# Leader-Follower Cooperative Attitude Control of Multiple Rigid Bodies

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**Abstract**

In this paper we extend our previous results on coordinated control of rotating rigid bodies to the case of teams with heterogenous agents. We assume that only a certain subgroup of the agents (the leaders) are vested with the main control objective, that is, maintain constant relative orientation amongst themselves. The other members of the team must meet relaxed control specifications, namely, maintain their respective orientations within certain bounds, dictated by the orientation of the leaders. The proposed control laws respect the limited information each rigid body has with respect to the rest of its peers (leaders or followers), as well as with the rest of the team. Each rigid body is equipped with a feedback control law that utilizes the Laplacian matrix of the associated communication graph, and which encodes the limited communication capabilities between the team members. Similarly to the single integrator case, the convergence of the system relies on the connectivity of the communication graph.

*Key words:* Decentralized Control, Autonomous Systems, Spacecraft Control.

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## 1 Introduction

Cooperative distributed control strategies for multiple vehicles have gained increased attention in recent years in the control community, owing to the fact that such strategies provide attractive solutions to large-scale multi-agent problems, both in terms of complexity in the formulation of the problem, as well as in terms of the computational load required for its solution.

A typical control objective for a team of agents is the state-agreement or consensus problem. This control objective has been extensively pursued in recent years. Several results are based on treating the vehicle as a single integrator [12,1,6,14] or double integrator [17,11,7]. A recent review of the various approaches for solving the consensus problem when the underlying dynamics are linear can be found in [15]. A common analysis tool that is used to model these distributed systems is algebraic graph theory [5].

Extending the previous results to systems whose dynamics are nonlinear is, in general, a nontrivial task. A large and important class (in terms of applications) of systems whose dynamics are nonlinear are systems of rotating rigid bodies. Motivated by the fact that – despite the nonlinear dynamics – linear controllers can stabilize a single rigid body [20], in this paper we propose a control strategy that exploits graph-theoretic tools for cooperative control of multiple rigid bodies. We extend our previous work in this area [2] to address the case of teams with heterogenous agents. For some applications (i.e., Earth monitoring or stellar observation using a satellite cluster with a large baseline) it may be necessary for some satellites to acquire and maintain a certain (perhaps nonzero) relative orientation among themselves. A primary control objective is therefore to stabilize a subgroup of the team (leaders) to certain relative orientations. The orientations of the rest of the team (followers) are to remain within a certain orientation boundary, determined–in this case–by the convex hull of the leaders' orientations. At the same time, each agent is allowed to communicate its state (orientation and angular velocity) only with certain members of the team. These constraints limit the information exchange between the agents. The control laws for each agent proposed in this paper respect this limited information each rigid body has with respect to the rest of the team (leader or followers). A preliminary version of the paper appeared in [3].

We should mention that cooperative control of multiple rigid bodies has been addressed recently by many authors, notably [8,22,9,10]. While these papers use distributed consensus algorithms to achieve the desired objective, the specific algebraic graph theoretic framework (that is, the use of graph Laplacians) encountered in this work has not been considered in these papers. Recall that the Laplacian matrix encodes the limited communication capabilities between team members. Similarly to the linear case, the convergence of the multi-agent system relies on the connectivity of the communication graph.

## 2 System and Problem Definition

We consider a team of $N$ rigid bodies (henceforth called agents) indexed by the set $\mathcal{N} = \{1, \ldots, N\}$ . The dynamics of the $i$-th agent are given by [20]:

$$J_i \dot{\omega}_i = S(\omega_i) J_i \omega_i + u_i, \quad i \in \mathcal{N}, \tag{1}$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity vector, $u_i \in \mathbb{R}^3$ is the external torque vector, and $J_i \in \mathbb{R}^{3\times 3}$ is the symmetric inertia matrix of the $i$-th agent, all expressed in the $i$-th agent's body-fixed frame. The matrix $S(\cdot)$ denotes a skew-symmetric matrix representing the cross product between two vectors, i.e., $S(v_1)v_2 = -v_1 \times v_2$.

In this paper, the orientation of the rigid bodies with respect to the inertial frame will be described in terms of the Modified Rodriguez Parameters (MRPs)[16,18]. The MRP vector $\sigma \in \mathbb{R}^3$ is defined by

$$\sigma = \eta \tan \frac{\phi}{4}, \qquad -2\pi < \phi < 2\pi, \tag{2}$$

where $\eta$ is the eigenaxis unit vector and $\phi$ is the eigenangle corresponding to the given orientation (via Euler's theorem).

We hasten to point out that this choice can be done without loss of generality. If necessary, the analysis in terms of quaternions can be carried out by the interested reader *mutatis mutandis* following the developments below. The use of the MRPs, nonetheless, simplifies the analysis and the ensuing formulas, since there is no additional equality constraint to worry about, as for the quaternion case. Another advantage of the MRPs is the fact that they can parameterize eigenaxis rotations up to 360 deg, as it is evident from (2). In contrast, other three-dimensional parameterizations are limited to eigenaxis rotations of less than 180 deg; see [19,16] for more details.

Let $\sigma \in \mathbb{R}^3$ denote the MRP parameter vector that represents the orientation of a rigid body with respect to the inertial frame. The corresponding rotation matrix that relates inertial vector coordinates to the body-fixed coordinates is given by [16,21]

$$R(\sigma) = I_3 + 4\frac{(1 - \sigma^{\mathsf{T}}\sigma)}{(1 + \sigma^{\mathsf{T}}\sigma)^2} S(\sigma) + \frac{8}{(1 + \sigma^{\mathsf{T}}\sigma)^2} S^2(\sigma). \tag{3}$$

Using the MRPs the kinematics of the $i$-th agent are given by:

$$\dot{\sigma}_i = G(\sigma_i)\,\omega_i, \quad i \in \mathcal{N}, \tag{4}$$

where $G : \mathbb{R}^3 \mapsto \mathbb{R}^{3\times 3}$ is given by

$$G(\sigma_i) := \frac{1}{2}\left(\frac{1-\sigma_i^\intercal\sigma_i}{2} I_3 - S(\sigma_i) + \sigma_i\sigma_i^\intercal\right).$$

The matrix $G(\sigma_i)$ has the following properties [20]:

$$\sigma_i^\intercal G(\sigma_i)\,\omega_i = \left(\frac{1+\sigma_i^\intercal\sigma_i}{4}\right)\sigma_i^\intercal\omega_i, \tag{5}$$

$$G(\sigma_i)\,G^\intercal(\sigma_i) = \left(\frac{1+\sigma_i^\intercal\sigma_i}{4}\right)^2 I_3. \tag{6}$$

We assume that the team admits a leader-follower architecture. Specifically, we assume that the agents belong to either one of the two following subsets, namely, the subset of leaders $\mathcal{N}^l$, or the subset of followers $\mathcal{N}^f$. Subsequently, $\mathcal{N}^l \cap \mathcal{N}^f = \varnothing$ and $\mathcal{N}^l \cup \mathcal{N}^f = \mathcal{N}$.

A primary objective of each *leader* is to converge to a desired *relative* orientation with respect to the rest of the leaders. We assume that each leader $i \in \mathcal{N}^l$ is assigned a specific subset $\mathcal{N}_i^l \subseteq \mathcal{N}^l$ from the rest of the leaders, called the $i$-th's (leader) agent *leader communication set*. This is the set of leaders the $i$-th leader can communicate with, in order to achieve the desired objective, namely, to be stabilized in desired relative orientations $\sigma_{ij}^d$ with respect to each member $j \in \mathcal{N}_i^l$. It is assumed that the communication topology with respect to $\mathcal{N}_i^l$ for all $i \in \mathcal{N}^l$ is bidirectional, in the sense that $j \in \mathcal{N}_i^l$ if and only if $i \in \mathcal{N}_j^l$ for all $i, j \in \mathcal{N}^l,\ i \neq j$.

A secondary objective is for the leaders to "drag" the followers along, so that, at the final leader configuration, the latter are "contained" within the convex hull of the leader orientations. This is a sub-case of the containment control problem dealt with in multi-agent systems, which has been encountered in [4]. The reader is referred to that reference for a discussion on specific applications of the containment problem. For this objective, both the leaders and the followers are assigned to a specific subset $\mathcal{N}_i \subseteq \mathcal{N}$ from the rest of the team called $i$-th's (leader or follower) agent *leader-follower communication set*. This is the set of other agents the $i$-th agent can communicate with, in order to achieve the desired objective (that is, containment of the followers' final orientations in the convex hull of the leaders' orientations). For this case we assume that the sets $\mathcal{N}_i, \mathcal{N}_i^l$ are disjoint, i.e. $\mathcal{N}_i \cap \mathcal{N}_i^l = \varnothing$, for all $i \in \mathcal{N}^l$. Hence, for the containment objective the leader-follower communication set of each leader contains only followers. However, the leader-follower communication set of each follower may contain both leaders and followers.

The previous two control objectives can be encoded by two different communication graphs, which are defined with respect to the limited communication of all the agents as follows:

(1) The *leader communication graph* $G^l := \{V^l, E^l, C\}$ is the undirected graph consisting of: (i) the set of vertices $V^l = \mathcal{N}^l$ indexed by the leaders of the multi-agent team, (ii) a set of edges, $E^l = \{(i,j) \in V^l \times V^l | i \in \mathcal{N}_j^l\}$ containing pairs of nodes that represent inter-leader formation specifications, and (iii) the set of labels $C = \{\sigma_{ij}^d\}$, where $(i,j) \in E^l$, that specify the desired inter-agent relative orientations in the leader formation configuration.
(2) The *leader-follower communication graph* $G := \{V, E\}$ is the undirected graph that consists of (i) a set of vertices $V = \mathcal{N}$ indexed by the team members, and (ii) a set of edges, $E = \{(i,j) \in V \times V | i \in \mathcal{N}_j\}$ containing the pairs of nodes that represent inter-agent communication specifications.

As an example, suppose that for a seven-agent team whose members are indexed by $\mathcal{N} = 1, \ldots, 7$, we have $\mathcal{N}^l = \{1,2,3\}$, $\mathcal{N}^f = \{4,5,6,7\}$ and the communication sets are defined as $\mathcal{N}_1^l = \{2\}$, $\mathcal{N}_2^l = \{1,3\}$, $\mathcal{N}_3^l = \{2\}$ and $\mathcal{N}_1 = \{4,5\}$, $\mathcal{N}_2 = \{5\}$, $\mathcal{N}_3 = \{6,7\}$, $\mathcal{N}_4 = \{1\}$, $\mathcal{N}_5 = \{1,2,6\}$, $\mathcal{N}_6 = \{3,5,7\}$, $\mathcal{N}_7 = \{3,6\}$. The leader communication graph and the leader-follower communication graph corresponding to these communication sets are shown in Figure 1.

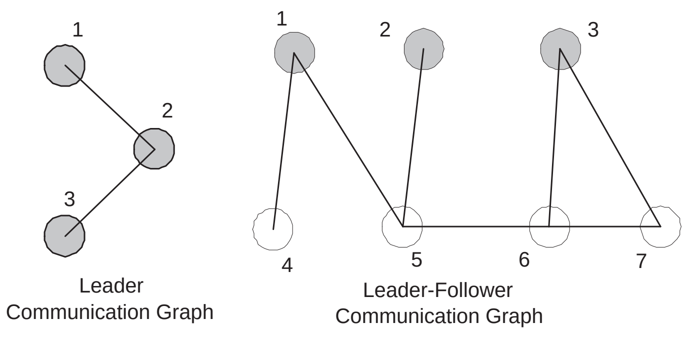


Fig. 1. Leader communication graph and leader-follower communication graph of a seven agent team with $\mathcal{N}^l = \{1,2,3\}$, $\mathcal{N}^f = \{4,5,6,7\}$ and communication sets $\mathcal{N}_1^l = \{2\}$, $\mathcal{N}_2^l = \{1,3\}$, $\mathcal{N}_3^l = \{2\}$, $\mathcal{N}_1 = \{4,5\}$, $\mathcal{N}_2 = \{5\}$, $\mathcal{N}_3 = \{6,7\}$, $\mathcal{N}_4 = \{1\}$, $\mathcal{N}_5 = \{1,2,6\}$, $\mathcal{N}_6 = \{3,5,7\}$, $\mathcal{N}_7 = \{3,6\}$. The leaders are shown in darker color.

## 3 Control Design and Stability Analysis

### *3.1 Tools from Algebraic Graph Theory*

In this subsection we review some tools from algebraic graph theory [5] that we will use in the sequel.

For an undirected graph $G$ with $n$ vertices, the *adjacency matrix* $A = A(G) = (a_{ij})$ is the $n \times n$ symmetric matrix given by $a_{ij} = 1$, if $(i, j) \in E$ and $a_{ij} = 0$, otherwise. If there is an edge connecting two vertices $i, j$, that is, $(i, j) \in E$, then $i, j$ are *adjacent*. A *path* of length $r$ from a vertex $i$ to a vertex $j$ is a sequence of $r + 1$ distinct vertices starting from $i$ and ending at $j$, such that consecutive vertices are adjacent. If there is a path between any two vertices of $G$, then $G$ is *connected*. Otherwise, it is *disconnected*. The *degree* $d_i$ of vertex $i$ is the number of its neighboring vertices, that is, $d_i = \{\#j : (i, j) \in E\}$. Let $\Delta$ be the $n \times n$ diagonal matrix with elements $d_i$ on the diagonal. The (combinatorial) *Laplacian* of $G$ is the symmetric positive semidefinite matrix $L := \Delta - A$. The Laplacian matrix $L$ captures many topological properties of the graph. Of particular interest is the fact that for a connected graph, the Laplacian has a single zero eigenvalue, and the corresponding eigenvector is the vector with all its elements equal to one, denoted by $\overrightarrow{\mathbf{1}}$.

### *3.2 Multiple Stationary Leaders*

We assume that the leaders are responsible for a global objective, and their time evolution is independent of the followers' motion. In this section, we first assume that the leaders have converged to some desired final orientations with zero angular velocity, i.e., we have

$$\sigma_i = \sigma_i^d, \quad \omega_i = 0, \qquad i \in \mathcal{N}^l. \tag{7}$$

Consider the case when the leaders must "drag" the followers to a configuration where the orientations of the latter are "contained" within the convex hull of the leader orientations in the final formation configuration. In the multiple satellite scheme this case implies, for instance, coverage of a specific area. In this case, the leaders' orientations dictate the "boundary" of the area to be covered.

The control law of the followers is given by:

$$u_i = -G^{\mathsf{T}}(\sigma_i) \sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) - \sum_{j \in \mathcal{N}_i} (\omega_i - \omega_j), \quad i \in \mathcal{N}^f. \tag{8}$$

The following Theorem then holds:

**Theorem 1** *Assume that the leader-follower communication graph is connected and that the subset of leaders is nonempty. Then the control law (8) drives the followers to the convex hull of the leaders' orientations with zero angular velocities.*

**Proof:** Let $u, \omega, \sigma \in \mathbb{R}^{3N}$ be the stack vectors of all the control inputs, the angular velocities and the orientations of the multi-agent team, respectively.

Consider the nonnegative function

$$V(\sigma, \omega) := \sum_{i=1}^{N} \left( \frac{1}{2} \omega_i^\intercal J_i \omega_i \right) + \frac{1}{2} \sigma^\intercal \left( L \otimes I_3 \right) \sigma$$

as a candidate Lyapunov function, where $L$ is the Laplacian of leader-follower communication graph $G$. From (1) we have that

$$\omega_i^\intercal J_i \dot{\omega}_i = \omega_i^\intercal S\left(\omega_i\right) J_i \omega_i + \omega_i^\intercal u_i = \omega_i^\intercal u_i$$

for all $i \in \mathcal{N}$, due to the definition of $S(\omega_i)$. The time derivative of $V$ is given by

$$\begin{aligned} \dot{V}\left(\sigma, \omega\right) &= \sum_{i=1}^{N} \left( \omega_i^\intercal J_i \dot{\omega}_i \right) + \sigma^\intercal \left( L \otimes I_3 \right) \dot{\sigma} \\ &= \sum_{i=1}^{N} \left( u_i^\intercal \omega_i + \sum_{j \in \mathcal{N}_i} \left( \sigma_i - \sigma_j \right)^\intercal G\left(\sigma_i\right) \omega_i \right), \end{aligned}$$

and since $\omega_i = 0$, for all $i \in \mathcal{N}^l$, substituting $u_i$ from (8) we get

$$\dot{V}\left(\sigma, \omega\right) = \sum_{i \in \mathcal{N}^f} \omega_i^\intercal \left( u_i + G^\intercal\left(\sigma_i\right) \sum_{j \in \mathcal{N}_i} \left( \sigma_i - \sigma_j \right) \right) = - \sum_{i \in \mathcal{N}^f} \omega_i^\intercal \sum_{j \in \mathcal{N}_i} \left( \omega_i - \omega_j \right).$$

Quoting again the fact that $\omega_i = 0$, for all $i \in \mathcal{N}^l$, we get

$$\sum_{i \in \mathcal{N}^f} \omega_i^\intercal \sum_{j \in \mathcal{N}_i} \left( \omega_i - \omega_j \right) = \sum_{i=1}^{N} \omega_i^\intercal \sum_{j \in \mathcal{N}_i} \left( \omega_i - \omega_j \right) = \omega^\intercal \left( L \otimes I_3 \right) \omega$$

so that

$$\dot{V}\left(\sigma, \omega\right) = -\omega^\intercal \left( L \otimes I_3 \right) \omega \leq 0.$$

The last inequality implies, in particular, that $V$ remains bounded. We first show that the level sets of $V$ define compact sets in the product space of the angular velocities and relative orientations of the agents [1]. Specifically, the set

[1] Note that $\sigma_{ij} := \sigma_i - \sigma_j$ is *not* the MRP vector corresponding to the rotation

$\Omega_c = \{(\omega, \sigma) : V(\sigma, \omega) \le c\}$ for $c > 0$ is closed by the continuity of $V$. For all $(\omega, \sigma) \in \Omega_c$ we have

$$\omega_i^\intercal J_i \omega_i \le 2c \Rightarrow \|\omega_i\| \le \sqrt{\frac{2c}{\lambda_{\min}(J_i)}}.$$

Furthermore, we also have

$$\sigma^\intercal (L \otimes I_3)\, \sigma \le 2c \Rightarrow \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \|\sigma_i - \sigma_j\|^2 \le 2c.$$

Hence,

$$\|\sigma_i - \sigma_j\|^2 \le 4c, \qquad \forall\, (i, j) \in E,$$

where $E$ is the edge set of the leader-follower communication graph $G$. Connectivity of $G$ ensures that the maximum length of a path connecting two vertices of the graph is at most $N-1$. Hence $\|\sigma_i - \sigma_j\| \le 2\sqrt{c\,(N-1)}$, for all $i, j \in \mathcal{N}$.

Using now the fact that $\sigma_i = \sigma_i^d$ for all $i \in \mathcal{N}^l$ we can show that the set $\Omega_c$ is also bounded in the space of *absolute* orientations. This is obviously true for the leaders which are static, while for each follower $j \in \mathcal{N}^f$ and any leader $i \in \mathcal{N}^l$ we have $\left\|\sigma_j - \sigma_i^d\right\| \le 2\sqrt{c\,(N-1)}$ which implies that

$$\|\sigma_j\| \le 2\sqrt{c\,(N-1)} + \sigma^{d*}, \forall j \in \mathcal{N}^f,$$

where

$$\sigma^{d*} \triangleq \max_{i \in \mathcal{N}^l} \left\{ \left\|\sigma_i^d\right\| \right\}.$$

Hence the stack vector $\sigma$ remains bounded. In essence, the set $\Omega_c$ is closed and bounded in the product space of the agents' angular velocities and *absolute* orientations.

By LaSalle's invariance principle, the system converges to the largest invariant set inside the set

$$M := \{(\omega, \sigma) : \omega^\intercal (L \otimes I_3)\, \omega = 0\}.$$

Since $L \otimes I_3$ is positive semidefinite, if follows that $(L \otimes I_3)\omega = 0$, which implies that

$$L\omega^1 = L\omega^2 = L\omega^3 = 0, \tag{9}$$

where $\omega^1, \omega^2, \omega^3 \in \mathbb{R}^N$ are the stack vectors of the three coefficients of the agents' angular velocities, respectively. Connectivity of the leader-follower

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matrix $R(\sigma_j)R^\intercal(\sigma_i)$. The latter can be easily computed, nonetheless, from $\sigma_{ij}$ and the knowledge of either $\sigma_i$ or $\sigma_j$ via the use of (3). Similarly, given the rotation matrix between the desired body frames of agents $i$ and $j$, along with $\sigma_i$ or $\sigma_j$, the desired "relative attitude" $\sigma_{ij}$ can be readily computed.

communication graph implies that $L$ has a simple zero eigenvalue with corresponding eigenvector $\overrightarrow{\mathbf{1}}$. Equation (9) now implies that $\omega^1, \omega^2, \omega^3$ are eigenvectors of the matrix $L$ that correspond to the zero eigenvalue, thus they belong to span$\{\overrightarrow{\mathbf{1}}\}$. Hence $\omega_i = \omega_j$ for all $i, j \in \mathcal{N}$, implying that all $\omega_i$'s converge to a common value $\omega^*$ at steady state. Since $\omega_i = 0$ for all $i \in \mathcal{N}^l$, we have that $\omega^* = 0$, and hence all agents assume zero angular velocities.

By virtue of (1), the control inputs of all followers tend to zero, and

$$u_i = -G^{\intercal}(\sigma_i) \sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) = 0,$$

which implies that

$$G(\sigma_i) G^{\intercal}(\sigma_i) \sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) = 0$$

or

$$\left(\frac{1 + \sigma_i^{\intercal}\sigma_i}{4}\right)^2 \sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) = 0$$

and finally,

$$\sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) = 0, \quad \forall i \in \mathcal{N}^f.$$

Hence, we deduce that the agents' orientations, at steady state, satisfy:

$$\begin{gathered} (L\sigma^1)_i = (L\sigma^2)_i = (L\sigma^3)_i, \quad i \in \mathcal{N}^f \\ \sigma_i = \sigma_i^d, \quad i \in \mathcal{N}^l \end{gathered} \tag{10}$$

The solutions of (10) have been studied in [4]. In particular, Theorem 2 in [4] states that for a connected leader-follower communication graph and a nonempty set of leaders, the orientation of each follower, as given by the solution of (10), lies in the convex hull of the leaders' orientations. $\diamondsuit$

## 4 Leader Relative Orientation Control Design

In this section we present a control algorithm that drives the team of leaders to the desired relative orientations. This is a problem that resembles the formation control problem in multi-vehicle systems. The relative orientation for each pair of leaders may be different, and is dictated by the mission requirements. We impose that, for each pair $(i, j) \in E^l$, there exists a desired relative orientation $\sigma_{ij}^d \in \mathbb{R}^3$, to which the corresponding pair of leaders $(i, j) \in E^l$ must converge (see again footnote on page 7 on our non-standard definition of the relative orientation between two agents.). In the sequel, we provide a control law that respects the limited communication requirements dictated by the leader communication graph $G^l$ in order to achieve this objective.

We first assume that a leader $\alpha \in \mathcal{N}^l$ plays the role of a reference attitude with respect to which the desired relative orientations should be fulfilled. This may represent the case of a satellite that is initially aware of the desired target area and has already converged to it. The rest of the leaders will attain desired relative orientations with respect to this leader. We assume that this satellite has already been stabilized to a desired equilibrium point, that is,

$$\sigma_\alpha = \sigma_\alpha^d, \quad \omega_\alpha = 0. \tag{11}$$

The control design for the case of single rigid body stabilization using MRP's can be found in [20]. The main result of this section is summarized in the following theorem:

**Theorem 2** *Assume that the leader communication graph $G^l$ is connected and that (11) holds. Then the control strategy*

$$u_i = -G^\intercal(\sigma_i) \sum_{j \in \mathcal{N}_i^l} \left(\sigma_i - \sigma_j - \sigma_{ij}^d\right) - \sum_{j \in \mathcal{N}_i^l} (\omega_i - \omega_j), \quad i \in \mathcal{N}^l \backslash \{\alpha\}, \tag{12}$$

*drives the remaining leaders to the desired relative orientations.*

**Proof:** For each leader $i \in \mathcal{N}^l$, we define the "cost function"

$$\gamma_i := \sum_{j \in \mathcal{N}_i^l} \left\| \sigma_i - \sigma_j - \sigma_{ij}^d \right\|^2,$$

and we introduce

$$V(\sigma, \omega) := \sum_{i \in \mathcal{N}^l} \left( \frac{1}{2} \omega_i^\intercal J_i \omega_i \right) + \sum_{i \in \mathcal{N}^l} \left( \frac{1}{2} \gamma_i \right)$$

as a candidate Lyapunov function. We then have

$$\dot{V}(\sigma, \omega) = \sum_{i \in \mathcal{N}^l} (\omega_i^\intercal J_i \dot{\omega}_i) + \frac{1}{2} \left[ \sum_{i \in \mathcal{N}^l} (\nabla \gamma_i)^\intercal \right] \dot{\sigma}. \tag{13}$$

Without loss of generality, we denote the leaders' indices by $1, \ldots, |\mathcal{N}^l|$ and we also note that in this section, the notation $\sigma, \omega$ refers to the stack vectors of the *leaders'* orientations and angular velocities, respectively. With a slight abuse of notation, we can now write the term in brackets in equation (13) as

$$\nabla \gamma_i = \left[ \frac{\partial^\intercal \gamma_i}{\partial \sigma_1} \cdots \frac{\partial^\intercal \gamma_i}{\partial \sigma_{|\mathcal{N}^l|}} \right],$$

where

$$\frac{\partial \gamma_i}{\partial \sigma_j} = \begin{cases} \sum_{j\in\mathcal{N}_i^l} (\sigma_i - \sigma_j) + \sigma_{ii}^d, & i = j, \\ -\left(\sigma_i - \sigma_j - \sigma_{ij}^d\right), & j \in \mathcal{N}_i^l,\ j \neq i, \\ 0, & j \notin \mathcal{N}_i^l, \end{cases}$$

where we have defined $\sigma_{ii}^d := -\sum_{j\in\mathcal{N}_i^l} \sigma_{ij}^d$. Hence,

$$\begin{aligned} \sum_{i\in\mathcal{N}^l} \frac{\partial \gamma_i}{\partial \sigma_j} &= \frac{\partial \gamma_j}{\partial \sigma_j} + \sum_{i\in\mathcal{N}_j^l} \frac{\partial \gamma_i}{\partial \sigma_j} \\ &= \sum_{i\in\mathcal{N}_j^l} (\sigma_j - \sigma_i) + \sigma_{jj}^d + \sum_{i\in\mathcal{N}_j^l} (-\sigma_i + \sigma_j + \sigma_{ij}^d) \\ &= 2\sum_{i\in\mathcal{N}_j^l} \sigma_j - 2\sum_{i\in\mathcal{N}_j^l} \sigma_i + 2\sigma_{jj}^d \\ &= 2d_j\sigma_j - 2\sum_{i\in\mathcal{N}_j^l} \sigma_i + 2\sigma_{jj}^d, \end{aligned}$$

where $d_j$ is the degree of vertex $j$ in the leader communication graph $G^l$. It follows that

$$\begin{aligned} \sum_{i\in\mathcal{N}^l} \nabla\gamma_i &= \sum_{i\in\mathcal{N}^l} \left[\frac{\partial \gamma_i}{\partial \sigma_1} \cdots \frac{\partial \gamma_i}{\partial \sigma_{|\mathcal{N}^l|}}\right] \\ &= 2\left[d_1\sigma_1 \cdots d_{|\mathcal{N}^l|}\sigma_{|\mathcal{N}^l|}\right] - 2\left[\sum_{j\in\mathcal{N}_1^l} \sigma_j \cdots \sum_{j\in\mathcal{N}_{|\mathcal{N}^l|}^l} \sigma_j\right] \\ &\quad + 2\left[\sigma_{11}^d \cdots \sigma_{|\mathcal{N}^l||\mathcal{N}^l|}^d\right], \end{aligned}$$

and finally,

$$\sum_{i\in\mathcal{N}^l} \nabla\gamma_i = 2\left(\left(L^l \otimes I_3\right)\sigma + \sigma^*\right)^\intercal, \tag{14}$$

where $\sigma^* = \left[\sigma_{11}^d \cdots \sigma_{|\mathcal{N}^l||\mathcal{N}^l|}^d\right]^\intercal$ and $L^l$ is the Laplacian matrix of the leader communication graph $G^l$. Using (14), $\dot{V}$ can be written as

$$\dot{V}(\sigma, \omega) = u^\intercal \omega + \left(\left(L^l \otimes I_3\right)\sigma + \sigma^*\right)^\intercal \mathsf{G}(\sigma)\,\omega, \tag{15}$$

where

$$\mathsf{G}(\sigma) := \operatorname{blockdiag}\left(G(\sigma_1), \ldots, G\left(\sigma_{|\mathcal{N}^l|}\right)\right).$$

Substituting (12) in (15), we have

$$\dot{V} = \sum_{i\in\mathcal{N}^l} \omega_i^\intercal \left(u_i + G^\intercal(\sigma_i) \sum_{j\in\mathcal{N}_i^l} \left(\sigma_i - \sigma_j - \sigma_{ij}^d\right)\right),$$

from which it follows that

$$\dot{V} = -\sum_{i\in\mathcal{N}^l\backslash\{\alpha\}} \omega_i^\mathsf{T} \left(\sum_{j\in\mathcal{N}_i^l} (\omega_i - \omega_j)\right),$$

which, using the fact that $\omega_\alpha = 0$, yields

$$\dot{V} = -\omega^\mathsf{T} \left(L^l \otimes I_3\right) \omega \leq 0.$$

Similarly to the proof of Theorem 1, the level sets of $V$ define compact sets in the product space of the leaders' angular velocities and orientations. In particular, $\Omega_c = \{(\omega, \sigma) : V(\sigma, \omega) \leq c\}$ for $c > 0$ is closed by the continuity of $V$. For all $(\omega, \sigma) \in \Omega_c$ we have

$$\omega_i^\mathsf{T} J_i \omega_i \leq 2c \Rightarrow \|\omega_i\| \leq \sqrt{\frac{2c}{\lambda_{\min}(J_i)}}.$$

Furthermore, for each $i \in \mathcal{N}^l$ we have $\gamma_i \leq 2c$, which implies that

$$\left\|\sigma_i - \sigma_j - \sigma_{ij}^d\right\|^2 \leq 4c \Rightarrow \|\sigma_i - \sigma_j\| \leq 2\sqrt{c} + \left\|\sigma_{ij}^d\right\|$$

for all $(i, j) \in E^l$. Using the notation $\xi \triangleq 2\sqrt{c} + \max_{(i,j)\in E^l} \left\{\left\|\sigma_{ij}^d\right\|\right\}$ and since the leader communication graph $G^l$ is connected we get $\|\sigma_i - \sigma_j\| \leq \left(\left|\mathcal{N}^l\right| - 1\right)\xi$, for all $(i, j) \in E^l$. Finally, since $\sigma_\alpha = \sigma_\alpha^d$ for leader $\alpha$, we have that $\|\sigma_i\| \leq \left(\left|\mathcal{N}^l\right| - 1\right)\xi + \|\sigma_\alpha\|$ for all leaders $i \in \mathcal{N}^l$, and thus, the set $\Omega_c$ is compact with respect to the leaders' angular velocities and orientations.

Using similar arguments as with the proof of Theorem 1, we conclude that since the leader communication graph is connected, all leaders attain the same angular velocities at steady state. Since $\omega_\alpha = 0$, this common angular velocity is zero. We thus have shown that $\omega_i = 0$ for all $i \in \mathcal{N}^l$ at steady state. This in turn implies that $u_i = 0$ for all $i \in \mathcal{N}^l$, and following again the arguments of the proof of Theorem 1, we get

$$\sum_{j\in\mathcal{N}_i^l} \left(\sigma_i - \sigma_j - \sigma_{ij}^d\right) = 0, \quad \forall i \in \mathcal{N}^l \backslash \{\alpha\}$$

at steady state. This implies that the leaders' orientations satisfy the following equations at steady state:

$$\begin{aligned} \left(\left(L^l \otimes I_3\right)\sigma + \sigma^*\right)_i &= 0, \quad \forall i \in \mathcal{N}^l \backslash \{\alpha\} \\ \sigma_\alpha &= \sigma_\alpha^d, \end{aligned} \tag{16}$$

where the notation $(\cdot)_i$ denotes the $i$-th element of a vector.

For all $i \in \mathcal{N}^l \backslash \{\alpha\}$, let $\sigma_i^d$ denote the desired orientation of the $i$-th leader with respect to the global coordinate frame. It is then obvious that $\sigma_{ij}^d = \sigma_i^d - \sigma_j^d$ for all $(i,j) \in E^l$ for all possible desired final orientations. Define $\sigma_i - \sigma_j - \sigma_{ij}^d = \sigma_i - \sigma_j - (\sigma_i^d - \sigma_j^d) := \tilde{\sigma}_i - \tilde{\sigma}_j$. The condition $\left(\left(L^l \otimes I_3\right)\sigma + \sigma^*\right)_i = 0$ for all $i \in \mathcal{N}^l \backslash \{\alpha\}$, along with the fact that $\sigma_\alpha = \sigma_\alpha^d$, then implies that $\left(L^l \otimes I_3\right)\tilde{\sigma} = 0$, equivalently,

$$L^l \tilde{\sigma}_1 = L^l \tilde{\sigma}_2 = L^l \tilde{\sigma}_3 = 0,$$

where $\tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\sigma}^3$ are the stack vectors of each of the three coefficients of $\tilde{\sigma}$ of the leaders' orientations, respectively. The fact that the leader communication graph $G^l$ is connected implies that the matrix $L^l$ has a simple zero eigenvalue with corresponding eigenvector the vector of ones. This guarantees that each one of the vectors $\tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\sigma}^3$ is an eigenvector of $L^l$ belonging to span$\{\overrightarrow{\mathbf{1}}\}$. Therefore, all $\tilde{\sigma}_i$ are equal to a common vector value, say $c$. Hence $\tilde{\sigma}_i = c$ for all $i \in \mathcal{N}^l$, which implies that $\sigma_i - \sigma_j = \sigma_{ij}^d$, where $j \in \mathcal{N}_i^l$, for all $i \in \mathcal{N}^l$. We conclude that the leaders converge to the desired, specified configuration of relative orientations. $\diamondsuit$

## 5 The Case of Lack of a Global Objective

In this section we assume that no global objective is imposed by the team of leaders. In particular, we assume that $\mathcal{N}^l = \varnothing$. The objective is to build distributed algorithms that drive the team of multiple rigid bodies to a common constant orientation with zero angular velocities.

In order to ensure that all agents converge to the same constant orientation, in this section we show that it is sufficient that one agent has a damping element on the angular velocity. Without loss of generality, we assume that this is agent $i = 1$. In contrast, the control design of [13] assumes that all agents have a damping element in their angular velocity. The following theorem is the main result of this section:

**Theorem 3** *Assume that the leader-follower communication graph is connected. Then the control strategy*

$$u_i = -G^{\mathsf{T}}(\sigma_i) \sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) - \sum_{j \in \mathcal{N}_i} (\omega_i - \omega_j) - a_i \omega_i, \tag{17}$$

*where $i = 1, \ldots, N$ and $a_i = 1$ if $i = 1$, and $a_i = 0$ otherwise, drives the rigid bodies to the same constant orientation with zero angular velocities.*

**Proof:** We choose again

$$V(\sigma, \omega) := \sum_{i=1}^{N} \left( \frac{1}{2} \omega_i^\intercal J_i \omega_i \right) + \frac{1}{2} \sigma^\intercal \left( L \otimes I_3 \right) \sigma$$

as a candidate Lyapunov function. Differentiating with respect to time, and after some algebraic manipulation, we obtain

$$\dot{V}(\sigma, \omega) = -\omega^\intercal \left( L \otimes I_3 \right) \omega - \|\omega_1\|^2 \leq 0.$$

It follows that $\omega$ remains bounded. By LaSalle's invariance principle, the system converges to the largest invariant set inside the set

$$M := \{(\sigma, \omega) : (\omega^\intercal \left( L \otimes I_3 \right) \omega = 0) \wedge (\omega_1 = 0)\}.$$

Similarly to the proof of Theorem 1, the condition $\omega^\intercal \left( L \otimes I_3 \right) \omega = 0$ guarantees that all $\omega_i$'s converge to a common value. Since $\omega_1 = 0$, this common value is zero. Following now the same steps as in the last part of the proof of Theorem 1, we conclude that the system reaches a configuration in which $\sum_{j \in \mathcal{N}_i} (\sigma_i - \sigma_j) = 0$ for all $i \in \mathcal{N}$, and thus

$$\left( L \otimes I_3 \right) \sigma = 0.$$

Connectivity of the leader follower communication graph implies now that, at steady state, the agents attain a common constant orientation. $\diamondsuit$

## 6 Numerical Example

In this section we present a numerical example that supports the theoretical developments.

The simulation involves four rigid bodies indexed from 1 to 4. We assume that there are two leaders $\mathcal{N}^l = \{1, 2\}$ and two followers $\mathcal{N}^f = \{3, 4\}$. We further assume that leader 1 is the reference point, and according to (11) we have $\sigma_1 = \sigma_1^d = [1.02, -1.12, 0.4]$, and $\omega_1 = 0$. The reference point $\sigma_1^d$ was randomly produced for this example. We also have $\mathcal{N}_2^l = \{1\}$ and $\sigma_{12}^d = [1, -1, 1]$. The control law of leader 2 is given by (12). The communication sets of the followers are given by $\mathcal{N}_3 = \{1, 4\}$ and $\mathcal{N}_4 = \{2, 3\}$ and their control laws by (8). The inertia matrices of the four rigid bodies have been chosen here as $J_1 = \operatorname{diag}(18, 12, 10)$, $J_2 = \operatorname{diag}(22, 16, 12)$, $J_3 = \operatorname{diag}(17, 14, 12)$ and $J_4 = \operatorname{diag}(15, 13, 8)$.

Figure 2 shows the plots of the angular velocities and orientations of the four rigid bodies with respect to time in all three coordinates. We observe that the system behaves as expected. The angular velocities converge to zero. The orientations of the leaders converge to the desired relative value while the orientations of the followers converge to the convex hull of the leaders' final orientations. For this example, this implies that the final orientations of the followers 3,4 converge to values that are between the final values of the two leaders 1,2 in all three orientation coordinates.

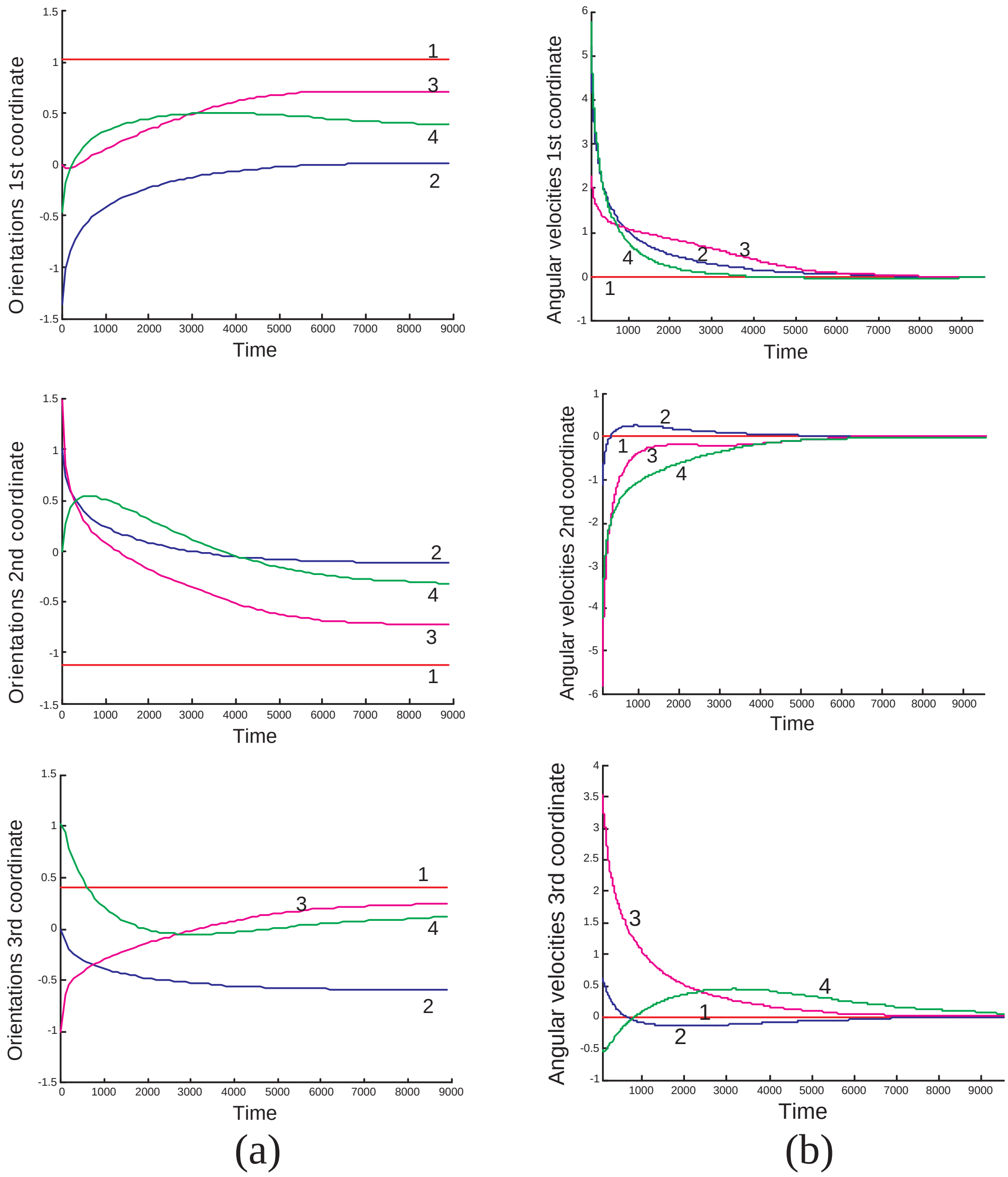


Fig. 2. Time histories of the angular velocities (right) and orientations (left) for the four rigid bodies using the leader-follower structure.

## 7 Conclusions

We propose distributed control strategies that exploit graph theoretic tools for cooperative rotational control of multiple rigid bodies. We assume that the agents are divided into leaders and followers. The leaders must maintain certain relative orientations with respect to each other, while the followers' orientations are to remain within a certain region that is dictated by the orientations of the leaders. Similarly to the case with linear agent dynamics, the convergence of the system was shown to rely on the connectivity of the communication graph. In the case of absence of any leaders, we have constructed control laws that drive all the members of the team to a common orientation with zero angular velocities. Results from numerical simulations were also included that illustrate the theory. Further research efforts will concentrate to the case of switching interconnection topology, as well as the case of unidirectional communication.

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