Layer Solutions in a Half-Space for Boundary Reactions

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1 Introduction

This article is concerned with the nonlinear problem

$$(1.1) \qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

where $n \geq 2$, $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0\}$ is a half-space, $\partial \mathbb{R}^n_+ = \{x = 0\}$, u is real-valued, and $\partial u / \partial \nu = -u_x$ is the exterior normal derivative of u. Points in $\mathbb{R}^{n-1}$ are denoted by $y = (y_1, \ldots, y_{n-1})$.

Our main goal is to study bounded solutions of (1.1) that are monotone increasing, say from -1 to 1, in one of the y-variables. We call them *layer solutions* of (1.1), and we study their existence, uniqueness, symmetry, and variational properties, as well as their asymptotic behavior.

The interest in such increasing solutions comes from some models of boundary phase transitions. When the nonlinearity f is given by $f(u) = \sin(cu)$ for some constant c, problem (1.1) in a half-plane is called the Peierls-Nabarro problem, and it appears as a model of dislocations in crystals (see [21, 36]). The Peierls-Nabarro problem is also central to the analysis of boundary vortices in the paper [28], which studies a model for soft thin films in micromagnetism recently derived by Kohn and Slastikov [26] (see also [27]).

Our main result, Theorem 1.2, characterizes the nonlinearities f for which there exists a layer solution of (1.1) in dimension $n = 2$. We prove that the necessary and sufficient condition is that the potential G (defined by $G' = -f$) has two, and only two, absolute minima in the interval $[-1, 1]$, located at ± 1. Under the additional hypothesis $G''(\pm 1) > 0$, we also establish the uniqueness of a layer solution up to translations in the y-variable.

The proofs of both the necessity and the sufficiency of the condition on G for existence use new ingredients, which we develop in this article. A first one is a nonlocal estimate, as well as a conserved or Hamiltonian quantity, satisfied by every layer solution in dimension 2 (see Theorem 1.3). The estimate can be seen as

an analogue of the pointwise Modica estimate [30] for entire solutions of equations with reaction in the interior.

Another important tool throughout the paper consists of establishing relations between layer solutions and two other classes of solutions: local minimizers and stable solutions of (1.1). This is in the spirit of similar ideas carried out for interior reactions by Alberti, Ambrosio, and one of the authors in [1]. We prove that every layer solution is a local minimizer in any dimension n (see Theorem 1.4). Another result, Theorem 1.5, establishes that stable solutions (and in particular local minimizers) are necessarily monotone functions of y for $n = 2$. For $n = 3$, we prove that stable solutions (and hence also local minimizers and layer solutions) are functions of only two variables: x and a linear combination of y_1 and y_2. This statement on two-dimensional symmetry is closely related to a conjecture of De Giorgi on one-dimensional symmetry for interior reactions, partially proven in [1, 4, 9, 22, 23] and completely settled by Savin [34].

Problem (1.1) in a half-space appears naturally after blowup when studying solutions of

$$(1.2) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = \dfrac{1}{\varepsilon} f(z, u) & \text{on } \partial \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, $z \in \partial \Omega$, and $\varepsilon > 0$ is a parameter. As $\varepsilon \to 0$, certain solutions of (1.2) develop sharp transition layers on some parts of $\partial \Omega$. In the limit $\varepsilon \to 0$, one obtains a discontinuous function on $\partial \Omega$ taking a finite number of values, while in the interior one still has a smooth function— the harmonic extension of the discontinuous limit function on $\partial \Omega$. The transition profiles for (1.2) near a discontinuity point $z_0 \in \partial \Omega$ can be studied using the standard blowup technique. If $f(z, u)$ is continuous in z at z_0, this leads naturally to problem (1.1) in a half-space and to the notion of layer or increasing solution.

One may consider solutions of (1.2) that are local minima of the associated energy

$$(1.3) \qquad E_\varepsilon(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \int_{\partial \Omega} \frac{1}{\varepsilon} G(z, u)$$

(here $G_u = -f$), and solutions of (1.2) that are stable under the associated parabolic problem. After blowup, we then obtain the classes of local minimizers and of stable solutions of (1.1); see Definition 1.1.

Regarding problem (1.2) in bounded domains, the first important question is whether it admits nonconstant stable solutions. It is a subtle task to study their existence when $f = f(u)$ does not depend on z and $\partial \Omega$ is connected. In this direction, Cónsul and one of the authors [19, 20], and also Carvalho and Lozada-Cruz [17], have proven the existence of nonconstant stable solutions for some nonlinearities $f = f(u)$ and certain bounded domains, for instance, certain dumbbells. When enforced by a certain dependence of $f(z, u)$ on the variable z, existence has been

proven in the articles [6, 7], which also establish estimates and properties for minimizers of E_ε.

Another approach to study problem (1.2) for ε small is of variational nature. Alberti, Bouchitté, and Seppecher [2, 3] have found the Γ-limit of $E_\varepsilon/|\log\varepsilon|$ (note the presence of the factor $1/|\log\varepsilon|$ here) for $n \leq 3$, when $G = G(u)$ is a double-well potential. Up to multiplicative universal constants, the Γ-limit is the counting measure of the discontinuity points on $\partial\Omega$ for $n = 2$, and the length of the curve of discontinuities on $\partial\Omega$ for $n = 3$. As a consequence, for $n = 3$ the limit of local minima as $\varepsilon \to 0$ is expected to jump on a geodesic of the surface $\partial\Omega$. Up to a great extent, this locates the set of discontinuities of the limiting function. However, this is not the case in dimension 2, where every couple of points have the same counting measure.

In a work in progress by Cónsul and one of the authors [15], the next term in the expansion of E_ε as a function of ε is found for bounded domains in $\mathbb{R}^2$ and with $G = G(u)$ an arbitrary double-well potential in (1.3). This is a nonlocal renormalized energy in the spirit of the renormalized energy of Bethuel, Brezis, and Hélein [14] for the complex Ginzburg-Landau equation. The next term in the expansion of E_ε in ε has also been found, independently of [15], by Kurzke [28] when $G(z, u) = \sin^2(u - g(z))$ in (1.3) and $e^{ig} : \partial\Omega \to \mathbb{S}^1$ is a map of nonzero degree. The renormalized energy is used in [15] to prove existence of nonconstant stable solutions of (1.2) for certain convex domains and bistable nonlinearities $f = f(u)$, as well as to locate the points of jump (or vortices) on the boundary as $\varepsilon \to 0$. The work on bounded domains [15] uses the methods and results developed in the present article for problem (1.1) in the half-plane. In particular, the uniqueness of a layer solution in the half-plane, Theorem 1.2(b) below, is crucial when computing the renormalized energy for (1.2).

To our knowledge, layer or increasing solutions of (1.1) in a half-plane have been studied only for the Peierls-Nabarro problem, that is, when

$$f(u) = \frac{1}{\pi a} \sin(\pi u),$$

and $a > 0$ is a constant. In an interesting article, Toland [36] has found all bounded solutions of the Peierls-Nabarro problem when $n = 2$, establishing that they are given by the layer solution

$$(1.4) \qquad u = \phi^a(x, y) = \frac{2}{\pi} \arctan \frac{y}{x + a}, \quad x \geq 0, \ y \in \mathbb{R},$$

which is the unique layer (up to translations in the y-variable), and by a family of periodic (in y) solutions that can all be written explicitly. The proof in [36] relies in a clever nonlocal transformation relating solutions of the Peierls-Nabarro problem with solutions of the Benjamin-Ono problem in hydrodynamics, which is problem (1.1) in $\mathbb{R}^2_+$ with $f(u) = -u + u^2$. Then, [36] uses a complete classification of all bounded solutions of the Benjamin-Ono equation due to Amick and Toland [5]. For this last equation all solutions can also be written explicitly and

are: a unique solitary (or ground state) solution and a family of periodic solutions. Such classification is obtained in [5] with the use of a complex variable method that allows the Benjamin-Ono equation to be integrated. The uniqueness of this solitary solution is a relevant fact since, for instance, it corresponds to a traveling wave of a time-dependent problem. In addition, it leads to the uniqueness of the layer solution (1.4) for the Peierls-Nabarro problem. Theorem 1.2(b) below extends Toland's result on uniqueness of the layer solution to the case of general nonlinearities f. Our proof of uniqueness uses the sliding method of Berestycki and Nirenberg [12].

Notice that up to a multiplicative constant, the layer solution (1.4) is simply the angle with the horizontal axis, taking the point $(-a, 0)$ as the origin. Each of its level sets is a straight half-line.

Positive solutions of radial (or ground state) type for problem (1.1) in a half-space have also been studied in [18, 29], among other papers. They establish results on existence, nonexistence, uniqueness, and radial symmetry for this type of solution and certain nonlinearities f, also including certain reaction terms in the interior $\mathbb{R}^n_+$. This is also the case of the recent work [13], where the reaction on the boundary is linear (Robin boundary conditions) and deals with spike-type solutions—see [13, 33] for other references on this class of solutions in the case of homogeneous Neumann boundary conditions.

Throughout the paper, we assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1,\alpha}$ for some $0 < \alpha < 1$. We denote by G the function satisfying

$$G' = -f,$$

which is uniquely defined up to an additive constant.

Regarding half-balls, we use the notation

$$B_R^+ = \{(x, y) \in \mathbb{R}^n : x > 0, |(x, y)| < R\},$$
$$\Gamma_R^0 = \{(0, y) \in \partial\mathbb{R}^n_+ : |y| < R\},$$
$$\Gamma_R^+ = \{(x, y) \in \mathbb{R}^n : x \geq 0, |(x, y)| = R\}.$$

To define the three classes of solutions considered throughout the paper, we introduce the energy functional

(1.5)
$$E_{B_R^+}(v) = \int_{B_R^+} \frac{1}{2}|\nabla v|^2 + \int_{\Gamma_R^0} G(v).$$

DEFINITION 1.1

(a) We say that u is a *layer solution* of (1.1) if it satisfies (1.1),

(1.6)
$$u_{y_1} > 0 \quad \text{on } \partial\mathbb{R}^n_+$$

and

(1.7)
$$\lim_{y_1 \to \pm\infty} u(0, y) = \pm 1 \quad \text{for every } (y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}.$$

(b) Assume that u is a C^1 function in $\overline{\mathbb{R}^n_+}$ satisfying $-1 < u < 1$ in $\overline{\mathbb{R}^n_+}$. We say that u is a *local minimizer* of problem (1.1) if

$$E_{B_R^+}(u) \leq E_{B_R^+}(u + \psi)$$

for every $R > 0$ and every C^1 function ψ in $\overline{\mathbb{R}^n_+}$ with compact support in $B_R^+ \cup \Gamma_R^0$ and such that $-1 \leq u + \psi \leq 1$ in B_R^+. To emphasize this last condition, on some occasions we will say that u is a local minimizer relative to perturbations in $[-1, 1]$.

(c) We say that u is a *stable solution* if it satisfies (1.1) and if

$$(1.8) \qquad \int_{\mathbb{R}^n_+} |\nabla \xi|^2 - \int_{\partial \mathbb{R}^n_+} f'(u)\, \xi^2 \geq 0$$

for every function $\xi \in C^1(\overline{\mathbb{R}^n_+})$ with compact support in $\overline{\mathbb{R}^n_+}$.

A solution u for $n = 2$ is a layer solution simply if the function $u(0, y)$, of one real variable, is increasing and has limits ± 1 at $\pm\infty$. For $n \geq 3$, the limits in (1.7) are taken for $(y_2, \ldots, y_{n-1})$ fixed. We do not assume these limits to be uniform in $(y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}$.

In Lemma 2.3 we establish regularity, as well as first and second derivative estimates, for weak solutions of (1.1). There we introduce a useful technique that allows us to deduce Schauder and Calderón-Zygmund bounds for the Neumann problem from the corresponding estimates for the Dirichlet case, avoiding in this way the use of Green's functions as in other references. These bounds and the maximum principle give that every layer solution satisfies $u_{y_1} > 0$ not only on $\partial \mathbb{R}^n_+$ but also in all of $\mathbb{R}^n_+$; see (2.27) in Section 2.4. The bounds also imply that every layer solution satisfies $u(x, y) \to \pm 1$ as $y_1 \to \pm\infty$ for every $(y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}$ and every $x \geq 0$; see (2.13) in Lemma 2.4. Note that in the definition of layer solution, (1.7) only assumes the existence of these limits when $x = 0$.

Every local minimizer of problem (1.1) is a stable solution. This follows immediately from the fact that the quadratic form in the stability condition (1.8) is the second variation of energy. Note that in the definition of local minimizer, the function ψ vanishes on the spherical part Γ_R^+ of the boundary of B_R^+, but not necessarily on the flat part Γ_R^0. We also emphasize that, as stated in Definition 1.1(b), from now on by local minimizer we mean an *absolute* minimizer u in every half-ball B_R^+ among functions taking values in $[-1, 1]$ and agreeing with u on the spherical part Γ_R^+ of ∂B_R^+. Hence, the word *local* stands for "locally in the half-space" and not for minimizer under small perturbations.

The following is our main result. It characterizes the nonlinearities f for which problem (1.1) admits a layer solution in dimension 2. In addition, it contains a result on the uniqueness of the layer solution. The proof of the theorem exploits strongly certain relations between layer solutions and local minimizers.

THEOREM 1.2 *Let* $n = 2$ *and* f *be any* $C^{1,\alpha}$ *function with* $0 < \alpha < 1$. *Let* $G' = -f$. *Then*:

(a) *There exists a layer solution of* (1.1) *if and only if*

$$(1.9) \qquad G'(-1) = G'(1) = 0 \quad and \quad G > G(-1) = G(1) \quad in \ (-1, 1).$$

(b) *If* $f'(\pm 1) < 0$, *then a layer solution of* (1.1) *is unique up to translations in the y-variable.*

(c) *If* f *is odd and* $f'(\pm 1) < 0$, *then every layer solution u of* (1.1) *is odd in y with respect to some half-axis. That is,* $u(x, y + b) = -u(x, -y + b)$ *for some* $b \in \mathbb{R}$.

Note that the equality $G(-1) = G(1)$ in (1.9) is equivalent to

$$\int_{-1}^{1} f(s)ds = 0,$$

which is usually rephrased as saying that f is balanced in $(-1, 1)$.

Notice that while part (a) of the theorem applies to general f, parts (b) and (c) require the additional hypothesis $f'(\pm 1) < 0$. In fact, as will be seen along the proof, Theorem 1.2(b,c) also hold if, instead of $f'(\pm 1) < 0$, we make the weaker hypothesis

$$f \text{ is nonincreasing in } (-1, -\tau) \cup (\tau, 1) \text{ for some } \tau \in (0, 1).$$

Note that G may have one or several local minima in $(-1, 1)$ with higher energy than -1 and 1 and still satisfy condition (1.9). Such G will therefore admit a layer solution, and hence a solution with limits -1 and 1 at infinity. Instead, such a layer solution will not exist if G has a minimum at some point in $(-1, 1)$ with the same height as -1 and 1. In particular, when G is periodic (as in the Peierls-Nabarro problem $f(u) = \sin(cu)$), there is no increasing solution connecting two nonconsecutive absolute minima of G.

It is worth noting that (1.9) is also a necessary and sufficient condition for the existence of an increasing solution of the interior reaction equation $-u'' = f(u)$ in all of $\mathbb{R}$ with limits -1 and 1 at infinity. This fact can easily be seen by integrating the equation. At the same time, we recall that $-u'' = f(u)$ may admit only three types of nonconstant bounded solutions in all of $\mathbb{R}$: monotone solutions (increasing or decreasing), radially increasing or decreasing solutions (these are even functions with respect to a point), and periodic solutions. By integrating the equation, one can easily characterize the nonlinearities that admit a solution of any of these types. We do not know if, for every bounded solution u of problem (1.1) in dimension $n = 2$ (whatever the nonlinearity f is), $u(0, \cdot)$ must be of one of the three types above.

The necessity of the conditions on G in (1.9) for the existence of a layer solution is a consequence of the following result:

THEOREM 1.3 *Let $n = 2$ and u be a layer solution of* (1.1). *Then $G(-1) = G(1)$ and*

$$(1.10) \qquad \int_0^x \frac{1}{2} \left\{ u_y^2(t, y) - u_x^2(t, y) \right\} dt < G(u(0, y)) - G(1)$$

for all $x \geq 0$ and $y \in \mathbb{R}$. In addition, $\int_0^{+\infty} |\nabla u(t, y)|^2 \, dt < \infty$ and

$$(1.11) \qquad \int_0^{+\infty} \frac{1}{2} \left\{ u_y^2(t, y) - u_x^2(t, y) \right\} dt = G(u(0, y)) - G(1) \quad \text{for all } y \in \mathbb{R}.$$

The proof of (1.10) is based on the maximum principle. The estimate is reminiscent of the pointwise Modica inequality $|\nabla u|^2/2 \leq G(u)$ if $G \geq 0$, satisfied by every bounded entire solution of the equation $-\Delta u = f(u)$ in $\mathbb{R}^n$ (see [30]). The main difference is, of course, the nonlocal character of (1.10). Indeed, this is a common fact throughout the paper: there are strong analogies between problem (1.1) and the one just mentioned on interior reactions, while an important difference is the nonlocal character of (1.1)—which often makes it more delicate. Problem (1.1) could also be set in a precise way as a nonlocal or integrodifferential problem in all of $\mathbb{R}^{n-1}$ looking at it as an equation for the trace of u, or of u_ν, on $\partial \mathbb{R}_+^n$. In this paper we do not follow this approach.

The quantity appearing in (1.11) arises naturally when looking at problem (1.1) for $n = 2$ in a formal way as a Hamiltonian system in infinite dimensions. Here the time variable is $\tau = y$, the position q is the function $u(\cdot, y) = u(\cdot, \tau)$ in the half-line $\{x \geq 0\}$, and the momentum $p = q' = u_y(\cdot, \tau)$. From the action (that is, the energy functional (1.5) in PDE terminology), which we already know, one finds the Lagrangian $\frac{1}{2}\|p\|_2^2 + W(q)$, with $W(q) = \frac{1}{2}\|\partial_x q\|_2^2 + G(q(0))$. Its Legendre transform with respect to p gives the Hamiltonian $\frac{1}{2}\|p\|_2^2 - W(q)$, which is precisely the quantity involved in (1.11). One can easily check that its associated Hamiltonian system

$$\begin{cases} q' = p \\ p' = W_q \end{cases}$$

is formally problem (1.1). In this paper we do not use a possible rigorous Hamiltonian setting. However, (1.11) establishes that the Hamiltonian is a well-defined and conserved quantity for every layer solution.

The proof of existence of a layer solution, as stated in Theorem 1.2(a), is rather nontrivial and uses variational techniques. Only in the case of some odd nonlinearities f have we found a simpler proof, based on sub- and supersolutions. The reason is that, by the expected odd symmetry of the solution, we can take the zero level set of u to be a half-line. We then construct the solution in a quarter-plane using a barrier, and finally we reflect it in odd manner. Note that, if f is not odd, the zero level set of a solution is a nontrivial curve in the half-plane.

To prove existence of a layer solution (for f not necessarily odd), we construct a sequence of local minimizers, each one increasing in y and defined in a half-ball. The first difficulty is to take all the minimizers vanishing at the origin and, at

the same time, such that their corresponding half-balls fill in the whole half-plane. Once this is accomplished, the limit is a local minimizer u of (1.1) vanishing at the origin. The crucial point is then to deduce that the limits of u at infinity are ± 1. For this, we use the local minimality of u and the hypothesis in (1.9) stating that every value in $(-1, 1)$ has a higher potential energy than ± 1 (see Proposition 3.2 for the precise variational result used to guarantee the limits ± 1).

The following result states that every layer solution in $\mathbb{R}^n_+$ is a local minimizer and, in particular, a stable solution. This holds in any dimension n. Among other necessary conditions on the nonlinearity f, the result also establishes that f must be balanced for a layer solution in $\mathbb{R}^n_+$ to exist.

THEOREM 1.4 *Assume that problem* (1.1) *admits a layer solution u in* $\mathbb{R}^n_+$. *Then*

(a) *u is a local minimizer of problem* (1.1), *and*

(b) *the potential G satisfies*

(1.12) $\qquad G'(-1) = G'(1) = 0 \quad and \quad G \geq G(-1) = G(1) \quad in\ (-1, 1).$

It is an open question whether the strict inequality $G > G(\pm 1)$ in $(-1, 1)$ necessarily holds whenever a layer solution exists in $\mathbb{R}^n_+$. We recall that this is the case for $n = 2$, by Theorem 1.2(a).

The following result establishes the monotonicity of every stable solution in $\mathbb{R}^2_+$ and the two-dimensional symmetry of every stable solution in $\mathbb{R}^3_+$. Since the result holds for stable solutions, it also applies to local minimizers and, by Theorem 1.4(a), to layer solutions.

THEOREM 1.5 *Let u be a bounded stable solution of* (1.1) *in* $\mathbb{R}^n_+$.

(a) *If* $n = 2$, *then either* $u_y > 0$ *in* $\overline{\mathbb{R}^2_+}$, $u_y < 0$ *in* $\overline{\mathbb{R}^2_+}$, *or u is identically constant.*

(b) *If* $n = 3$, *then u is a function of two variables. More precisely,*

$$u(x, y_1, y_2) = u_0(x, (\cos \theta)y_1 + (\sin \theta)y_2) \quad in\ \mathbb{R}^3_+$$

for some angle θ *and some solution* $u_0(x, y)$ *of the two-dimensional problem with the same nonlinearity* f, *and with either* $\partial_y u_0 > 0$ *everywhere or* u_0 *identically constant.*

A solution u in $\mathbb{R}^n_+$ depending only on two variables (the x-variable and a linear combination of the y-variables, as in Theorem 1.5(b)), is what we call a two-dimensional solution. Theorem 1.5(b) is the analogue of the one-dimensional symmetry result proven in [4] concerning a conjecture of De Giorgi for interior reactions. Their proofs use the same method.

A simpler task consists of studying solutions u of (1.1) with $|u| \leq 1$ and satisfying the limits (1.7) *uniformly* in $(y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}$. Under the hypothesis $f'(\pm 1) < 0$ and in every dimension n, we establish that these solutions depend only on the x- and y_1-variables and are monotone in y_1 (see Theorem 5.1). Here,

by the uniform limits hypothesis, the y-variable on which the solution finally depends is known a priori, in contrast with the situation of Theorem 1.5(b). Our result on uniform limits is the exact analogue of a result for interior reactions due to Berestycki, Hamel, and Monneau [11]. Its proof uses the sliding method and also leads to the uniqueness of the two-dimensional layer solution as stated in Theorem 1.2(b).

For interior reactions, Barlow, Bass, and Gui [8] have used probabilistic tools to prove a difficult one-dimensional symmetry result. Instead of the uniform limits hypothesis, they make the weaker assumption that one level set is a globally Lipschitz graph.

Theorem 1.5(a) is a partial converse in dimension two of Theorem 1.4(a) in the sense that it establishes the monotonicity of stable solutions and, in particular, of local minimizers. The remaining property for being a layer solution (i.e., having limits ± 1 at infinity) requires additional hypotheses on G. In Proposition 6.1 we have collected several assumptions on G to guarantee that a stable solution, or a local minimizer, is necessarily a layer solution.

The following result states the precise behavior at infinity of every two-dimensional layer solution as well as of its kinetic and potential energies. Such decay rates are nonlinear phenomena; that is, they are consequences of the boundary condition $u_\nu = f(u)$ under hypothesis $f'(\pm 1) < 0$.

THEOREM 1.6 *Assume that $n = 2$, $f'(\pm 1) < 0$, and that u is a layer solution of* (1.1). *Then,*

$$(1.13) \qquad 0 < \frac{c}{1 + y^2} \leq u_y(0, y) \leq \frac{C}{1 + y^2} \quad \text{for all } y \in \mathbb{R}$$

and

$$(1.14) \qquad |\nabla u(x, y)| \leq \frac{C}{1 + |(x, y)|} \quad \text{for all } x \geq 0, \ y \in \mathbb{R},$$

for some constants $0 < c \leq C$. Consequently, we have $|\pm 1 - u(0, y)| \leq C/|y|$ as $y \to \pm\infty$,

$$(1.15) \qquad \int_{B_R^+} |\nabla u|^2 \, d\mathcal{H}_2 \leq C \log R \quad \text{for all } R > 2$$

and

$$(1.16) \qquad \int_{-\infty}^{+\infty} \{G(u(0, y)) - G(1)\} dy < \infty.$$

In addition, for all real numbers m and p, we have

$$(1.17) \qquad \lim_{x \to +\infty} u(x, mx + p) = \frac{2}{\pi} \arctan m.$$

Moreover, for each $s \in (-1, 1)$, the s-level set of u is a globally Lipschitz graph $\{u = s\} = \{y = \varphi^s(x), x \geq 0\}$ *with*

$$(1.18) \qquad \lim_{x \to +\infty} \frac{\varphi^s(x)}{x} = m_s,$$

where m_s is defined by $s = (2/\pi) \arctan m_s$.

All the previous decay and energy bounds for two-dimensional layer solutions are optimal in the sense that they give the exact rates for the explicit solutions (1.4) (see Section 2.1).

The different behavior of the kinetic and potential energies for large radii creates a technical difficulty in the study of (1.1) that is not present when studying layer solutions for interior reactions

$$(1.19) \qquad -\Delta u = f(u) \quad \text{in } \mathbb{R}^n.$$

The kinetic and potential energies in B_R of any one-dimensional increasing solution of (1.19) coincide and behave as cR^{n-1} for large R, where c is a constant. In addition, every layer solution u of (1.19) has total energy $I_{B_R}(u)$ in B_R bounded above and below by constants times R^{n-1} for $R > 1$. This fact holds in every dimension n. The upper bound has been proven by two different methods: a quite elementary one in [4] and a variational one in [1] that uses the local minimality property of layer solutions. The lower bound is a consequence of Modica's monotonicity formula [31] for $I_{B_R}(u)/R^{n-1}$. It is an open question whether some type of monotonicity formula concerning the energy of layer solutions of problem (1.1) may hold.

It is also an open problem whether

$$(1.20) \qquad \int_{B_R^+} |\nabla u|^2 \leq CR^{n-2} \log R$$

and

$$(1.21) \qquad \int_{\Gamma_R^0} \{G(u(0, y)) - G(1)\} dy \leq CR^{n-2}$$

for all $R > 2$, or the weaker estimate $E_{B_R^+}(u) \leq CR^{n-2} \log R$, holds for every layer solution u of (1.1) in $\mathbb{R}_+^n$ when $n \geq 4$. Note that (1.20) and (1.21) hold when $n = 3$ as a consequence of Theorems 1.5(b) and 1.6. Obviously, the bounds also hold for any two-dimensional solution placed in $\mathbb{R}_+^n$. In the present paper we simply use the nonoptimal bound $\int_{B_R^+} |\nabla u|^2 = \int_{\partial B_R^+} u u_\nu \leq CR^{n-1}$ (since $u u_\nu$ is bounded), which allows us to prove two-dimensional symmetry up to $n \leq 3$. Any further development on two-dimensional symmetry of layer solutions of (1.1) will probably need estimates (1.20) and (1.21) or close versions of them. One would think that at least one of the proofs in [1] or [4] mentioned in the previous paragraph

could be adapted to reactions on the boundary (see the proof of Proposition 3.2 for a nonoptimal energy bound using the minimality property of u, as in [1]).

As a final remark, we note that if (1.20) were true for layer solutions in $\mathbb{R}^4_+$, then every layer solution for $n = 4$ would necessarily be a two-dimensional solution. The proof of this would proceed as the proof for Theorem 1.5(b) for $n = 3$ (see the end of Section 4 and Lemma 4.2). Now, $(\varphi\sigma)^2 \leq |\nabla u|^2$ would not satisfy hypothesis (2.20) of the Liouville property of Lemma 2.6. However, one could apply an improved Liouville theorem due to Moschini [32], in which hypothesis (2.20) is replaced by the optimal growth condition $\int_{B_R^+}(\varphi\sigma)^2 \leq C R^2 \log R$.

The paper is organized as follows:

• In Section 2 we present the useful family of explicit layer solutions and their corresponding nonlinearities. We also establish the $C^{2,\alpha}$ regularity of weak solutions of (1.1), a Harnack inequality, a Liouville theorem, some maximum principles, and the existence of minimizers in bounded domains. All of these results are related to problem (1.1).

• In Section 3 we prove the local minimality property of layer solutions and its consequences, as stated in Theorem 1.4.

• In Section 4 we consider stable solutions and prove their monotonicity in $\mathbb{R}^2_+$ and their two-dimensional symmetry in $\mathbb{R}^3_+$ (Theorem 1.5).

• Section 5 studies solutions that have uniform limits at infinity and establishes their monotonicity and two-dimensional symmetry (see Theorem 5.1). In that section, we also prove the uniqueness of layer solution in $\mathbb{R}^2_+$, Theorem 1.2(b).

• Section 6, where we always have $n = 2$, deals with the Modica-type estimate and the Hamiltonian quantity of Theorem 1.3, the existence result in Theorem 1.2, and the decay estimates of Theorem 1.6. This last section also contains a result, Proposition 6.1, where we collect different hypotheses on G to guarantee that every local minimizer, or every stable solution, is necessarily a layer solution.

2 Explicit Solutions and Preliminary Results

After Section 2.1, the reader may proceed to Section 3, since Sections 2.2 through 2.5 contain auxiliary results that can be read when needed in future sections.

2.1 Explicit Layer Solutions

Here we give the family of explicit layer solutions corresponding to the Peierls-Nabarro nonlinearities; see [36]. Besides providing us with a useful initial guide in the study of layer solutions, they will be used in the proof of the decay estimate (1.13) for other nonlinearities f.

LEMMA 2.1 *For $a > 0$, the functions*

$$\phi^a(x, y) = \frac{2}{\pi} \arctan \frac{y}{x + a}, \quad x \geq 0, \ y \in \mathbb{R},$$

are explicit solutions of (1.1) *with* $n = 2$ *and*

$$f(s) = f_a(s) = \frac{1}{\pi a} \sin(\pi s),$$

and they satisfy (1.6) *and* (1.7). *Their y-derivatives*

$$(2.1) \qquad \phi_y^a(x, y) = \frac{2}{\pi} \frac{x + a}{(x + a)^2 + y^2}$$

satisfy

$$(2.2) \qquad \begin{cases} \Delta \phi_y^a = 0 & in \ \mathbb{R}_+^2 \\ \dfrac{\partial \phi_y^a}{\partial \nu} + \dfrac{1}{a} \phi_y^a \geq 0 & on \ \partial \mathbb{R}_+^2. \end{cases}$$

The proof of this lemma is just a simple calculation. Note that ϕ^a is the imaginary part of the analytic function $(2/\pi) \log(x + a + iy)$, and hence we have that

$$(2.3) \qquad |\nabla \phi^a| = \frac{2}{\pi} \frac{1}{|(x + a, y)|}.$$

In particular,

$$(2.4) \qquad \int_{B_R^+} |\nabla \phi^a|^2 \, d\mathcal{H}_2 \simeq C \log R \quad \text{for } R \gg 1$$

in the sense that, for some constant C, the difference of the right- and left-hand sides is of smaller order than $\log R$ for R large. For the potential energy we have

$$(2.5) \qquad \int_{-R}^{R} G_a(\phi^a(0, y)) dy \simeq \tilde{C} \quad \text{for } R \gg 1$$

in the sense that $G_a(\phi^a(0, y))$ is in fact integrable in all $\mathbb{R}$.

Note that (2.1), (2.3), (2.4), and (2.5) show that estimates (1.13), (1.14), (1.15), and (1.16) for an arbitrary layer solution and nonlinearity f are optimal.

2.2 Regularity of Weak Solutions

Lemma 2.3 below establishes that bounded weak solutions of (1.1) are $C^{2,\alpha}$ up to the boundary $\partial \mathbb{R}_+^n$, and that ∇u and $D^2 u$ belong to $L^\infty(\mathbb{R}_+^n)$. In addition, statement (c) of the lemma applied with $d(y) = -f'(u(0, y))$ states that bounded weak solutions of the linearized problem are $C^{1,\alpha}$ up to the boundary $\partial \mathbb{R}_+^n$.

In this subsection we also consider solutions with limits in one direction y_1 on $\partial \mathbb{R}_+^n$. Lemma 2.4 establishes the existence of limits in the whole $\mathbb{R}_+^n$ and the convergence towards 0 as $y_1 \to \pm\infty$ for the gradient. In addition, we show that, in the presence of such a solution, its limits must be zeroes of f.

Lemma 2.3 is concerned with Calderón-Zygmund and Schauder estimates for linear Neumann boundary value problems. These estimates have been established in the existing literature in different ways, such as from integral representations

with the Green's function for the Neumann problem—see, for example, theorem 4.1 of [18] and section 6.7 of [24]—or from a Hilbert transform representation [35]. Here we deduce these bounds in a simple and fast way from the well-known corresponding estimates for the Dirichlet problem that we apply to the auxiliary function

$$(2.6) \qquad v(x, y) = \int_0^x u(t, y)dt.$$

To describe this in more detail, let us first give the following definition:

DEFINITION 2.2 Given $R > 0$ and a function $h \in L^1(\Gamma_R^0)$, we say that u is a *weak solution of*

$$(2.7) \qquad \begin{cases} \Delta u = 0 & \text{in } B_R^+ \subset \mathbb{R}_+^n \\ \dfrac{\partial u}{\partial \nu} = h(y) & \text{on } \Gamma_R^0 \end{cases}$$

if $u \in H^1(B_R^+)$ and

$$(2.8) \qquad \int_{B_R^+} \nabla u \nabla \xi - \int_{\Gamma_R^0} h(y)\xi = 0$$

for all $\xi \in C^1(\overline{B_R^+})$ such that $\xi \equiv 0$ on Γ_R^+.

Define now v by (2.6) for $(x, y) \in B_R^+$. Assume first that u is a classical solution of (2.7). Then, since $(\Delta v)_x = 0$, we have that Δv is a function of y alone. Hence, it is enough to compute it on $\{x = 0\}$. On $\{x = 0\}$ we have $\Delta v = v_{xx} = u_x$, and hence v is a solution of the Dirichlet problem

$$(2.9) \qquad \begin{cases} \Delta v(x, y) = -h(y) & \text{in } B_R^+ \\ v(0, y) = 0 & \text{on } \Gamma_R^0. \end{cases}$$

It is easy to see that if u is a weak solution of (2.7), then v is a weak solution of (2.9). The weak meaning of the Dirichlet condition on (2.9) is that $v \in H^1(B_R^+)$ can be approximated in $H^1(B_R^+)$ by a sequence of C^1 functions vanishing in a neighborhood of Γ_R^0 in $B_R^+ \cup \Gamma_R^0$. This is easily verified for the function v defined by (2.6). To prove that the equation in (2.9) holds in the weak sense, we take a C^1 function η with compact support in B_R^+. For every $t > 0$, we write (2.8) for $\xi(x, y) = \eta(x + t, y)$. Integrating these equalities with respect to t, we obtain the weak sense for problem (2.9).

LEMMA 2.3 *Let* $\alpha \in (0, 1)$.

(a) *Let* $R > 0$ *and* $u \in L^\infty(B_{4R}^+) \cap H^1(B_{4R}^+)$ *be a weak solution of*

$$\begin{cases} \Delta u = 0 & \text{in } B_{4R}^+ \subset \mathbb{R}_+^n \\ \dfrac{\partial u}{\partial \nu} = f(u) & \text{on } \Gamma_{4R}^0. \end{cases}$$

If f is bounded, then $u \in W^{1,p}(B_R^+)$ for all $p < \infty$.

If f is Lipschitz, then $u \in C^{1,\beta}(\overline{B_R^+})$ for all $\beta \in (0, 1)$.

If f is $C^{1,\alpha}$, then $u \in C^{2,\alpha}(\overline{B_R^+})$ and

$$\|u\|_{C^{2,\alpha}(\overline{B_R^+})} \leq C_R$$

for a constant C_R depending only on n, α, and R and on upper bounds for $\|f\|_{C^{1,\alpha}}$ and $\|u\|_{L^\infty(B_{4R}^+)}$.

(b) *If f is $C^{1,\alpha}$ and $u \in L^\infty(\mathbb{R}_+^n)$ is weak solution of (1.1), then $|\nabla u| \in L^\infty(\mathbb{R}_+^n)$ and $\|D^2 u\| \in L^\infty(\mathbb{R}_+^n)$.*

(c) *Let $R > 0$ and $\varphi \in L^\infty(B_{4R}^+) \cap H^1(B_{4R}^+)$ be a weak solution of*

$$\begin{cases} \Delta\varphi = 0 & \text{in } B_{4R}^+ \subset \mathbb{R}_+^n \\ \dfrac{\partial\varphi}{\partial\nu} + d(y)\varphi = 0 & \text{on } \Gamma_{4R}^0. \end{cases}$$

If $d \in C^\alpha(\Gamma_{4R}^0)$, then $\varphi \in C^{1,\alpha}(\overline{B_R^+})$ and

$$\|\varphi\|_{C^{1,\alpha}(\overline{B_R^+})} \leq C_R$$

for some constant C_R that depends only on n, α, and R and on upper bounds for $\|\varphi\|_{L^\infty(B_{4R}^+)}$ and $\|d\|_{C^\alpha(\Gamma_{4R}^0)}$.

PROOF: To prove statement (a), define v by (2.6) for $(x, y) \in B_{4R}^+$. We know that v is a weak solution of

$$(2.10) \qquad \begin{cases} \Delta v(x, y) = -f(u(0, y)) & \text{in } B_{4R}^+ \\ v(0, y) = 0 & \text{on } \Gamma_{4R}^0. \end{cases}$$

For every $p < \infty$, we use $W^{2,p}$ boundary regularity for problem (2.10), which follows easily from $W^{2,p}$ interior regularity after considering the odd reflection of v across $\{x = 0\}$ (see the proof of lemma 9.12 of [24]). Since $v(x, y)$ and $f(u(0, y))$ belong to $L^\infty(B_{4R}^+) \subset L^p(B_{4R}^+)$, we obtain that $v \in W^{2,p}(B_{3R}^+) \subset C^{1,\beta}(\overline{B_{3R}^+})$ (here, given $\beta \in (0, 1)$, we have chosen p large enough to have $W^{2,p}(B_{3R}^+) \subset C^{1,\beta}(\overline{B_{3R}^+})$). Hence, using $v_x = u$, we get $u \in W^{1,p}(\overline{B_{3R}^+}) \subset C^\beta(\overline{B_{3R}^+})$ and $\|u\|_{C^\beta(\overline{B_{3R}^+})} \leq C_R$, for some constant C_R depending only on n, β, R, $\|u\|_{L^\infty}$, and $\|f\|_{L^\infty}$.

Next, if f is Lipschitz, since $v \in C^{1,\beta} \subset C^\beta$ in $\overline{B_{3R}^+}$ and

$$\|f(u(0, y))\|_{C^\beta} \leq \|f'\|_{L^\infty}\|u\|_{C^\beta} \leq C_R,$$

boundary $C^{2,\beta}$ regularity for (2.10) (see [24, theorem 4.11]) leads to $v \in C^{2,\beta}(\overline{B_{2R}^+})$ and a corresponding estimate. In particular, $\|u\|_{C^{1,\beta}(\overline{B_{2R}^+})} \leq C_R$, for some constant C_R depending only on n, β, R, $\|u\|_{L^\infty}$, $\|f\|_{L^\infty}$, and $\|f'\|_{L^\infty}$.

Let now $f \in C^{1,\alpha}$ and consider the problem satisfied in the weak sense by $v_{y_i} \in C^{1,\alpha}(\overline{B_{2R}^+})$, $1 \le i \le n - 1$:

(2.11)
$$\begin{cases} \Delta v_{y_i}(x, y) = -f'(u(0, y))u_{y_i}(0, y) & \text{in } B_{2R}^+ \\ v_{y_i}(0, y) = 0 & \text{on } \Gamma_{2R}^0. \end{cases}$$

Since

$$\|f'(u(0, y))u_{y_i}(0, y)\|_{C^\alpha(\overline{B_{2R}^+})} \le \|f\|_{C^1}\|u\|_{C^{1,\alpha}} + \|f\|_{C^{1,\alpha}}\|u\|_{C^1}^{1+\alpha} \le C_R$$

and $\|v_{y_i}\|_{C^{1,\alpha}(\overline{B_{2R}^+})} \le C_R$, $C^{2,\alpha}$ regularity for (2.11) along the boundary leads to $v_{y_i} \in C^{2,\alpha}(\overline{B_R^+})$ and a corresponding estimate. Hence $u_{y_i} = v_{xy_i} \in C^{1,\alpha}$ and

$$-u_{xx} = \sum_{i=1}^{n-1} u_{y_i y_i} = \sum_{i=1}^{n-1}(v_{y_i})_{xy_i} \in C^\alpha.$$

We conclude that $u \in C^{2,\alpha}(\overline{B_R^+})$ and $\|u\|_{C^{2,\alpha}(\overline{B_R^+})} \le C_R$ for some constant C_R depending only on n, α, R, $\|u\|_{L^\infty}$, and $\|f\|_{C^{1,\alpha}}$.

Next, to establish part (b), we use the previous estimates in every half-ball $B_4^+(0, b)$ centered at a point $b \in \mathbb{R}^{n-1}$. We obtain uniform bounds (independent of b); note that here $R = 1$. Therefore ∇u and $D^2 u$ are bounded in $\overline{\mathbb{R}_+^n} \cap \{0 \le x \le 1\}$. These bounds, together with interior estimates for harmonic functions, give that ∇u and $D^2 u$ belong to $L^\infty(\overline{\mathbb{R}_+^n})$.

Finally, to prove part (c), we proceed as in the first two steps of part (a), applying $W^{2,p}$ and $C^{2,\alpha}$ estimates to problem (2.9), where now u is replaced by φ in the definition (2.6) of v, and $-h(y) = d(y)\varphi(0, y)$. $\square$

We now consider solutions with limits in one direction y_1 on $\partial\mathbb{R}_+^n$. We use the previous estimates to prove the existence of limits in the whole $\mathbb{R}_+^n$ and the convergence towards 0 as $y_1 \to \pm\infty$ for the gradient. This will be useful in several of our future arguments, for instance, to obtain energy estimates. We use the notation $B_R^+(0, y) = (0, y) + B_R^+ = (0, y) + B_R^+(0, 0)$, where $y \in \mathbb{R}^{n-1}$.

LEMMA 2.4 *Let u be a bounded solution of* (1.1) *such that*

(2.12)
$$\lim_{y_1 \to \pm\infty} u(0, y) = L^\pm \quad \text{for every } (y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-2}$$

for some constants $L^\pm$. Then

$$f(L^-) = f(L^+) = 0.$$

Moreover, for every fixed $R > 0$ and $(y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-2}$, we have

(2.13)
$$\|u - L^\pm\|_{L^\infty(B_R^+(0,y))} \to 0 \quad \text{as } y_1 \to \pm\infty$$

and

(2.14)
$$\|\nabla u\|_{L^\infty(B_R^+(0,y))} \to 0 \quad \text{as } y_1 \to \pm\infty.$$

PROOF: Let η be a nonnegative C^∞ function with compact support in $B_1^+ \cup \Gamma_1^0$ and with $\int_{\Gamma_1^0} \eta > 0$. For $R > 0$, let $\eta_R(x, y) = \eta(x/R, y/R)$. For $t \in \mathbb{R}$, consider

$$u^t(x, y) = u(x, y_1 + t, y_2, \ldots, y_n),$$

which is also a solution of (1.1). We multiply the problem for u^t by η_R and integrate by parts in B_R^+. We get

$$0 = \int_{B_R^+} \Delta u^t \, \eta_R = \int_{\Gamma_R^0} f(u^t) \eta_R - \int_{B_R^+} \nabla u^t \nabla \eta_R$$

$$= \int_{\Gamma_R^0} f(u^t) \eta_R + \int_{\Gamma_R^0} u^t \partial_x \eta_R + \int_{B_R^+} u^t \Delta \eta_R.$$

Note that the last two integrals are bounded by CR^{n-2} uniformly in t. On the other hand, by hypothesis (2.12) and the dominated convergence theorem, the first integral $\int_{\Gamma_R^0} f(u^t) \eta_R$ converges as $t \to +\infty$ to $f(L^+) R^{n-1} \int_{\Gamma_1^0} \eta$. Hence $|f(L^+)| \leq C/R$ for every R. Letting $R \to +\infty$, we conclude $f(L^+) = 0$. Similarly, one shows that $f(L^-) = 0$.

To prove (2.13) and (2.14), which is equivalent to proving that

$$\|u^t - L^\pm\|_{L^\infty(B_R^+(0,y))} + \|\nabla u^t\|_{L^\infty(B_R^+(0,y))} \to 0$$

as $t \to \pm\infty$ for every fixed R and $y \in \mathbb{R}^{n-1}$, we use a simple compactness argument. Arguing by contradiction, assume that there exist $R > 0$, $y \in \mathbb{R}^{n-1}$, $\varepsilon > 0$, and a sequence $t_m \to +\infty$ such that

$$(2.15) \qquad \|u^{t_m} - L^+\|_{L^\infty(B_R^+(0,y))} + \|\nabla u^{t_m}\|_{L^\infty(B_R^+(0,y))} \geq \varepsilon$$

for every m. Since u^{t_m} are all solutions of (1.1) uniformly bounded in all the half-space independently of t_m, Lemma 2.3 gives $C^{2,\alpha}(\overline{B_S^+})$ estimates for u^{t_m} uniform in m for every $S > 0$. Hence, for a subsequence that we still denote by (u^{t_m}), we have that u^{t_m} converges as $m \to \infty$ in $C^2_{\text{loc}}(\overline{\mathbb{R}_+^n})$ to a bounded harmonic function $v \in C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}_+^n})$. By hypothesis (2.12), we have that $v \equiv L^+$ on $\partial \mathbb{R}_+^n$. By the maximum principle—see (2.26) in Section 2.4—we deduce that $v \equiv L^+$ in all of $\mathbb{R}_+^n$. This contradicts (2.15), by the C^1 convergence in compact sets of u^{t_m} towards $v \equiv L^+$.

As a final remark, note that the previous compactness argument gives an alternative proof of the fact that $f(L^+) = 0$, since the limit $v \equiv L^+$ is a solution of (1.1). $\qquad \square$

2.3 A Harnack Inequality and a Liouville Theorem

The following Harnack inequality for linear Neumann problems will be useful in the study of stable solutions of (1.1).

LEMMA 2.5 *Let $\varphi \in C^1(\overline{B_{4R}^+}) \cap C^2(B_{4R}^+)$ be a nonnegative solution of*

$$\begin{cases} \Delta\varphi = 0 & \text{in } B_{4R}^+ \subset \mathbb{R}_+^n \\ \dfrac{\partial\varphi}{\partial\nu} + d(y)\varphi = 0 & \text{on } \Gamma_{4R}^0 \end{cases}$$

where d is a bounded function in Γ_{4R}^0. Then

(2.16)
$$\sup_{B_R^+} \varphi \le C_R \inf_{B_R^+} \varphi$$

for some constant C_R depending only on n and $R\|d\|_{L^\infty(\Gamma_{4R}^0)}$.

We give a short proof of this result using a reflection argument and a strong tool: the De Giorgi–Nash–Moser Harnack inequality for elliptic equations with bounded, measurable coefficients.

PROOF OF LEMMA 2.5: By scaling the variables (x, y), we may assume that $R = 1$. Let

$$A = \|d\|_{L^\infty(\Gamma_4^0)}.$$

Throughout the rest of the proof, C will denote a positive constant depending only on n and A.

Consider the function

$$\varphi^a(x, y) = e^{ax}\varphi(x, y)$$

for $(x, y) \in B_4^+ \cup \Gamma_4^0$, where a is a given real number. A direct calculation gives that φ^a is a nonnegative solution of

$$\begin{cases} -\Delta\varphi^a + 2a\varphi_x^a - a^2\varphi^a = 0 & \text{in } B_4^+ \\ -\varphi_x^a = -(a + d(y))\varphi^a & \text{on } \Gamma_4^0. \end{cases}$$

We consider the even extension of φ^a across Γ_4^0, defined by

$$\tilde{\varphi}^a(x, y) = \varphi^a(-x, y) \quad \text{for } (x, y) \in B_4, \ x \le 0.$$

Note that $\tilde{\varphi}^a$ is a $W^{1,\infty}$ function (that is, a Lipschitz function) in B_4. Taking $a = A = \|d\|_{L^\infty}$, we have that $\varphi_x^A \ge 0$ in Γ_4^0, and hence $\tilde{\varphi}^A$ satisfies

(2.17)
$$-\Delta\tilde{\varphi}^A + 2As(x)\tilde{\varphi}_x^A - A^2\tilde{\varphi}^A \le 0 \quad \text{in } B_4$$

in the weak sense. Here $s(x)$ is the bounded discontinuous function

$$s(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Recall that the weak sense in which $\tilde{\varphi}^A$ is a subsolution of (2.17) is that

$$\int_{B_4} \nabla\tilde{\varphi}^A \nabla\xi + 2As(x)\tilde{\varphi}_x^A\xi - A^2\tilde{\varphi}^A\xi \le 0$$

for all C^1 nonnegative function ξ with compact support in B_4. The previous inequality holds since, for every such function ξ and every $a \in \mathbb{R}$, we have

$$\int_{B_4} \nabla \tilde{\varphi}^a \nabla \xi + 2as(x) \tilde{\varphi}_x^a \xi - a^2 \tilde{\varphi}^a \xi = \int_{\Gamma_4^0} -2\varphi_x^a \xi,$$

and the last integral is nonpositive when $a = A$, since $\varphi_x^A \geq 0$ in Γ_4^0.

Next, taking $a = -A$ we obtain $\varphi_x^{-A} \leq 0$ in Γ_4^0, and arguing as before, we deduce that $\tilde{\varphi}^{-A} \geq 0$ satisfies

$$(2.18) \qquad -\Delta \tilde{\varphi}^{-A} - 2As(x) \tilde{\varphi}_x^{-A} - A^2 \tilde{\varphi}^{-A} \geq 0 \quad \text{in } B_4$$

in the weak sense.

Note that the two elliptic operators in (2.17) and (2.18) have bounded measurable coefficients, and therefore we may use the De Giorgi–Nash–Moser theory. The L^∞ norms of the coefficients are controlled by the constant $A^2 + 2A$, and hence all constants from now on will depend only on n and A.

We choose an exponent $p \in (1, n/(n-2))$. We first apply the local maximum principle (see [24, theorem 8.17]) to $\tilde{\varphi}^A$ in B_4, a subsolution of (2.17). We obtain

$$\sup_{B_1} \tilde{\varphi}^A \leq C \|\tilde{\varphi}^A\|_{L^p(B_2)}.$$

Next, we apply the weak Harnack inequality (see [24, theorem 8.18]) to $\tilde{\varphi}^{-A}$ in B_4, a nonnegative supersolution of (2.18). We get

$$\|\tilde{\varphi}^{-A}\|_{L^p(B_2)} \leq C \inf_{B_1} \tilde{\varphi}^{-A}.$$

Since $\tilde{\varphi}^A \leq e^{4A} \tilde{\varphi}^{-A}$ in B_2, we can put together the last two displayed inequalities. Using also that $\tilde{\varphi}^{-A} \leq \varphi \leq \tilde{\varphi}^A$ in B_1^+, we conclude that (2.16) holds. $\qquad \square$

The following Liouville theorem will be a key tool to establish the monotonicity of stable solutions in $\mathbb{R}_+^2$ and their two-dimensional symmetry in $\mathbb{R}_+^3$. It is an analogue of a Liouville theorem from [4], used in that paper to prove a conjecture of De Giorgi for reactions in the interior when $n = 3$.

LEMMA 2.6 *Let $\varphi \in L^\infty_{\text{loc}}(\overline{\mathbb{R}_+^n})$ be a positive function, not necessarily bounded in all of $\mathbb{R}_+^n$. Suppose that $\sigma \in H^1_{\text{loc}}(\overline{\mathbb{R}_+^n})$ satisfies*

$$(2.19) \qquad \begin{cases} -\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \leq 0 & \text{in } \mathbb{R}_+^n \\ \sigma \dfrac{\partial \sigma}{\partial \nu} \leq 0 & \text{on } \partial \mathbb{R}_+^n \end{cases}$$

in the weak sense. Assume that, for every $R > 1$,

$$(2.20) \qquad \int_{B_R^+} (\varphi \sigma)^2 \leq CR^2$$

for some constant C independent of R. Then σ is constant.

PROOF: Let ζ be a C^∞ function on $\mathbb{R}^+$ such that $0 \leq \zeta \leq 1$ and

$$\zeta = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t \geq 2. \end{cases}$$

For $R > 1$, let $\zeta_R(x, y) = \zeta(|(x, y)|/R)$.

Multiplying (2.19) by ζ_R^2 and integrating by parts in $\mathbb{R}_+^n$, we obtain

$$\int_{\mathbb{R}_+^n} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \leq -2 \int_{\mathbb{R}_+^n} \zeta_R \varphi^2 \sigma \nabla \zeta_R \nabla \sigma$$

$$\leq 2 \left[\int_{\mathbb{R}_+^n \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \right]^{1/2} \left[\int_{\mathbb{R}_+^n} \varphi^2 \sigma^2 |\nabla \zeta_R|^2 \right]^{1/2}$$

$$\leq C \left[\int_{\mathbb{R}_+^n \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \right]^{1/2} \left[\frac{1}{R^2} \int_{B_{2R}^+} (\varphi \sigma)^2 \right]^{1/2},$$

for some constant C independent of R. Using hypothesis (2.20), we infer that

$$(2.21) \qquad \int_{\mathbb{R}_+^n} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \leq C \left[\int_{\mathbb{R}_+^n \cap \{R < r < 2R\}} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \right]^{1/2},$$

again with C independent of R. This implies that $\int_{\mathbb{R}_+^n} \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \leq C$ and, letting $R \to \infty$, we deduce $\int_{\mathbb{R}_+^n} \varphi^2 |\nabla \sigma|^2 \leq C$. It follows that the right-hand side of (2.21) tends to 0 as $R \to \infty$, and therefore $\int_{\mathbb{R}_+^n} \varphi^2 |\nabla \sigma|^2 = 0$ by (2.21). We conclude that σ is constant. □

2.4 Maximum Principles

Here we present several maximum principles related to problem (1.1). We assume that v is a bounded function in $\mathbb{R}_+^n$ satisfying

$$(2.22) \qquad \begin{cases} -\Delta v \geq 0 & \text{in } \mathbb{R}_+^n \\ \dfrac{\partial v}{\partial \nu} + d(y)v \geq 0 & \text{on } \partial \mathbb{R}_+^n. \end{cases}$$

For simplicity, we suppose that v is C^2 in $\mathbb{R}_+^n$ and C^1 up to the boundary $\partial \mathbb{R}_+^n$, and that d is a bounded function.

We use a standard procedure in maximum principles and consider the new function

$$w = \frac{v}{\psi},$$

where ψ is a certain harmonic function in $\mathbb{R}^n_+$, continuous up to the boundary $\partial\mathbb{R}^n_+$, and with $\psi > 0$ in $\overline{\mathbb{R}^n_+}$. It is simple to verify that w satisfies

(2.23)
$$\begin{cases} -\Delta w - 2\dfrac{\nabla\psi}{\psi} \cdot \nabla w \geq 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial w}{\partial \nu} + \left(d - \dfrac{\psi_x}{\psi}\right) w \geq 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

The following classical choice of ψ will be useful in several arguments. For a given constant $a > 1$, let

$$\rho = \sqrt{(x+a)^2 + |y|^2}, \qquad r = \sqrt{x^2 + |y|^2},$$

$S = \{(x,y) \in \mathbb{R}^n : \rho = 1\}$ be a unit sphere, $\mu > 0$, and $\varphi((x+a)/\rho, y/\rho) > 0$ be the first eigenvalue and eigenfunction of the Laplace-Beltrami operator, with Dirichlet boundary conditions, on the subset

$$\left\{(x,y) \in \mathbb{R}^n : \rho = 1, x + a > -\frac{1}{2}\right\}$$

of S. Choosing $\alpha > 0$ to be the solution of $\mu = \alpha(n - 2 + \alpha)$, the function

(2.24)
$$\psi(x,y) = \rho^\alpha \varphi\left(\frac{x+a}{\rho}, \frac{y}{\rho}\right)$$

is harmonic and positive in $\{(x+a)/\rho > -\frac{1}{2}\}$. In particular, ψ is a positive harmonic function in $\overline{\mathbb{R}^n_+}$. Note that if $x \geq 0$, then $(x+a)/\rho \geq 0$. In addition, $\varphi \geq c > 0$ in the hemisphere $\{(x+a)/\rho \geq 0\}$ for a positive constant c independent of a. Hence, we have that

(2.25)
$$\psi \geq c\rho^\alpha \geq cr^\alpha \quad \text{in } \mathbb{R}^n_+$$

for a positive constant c independent of a.

We can now recall and prove a simple Phragmen-Lindelöf-type result. If v is a bounded superharmonic function in $\mathbb{R}^n_+$ that is continuous up to the boundary $\partial\mathbb{R}^n_+$, then

(2.26)
$$\inf_{\mathbb{R}^n_+} v = \inf_{\partial\mathbb{R}^n_+} v.$$

Indeed, subtracting a constant from v, we may assume that v is nonnegative on $\partial\mathbb{R}^n_+$, and we need to show that $v \geq 0$ in $\mathbb{R}^n_+$. For this, consider $w = v/\psi$, where ψ is given by (2.24) with, for instance, $a = 2$. Note that w has the same sign as v. In addition, by (2.25), $w(x,y) \to 0$ as $r = |(x,y)| \to \infty$, $(x,y) \in \overline{\mathbb{R}^n_+}$. Hence, if w were negative at some point in $\mathbb{R}^n_+$, its negative minimum would be achieved at some point of $\mathbb{R}^n_+$. This contradicts the first inequality in (2.23) satisfied by w.

If u is a layer solution of (1.1), we can apply the previous maximum principle to u_{y_1}, which is bounded, harmonic, and continuous up to the boundary by Lemma 2.3. We conclude that every layer solution satisfies

(2.27)
$$u_{y_1} > 0 \quad \text{in } \overline{\mathbb{R}^n_+}.$$

The following maximum principle is extremely simple:

LEMMA 2.7 *Let $n = 2$, and v be a bounded function in $\mathbb{R}^2_+$ satisfying (2.22) and*

(2.28) $v(0, y) \to 0 \quad as \ |y| \to \infty.$

Assume that there exists a nonempty set $H \subset \mathbb{R}$ such that $v(0, y) > 0$ for $y \in H$ and $d(y) \geq 0$ for $y \notin H$. Then, $v > 0$ in $\overline{\mathbb{R}^2_+}$.

PROOF: Suppose that $\inf_{\mathbb{R}^2_+} v = \inf_{y \in \mathbb{R}} v(0, y) < 0$. Then, by assumption (2.28), this infimum must be achieved at some point $(0, y_0)$. We necessarily have $y_0 \notin H$. Since v is not constant (v is positive in H and negative in $(0, y_0)$), Hopf's boundary lemma gives $-v_x(0, y_0) < 0$, a contradiction with the boundary inequality since $d(y_0) \geq 0$ and $v(0, y_0) < 0$.

Hence, $v \geq 0$ in $\mathbb{R}^2_+$. Now, since $v \not\equiv 0$, Hopf's boundary lemma gives $v > 0$ in $\overline{\mathbb{R}^2_+}$.   □

Next, we present two maximum principles where we assume that the coefficient $d(y)$ is greater than a positive constant in all or part of the boundary $\partial \mathbb{R}^n_+$.

LEMMA 2.8 *Let $n = 2$, and v be a bounded function in $\mathbb{R}^2_+$ satisfying (2.22). Assume that for some $\varepsilon > 0$, $d(y) \geq \varepsilon$ for all $y \in \mathbb{R}$. Then $v > 0$ in $\overline{\mathbb{R}^2_+}$ unless $v \equiv 0$.*

PROOF: For $a > 1$ let us define $\rho = \sqrt{(x + a)^2 + y^2}$ and the auxiliary function $w = v / \log \rho$. Then, w has the same sign as v, tends to 0 at infinity, and satisfies

$$\begin{cases} -\Delta w - 2 \dfrac{\nabla \rho}{\rho \log \rho} \cdot \nabla w \geq 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial w}{\partial \nu} + \left(d - \dfrac{\rho_x}{\rho \log \rho} \right) w \geq 0 & \text{on } \partial \mathbb{R}^2_+ \end{cases}$$

by (2.23). If a is chosen large enough, depending on ε, then on $\partial \mathbb{R}^2_+$ we have that $d - \rho_x / (\rho \log \rho) \geq 0$. Hence, the result follows by the application of Hopf's maximum principle to a possible nonpositive global minimum of w.   □

Our last maximum principle is valid in all dimensions $n \geq 2$.

LEMMA 2.9 *Let v be a bounded function in $\mathbb{R}^n_+$ satisfying (2.22). Assume that there exist a set $H \subset \mathbb{R}^{n-1}$ (possibly empty) and a constant $\varepsilon > 0$ such that*

$$v(0, y) > 0 \ for \ y \in H \quad and \quad d(y) \geq \varepsilon \ for \ y \notin H.$$

Then $v > 0$ in $\overline{\mathbb{R}^n_+}$ unless $v \equiv 0$.

PROOF: For some $a > 1$ to be chosen later, consider the function ψ defined in (2.24). Then, $w = v / \psi$ has the same sign as v and satisfies (2.23). Note that, by (2.25), $w(x, y) \to 0$ as $r = |(x, y)| \to \infty$, $(x, y) \in \overline{\mathbb{R}^n_+}$.

Moreover,

$$\left| \frac{\psi_x}{\psi}(x, y) \right| \leq \frac{|\psi_x(x, y)|}{c\rho^\alpha} \leq \frac{C}{\rho} \leq \frac{C}{a} \quad \text{if } x \geq 0$$

for some constant C independent of a. Recall that, by hypothesis, the coefficient $d(y)$ satisfies $d \geq \varepsilon$ in $\partial \mathbb{R}^n_+ \setminus H$. Hence, taking a large enough, we have that $|\psi_x/\psi| \leq \varepsilon$ in $\overline{\mathbb{R}^n_+}$ and, as a consequence,

$$(2.29) \qquad -w_x + \tilde{d}(y)w \geq 0 \quad \text{and} \quad \tilde{d}(y) \geq 0 \quad \text{on } \partial \mathbb{R}^n_+ \setminus H,$$

where $\tilde{d} = d - \psi_x/\psi$.

Finally, if $w \leq 0$ somewhere in $\mathbb{R}^n_+$, then w would achieve a nonpositive global minimum at some point $(0, y_0)$ with $y_0 \notin H$. At this point, Hopf's boundary lemma gives a contradiction to (2.29). $\square$

2.5 Minimizers of the Dirichlet-Neumann Problem in Bounded Domains

Let $\Omega \subset \mathbb{R}^n_+$ be a bounded domain. We define the following subsets of $\partial \Omega$:

$$(2.30) \qquad \partial^0 \Omega = \{(0, y) \in \partial \mathbb{R}^n_+ : B^+_\varepsilon(0, y) \subset \Omega \text{ for some } \varepsilon > 0\}$$

and

$$(2.31) \qquad \partial^+ \Omega = \overline{\partial \Omega \cap \mathbb{R}^n_+}.$$

In future sections we will use the existence of an absolute minimizer of the energy in some bounded domains $\Omega \subset \mathbb{R}^n_+$ with given Dirichlet boundary conditions on $\partial^+ \Omega$. We will need this existence result both for Ω a half-ball in $\mathbb{R}^n_+$ and for Ω a rectangle in $\mathbb{R}^2_+$. For the sake of completeness, in this subsection we show the existence of such minimizer. We also study its global regularity (for certain Dirichlet boundary values) in the case of half-balls and rectangles.

Let u be a $C^1(\overline{\Omega})$ function with $|u| \leq 1$. We consider the energy functional

$$(2.32) \qquad E_\Omega(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + \int_{\partial^0 \Omega} G(v)$$

in the class

$$\mathcal{C}_u(\Omega) = \{v \in H^1(\Omega) : -1 \leq v \leq 1 \text{ a.e. in } \Omega \text{ and } v \equiv u \text{ on } \partial^+ \Omega\},$$

which clearly contains u and hence is nonempty. This class is a closed convex subset of the affine space

$$(2.33) \qquad H_u(\Omega) = \{v \in H^1(\Omega) : v \equiv u \text{ on } \partial^+ \Omega\},$$

where the last condition should be understood to mean that $v - u$ vanishes on $\partial^+ \Omega$ in the weak sense. That is, $H_u(\Omega)$ is the set of functions $v \in H^1(\Omega)$ such that $v - u$ belongs to the closure in $H^1(\Omega)$ of $C^1(\overline{\Omega})$ functions with compact support in $\Omega \cup \partial^0 \Omega$.

LEMMA 2.10 *Let* $\Omega \subset \mathbb{R}^n_+$ *be a bounded Lipschitz domain. Let u be a $C^1(\overline{\Omega})$ function with $|u| \leq 1$. Assume that*

$$(2.34) \qquad\qquad f(1) \leq 0 \leq f(-1).$$

Then the functional E_Ω admits an absolute minimizer w in $C_u(\Omega)$. In particular, w is a weak solution of

$$(2.35) \qquad \begin{cases} \Delta w = 0 & \text{in } \Omega \\ \dfrac{\partial w}{\partial \nu} = f(w) & \text{on } \partial^0 \Omega \\ w = u & \text{on } \partial^+ \Omega. \end{cases}$$

Moreover, w is a stable solution of (2.35) *in the sense that*

$$(2.36) \qquad \int_\Omega |\nabla \xi|^2 - \int_{\partial^0 \Omega} f'(w)\xi^2 \geq 0$$

for every $\xi \in H^1(\Omega)$ such that $\xi \equiv 0$ on $\partial^+\Omega$ in the weak sense.

Note that the solution w satisfies the Neumann boundary condition in (2.35) in the classical sense. This follows from the definition of $\partial^0 \Omega$ and from the regularity result of Lemma 2.3 applied in small half-balls centered at points on $\partial^0 \Omega$ (see more details at the end of the following proof).

Hypothesis (2.34) simply states that -1 and 1 are a subsolution and a supersolution, respectively, of (2.35).

PROOF OF LEMMA 2.10: It is useful to consider the following continuous extension $\tilde{f}$ of f outside $[-1, 1]$:

$$\tilde{f}(t) = \begin{cases} f(-1) & \text{if } s \leq -1 \\ f(s) & \text{if } -1 \leq s \leq 1 \\ f(1) & \text{if } 1 \leq s. \end{cases}$$

Let

$$\tilde{G}(s) = - \int_0^s \tilde{f},$$

and consider the new functional

$$\tilde{E}_\Omega(v) = \int_\Omega \frac{1}{2} |\nabla v|^2 + \int_{\partial^0 \Omega} \tilde{G}(v)$$

in the affine space $H_u(\Omega)$ defined by (2.33).

Note that $\tilde{G} = G$ in $[-1, 1]$ up to an additive constant. Therefore, any minimizer w of $\tilde{E}_\Omega$ in $H_u(\Omega)$ such that $-1 \leq w \leq 1$ is also a minimizer of E_Ω in $C_u(\Omega)$.

To show that $\tilde{E}_\Omega$ admits a minimizer in $H_u(\Omega)$, we first recall that the inclusion $H_u(\Omega) \Subset L^2(\partial^0 \Omega)$ is compact. Indeed, let $v \in H_u(\Omega)$. Since $v - u \equiv 0$ on $\partial^+\Omega$,

we can extend $v - u$ to be identically 0 in $\mathbb{R}^n_+ \setminus \Omega$, and we have $v - u \in H^1(\mathbb{R}^n_+)$. We have

$$\int_{\partial^0\Omega} |v(0, y) - u(0, y)|^2 \, dy = -\int_{\mathbb{R}^n_+} \partial_x(|v - u|^2) = -2\int_{\mathbb{R}^n_+} (v - u)\partial_x(v - u)$$

$$\leq C\|v - u\|_{L^2(\Omega)}\|v - u\|_{H^1(\Omega)}.$$

Now, the compactness of the inclusion

$$H_u(\Omega) \Subset L^2(\partial^0\Omega)$$

follows from the well-known fact that, since Ω is Lipschitz, $H^1(\Omega) \Subset L^2(\Omega)$ is compact.

Now, since $H_u(\Omega) \subset L^2(\partial^0\Omega)$ and $\tilde{G}$ has linear growth at infinity, it follows that $\tilde{E}_\Omega$ is well-defined, bounded below, and coercive in $H_u(\Omega)$. Hence, using the compactness of the inclusion $H_u(\Omega) \Subset L^2(\partial^0\Omega)$, taking a minimizing sequence in $H_u(\Omega)$ and a subsequence convergent in $L^2(\partial^0\Omega)$, we conclude that $\tilde{E}_\Omega$ admits an absolute minimizer w in $H_u(\Omega)$.

Since $\tilde{f}$ is a continuous function, $\tilde{E}$ is a C^1 functional in $H_u(\Omega)$. Making first- and second-order variations of $\tilde{E}$ at the minimum w, we obtain that w is a weak solution of (2.35) that satisfies (2.36), with f and f' replaced by $\tilde{f}$ and $\tilde{f}'$, respectively, in both (2.35) and (2.36).

Therefore, it only remains to show that $-1 \leq w \leq 1$ a.e. in Ω. To do this, we simply use that -1 and 1 are natural barriers for (2.35). That is, we use that -1 and 1 are, respectively, a subsolution and a supersolution of (2.35) due to hypothesis (2.34). We proceed as follows: We use that the first variation of $\tilde{E}_\Omega$ at w in the direction $(w - 1)^+$ (the positive part of $w - 1$) is 0. Since $|w| = |u| \leq 1$ on $\partial^+\Omega$ and hence $(w - 1)^+$ vanishes on $\partial^+\Omega$, we have that $w + \varepsilon(w - 1)^+ \in H_u(\Omega)$ for every ε. We deduce

$$0 = \int_\Omega \nabla w \nabla(w - 1)^+ - \int_{\partial^0\Omega} \tilde{f}(w)(w - 1)^+$$

$$= \int_{\Omega \cap \{w \geq 1\}} |\nabla(w - 1)^+|^2 - \int_{\partial^0\Omega \cap \{w \geq 1\}} f(1)(w - 1)^+ \geq \int_\Omega |\nabla(w - 1)^+|^2,$$

where we have used that $\tilde{f}(s) = f(1)$ for $s \geq 1$ and that $f(1) \leq 0$ by hypothesis. We conclude that $(w - 1)^+$ is constant and hence identically 0. Therefore, $w \leq 1$ a.e.; the inequality $w \geq -1$ can be proven in the same way by using $f(-1) \geq 0$.

Finally, we point out that the two first equalities in (2.35) are satisfied in the classical sense. Indeed, we first consider the first variation of $\tilde{E}$ at w with a perturbation compactly supported in Ω. We obtain that w is harmonic in Ω. Next, given a point on $\partial^0\Omega$, we consider a small half-ball B_ε^+ centered at this point and contained in Ω. By the regularity result of Lemma 2.3, we know that w is $C^{2,\alpha}$ in $\overline{B_\varepsilon^+}$. We now consider perturbations vanishing on Γ_ε^+. Integrating by parts and

using that w is harmonic, we find that $-w_x - f(w) = -w_x - \tilde{f}(w) \equiv 0$ on Γ_ε^0 in the classical sense. $\square$

In future sections we will need to know further regularity for the weak solution w of (2.35) in the case when Ω is a half-ball in $\mathbb{R}_+^n$ or a rectangle in $\mathbb{R}_+^2$ and for certain Dirichlet boundary values u. The main point is to have regularity up to the corners $\partial^+\Omega \cap \partial\mathbb{R}_+^n$, where the Dirichlet condition changes to a Neumann condition. Even if more general results could be established, to simplify the proof we state the regularity result in the particular domains just mentioned.

LEMMA 2.11 *Let $f \in C^{1,\alpha}$ for some $\alpha \in (0,1)$, and $w \in H^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of* (2.35).

(a) *Assume that $\Omega = B_R^+ \subset \mathbb{R}_+^n$ and that u is a bounded solution of problem* (1.1) *in all of $\mathbb{R}_+^n$. Then $w \in C^{2,\alpha}(\overline{\Omega})$.*

(b) *Assume that $\Omega = (0,a) \times (b^-, b^+) \subset \mathbb{R}_+^2$, $f(-1) = f(1) = 0$, and that $u = u(y)$ is a $C^2([b^-, b^+])$ function with $u(b^-) = -1$ and $u(b^+) = 1$. Then one has that $w \in W^{2,p}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for all $p < \infty$ and all $\beta \in (0,1)$. In addition, for every $\varepsilon > 0$, one has $w \in C^{2,\alpha}([0, a-\varepsilon] \times [b^-, b^+])$.*

In case (b), if we assume in addition $u \in C^{2,\alpha}([b^-, b^+])$ and the compatibility conditions $u_{yy}(b^-) = u_{yy}(b^+) = 0$, we then have $w \in C^{2,\alpha}(\overline{\Omega})$.

PROOF OF LEMMA 2.11:

(a) The function $v = w - u$ is a weak solution of

$$\begin{cases} \Delta v = 0 & \text{in } B_R^+ \\ \dfrac{\partial v}{\partial \nu} = h(y) & \text{on } \Gamma_R^0 \\ v = 0 & \text{on } \Gamma_R^+, \end{cases}$$

where $h(y) = f(w(0,y)) - f(u(0,y))$. Consider the odd reflection of v through Γ_R^+ given by minus its Kelvin transform

$$v(x,y) = -\frac{r^{2-n}}{R^{2-n}}\, v\left(R^2 \frac{(x,y)}{r^2}\right) \quad \text{for } (x,y) \in \overline{\mathbb{R}_+^n} \setminus \overline{B_R^+},$$

where $r = |(x,y)|$. Since u is $C^{2,\alpha}$ in $\overline{\mathbb{R}_+^n}$ by Lemma 2.3, it suffices to show that v is $C^{2,\alpha}$ in every half-ball of $\overline{\mathbb{R}_+^n}$.

Away from any neighborhood of $\{x = 0, |y| = R\}$ in $\overline{B_R^+}$, we know that w, and hence v, are $C^{2,\alpha}$. This follows from Lemma 2.3 applied at points on Γ_R^0, and from boundary regularity for harmonic functions with a smooth Dirichlet boundary condition on $\{x > 0, |(x,y)| = R\}$. In addition, v is harmonic in $\overline{\mathbb{R}_+^n} \setminus \overline{B_R^+}$, and the normal derivatives of v on Γ_R^+ from both sides coincide. Hence, away from any

neighborhood of $\{x = 0, |y| = R\}$ in $\overline{\mathbb{R}^n_+}$, the extended function v satisfies

(2.37)
$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial v}{\partial \nu} = \tilde{h}(y) & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

in the classical sense, where

(2.38) $$\tilde{h}(y) = \begin{cases} h(y) = f(w(0, y)) - f(u(0, y)) & \text{if } |y| < R \\ -R^n |y|^{-n} h(R^2 y/|y|^2) & \text{if } |y| > R. \end{cases}$$

We deduce that

(2.39)
$$\int_{\mathbb{R}^n_+} \nabla v \nabla \xi - \int_{\partial \mathbb{R}^n_+} \tilde{h}(y) \xi = 0$$

for all $\xi \in C^1$ with compact support in $\overline{\mathbb{R}^n_+}$ and vanishing in a neighborhood in $\overline{\mathbb{R}^n_+}$ of $\{x = 0, |y| = R\}$.

We claim that (2.39) also holds for every $\xi \in C^1$ with compact support in $\overline{\mathbb{R}^n_+}$, and hence v is a weak solution of (2.37). This is easily seen by writing (2.39) with ξ replaced by $\tilde{\xi} = \xi(x, y)\{1 - \eta((|y| - R)/\varepsilon)\eta(x/\varepsilon)\}$, where η is a smooth function with $\eta(s) \equiv 1$ for $|s| \leq \frac{1}{2}$ and $\eta(s) \equiv 0$ for $|s| \geq 1$. Since $|\nabla \tilde{\xi}| \leq C/\varepsilon$ on the set $\{\tilde{\xi} \neq \xi\}$, which has measure not larger than $C\varepsilon^2$, letting $\varepsilon \to 0$ we deduce the claim.

Now, we proceed as in the method of the proof of Lemma 2.3. We consider the auxiliary function $V(x, y) = \int_0^x v(t, y)dt$ and the Dirichlet problem that it solves. First, since $\tilde{h}$ is a bounded function, we obtain that v is $W^{1,p}$ for all $p < \infty$ and hence is C^α in every half-ball of $\mathbb{R}^n_+$. We deduce that $w \in C^\alpha(\overline{B_R^+})$ and, in particular, $w \equiv u$ in the classical sense on $\{x = 0, |y| = R\}$. It follows that h is C^α on $\{x = 0, |y| \leq R\}$ and that $h \equiv 0$ on $\{x = 0, |y| = R\}$. Using (2.38), we easily deduce that $\tilde{h}$ is a C^α function on $\partial \mathbb{R}^n_+$. Now Schauder estimates give that v, and hence w, are $C^{1,\alpha}$ in every half-ball of $\mathbb{R}^n_+$.

As a consequence, we obtain that $h \in C^{1,\alpha}(\overline{B_R^+})$. Using the form (2.38) of the extension $\tilde{h}$, we deduce that $\tilde{h}$ is $C^{1,\alpha}$ in every half-ball of $\mathbb{R}^n_+$. Hence, considering the Dirichlet problem satisfied by each tangential derivative V_{y_i}, as in the proof of Lemma 2.3, we conclude that $v \in C^{2,\alpha}$ in every half-ball of $\mathbb{R}^n_+$.

(b) We only need to study the regularity of w in a neighborhood of each of the four corners of the rectangle. It suffices to consider the corners on $\{y = b^+\}$; the other two are treated in the same way. Recall that w is smooth and identically 1 on $\{0 < x < a, y = b^+\}$. We consider the "odd" reflection of w across this segment, given by $w(x, y) = 2 - w(x, 2b^+ - y)$ for $b^+ < y < 2b^+ - b^-$. As in case (a), it

is easy to check that we obtain a weak solution of

$$\begin{cases} \Delta w = 0 & \text{in } (0, a) \times (b^-, 2b^+ - b^-) \\ \dfrac{\partial w}{\partial v} = \tilde{h}(y) & \text{on } \{x = 0, b^- < y < 2b^+ - b^-\} \\ w = \tilde{u}(y) & \text{on } \{x = a, b^- < y < 2b^+ - b^-\}, \end{cases}$$

where

$$\tilde{h}(y) = \begin{cases} f(w(0, y)) & \text{if } b^- < y < b^+ \\ -f(w(0, 2b^+ - y)) & \text{if } b^+ < y < 2b^+ - b^-, \end{cases}$$

and $\tilde{u}$ denotes the "odd" reflection of u through $\{y = b^+\}$.

To obtain regularity at (a, b^+), we simply consider the function $w - \tilde{u}$, which vanishes on $\{x = a, b^- < y < 2b^+ - b^-\}$ and has bounded Laplacian on the left of this segment, since the reflected $\tilde{u}$ is a $W^{2,\infty}$ function. Boundary regularity for the Dirichlet problem gives that $w - \tilde{u}$, and hence w, are $W^{2,p}$ for all $p < \infty$, and in particular $C^{1,\beta}$ for all $\beta \in (0, 1)$ in a neighborhood of (a, b^+) in $\{x \le a\}$. If, in addition, $u \in C^{2,\alpha}([b^-, b^+])$ and $u_{yy}(b^-) = u_{yy}(b^+) = 0$, then the reflected function $\tilde{u}$ is $C^{2,\alpha}$, and hence its Laplacian is C^α. We obtain in this case that w is $C^{2,\alpha}$ in a neighborhood of (a, b^+) in $\{x \le a\}$.

To deal with the left corner $(0, b^+)$, we proceed as in the proof of Lemma 2.3. We consider the function $W(x, y) = \int_0^x w(t, y) dt$ and the Dirichlet problem that it solves. Since $\tilde{h}$ is bounded, we first obtain that w is C^α up to $\{x = 0, b^- < y < 2b^+ - b^-\}$. This leads to $w(0, b^+) = 1$ in the classical sense, and using the hypothesis that $f(1) = 0$, we deduce that $\tilde{h}$ is a C^α function. Schauder estimates now give that w is $C^{1,\alpha}$ up to $\{x = 0, b^- < y < 2b^+ - b^-\}$.

As a consequence, we obtain that $f(w(0, y))$ is $C^{1,\alpha}([b^-, b^+])$. Hence, its extension $\tilde{h}$ is $C^{1,\alpha}(b^-, 2b^+ - b^-)$. Hence, considering the Dirichlet problem satisfied by each tangential derivative W_{y_i} as in the proof of Lemma 2.3, we conclude that $w \in C^{2,\alpha}$ in a neighborhood of $(0, b^+)$ in $\{x \ge 0\}$. $\qquad\square$

3 Layer Solutions: Local Minimality, Consequences, and the Proof of Theorem 1.4

The fact that, for reactions in the interior, layer solutions are necessarily local minimizers was recently found in [1]. The proof in that paper used the variational theory of calibrations. An alternative, more elementary proof was later given in [16, 25]. Here we follow this simpler proof, adapted to reactions on the boundary. Its key point lies in the following uniqueness result for the Dirichlet-Neumann problem with Dirichlet boundary conditions equal to a certain layer.

LEMMA 3.1 *Let u be a layer solution of* (1.1). *Then, for every $R > 0$, u is the unique weak solution of the problem*

(3.1)
$$\begin{cases} \Delta w = 0 & \text{in } B_R^+ \subset \mathbb{R}_+^n \\ -1 \leq w \leq 1 & \text{in } B_R^+ \\ \dfrac{\partial w}{\partial \nu} = f(w) & \text{on } \Gamma_R^0 \\ w = u & \text{on } \Gamma_R^+. \end{cases}$$

We emphasize that the previous lemma makes no assumption on the nonlinearity f and that uniqueness is a consequence of the presence in the whole half-space of a layer solution.

The local minimality of every layer solution u follows automatically from the previous lemma, since by uniqueness u must agree in B_R^+ with the absolute minimizer w of problem (3.1).

PROOF OF LEMMA 3.1: Since u is a layer solution, we know by Lemma 2.4 that

(3.2)
$$f(-1) = f(1) = 0.$$

Let w be a weak solution of (3.1). By Lemma 2.11(a), we know that $w \in C^2(\overline{B_R^+})$. Hence, by Hopf's boundary lemma and (3.2), we deduce that

$$-1 < w < 1 \quad \text{in } \overline{B_R^+};$$

recall that w cannot be identically -1 or 1 since it agrees with $u \in (-1, 1)$ on Γ_R^+.

To establish $w \equiv u$, we first prove that $w \leq u$ in B_R^+. For this, we slide u in the y_1-direction. That is, for $t > 0$, consider

$$u^t(x, y_1, y_2, \ldots, y_{n-1}) = u(x, y_1 + t, y_2, \ldots, y_{n-1}) \quad \text{for } (x, y) \in \overline{B_R^+}.$$

Note that $u^t \to 1$ as $t \to +\infty$ uniformly in $\overline{B_R^+}$, as a consequence of (2.13). Moreover, $-1 < w < 1$ is a continuous function in the compact set $\overline{B_R^+}$. We deduce that $w < u^t$ in $\overline{B_R^+}$ for $t > 0$ large enough.

Next, we claim that the same inequality is true for all $t > 0$. This will conclude the proof of $w \leq u$, by letting $t > 0$ tend to 0.

Note that if $w < u^t$ holds for some t_0, then it also holds for every $t \geq t_0$. Hence, we suppose that $s > 0$ is the infimum of those $t > 0$ such that $w < u^t$ in $\overline{B_R^+}$, and we need to arrive at a contradiction. We have that $w \leq u^s$ in $\overline{B_R^+}$. At the same time, on Γ_R^+ we have $w = u < u^s$, since $s > 0$. In particular, $w \not\equiv u^s$.

Since s is supposed to be such an infimum value of the parameter t, we deduce that $w = u^s$ at some point (x_0, y_0) in $B_R^+ \cup \Gamma_R^0$. But since $w \leq u^s$ in $\overline{B_R^+}$ and since both w and u^s are solutions of the same nonlinear boundary value problem, Hopf's maximum principle implies that $w \equiv u^s$, a contradiction. Here we have used the strong maximum principle if $x_0 > 0$ and Hopf's boundary lemma if $x_0 = 0$.

To prove the reversed inequality, $u \leq w$ in Ω^+, we use the same sliding method but now with $t < 0$. □

The following result, which we prove with variational techniques, will be useful in several future arguments. For instance, we will use it to establish existence of a two-dimensional layer solution when proving Theorem 1.2(a).

PROPOSITION 3.2 *Let u be a solution of* (1.1) *such that $|u| < 1$ and*

$$\lim_{y_1 \to \pm\infty} u(0, y) = L^\pm \quad \text{for every } (y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}$$

for some constants L^- and L^+ (that could be equal). Assume that u is a local minimizer relative to perturbations in $[-1, 1]$. Then

$$G \geq G(L^-) = G(L^+) \quad \text{in } [-1, 1].$$

PROOF: It suffices to show that $G \geq G(L^-)$ and $G \geq G(L^+)$ in $[-1, 1]$. It then follows that $G(L^-) = G(L^+)$. By symmetry, it is enough to establish that $G \geq G(L^+)$ in $[-1, 1]$. Note that this inequality, as well as the notion of local minimizer, is independent of adding a constant to G. Hence, we may assume that

$$G(s) = 0 < G(L^+) \quad \text{for some } s \in [-1, 1],$$

and we need to obtain a contradiction.

Since $G(L^+) > 0$, we have that $G(t) \geq \varepsilon > 0$ for t in a neighborhood of L^+. Consider the points $(0, b, 0) = (0, b, 0, \ldots, 0)$ on $\partial\mathbb{R}^n_+$. Since for $R > 0$,

$$E_{B_R^+(0,b,0)}(u) \geq \int_{\Gamma_R^0(b,0)} G(u(0, y))dy$$

and $u(0, y) \xrightarrow[y_1 \to +\infty]{} L^+$, we deduce

$$(3.3) \qquad \lim_{b \to +\infty} E_{B_R^+(0,b,0)}(u) \geq c\varepsilon R^{n-1} \quad \text{for all } R > 0.$$

Recall that the definition of the energy functional E_Ω in a bounded set Ω was given in (2.32). Throughout the proof, c and C denote positive constants independent of R and b.

The lower bound (3.3) will be a contradiction with an upper bound for the energy of u that we obtain using the local minimality of u, as follows:

For $R > 2$, consider

$$\xi_R(x, y) = \begin{cases} \dfrac{\log R - \log r}{\log R - \log(R - R^{1/2})} & \text{if } r = |(x, y)| \in [R - R^{1/2}, R] \\ 1 & \text{if } r \leq R - R^{1/2}. \end{cases}$$

Note that ξ_R is Lipschitz-continuous in $\overline{B_R^+}$, $0 \leq \xi_R \leq 1$, and $\xi_R \equiv 0$ on Γ_R^+. Consider

$$\xi_{R,b}(x, y) = \xi_R(x, y_1 - b, y_2, \ldots, y_{n-1}).$$

We have that $\xi_{R,b} \equiv 0$ on $\Gamma_R^+(0, b, 0)$. Hence, the comparison function

$$v_{R,b} = (1 - \xi_{R,b})u + \xi_{R,b}s$$

takes values in $[-1, 1]$ and agrees with u on $\Gamma_R^+(0, b, 0)$. Note also that

$$(3.4) \qquad v_{R,b} \equiv s \quad \text{in } B_{R-R^{1/2}}^+(0, b, 0).$$

Let $S > 0$ be large enough such that $B_R^+(0, b, 0) \subset B_S^+ = B_S^+(0)$, and extend $v_{R,b}$ to be identically u outside $B_R^+(0, b, 0)$. Clearly we can approximate ξ_R in $H^1(B_R^+)$ by C^1 functions taking values in $[0, 1]$ and with compact support in $B_R^+ \cup \Gamma_R^0$. Therefore, $v_{R,b} = u + \xi_{R,b}(s - u)$ can be approximated in $H^1(B_S^+)$ by C^1 functions taking values in $[-1, 1]$ and such that $v_{R,b} - u$ has compact support in $B_S^+ \cup \Gamma_S^0$. Hence, since u is a local minimizer, we deduce $E_{B_S^+}(u) \leq E_{B_S^+}(v_{R,b})$.

This is equivalent to $E_{B_R^+(0,b,0)}(u) \leq E_{B_R^+(0,b,0)}(v_{R,b})$ since u and $v_{R,b}$ agree outside $B_R^+(0, b, 0)$. Using also (3.4) and that $G(s) = 0$, we obtain

$$E_{B_R^+(0,b,0)}(u)$$

$$\leq E_{B_R^+(0,b,0)}(v_{R,b})$$

$$= \int_{B_R^+(0,b,0)\backslash B_{R-R^{1/2}}^+(0,b,0)} \frac{1}{2}|\nabla v_{R,b}|^2 \, dx\, dy + \int_{\Gamma_R^0(b,0)\backslash \Gamma_{R-R^{1/2}}^0(b,0)} G(v_{R,b}(0, y))dy$$

$$\leq \int_{B_R^+(0,b,0)\backslash B_{R-R^{1/2}}^+(0,b,0)} \frac{1}{2}|\nabla v_{R,b}|^2 \, dx\, dy + C\{R^{n-1} - (R - R^{1/2})^{n-1}\}$$

$$\leq \int_{B_R^+(0,b,0)\backslash B_{R-R^{1/2}}^+(0,b,0)} \frac{1}{2}|\nabla v_{R,b}|^2 \, dx\, dy + C R^{n-3/2}.$$

Next, we have that $\nabla v_{R,b} = (1 - \xi_{R,b})\nabla u + (s - u)\nabla \xi_{R,b}$ and hence $|\nabla v_{R,b}| \leq 2(|\nabla u| + |\nabla \xi_{R,b}|)$. Inserting this bound in the previous estimate for the energy, we deduce

$$\varlimsup_{b \to +\infty} E_{B_R^+(0,b,0)}(u)$$

$$\leq \varlimsup_{b \to +\infty} C \left\{ R^n \|\nabla u\|_{L^\infty(B_R^+(0,b,0))}^2 \right.$$

$$+ \int_{B_R^+(0,b,0)\backslash B_{R-R^{1/2}}^+(0,b,0)} |\nabla \xi_{R,b}|^2 \, dxdy + R^{n-3/2} \right\}$$

$$\leq C \int_{B_R^+\backslash B_{R-R^{1/2}}^+} |\nabla \xi_R|^2 \, dx\, dy + C R^{n-3/2},$$

where we have used (2.14) from Lemma 2.4.

We have that

$$|\nabla \xi_R| = \{\log R - \log(R - R^{1/2})\}^{-1} r^{-1}$$

in $B_R^+ \setminus B_{R-R^{1/2}}^+$, and hence

$$\int_{B_R^+ \setminus B_{R-R^{1/2}}^+} |\nabla \xi_R|^2 \, dx \, dy = \frac{C}{\{\log R - \log(R - R^{1/2})\}^2} \int_{R-R^{1/2}}^R \frac{1}{r^2} r^{n-1} \, dr.$$

Note that $\int_{R-R^{1/2}}^R r^{n-3} \, dr$ is bounded by $C\{\log R - \log(R - R^{1/2})\}$ if $n = 2$, and by $C R^{n-5/2}$ if $n \geq 3$. Using that

$$\log R - \log(R - R^{1/2}) = \log \frac{R}{R - R^{1/2}} = -\log(1 - R^{-1/2}) \geq R^{-1/2},$$

we finally arrive at

$$\overline{\lim_{b \to +\infty}} \, E_{B_R^+(0,b,0)}(u) \leq C R^{n-3/2}$$

for every dimension n. For R large enough, this contradicts (3.3). $\square$

We can now give the following proof:

PROOF OF THEOREM 1.4: To prove statement (a), let u be a layer solution of (1.1) and let $R > 0$. By Lemma 2.10, we know that there exists an absolute minimizer w of the energy $E_{B_R^+}$ in the set $\mathcal{C}_u(B_R^+)$ defined in Lemma 2.10. Moreover, w is a weak solution of (3.1) and hence, by Lemma 3.1, w must agree with u. Therefore, u is the absolute minimizer of $E_{B_R^+}$ in $\mathcal{C}_u(B_R^+)$. Since R is arbitrary, we conclude that u is a local minimizer of problem (1.1) relative to perturbations in $[-1, 1]$ in the sense of Definition 1.1.

Now we establish part (b). We are assuming that problem (1.1) admits a layer solution u. By Lemma 2.4, we deduce that $f(-1) = f(1) = 0$. This is the first statement of (1.12). Next, we already know that u is a local minimizer relative to perturbations in $[-1, 1]$. Hence we can apply Proposition 3.2 and, since $L^\pm = \pm 1$ here, its conclusion gives the second statement of (1.12). $\square$

4 Stable Solutions: Monotonicity, Two-Dimensional Symmetry, and the Proof of Theorem 1.5

To prove Theorem 1.5, we need two lemmas. The following one, applied with $d(y) = -f'(u(0, y))$, establishes an alternative criterion for a solution u of (1.1) to be stable.

LEMMA 4.1 *Let d be a bounded, Hölder-continuous function on $\partial \mathbb{R}_+^n$. Then*

$$(4.1) \qquad \int_{\mathbb{R}_+^n} |\nabla \xi|^2 + \int_{\partial \mathbb{R}_+^n} d(y)\xi^2 \geq 0$$

for every function $\xi \in C^1(\overline{\mathbb{R}^n_+})$ with compact support in $\overline{\mathbb{R}^n_+}$ if and only if there exists a function $\varphi \in C^1_{\text{loc}}(\overline{\mathbb{R}^n_+}) \cap C^2(\mathbb{R}^n_+)$ such that $\varphi > 0$ in $\overline{\mathbb{R}^n_+}$ and

(4.2)
$$\begin{cases} \Delta\varphi = 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial\varphi}{\partial\nu} + d(y)\varphi = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

PROOF: First, assume the existence of a positive solution φ of (4.2), as in the statement of the lemma. Let $\xi \in C^1(\overline{\mathbb{R}^n_+})$ have compact support in $\overline{\mathbb{R}^n_+}$. Multiplying $\Delta\varphi = 0$ by ξ^2/φ, integrating by parts, and using the Cauchy-Schwarz inequality, one can easily obtain (4.1).

The other implication is the one we will need in the sequel, and it is more delicate to prove. For the sake of completeness, we give all details of the proof—which are standard by now in the case of reactions in the interior.

Assume that (4.1) holds for every $\xi \in C^1(\overline{\mathbb{R}^n_+})$ with compact support in $\overline{\mathbb{R}^n_+}$. For every $R > 0$, let λ_R be the infimum of the quadratic form

(4.3)
$$Q_R(\xi) = \int_{B_R^+} |\nabla\xi|^2 + \int_{\Gamma_R^0} d(y)\xi^2$$

among functions in the class S_R, defined by

$$S_R = \left\{ \xi \in H^1(B_R^+) : \xi \equiv 0 \text{ on } \Gamma_R^+ \text{ and } \int_{\Gamma_R^0} \xi^2 = 1 \right\}$$

$$\subset H_0(B_R^+) = \left\{ \xi \in H^1(B_R^+) : \xi \equiv 0 \text{ on } \Gamma_R^+ \right\}.$$

By our assumption, $\lambda_R \geq 0$ for every R. By definition it is clear that λ_R is a nonincreasing function of R. Next, we show that λ_R is indeed a decreasing function of R. As a consequence, we deduce that $\lambda_R > 0$ for every R, and this will be important in the sequel.

To show that λ_R is decreasing in R, note first that since d is assumed to be a bounded function, the functional Q_R is bounded below in the class S_R. For the same reason, any minimizing sequence (ξ^k) has $(\nabla\xi^k)$ uniformly bounded in $L^2(B_R^+)$. Hence, by the compact inclusion $H_0(B_R^+) \Subset L^2(\Gamma_R^0)$ (see Section 2.5), we conclude that the infimum of Q_R in S_R is achieved by a function $\phi_R \in S_R$. Moreover, we may take $\phi_R \geq 0$, since $|\phi|$ is a minimizer whenever ϕ is a minimizer.

Note that $\phi_R \geq 0$ is a solution of

$$\begin{cases} \Delta\phi_R = 0 & \text{in } B_R^+ \\ \dfrac{\partial\phi_R}{\partial\nu} + d(y)\phi_R = \lambda_R\phi_R & \text{on } \Gamma_R^0 \\ \phi_R = 0 & \text{on } \Gamma_R^+. \end{cases}$$

It follows from the strong maximum principle that $\phi_R > 0$ in B_R^+.

We can now easily prove that λ_R is decreasing in R. Indeed, arguing by contradiction, assume that $R_1 < R_2$ and $\lambda_{R_1} = \lambda_{R_2}$. Multiply $\Delta\phi_{R_1} = 0$ by ϕ_{R_2}, integrate by parts, and use the equalities satisfied by ϕ_{R_1} and ϕ_{R_2} and also the assumption $\lambda_{R_1} = \lambda_{R_2}$. We obtain

$$\int_{\Gamma_{R_1}^+} \frac{\partial\phi_{R_1}}{\partial\nu}\phi_{R_2} = 0,$$

and this is a contradiction since, on $\Gamma_{R_1}^+$, we have $\phi_{R_2} > 0$ and the normal derivative $\partial\phi_{R_1}/\partial\nu < 0$.

Next, using that $\lambda_R > 0$ we obtain

$$\int_{B_R^+} |\nabla\xi|^2 + \int_{\Gamma_R^0} d(y)\xi^2 \geq \lambda_R \int_{\Gamma_R^0} \xi^2 \geq -\delta_R \int_{\Gamma_R^0} d(y)\xi^2$$

for all $\xi \in H_0(B_R^+)$, where δ_R is taken such that $0 < \delta_R \leq \lambda_R/\|d\|_{L^\infty}$. From the last inequality, we deduce that

$$(4.4) \qquad \int_{B_R^+} |\nabla\xi|^2 + \int_{\Gamma_R^0} d(y)\xi^2 \geq \varepsilon_R \int_{B_R^+} |\nabla\xi|^2$$

for all $\xi \in H_0(B_R^+)$, for $\varepsilon_R > 0$ given by $\varepsilon_R = 1 - 1/(1 + \delta_R)$.

It is now easy to prove that, for every constant $c_R > 0$, there exists a solution φ_R of

$$(4.5) \qquad \begin{cases} \Delta\varphi_R = 0 & \text{in } B_R^+ \\ \dfrac{\partial\varphi_R}{\partial\nu} + d(y)\varphi_R = 0 & \text{on } \Gamma_R^0 \\ \varphi_R = c_R & \text{on } \Gamma_R^+. \end{cases}$$

Indeed, rewriting this problem for the function $\psi_R = \varphi_R - c_R$, we need to solve

$$\begin{cases} \Delta\psi_R = 0 & \text{in } B_R^+ \\ \dfrac{\partial\psi_R}{\partial\nu} + d(y)\psi_R + c_R d(y) = 0 & \text{on } \Gamma_R^0 \\ \psi_R = 0 & \text{on } \Gamma_R^+. \end{cases}$$

This problem can be solved by minimizing the functional

$$\int_{B_R^+} \frac{1}{2}|\nabla\xi|^2 + \int_{\Gamma_R^0} \left\{\frac{1}{2}d(y)\xi^2 + c_R d(y)\xi\right\}$$

in the space $H_0(B_R^+)$. Note that the functional is bounded below and coercive, thanks to inequality (4.4). Finally, the compact inclusion $H_0(B_R^+) \Subset L^2(\Gamma_R^0)$ gives the existence of a minimizer.

Next, we claim that

$$\varphi_R > 0 \quad \text{in } \overline{B_R^+}.$$

Indeed, the negative part φ_R^- of φ_R vanishes on Γ_R^+. Using this, (4.5), and the definition (4.3) of Q_R, it is easy to verify that $Q_R(\varphi_R^-) = 0$. By definition of the first eigenvalue λ_R and the fact that $\lambda_R > 0$, this implies that $\varphi_R^- \equiv 0$, i.e., $\varphi_R \geq 0$. Now, Hopf's maximum principle gives $\varphi_R > 0$ up to the boundary.

Finally, we choose the constant $c_R > 0$ in (4.5) such that $\varphi_R(0) = 1$. Then, by the Harnack inequality of Lemma 2.5, we deduce

$$\sup_{B_R^+} \varphi_S \leq C_R \quad \text{for all } S > 4R.$$

Hence, using the $C^{1,\alpha}$ estimate of Lemma 2.3(c), a subsequence of (φ_S) converges locally in $\overline{\mathbb{R}_+^n}$ to a $C_{\text{loc}}^1(\overline{\mathbb{R}_+^n}) \cap C^2(\mathbb{R}_+^n)$ solution $\varphi \geq 0$ of (4.2) with $\varphi(0) = 1$. It follows that $\varphi > 0$. □

Observe that the previous lemma provides a direct proof of the fact that every layer solution u of (1.1) is stable —something that we already knew from the local minimality property of layer solutions proven in Section 3. Indeed, we simply note that $\varphi = u_{y_1}$ is strictly positive and solves the linearized problem (4.2), with $d(y) = -f'(u(0, y))$. Hence the stability of u follows from Lemma 4.1.

We use now the previous lemma to establish a result that easily leads to the monotonicity and the two-dimensional symmetry of stable solutions in dimensions 2 and 3.

LEMMA 4.2 *Assume that $n \leq 3$ and that u is a bounded stable solution of* (1.1). *Then there exists a function $\varphi \in C_{\text{loc}}^1(\overline{\mathbb{R}_+^n}) \cap C^2(\mathbb{R}_+^n)$ with $\varphi > 0$ in $\overline{\mathbb{R}_+^n}$ and such that, for every $i = 1, \ldots, n - 1$,*

$$u_{y_i} = c_i \varphi \quad \text{in } \mathbb{R}_+^n$$

for some constant c_i.

PROOF: Since u is assumed to be a stable solution, then (4.1) holds with $d(y) = -f'(u(0, y))$. Note that $d \in C^\alpha$ by Lemma 2.3. Hence, by Lemma 4.1, there exists a function $\varphi \in C_{\text{loc}}^1(\overline{\mathbb{R}_+^n}) \cap C^2(\mathbb{R}_+^n)$ such that $\varphi > 0$ in $\overline{\mathbb{R}_+^n}$ and

$$\begin{cases} \Delta \varphi = 0 & \text{in } \mathbb{R}_+^n \\ \dfrac{\partial \varphi}{\partial \nu} - f'(u(0, y))\varphi = 0 & \text{on } \partial \mathbb{R}_+^n. \end{cases}$$

For $i = 1, \ldots, n - 1$ fixed, consider the function

$$\sigma = \frac{u_{y_i}}{\varphi}.$$

The goal is to prove that σ is constant in $\mathbb{R}_+^n$.

Since

$$\varphi^2 \nabla \sigma = \varphi \nabla u_{y_i} - u_{y_i} \nabla \varphi,$$

we have that

$$\text{div}(\varphi^2 \nabla \sigma) = 0 \quad \text{in } \mathbb{R}_+^n.$$

Moreover, the normal derivative of σ on $\partial\mathbb{R}^n_+$ is 0. Indeed, on $\partial\mathbb{R}^n_+$ we have

$$\varphi^2\sigma_x = \varphi u_{xy_i} - u_{y_i}\varphi_x = 0,$$

since both u_{y_i} and φ satisfy the same boundary condition $-u_{xy_i} - f'(u)u_{y_i} = 0$, $-\varphi_x - f'(u)\varphi = 0$.

We can use the Liouville property of Lemma 2.6 and deduce that σ is constant provided that the growth condition

$$\int\limits_{B_R^+} (\varphi\sigma)^2 \leq CR^2 \quad \text{for all } R > 1$$

holds for some constant C independent of R. But note that $\varphi\sigma = u_{y_i}$, and therefore

$$\int\limits_{B_R^+} (\varphi\sigma)^2 \leq \int\limits_{B_R^+} |\nabla u|^2 = \int\limits_{\partial B_R^+} uu_\nu \leq CR^{n-1} \leq CR^2$$

since uu_ν is bounded and $n \leq 3$. This finishes the proof of the lemma. $\qquad\square$

We can now give the following proof:

PROOF OF THEOREM 1.5: Let u be a bounded stable solution of (1.1).

To prove part (a), let $n = 2$. The conclusion of Lemma 4.2 establishes that $u_y = c\varphi$ for some function $\varphi > 0$ in $\overline{\mathbb{R}^2_+}$ and some constant c. Therefore, depending on the sign of the constant c, we have that either $u_y \equiv 0$, $u_y > 0$ everywhere, or $u_y < 0$ everywhere. In the case $u_y \equiv 0$, we deduce that u is a bounded harmonic function that depends only on the x-variable. Hence, u must be constant.

To prove part (b), let $n = 3$. Lemma 4.2 establishes that $u_{y_i} \equiv c_i\varphi$ for some constants c_i, $i = 1, 2$. If $c_1 = c_2 = 0$, then u is constant. Otherwise we have that $c_2 u_{y_1} - c_1 u_{y_2} \equiv 0$, and we conclude that u depends only on x and on the variable parallel to $(0, c_1, c_2)$. That is,

$$u(x, y_1, y_2) = u_0\left(x, \frac{c_1 y_1 + c_2 y_2}{(c_1^2 + c_2^2)^{1/2}}\right) = u_0(x, y),$$

where y denotes the variable parallel to $(0, c_1, c_2)$ and u_0 is a solution for $n = 2$.

In particular, $\partial_y u_0 = (c_1^2 + c_2^2)^{1/2}\varphi$, and hence $\partial_y u_0 > 0$ everywhere. This finishes the proof of the theorem. $\qquad\square$

5 Solutions with Uniform Limits: Monotonicity, Two-Dimensional Symmetry, and Uniqueness

The following result establishes the two-dimensional symmetry of solutions that have limits ± 1 as $y_1 \to \pm\infty$ on $\partial\mathbb{R}^n_+$, uniformly in $(y_2, \ldots, y_{n-1})$. This is the exact analogue of a result for interior reactions due to Berestycki, Hamel, and Monneau [11].

THEOREM 5.1 *Assume $f'(\pm 1) < 0$. Let u be a solution of (1.1) satisfying $|u| \leq 1$ in $\mathbb{R}^n_+$ and*

$$(5.1) \qquad \lim_{y_1 \to \pm\infty} u(0, y) = \pm 1 \quad \text{uniformly in } (y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-2}.$$

Then u is necessarily a function of the two variables x and y_1 alone, that is, $u = u(x, y_1)$. Moreover, $u_{y_1} > 0$ and u is unique up to translations in the y_1-variable.

Due to the hypothesis of uniform limits, the y-variable on which the solution depends is exactly y_1. This is in contrast to the two-dimensional symmetry result of Theorem 1.5(b) in dimension 3, where the y-variable on which u depends is not known a priori and can be any combination of y_1 and y_2. This reflects the greater difficulty of proving the two-dimensional symmetry of solutions with no hypothesis on uniform limits.

We start by treating the case $n = 2$ and later prove Theorem 5.1 in any dimension. The following lemma will imply the monotonicity of solutions with limits, as well as the uniqueness of the layer solution in dimension 2. The statement of the lemma contains the main features of its proof, which is based on a very useful technique introduced by Berestycki and Nirenberg in [12]: the sliding method.

LEMMA 5.2 *Assume that $n = 2$ and that*

$$(5.2) \qquad f \text{ is nonincreasing in } (-1, -\tau) \cup (\tau, 1) \text{ for some } \tau \in (0, 1).$$

Let u_1 and u_2 be two solutions of (1.1) such that, for $i = 1, 2$,

$$(5.3) \qquad u_i(0, 0) = 0, \quad |u_i| \leq 1, \quad and \quad \lim_{y \to \pm\infty} u_i(0, y) = \pm 1.$$

For $t > 0$, consider

$$u_2^t(x, y) = u_2(x, y + t).$$

Then, for every $t > 0$, $u_1 \leq u_2^t$ in $\overline{\mathbb{R}^2_+}$.

PROOF: First, note that $f(-1) = f(1) = 0$ by Lemma 2.4. Therefore, since $|u_i| \leq 1$, we have that $|u_i| < 1$ for $i = 1, 2$. Note that the u_i are not identically constant by the assumption in (5.3) about their limits as $y \to \pm\infty$.

By hypothesis (5.3), there exists a compact interval $[a, b]$ in $\mathbb{R}$ such that, for $i = 1, 2$,

$$\begin{cases} u_i(0, y) \in (-1, -\tau) & \text{if } y \leq a \\ u_i(0, y) \in (\tau, 1) & \text{if } y \geq b. \end{cases}$$

Note that u_2^t is also a solution of (1.1), and hence

$$\begin{cases} \Delta(u_2^t - u_1) = 0 & \text{in } \mathbb{R}^2_+ \\ -(u_2^t - u_1)_x = -d^t(y)(u_2^t - u_1) & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

where

$$d^t(y) = -\frac{f(u_2^t) - f(u_1)}{u_2^t - u_1}(0, y)$$

if $(u_2^t - u_1)(0, y) \neq 0$, and $d^t(y) = 0$ otherwise. Note that d^t is a bounded function since f is Lipschitz. Note that we also have

$$(u_2^t - u_1)(0, y) \to 0 \quad \text{as } |y| \to \infty.$$

We finish the proof in three steps.

Step 1. We claim that $u_1 < u_2^t$ for $t > 0$ large enough.

To show this, we take $t > 0$ sufficiently large such that $u_1(0, y) < u_2^t(0, y)$ for $y \in [a, b]$. This is possible since $u_1 < 1$ and $u_2(0, y + t) \to 1$ as $t \to +\infty$. We apply Lemma 2.7 to $v = u_2^t - u_1$, with

$$H = (a, b) \cup \{y \in \mathbb{R} : (u_2^t - u_1)(0, y) > 0\}$$
$$= \{y \in \mathbb{R} : (u_2^t - u_1)(0, y) > 0\}.$$

Clearly, $v(0, y) > 0$ in H.

To show that $d^t \geq 0$ in $\mathbb{R} \setminus H$, let $y \notin H$. There are two possibilities. First, if $y \geq b$ then $y + t \geq b$ also. Therefore, $u_1(0, y) \geq \tau$ and $u_2^t(0, y) = u_2(0, y+t) \geq \tau$. We conclude that $d^t(y) \geq 0$ by (5.2).

The other possibility is that $y \leq a$. In this case, we have $u_1(0, y) \leq -\tau$, and since $y \notin H$, then $(u_2^t - u_1)(0, y) \leq 0$. Therefore $u_2^t(0, y) \leq u_1(0, y) \leq -\tau$, and we conclude $d^t(y) \geq 0$, again by (5.2).

Lemma 2.7 gives that $u_2^t - u_1 > 0$ in $\overline{\mathbb{R}_+^2}$.

Step 2. Claim: If $u_1 \leq u_2^t$ for some $t > 0$, then $u_1 \leq u_2^{t+\mu}$ for every μ small enough (with μ either positive or negative).

This statement will finish the proof of the lemma, since then $\{t > 0 : u_1 \leq u_2^t\}$ is a nonempty, closed and open set in $(0, \infty)$, and hence equal to this interval. We conclude $u_1 \leq u_2^t$ for all $t > 0$.

To prove the claim of Step 2, we will show in Step 3 that

(5.4)   if $t > 0$ and $u_1 \leq u_2^t$, then $u_1 \not\equiv u_2^t$.

Once (5.4) is known, we can finish the proof of the claim as follows: First, by Hopf's maximum principle, $u_1 < u_2^t$ in $\overline{\mathbb{R}_+^2}$. Let K_t be a compact interval such that, for $y \notin K_t$, $|u_1(0, y)| > 1 - \tau/2$ and $|u_2^t(0, y)| > 1 - \tau/2$. Recall that $(u_2^t - u_1)(0, y) > 0$ in the compact set K_t. By continuity and the existence of limits at infinity, we have that if $|\mu|$ is small enough, then $(u_2^{t+\mu} - u_1)(0, y) > 0$ for $y \in K_t$ and $|u_2^{t+\mu}(0, y)| > 1 - \tau$ for $y \notin K_t$. Hence, we can apply Lemma 2.7 to $v = u_2^{t+\mu} - u_1$ with $H = K_t$, since $d_{t+\mu} \geq 0$ outside K_t. We therefore conclude $u_2^{t+\mu} - u_1 > 0$.

Step 3. Here we establish (5.4), therefore completing the proof of Step 2 and of the lemma. That is, we assume that $t > 0$ and $u_1 \leq u_2^t$, and we need to show that $u_1 \not\equiv u_2^t$.

To prove this, consider first the case when both solutions in the lemma are the same, that is, $u_1 \equiv u_2$. Assume that $t > 0$ and $u_1 \equiv u_1^t$. Then, the function $u_1(0, y)$ is t-periodic. But this contradicts the hypothesis $u_1(0, y) \to \pm 1$ as $y \to \pm\infty$ in

the lemma. Therefore, in the case $u_1 \equiv u_2$, the two steps above can be carried out. We conclude that, for every solution u_1 as in the lemma, we have $u_1 \leq u_1^t$ for every $t > 0$. In particular, $\partial_y u_1 \geq 0$ and, by the strong maximum principle, u_1 is increasing in y.

Finally, consider the general case of two solutions u_1 and u_2. Assume that $t > 0$ and $u_1 \equiv u_2^t$. Then, by (5.3), $u_1(0, -t) = u_2^t(0, -t) = u_2(0, 0) = 0$. Moreover, $u_1(0, 0) = 0$ by hypothesis. Hence, both $(0, -t)$ and $(0, 0)$ are zeroes of u_1. This is a contradiction, since we have already established that u_1 is increasing in y. □

We now give the following proof:

PROOF OF THEOREM 5.1: First, note that $f(-1) = f(1) = 0$ by Lemma 2.4. Therefore, since $|u| \leq 1$, we conclude that $|u| < 1$ by Hopf's boundary lemma. Note that u is not identically constant by assumption (5.1).

Since $f'(\pm 1) < 0$, we have that

$$(5.5) \qquad\qquad f' \leq -\varepsilon \quad \text{in } (-1, -\tau) \cup (\tau, 1)$$

for some $\varepsilon > 0$ and $0 < \tau < 1$. By hypothesis (5.1) on uniform limits, there exists a compact interval $[a, b]$ in $\mathbb{R}$ such that

$$\begin{cases} u(0, y) \in (-1, -\tau) & \text{if } y_1 \leq a \\ u(0, y) \in (\tau, 1) & \text{if } y_1 \geq b. \end{cases}$$

We claim that

$$(5.6) \qquad\qquad \sup_{y \in [a,b] \times \mathbb{R}^{n-2}} u(0, y) < 1.$$

We prove this arguing by contradiction. Suppose that there exists a sequence (y^k) of points in $[a, b] \times \mathbb{R}^{n-2}$ such that $u(0, y^k) \to 1$ as $k \to \infty$. Set $u_k(x, y) = u(x, y + y^k)$. By the estimates of Lemma 2.3, up to extraction of a subsequence the solutions u_k tend locally to a classical solution u_∞ of (1.1). Since $y_1^k \in [a, b]$ for all k, u_∞ also satisfies condition (5.1) on uniform limits and, in particular, u_∞ is not constant. But $u_\infty \leq 1$ and $u_\infty(0, 0) = 1$. Since $f(1) = 0$, we deduce that $u_\infty \equiv 1$, a contradiction.

Next, we prove that u is increasing in any direction $\nu = (0, \nu_1, \ldots, \nu_{n-1})$ with $\nu_1 > 0$. For this, we define the function

$$u^t(x, y) = u((x, y) + t\nu)$$

for every $t > 0$.

Note that u^t is also a solution of (1.1), and hence

$$(5.7) \qquad\qquad \begin{cases} \Delta(u^t - u) = 0 & \text{in } \mathbb{R}_+^n \\ -(u^t - u)_x = -d^t(y)(u^t - u) & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

where

$$(5.8) \qquad d^t(y) = -\frac{f(u^t) - f(u)}{u^t - u}(0, y)$$

if $(u^t - u)(0, y) \neq 0$, and $d^t(y) = 0$ otherwise. Note that d^t is a bounded function since f is Lipschitz.

First, we claim that $u < u^t$ for $t > 0$ large enough.

To show this, we take $t > 0$ sufficiently large such that $u(0, y) < u^t(0, y)$ for $y_1 \in [a, b]$. This is possible since u satisfies (5.6) and $u^t(0, y) \to 1$ as $t \to +\infty$ uniformly in $[a, b] \times \mathbb{R}^{n-2}$. We apply Lemma 2.9 to $v = u^t - u$, with

$$H = [(a, b) \times \mathbb{R}^{n-2}] \cup \{y \in \mathbb{R}^{n-1} : (u^t - u)(0, y) > 0\}$$
$$= \{y \in \mathbb{R}^{n-1} : (u^t - u)(0, y) > 0\}.$$

Clearly, $v(0, y) > 0$ in H. To show that $d^t \geq \varepsilon$ in $\mathbb{R}^{n-1} \setminus H$, let $y \notin H$. There are two possibilities. First, if $y_1 \geq b$, then $y_1 + t v_1 \geq b$ also. Therefore, $u(0, y) \geq \tau$ and $u^t(0, y) \geq \tau$, and we conclude that $d^t(y) \geq \varepsilon$ by (5.5). The other possibility is that $y_1 \leq a$. In this case, we have $u(0, y) \leq -\tau$, and since $y \notin H$, then $(u^t - u)(0, y) \leq 0$. Therefore $u^t(0, y) \leq u(0, y) \leq -\tau$, and we conclude again $d^t(y) \geq \varepsilon$. Lemma 2.9 gives that $u^t - u > 0$ in $\overline{\mathbb{R}^n_+}$.

Next, we claim that if $u \leq u^t$ for some $t > 0$, then $u \leq u^{t+\mu}$ for every μ small enough (with μ either positive or negative). This statement will prove that u is nondecreasing in the direction v, since then the set $\{t > 0 : u \leq u^t\}$ is nonempty, closed, and open in $(0, \infty)$, and hence equal to this interval.

To prove the previous claim, we assume that $t > 0$ and $u \leq u^t$. We first show that

$$(5.9) \qquad \inf_{y \in [a,b] \times \mathbb{R}^{n-2}} (u^t - u)(0, y) > 0.$$

Indeed, if this were not the case, there would be a sequence (y^k) of points in $[a, b] \times \mathbb{R}^{n-2}$ such that $(u^t - u)(0, y^k) \to 0$ as $k \to \infty$. Set $u_k(x, y) = u(x, y + y^k)$. By Lemma 2.3, up to extraction of a subsequence the solutions u_k tend locally to a solution u_∞ of (1.1). Therefore, $u_\infty^t - u_\infty$ satisfies (5.7) and (5.8), with u replaced by u_∞ throughout these three equalities. Moreover, by construction, $u_\infty^t - u_\infty \geq 0$ and $(u_\infty^t - u_\infty)(0, 0) = 0$. We deduce that $u_\infty^t - u_\infty \equiv 0$, and hence u_∞ is periodic with respect to the vector $t v$. This is a contradiction, since $t v_1 > 0$ and $y_1^k \in [a, b]$ for all k imply that u_∞ also satisfies condition (5.1) on uniform limits.

Since u is globally Lipschitz (recall that $\nabla u \in L^\infty(\mathbb{R}^n_+)$ by Lemma 2.3), (5.9) implies that

$$\inf_{y \in [a,b] \times \mathbb{R}^{n-2}} (u^{t+\mu} - u)(0, y) > 0$$

for every μ small enough (with $t + \mu > 0$). We finally apply Lemma 2.9 to $v = u^{t+\mu} - u$, with

$$H = [(a, b) \times \mathbb{R}^{n-2}] \cup \{y \in \mathbb{R}^{n-1} : (u^{t+\mu} - u)(0, y) > 0\}$$
$$= \{y \in \mathbb{R}^{n-1} : (u^{t+\mu} - u)(0, y) > 0\}.$$

Clearly, $v(0, y) > 0$ in H. To show that $d^{t+\mu} \geq \varepsilon$ in $\mathbb{R}^{n-1} \setminus H$, let $y \notin H$. We argue as before. There are two possibilities. First, if $y_1 \geq b$, then $y_1 + (t + \mu)\nu_1 \geq b$ also. Therefore, $u(0, y) \geq \tau$ and $u^{t+\mu}(0, y) \geq \tau$, and we conclude that $d^{t+\mu}(y) \geq \varepsilon$. The other possibility is that $y_1 \leq a$. In this case, we have $u(0, y) \leq -\tau$, and since $y \notin H$, then $(u^{t+\mu} - u)(0, y) \leq 0$. Therefore $u^{t+\mu}(0, y) \leq u(0, y) \leq -\tau$, and we conclude again $d^{t+\mu}(y) \geq \varepsilon$. Lemma 2.9 gives that $u^{t+\mu} - u > 0$ in $\overline{\mathbb{R}_+^n}$.

Hence, we have proven that u is a nondecreasing function in every direction $\nu = (0, \nu_1, \ldots, \nu_n)$ with $\nu_1 > 0$. That is, we have $\partial_\nu u \geq 0$. Letting ν_1 decrease to 0, we deduce that $\partial_\xi u \geq 0$ for every direction $\xi = (0, 0, \nu_2, \ldots, \nu_n)$. In particular, considering the direction $(0, 0, -\nu_2, \ldots, -\nu_n)$, we also have $\partial_{-\xi} u \geq 0$. Therefore, $\partial_\xi u \equiv 0$ for every $\xi = (0, 0, \nu_2, \ldots, \nu_n)$, and this implies that u is a function of x and y_1 alone. In addition, we have proven that $u_{y_1} \geq 0$ and hence, since u is not identically constant, $u_{y_1} > 0$.

Finally, to establish that u is unique up to translations in the y_1-variable, we use Lemma 5.2. Take two solutions u_1 and u_2 as in the theorem. We already know that they are functions of x and y_1 alone. Slide them so that $u_1(0, 0) = u_2(0, 0) = 0$. Now, letting $t \searrow 0$ in the conclusion of Lemma 5.2, we obtain $u_1 \leq u_2$ in $\mathbb{R}_+^2$. Interchanging u_1 and u_2, we conclude $u_1 \equiv u_2$. $\square$

6 Existence and Properties of Two-Dimensional Solutions: Proof of Theorems 1.2, 1.3, and 1.6

In this section we always consider $n = 2$. First we prove the Modica-type estimate and the Hamiltonian conserved quantity of Theorem 1.3. The argument on its proof establishes the necessary condition $G(-1) = G(1)$ for existence of a layer solution. This is an alternative proof to the variational one presented in Proposition 3.2. More importantly, the Modica-type estimate leads to the strict inequality $G > G(\pm 1)$ in $(-1, 1)$ in the presence of a layer solution. This is a part of the necessary and sufficient condition of our main result, Theorem 1.2(a).

The next and main part of the section is dedicated to proving the existence of a layer solution, as stated in Theorem 1.2(a). The proof is based on the monotonicity and variational properties of an appropriate sequence of solutions in larger and larger half-balls (see Lemma 6.2 below).

This section also contains the proof of the decay estimates at infinity for two-dimensional layer solutions, Theorem 1.6.

We also include the following result, where we have collected different assumptions on G that guarantee for $n = 2$ that a local minimizer, a solution with limits, or a stable solution is necessarily a layer solution.

PROPOSITION 6.1 *Let $n = 2$ and u be a function such that*

$$|u| < 1 \quad in \ \overline{\mathbb{R}^2_+}.$$

(a) *Assume that $G > G(-1) = G(1)$ in $(-1, 1)$, and let u be a local minimizer of problem (1.1) relative to perturbations in $[-1, 1]$. Then, either $u = u(x, y)$ or $v = v(x, y) = u(x, -y)$ is a layer solution of (1.1).*

(b) *Assume $G''(\pm 1) > 0$, and let u be a solution of (1.1) with*

$$\lim_{y \to \pm\infty} u(0, y) = \pm 1.$$

Then u is a layer solution of (1.1).

(c) *Assume that G satisfies*

> *if $-1 \leq L^- < L^+ \leq 1$, $G'(L^\pm) = 0$, and $G > G(L^-) = G(L^+)$*
> *in (L^-, L^+), then $L^- = -1$ and $L^+ = 1$.*

Let u be a nonconstant stable solution of (1.1). Then either $u = u(x, y)$ or $v = v(x, y) = u(x, -y)$ is a layer solution of (1.1).

Note that an identically constant function $u \equiv s$ is a stable solution of (1.1) if and only if $G'(s) = 0$ and $G''(s) \geq 0$. This follows easily from definition (1.8) of stability. Therefore, regarding part (c) of the proposition, a way to guarantee that a stable solution u is nonconstant is that $u = s \in (-1, 1)$ at some point, and either $G'(s) \neq 0$ or $G''(s) < 0$.

This section is organized as follows: In Section 6.1, we prove Theorem 1.3, establishing the Modica-type estimate (1.10). Section 6.2 contains the proof of our main result on existence of a layer solution, namely Theorem 1.2(a), and also of Proposition 6.1. Finally, Section 6.3 is devoted to the proof of the decay and asymptotic estimates stated in Theorem 1.6.

6.1 A Modica-Type Estimate: Proof of Theorem 1.3

PROOF OF THEOREM 1.3: Let $n = 2$ and u be a layer solution. By interior gradient estimates applied to the harmonic function u in the ball $B_t(t, y) \subset \mathbb{R}^2_+$, we have that

$$(6.1) \qquad |\nabla u(t, y)| \leq \frac{C}{t} \|u\|_{L^\infty} \leq \frac{C}{t} \quad \text{for all } t > 0, \ y \in \mathbb{R}.$$

Since, in addition, ∇u is bounded in all $\mathbb{R}^2_+$ (see Lemma 2.3(b)), we deduce

$$(6.2) \qquad |\nabla u(t, y)| \leq \frac{C}{1+t} \quad \text{for all } t > 0, \ y \in \mathbb{R}.$$

We use the same argument applied now to the partial derivatives of u (instead of u). Recall that $D^2 u$ is bounded in all $\mathbb{R}^2_+$ (again by Lemma 2.3(b)). Using (6.2) to control $t^{-1} \|\nabla u\|_{L^\infty}$, we conclude

$$(6.3) \qquad \|D^2 u(t, y)\| \leq \frac{C}{1+t^2} \quad \text{for all } t > 0, \ y \in \mathbb{R}.$$

We start establishing equality (1.11). Consider the function

$$(6.4) \qquad v(y) = \int_0^{+\infty} \frac{1}{2}\{u_y^2(t, y) - u_x^2(t, y)\}dt,$$

which is well-defined by (6.2). Note that

$$\partial_y \frac{(u_y^2 - u_x^2)(t, y)}{2} = (u_{yy}u_y - u_{xy}u_x)(t, y),$$

which is bounded in absolute value by $C/(1 + t^3)$ from (6.2) and (6.3). Hence, using the intermediate value property to estimate $\{v(y + h) - v(y)\}/h$ and the dominated convergence theorem, we see that (6.4) can be differentiated under the integral sign. Therefore,

$$\frac{d}{dy}v(y) = \int_0^{+\infty} (u_{yy}u_y - u_{xy}u_x)(t, y)dt$$

$$= \int_0^{+\infty} (-u_{xx}u_y - u_{xy}u_x)(t, y)dt = (u_x u_y)(0, y) = \frac{d}{dy}G(u(0, y)),$$

where we have integrated by parts in the second integral.

Hence, the function $v(y) - \{G(u(0, y)) - G(1)\}$ is constant in y. We need to show that this function is identically 0, and hence we simply look at its limits at $\pm\infty$.

For this, let $R > 0$ and use (2.14) to deduce that

$$\overline{\lim_{y \to \pm\infty}} |v(y)| = \overline{\lim_{y \to \pm\infty}} \left| \int_R^{+\infty} \frac{1}{2}\{u_y^2(t, y) - u_x^2(t, y)\}dt \right|.$$

Now, by (6.2), this last integral is bounded by C/R. Letting $R \to \infty$ we conclude that $v \to 0$ as $y \to \pm\infty$. Therefore, $v - \{G(u(0, \cdot)) - G(1)\} \equiv 0 \equiv G(1) - G(-1)$. We conclude equality (1.11) and, at the same time, the necessary condition

$$(6.5) \qquad\qquad\qquad G(-1) = G(1)$$

for the existence of a layer solution.

Note that the previous argument gives an alternative proof of (6.5), a property of G that we have already established in the paper by variational methods. Indeed, by Theorem 1.4 we know that u is a local minimizer relative to perturbations in $[-1, 1]$, and then (6.5) follows from Proposition 3.2.

Next, to prove estimate (1.10) we essentially use equality (6.5) and the maximum principle, as follows: Consider the harmonic function $(u_y^2 - u_x^2)/2$, and define the function

$$w(x, y) = \int_0^x \frac{1}{2}\{u_y^2(t, y) - u_x^2(t, y)\}dt,$$

which is bounded in all $\mathbb{R}_+^2$ by (6.2).

Note that $(\Delta w)_x = 0$, and hence Δw depends only on y. It is enough then to compute Δw on $\{x = 0\}$. We obtain $\Delta w(x, y) = w_{xx}(0, y) = (u_y u_{xy} - u_x u_{xx})(0, y)$. But we observe that

$$\frac{d^2}{dy^2} G(u(0, y)) = \frac{d}{dy}(u_x(0, y) u_y(0, y))$$
$$= (u_{xy} u_y + u_x u_{yy})(0, y).$$

Hence, the function $G(u(0, y)) - G(1) - w(x, y)$ is a bounded harmonic function in $\mathbb{R}_+^2$, and its restriction to $\{x = 0\}$ is $G(u(0, y)) - G(1)$. We need to show that

$$G(u(0, y)) - G(1) - w(x, y) > 0 \quad \text{in } \overline{\mathbb{R}_+^2}.$$

We argue by contradiction and assume that $G(u(0, y)) - G(1) - w(x, y)$ is nonpositive at some point in $\overline{\mathbb{R}_+^2}$. Since the harmonic function $G(u(0, y)) - G(1) - w(x, y)$ is bounded in $\mathbb{R}_+^2$, it then follows that $G(u(0, y)) - G(1) - w(x, y)$ is nonpositive somewhere in $\{x = 0\}$. That is, $G(u(0, y_0)) - G(1) \leq 0$ for some y_0.

Since $G(-1) = G(1)$, we deduce that $G(u(0, y)) - G(1) \to 0$ as $y \to \pm\infty$. Hence we may assume that y_0 is a global minimum of $G(u(0, y)) - G(1)$. Then

$$0 = \frac{d}{dy} G(u(0, y)) \Big|_{y=y_0} = -f(u(0, y_0)) u_y(0, y_0),$$

and therefore

(6.6) $$0 = -f(u(0, y_0)) = u_x(0, y_0).$$

On the other hand, since y_0 is a global minimum for $G(u(0, y)) - G(1)$, it is also a global minimum of its bounded harmonic extension, namely, $G(u(0, y)) - G(1) - w(x, y)$. Using (6.6), we conclude

$$0 \geq -\partial_x \{G(u(0, y_0)) - w(x, y_0)\}\big|_{x=0} = \partial_x w(x, y_0)\big|_{x=0} =$$
$$\frac{1}{2}\{u_y^2(0, y_0) - u_x^2(0, y_0)\} = \frac{1}{2} u_y^2(0, y_0) > 0,$$

a contradiction. $\square$

6.2 Existence of a Layer Solution: Proofs of Theorem 1.2 and Proposition 6.1

In this section we prove the existence of a layer solution in dimension 2 whenever the nonlinearity satisfies the conditions of Theorem 1.2(a). This existence result is contained in the following key lemma, whose statement explains (at least partially) how the layer solution is constructed.

LEMMA 6.2 *Assume that* $n = 2$ *and that*

$$G'(-1) = G'(1) = 0 \quad \text{and} \quad G > G(-1) = G(1) \quad \text{in } (-1, 1).$$

Then, for every $R > 0$, there exists a function $u_R \in C^1(\overline{B_R^+})$ such that

$$-1 < u_R < 1 \quad in \ \overline{B_R^+},$$
$$u_R(0) = 0,$$
$$\partial_y u_R \geq 0 \quad in \ B_R^+,$$

and u_R is a minimizer of the energy in B_R^+ in the sense that

$$E_{B_R^+}(u_R) \leq E_{B_R^+}(u_R + \psi)$$

for every $\psi \in H^1(\overline{B_R^+})$ with compact support in $B_R^+ \cup \Gamma_R^0$ and such that $-1 \leq u_R + \psi \leq 1$ in B_R^+. Moreover, as a consequence of the previous statements, we will deduce that a subsequence of (u_R) converges in $C^2_{loc}(\mathbb{R}^2_+)$ to a layer solution u of (1.1).

PROOF: For $R > 0$, let

$$Q_R^+ = (0, R^{1/4}) \times (-R, R).$$

Consider the function

$$v^R(x, y) = v^R(y) = \frac{\arctan y}{\arctan R} \quad for \ (x, y) \in \overline{Q_R^+}.$$

Note that $-1 \leq v^R \leq 1$ in Q_R^+.

Let u^R be an absolute minimizer of $E_{Q_R^+}$ in the set of functions $v \in H^1(Q_R^+)$ such that $|v| \leq 1$ in Q_R^+ and $v \equiv v^R$ on $\partial^+ Q_R^+$ in the weak sense; recall (2.30) and (2.31) for the definition of ∂^0 and ∂^+. Since we are assuming $G'(-1) = G'(1) = 0$, the existence of such a minimizer was proven in Lemma 2.10. We have that u^R is a weak solution of

$$\begin{cases} \Delta u^R = 0 & in \ Q_R^+ \\ \dfrac{\partial u^R}{\partial \nu} = f(u^R) & on \ \partial^0 Q_R^+ \\ u^R = v^R & on \ \partial^+ Q_R^+, \end{cases}$$

and, by the strong maximum principle,

$$|u^R| < 1 \quad in \ Q_R^+.$$

We proceed in three steps. First we show that, for $R > 1$,

(6.7)   Claim 1: $E_{Q_R^+}(u^R) \leq CR^{1/4}$

for some constant C independent of R. Here we take $G - G(1) = G - G(-1)$ as boundary energy potential. We will use this energy bound to prove in a second step that, for R large enough,

(6.8)   Claim 2: $\left| \left\{ u^R(0, y) > \dfrac{1}{2} \right\} \right| \geq R^{3/4}$ and $\left| \left\{ u^R(0, y) < -\dfrac{1}{2} \right\} \right| \geq R^{3/4}$.

Finally, in a third step independent of the two previous ones, we prove that

(6.9)   Claim 3: $u_y^R = \partial_y u^R \geq 0$ in Q_R^+.

With the above three claims, we can easily finish the proof of the lemma, as follows. Since $u^R(0, y)$ is *nondecreasing* and continuous in $(-R, R)$, we deduce from (6.8) that for R large enough,

$$u^R(0, y_R) = 0 \quad \text{for some } y_R \text{ such that } |y_R| \leq R - R^{3/4}.$$

Since $|y_R| \leq R - R^{3/4} < R - R^{1/4}$, we have that

$$\overline{B_{R^{1/4}}^+}(0, y_R) \subset [0, R^{1/4}] \times (-R, R) \subset \overline{Q_R^+}.$$

We slide u^R and define

$$u_{R^{1/4}}(x, y) = u^R(x, y + y_R) \quad \text{for } (x, y) \in \overline{B_{R^{1/4}}^+}(0, 0).$$

Then, relabeling the index by setting $S = R^{1/4}$, we have that $u_S \in C^1(\overline{B_S^+}(0, 0))$, $-1 < u_S < 1$ in $\overline{B_S^+}(0, 0)$, $u_S(0) = 0$, and $\partial_y u_S \geq 0$ in $B_S^+(0, 0)$. Moreover, u_S is a minimizer in $B_S^+(0, 0)$ in the sense of Lemma 6.2. This follows from extending a given H^1 function ψ with compact support in $(B_S^+ \cup \Gamma_S^0)(0, y_R)$, and with $|u + \psi| \leq 1$ in $B_S^+(0, y_R)$, by 0 in $Q_R^+ \setminus B_S^+(0, y_R)$. Hence ψ is a $H^1(Q_R^+)$ function. Then one uses the minimality of u^R in Q_R^+ and the fact that the energies of u^R and $u^R + \psi$ coincide in $Q_R^+ \setminus B_S^+(0, y_R)$ to deduce the desired relation between the energies in $B_S^+(0, y_R)$.

Now we prove the last statement of the lemma: a subsequence of (u_R) converges to a layer solution. Note that we use the sequence (u_R) just constructed, and not the sequence (u^R) in the beginning of the proof.

Let $S > 0$. Since $|u_R| < 1$, Lemma 2.3 gives $C^{2,\alpha}(\overline{B_S^+})$ estimates for u_R, uniform for $R \geq 4S$. Hence, for a subsequence (that we still denote by u_R), we have that u_R converges as $R \to \infty$ in $C^2_{\text{loc}}(\overline{\mathbb{R}_+^2})$ to some function $u \in C^{2,\alpha}_{\text{loc}}(\overline{\mathbb{R}_+^2})$. Automatically, u is a solution of (1.1), $|u| \leq 1$,

$$u(0) = 0 \quad \text{and} \quad u_y \geq 0 \quad \text{in } \mathbb{R}_+^2.$$

Since $u(0) = 0$, we have $|u| \not\equiv 1$ and hence $|u| < 1$ by the strong maximum principle. Note that ± 1 are solutions of the problem since, by hypothesis, $G'(\pm 1) = f(\pm 1) = 0$.

Let us now show that u is a local minimizer relative to perturbations in $[-1, 1]$. Indeed, let $S > 0$ and ψ be a C^1 function with compact support in $B_S^+ \cup \Gamma_S^0$ and such that $|u + \psi| \leq 1$ in B_S^+. Extend ψ to be identically 0 outside B_S^+ so that $\psi \in H^1_{\text{loc}}(\overline{\mathbb{R}_+^2})$. Note that, since $-1 < u < 1$ and $-1 \leq u + \psi \leq 1$, we have that $-1 < u + (1 - \varepsilon)\psi < 1$ in $\overline{B_S^+}$ for every $0 < \varepsilon < 1$. Hence, by the local convergence of (u_R) towards u, for R large enough we have $-1 \leq u_R + (1-\varepsilon)\psi \leq 1$ in B_S^+, and hence also in B_R^+. Then, since u_R is a minimizer in B_R^+, we have

$E_{B_R^+}(u_R) \leq E_{B_R^+}(u_R + (1 - \varepsilon)\psi)$ for R large. Since ψ has support in $B_S^+ \cup \Gamma_S^0$, this is equivalent to

$$E_{B_S^+}(u_R) \leq E_{B_S^+}(u_R + (1 - \varepsilon)\psi) \quad \text{for } R \text{ large.}$$

Letting $R \to \infty$, we deduce $E_{B_S^+}(u) \leq E_{B_S^+}(u + (1 - \varepsilon)\psi)$. We conclude now by letting $\varepsilon \to 0$.

Finally, since $u_y \geq 0$, the limits $L^{\pm} = \lim_{y \to \pm \infty} u(0, y)$ exist. To establish that u is a layer solution, it remains only to prove that $L^{\pm} = \pm 1$. For this, note that we can apply Proposition 3.2 to u, a local minimizer relative to perturbations in $[-1, 1]$, and deduce that

$$G \geq G(L^-) = G(L^+) \quad \text{in } [-1, 1].$$

Since $G > G(-1) = G(1)$ in $(-1, 1)$, we infer that $|L^{\pm}| = 1$. Since $u(0) = 0$, u cannot be identically 1 or -1. We conclude that $L^- = -1$ and $L^+ = 1$, and therefore u is a layer solution.

We now go back to the functions u^R defined in the beginning of the proof and proceed to establish the three claims made above.

Step 1. Here we prove (6.7) for $R > 1$ and for some constant C independent of R. We take $G - G(1) = G - G(-1)$ as the boundary energy potential.

Since $E_{Q_R^+}(u^R) \leq E_{Q_R^+}(v^R)$, we simply need to bound the energy of v^R. We have

$$|\nabla v^R| = |\partial_y v^R| = \frac{1}{\arctan R} \frac{1}{1 + y^2} \leq C \frac{1}{1 + y^2},$$

and hence

$$\int_{Q_R^+} |\nabla v^R|^2 \leq C R^{1/4} \int_{-R}^{R} \frac{dy}{(1 + y^2)^2} \leq C R^{1/4}.$$

Next, since $G'(-1) = G'(1) = 0$ and $G(-1) = G(1)$, we have that

$$G(s) - G(1) \leq C(1 + \cos(\pi s)) \quad \text{for all } s \in [-1, 1]$$

for some constant $C > 0$. Therefore, using that $\pi/(\arctan R) > 2$, we have

$$G(v^R(0, y)) - G(1) \leq C\left\{1 + \cos\left(\pi \frac{\arctan y}{\arctan R}\right)\right\}$$

$$\leq C\left(1 + \cos(2 \arctan y)\right) = C 2 \cos^2(\arctan y) = \frac{2C}{1 + y^2}.$$

We conclude that

$$\int_{-R}^{R} \{G(v^R(0, y)) - G(1)\} \, dy \leq C \int_{-R}^{R} \frac{dy}{1 + y^2} \leq C.$$

This, together with the above bound for the Dirichlet energy, proves (6.7).

Step 2. Here we prove (6.8) for R large enough.

Since $u^R \equiv v^R$ on $\{x = R^{1/4}\}$ and $\int_{-R}^{R} v^R(y)\, dy = 0$, we have

$$\int_{-R}^{R} u^R(0, y)dy = \int_{-R}^{R} u^R(0, y)dy - \int_{-R}^{R} u^R(R^{1/4}, y)dy = -\int_{Q_R^+} u_x^R.$$

Hence, using the energy bound (6.7) and the hypothesis that $G - G(1) \geq 0$, we have

$$\left| \int_{-R}^{R} u^R(0, y)dy \right| \leq \int_{Q_R^+} |u_x^R| \leq \left\{ |Q_R^+| \cdot \int_{Q_R^+} |\nabla u^R|^2 \right\}^{1/2}$$

(6.10)

$$\leq C\{RR^{1/4}R^{1/4}\}^{1/2} = CR^{3/4}.$$

Next, by (6.7) we know that $\int_{-R}^{R}\{G(u^R(0, y)) - G(1)\}dy \leq CR^{1/4} \leq CR^{3/4}$. On the other hand, $G(s) - G(1) \geq \varepsilon > 0$ if $s \in [-\frac{1}{2}, \frac{1}{2}]$, for some $\varepsilon > 0$ independent of R. Moreover, $G - G(1) \geq 0$ in $(-1, 1)$. We deduce

$$\varepsilon \left| \left\{ |u^R(0, y)| \leq \frac{1}{2} \right\} \right| \leq \int_{-R}^{R} \{G(u^R(0, y)) - G(1)\}dy \leq CR^{3/4},$$

and therefore $|\{|u^R(0, y)| \leq \frac{1}{2}\}| \leq CR^{3/4}$. This combined with (6.10) leads to

(6.11)

$$\left| \int_{(-R,R)\cap\{|u^R(0,y)|>1/2\}} u^R(0, y)dy \right| \leq CR^{3/4}.$$

We claim that

$$\left| \left\{ u^R(0, y) > \frac{1}{2} \right\} \right| \geq R^{3/4} \quad \text{for } R \text{ large enough.}$$

Suppose not. Then, using (6.11) and $|\{u^R(0, y) > 1/2\}| \leq R^{3/4}$, we obtain

$$\frac{1}{2} \left| \left\{ u^R(0, y) < -\frac{1}{2} \right\} \right| \leq \left| \int_{(-R,R)\cap\{u^R(0,y)<-1/2\}} u^R(0, y)dy \right| \leq CR^{3/4}.$$

Hence, all three sets $\{|u^R(0, y)| \leq \frac{1}{2}\}$, $\{u^R(0, y) > \frac{1}{2}\}$, and $\{u^R(0, y) < -\frac{1}{2}\}$ would have length smaller than $CR^{3/4}$. This is a contradiction for R large, since these sets fill $(-R, R)$.

Step 3. Here we establish (6.9). We prove this monotonicity result using the sliding method. There are, however, three other ways to obtain the same result, which use different methods. Because of their independent interest, we present these three alternative proofs below.

To use the sliding method, extend u^R to be identically 1 on $(0, R^{1/4}) \times [R, +\infty)$. For $t > 0$, consider

(6.12) $$u^{R,t}(x, y) = u^R(x, y + t) \quad \text{for } (x, y) \in \overline{Q_R^+}.$$

For $0 < \varepsilon < 1$, let
$$Q_{R,\varepsilon}^+ = (0, R^{1/4}) \times (-R, R - \varepsilon).$$
With ε fixed, we are going to prove that

(6.13) $\qquad\qquad u^R \leq u^{R,t} \quad \text{in } \overline{Q_{R,\varepsilon}^+} \text{ for every } t \geq \varepsilon.$

Then, given $(x, y) \in Q_R^+$, we have $(x, y) \in Q_{R,\varepsilon}^+$ for every ε small enough. From (6.13) applied with $t = \varepsilon$, we obtain $u^R(x, y) \leq u^{R,\varepsilon}(x, y)$ for every small $\varepsilon > 0$. Letting ε decrease to 0, we deduce that $u_y^R(x, y) \geq 0$, as claimed.

To establish (6.13), note first that by Lemma 2.11(b), we know that u^R is a continuous function in $\overline{Q_R^+}$. Hence, u^R and $u^{R,t}$ are continuous in $\overline{Q_{R,\varepsilon}^+}$ for all $t \geq \varepsilon$. We also know that $-1 < u^R < 1$ in $[0, R^{1/4}] \times (-R, R)$. Hence, using also the structure of the Dirichlet boundary value v^R, it is easy to check that

(6.14) $\qquad\qquad u^R < u^{R,t} \quad \text{on } \partial^+ Q_{R,\varepsilon}^+ \text{ for every } t \geq \varepsilon.$

Since $u^{R,t} \equiv 1$ in $Q_{R,\varepsilon}^+$ for t large, we have that (6.13) holds for t large enough. We now consider the set of t's such that $t \geq \varepsilon$ and (6.13) holds. This is clearly a closed set. We only need to show that it is also open. For this, assume that $u^R \leq u^{R,t_0}$ in $\overline{Q_{R,\varepsilon}^+}$ for some $t_0 \geq \varepsilon$.

Suppose that $u^R = u^{R,t_0}$ at some point $(x_0, y_0) \in \overline{Q_{R,\varepsilon}^+}$. Then, by (6.14), $(x_0, y_0) \in Q_{R,\varepsilon}^+ \cup \partial^0 Q_{R,\varepsilon}^+$ and, in particular, $u^{R,t_0}(x_0, y_0) = u^R(x_0, y_0) \in (-1, 1)$. Hence both u^R and u^{R,t_0} are solutions of the same Neumann problem in a neighborhood in $\overline{\mathbb{R}_+^2}$ of (x_0, y_0), a point where they agree, and $u^R \leq u^{R,t_0}$. Hence, they must agree everywhere, which contradicts (6.14). We conclude that $u^R < u^{R,t_0}$ in $\overline{Q_{R,\varepsilon}^+}$. Hence, by continuity, the same inequality is true for every $t \geq \varepsilon$ in a neighborhood of t_0.

This concludes the proof of Step 3 and of the lemma. $\qquad\qquad\square$

In the case of our problem in the rectangle, the monotonicity in y of the minimizer u^R can be proven in three other ways, which we present next. Each proof uses a different method. For other nonlinear problems it may happen that only some of the four methods can be applied.

The second proof is based on the stability of the minimizer, as follows: Since u^R is an absolute minimizer, we know that

(6.15) $\qquad Q(\xi) = \int_{Q_R^+} |\nabla \xi|^2 - \int_{-R}^R f'(u^R(0, y))\xi^2(0, y)dy \geq 0$

for all $\xi \in H^1(Q_R^+)$ with $\xi \equiv 0$ on $\partial^+ Q_R^+$ in the weak sense; see Section 2.5 and (2.36). By Lemma 2.11(b), we know that $u^R \in H^2(Q_R^+) \cap C^1(\overline{Q_R^+})$. Using that the Dirichlet boundary value v^R is increasing in y and also Hopf's boundary lemma, we have that $u_y^R \in H^1(Q_R^+) \cap C(\overline{Q_R^+})$ satisfies $u_y^R > 0$ on $\partial^+ Q_R^+ \cap \{x > 0\}$. Therefore, the negative part of u_y^R, $(u_y^R)^- \geq 0$, can be approximated in H^1 by

C^1 functions with compact support in $Q_R^+ \cup \{x = 0, -R < y < R\}$, and hence it is an admissible test function on (6.15). Since $(u_y^R)^-$ is harmonic on its support, we have

$$Q((u_y^R)^-) = \int_{Q_R^+} |\nabla(u_y^R)^-|^2 - \int_{-R}^R f'(u^R(0, y))((u_y^R)^-)^2(0, y)dy$$

$$= \int_{-R}^R \{[-\partial_x(u_y^R)^- - f'(u^R)(u_y^R)^-](u_y^R)^-\}(0, y)dy = 0,$$

since $-\partial_x(u_y^R)^- = f'(u^R)(u_y^R)^-$ on $\{x = 0\} \cap \{u_y^R < 0\}$, an open set of $\{x = 0\}$.

We conclude that $(u_y^R)^-$ minimizes the quadratic form Q on the space of functions vanishing on $\partial^+ Q_R^+$. Therefore, if $(u_y^R)^-$ is not identically 0, then it must be the first eigenfunction of the linearized problem at u^R (see the proof of Lemma 4.1 for this type of argument). In particular, $(u_y^R)^-$ is harmonic in all Q_R^+. But $(u_y^R)^-$ vanishes in a neighborhood of $\partial^+ Q_R^+ \cap \{x > 0\}$. Therefore $(u_y^R)^- \equiv 0$ in Q_R^+, and hence $u_y^R \geq 0$ in Q_R^+.

The third proof of monotonicity is variational and has been employed in the literature for other problems. The key idea is that, under appropriate boundary conditions, the graphs of two minimizers of the same problem cannot intersect each other. The details go as follows: Since $u^R - v^R$ can be approximated in $H^1(Q_R^+)$ by C^1 functions with compact support in $Q_R^+ \cup \partial^0 Q_R^+$, we deduce (here no global regularity result is needed) that, extending u^R by 1 for $y \geq R$ and by -1 for $y \leq -R$, we have $u^R \in H^1_{loc}((0, R^{1/4}) \times \mathbb{R})$. We now define $u^{R,t}$ by (6.12) for every $t > 0$. Consider the H^1_{loc} functions

$$\underline{w} = \min(u^R, u^{R,t}) \quad \text{and} \quad \overline{w} = \max(u^R, u^{R,t}).$$

Recall that $-1 \leq u^R \leq 1$ a.e. and that v^R is increasing in y. It follows that $\underline{w} = v^R$ on $\partial^+ Q_R^+$ in the weak sense and that $\overline{w} = v^{R,t}$ on $\partial^+\{(0, -t) + Q_R^+\}$ in the weak sense. Since $u^{R,t}$ is a minimizer in $(0, -t) + Q_R^+$ with $v^{R,t}$ as Dirichlet boundary value, we deduce that

$$E_{(0,-t)+Q_R^+}(u^{R,t}) \leq E_{(0,-t)+Q_R^+}(\overline{w}).$$

This is equivalent to $E_{\{u^R > u^{R,t}\}}(u^{R,t}) \leq E_{\{u^R > u^{R,t}\}}(u^R)$. But this implies that

$$E_{Q_R^+}(\underline{w}) \leq E_{Q_R^+}(u^R),$$

and hence both u^R and $\underline{w}$ are absolute minimizers in Q_R^+ with v^R as the Dirichlet boundary value. Now, the interior regularity of any minimizer together with $\underline{w} \leq u^R$ leads to $\underline{w} \equiv u^R$. That is, $u^R \leq u^{R,t}$. Therefore, u^R is monotone in y.

A fourth proof of the monotonicity result can be given using the moving plane method. As pointed out in [12], it is possible for some problems to apply both the sliding and the moving plane method, yielding the same result. This is our case. We simply recall that with the moving plane method one establishes $u^R \leq u^{\lambda,R}$ in

$[0, R^{1/4}] \times [-R, \lambda]$, where here $u^{\lambda, R}$ is the even reflection of u^R across $\{y = \lambda\}$ (after extending u^R by 1 for $y \geq R$). One starts with λ close to $-R$ and goes all the way up to $\lambda = R$. To deal with the corners, one needs to use the improved version of the moving plane method due to Berestycki and Nirenberg [12], based on maximum principles in small domains (adapted here to Neumann-Dirichlet problems). See [18] for an application of the moving plane method to positive solutions of "ground state" type for a Neumann-Dirichlet problem.

We now present the following proof:

PROOF OF THEOREM 1.2: The existence part of statement (a) follows from Lemma 6.2. Now we prove the necessary conditions on G. First, by Lemma 2.4, we have that $G'(\pm 1) = -f(\pm 1) = 0$. Next, by Theorem 1.3, we have that $G(-1) = G(1)$ and, taking $x = 0$ in (1.10), that $G > G(1)$ in $(-1, 1)$.

Let us now prove part (b), i.e., that problem (1.1) admits at most one layer solution up to translations in the y-variable under hypothesis $f'(\pm 1) < 0$. Take two layer solutions u_1 and u_2, and slide them so that (abusing notation) $u_1(0, 0) = u_2(0, 0) = 0$. Now, let $t \searrow 0$ in the conclusion of Lemma 5.2 to obtain $u_1 \leq u_2$ in $\mathbb{R}_+^2$. Interchanging u_1 and u_2, we conclude $u_1 \equiv u_2$.

Finally, to prove statement (c) of Theorem 1.2 concerning odd nonlinearities f, simply note that if u satisfies (1.1), $|u| \leq 1$, and $\lim_{y \to \pm \infty} u(0, y) = \pm 1$, then $v(x, y) = -u(x, -y)$ also satisfies these three conditions, since f is odd. Hence, by the uniqueness result above, $-u(x, -y) = v(x, y) = u(x, y + a)$ for some $a \in \mathbb{R}$. Replacing y by $y - a/2$, we obtain $u(x, y + a/2) = -u(x, -y + a/2)$, as stated. $\square$

PROOF OF PROPOSITION 6.1: To prove part (a), let u be a local minimizer relative to perturbations in $[-1, 1]$. It follows that u is a stable solution and hence, by Theorem 1.5(a), u is either identically constant, increasing in y, or decreasing in y. In particular, the limits $\lim_{y \to \pm \infty} u(0, y) = L^{\pm}$ exist. Now we can apply Proposition 3.2 to u and deduce that

$$G \geq G(L^-) = G(L^+) \quad \text{in } [-1, 1].$$

Since by hypothesis $G > G(-1) = G(1)$ in $(-1, 1)$, we infer that $|L^{\pm}| = 1$. Since $|u| < 1$ by hypothesis, we conclude that $\{L^-, L^+\} = \{-1, 1\}$. Therefore, either u or $u(x, -y)$ is a layer solution.

Statement (b), i.e., that every solution as in Proposition 6.1(b) is increasing in y, follows by taking $u_1 = u_2 = u$ in Lemma 5.2 after a translation in y to have $u(0, 0) = 0$. Its conclusion ($u < u^t$ for $t > 0$) leads to $u_y \geq 0$. Since u has limits ± 1, u is not identically constant, and hence we conclude $u_y > 0$.

Finally, we prove (c). Let u be a nonconstant, stable solution. By Theorem 1.5(a), u is either increasing or decreasing in y. Hence, the limits $L^{\pm} = \lim_{y \to \pm \infty} u(0, y)$ exist. Up to a change of u by $u(x, -y)$ (in case u is decreasing in y), we may assume that $L^- < L^+$.

Now note that $\{u - (L^- + L^+)/2\}/\{(L^+ - L^-)/2\}$ is a layer solution for a new nonlinearity. Using Theorem 1.2(a) and restating the conclusion for u, we have that $G'(L^\pm) = 0$ and $G > G(L^-) = G(L^+)$ in (L^-, L^+). Finally, the hypothesis made on G implies that $L^\pm = \pm 1$, and hence u is a layer solution. $\qquad\square$

6.3 Decay Estimates: Proof of Theorem 1.6

We now prove the precise decay estimates for layer solutions in dimension 2 when $f'(\pm 1) < 0$, as stated in Theorem 1.6. The proof uses the explicit solutions presented in Section 2.1.

PROOF OF THEOREM 1.6: Assume $n = 2$ and $f'(\pm 1) < 0$, and let u be a layer solution of (1.1). Set $r = \sqrt{x^2 + y^2}$.

We start with the upper bound for u_y. We have

$$\begin{cases} \Delta u_y = 0 & \text{in } \mathbb{R}_+^2 \\ -(u_y)_x = f'(u(0, y))u_y & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Since $u(0, y) \to \pm 1$ as $y \to \pm\infty$ and $f'(\pm 1) < 0$, we can write $f'(u(0, y)) = \tilde{d}(y) - d(y)$ for some function $\tilde{d}$ with compact support and some function $d(y) \geq \varepsilon$, where $\varepsilon > 0$. Then the function $\tilde{d}(y)u_y(0, y)$ has also compact support, and hence it can be bounded in the form

$$\tilde{d}(y)u_y(0, y) \leq K\phi_y^{2/\varepsilon}(0, y)$$

for some $K > 0$ (recall that the functions ϕ^a were defined in Lemma 2.1).

Note that the boundary inequality in (2.2) can be rewritten as $-(a\phi_y^a)_x + 2\phi_y^a \geq \phi_y^a$. Using this, we obtain the following inequalities, where all the functions are evaluated at points $(0, y)$ on $\partial\mathbb{R}_+^2$:

$$\begin{aligned} -(u_y)_x + d(y)u_y = \tilde{d}(y)u_y \leq K\phi_y^{2/\varepsilon} &\leq K\left\{ -\left(\frac{2}{\varepsilon}\phi_y^{2/\varepsilon}\right)_x + 2\phi_y^{2/\varepsilon} \right\} \\ &= -\left(\frac{2K}{\varepsilon}\phi_y^{2/\varepsilon}\right)_x + \varepsilon\frac{2K}{\varepsilon}\phi_y^{2/\varepsilon} \\ &\leq -\left(\frac{2K}{\varepsilon}\phi_y^{2/\varepsilon}\right)_x + d(y)\frac{2K}{\varepsilon}\phi_y^{2/\varepsilon}. \end{aligned}$$

Looking only at the first and last terms in this chain of inequalities and using Lemma 2.8, we conclude that

$$(6.16) \qquad u_y(x, y) \leq \frac{2K}{\varepsilon}\phi_y^{2/\varepsilon} = \frac{2K}{\varepsilon}\frac{2}{\pi}\frac{x + 2/\varepsilon}{(x + 2/\varepsilon)^2 + y^2}$$

for all $(x, y) \in \mathbb{R}_+^2$. Since $u_y > 0$, the desired bound $|u_y| \leq C/(1 + r)$ in $\mathbb{R}_+^2$ for u_y follows. In addition, evaluating (6.16) at $x = 0$, we see that

$$(6.17) \qquad u_y(0, y) \leq \frac{C}{1 + y^2} \quad \text{for all } y \in \mathbb{R},$$

for some constant $C > 0$. This is the upper bound in (1.13). Throughout the proof, c and C will denote positive constants that may change in every inequality.

Note that (6.16) could also be proven using Lemma 2.7 instead of Lemma 2.8, since we know that $u_y(0, y) \to 0$ as $y \to \pm\infty$ by (2.14). Now, to prove the lower bound on u_y, we simply note that we can interchange the roles of u_y and ϕ^a. More precisely, let $-f' \leq (2a)^{-1}$ in $(-1, 1)$ for a constant $a > 0$. We have $-(u_y)_x + (2a)^{-1} u_y \geq 0 \geq -(\phi_y^a)_x + (2a)^{-1}\phi_y^a$ for $|y|$ large, since $-f_a' \geq (2a)^{-1}$ near ± 1 (recall Lemma 2.1 for the definition of f_a). Lemma 2.7 gives that $Cu_y - \phi_y^a > 0$ in $\mathbb{R}_+^2$ if C is chosen large enough. That is,

$$(6.18) \qquad\qquad u_y(x, y) \geq c\frac{x + a}{(x + a)^2 + y^2}$$

in $\mathbb{R}_+^2$. Taking $x = 0$, we deduce the lower bound in (1.13).

It follows from (6.17) that

$$(6.19) \qquad\qquad |\pm 1 - u(0, y)| \leq \frac{C}{|y|} \quad \text{as } y \to \pm\infty,$$

and hence that

$$(6.20) \qquad\qquad |f(u(0, y))| \leq \frac{C}{1 + |y|} \quad \text{for all } y \in \mathbb{R},$$

an inequality that will be important below.

To obtain estimate (1.14) for the gradient, it remains to bound $|u_x|$. First, by (6.2) we have $|u_x| \leq C/(1 + x) \leq C/(1 + r)$ in the sector $\{|y| \leq x\}$.

To estimate $|u_x|$ in $\{|y| > x\}$, we use the maximum principle in the set $S = \{|y| > x > 0, r > 1\}$. We know that $|u_x| \leq C/x$ on $\{|y| = x\}$ by (6.1), and that $|u_x(0, y)| = |f(u(0, y))| \leq C/(1 + |y|)$ for all y by (6.20). We deduce

$$(6.21) \qquad\qquad |u_x| \leq C\frac{|y|}{x^2 + y^2} \quad \text{on } \partial S = \partial\{|y| > x > 0, r > 1\}$$

for some constant C. Since both functions u_x and $Cy/(x^2 + y^2)$ are harmonic and bounded in S (recall that $\nabla u \in L^\infty(\mathbb{R}_+^2)$ by Lemma 2.3(b)), the maximum principle implies that the inequality in (6.21) also holds in S. Here we have used Lemma 2.1 of [10] (among other possibilities); this is a version of the maximum principle for bounded subsolutions in unbounded domains (such as S) that admit an infinite open exterior cone. The proof in [10] of such a maximum principle simply uses a comparison function similar to the one used in the beginning of Section 2.4.

Hence, $|u_x| \leq C/(1 + r)$ in the sector $\{|y| > x\}$, and this concludes the proof of (1.14).

Now, estimate (1.15) for the Dirichlet energy follows from the gradient bound (1.14). Estimate (1.16) follows from bound (6.19) and the fact that $G(s) - G(1) \leq C(1 - s)^2$ (since $G'(1) = G'(-1) = 0$ by Lemma 2.4), and $G(s) - G(1) = G(s) - G(-1) \leq C(1 + s)^2$.

Next, $(u - \phi^1)^2$ is a nonnegative, bounded subharmonic function in $\mathbb{R}_+^2$ less than or equal to $C/(1 + y^2)$ on $\{x = 0\}$—since both u and ϕ^1 satisfy (6.19). Hence the maximum principle gives $(u - \phi^1)^2 \le C(x + 1)/((x + 1)^2 + y^2)$ in $\mathbb{R}_+^2$, since this last function is harmonic. Therefore

$$(6.22) \qquad \left| u(x, y) - \frac{2}{\pi} \arctan \frac{y}{x + 1} \right| = |(u - \phi^1)(x, y)| \le \frac{C}{\sqrt{r}}$$

in $\mathbb{R}_+^2$. From this, (1.17) follows immediately.

Since $u_y > 0$, every level set of u is a graph of y as a function of x. Let $\{u = s\} = \{y = \varphi^s(x), x \ge 0\}$. Evaluating (6.22) at $y = \varphi^s(x)$, we obtain (1.18).

Finally, we prove that $\{u = s\}$ is globally Lipschitz; i.e., $|(\varphi^s)'|$ is bounded. Note that at points $(x, \varphi^s(x))$ on the s-level set we have

$$|(\varphi^s)'| = \frac{|u_x|}{u_y} \le \frac{C}{(1 + r)u_y} \le C\frac{(1 + r)^2}{(1 + r)(1 + x)} \le C\frac{1 + x + |\varphi^s(x)|}{1 + x},$$

where we have used the upper bound (1.14) for the gradient in the first inequality and the lower bound (6.18) for u_y in the second one. Finally, the limit (1.18) implies that $|\varphi^s(x)|/(1+x)$ is bounded, and hence φ^s is a globally Lipschitz graph. $\qquad \square$

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