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# LAWS OF THE ITERATED LOGARITHM FOR ORDER STATISTICS OF UNIFORM SPACINGS

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Let $X_1, X_2, \cdots$ be a sequence of independent uniformly distributed random variables on [0, 1], and let $K_n$ be the $k$th largest spacing induced by the order statistics of $X_1, \cdots, X_{n-1}$. We show that

$$\limsup (nK_n - \log n)/2 \log_2 n = 1/k \quad \text{almost surely,}$$

and

$$\liminf (nK_n - \log n + \log_3 n) = c \quad \text{almost surely,}$$

where $-\log 2 \le c \le 0$, and $\log_j$ is the $j$ times iterated logarithm.

**1. Introduction.** Consider a sequence $X_1, X_2, \cdots$ of independent identically distributed random variables with a uniform distribution on [0, 1]. If $X_{(1)} < X_{(2)} < \cdots < X_{(n-1)}$ are the order statistics corresponding to $X_1, \cdots, X_{n-1}$, then the *maximal uniform spacing* (or, the *maximal gap*) $M_n$ is defined by

$$M_n = \max_{1\le i\le n} S_i$$

where $S_1 = X_{(1)}$, $S_i = X_{(i)} - X_{(i-1)}$ for $1 < i < n$, and $S_n = 1 - X_{(n-1)}$. The $S_i$'s are called the *spacings*; see Pyke (1965).

Slud (1978) showed that $nM_n - \log n = O(\log_2 n)$ a.s.; we will refine Slud's result and show that

$$\limsup (nM_n - \log n)/2 \log_2 n = 1 \quad \text{a.s.} \tag{1.1}$$

and that

$$\liminf nM_n - \log n + \log_3 n = c \quad \text{a.s.} \tag{1.2}$$

where $-\log 2 \le c \le 0$. Along the way, we will obtain a few large deviation results for $M_n$. In Section 2, we state without proof a few known results about the distribution and the weak convergence of $M_n$. In Sections 4 and 5, we will establish (1.1) and (1.2) for $K_n$, the $k$th largest spacing among $S_1, \cdots, S_n$, when the constant "1" in (1.1) is replaced by $1/k$.

**2. Auxiliary results.** It is well-known that $(S_1, \cdots, S_n)$ is uniformly distributed on the simplex $\{(x_1, \cdots, x_n) \mid x_i \ge 0; \sum x_i = 1\}$, and that, therefore

$$\begin{aligned} P(S_1 > a_1; \cdots; S_n > a_n) &= (1 - \textstyle\sum_{i=1}^n a_i)^{n-1}, \qquad &\textstyle\sum_{i=1}^n a_i < 1 \\ &= 0, &\text{otherwise,} \end{aligned}$$

where $a_1, \cdots, a_n$ are nonnegative numbers. From this, one can get Whitworth's formula (Whitworth (1897); see also Kendall and Moran (1963)):

$$\begin{aligned} P(M_n > x) &= P(\cup_{i=1}^n [S_i > x]) = \textstyle\sum_i P(S_i > x) - \sum_{i<j} P(S_i > x; S_j > x) + \cdots \\ &= \textstyle\sum_{k\ge1; kx<1} (-1)^{k+1}(1 - kx)^{n-1}\binom{n}{k}, \qquad \text{all} \quad x > 0. \end{aligned}$$

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A very useful property of uniform spacings is the following.

LEMMA 2.1. *If $Y_1, \cdots, Y_n$ are independent identically distributed exponential random variables, and if $T_n = \sum Y_i$, then $(S_1, \cdots, S_n)$ is distributed as $(Y_1/T_n, \cdots, Y_n/T_n)$. In particular, $M_n$ is distributed as $L_n/T_n$ where $L_n = \max(Y_i)$.*

For a proof of Lemma 2.1, see Pyke (1965).

LEMMA 2.2. (Sukhatme, 1937). *If $Y_1, \cdots, Y_n$ are independent identically distributed exponential random variables with corresponding order statistics $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$, then the following random variables are also independent and exponentially distributed*:

$$nY_{(1)},\ (n-1)(Y_{(2)} - Y_{(1)}),\ \cdots,\ 2(Y_{(n-1)} - Y_{(n-2)}),\ Y_{(n)} - Y_{(n-1)}.$$

An immediate consequence of Lemma 2.2 is the following.

LEMMA 2.3. *$M_n$ is distributed as*

$$\textstyle\sum_{i=1}^n (Y_i/i)/\sum_{i=1}^n Y_i$$

*where $Y_1, \cdots, Y_n$ are independent exponentially distributed random variables.*

The limit distribution of $M_n$ was found by Levy (1939) and was rederived later by Darling (1952, 1953) and others.

LEMMA 2.4. *For all $x \in R$, $P(nM_n < \log n + x) \to \exp(-\exp(-x))$ as $n \to \infty$.*

LEMMA 2.5. *$nM_n/\log n \to 1$ in probability as $n \to \infty$.*

*Note.* If $G_n$ is the distribution function of $nM_n - \log n$ and $G(x) = \exp(-\exp(-x))$, and if $a_n \log n \to \infty$ as $n \to \infty$, then

$$\begin{aligned} P(|nM_n/\log n - 1| > a_n) &= G_n(-a_n \log n) + 1 - G_n(a_n \log n) \\ &\leq 2\sup_x |G_n(x) - G(x)| \\ &\quad + G(-a_n \log n) + 1 - G(a_n \log n) \to 0. \end{aligned} \tag{2.1}$$

The distribution function $G(x) = \exp(-\exp(-x))$ has mean $\gamma = 0.5772157\ldots$ (the Euler constant) and variance $\pi^2/6$; see Gnedenko (1943), Gumbel (1958), Barndorff-Nielsen (1963) and David (1970) for a closer analysis of its properties. A careful application of Lemma 2.3 also gives

LEMMA 2.6. *$E(nM_n - \log n) \to \gamma$ as $n \to \infty$, and $\mathrm{Var}(nM_n) \to \pi^2/6$ as $n \to \infty$.*

**3. Large deviation results.** We will first derive exponential estimates for the probability in the tail of the gamma density. We recall here that the sum $T_n$ of $n$ independent exponentially distributed random variables has the gamma density $g_n(x) = x^{n-1}e^{-x}/(n-1)!$, $x \geq 0$.

LEMMA 3.1. *For all $x > 0$,*

$$P(T_n/n - 1 > x) \leq \exp(-nx^2(1-x)/2)$$

*and*

$$P(T_n/n - 1 < -x) \leq \exp(-nx^2/2).$$

PROOF. Here and throughout the paper we will use these analytic inequalities, valid for all $x \ge 0$:

$$e^{x-x^2/2} \le 1 + x \le e^x \le 1 + x + x^2e^x/2 \tag{3.1}$$

$$1 - x \le e^{-x-x^2/2-x^3/3} \le e^{-x-x^2/2} \le e^{-x} \le 1 - x + x^2/2. \tag{3.2}$$

Lemma 3.1 is now easily proved by Chernoff's classical technique (Chernoff, 1952). For any $0 < s < 1$, we have $P(T_n/n - 1 > x) \le e^{-snx}E(e^{s(T_n-n)}) = e^{-sn(1+x)}(1 - s)^{-n}$. This expression is minimal when $1 - s = 1/(1 + x)(s = x/(1 + x))$, so that the said probability is not greater than $(e^{-x}(1 + x))^n \le ((1 - x + x^2/2)(1 + x))^n = (1 - x^2/2 + x^3/2)^n \le e^{-nx^2(1-x)/2}$. Similarly, for all $s > 0$, $P(T_n/n - 1 < -x) \le e^{-snx}E(e^{-s(T_n-n)}) = e^{sn(1-x)}(1 + s)^{-n} = (e^x(1 - x))^n \le (e^{x-x-x^2/2})^n = e^{-nx^2/2}$ where we let $s = x/(1 - x)$ whenever $x < 1$. For $x \ge 1$, the result is trivially true.

LEMMA 3.2. *Let $k \ge 1$ be a fixed integer, and let $a_n \to 0$ and $a_n \log n \to \infty$. If $K_n$ is the $k$-th largest spacing among $S_1, \cdots, S_n$, then*

$$P(nK_n/\log n - 1 > a_n) \sim n^{-ka_n}/k!$$

*and*

$$P(nK_n/\log n - 1 \le -a_n) \sim n^{(k-1)a_n} \exp(-n^{a_n})/(k - 1)!.$$

PROOF. We will use the following fact about the tail of the binomial distribution. If $B$ is a binomial random variable with parameters $n$ and $p$, then $np \to 0$ implies $P(B \ge k) \sim P(B = k)$, and $np \to \infty$ implies $P(B < k) \sim P(B = k - 1)$ (Feller, 1957, page 140).

$K_n$ is distributed as $L_n'/T_n$ where $L_n'$ is the $k$th largest of $n$ independent identically distributed random variables with exponential density and whose sum is $T_n$ (Lemma 2.1). For arbitrary $a$, $b > 0$ we have

$$\begin{aligned} P(L_n' < (1 - a - b)\log n) - P(T_n < n(1 - b)) &\le P(nK_n/\log n < 1 - a) \\ &\le P(L_n' < (1 - a + b) \log n) + P(T_n \ge n(1 + b)) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P(L_n' > (1 + a + b)\log n) - P(T_n > n(1 + b)) &\le P(nK_n/\log n > 1 + a) \\ &\le P(L_n' > (1 + a - b)\log n) + P(T_n \le n(1 - b)). \end{aligned} \tag{3.4}$$

Let us take $a = a_n$ and $b = n^{-1/4}$. Lemma 3.2 follows if we can show the following things:

(i) $P(L_n' < (1 - a)\log n) \sim \exp(-n^a)n^{(k-1)a}/(k - 1)!$;
(ii) $P(L_n' > (1 + a)\log n) \sim n^{-ka}/k!$;
(iii) $P(|T_n - n| > bn)/\min(P(L_n' < (1 - a)\log n),\ \ P(L_n' > (1 + a)\log n)) \to 0$;
(iv) $P(L_n' < (1 - a - b)\log n) \sim P(L_n' < (1 - a + b)\log n)$;
(v) $P(L_n' > (1 + a + b)\log n) \sim P(L_n' > (1 + a - b)\log n)$.

Clearly, $P(L_n' < (1 - a)\log n) = P(B < k)$ where $B$ is binomial with parameters $n$ and $p = \exp(-(1 - a)\log n) = n^a/n$. Since $np \to \infty$, we have $P(B < k) \sim P(B = k - 1) = \binom{n}{k-1}p^{k-1}(1 - p)^{n-k+1} \sim (np)^{k-1} \exp(-np)/(k - 1)! = n^{(k-1)a} \exp(-n^a)/(k - 1)!$. Similarly, $P(L_n' > (1 + a)\log n) = P(B \ge k)$ where now $B$ is binomial with parameters $n$ and $p = \exp(-(1 + a)\log n) = 1/n^{1+a}$. Since $np \to 0$, we have $P(B \ge k) \sim P(B = k) \sim 1/n^{ka}k!$. This proves (i) and (ii). The same asymptotic results are valid if in (i) and (ii) we replace $a$ by $(a + b)$ or $(a - b)$ on both sides. The ratio of the two terms of (v) (left divided by right) is $\sim n^{-2kb} \sim 1$. The ratio of the two terms of (iv) is $\sim n^{2(k-1)b} \exp(n^{(a-b)} - n^{(a+b)}) \sim 1$.

To prove (iii) we first use Lemma 3.1: $P(|T_n - n| > bn) \le 2\exp(-nb^2/4)$ for $n$ large enough. It remains to check that $n^{ka}\exp(-nb^2/4) \to 0$ and that $n^{(k-1)a}\exp(n^a - nb^2/4) \to 0$. This follows from $a \to 0$.

**4. Outer Bounds.** In 1961 Barndorff-Nielsen (and independently Robbins and Siegmund (1970) and Deheuvels (1974)) established laws of the iterated logarithm for $Z_n = \min(X_1, \cdots, X_n)$ where $X_1, \cdots, X_n$ is a sequence of independent uniform [0, 1] random variables. These results can be summarized as follows. Let $a_n$ be positive and nonincreasing. Then,

(i) $Z_n < a_n$ i.o. (f.o.) when $\sum a_n = \infty$ ($\sum a_n < \infty$). See Geffroy (1958) for the first proof.

(ii) $Z_n > a_n$ i.o. (f.o.) when $\sum a_n \exp(na_n) = \infty$ ($\sum a_n \exp(-na_n) < \infty$) under the assumption that $na_n$ is ultimately non-decreasing (Robbins and Siegmund, 1970). Barndorff-Nielsen's result uses the series $\sum \log_2 n(1 - a_n)^n/n$ instead of $\sum a_n \exp(-na_n)$. For related work, see Frankel (1972) and Wichura (1973). For a short proof of the first order result: $Z_n > (1+\varepsilon)\log_2 n/n$ i.o. (f.o.) when $\varepsilon = 0$ ($\varepsilon > 0$), see Kiefer (1970). For a survey, with proofs, see Galambos (1978).

In this section we derive sufficient conditions (of the summability type) for $nK_n > (1 + a_n)\log n$ finitely often a.s. and $nK_n < (1 - a_n)\log n$ finitely often a.s.

LEMMA 4.1. *Let $A_1, A_2, \cdots$ be a sequence of events with $P(A_n) \to 0$ as $n \to \infty$. If either $\sum P(A_n^c \cap A_{n+1}) < \infty$ or $\sum P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n$ f.o.$) = 1$.*

PROOF. See Barndorff-Nielsen (1961).

THEOREM 4.1. *Let $a_n \to 0$ and $a_n \log n \to \infty$ as $n \to \infty$ such that $(1 + a_n)\log n/n$ is ultimately nonincreasing. Then, $P(nK_n > (1 + a_n)\log n$ i.o.$) = 0$ when*

$$\sum_{n=1}^{\infty} \log n/n^{1+ka_n} < \infty. \tag{4.1}$$

PROOF. Let $A_n$ be the event $nK_n > (1 + a_n)\log n$. By (2.1), $P(A_n) \to 0$ as $n \to \infty$. Then, for $n$ large enough,

$$\begin{aligned} P(A_n \cap A_{n+1}^c) &\le P(nK_n > (1 + a_n)\log n)2k(1 + a_{n+1})(\log(n+1)/(n+1)) \\ &= 2k(1 + o(1))n^{-ka_n}k!^{-1}\log n/n, \end{aligned}$$

from which Theorem 4.1 follows after applying Lemma 4.1.

THEOREM 4.2. *Let $a_n \to 0$ and $a_n \log n \to \infty$ as $n \to \infty$ such that $(1 - a_n)\log n/n$ is ultimately nonincreasing. Then, $P(nK_n < (1 - a_n)\log n$ i.o.$) = 0$ when*

$$\sum_{n=1}^{\infty} (\log n/n)n^{ka_n}\exp(-n^{a_n}) < \infty. \tag{4.2}$$

PROOF. Let $A_n$ be the event $nK_n < (1 - a_n)\log n$. Once again, we will use Lemma 4.1. Obviously, $P(A_n) \sim n^{(k-1)a_n}\exp(-n^{a_n})/(k-1)! \to 0$ as $n \to \infty$. Also, if $K_n'$ is the $(k+1)$st largest spacing among $S_1, \cdots, S_n$, then for $n$ large,

$$\begin{aligned} P(A_n^c \cap A_{n+1}) &= P(A_n^c \cap A_{n+1} \cap [K_n' < (1 - a_{n+1})\log(n+1)/(n+1)]) \\ &\le P(K_n' < (1 - a_n)\log n/n)2k\log n/n \\ &= 2k(1 + o(1))n^{ka_n}\exp(-n^{a_n})k!^{-1}\log n/n. \end{aligned}$$

REMARK 4.1. It follows trivially from Theorems 4.1 and 4.2 that $nK_n/\log n \to 1$ a.s. as $n \to \infty$. Of course, we have done too much work by invoking Lemma 3.2. For a short proof of $nM_n/\log n \to 1$ a.s., see Slud (1978) or Devroye (1979).

REMARK 4.2. Condition (4.1) is satisfied if for some $\delta > 0$, $J \ge 2$, we have

$$a_n = (k\log n)^{-1}(\log_2 n + \textstyle\sum_{j=2}^{J}\log_j n + \delta\log_J n).$$

In particular, it is satisfied if we take $a_n = (2 + \delta)\log_2 n/(k \log n)$, $\delta > 0$. Hence,

$$\limsup(nK_n - \log n)/2 \log_2 n \le 1/k \text{ a.s.} \tag{4.3}$$

REMARK 4.3. Condition (4.2) is satisfied if for some $J \ge 3$, $\delta > 0$, we have

$$a_n = (\log n)^{-1}(\log(2 \log_2 n + k \log_3 n + \textstyle\sum_{j=3}^{J} \log_j n + \delta \log_J n)),$$

or when for some $\delta > 0$, $a_n = \log((2 + \delta)\log_2 n)/\log n$. Hence,

$$\liminf(nK_n - \log n + \log_3 n) \ge -\log 2 \quad \text{a.s.,} \tag{4.4}$$

independent of $k$. The influence of $k$ on the lower outer bound is only in the second order term of the sequence $a_n$. In other words, whenever $M_n$ is small, it is very likely that the second and third largest spacings are very close in magnitude to $M_n$.

**5. Inner Bounds.** In this section we will prove the following theorems:

THEOREM 5.1. $\limsup(nK_n - \log n)/2 \log_2 n = 1/k$ a.s.

THEOREM 5.2. $\liminf(nK_n - \log n + \log_3 n) = c$ a.s. *for some* $c \in [-\log 2, 0]$.

We will use the notation $[\cdot]$ for the integer part of a number. Furthermore; we will need two lemmas.

LEMMA 5.1. *If* $b_j = \exp(a\sqrt{j} \log j)$, *where* $a > 0$, *then*

$$(b_{j+1} - b_j)/b_j \sim a \log j/2\sqrt{j} \qquad \text{as} \quad j \to \infty.$$

*The same is true for* $c_j = [b_j]$.

PROOF. In view of $(\sqrt{j+1} - \sqrt{j}) \sim \frac{1}{2}\sqrt{j}$ and $\log(1 + 1/j) \sim 1/j$, we have $(b_{j+1} - b_j)/b_j \sim a(\sqrt{j+1} \log(j+1) - \sqrt{j} \log j) \sim a \log j/2\sqrt{j}$.

LEMMA 5.2. *If* $b_j = \exp(j \log j)$, *then*

$$b_j/b_{j+1} \sim 1/ej \qquad \text{as} \quad j \to \infty.$$

*The same is true for* $c_j = [b_j]$.

PROOF. By (3.1) and (3.2) we have $b_{j-1}/b_j = (j - 1)^{-1} \exp(j \log(1 - 1/j)) \le 1/(e(j - 1))$, and $b_{j-1}/b_j \ge (j - 1)^{-1} \exp(-1 - 1/j) \ge (j - 1)^{-1}e^{-1}(1 - 1/j) = 1/ej$.

PROOF OF THEOREM 5.1. In view of (4.3) we need only show that $nK_n - \log n > (2/k - \delta)\log_2 n$ i.o. almost surely, for all $\delta > 0$. We define the following sequences:

$$n_j = [\exp(\sqrt{j} \log j)],$$

$$t_j = [n_j(2/k - \delta/2)\log_2 n_j/\log n_j],$$

$$a_j = (2/k - \delta)\log_2 j/\log j,$$

$$d_j = (1 + a_j)\log j/j,$$

$$d_j' = (1 + (3/k)\log_2 n_j/\log n_j)\log n_j/n_j,$$

$$d_j'' = (1 - \log(3 \log_2 n_j)/\log n_j)\log n_j/n_j.$$

Let us define the following events: $A_N$ is the event that $K_{n_j} \in (d_j'', d_j')$ for all $j \ge N$; $B_N$ is the event that for some $j \ge N$, none of the random variables $X_{n_j}, \cdots, X_{n_j+t_j-1}$ belong to the set $C_j$, where $C_j$ is the union of $k$ intervals of length $d_j'$ each, with the restriction that the leftmost point of each interval coincides with the leftmost point of one of the $k$ largest spacings.

We will see that $t_j + n_j < n_{j+1}$ for all $j$ large enough, and that $d_j'' > d_{n_j+t_j}$ for all $j$ large enough. Thus, $A_N \cap B_N \subseteq [K_{n_j+t_j} > d_{n_j+t_j}$ for some $j \ge N]$. The theorem now follows if we can show that $P(A_N^c) + P(B_N^c) \to 0$ as $N \to \infty$. From Theorems 4.1 and 4.2 we deduce that $P(A_N^c) \to 0$ as $N \to \infty$. Furthermore,

$$P(B_N^c) \le \prod_{j=N}^{\infty} (1 - (1 - kd_j')^{t_j}) \le \exp(-\sum_{j=N}^{\infty} (1 - kd_j')^{t_j}) = 0$$

whenever

$$\sum_{j=1}^{\infty} (1 - kd_j')^{t_j} = \infty.$$

Because $(1 - kd_j')^{t_j} \ge \exp(-d_j' k t_j - k^2 d_j'^2 t_j/2)$ and $d_j'^2 t_j \to 0$, it suffices to check whether $\sum \exp(-kd_j' t_j) = \infty$. We have $\exp(-kd_j' t_j) \sim \exp(-(2 - \delta k/2)\log_2 n_j.\ (1 + (3/k)\log_2 n_j/\log n_j)) \sim \exp(-(2 - \delta k/2)\log_2 n_j) \sim (\sqrt{j}\ \log j)^{2-\delta k/2}$, which is not summable with respect to $j$.

We will now show that $n_j + t_j < n_{j+1}$ for all $j$ large enough. Indeed, $n_{j+1} - n_j \sim n_j \log j/2\sqrt{j}$ (Lemma 5.1), while $t_j \sim (1/k - \delta/4)n_j/\sqrt{j}$.

Finally, let us establish that $d_j'' > d_{n_j+t_j}$ for all $j$ large enough. Clearly,

$$\begin{aligned} d_{n_j+t_j} &= \log(n_j + t_j)/(n_j + t_j) + (2/k - \delta)\log_2(n_j + t_j)/(n_j + t_j) \\ &< \log n_j/(n_j + t_j) + t_j/n_j^2 + (2/k - \delta)\log_2 n_j/n_j \\ &< (\log n_j/n_j)(1 - (1 + o(1))t_j/n_j) + o(1)/n_j + (2/k - \delta)\log_2 n_j/n_j \\ &< \log n_j/n_j - ((2/k - \delta/2)(1 + o(1))\log_2 n_j - (2/k - \delta)\log_2 n_j)/n_j \\ &= \log n_j/n_j - (\delta/2)(1 + o(1))\log_2 n_j/n_j. \end{aligned}$$

Also, $d_j'' = \log n_j/n_j - \log(3 \log_2 n_j)/n_j > d_{n_j+t_j}$ for all $j$ large enough.

Proof of Theorem 5.2. We will show that for all $\delta > 0$, the inequality $nK_n < \log n - \log_3 n + \delta$ is satisfied i.o. almost surely, that is, a.s. $\liminf(nK_n - \log n + \log_3 n) \le 0$. This result together with (4.4) imply the statement of Theorem 5.2.

For given $\delta > 0$, define $n_j = [\exp(2j \log j)]$, $d_j = (\log n_j - \log_3 n_j + \delta)/n_j$, $t_j = n_j - n_{j-1}$ and $a_j = (\log_3 n_j - \delta/2)/\log n_j$. Let further $N_j$ be the $k$th largest gap defined by $X_{n_{j-1}}, \cdots, X_{n_j-1}$ on [0, 1]. Obviously, $N_j < d_j$ i.o. implies that $K_{n_j} < d_j$ i.o. Since the $N_j$'s are independent, $N_j < d_j$ i.o. almost surely whenever $\sum P(N_j < d_j) = \infty$. By Lemma 3.2,

$$P(N_j < (\log t_j/t_j)(1 - a_j)) \sim t_j^{(k-1)a_j} \exp(-t_j^{a_j})/(k - 1)!$$

because $a_j \log t_j \to \infty$. Also, $\exp(-t_j^{a_j}) \ge \exp(-n_j^{a_j}) = \exp(-c' \log_2 n_j) \sim (2j \log j)^{-c'}$ for some $c' < 1$. Thus, $\sum P(N_j < d_j) = \infty$ if $d_j > (\log t_j/t_j)(1 - a_j)$ for all $j$ large enough. Now,

$$d_j t_j/\log n_j \ge (t_j/n_j)(1 - (\log_3 n_j - \delta)/\log n_j) = (1 - O(j^{-2}))(1 - (\log_3 n_j - \delta)/\log n_j)$$

which is greater than $1 - a_j = 1 - (\log_3 n_j - \delta/2)/\log n_j$ for all $j$ large enough.

## 6. Applications.

Example 6.1. *Random covers.* Assume that we try to cover [0, 1] by intervals of length $\ell_n$ centered at $X_1, \cdots, X_{n-1}$ (where the $X_i$'s are independent and uniformly distributed on [0, 1]). Let $A_n$ be the event [[0, 1] is entirely covered]. Then, if $n\ell_n = \log n - \log_3 n + \delta$,

$$P(A_n \text{ i.o.}) = \begin{bmatrix} 1, & \text{if } \delta > 0 \\ 0, & \text{if } \delta + \log 2 < 0. \end{bmatrix}$$

If $n\ell_n = \log n + (2 + \delta)\log_2 n$, we have

$$P(A_n^c \text{ i.o.}) = \begin{bmatrix} 1, & \text{if } \delta < 0 \\ 0, & \text{if } \delta > 0. \end{bmatrix}$$

It is perhaps interesting to compare this result with Shepp's covering theorem (1972): let $\ell_1 \geq \ell_2 \geq \cdots \geq 0$ be the lengths of arcs thrown at random on the circle with unit circumference ($\ell_1 < 1$). Then the circle is covered almost surely if and only if

$$\sum_{n=1}^{\infty} n^{-2} \exp(\ell_1 + \cdots + \ell_n) = \infty.$$

If $\ell_n = (1/n)(1 - (1 + \delta)/\log n)$, then this condition is satisfied when $\delta \leq 0$ and is violated when $\delta > 0$.

EXAMPLE 6.2. *Uniform convergence of nonparametric estimates.* Assume that $f$ is a uniformly continuous function on [0, 1], and that $f$ is estimated by

$$f_n(x) = \sum_{i=1}^{n} f(X_i)K\left(\frac{X_i - x}{\ell_n}\right) \Big/ \sum_{i=1}^{n} K\left(\frac{X_i - x}{\ell_n}\right)$$

where $X_1, \cdots, X_n$ are independent identically distributed uniform [0, 1] random variables, and $K(u)$ is a nonincreasing nonnegative function of $u$ when $u > 0$, and a nondecreasing nonnegative function of $u$ when $u < 0$. Let the support of $K$ be a compact set $[a, b]$ (clearly, $a \leq 0 \leq b$) with $a < b$.

It is clear that $\sup_x |f_n(x) - f(x)| \to 0$ a.s. for all uniformly continuous $f$ if and only if $M_n > (b - a)\ell_n$ f.o. almost surely. Now, if we take $n(b - a)\ell_n = \log n + (2 + \delta)\log_n n$, then

$$\sup_x |f_n(x) - f(x)| \to 0 \text{ a.s.} \quad \text{as} \quad n \to \infty$$

for all uniformly continuous $f$ if $\delta > 0$; the statement is false if $\delta < 0$.

EXAMPLE 6.3. *Estimating the minimum of a density.* Let $f$ be a uniformly continuous density on [0, 1], and let $z$ be the unique point with the property that $f(z) = \min_x f(x)$. Assume that $X_1, X_2, \cdots$ is an independent sample from $f$, and that $z$ is estimated by $Z_n$, the midpoint of the largest interval created by $X_1, \cdots, X_n$. From $nM_n/\log n \to 1$ a.s. for uniform distributions, one can show that $Z_n \to z$ a.s. as $n \to \infty$. For the study of laws of the iterated logarithm of $M_n$ in the non-uniform case, additional assumptions about the rate of increase of $f$ near $z$ seem necessary. Notice also that if the maximum of $f$ were estimated by the midpoint of the smallest interval, then one would *not* obtain almost sure convergence as in the case of $Z_n$.

EXAMPLE 6.4. *Rate of convergence of nearest neighbor estimates.* Let $f$ and $X_1, X_2, \cdots$ be as in Example 6.2, but consider now the nearest neighbor estimate $f_n(x) = f(X_n^N(x))$ where $X_n^N(x)$ is the nearest neighbor to $x$ among $X_1, \cdots, X_n$. If $f$ is Lipschitz with constant $C$, then $\sup_x |f_n(x) - f(x)| \leq \max(CM_{n+1}/2; CX_{(1)}; C(1 - X_{(n)}))$ where $X_{(1)} < \cdots < X_{(n)}$ are the order statistics obtained from $X_1, \cdots, X_n$. From the properties of $X_{(1)}$ and $M_n$ (Theorem 4.1) we have the following rate of convergence result:

$$\sup_x |f_n(x) - f(x)| (2n/C \log n) > 1 + a_n \qquad \text{f.o. a.s.}$$

when $a_n \log n \to \infty$, $(1 + a_n)\log n/n$ is ultimately nonincreasing and $\sum_{n=1}^{\infty} \log n/n^{1+a_n} < \infty$. On the other hand, if $f(x) = Cx$, then the supremum is equal to the maximum of the three given terms, so that we may conclude, by Theorem 5.1, that there exists a Lipschitz function with constant $C$ such that

$$\sup_x |f_n(x) - f(x)| 2n/(C \log n) > 1 + (2 - \delta)\log_2 n/\log n \qquad \text{i.o. a.s. for all } \delta > 0.$$

In other words, in the class $\mathrm{Lip}(C)$, we have

$$\limsup((2n/C)\sup_x |f_n(x) - f(x)| - \log n)/\log_2 n \leq 2 \qquad \text{a.s.} \tag{6.1}$$

but there always exists an $f$ in $\mathrm{Lip}(C)$ for which (6.1) is valid with equality.

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