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LAYOUT OF GRAPHS WITH BOUNDED TREE-WIDTH*

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Abstract. A *queue layout* of a graph consists of a total order of the vertices, and a partition of the edges into *queues*, such that no two edges in the same queue are nested. The minimum number of queues in a queue layout of a graph is its *queue-number*. A *three-dimensional* (*straight-line grid*) *drawing* of a graph represents the vertices by points in $\mathbb{Z}^3$ and the edges by noncrossing line-segments. This paper contributes three main results:

(1) It is proved that the minimum volume of a certain type of three-dimensional drawing of a graph G is closely related to the queue-number of G. In particular, if G is an n-vertex member of a proper minor-closed family of graphs (such as a planar graph), then G has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing if and only if G has a $\mathcal{O}(1)$ queue-number.

(2) It is proved that the queue-number is bounded by the tree-width, thus resolving an open problem due to Ganley and Heath [*Discrete Appl. Math.*, 109 (2001), pp. 215–221] and disproving a conjecture of Pemmaraju [*Exploring the Powers of Stacks and Queues via Graph Layouts*, Ph. D. thesis, Virginia Polytechnic Institute and State University, Blacksburg, VA, 1992]. This result provides renewed hope for the positive resolution of a number of open problems in the theory of queue layouts.

(3) It is proved that graphs of bounded tree-width have three-dimensional drawings with $\mathcal{O}(n)$ volume. This is the most general family of graphs known to admit three-dimensional drawings with $\mathcal{O}(n)$ volume.

The proofs depend upon our results regarding *track layouts* and *tree-partitions* of graphs, which may be of independent interest.

Key words. queue layout, queue-number, three-dimensional graph drawing, tree-partition, tree-partition-width, tree-width, k-tree, track layout, track-number, acyclic coloring, acyclic chromatic number

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1. Introduction.

A *queue layout* of a graph consists of a total order of the vertices, and a partition of the edges into *queues*, such that no two edges in the same queue are nested. The dual concept of a *stack layout*, introduced by Ollmann [71] and commonly called a *book embedding*, is defined similarly, except that no two edges in the same *stack* may cross. The minimum number of queues (respectively, stacks) in a queue (stack) layout of a graph is its *queue-number* (*stack-number*). Queue layouts have been extensively studied [41, 53, 54, 58, 74, 78, 84, 86] with applications in parallel process scheduling, fault-tolerant processing, matrix computations, and sorting networks (see [74] for a survey). Queue layouts of directed acyclic graphs [9, 56, 57, 74] and posets [55, 74] have also been investigated. Our motivation for studying queue layouts is a connection with three-dimensional graph drawing.

Graph drawing is concerned with the automatic generation of aesthetically pleasing geometric representations of graphs. Graph drawing in the plane is well studied

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(see [23, 64]). Motivated by experimental evidence suggesting that displaying a graph in three dimensions is better than in two [88, 89], and applications including information visualisation [88], VLSI circuit design [66], and software engineering [90], there is a growing body of research in three-dimensional graph drawing. In this paper we study *three-dimensional straight-line grid drawings*, or *three-dimensional drawings* for short. In this model, vertices are positioned at grid-points in $\mathbb{Z}^3$, and edges are drawn as straight line-segments with no crossings [16, 20, 24, 26, 27, 42, 53, 76, 73]. We focus on the problem of producing three-dimensional drawings with small volume. Three-dimensional drawings with vertices in $\mathbb{R}^3$ have also been studied [39, 47, 18, 15, 17, 61, 21, 63, 60, 62, 68, 72]. Aesthetic criteria besides volume that have been considered include symmetry [60, 61, 62, 63], aspect ratio [18, 47], angular resolution [47, 18], edge-separation [18, 47], and convexity [17, 18, 39, 85].

The first main result of this paper (Theorem 2.10) reduces the question of whether a graph has a three-dimensional drawing with small volume to a question regarding queue layouts. In particular, we prove that every n-vertex graph from a proper minor-closed graph family $\mathcal{G}$ has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing if and only if $\mathcal{G}$ has a $\mathcal{O}(1)$ queue-number, and this result holds true when replacing $\mathcal{O}(1)$ by $\mathcal{O}(\operatorname{polylog} n)$. Consider the family of planar graphs, which are minor-closed. (In the conference version of their paper) Felsner, Liotta, and Wismath [42] asked whether every planar graph has a three-dimensional drawing with $\mathcal{O}(n)$ volume. Heath and Rosenberg [58] and Heath Leighton, and Rosenberg [54] asked whether every planar graph has a $\mathcal{O}(1)$ queue-number. By our result, these two open problems are almost equivalent in the following sense. If every planar graph has $\mathcal{O}(1)$ queue-number, then every planar graph has a three-dimensional drawing with $\mathcal{O}(n)$ volume. Conversely, if every planar graph has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing, then every planar graph has $\mathcal{O}(1)$ queue-number. It is possible, however, that planar graphs have unbounded queue-number, yet have, say, $\mathcal{O}(n^{1/3}) \times \mathcal{O}(n^{1/3}) \times \mathcal{O}(n^{1/3})$ drawings.

Our other main results regard three-dimensional drawings and queue layouts of graphs with bounded tree-width. Tree-width, first defined by Halin [50], although largely unnoticed until independently rediscovered by Robertson and Seymour [79] and Arnborg and Proskurowski [7], is a measure of the similarity of a graph to a tree (see section 2.1 for the definition). Tree-width (or its special case, path-width) has been previously used in the context of graph drawing by Dujmović et al. [32], Hliněný [59], and Peng [75], for example.

The second main result (Corollary 2.8) is that the queue-number of a graph is bounded by its tree-width. This solves an open problem due to Ganley and Heath [45], who proved that stack-number is bounded by tree-width and asked whether a similar relationship holds for queue-number. This result has significant implications for the above open problem (does every planar graph have $\mathcal{O}(1)$ queue-number), and the more general question (since planar graphs have stack-number at most four [93]) of whether queue-number is bounded by stack-number. Heath and colleagues [58, 54] originally conjectured that both of these questions have an affirmative answer. More recently, however, Pemmaraju [74] conjectured that the "stellated K_3," a planar 3-tree, has $\Theta(\log n)$ queue-number, and provided evidence to support this conjecture (also see [45]). This suggested that the answer to both of the above questions was negative. In particular, Pemmaraju [74] and Heath [private communication, 2002] conjectured that planar graphs have $\mathcal{O}(\log n)$ queue-number. However, our result provides a queue layout of *any* 3-tree, and thus the stellated K_3, with $\mathcal{O}(1)$ queues. Hence our result disproves the first conjecture of Pemmaraju [74] mentioned above and renews hope in an affirmative answer to the above open problems.

The third main result is that every graph of bounded tree-width has a three-dimensional drawing with $\mathcal{O}(n)$ volume. The family of graphs of bounded tree-width includes most of the graphs previously known to admit three-dimensional drawings with $\mathcal{O}(n)$ volume (for example, outerplanar graphs), and also includes many graph families for which the previous best volume bound was $\mathcal{O}(n^2)$ (for example, series-parallel graphs). Many graphs arising in applications of graph drawing do have small tree-width. Outerplanar and series-parallel graphs are the obvious examples. Another example arises in software engineering applications. Thorup [87] proved that the control-flow graphs of go-to free programs in many programming languages have tree-width bounded by a small constant; in particular, 3 for Pascal and 6 for C. Other families of graphs having bounded tree-width (for constant k) include almost trees with parameter k, graphs with a feedback vertex set of size k, band-width k graphs, cut-width k graphs, planar graphs of radius k, and k-outerplanar graphs. If the size of a maximum clique is a constant k, then chordal, interval, and circular arc graphs also have bounded tree-width. Thus, by our result, all of these graphs have three-dimensional drawings with $\mathcal{O}(n)$ volume, and $\mathcal{O}(1)$ queue-number.

To prove our results for graphs of bounded tree-width, we employ a related structure called a tree-partition, introduced independently by Seese [83] and Halin [51]. A *tree-partition* of a graph is a partition of its vertices into "bags" such that contracting each bag to a single vertex gives a forest (after deleting loops and replacing parallel edges by a single edge). In a result of independent interest, we prove that every k-tree has a tree-partition such that each bag induces a connected $(k-1)$-tree, amongst other properties. The second tool that we use is a *track layout*, which consists of a vertex-coloring and a total order of each color class, such that between any two color classes no two edges cross.

The remainder of the paper is organized as follows. In section 2 we introduce the required background material, state our results regarding three-dimensional drawings and queue layouts, and compare these with results in the literature. In section 3 we establish a number of results concerning track layouts. That three-dimensional drawings and queue-layouts are closely related stems from the fact that three-dimensional drawings and queue layouts are both closely related to track layouts, as proved in section 4 and section 5, respectively. In section 6 we prove the above-mentioned theorem for tree-partitions of k-trees, which is used in section 7 to construct track layouts of graphs with bounded tree-width. We conclude in section 8 with a number of open problems.

2. Background and results. Throughout this paper all graphs G are undirected, simple, and finite with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and the maximum degree of G are respectively denoted by $n = |V(G)|$ and $\Delta(G)$. The subgraph induced by a set of vertices $A \subseteq V(G)$ is denoted by $G[A]$. For all disjoint subsets $A, B \subseteq V(G)$, the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$ is denoted by $G[A, B]$.

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A family of graphs closed under taking minors is *proper* if it is not the class of all graphs.

A *graph parameter* is a function α that assigns to every graph G a nonnegative integer $\alpha(G)$. Let $\mathcal{G}$ be a family of graphs. By $\alpha(\mathcal{G})$ we denote the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum of $\alpha(G)$ taken over all n-vertex graphs $G \in \mathcal{G}$. We say that $\mathcal{G}$ has *bounded* α if $\alpha(\mathcal{G}) \in \mathcal{O}(1)$. A graph parameter α is *bounded by* a graph parameter β (for some graph family $\mathcal{G}$), if there exists a function g such that $\alpha(G) \leq g(\beta(G))$ for every graph G (in $\mathcal{G}$).

2.1. Tree-width. Let G be a graph and let T be a tree. An element of $V(T)$ is called a *node*. Let $\{T_x \subseteq V(G) : x \in V(T)\}$ be a set of subsets of $V(G)$ indexed by the nodes of T. Each T_x is called a *bag*. The pair $(T, \{T_x : x \in V(T)\})$ is a *tree-decomposition* of G if

1. $\bigcup_{x \in V(T)} T_x = V(G)$ (that is, every vertex of G is in at least one bag),
2. $\forall$ edges vw of G, $\exists$ node x of T such that $v \in T_x$ and $w \in T_x$, and
3. $\forall$ nodes x, y, z of T, if y is on the path from x to z in T, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree-decomposition is one less than the maximum cardinality of a bag. A *path-decomposition* is a tree-decomposition where the tree T is a path $T = (x_1, x_2, \ldots, x_m)$, which is simply identified by the sequence of bags $T_1, T_2, \ldots, T_m$ where each $T_i = T_{x_i}$. The *path-width* (respectively, *tree-width*) of a graph G, denoted by $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$), is the minimum width of a path- (tree-) decomposition of G. Graphs with tree-width at most one are precisely the forests. Graphs with tree-width at most two are called *series-parallel*,[1] and are characterized as those graphs with no K_4 minor (see [10]).

A *k-tree* for some $k \in \mathbb{N}$ is defined recursively as follows. The empty graph is a k-tree, and the graph obtained from a k-tree by adding a new vertex adjacent to each vertex of a clique with at most k vertices is also a k-tree. This definition of a k-tree is by Reed [77]. The following more restrictive definition of a k-tree, which we call "strict," was introduced by Arnborg and Proskurowski [7], and is more often used in the literature. A k-clique is a *strict k-tree*, and the graph obtained from a strict k-tree by adding a new vertex adjacent to each vertex of a k-clique is also a strict k-tree. Obviously the strict k-trees are a proper subclass of the k-trees. A subgraph of a k-tree is called a *partial k-tree*, and a subgraph of a strict k-tree is called a *partial strict k-tree*. The following result is well known (see, for example, [10, 77]). A *chord* of a cycle C is an edge not in C whose end-vertices are both in C. A graph is *chordal* if every cycle on at least four vertices has a chord.

LEMMA 2.1. *Let G be a graph. The following are equivalent:*

1. *G has tree-width $\mathsf{tw}(G) \leq k$,*
2. *G is a partial k-tree,*
3. *G is a partial strict k-tree,*
4. *G is a subgraph of a chordal graph that has no clique on $k+2$ vertices.*

Proof. Scheffler [81] proved that (1) and (3) are equivalent. That (1) and (4) are equivalent is due to Robertson and Seymour [79]. That (2) and (4) are equivalent is the characterization of chordal graphs in terms of "perfect elimination" vertex-orderings due to Fulkerson and Gross [44]. ◻

2.2. Tree-partitions. As in the definition of a tree-decomposition, let G be a graph and let $\{T_x \subseteq V(G) : x \in V(T)\}$ be a set of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree T. The pair $(T, \{T_x : x \in V(T)\})$ is a *tree-partition* of G if

1. $\forall$ distinct nodes x and y of T, $T_x \cap T_y = \emptyset$, and
2. $\forall$ edges vw of G, either
   (i) $\exists$ node x of T with $v \in T_x$ and $w \in T_x$ (vw is called an *intrabag* edge), or

[1] "Series-parallel digraphs" are often defined in terms of certain "series" and "parallel" composition operations. The underlying undirected graph of such a digraph has tree-width at most two (see [10]).

(ii) $\exists$ edge xy of T with $v \in T_x$ and $w \in T_y$ (vw is called an *interbag* edge).

The main property of tree-partitions that has been studied in the literature is the maximum cardinality of a bag, called the *width* of the tree-partition [11, 51, 83, 30, 31]. The minimum width over all tree-partitions of a graph G is the *tree-partition-width*[2] of G, denoted by $\mathsf{tpw}(G)$. A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [31]. In particular, for every graph G, Ding and Oporowski [30] proved that $\mathsf{tpw}(G) \leq 24\,\mathsf{tw}(G)\,\Delta(G)$ (assuming $\Delta(G) \geq 1$), and Seese [83] proved that $\mathsf{tw}(G) \leq 2\,\mathsf{tpw}(G) - 1$.

Theorem 6.1 provides a tree-partition of a k-tree G with additional features besides small width. First, the subgraph induced by each bag is a connected $(k-1)$-tree. This allows us to perform induction on k. Second, in each nonroot bag T_x the set of vertices in the parent bag of x with a neighbor in T_x form a clique. This feature is crucial in the intended application (Theorem 7.3). Finally the tree-partition has width at most $\max\{1, k(\Delta(G)-1)\}$, which represents a constant-factor improvement over the above result by Ding and Oporowski [30] in the case of k-trees.

2.3. Track layouts. Let G be a graph. A *coloring* of G is a partition $\{V_i : i \in I\}$ of $V(G)$, where I is a set of *colors*, such that for every edge vw of G, if $v \in V_i$ and $w \in V_j$, then $i \neq j$. Each set V_i is called a *color class*. A coloring of G with c colors is a *c-coloring*, and we say that G is *c-colorable*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum c such that G is c-colorable.

If $<_i$ is a total order of a color class V_i, then we call the pair $(V_i, <_i)$ a *track*. If $\{V_i : i \in I\}$ is a coloring of G and $(V_i, <_i)$ is a track for each color $i \in I$, then we say $\{(V_i, <_i) : i \in I\}$ is a *track assignment* of G *indexed by* I. Note that at times it will be convenient to also refer to a color $i \in I$ and the color class V_i as a *track*. The precise meaning will always be clear from the context. A *t-track assignment* is a track assignment with t tracks.

As illustrated in Figure 2.1, an *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$ for distinct tracks V_i and V_j. A t-track assignment with no X-crossing is called a *t-track layout*. The *track-number* of a graph G, denoted by $\mathsf{tn}(G)$, is the minimum t such that G has a t-track layout.

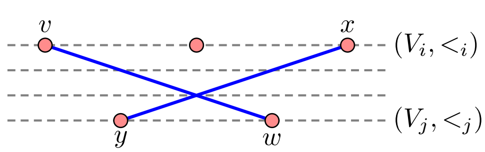

FIG. 2.1. *An example of an X-crossing in a track assignment.*

Let $\{(V_i, <_i) : i \in I\}$ be a t-track layout of a graph G. The *span* of an edge vw of G, with respect to a numbering of the tracks $I = \{1, 2, \ldots, t\}$, is defined to be $|i - j|$, where $v \in V_i$ and $w \in V_j$.

Track layouts will be central in most of our proofs. To enable comparison of our results to those in the literature we now introduce the notion of an "improper" track layout. A *improper coloring* of a graph G is simply a partition $\{V_i : i \in I\}$ of $V(G)$. Here adjacent vertices may be in the same color class. A track of an improper coloring is defined as above. Suppose $\{V_i : i \in I\}$ is an improper coloring of G and $(V_i, <_i)$ is a track for each color $i \in I$. An edge with both end-vertices in the same

[2]Tree-partition-width has also been called *strong tree-width* [83, 11].

track is called an *intratrack* edge; otherwise it is called an *intertrack* edge. We say that $\{(V_i, <_i) : i \in I\}$ is an *improper track assignment* of G if, for all intratrack edges $vw \in E(G)$ with $v \in V_i$ and $w \in V_i$ for some $i \in I$, there is no vertex x with $v <_i x <_i w$. That is, adjacent vertices in the same track are consecutive in that track. An improper t-track assignment with no X-crossing is called an *improper t-track layout.*[3]

LEMMA 2.2. *If a graph G has an improper t-track layout, then G has a $2t$-track layout.*

Proof. For every track V_i of an improper t-track layout of G, let V_i' be a new track. Move every second vertex from V_i to V_i' such that V_i' inherits its total order from the original V_i. Clearly there is no intratrack edge and no X-crossing. Thus we obtain a $2t$-track layout of G. □

Hence the track-number of a graph is at most twice its "improper track-number." The following lemma, which was jointly discovered with Giuseppe Liotta, gives a compelling reason to only consider proper track layouts. Similar ideas can be found in [42, 26]. Let vw be an edge of a graph G. Let G' be the graph obtained from G by adding a new vertex x only adjacent to v and w. We say x is an *ear*, and G' is obtained from G by *adding an ear to vw*.

LEMMA 2.3. *Let $\mathcal{G}$ be a class of graphs closed under the addition of ears* (*for example, series-parallel graphs or planar graphs*)*. If every graph in $\mathcal{G}$ has an improper t-track layout for some constant t, then every graph in $\mathcal{G}$ has a* (*proper*) *t-track layout.*

Proof. For any graph $G \in \mathcal{G}$, let G' be the graph obtained from G by adding t ears to every edge of G. By assumption, G' has an improper t-track layout. Suppose that there is an edge vw of G such that v and w are in the same track. None of the ears added to vw are on the same track, as otherwise adjacent vertices would not be consecutive in that track. Thus there is a track containing at least two of the ears added to vw. However, this implies that there is an X-crossing, which is a contradiction. Thus the end-vertices of every edge of G are in distinct tracks. Hence the improper t-track layout of G' contains a t-track layout of G. □

Lemmas 2.2 and 2.3 imply that only for relatively small classes of graphs will the distinction between track layouts and improper track layouts be significant. We therefore chose to work with the less cumbersome notion of a track layout. The following theorem summarizes our bounds on the track-number of a graph.

THEOREM 2.4. *Let G be a graph with maximum degree $\Delta(G)$, path-width $\mathsf{pw}(G)$, tree-partition-width $\mathsf{tpw}(G)$, and tree-width $\mathsf{tw}(G)$. The track-number of G satisfies*

(a) $\mathsf{tn}(G) \leq \mathsf{pw}(G) + 1 \leq 1 + (\mathsf{tw}(G) + 1) \log n$,

(b) $\mathsf{tn}(G) \leq 3\,\mathsf{tpw}(G) \leq 72\,\mathsf{tw}(G)\,\Delta(G)$ (*assuming* $\Delta(G) \geq 1$),

(c) $\mathsf{tn}(G) \leq 3^{\mathsf{tw}(G)} \cdot 6^{(4^{\mathsf{tw}(G)} - 3\,\mathsf{tw}(G) - 1)/9}$.

Proof. Part (a) follows from Lemma 3.2 and the fact that $\mathsf{pw}(G) \leq (\mathsf{tw}(G) + 1) \log n$ (see [10]). Note that $\mathsf{tn}(G) \leq 1 + (\mathsf{tw}(G)+1) \log n$ can be proved directly using a separator-based approach similar to that used to prove $\mathsf{pw}(G) \leq (\mathsf{tw}(G) + 1) \log n$. Part (b) follows from Lemma 3.3 in section 3 and the result of Ding and Oporowski [30] discussed in section 2.2. Part (c) is Theorem 7.3. □

2.4. Vertex-orderings. Let G be a graph. A total order $\sigma = (v_1, v_2, \ldots, v_n)$ of $V(G)$ is called a *vertex-ordering* of G. Suppose that G is connected. The *depth* of a vertex v_i in σ is the graph-theoretic distance between v_1 and v_i in G. We say that

[3]In [33, 35, 91] we called a track layout an *ordered layering with no X-crossing and no intralayer edges*, and an improper track layout was called an *ordered layering with no X-crossing.*

σ is a *breadth-first* vertex-ordering if for all vertices v and w with $v <_\sigma w$ the depth of v in σ is no more than the depth of w in σ. Vertex-orderings, and in particular, vertex-orderings of trees, will be used extensively in this paper. Consider a breadth-first vertex-ordering σ of a tree T such that vertices at depth $d \geq 1$ are ordered with respect to the ordering of vertices at depth $d-1$. In particular, if v and x are vertices at depth d with respective parents w and y at depth $d-1$ with $w <_\sigma y$, then $v <_\sigma x$. Such a vertex-ordering is called a *lexicographical* breadth-first vertex-ordering of T, and is illustrated in Figure 2.2.

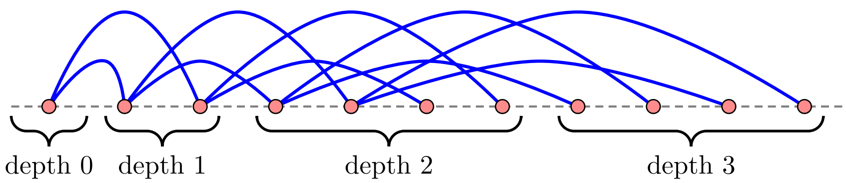

FIG. 2.2. *A lexicographical breadth-first vertex-ordering of a tree.*

2.5. Queue layouts. A *queue layout* of a graph G consists of a vertex-ordering σ of G and a partition of $E(G)$ into *queues* such that no two edges in the same queue are *nested* with respect to σ. That is, there are no edges vw and xy in a single queue with $v <_\sigma x <_\sigma y <_\sigma w$. The minimum number of queues in a queue layout of G is called the *queue-number* of G and is denoted by $\mathsf{qn}(G)$. A similar concept is that of a *stack layout* (or *book embedding*), which consists of a vertex-ordering σ of G and a partition of $E(G)$ into *stacks* (or *pages*) such that there are no edges vw and xy in a single stack with $v <_\sigma x <_\sigma w <_\sigma y$. The minimum number of stacks in a stack layout of G is called the *stack-number* (or *page-number* or *book-thickness*) of G and is denoted by $\mathsf{sn}(G)$. A queue (respectively, stack) layout with k queues (stacks) is called a *k-queue* (*k-stack*) *layout*, and a graph that admits a k-queue (k-stack) layout is called a *k-queue* (*k-stack*) *graph*.

Heath and Rosenberg [58] characterized 1-queue graphs as the "arched levelled planar" graphs, and proved that it is $\mathcal{NP}$-complete to recognize such graphs. This result is in contrast to the situation for stack layouts—1-stack graphs are precisely the outerplanar graphs [8], which can be recognized in polynomial time. Heath, Leighton, and Rosenberg [54] proved that 1-stack graphs are 2-queue graphs (rediscovered by Rengarajan and Veni Madhavan [78]), and that 1-queue graphs are 2-stack graphs.

While it is $\mathcal{NP}$-hard to minimize the number of stacks in a stack layout given a fixed vertex-ordering [46], the analogous problem for queue layouts can be solved as follows. A *k-rainbow* in a vertex-ordering σ consists of a matching $\{v_i w_i : 1 \leq i \leq k\}$ such that $v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1$, as illustrated in Figure 2.3.

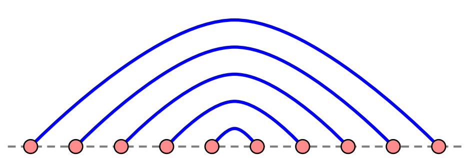

FIG. 2.3. *A rainbow of five edges in a vertex-ordering.*

A vertex-ordering containing a k-rainbow needs at least k queues. A straightforward application of Dilworth's Theorem [29] proves the converse. That is, a fixed vertex-ordering admits a k-queue layout, where k is the size of the largest rainbow. (Heath and Rosenberg [58] describe a $\mathcal{O}(m \log\log n)$ time algorithm to compute the queue assignment.) Thus determining $\mathsf{qn}(G)$ can be viewed as the following vertex-ordering problem.

LEMMA 2.5 (see [58]). *The queue-number* $\mathsf{qn}(G)$ *of a graph* G *is the minimum, taken over all vertex-orderings* σ *of* G*, of the maximum size of a rainbow in* σ*.* □

Stack and/or queue layouts of k-trees have previously been investigated in [19, 78, 45]. A 1-tree is a 1-queue graph, since in a lexicographical breadth-first vertex-ordering of a tree no two edges are nested (see Figure 2.2). Chung, Leighton, and Rosenberg [19] proved that in a depth-first vertex-ordering of a tree no two edges cross. Thus 1-trees are 1-stack graphs. Rengarajan and Veni Madhavan [78] proved that graphs with tree-width at most two (the series parallel graphs) are 2-stack and 3-queue graphs.[4] Improper track layouts are implicit in the work of Heath, Leighton, and Rosenberg [54] and Rengarajan and Veni Madhavan [78]. In section 5 we prove the following fundamental relationship between queue and track layouts.

THEOREM 2.6. *For every graph* G*,* $\mathsf{qn}(G) \le \mathsf{tn}(G) - 1$*. Moreover, if* $\mathcal{G}$ *is any proper minor-closed graph family, then* $\mathcal{G}$ *has queue-number* $\mathsf{qn}(\mathcal{G}) \in \mathcal{F}(n)$ *if and only if* $\mathcal{G}$ *has track-number* $\mathsf{tn}(\mathcal{G}) \in \mathcal{F}(n)$*, where* $\mathcal{F}(n)$ *is any family of functions closed under multiplication* (*such as* $\mathcal{O}(1)$ *or* $\mathcal{O}(\text{polylog}\, n)$)*.*

Ganley and Heath [45] proved that every graph G has stack-number $\mathsf{sn}(G) \le \mathsf{tw}(G) + 1$ (using a depth-first traversal of a tree-decomposition), and asked whether queue-number is bounded by tree-width. One of the principal results of this paper is to solve this question in the affirmative. Applying Theorems 2.4 and 2.6, we have the following.

THEOREM 2.7. *Let* G *be a graph with maximum degree* $\Delta(G)$*, path-width* $\mathsf{pw}(G)$*, tree-partition-width* $\mathsf{tpw}(G)$*, and tree-width* $\mathsf{tw}(G)$*. The queue-number* $\mathsf{qn}(G)$ *satisfies*[5]

(a) $\mathsf{qn}(G) \le \mathsf{pw}(G) \le (\mathsf{tw}(G)+1)\log n$,

(b) $\mathsf{qn}(G) \le 3\,\mathsf{tpw}(G) - 1 \le 72\,\mathsf{tw}(G)\,\Delta(G) - 1$ (*assuming* $\Delta(G) \ge 1$),

(c) $\mathsf{qn}(G) \le 3^{\mathsf{tw}(G)} \cdot 6^{(4^{\mathsf{tw}(G)} - 3\,\mathsf{tw}(G) - 1)/9} - 1$. □

A similar upper bound to Theorem 2.7(a) was obtained by Heath and Rosenberg [58], who proved that every graph G has $\mathsf{qn}(G) \le \lceil \frac{1}{2}\mathsf{bw}(G) \rceil$, where $\mathsf{bw}(G)$ is the band-width of G. In many cases this result is weaker than Theorem 2.7(a) since $\mathsf{pw}(G) \le \mathsf{bw}(G)$ (see [28]). More importantly, we have the following corollary of Theorem 2.7(c).

COROLLARY 2.8. *Queue-number is bounded by tree-width, and hence graphs with bounded tree-width have bounded queue-number.* □

2.6. Three-dimensional drawings. A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *three-dimensional drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$ (called *grid-points*) and represents each edge as a line-segment between its end-vertices, such that edges intersect only at common end-vertices, and an edge only intersects a vertex that is an end-vertex of that edge.

In contrast to the case in the plane, a folklore result states that every graph has a three-dimensional drawing. Such a drawing can be constructed using the "moment

[4] In [35] we give a simple proof based on Theorem 6.1 for the result by Rengarajan and Veni Madhavan [78] that every series-parallel graph has a 3-queue layout.

[5] In [91] we obtained an alternative proof that $\mathsf{qn}(G) \le \mathsf{pw}(G)$ using the "vertex separation number" of a graph (which equals its path-width); applying Lemma 2.5 directly, we proved that $\mathsf{qn}(G) \le \frac{3}{2}\,\mathsf{tpw}(G)$, and thus $\mathsf{qn}(G) \le 36\,\Delta(G)\,\mathsf{tw}(G)$.

curve" algorithm in which vertex v_i, $1 \leq i \leq n$, is represented by the grid-point (i, i^2, i^3). It is easily seen—compare with Lemma 4.2—that no two edges cross. (Two edges *cross* if they intersect at some point other than a common end-vertex.)

Since every graph has a three-dimensional drawing, we are interested in optimizing certain measures of the aesthetic quality of a drawing. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X-1$, $Y-1$, and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$ and *aspect ratio* $\max\{X, Y, Z\}/\min\{X, Y, Z\}$. This paper considers the problem of producing a three-dimensional drawing of a given graph with small volume, and with small aspect ratio as a secondary criterion.

Observe that the drawings produced by the moment curve algorithm have $\mathcal{O}(n^6)$ volume. Cohen et al. [20] improved this bound by proving that if p is a prime with $n < p \leq 2n$, and each vertex v_i is represented by the grid-point $(i, i^2 \bmod p, i^3 \bmod p)$, then there is still no crossing. This construction is a generalization of an analogous two-dimensional technique due to Erdős [40]. Furthermore, Cohen et al. [20] proved that the resulting $\mathcal{O}(n^3)$ volume bound is asymptotically optimal in the case of the complete graph K_n. It is therefore of interest to identify fixed graph parameters that allow for three-dimensional drawings with small volume.

The first such parameter to be studied was the chromatic number [16, 73]. Calamoneri and Sterbini [16] proved that every 4-colorable graph has a three-dimensional drawing with $\mathcal{O}(n^2)$ volume. Generalizing this result, Pach, Thiele, and Tóth [73] proved that graphs of bounded chromatic number have three-dimensional drawings with $\mathcal{O}(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. If p is a suitably chosen prime, the main step of this algorithm represents the vertices in the ith color class by grid-points in the set $\{(i, t, it) : t \equiv i^2 \pmod{p}\}$. It follows that the volume bound is $\mathcal{O}(k^2 n^2)$ for k-colorable graphs.

The lower bound of Pach, Thiele, and Tóth [73] for the complete bipartite graph was generalized by Bose et al. [14] for all graphs. They proved that every three-dimensional drawing with n vertices and m edges has volume at least $\frac{1}{8}(n+m)$. In particular, the maximum number of edges in an $X \times Y \times Z$ drawing is exactly $(2X-1)(2Y-1)(2Z-1) - XYZ$. For example, graphs admitting three-dimensional drawings with $\mathcal{O}(n)$ volume have $\mathcal{O}(n)$ edges.

The first nontrivial $\mathcal{O}(n)$ volume bound was established by Felsner, Liotta, and Wismath [42] for outerplanar graphs. Their elegant algorithm "wraps" a two-dimensional drawing around a triangular prism to obtain an improper 3-track layout (see Lemmas 3.1 and 3.4 for more on this method). Poranen [76] proved that series-parallel digraphs have upward three-dimensional drawings with $\mathcal{O}(n^3)$ volume, and that this bound can be improved to $\mathcal{O}(n^2)$ and $\mathcal{O}(n)$ in certain special cases. Di Giacomo, Liotta, and Wismath [26] proved that series-parallel graphs with maximum degree three have three-dimensional drawings with $\mathcal{O}(n)$ volume.

In section 4 we prove the following intrinsic relationship between three-dimensional drawings and track layouts.

THEOREM 2.9. *Every graph G has a $\mathcal{O}(\mathsf{tn}(G)) \times \mathcal{O}(\mathsf{tn}(G)) \times \mathcal{O}(n)$ drawing. Moreover, G has an $\mathcal{F}(n) \times \mathcal{F}(n) \times \mathcal{O}(n)$ drawing if and only if G has track-number $\mathsf{tn}(G) \in \mathcal{F}(n)$, where $\mathcal{F}(n)$ is a family of functions closed under multiplication.*

Of course, every graph has an n-track layout—simply place a single vertex on each track. Thus Theorem 2.9 matches the $\mathcal{O}(n^3)$ volume bound discussed in section 2.6. In fact, the drawings of K_n produced by our algorithm, with each vertex in a distinct track, are identical to those produced by the algorithm of Cohen et al. [20].

Theorems 2.6 and 2.9 immediately imply the following result, which reduces the problem of producing a three-dimensional drawing with small volume to that of producing a queue layout of the same graph with few queues.

THEOREM 2.10. *Let $\mathcal{G}$ be a proper minor-closed family of graphs, and let $\mathcal{F}(n)$ be a family of functions closed under multiplication. The following are equivalent:*

(a) *every n-vertex graph in $\mathcal{G}$ has an $\mathcal{F}(n) \times \mathcal{F}(n) \times \mathcal{O}(n)$ drawing,*

(b) *$\mathcal{G}$ has track-number $\mathsf{tn}(\mathcal{G}) \in \mathcal{F}(n)$, and*

(c) *$\mathcal{G}$ has queue-number $\mathsf{qn}(\mathcal{G}) \in \mathcal{F}(n)$.* □

Graphs with constant queue-number include de Bruijn graphs, FFT, and Beneš network graphs [58]. By Theorem 2.10, these graphs have three-dimensional drawings with $\mathcal{O}(n)$ volume. Applying Theorems 2.4 and 2.9, we have the following result.

THEOREM 2.11. *Let G be a graph with maximum degree $\Delta(G)$, path-width $\mathsf{pw}(G)$, tree-partition-width $\mathsf{tpw}(G)$, and tree-width $\mathsf{tw}(G)$. Then G has a three-dimensional drawing with the following dimensions:*

(a) $\mathcal{O}(\mathsf{pw}(G)) \times \mathcal{O}(\mathsf{pw}(G)) \times \mathcal{O}(n)$, *which is* $\mathcal{O}(\mathsf{tw}(G)\, \log n) \times \mathcal{O}(\mathsf{tw}(G)\, \log n) \times \mathcal{O}(n)$,

(b) $\mathcal{O}(\mathsf{tpw}(G)) \times \mathcal{O}(\mathsf{tpw}(G)) \times \mathcal{O}(n)$, *which is* $\mathcal{O}(\Delta(G)\, \mathsf{tw}(G)) \times \mathcal{O}(\Delta(G)\, \mathsf{tw}(G)) \times \mathcal{O}(n)$,

(c) $\mathcal{O}(3^{\,\mathsf{tw}(G)} \cdot 6^{(4^{\,\mathsf{tw}(G)} - 3\,\mathsf{tw}(G) - 1)/9}) \times \mathcal{O}(3^{\,\mathsf{tw}(G)} \cdot 6^{(4^{\,\mathsf{tw}(G)} - 3\,\mathsf{tw}(G) - 1)/9}) \times \mathcal{O}(n)$. □

Most importantly, we have the following corollary of Theorem 2.11(c).

COROLLARY 2.12. *Every graph with bounded tree-width has a three-dimensional drawing with $\mathcal{O}(n)$ volume.* □

Note that bounded tree-width is not necessary for a graph to have a three-dimensional drawing with $\mathcal{O}(n)$ volume. The $\sqrt{n} \times \sqrt{n}$ plane grid graph has $\Theta(\sqrt{n})$ tree-width, and has a $\sqrt{n} \times \sqrt{n} \times 1$ drawing with n volume. It also has a 3-track layout, and thus, by Lemma 4.2, has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing.

Since a planar graph is 4-colorable, by the results of Calamoneri and Sterbini [16] and Pach, Thiele, and Tóth [73] discussed above, every planar graph has a three-dimensional drawing with $\mathcal{O}(n^2)$ volume. This result also follows from the classical algorithms of de Fraysseix, Pach, and Pollack [22] and Schnyder [82] for producing $\mathcal{O}(n) \times \mathcal{O}(n)$ plane grid drawings. All of these methods produce $\mathcal{O}(n) \times \mathcal{O}(n) \times \mathcal{O}(1)$ drawings, which have $\Theta(n)$ aspect ratio. Since every planar graph G has $\mathsf{pw}(G) \in \mathcal{O}(\sqrt{n})$ [10], we have the following corollary of Theorem 2.11(a).

COROLLARY 2.13. *Every planar graph has a three-dimensional drawing with $\mathcal{O}(n^2)$ volume and $\Theta(\sqrt{n})$ aspect ratio.* □

This result matches the above $\mathcal{O}(n^2)$ volume bounds with an improvement in the aspect ratio by a factor of $\Theta(\sqrt{n})$. Our final result regarding three-dimensional drawings, which is proved in section 4, examines the apparent trade-off between aspect ratio and volume.

THEOREM 2.14. *For every graph G and for every r, $1 \leq r \leq n/\mathsf{tn}(G)$, G has a three-dimensional drawing with $\mathcal{O}(n^3/r^2)$ volume and aspect ratio $2r$.*

3. Track layouts.

In this section we describe a number of methods for producing and manipulating track layouts. The following result is implicit in the proof by Felsner, Liotta, and Wismath [42] that every outerplanar graph has an improper 3-track layout.

LEMMA 3.1 (see [42]). *Every tree T has a 3-track layout.*

Proof. Root T at an arbitrary node r. Let σ be a lexicographical breadth-first vertex-ordering of T starting at r, as described in section 2.4. For $i \in \{0, 1, 2\}$, let V_i be the set of nodes of T with depth $d \equiv i \pmod 3$ in σ. With each V_i ordered by σ, we have a 3-track assignment of T. Clearly adjacent vertices are on distinct tracks. Since no two edges are nested in σ, there is no X-crossing (see Figure 3.1). ∎

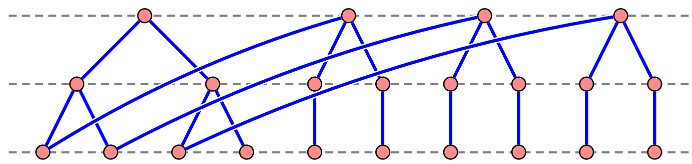

FIG. 3.1. *A 3-track layout of a tree.*

LEMMA 3.2. *Every graph G with path-width $\mathsf{pw}(G)$ has track-number $\mathsf{tn}(G) \leq \mathsf{pw}(G) + 1$.*

Proof. Let $k = \mathsf{pw}(G)+1$. It is well known that G is the subgraph of a k-colorable interval graph [10, 48]. That is, there is a set of intervals $\{[\ell(v), r(v)] \subseteq \mathbb{R} : v \in V(G)\}$ such that $[\ell(v), r(v)] \cap [\ell(w), r(w)] \neq \emptyset$ for every edge vw of G. Let $\{V_i : 1 \leq i \leq k\}$ be a k-coloring of G. Consider each color class V_i to be an ordered track $(v_1, v_2, \dots, v_p)$, where $\ell(v_1) < r(v_1) < \ell(v_2) < r(v_2) < \cdots < \ell(v_p) < r(v_p)$, as illustrated in Figure 3.2. Suppose there is an X-crossing between edges vw and xy with $v, x \in V_i$ and $w, y \in V_j$ for some pair of tracks V_i and V_j. Without loss of generality, $r(v) < \ell(x)$ and $r(y) < \ell(w)$. Since vw is an edge, $\ell(w) \leq r(v)$. Thus $r(y) < \ell(w) \leq r(v) < \ell(x)$, which implies that xy is not an edge of G. This contradiction proves that there is no X-crossing, and G has a k-track layout. ∎

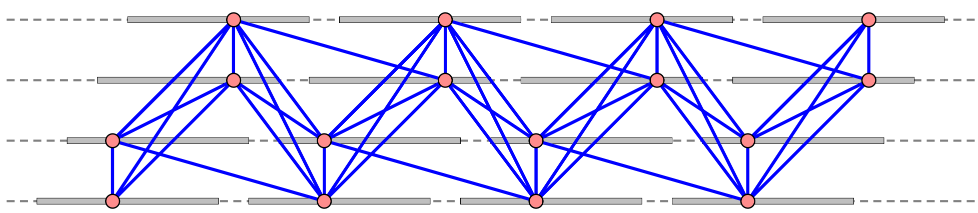

FIG. 3.2. *A 4-track layout of a 4-colorable interval graph.*

The next lemma uses a tree-partition to construct a track layout.

LEMMA 3.3. *Every graph G with maximum degree $\Delta(G) \geq 1$, tree-width $\mathsf{tw}(G)$, and tree-partition-width $\mathsf{tpw}(G)$, has track-number $\mathsf{tn}(G) \leq 3\,\mathsf{tpw}(G) \leq 72\,\Delta(G)\mathsf{tw}(G)$.*

Proof. Let $(T, \{T_x : x \in V(T)\})$ be a tree-partition of G with width $\mathsf{tpw}(G)$. By Lemma 3.1, T has a 3-track layout. Replace each track by $\mathsf{tpw}(G)$ "subtracks," and for each node x in T place the vertices in bag T_x on the subtracks replacing the track containing x, with at most one vertex in T_x in a single track. For all nodes x and y of T, if $x < y$ in a single track of the 3-track layout of T, then for all vertices $v \in T_x$ and $w \in T_y$, $v < w$ whenever v and w are assigned to the same track. There is no X-crossing, since in the track layout of T, adjacent nodes are on distinct tracks and there is no X-crossing. Thus we have a track layout of G. The number of tracks is $3\,\mathsf{tpw}(G)$, which is at most $72\,\Delta(G)\mathsf{tw}(G)$ by the theorem of Ding and Oporowski [30] discussed in section 2.2. ∎

In the remainder of this section, we prove two results that show how track layouts can be manipulated without introducing an X-crossing. The first is a generalization of the "wrapping" algorithm of Felsner, Liotta, and Wismath [42], who implicitly proved the case $s = 1$.

LEMMA 3.4. *If a graph G has an (improper) track layout $\{(V_i, <_i) : 1 \leq i \leq t\}$ with maximum edge span s, then G has an (improper) $(2s+1)$-track layout.*

Proof. Let $\ell = 2s + 1$. Construct an ℓ-track assignment of G by merging the tracks $\{V_i : i \equiv j \pmod t\}$ for each j, $0 \leq j \leq t-1$, with vertices in V_α appearing before vertices in V_β in the new track j for all $\alpha, \beta \equiv j \pmod t$ with $\alpha < \beta$. The given order of each V_i is preserved in the new tracks. It remains to prove that there is no X-crossing. Consider two edges vw and xy. Let i_1 and i_2, $1 \leq i_1 < i_2 \leq t$, be the minimum and maximum tracks containing v, w, x, or y in the given t-track layout of G.

First consider the case that $i_2 - i_1 > 2s$. Then without loss of generality v is in track i_2 and y is in track i_1. Thus w is in a greater track than x, and even if x (or y) appear on the same track as v (or w) in the new ℓ-track assignment, x (or y) will be to the left of v (or w). Thus these edges do not form an X-crossing in the ℓ-track assignment. Otherwise $i_2 - i_1 \leq 2s$. Thus any two of v, w, x, or y will appear on the same track in the ℓ-track assignment if and only if they are on the same track in the given t-track layout (since $\ell > 2s$). Hence the only way for these four vertices to appear on exactly two tracks in the ℓ-track assignment is if they were on exactly two layers in the given t-track layout, in which case, by assumption, vw and xy do not form an X-crossing. Therefore there is no X-crossing, and we have an ℓ-track layout of G. ∎

The next result shows that the number of vertices in different tracks of a track layout can be balanced without introducing an X-crossing. The proof is based on an idea due to Pach, Thiele, and Tóth [73] for balancing the size of the color classes in a coloring.

LEMMA 3.5. *If a graph G has an (improper) t-track layout, then for every $t' > 0$, G has an (improper) $\lfloor t + t' \rfloor$-track layout with at most $\lceil \frac{n}{t'} \rceil$ vertices in each track.*

Proof. For each track with $q > \lceil \frac{n}{t'} \rceil$ vertices, replace it by $\lceil q / \lceil \frac{n}{t'} \rceil \rceil$ "subtracks" each with exactly $\lceil \frac{n}{t'} \rceil$ vertices except for at most one subtrack with $q \bmod \lceil \frac{n}{t'} \rceil$ vertices, such that the vertices in each subtrack are consecutive in the original track and the original order is maintained. There is no X-crossing between subtracks from the same original track as there is at most one edge between such subtracks. There is no X-crossing between subtracks from different original tracks as otherwise there would be an X-crossing in the original. There are at most $\lfloor t' \rfloor$ tracks with $\lceil \frac{n}{t'} \rceil$ vertices. Since there are at most t tracks with less than $\lceil \frac{n}{t'} \rceil$ vertices, one for each of the original tracks, there is a total of at most $\lfloor t + t' \rfloor$ tracks. ∎

4. Three-dimensional drawings and track layouts. In this section we prove Theorem 2.9, which states that three-dimensional drawings with small volume are closely related to track layouts with few tracks.

LEMMA 4.1. *If a graph G has an $A \times B \times C$ drawing, then G has an improper AB-track layout, and G has a $2AB$-track layout.*

Proof. Let $V_{x,y}$ be the set of vertices of G with an X-coordinate of x and a Y-coordinate of y, where without loss of generality $1 \leq x \leq A$ and $1 \leq y \leq Y$. With each set $V_{x,y}$ ordered by the Z-coordinates of its elements, $\{V_{x,y} : 1 \leq x \leq A, 1 \leq y \leq Y\}$ is an improper AB-track assignment. There is no X-crossing, as otherwise there would be a crossing in the original drawing, and hence we have an improper AB-track layout. By Lemma 2.2, G has a $2AB$-track layout. ∎

We now prove the converse of Lemma 4.1. The proof is inspired by the generalizations of the moment curve algorithm by Cohen et al. [20] and Pach, Thiele, and Tóth [73], described in section 2.6. Loosely speaking, Cohen et al. [20] allow three "free" dimensions, whereas Pach, Thiele, and Tóth [73] use the assignment of vertices to color classes to "fix" one dimension with two dimensions free. We use an assignment of vertices to tracks to fix two dimensions with one dimension free. The style of three-dimensional drawing produced by our algorithm, where tracks are drawn vertically, is illustrated in Figure 4.1.

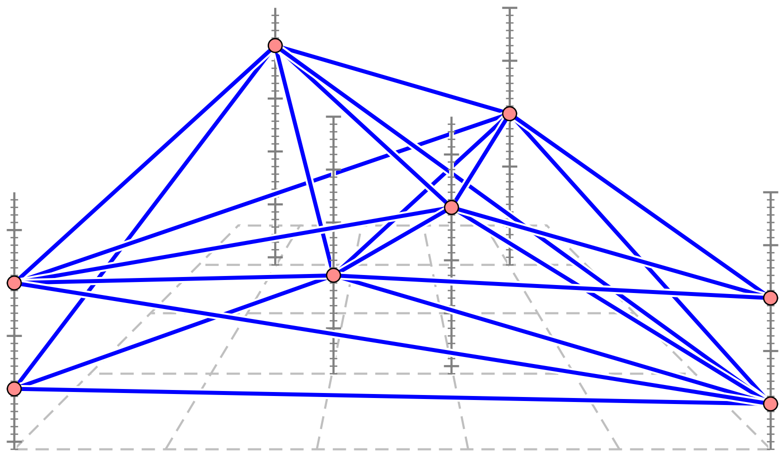

FIG. 4.1. *A three-dimensional drawing produced from a track layout.*

LEMMA 4.2. *If a graph G has a (possibly) improper k-track layout, then G has a $k \times 2k \times 2k \cdot n'$ three-dimensional drawing, where n' is the maximum number of vertices in a track.*

Proof. Suppose that $\{(V_i, <_i) : 1 \le i \le k\}$ is the given improper k-track layout. Let p be the smallest prime such that $p > k$. Then $p \le 2k$ by Bertrand's postulate. For each i, $1 \le i \le k$, represent the vertices in V_i by the grid-points

$$\{(i, i^2 \bmod p, t) : 1 \le t \le p \cdot |V_i|, \, t \equiv i^3 \pmod p\}$$

such that the Z-coordinates respect the given total order $<_i$. Draw each edge as a line-segment between its end-vertices. Suppose that two edges e and e' cross such that their end-vertices are at distinct points $(i_\alpha, i_\alpha^2 \bmod p, t_\alpha)$, $1 \le \alpha \le 4$. Then these points are coplanar, and if M is the matrix

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \bmod p & t_1 \\ 1 & i_2 & i_2^2 \bmod p & t_2 \\ 1 & i_3 & i_3^2 \bmod p & t_3 \\ 1 & i_4 & i_4^2 \bmod p & t_4 \end{pmatrix},$$

then the determinant $\det(M) = 0$. We proceed by considering the number of distinct tracks $N = |\{i_1, i_2, i_3, i_4\}|$.

- $N = 1$: By the definition of an improper track layout, e and e' do not cross.
- $N = 2$: If either edge is intratrack, then e and e' do not cross. Otherwise neither edge is intratrack, and since there is no X-crossing, e and e' do not cross.

• $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $\det(M) = (t_2 - t_1) \cdot \det(M')$, where

$$M' = \begin{pmatrix} 1 & i_2 & i_2^2 \bmod p \\ 1 & i_3 & i_3^2 \bmod p \\ 1 & i_4 & i_4^2 \bmod p \end{pmatrix}.$$

Since $t_1 \neq t_2$, $\det(M') = 0$. However, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is nonzero since i_2, i_3, and i_4 are distinct and p is a prime, a contradiction.

• $N = 4$: Let M' be the matrix obtained from M by taking each entry modulo p. Then $\det(M') = 0$. Since $t_\alpha \equiv i_\alpha^3 \pmod{p}$, $1 \leq \alpha \leq 4$,

$$M' \equiv \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} \pmod{p}.$$

Since each $i_\alpha < p$, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is nonzero since $i_\alpha \neq i_\beta$ and p is a prime. This contradiction proves there are no edge crossings. The produced drawing is at most $k \times 2k \times 2k \cdot n'$. ∎

Proof of Theorem 2.9. Let $\mathcal{F}(n)$ be a family of functions closed under multiplication. Let G be an n-vertex graph with a t-track layout, where $t \in \mathcal{F}(n)$. By Lemma 3.5 with $t' = t$, G has a $2t$-track layout with at most $\lceil \frac{n}{t} \rceil$ vertices in each track. By Lemma 4.2, G has a $2t \times 4t \times 4t \cdot \lceil \frac{n}{t} \rceil$ drawing, which is $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$. Conversely, suppose that an n-vertex graph G has an $A \times B \times \mathcal{O}(n)$ drawing, where $A, B \in \mathcal{F}(n)$. By Lemma 4.1, G has a track layout with $2AB \in \mathcal{F}(n)$ tracks. ∎

Proof of Theorem 2.14. Let $t = \mathsf{tn}(G)$, and suppose $1 \leq r \leq n/t$. By Lemma 3.5 with $t' = \frac{n}{r}$, G has a $\lfloor \frac{n}{r} + t \rfloor$-track layout with at most r vertices in each track. By assumption $t \leq \frac{n}{r}$, and the number of tracks is at most $\frac{2n}{r}$. By Lemma 4.2, G has a $\frac{2n}{r} \times \frac{4n}{r} \times 4n$ three-dimensional drawing, which has volume $32n^3/r^2$ and aspect ratio $2r$. ∎

5. Queue layouts and track layouts. In this section we prove Theorem 2.6, which states that track and queue layouts are closely related. Our first lemma highlights this fact—its proof follows immediately from the definitions (see Figure 5.1).

LEMMA 5.1. *A bipartite graph $G = (A, B; E)$ has a 2-track layout with tracks A and B if and only if G has a 1-queue layout such that in the corresponding vertex-ordering, the vertices in A appear before the vertices in B.* ∎

We now show that a queue layout can be obtained from a track layout. This result can be viewed as a generalization of the construction of a 2-queue layout of an outerplanar graph by Heath, Leighton, and Rosenberg [54] and Rengarajan and Veni Madhavan [78] (with $s = 1$).

LEMMA 5.2. *If a graph G has a (possibly) improper t-track layout $\{(V_i, <_i) : 1 \leq i \leq t\}$ with maximum edge span s $(\leq t-1)$, then $\mathsf{qn}(G) \leq s+1$, and if the given track layout is not improper, then $\mathsf{qn}(G) \leq s$.*

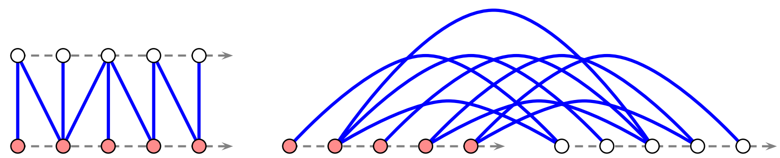

FIG. 5.1. *A 2-track layout and a 1-queue layout of a bipartite graph.*

Proof. First suppose that there are no intratrack edges. Let σ be the vertex ordering $(V_1, V_2, \dots, V_t)$ of G. Let E_α be the set of edges with span α in the given track layout. As in Lemma 5.1, two edges from the same pair of tracks are nested in σ if and only if they form an X-crossing in the track layout. Since no two edges form an X-crossing in the track layout, no two edges that are between the same pair of tracks are nested in σ. If two edges not from the same pair of tracks have the same span, then they are not nested in σ. (This idea is due to Heath and Rosenberg [58].) Thus no two edges are nested in each E_α, and we have an s-queue layout of G. If there are intra-track edges, then they all form one additional queue in σ. ∎

We now set out to prove the converse of Lemma 5.2. It is well known that the subgraph induced by any two tracks of a track layout is a forest of caterpillars [52]. A coloring of a graph is *acyclic* if every bichromatic subgraph is a forest; that is, every cycle receives at least three distinct colors. Thus a t-track layout of a graph G defines an acyclic t-coloring of G. The minimum number of colors in an acyclic coloring of G is the *acyclic chromatic number* of G, denoted by $\chi_{\mathrm{a}}(G)$. Thus,

$$\chi_{\mathrm{a}}(G) \;\leq\; \mathsf{tn}(G).$$

Acyclic colorings were introduced by Grünbaum [49], who proved that every planar graph is acyclically 9-colorable. This result was steadily improved [1, 65, 67] until Borodin [12] proved that every planar graph is acyclically 5-colorable, which is the best possible bound. Many other graph families have bounded acyclic chromatic number, including graphs embeddable on a fixed surface [2, 3, 6], 1-planar graphs [13], graphs with bounded maximum degree [5], and graphs with bounded tree-width. A folklore result states that $\chi_{\mathrm{a}}(G) \leq \mathsf{tw}(G) + 1$ (see [43]). More generally, Nešetřil and Ossona de Mendez [69] proved that every proper minor-closed graph family has bounded acyclic chromatic number. In fact, they proved that every graph G has a *star* k-coloring (every bichromatic subgraph is a forest of stars), where k is a (small) quadratic function of the maximum chromatic number of a minor of G.

LEMMA 5.3. *Every graph G with acyclic chromatic number $\chi_{\mathrm{a}}(G) \leq c$ and queue-number $\mathsf{qn}(G) \leq q$ has track-number $\mathsf{tn}(G) \leq c\,(2q)^{c-1}$.*

Proof. Let $\{V_i : 1 \leq i \leq c\}$ be an acyclic coloring of G. Let σ be the vertex-ordering in a q-queue layout of G. Consider an edge vw with $v \in V_i$, $w \in V_j$, and $i < j$. If $v <_\sigma w$, then vw is *forward*, and if $w <_\sigma v$, then vw is *backward*. Consider the edges to be colored with $2q$ colors, where each color class consists of the forward edges in a single queue, or the backward edges in a single queue.

Alon and Marshall [4] proved that given a (not necessarily proper) edge k-coloring of a graph G, any acyclic c-coloring of G can be refined to a ck^{c-1}-coloring so that the edges between any pair of (vertex) color classes are monochromatic, and each (vertex) color class is contained in some original color class. (Nešetřil and Raspaud [70] generalized this result for colored mixed graphs.) Apply this result with the given acyclic c-coloring of G and the edge $2q$-coloring discussed above. Consider the resulting $c(2q)^{c-1}$ color classes to be tracks ordered by σ. The edges between any two tracks are from a single queue, and are all forward or all backward.

Suppose that there are edges vw and xy that form an X-crossing. Since each track is a subset of some V_i, we can assume that $v, x \in V_i$, $w, y \in V_j$, and $i < j$. Suppose that vw and xy are both forward. The case in which vw and xy are both backward is symmetric. Thus $v <_\sigma w$ and $x <_\sigma y$. Since vw and xy form an X-crossing, and the tracks are ordered by σ, we have $v <_\sigma x$ and $y <_\sigma w$. Hence $v <_\sigma x <_\sigma y <_\sigma w$. That is, vw and xy are nested. This is the desired contradiction, since edges between any pair of tracks are from a single queue. Thus we have a $c(2q)^{c-1}$-track layout of G. □

Proof of Theorem 2.6. Let $\mathcal{F}(n)$ be a family of functions closed under multiplication. Let G be an n-vertex graph from a proper minor-closed graph family $\mathcal{G}$. First, suppose that G has a t-track layout, where $t \in \mathcal{F}(n)$. By Lemma 5.2, G has queue-number $\mathsf{qn}(G) \leq t - 1 \in \mathcal{F}(n)$. Conversely, suppose G has queue-number $\mathsf{qn}(G) = q \in \mathcal{F}(n)$. By the above-mentioned result of Nešetřil and Ossona de Mendez [69], G has bounded acyclic chromatic number $\chi_{\mathrm{a}}(G) \leq c \in \mathcal{O}(1)$. By Lemma 5.3, G has a t-track layout, where $t \leq c(2q)^{c-1} \in \mathcal{F}(n)$. □

6. Tree-partitions of k-trees. In this section we prove our theorem mentioned in section 2.2 regarding tree-partitions of k-trees. This result forms the cornerstone of the proof of Theorem 7.3.

THEOREM 6.1. *Let G be a k-tree with maximum degree Δ. Then G has a rooted tree-partition $(T, \{T_x : x \in V(T)\})$ such that for all nodes x of T,*

(a) *if x is a nonroot node of T and y is the parent node of x, then the set of vertices in T_y with a neighbor in T_x forms a clique C_x of G, and*

(b) *the induced subgraph $G[T_x]$ is a connected $(k-1)$-tree.*

Furthermore the width of $(T, \{T_x : x \in V(T)\})$ is at most $\max\{1, k(\Delta - 1)\}$.

Proof. We assume that G is connected, since if G is not connected, then a tree-partition of G that satisfies the theorem can be determined by adding a new root node with an empty bag, adjacent to the root node of a tree-partition of each connected component of G.

It is well known that G is a connected k-tree if and only if G has a vertex-ordering $\sigma = (v_1, v_2, \ldots, v_n)$, such that for all $i \in \{1, 2, \ldots, n\}$,

(i) if G^i is the induced subgraph $G[\{v_1, v_2, \ldots, v_i\}]$, then G^i is connected and the vertex-ordering of G^i induced by σ is a breadth-first vertex-ordering of G^i, and

(ii) the neighbors of v_i in G^i form a clique $C_i = \{v_j : v_i v_j \in E(G), j < i\}$ with $1 \leq |C_i| \leq k$ (unless $i = 1$, in which case $C_i = \emptyset$).

In the language of chordal graphs, σ is a (reverse) "perfect elimination" vertex-ordering and can be determined, for example, by the Lex-BFS algorithm by Rose, Tarjan, and Leuker [80] (also see [48]). Moreover, we can choose v_1 to be any vertex in G.

Let r be a vertex of minimum degree[6] in G. Then $\deg(r) \leq k$. Let $\sigma = (v_1, v_2, \ldots, v_n)$ be a vertex-ordering of G with $v_1 = r$ and satisfying (i) and (ii). By (i), the depth of each vertex v_i in σ is the same as the depth of v_i in the vertex-ordering of G^j induced by σ for all $j \geq i$. We therefore simply speak of *the* depth of v_i. Let V_d be the set of vertices of G at depth d.

[6] We choose r to have minimum degree to obtain a slightly improved bound on the width of the tree-partition. If we choose r to be an arbitrary vertex, then the width is at most $\max\{1, \Delta, k(\Delta-1)\}$, and the remainder of Theorem 6.1 holds.

CLAIM 1. *For all $d \geq 1$, and for every connected component Z of $G[V_d]$, the set of vertices at depth $d-1$ with a neighbor in Z form a clique of G.*

Proof. The claim is trivial for $d = 1$ or $d = 2$. Now suppose that $d \geq 3$. Assume for the sake of contradiction that there are two nonadjacent vertices x and y at depth $d-1$ such that x has a neighbor in Z and y has a neighbor in Z. Let P_1 be a shortest path between x and y with its interior vertices in Z. Let P_2 be a shortest path between x and y with its interior vertices at depth at most $d-2$. Since the interior vertices of P_1 are at depth d, there is no edge between an interior vertex of P_1 and an interior vertex of P_2. Thus $P_1 \cup P_2$ is a chordless cycle of length at least four, contradicting the fact that G is chordal (by Lemma 2.1). ▯

Define a graph T and a partition $\{T_x : x \in V(T)\}$ of $V(G)$ indexed by the nodes of T as follows. There is one node x in T for every connected component of each $G[V_d]$, whose bag T_x is the vertex-set of the corresponding connected component. We say x and T_x are at *depth d*. Clearly a vertex in a depth-d bag is also at depth d. The (unique) node of T at depth zero is called the *root* node. Let two nodes x and y of T be connected by an edge if there is an edge vw of G with $v \in T_x$ and $w \in T_y$. Thus $(T, \{T_x : x \in V(T)\})$ is a "graph-partition."

We now prove that in fact T is a tree. First observe that T is connected since G is connected. By definition, nodes of T at the same depth d are not adjacent. Moreover, nodes of T can be adjacent only if their depths differ by one. Thus T has a cycle only if there is a node x in T at some depth d such that x has at least two distinct neighbors in T at depth $d-1$. However this is impossible since, by Claim 1, the set of vertices at depth $d-1$ with a neighbor in T_x form a clique (which we call C_x) and are hence in a single bag at depth $d-1$. Thus T is a tree, and $(T, \{T_x : x \in V(T)\})$ is a tree-partition of G (see Figure 6.1).

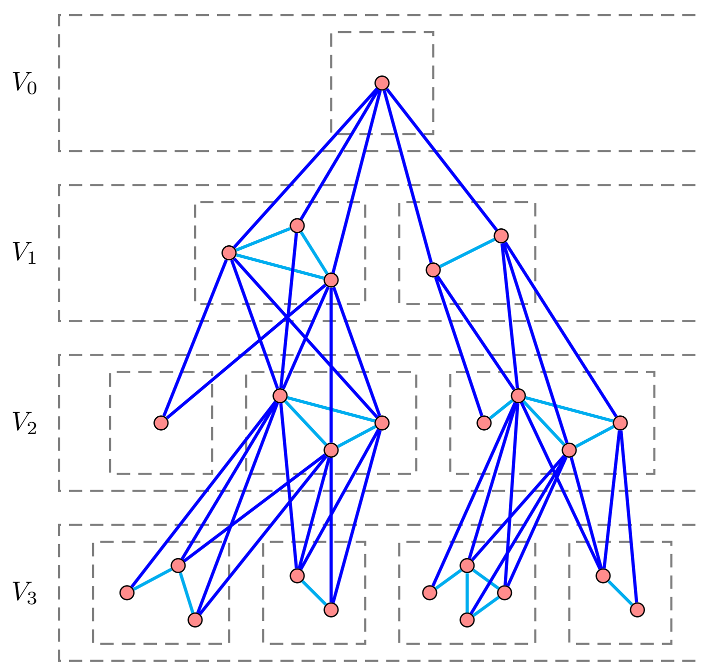

FIG. 6.1. *Illustration for Theorem* 6.1 *in the case of* $k = 3$.

We now prove that each bag T_x induces a connected $(k-1)$-tree. This is true for the root node which only has one vertex. Suppose x is a nonroot node of T at depth d. Each vertex in T_x has at least one neighbor at depth $d-1$. Thus in the vertex-ordering of T_x induced by σ, each vertex $v_i \in T_x$ has at most $k-1$ neighbors $v_j \in T_x$ with $j < i$. Thus the vertex-ordering of T_x induced by σ satisfies (i) and (ii) for $k-1$, and $G[T_x]$ is $(k-1)$-tree. By definition, each $G[T_x]$ is connected.

Finally, consider the cardinality of a bag in T. We claim that each bag contains at most $\max\{1, k(\Delta-1)\}$ vertices. The root bag has one vertex. Let x be a nonroot node of T with parent node y. Suppose that y is the root node. Then $T_y = \{r\}$, and thus $|T_x| \leq \deg(r) \leq k \leq k(\Delta-1)$, assuming $\Delta \geq 2$. If $\Delta \leq 1$, then all bags have one vertex. Now assume that y is a nonroot node. The set of vertices in T_y with a neighbor in T_x forms the clique C_x. Let $k' = |C_x|$. Thus $k' \geq 1$, and since $C_x \subseteq T_y$ and $G[T_y]$ is a $(k-1)$-tree, $k' \leq k$. A vertex $v \in C_x$ has $k'-1$ neighbors in C_x and at least one neighbor in the parent bag of y. Thus v has at most $\Delta - k'$ neighbors in T_x. Hence the number of edges between C_x and T_x is at most $k'(\Delta-k')$. Every vertex in T_x is adjacent to a vertex in C_x. Thus $|T_x| \leq k'(\Delta-k') \leq k(\Delta-1)$. This completes the proof. ◻

7. Tree-width and track layouts. In this section we prove that track-number is bounded by tree-width. Let $\{(V_i, <_i) : i \in I\}$ be a track layout of a graph G. We say a clique C of G *covers* the set of tracks $\{i \in I : C \cap V_i \neq \emptyset\}$. Let S be a set of cliques of G. Suppose that there exists a total order $\preceq$ on S such that for all cliques $C_1, C_2 \in S$, if there exists a track $i \in I$, and vertices $v \in V_i \cap C_1$ and $w \in V_i \cap C_2$ with $v <_i w$, then $C_1 \prec C_2$. In this case, we say $\preceq$ is *nice*, and S is *nicely ordered* by the track layout.

LEMMA 7.1. *Let $L \subseteq I$ be a set of tracks in a track layout $\{(V_i, <_i) : i \in I\}$ of a graph G. If S is a set of cliques each of which covers L, then S is nicely ordered by the given track layout.*

Proof. Define a relation $\preceq$ on S as follows. For every pair of cliques $C_1, C_2 \in S$, define $C_1 \preceq C_2$ if $C_1 = C_2$ or there exists a track $i \in L$ and vertices $v \in C_1$ and $w \in C_2$ with $v <_i w$. Clearly all cliques in S are comparable.

Suppose that $\preceq$ is not antisymmetric; that is, there exist distinct cliques $C_1, C_2 \in S$, distinct tracks $i, j \in L$, and distinct vertices $v_1, w_1 \in C_1$ and $v_2, w_2 \in C_2$ such that $v_1 <_i v_2$ and $w_2 <_j w_1$. Since C_1 and C_2 are cliques, the edges v_1w_1 and v_2w_2 form an X-crossing, which is a contradiction. Thus $\preceq$ is antisymmetric.

We claim that $\preceq$ is transitive. Suppose that there exist cliques $C_1, C_2, C_3 \in S$ such that $C_1 \preceq C_2$ and $C_2 \preceq C_3$. We can assume that C_1, C_2, and C_3 are pairwise distinct. Thus there are vertices $u_1 \in C_1$, $u_2 \in C_2$, $v_2 \in C_2$, and $v_3 \in C_3$ such that $u_1 <_i u_2$ and $v_2 <_j v_3$ for some pair of (not necessarily distinct) tracks $i, j \in L$. Since C_3 has a vertex in V_i and since $C_3 \not\preceq C_2$, there is a vertex $u_3 \in C_3$ with $u_2 \leq_i u_3$. Thus $u_1 <_i u_3$, which implies that $C_1 \preceq C_3$. Thus $\preceq$ is transitive.

Hence $\preceq$ is a total order on S, which by definition is nice. ◻

Consider the problem of partitioning the cliques of a graph into sets such that each set is nicely ordered by a given track layout. The following immediate corollary of Lemma 7.1 says that there exists such a partition where the number of sets does not depend upon the size of the graph.

COROLLARY 7.2. *Let G be a graph with maximum clique size k. Given a t-track layout of G, there is a partition of the cliques of G into $\sum_{i=1}^{k} \binom{t}{i}$ sets, each of which is nicely ordered by the given track layout.* ◻

We do not actually use Corollary 7.2 in the following result, but the idea of partitioning the cliques into nicely ordered sets is central to its proof.

THEOREM 7.3. *For every integer $k \geq 0$, there is a constant $t_k = 3^k \cdot 6^{(4^k-3k-1)/9}$ such that every graph G with tree-width $\mathsf{tw}(G) \leq k$ has a t_k-track layout.*

Proof. If the input graph G is not a k-tree, then add edges to G to obtain a k-tree containing G as a subgraph. It is well known that a graph with tree-width at most k is a *spanning* subgraph of a k-tree. These extra edges can be deleted once we are done. We proceed by induction on k with the following hypothesis:

For all $k \in \mathbb{N}$, there exists a constant s_k and sets $\mathcal{I}_k$ and $\mathcal{S}_k$ such that

1. $|\mathcal{I}_k| = t_k$ *and* $|\mathcal{S}_k| = s_k$,
2. *each element of $\mathcal{S}_k$ is a subset of $\mathcal{I}_k$, and*
3. *every k-tree G has a t_k-track layout indexed by $\mathcal{I}_k$, such that for every clique C of G, the set of tracks that C covers is in $\mathcal{S}_k$.*

Consider the base case with $k = 0$. A 0-tree G has no edges and thus has a 1-track layout. Let $\mathcal{I}_0 = \{1\}$, and order $V_1 = V(G)$ arbitrarily. Thus $t_0 = 1$, $s_0 = 1$, and $\mathcal{S}_0 = \{\{1\}\}$ satisfy the hypothesis for every 0-tree. Now suppose that the result holds for $k - 1$, and G is a k-tree.

Let $(T, \{T_x : x \in V(T)\})$ be a tree-partition of G described in Theorem 6.1, where T is rooted at r. Each induced subgraph $G[T_x]$ is a $(k-1)$-tree. Thus, by induction, there are sets $\mathcal{I}_{k-1}$ and $\mathcal{S}_{k-1}$ with $|\mathcal{I}_{k-1}| = t_{k-1}$ and $|\mathcal{S}_{k-1}| = s_{k-1}$ such that for every node x of T the induced subgraph $G[T_x]$ has a t_{k-1}-track layout indexed by $\mathcal{I}_{k-1}$. For every clique C of $G[T_x]$, if C covers $L \subseteq \mathcal{I}_{k-1}$, then $L \in \mathcal{S}_{k-1}$. Assume $\mathcal{I}_{k-1} = \{1, 2, \ldots, t_{k-1}\}$ and $\mathcal{S}_{k-1} = \{X_1, X_2, \ldots, X_{s_{k-1}}\}$. By Theorem 6.1, for each nonroot node x of T, if p is the parent node of x, then the set of vertices in T_p with a neighbor in T_x form a clique C_x. Let $\alpha(x) = i$, where C_x covers X_i. For the root node r of T, let $\alpha(r) = 1$.

Track layout of *T*. To construct a track layout of G we first construct a track layout of the tree T indexed by the set $\{(d, i) : d \geq 0, 1 \leq i \leq s_{k-1}\}$, where the track $L_{d,i}$ consists of nodes x of T at depth d with $\alpha(x) = i$. Here the *depth* of a node x is the distance in T from the root node r to x. We order the nodes of T within the tracks by increasing depth. There is only one node at depth $d = 0$. Suppose that we have determined the orders of the nodes up to depth $d - 1$ for some $d \geq 1$.

Let $i \in \{1, 2, \ldots, s_{k-1}\}$. The nodes in $L_{d,i}$ are ordered primarily with respect to the relative positions of their parent nodes (at depth $d - 1$). More precisely, let $\rho(x)$ denote the parent node of each node $x \in L_{d,i}$. For all nodes x and y in $L_{d,i}$, if $\rho(x)$ and $\rho(y)$ are in the same track and $\rho(x) < \rho(y)$ in that track, then $x < y$ in $L_{d,i}$. For x and y with $\rho(x)$ and $\rho(y)$ on distinct tracks, the relative order of x and y is not important. It remains to specify the order of nodes in $L_{d,i}$ with a common parent.

Suppose that P is a set of nodes in $L_{d,i}$ with a common parent node p. By construction, for every node $x \in P$, the parent clique C_x covers X_i in the track layout of $G[T_p]$. By Lemma 7.1 the cliques $\{C_x : x \in P\}$ are nicely ordered by the track layout of $G[T_p]$. Let the order of P in track $L_{d,i}$ be specified by a nice ordering of $\{C_x : x \in P\}$, as illustrated in Figure 7.1.

This construction defines a partial order on the nodes in track $L_{d,i}$, which can be arbitrarily extended to a total order. Hence we have a track assignment of T. Since the nodes in each track are ordered primarily with respect to the relative positions of their parent nodes in the previous tracks, there is no X-crossing, and hence we have a track layout of T.

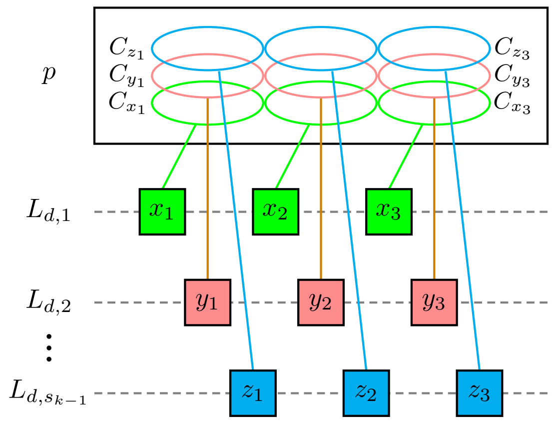

FIG. 7.1. *Track layout of nodes with a common parent p.*

Track layout of G. To construct a track assignment of G from the track layout of T, replace each track $L_{d,i}$ by t_{k-1} subtracks, and for each node x of T insert the track layout of $G[T_x]$ in place of x on the subtracks corresponding to the track containing x in the track layout of T. More formally, the track layout of G is indexed by the set

$$\{(d,i,j) : d \geq 0, 1 \leq i \leq s_{k-1}, 1 \leq j \leq t_{k-1}\}.$$

Each track $V_{d,i,j}$ consists of those vertices v of G such that, if T_x is the bag containing v, then x is at depth d in T, $\alpha(x) = i$, and v is in track j in the track layout of $G[T_x]$. If x and y are distinct nodes of T with $x < y$ in $L_{d,i}$, then $v < w$ in $V_{d,i,j}$ for all vertices $v \in T_x$ and $w \in T_y$ in track j. If v and w are vertices of G in track j in bag T_x at depth d, then the relative order of v and w in $V_{d,\alpha(x),j}$ is the same as in the track layout of $G[T_x]$.

Clearly adjacent vertices of G are in distinct tracks. Thus we have defined a track assignment of G. We claim there is no X-crossing. Clearly an intrabag edge of G is not in an X-crossing with an edge not in the same bag. By induction, there is no X-crossing between intrabag edges in a common bag. Since there is no X-crossing in the track layout of T, interbag edges of G which are mapped to edges of T without a common parent node are not involved in an X-crossing.

Consider a parent node p in T. For each child node x of p, the set of vertices in T_p adjacent to a vertex in T_x forms the clique C_x. Thus there is no X-crossing between a pair of edges both from C_x to T_x, since the vertices of C_x are on distinct tracks. Consider two child nodes x and y of p. For there to be an X-crossing between an edge from T_p to T_x and an edge from T_p to T_y, the nodes x and y must be on the same track in the track layout of T. Suppose $x < y$ in this track. By construction, C_x and C_y cover the same set of tracks, and $C_x \preceq C_y$ in the corresponding nice ordering. Thus for any track containing vertices $v \in C_x$ and $w \in C_y$, $v \leq w$ in that track. Since all the vertices in T_x are to the left of the vertices in T_y (in a common track), there is no X-crossing between an edge from T_p to T_x and an edge from T_p to T_y. Therefore there is no X-crossing, and hence we have a track layout of G.

Wrapped track layout of G. 0As illustrated in Figure 7.2, we now "wrap" the track layout of G in the spirit of Lemma 3.1. In particular, define a track assignment

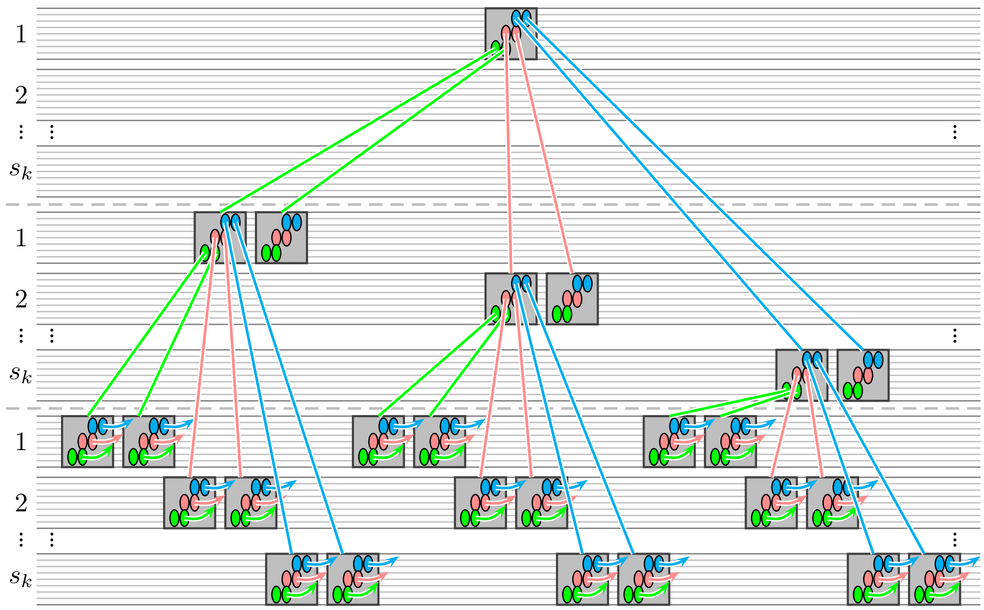

FIG. 7.2. *Wrapped track layout in Theorem* 7.3.

of G indexed by

$$\{(d', i, j) : d' \in \{0, 1, 2\}, 1 \leq i \leq s_{k-1}, 1 \leq j \leq t_{k-1}\},$$

where each track

$$W_{d',i,j} = \bigcup \{V_{d,i,j} : d \equiv d' \pmod 3\}.$$

If $v \in V_{d,i,j}$ and $w \in V_{d+3,i,j}$, then $v < w$ in the order of $W_{d',i,j}$ (where $d' = d \bmod 3$). The order of each $V_{d,i,j}$ is preserved in $W_{d',i,j}$. The set of tracks $\{W_{d',i,j} : d' \in \{0, 1, 2\}, 1 \leq i \leq s_{k-1}, 1 \leq j \leq t_{k-1}\}$ forms a track assignment of G.

For every edge vw of G, the depths of the bags in T containing v and w differ by at most one. Thus in the wrapped track assignment of G, adjacent vertices remain on distinct tracks, and there is no X-crossing. The number of tracks is $3 \cdot s_{k-1} \cdot t_{k-1}$.

Every clique C of G is either contained in a single bag of the tree-partition or is contained in two adjacent bags. Let

$$\mathcal{S}' = \{\{(d', i, h) : h \in X_j\} : d' \in \{0, 1, 2\}, 1 \leq i, j \leq s_{k-1}\}.$$

For every clique C of G contained in a single bag, the set of tracks containing C is in $\mathcal{S}'$. Let

$$\mathcal{S}'' = \{\{(d', i, \ell) : \ell \in X_j\} \cup \{((d'+1) \bmod 3, p, h) : h \in X_q\} : d' \in \{0, 1, 2\}, 1 \leq i, j, p, q \leq s_{k-1}\}.$$

For every clique C of G contained in two bags, the set of tracks containing C is in $\mathcal{S}''$. Observe that $\mathcal{S}' \cup \mathcal{S}''$ is independent of G. Hence $\mathcal{S}_k = \mathcal{S}' \cup \mathcal{S}''$ satisfies the hypothesis for k.

Now $|\mathcal{S}'| = 3s_{k-1}^2$ and $|\mathcal{S}''| = 3s_{k-1}^4$, and thus $|\mathcal{S}' \cup \mathcal{S}''| = 3s_{k-1}^2(s_{k-1}^2 + 1)$. Therefore any solution to the following set of recurrences satisfies the theorem:

$$s_0 \geq 1, \quad t_0 \geq 1, \quad s_k \geq 3s_{k-1}^2(s_{k-1}^2 + 1), \quad t_k \geq 3s_{k-1} \cdot t_{k-1}. \tag{7.1}$$

We claim that

$$s_k = 6^{(4^k-1)/3} \quad \text{and} \quad t_k = 3^k \cdot 6^{(4^k-3k-1)/9}$$

is a solution to (7.1). Observe that $s_0 = 1$ and $t_0 = 1$. Now

$$3s_{k-1}^2(s_{k-1}^2 + 1) \leq 6s_{k-1}^4$$

and

$$6(6^{(4^{k-1}-1)/3})^4 = 6^{1+4(4^{k-1}-1)/3} = 6^{(4^k-1)/3} = s_k.$$

Thus the recurrence for s_k is satisfied. Now

$$\begin{aligned} 3 \cdot s_{k-1} \cdot t_{k-1} &= 3 \cdot 6^{(4^{k-1}-1)/3} \cdot 3^{k-1} \cdot 6^{(4^{k-1}-3(k-1)-1)/9} \\ &= 3^k \cdot 6^{(3\cdot 4^{k-1}-3+4^{k-1}-3k+3-1)/9} \\ &= 3^k \cdot 6^{(4^k-3k-1)/9} \\ &= t_k. \end{aligned}$$

Thus the recurrence for t_k is satisfied. This completes the proof. ∎

In the proof of Theorem 7.3 we have made little effort to reduce the bound on t_k, beyond that it is a doubly exponential function of k. In [35] we describe a number of refinements that result in improved bounds on t_k. One such refinement uses strict k-trees. From an algorithmic point of view, the disadvantage of using strict k-trees is that at each recursive step, extra edges must be added to enlarge the graph from a partial strict k-tree into a strict k-tree, whereas when using (nonstrict) k-trees, extra edges need be added only at the beginning of the algorithm.

For small values of k, much-improved results can be obtained. For example, we prove that every series-parallel graph (that is, with tree-width at most two) has an 18-track layout [35], whereas $t_2 = 54$. This bound has recently been improved to 15 by Di Giacomo, Liotta, and Meijer [25]. Their method is based on Theorems 6.1 and 7.3, and in the general case still gives a doubly exponential upper bound on the track-number of graphs with tree-width k. For other particular classes of graphs, Di Giacomo [24] and Di Giacomo and Meijer [27] recently improved the constants in our results.

Our doubly exponential upper bound is probably not best possible. Di Giacomo, Liotta, and Meijer [25] constructed graphs with tree-width k and track-number at least $2k + 1$. The following construction establishes a quadratic lower bound. It is similar to a graph due to Albertson et al. [3], which gives a tight lower bound on the star chromatic number of graphs with tree-width k.

THEOREM 7.4. *For all $k \geq 0$, there is a graph G_k with tree-width at most k and track-number* $\mathsf{tn}(G_k) = \frac{1}{2}(k+1)(k+2)$.

Proof. Let $G_0 = K_1$. Obviously G_0 has tree-width 0. Construct G_k from G_{k-1} as follows. Start with a k-clique $\{v_1, v_2, \dots, v_k\}$. Let $n = 2(\frac{1}{2}(k+1)(k+2) - 1 - k) + 1$. Add n vertices $\{w_1, w_2, \dots, w_n\}$, each adjacent to every v_i. Let $H_1, H_2, \dots, H_n$ be

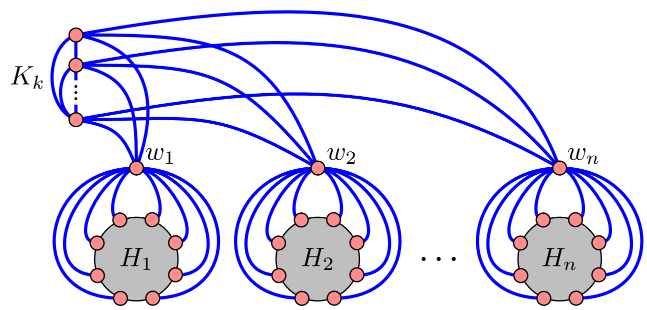

FIG. 7.3. *The graph* G_k.

copies of G_{k-1}. For all $1 \leq j \leq n$, add an edge between w_j and each vertex of H_j, as illustrated in Figure 7.3. It is easily seen that from a tree decomposition of G_{k-1} of width $k-1$ we can construct a tree decomposition of G_k of width k. Thus G_k has tree-width at most k.

To prove that $\mathsf{tn}(G_k) \geq \frac{1}{2}(k+1)(k+2)$, we proceed by induction on $k \geq 0$. Obviously $\mathsf{tn}(G_0) = 1$. Suppose that $\mathsf{tn}(G_{k-1}) \geq \frac{1}{2}k(k+1)$ but $\mathsf{tn}(G_k) \leq \frac{1}{2}(k+1)(k+2)-1$. Since $\{v_1, v_2, \dots, v_k\}$ is a clique, we can assume that v_i is in track i. Since each vertex w_j is adjacent to each v_i, no w_j is in tracks $\{1, 2, \dots, k\}$. There are $\frac{1}{2}(k+1)(k+2) - 1 - k$ remaining tracks. Since n is more than twice this number, there are at least three w_j vertices in a single track. Without loss of generality, $w_1 < w_2 < w_3$ in track $k+1$. No vertex x of H_2 is in track $i \in \{1, 2, \dots, k\}$, as otherwise xw_2 would form an X-crossing with v_iw_1 or v_iw_3. No vertex x of H_2 is in track $k+1$, since x and w_2 are adjacent, and w_2 is in track $k+1$. Thus all the vertices of H_2 are in tracks $\{k+2, k+3, \dots, \frac{1}{2}(k+1)(k+2)-1\}$. There are $\frac{1}{2}(k+1)(k+2) - 1 - (k+1) = \frac{1}{2}k(k+1) - 1$ such tracks. This contradicts the assumption that $\mathsf{tn}(G_{k-1}) \geq \frac{1}{2}k(k+1)$. Therefore $\mathsf{tn}(G_k) \geq \frac{1}{2}(k+1)(k+2)$.

It remains to prove that $\mathsf{tn}(G_k) \leq \frac{1}{2}(k+1)(k+2)$. Suppose we have a $\frac{1}{2}k(k+1)$-track layout of G_{k-1}. Thus each H_j has a $\frac{1}{2}k(k+1)$-track layout. Put each vertex v_i of G_k in track i. Put the vertices $\{w_1, w_2, \dots, w_n\}$ in track $k+1$ in this order. Put the track layout of each H_j in tracks $k+2, k+3, \dots, \frac{1}{2}(k+1)(k+2)$ such that the vertices of H_j precede the vertices of H_{j+1}. Clearly there are no X-crossings. ☐

Also note that Theorem 7.4 (for $k \geq 2$) can be extended using the proof technique of Lemma 2.3 to give the same lower bound for improper track layouts.

8. Open problems.

1. (In the conference version of their paper) Felsner, Liotta, and Wismath [42] asked whether every planar graph has a three-dimensional drawing with $\mathcal{O}(n)$ volume. By Theorem 2.9, this question has an affirmative answer if every planar graph has a $\mathcal{O}(1)$ track-number. Whether every planar graph has $\mathcal{O}(1)$ track-number is an open problem due to H. de Fraysseix [private communication, 2000] and, by Theorem 2.6, is equivalent to the following question.

2. Heath and colleagues [58, 54] asked whether every planar graph has a $\mathcal{O}(1)$ queue-number. The best known upper bound on the queue-number of a planar graph is $\mathcal{O}(\sqrt{n})$. In general, Dujmović and Wood [37] proved that every m-edge graph has queue-number at most $\boldsymbol{e}\sqrt{m}$, where $\boldsymbol{e}$ is the base of the natural logarithm.

3. Heath and colleagues [58, 54] also asked whether stack-number is bounded by queue-number (and vice-versa). Note that there is a family of graphs $\mathcal{G}$ with

$\mathsf{sn}(G) \in \Omega(3^{\Omega(\mathsf{qn}(G))-\epsilon})$ for all $G \in \mathcal{G}$ [54].

4. Is the queue-number of a graph bounded by a polynomial (or even singly exponential) function of its tree-width?

Note added in proof. Subsequent to this research, Dujmović and Wood [38] proved that graphs excluding a fixed graph as a minor, such as planar graphs, have three-dimensional drawings with $\mathcal{O}(n^{3/2})$ volume, as do graphs with bounded degree; Dujmović, Pór, and Wood [34] proved that track-number and queue-number are tied for all graphs; and Theorem 6.1 has been generalized (with a different proof) by Wood [92].

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