p c$ then $F \in \mathscr{E}_c$. In fact, by the equivalence of (A) and (B) convergence in probability can be replaced by almost sure convergence for the special sequence $k_n = [n^\alpha]$.
The analogous theorem is true for the class $\mathscr{R}_c$.

THEOREM 2. *Let $Y_1, Y_2, \cdots$, be a sequence of* i.i.d. *nonnegative random variables with common distribution G. There exists a finite positive constant c such that*

(A') *for some* $0 < \alpha < 1$, $S_n(\alpha) \to_p c$ *if and only if*
(B′) $S_n(\alpha) \to c$ a.s. *if and only if*
(C′) $S_n \to_p c$ *for every sequence* k_n *satisfying* (K) *if and only if*
(D′) $G \in \mathscr{R}_c$.

PROOF. Also postponed until Section 3.

Theorems 1 and 2 by no means exhaust the possible theorems of this sort.
Let g by any nondecreasing left continuous function defined on $(0, \infty)$ such that $g(x) \to \infty$ as $x \to \infty$, and let h denote the right continuous inverse of g. Define via g and any finite positive constant c the class of distributions:

$$\mathscr{G}_c = \left\{F: \lim_{x\to\infty} \int_x^\infty \frac{(1 - F(h(y)))}{1 - F(h(x))} dy = c\right\}.$$

For any sequence k_n satisfying (K) set

$$T_n(g) = k_n^{-1} \textstyle\sum_{i=1}^{k_n} (g(X_{n+1-1,n}) - g(X_{n-k_n,n}));$$

when $k_n = [n^\alpha]$ for some $0 < \alpha < 1$, set $T_n(g, \alpha) \equiv T_n(g)$.
The following corollary follows almost immediately from Theorem 1.

COROLLARY 1. *Let $X_1, X_2, \cdots$, be a sequence of independent nonnegative random variables with common distribution F. There exists a nondecreasing left continuous function g defined on* $(0, \infty)$ *such that* $g(x) \to \infty$ *as* $x \to \infty$ *and a finite positive constant c such that*

(Ag) *for some* $0 < \alpha < 1$, $T_n(g, \alpha) \to_p c$ *if and only if*
(Bg) $T_n(g,\alpha) \to c$ a.s. *if and only if*
(Cg) $T_n(g) \to_P c$ *for every sequence* k_n *satisfying* (K) *if and only if*
(Dg) $F \in \mathscr{G}_c$.

REMARK 3. Theorem 1 corresponds to the case when $g(x) = x$, and Theorem 2 to the case when $g(x) = \ln(x \vee 1)$. To see how other classes of tail distributions can be constructed by appropriate choices of g, consider for instance $g(x) = x^{\beta}$, where $0 < \beta < \infty$. In this case, the class $\mathscr{G}_c$ contains the Weibull distribution:

$$1 - F(x) = \exp(-x^{\beta}c^{-1}) \quad \text{for} \quad x \geq 0.$$

$\mathscr{G}_c$ is thus a class of distributions with Weibull-like upper tails with shape parameter β. For all sequences k_n satisfying (K)

$$T_n(g) = k_n^{-1} \textstyle\sum_{i=1}^{k_n} (X_{n+1-i,n}^{\beta} - X_{n-k_n,n}^{\beta})$$

is a consistent estimate of c if and only if the distribution of X is in $\mathscr{G}_c$.

3. Proofs of Theorems 1 and 2. It will be convenient to break up the proofs of Theorems 1 and 2 into a number of propositions, each of which may be of independent interest.

PROPOSITION 1. *Let k_n be any sequence of integers satisfying* (K).

(E) *If $Y_1, Y_2, \cdots,$ is a sequence of* i.i.d. *nonnegative unbounded random variables with common distribution $G \in \mathscr{R}_c$ then $S_n \to_p c$.*

(F) *If $X_1, X_2, \cdots,$ is a sequence of* i.i.d. *nonnegative unbounded random variables with common distribution $F \in \mathscr{E}_c$, then $T_n \to_p c$.*

PROOF. Let $G^{-1}(u) = \inf\{x : G(x) \geq u\}$. Since $1 - G$ is regularly varying with exponent $-c^{-1}$, $G^{-1}(1 - x^{-1})$ is regularly varying with exponent c. (See Corollary 1.2.1 of de Haan (1970).) Hence

(1) $$G^{-1}(1 - x^{-1}) = L(x)x^c,$$

where $L(x)$ is a slowly varying function. $L(x)$ can be represented as

(2) $$L(x) = a(x) \exp\left(\int_1^x b(u)u^{-1}\, du\right) \quad \text{for} \quad x \geq 1,$$

where $\lim_{x\to\infty} a(x) = a_0$, with $0 < a_0 < \infty$, and $\lim_{x\to\infty} b(x) = 0$.

Let $U_1, \cdots, U_n$ be independent Uniform (0, 1) random variables and let $U_{1n} \leq \cdots \leq U_{nn}$ denote the order statistics based on $U_1, \cdots, U_n$. Now

$$S_n =_d \textstyle\sum_{i=1}^{k_n} (\ln G^{-1}(U_{n+1-i,n}) - \ln G^{-1}(U_{n+1-k_n,n}))\, k_n^{-1},$$

which by (1) and (2) is equal to

(3) $$c \textstyle\sum_{i=1}^{k_n} (-\ln(1 - U_{n+1-i,n}) + \ln(1 - U_{n+1-k_n,n}))\, k_n^{-1} +$$

(4) $$\textstyle\sum_{i=1}^{k_n} (a((1 - U_{n+1-i,n})^{-1}) - a((1 - U_{n+1-k_n,n})^{-1}))k_n^{-1} +$$

(5) $$\sum_{i=1}^{k_n} \int_{(1-U_{n+1-k_n,n})^{-1}}^{(1-U_{n+1-i,n})^{-1}} b(u)u^{-1}\, du\, k_n^{-1}.$$

$-\ln(1 - U_{1n}) \leq \cdots \leq -\ln(1 - U_{nn})$ have the same distribution as the order statistics $E_{1n} \leq \cdots \leq E_{nn}$ of n independent exponential random variables $E_1, \cdots, E_n$ with mean 1. Hence expression (3) is equal in distribution to

(6) $$c \textstyle\sum_{i=1}^{k_n} (E_{n+1-i,n} - E_{n-k_n,n})\, k_n^{-1},$$

but since the random variables $E_{1n} \leq \cdots \leq E_{nn}$ are equal in distribution to $\sum_{j=1}^{i} (n + 1 - j)^{-1}E_{n+1-j}$ $i = 1, \cdots, n$ it is easy to show that expression (6) is equal in distribution to

(7) $$ck_n^{-1} \textstyle\sum_{j=1}^{k_n} E_j.$$

Since $k_n \to \infty$ the weak law of large numbers implies that expression (7) and hence expression (3) converges in probability to c. It is easy to see that expression (4) converges in probability to zero. Finally, expression (5) converges in probability to zero since the absolute value of expression (5) is

$$\le \max\{|b(u)|: u \ge (1 - U_{n+1-k_n,n})^{-1}\} k_n^{-1} \textstyle\sum_{i=1}^{k_n} \displaystyle\int_{(1-U_{n+1-k_n,n})^{-1}}^{(1-U_{n+1-i,n})^{-1}} u^{-1}\, du,$$

which obviously converges in probability to zero by the above remarks.

The proof of part (F) follows directly from part (E) and the following lemma.

LEMMA 1. *A nonnegative random variable X has a distribution $F \in \mathscr{E}_c$ if and only if* $\exp X$ *has a distribution $G \in \mathscr{R}_c$.*

PROOF. The proof is an elementary consequence of Theorem 1.2.1 of de Haan (1970). This completes the proof of Proposition 1. □

Proposition 1 proves the (D) implies (C) part of Theorem 1 and the (D′) implies (C′) part of Theorem 2. To complete the proofs of Theorems 1 and 2, we will require the following equivalent characterization of the class $\mathscr{E}_c$.

Let $F^{-1}(u) = \inf\{x: F(x) \ge u\}$ for $u \in (0, 1]$, and $F^{-1}(0) \equiv F^{-1}(0+)$.

PROPOSITION 2. *Let F be any distribution function. There exists a finite positive constant c such that*

$$\lim_{u\uparrow 1} \int_u^1 \frac{(1-v)}{1-u}\, dF^{-1}(v) = c \quad \textit{if and only if} \quad F \in \mathscr{E}_c.$$

PROOF. First assume that the above limit holds for some $0 < c < \infty$. Since $F^{-1}(F(x)) \le x$ and $F(F^{-1}(F(x))) = F(x)$

$$\int_x^\infty \frac{(1-F(y))}{1-F(x)}\, dy = \int_{F^{-1}(F(x))}^\infty \frac{(1-F(y))}{1-F(x)}\, dy - (x - F^{-1}(F(x))),$$

which by the change of variables $y = F^{-1}(v)$ equals

$$= \int_{F(x)}^1 \frac{(1-v)}{1-F(x)}\, dF^{-1}(v) - (x - F^{-1}(F(x))).$$

The proof will be completed by showing that $\lim_{x\to\infty} (x - F^{-1}(F(x))) = 0$. For this purpose we need the following lemma.

LEMMA 2. *If $\lim_{u\uparrow 1} \int_u^1 (1-v)/(1-u)\, dF^{-1}(v) = c$ for some $0 < c < \infty$, then*

$$\limsup_{\beta\uparrow 1} \sup_{\beta\le u<1} \{F^{-1}(u+) - F^{-1}(u)\} = 0.$$

PROOF. Choose any $\varepsilon > 0$ and $\beta \in (0, 1)$ such that for all $u \in [\beta, 1)$,

$$c + \varepsilon \ge \int_u^1 \frac{(1-v)}{1-u}\, dF^{-1}(v) \ge c - \varepsilon.$$

Now

$$c + \varepsilon \ge \int_u^1 \frac{(1-v)}{1-u}\, dF^{-1}(v) = F^{-1}(u+) - F^{-1}(u) + \int_{u+}^1 \frac{(1-v)}{1-u}\, dF^{-1}(v),$$

which for all $\delta > 0$ such that $u + \delta < 1$ is

$$\geq F^{-1}(u+) - F^{-1}(u) + \frac{(1-u-\delta)}{1-u}\int_{u+\delta}^{1}\frac{(1-v)}{1-u-\delta}\,dF^{-1}(v)$$

$$\geq F^{-1}(u+) - F^{-1}(u) + \frac{(1-u-\delta)}{1-u}(c-\varepsilon).$$

Hence, by letting $\delta \downarrow 0$, we have for all $u \in [\beta, 1)$ $c + \varepsilon \geq F^{-1}(u+) - F^{-1}(u) + c - \varepsilon$.
This last inequality implies that for all $u \in [\beta, 1)$, $2\varepsilon \geq F^{-1}(u+) - F^{-1}(u) \geq 0$. □

To complete the first half of Proposition 2, observe that $x - F^{-1}(F(x)) \leq F^{-1}(F(x)+) - F^{-1}(F(x))$, which by Lemma 2 converges to zero as $x \to \infty$. The second half of the proposition is proven in a very similar manner and is left to the reader. □

In what follows, it will be convenient to use an alternate representation for T_n.

Let $U_1, U_2, \cdots$, be a sequence of independent Uniform (0, 1) random variables and for each $n \geq 1$ let $U_{1n} \leq \cdots \leq U_{nn}$ denote the order statistics based on $U_1, \cdots, U_n$. It is well known that if X has distribution F, $F^{-1}(U_1), F^{-1}(U_2), \cdots$, has the same distribution as a process as the sequence of independent random variables $X_1, X_2, \cdots$, with common distribution F. For each $n \geq 1$ and $u \in (0, 1)$, let

$$G_n(u) = n^{-1}\sum_{i=1}^{n} I(U_i \leq u),$$

where $I(x \leq y) = 1$ if $x \leq y$ and 0 if $x > y$. G_n is called the uniform empirical distribution based on $U_1, \cdots, U_n$. Now

$$\begin{aligned} T_n &= k_n^{-1}\sum_{i=1}^{k_n}(X_{n+1-i,n} - X_{n-k_n,n}) \\ &=_d k_n^{-1}\sum_{i=1}^{k_n}(F^{-1}(U_{n+1-i,n}) - F^{-1}(U_{n+1-k_n,n})), \end{aligned}$$

which is not too difficult to show to be equal to

$$nk_n^{-1}\int_{U_{n-k_n,n}}^{1}(1 - G_n(u))\,dF^{-1}(u) \equiv T_n^* \quad \text{a.s.}$$

By the above remarks T_n, $n \geq 1$ is equal in distribution as a process to T_n^*, $n \geq 1$.

PROPOSITION 3. *Suppose for some* $0 < \alpha < 1$, $T_n(\alpha)$ *is bounded in probability, then all the positive moments of* X *are finite.*

PROOF. By the previous discussion

$$T_n(\alpha) =_d T_n^*(\alpha) \equiv nk_n^{-1}\int_{U_{n-k_n,n}}^{1}(1 - G_n(u))\,dF^{-1}(u), \quad \text{with} \quad k_n = [n^\alpha].$$

The proof will require the following lemma.

LEMMA 3. *Let* k_n *be any sequence of positive integers satisfying* (K). *For all* $1 > \varepsilon > 0$, *there exist* $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, $1 < \lambda_3 < \infty$ *and* $0 < n_0 < \infty$ *such that for all* $n \geq n_0$

(8) $$P\left(\inf\left\{\frac{(1 - G_n(u))}{1-u} : U_{n-k_n,n} \leq u < U_{nn}\right\} > \lambda_1\right) > 1 - \varepsilon,$$

(9) $$P(U_{n-k_n,n} < 1 - n^{-1}\lambda_2 k_n) > 1 - \varepsilon, \quad \textit{and}$$

(10) $$P(U_{nn} > 1 - n^{-1}\lambda_3) > 1 - \varepsilon.$$

PROOF. Choose $0 < \varepsilon < 1$. Let p equal to the left side of inequality (8).

$$1 - p = P(\sup_{U_{n-k_n,n} \le u < U_{nn}}\{(1 - G_n(u))^{-1}(1-u)\} \ge \lambda_1^{-1})$$
$$\le P(\sup_{0 \le u < U_{nn}}\{(1 - G_n(u))^{-1}(1-u)\} \ge \lambda_1^{-1}),$$

which is easily seen by Remark 1 of Wellner (1978) to be

$$\le e\lambda_1^{-1}e^{-\lambda_1^{-1}} \quad \text{for} \quad 0 < \lambda_1 < 1.$$

Now choose $\lambda_1 > 0$ sufficiently small, so that this last term is less than ϵ.
Choose any $0 < \lambda_2 < 1$. The left side of expression (9) is equal to

$$P(nk_n^{-1/2}(U_{n-k_n,n} - (1 - k_n n^{-1})) < k_n^{1/2}(1 - \lambda_2)).$$

Now since

$$\text{(N)} \qquad nk_n^{-1/2}(U_{n-k_n,n} - (1 - k_n n^{-1})) \to_d N(0, 1)$$

(see Page 18 of Balkema and de Haan (1974)) inequality (9) is true for all n sufficiently large.

The left side of expression (10) is equal to $1 - (1 - \lambda_3 n^{-1})^n \approx 1 - e^{-\lambda_3}$, which for $\lambda_3 > 1$ sufficiently large and all n sufficiently large is $> 1 - \varepsilon$. □

To finish the proof of Proposition 3, observe that

$$T_n^*(\alpha) \ge nk_n^{-1} \inf_{U_{n-k_n,n} \le u < U_{nn}}(1 - G_n(u))(1-u)^{-1} \int_{U_{n-k_n,n}}^{U_{nn}} (1-u)\, dF^{-1}(u).$$

Choose any $0 < \varepsilon < 1$. Since $T_n^*(\alpha)$ is bounded in probability, we have in combination with Lemma 3 that for appropriate $0 < M < \infty$, λ_1, λ_2, λ_3, and n_0 that for all $n \ge n_0$

$$P\left(M > T_n^*(\alpha) \ge nk_n^{-1}\lambda_1 \int_{1-n^{-1}k_n\lambda_2}^{1-n^{-1}\lambda_3} (1-u)\, dF^{-1}(u)\right) > 1 - \varepsilon,$$

which implies that for all $n \ge n_0$

$$\text{(11)} \qquad M \ge nk_n^{-1}\lambda_1 \int_{1-n^{-1}k_n\lambda_2}^{1-n^{-1}\lambda_3} (1-u)\, dF^{-1}(u).$$

Since $k_n = [n^\alpha]$, we can find a number $\lambda > 0$ such that the right side of expression (11) is

$$\ge n^{1-\alpha}\lambda_1 \int_{1-\lambda n^{-1+\alpha}}^{1-\lambda n^{-1+\alpha}2^{-1}} (1-u)\, dF^{-1}(u).$$

It is easy to see now that for every $\delta > 0$ such that $1 - \alpha > \delta > 0$ there exists a constant $B_\delta > 0$ such that for all $n \ge n_0$

$$\text{(12)} \qquad Mn^{-\delta} \ge B_\delta \int_{1-\lambda n^{-1+\alpha}}^{1-\lambda n^{-1+\alpha}2^{-1}} (1-u)^{\delta(1-\alpha)^{-1}}\, dF^{-1}(u).$$

To keep the notation simple, we will assume without loss of generality that $2^{(1-\alpha)^{-1}}$ is an integer. Observe that by means of inequality (12), we have

$$\infty > M \textstyle\sum_{k=k_0}^\infty 2^{-k\delta(1-\alpha)^{-1}} \ge \sum_{k=k_0}^\infty B_\delta \displaystyle\int_{1-\lambda 2^{-k}}^{1-\lambda 2^{-(k+1)}} (1-u)^{\delta(1-\alpha)^{-1}}\, dF^{-1}(u)$$

$$\text{(13)} \qquad = B_\delta \int_{1-\lambda 2^{-k_0}}^{1} (1-u)^{\delta(1-\alpha)^{-1}}\, dF^{-1}(u),$$

where k_0 is chosen so that $2^{k_0(1-\alpha)^{-1}} > n_0$. Since expression (13) is finite for all $1 - \alpha > \delta > 0$, we have

$$(14) \qquad \int_0^1 (1-u)^{\varepsilon}\, dF^{-1}(u) < \infty \quad \text{for every} \quad \varepsilon > 0,$$

but this is equivalent to all the positive moments of X being finite. □

Proposition 3 is the key to the proofs of Theorems 1 and 2.

PROPOSITION 4. *Suppose all the positive moments of X are finite, then for every $0 < \alpha < 1$*

$$T_n(\alpha) - E(T_n(\alpha) \mid U_{n-k_n,n}) \to 0 \qquad \text{a.s.}$$

PROOF. Choose any $0 < \alpha < 1$. We will show that

$$T_n^*(\alpha) - E(T_n^*(\alpha) \mid U_{n-k_n,n}) \to 0 \qquad \text{a.s.}$$

Observe that

$$(15) \qquad E(T_n^*(\alpha) \mid U_{n-k_n,n}) = E(k_n^{-1} \textstyle\sum_{i=1}^{k_n} (F^{-1}(U_{n+1-i,n}) - F^{-1}(U_{n-k_n,n})) \mid U_{n-k_n,n}).$$

Now since conditioned on $U_{n-k_n,n}$ being fixed $\sum_{i=1}^{k_n} F^{-1}(U_{n+1-i,n})$ has the same distribution as $\sum_{i=1}^{k_n} F^{-1}(V_i)$, where $V_1, \cdots, V_{k_n}$ are independent Uniform $(U_{n-k_n,n}, 1)$ random variables; expression (15) is equal to

$$\int_{U_{n-k_n,n}}^1 F^{-1}(v)(1 - U_{n-k_n,n})^{-1}\, dv - F^{-1}(U_{n-k_n,n}).$$

Hence, for each $p \ge 1$

$$(16) \qquad E(T_n^*(\alpha) - E(T_n^*(\alpha) \mid U_{n-k_n,n}))^{2p} = E(E(k_n^{-1} \textstyle\sum_{i=1}^{k_n} (F^{-1}(V_i) - EF^{-1}(V_1))^{2p} \mid U_{n-k_n,n}).$$

By the Marcinkiewicz and Zygmund inequality (see Page 149 of Stout (1974)), expression (16) is

$$\le k_n^{-p} A_p E(E(F^{-1}(V_1) - EF^{-1}(V_1))^{2p} \mid U_{n-k_n,n}),$$

for some constant $A_p > 0$ independent of F^{-1}, which in turn equals

$$(17) \qquad k_n^{-p} A_p E\left(\int_{U_{n-k_n,n}}^1 (1 - U_{n-k_n,n})^{-1} \left(\int_{U_{n-k_n,n}}^1 \frac{(F^{-1}(u) - F^{-1}(v))}{1 - U_{n-k_n,n}} dv\right)^{2p} du\right).$$

By Hölder's inequality and the c_r inequality expression (17) is

$$\le k_n^{-p} B_p E\left(\int_{U_{n-k_n,n}}^1 \int_{U_{n-k_n,n}}^1 \frac{(|F^{-1}(u)|^{2p} + |F^{-1}(v)|^{2p})}{(1 - U_{n-k_n,n})^2}\, du\, dv\right)$$

for some constant B_p independent of F^{-1}, which is

$$(18) \qquad \le 2k_n^{-p} B_p E X^{2p} E(1 - U_{n-k_n,n})^{-1}.$$

At this point we require a lemma.

LEMMA 4. *There exists a constant $0 < C < \infty$ such that for all $n \ge 2$ and $1 < i \le n$*

$$E(1 - U_{n+1-i,n})^{-1} \le Cni^{-1}.$$

PROOF. $E(1 - U_{n+1-i,n})^{-1} = \prod_{j=i}^n (1 - j^{-1})^{-1}$. Now standard approximations show that

$$-\textstyle\sum_{j=i}^n \ln(1 - j^{-1}) \le \sum_{j=i}^n j^{-1} + \tau_1 \le \ln(ni^{-1}) + \tau_2,$$

where τ_1 and τ_2 are finite constants independent of $n \geq 2$ and $1 < i \leq n$. Let $C = e^{\tau_2}$. □

Application of Lemma 4 along with the assumption that $k_n = [n^\alpha]$ for some $0 < \alpha < 1$ implies that expression (18) is

$$\leq D_p n^{-p\alpha-\alpha+1},$$

for some constant $0 < D_p < \infty$ independent of n.

By choosing p sufficiently large so that $\sum_{n=1}^{\infty} n^{-p\alpha-\alpha+1} < \infty$ and applying the Borel-Cantelli lemma in the usual way we complete the proof. □

PROPOSITION 5. *Suppose for some* $0 < \alpha < 1$, $T_n(\alpha) \to_p c$, *where c is a finite positive constant, then*

$$\text{(G)} \qquad \lim_{u\uparrow 1} \int_u^1 \frac{(1-v)}{1-u}\, dF^{-1}(v) = c, \textit{ and}$$

$$\text{(H)} \qquad T_n(\alpha) \to c \quad \text{a.s.}$$

PROOF. Since $T_n(\alpha) \to_p c$, all of the positive moments of X are finite, by Proposition 3. Hence by Proposition 4

$$\text{(19)} \qquad T_n^*(\alpha) - E(T_n^*(\alpha) |\, U_{n-k_n,n}) \to 0 \quad \text{a.s.}$$

which in turn implies that

$$E(T_n^*(\alpha) |\, U_{n-k_n,n}) \to_p c.$$

By integration by parts we see that

$$\text{(20)} \qquad E(T_n^*(\alpha) |\, U_{n-k_n,n}) = \int_{U_{n-k_n,n}}^1 \frac{(1-v)}{(1-U_{n-k_n,n})}\, dF^{-1}(v) \quad \text{a.s.}$$

Hence the right side of (20) converges in probability to c. To complete the proof, it will be sufficient to show that this last statement implies (G); since (G) combined with the fact that $U_{n-k_n,n} \to 1$ a.s. implies that

$$E(T_n^*(\alpha) |\, U_{n-k_n,n}) \to c \quad \text{a.s.,}$$

which in turn by (19) implies (H).

To prove that convergence in probability of the right side of (20) to c for some $0 < \alpha < 1$ implies (G), it will be enough to show that there exists a sequence of constants a_m such that (i) $0 < a_m < 1$, (ii) $a_m \to 1$, (iii) $(1 - a_m)(1 - a_{m+1})^{-1} \to 1$, and

$$\text{(iv)} \qquad \lim_{m\to\infty} \int_{a_m}^1 \frac{(1-v)}{1-a_m}\, dF^{-1}(v) = c.$$

We leave it to the reader to show that the existence of such a sequence implies (G). Statement (N) in the proof of Lemma 3 implies that

$$1 - U_{n-k_n,n} = \frac{k_n}{n} + O_p\left(\frac{\sqrt{k_n}}{n}\right).$$

Now since in this case $nk_n^{-1}(n+1)^{-1} k_{n+1} \to 1$, and

$$\int_{U_{n-k_n,n}}^1 (1-v)\, dF^{-1}(v)(1 - U_{n-k_n,n})^{-1} \to_p c,$$

a routine argument shows that we can extract a sequence a_m satisfying (i) through (iv).
This completes the proof of Proposition 5. □

We can now complete the proof of Theorems 1 and 2.

PROOF OF THEOREM 1. Propositions 2 and 5 give (A) implies (B), (B) implies (D), and (C) implies (D). Proposition 1 gives (D) implies (C). Finally, it is obvious that (B) implies (A), and (C) implies (A). All of these implications together show that (A), (B), (C) and (D) are equivalent.

PROOF OF THEOREM 2. Set $X_i = \ln(Y_i \vee 1)$ for $i = 1, 2, \cdots$, where $x \vee y = \max(x, y)$, and let F denote the distribution of X_1.
It is easy to see that

$$T_n = k_n^{-1} \sum_{i=1}^{k_n} (X_{n+1-i,n} - X_{n-k_n,n}) \rightarrow_p c$$

if and only if $S_n \rightarrow_p c$.
Hence statements (A′), (B′) and (C′) of Theorem 2 are equivalent to the corresponding statements (A), (B) and (C) of Theorem 1 for T_n. Now by Lemma 1 statement (D) is true for F if and only if the distribution of $Y_1 \vee 1$ is in $\mathscr{R}_c$, which is easy to see to be true if and only if the distribution G of Y_1 is in $\mathscr{R}_c$. □

4. Appendix. We will give an example of a distribution F such that $T_n \rightarrow_p c$ for a particular sequence k_n satisfying (K) and finite positive constant c, yet $F \notin \mathscr{E}_c$.

EXAMPLE. Let X be a nonnegative random variable with distribution F, where F is defined via its inverse as follows: Let

$$F^{-1}(1 - 2^{-m}) = m \quad \text{for} \quad m = 0, 1, 2, \cdots, \quad \text{and}$$

$$F^{-1}(u) = m + (u - (1 - 2^{-m}))2^{m+1} \quad \text{for} \quad 1 - 2^{-m} < u < 1 - 2^{-m-1}.$$

For each $m \geq 0$

$$\int_{1-2^{-m}}^{1} (1 - u)\, dF^{-1}(u) = \sum_{k=m}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} 2^{k+1}(1 - u)\, du = \frac{3}{2^{m+1}} \cdot$$

Hence, for each integer $m \geq 0$,

$$2^m \int_{1-2^m}^{1} (1 - u)\, dF^{-1}(u) = \frac{3}{2}. \tag{21}$$

However, for each integer $m \geq 0$

$$2^{m+2} 3^{-1} \int_{1-3/2^{m+2}}^{1} (1 - u)\, dF^{-1}(u) = \frac{17}{12}. \tag{22}$$

Equations (21) and (22) show that $F \notin \mathscr{E}_c$ for any $0 < c < \infty$.
For each fixed integer $m \geq 1$, let

$$T_{n,m} = \sum_{i=1}^{[n2^{-m}]} X_{n+1-i,n/[n2^{-m}]} - X_{n-[n2^{-m}],n}$$

for each integer $n \geq 2^{m+1}$.
Observe that since F^{-1} is continuous

$$X_{n-[n2^{-m}],n} \rightarrow m \quad \text{a.s.}$$

Hence by Remark 1, we have

$$T_{n,m} \rightarrow 2^m \int_{1-2^{-m}}^{1} F^{-1}(u)\, du - m = \frac{3}{2} \quad \text{a.s.} \quad \text{as} \quad n \rightarrow \infty.$$

Now choose $N_1 > 2$ such that $P(|T_{n,1} - 3/2| > 1/2) < 1/2$ for all $n \geq N_1$, $N_2 > \max(2^{2^2}, N_1)$ such that $P(|T_{n,2} - 3/2| > 1/4) < 1/4$ for all $n \geq N_2, \cdots$, and $N_m > \max(2^{m^2}, N_{m-1})$ such that $P(|T_{n,m} - 3/2| > 2^{-m}) < 2^{-m}$ for all $n \geq N_m$, etc. Let $k_n = 1$ for all $2 \leq n < N_1$, and for $n \geq N_1$ let $k_n = [n2^{-m}]$ whenever $N_m \leq n < N_{m+1}$ for $m \geq 1$.

By construction we see that k_n satisfies (K), and $T_n \to_p 3/2$ for this particular sequence k_n; however, $F \notin \mathscr{E}_{3/2}$.

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