

Leafwise 2-jet cohomology on foliated Finsler manifolds

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Abstract. The tangent space of a Finsler manifold (M, F) is a Riemannian manifold with respect to the induced Sasaki-Finsler metric and admits a natural foliated structure F_V given by the vertical distribution (A. Bejancu and H.R. Farran, 2006). It is known that the Levi-Civita connection on the slit tangent space TM^0 induces a connection in the structural bundle. In this paper the Vaisman connection on (TM^0, F_V) is introduced; this induces connections on the structural and the transversal bundles. It is shown that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle iff (M, F) is a Landsberg manifold and that (TM^0, F_V) is a Reinhart space iff (M, F) is a Riemannian manifold. Further, on the foliated manifold (TM^0, F_V) is introduced and studied the space $J^{\{1,2\}}(TM^0)$ of leafwise 2-jets. A decomposition of this space is obtained, and the 1-dimensional Čech cohomology group of (TM^0, F_V) with coefficients in the sheaf of basic functions is expressed in terms of fields of leafwise 2-jets. We define the leafwise Mastrogiacomco cohomology group with respect to the connection ∇ induced in the structural bundle by a connection on (TM^0, F_V) ; as well it is shown that the cohomology group is isomorphic with the 1-dimensional Čech cohomology group of TM^0 with coefficients in the sheaf Ω_∇ of germs of functions f on TM^0 , which satisfy $\nabla df = 0$. In particular, for the 4-dimensional 4-root space, it is proved that the sheaf Ω_∇ is isomorphic with the sheaf of basic functions on TM^0 .

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1 Preliminaries

Generally speaking, jets are equivalence classes of maps between manifolds, maps which have the same derivative up to a specified order, [1], [8]. The second order

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jets are elements of $J^2(M, N)$, where M, N are two differentiable manifolds. In this paper we consider only the case $N = R$, hence the jets of the real differentiable functions on M . Supposing that M has a foliated structure, in [5] we introduced the leafwise 2-jets and the transversal 2-jets on M . These new types of 2-jets define the Mastrogiacomco cohomology groups on M , groups which are related with some Cech cohomology groups of the manifold M .

The tangent bundle of a Finsler manifold (M, F) is a Riemannian manifold with respect to the Sasaki-Finsler metric G , and it has a natural foliated structure F_V given by the vertical distribution, [3]. It is known, [3], that the Levi-Civita connection on TM^0 induces a connection in the structural bundle. In this paper we introduce the Vaisman connection, [10], on (TM^0, F_V) , which induces connections on the structural and the transversal bundles. We show that the Levi-Civita and the Vaisman connections induce the same connections in the structural bundle iff (M, F) is a Landsberg manifold. Moreover, (TM^0, F_V) is a Reinhart space iff (M, F) is a Riemannian manifold.

On the foliated manifold (TM^0, F_V) we introduce and study the space of leafwise 2-jets, $J^{l,2}(TM^0)$. We obtain a decomposition of this space and we express the 1-dimensional Cech cohomology group of (TM^0, F_V) with coefficients in the sheaf of basic functions in terms of fields of leafwise 2-jets. If ∇ is the connection induced in the structural bundle by a connection on (TM^0, F_V) , then we define the leafwise Mastrogiacomco cohomology group with respect to ∇ . This cohomology group is isomorphic with the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Ω_∇ of germs of functions f on TM^0 which satisfy $\nabla df = 0$.

At the end we present a particular case, where M is a 4-dimensional 4-root space, and we compute the sheaf Ω_∇ , proving that it is isomorphic with the sheaf of basic functions on TM^0 .

2 A natural foliation on the tangent bundle of a Finsler manifold

In this section we follow [4], [3] to present some questions about Finsler manifolds. Let M be an n -dimensional paracompact manifold and TM its tangent bundle. If E is a bundle over M , we denote by $\Gamma(E)$ the module of its smooth sections. If $(x^i)_{i=\overline{1,n}}$ are the local coordinates on M and $(y^i)_{i=\overline{1,n}}$ are the fiber coordinates, then $(x^i, y^i)_{i=\overline{1,n}}$ are the local coordinates on TM . It is well-known that the transformation of local coordinates on TM are

$$(2.1) \quad \tilde{x}^{i_1} = \tilde{x}^{i_1}(x^1, \dots, x^n), \quad \tilde{y}^{i_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} y^i.$$

In this paper the indices take the values $i, j, k, i_1, j_1, \dots = \overline{1, n}$ and we use the Einstein convention for summation.

We assume that M is a Finsler manifold, so there exists a function $F : TM \rightarrow [0, \infty)$ which vanishes only on the zero section on TM and is smooth on $TM^0 = TM \setminus \{0\}$, such that it is positive 1-homogeneous in y ,

$$F(x, ky) = |k| F(x, y), \quad \forall k \in R,$$

and the matrix of coefficients

$$(2.2) \quad g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

is positive definite at any point of the domain of the local chart.

Let F_V be the foliation on TM^0 determined by the fibers of $\pi : TM^0 \rightarrow M$, called the *vertical foliation*. The leaves are exactly $\{T_x M^0\}_{x \in M}$. The local coordinates $(x^i, y^i)_i$ are adapted to this foliation, which means that the leaves are locally defined by $x^i = \text{constant}$. The sections of the *structural* (also called vertical) bundle VTM^0 are locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. We consider the functions

$$G^i = \frac{1}{4} g^{ik} \left(\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k} \right), \quad G_i^j = \frac{\partial G^j}{\partial y^i},$$

where the matrix (g^{ik}) is the inverse of the matrix described by (2.2).

There is a complementary bundle HTM^0 to VTM^0 in TTM^0 , called the *transversal* (or horizontal) bundle of the foliation, whose sections are locally spanned by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad i = \overline{1, n}.$$

We have the decomposition

$$(2.3) \quad TTM^0 = HTM^0 \oplus VTM^0,$$

so every vector field X on TM^0 has a vertical part VX and an horizontal part HX . The relations

$$G \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = G \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}, \quad G \left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) = 0,$$

define a Riemannian metric on TM^0 , called the *Sasaki-Finsler metric*.

Let $\tilde{\nabla}$ be the Levi-Civita connection of G . It induces a connection ∇ on VTM^0 given by

$$\nabla_X VY = V(\tilde{\nabla}_X VY),$$

where $X, Y \in \Gamma(TTM^0)$ and V is the projection morphism of TTM^0 on VTM^0 . Locally, ∇ has the following expression ([3]):

$$(2.4) \quad \nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} = C_{ij}^k \frac{\partial}{\partial y^k}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = F_{ij}^k \frac{\partial}{\partial y^k},$$

where

$$(2.5) \quad C_{ij}^k = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h}, \quad F_{ij}^k = \frac{1}{2} g^{hk} \left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right).$$

Remark 2.1. M being a paracompact manifold, there is a Riemannian structure on M , let $a = (a_{ij})$ be the coefficients of its Riemannian metric. We have $F = \sqrt{a_{ij} y^i y^j}$ (hence $g_{ij} = a_{ij}$ don't depend on y), and, in general, (M, F) is Riemannian, iff

$$C_{ij}^k = 0, \quad \forall i, j, k = \overline{1, n}.$$

The decomposition (2.3) shows that for an arbitrary connection on TM^0 , the torsion T has the decomposition

$$T(X, Y) = V(T(X, Y)) + H(T(X, Y)),$$

where H is the projection morphism of TTM^0 on HTM^0 .

In the following we consider some notions from [10], related to foliated manifolds, in our particular case, (TM^0, F_V, G) . First of these notions is the *Vaisman* connection ∇^v on a Riemannian foliated manifold. This is a connection on TM^0 , uniquely defined by the conditions:

- a) If $Y \in \Gamma(VTM^0)$ (respectively $\in \Gamma(HTM^0)$), then $\nabla_X^v Y \in \Gamma(VTM^0)$ (respectively $\in \Gamma(HTM^0)$) for every X ;
- b) if $X, Y, Z \in \Gamma(VTM^0)$ ($\Gamma(HTM^0)$), then $(\nabla_X^v G)(Y, Z) = 0$;
- c) $V(T(X, Y)) = 0$ if at least one of the arguments is in $\Gamma(VTM^0)$ and $H(T(X, Y)) = 0$ if at least one of the arguments is in $\Gamma(HTM^0)$.

In [10, p. 188], the coefficients of ∇^v in an adapted chart are given and it is proved that:

Proposition 2.1. *Let (M, F) be a Finsler manifold. Then the Vaisman connection ∇^v on (TM^0, G, F_V) , is locally expressed with respect to the adapted local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ as follows:*

$$(2.6) \quad \begin{aligned} \nabla_{\frac{\delta}{\delta x^j}}^v \frac{\delta}{\delta x^i} &= F_{ij}^k \frac{\delta}{\delta x^k}, & \nabla_{\frac{\partial}{\partial y^j}}^v \frac{\delta}{\delta x^i} &= 0, \\ \nabla_{\frac{\delta}{\delta x^j}}^v \frac{\partial}{\partial y^i} &= \frac{\partial G_j^k}{\partial y^i} \frac{\partial}{\partial y^k}, & \nabla_{\frac{\partial}{\partial y^j}}^v \frac{\partial}{\partial y^i} &= C_{ij}^k \frac{\partial}{\partial y^k}. \end{aligned}$$

An immediate consequence of the above proposition is:

Proposition 2.2. *If (M, F) is a Finsler manifold, then the Levi-Civita and the Vaisman connections on the foliated manifold (TM^0, G, F_V) induce the same connection on the structural bundle iff M is a Landsberg manifold.*

Proof: The relations (2.6) give the local expression of the induced connection on VTM^0 . Taking into account relations (2.6) and (2.4), we have that the Levi-Civita and the Vaisman connections on the foliated manifold (TM^0, G, F_V) induce the same connection on the structural bundle iff

$$(2.7) \quad \frac{\partial G_j^k}{\partial y^i} = F_{ij}^k.$$

The first member of the above equality is usually denoted by G_{ij}^k and a Finsler manifold where holds the relation (2.7), is a *Landsberg manifold* ([3]).

Remark 2.2. In [3, p. 137, theorem 2.2], it is proved that a Finsler manifold is a Landsberg manifold if and only if the vertical foliation F_V on the Riemannian manifold (TM^0, G) is totally geodesic (that means all leaves are geodesic submanifolds). So, we can say that the Levi-Civita and Vaisman connection induce the same connection on the structural bundle VTM^0 if and only if the foliation F_V is totally geodesic.

Definition 2.1. A Riemannian foliated manifold with Riemannian metric g is called a Reinhart space iff

$$(2.8) \quad (\nabla_X^v G)(Y, Z) = 0,$$

for all the sections X of the structural bundle and Y, Z sections of the transversal bundle, where the covariant derivative is taken with respect to the Vaisman connection of the manifold.

Proposition 2.3. Let (M, F) be a Finsler manifold. The foliated manifold (TM^0, G, F_V) is a Reinhart space if and only if (M, F) is a Riemannian manifold.

Proof. Let $X \in \Gamma(VTM^0)$ and $Y, Z \in \Gamma(HTM^0)$, and ∇^v be the Vaisman connection on (TM^0, G, F_V) . In an adapted local chart we have

$$X = X^i \frac{\partial}{\partial y^i}, \quad Y = Y^j \frac{\delta}{\delta x^j}, \quad Z = Z^k \frac{\delta}{\delta x^k}.$$

By the linearity of ∇^v we can consider $X = \frac{\partial}{\partial y^i}$. Then we compute

$$\begin{aligned} \left(\nabla_{\frac{\partial}{\partial y^i}}^v G \right) (Y, Z) &= \frac{\partial}{\partial y^i} (Y^j Z^k g_{jk}) - G \left(\nabla_{\frac{\partial}{\partial y^i}}^v Y^j \frac{\delta}{\delta x^j}, Z^k \frac{\delta}{\delta x^k} \right) - \\ &- G \left(Y^j \frac{\delta}{\delta x^j}, \nabla_{\frac{\partial}{\partial y^i}}^v Z^k \frac{\delta}{\delta x^k} \right) = Y^j Z^k \frac{\partial g_{jk}}{\partial y^i}. \end{aligned}$$

Hence the condition (2.8) for all $Y, Z \in \Gamma(HTM^0)$ is equivalent to $\frac{\partial g_{jk}}{\partial y^i} = 0$, so, by remark 2.1, it is equivalent to (M, F) Riemannian manifold.

Remark 2.3. In [3, p. 137, theorem 2.1], it is proved that a Finsler manifold is a Riemannian manifold if and only if the Sasaki-Finsler metric G is bundle-like for the vertical foliation F_V of TM^0 . So, we can say that the Sasaki-Finsler metric G is bundle-like for the vertical foliation F_V of TM^0 if and only if (TM^0, G, F_V) is a Reinhart space.

Let $\Omega^0(TM^0)$ be the ring of real differentiable functions on TM^0 and $\Omega^{p,q}(TM^0)$ be the module of (p, q) forms on the foliated manifold (TM^0, G, F_V) . Hence, $\theta \in \Omega^{p,q}(TM^0)$ is a $(p+q)$ -form on (TM^0, G) which is potentially different from zero only if p arguments are sections of HTM^0 and q arguments are sections of VTM^0 . By this definition, $\theta \in \Omega^{0,1}(TM^0)$ has the local expression

$$(2.9) \quad \theta = \theta_i \delta y^i,$$

where θ_i are differentiable functions on the domain of the local chart and $\{dx^i, \delta y^i\}$ is the dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$. Obviously, if U, \tilde{U} are two domains which overlap, then in their intersection

$$(2.10) \quad \tilde{\theta}_{i_1} = \frac{\partial \tilde{y}^i}{\partial y^{i_1}} \theta_i = \frac{\partial \tilde{x}^i}{\partial x^{i_1}} \theta_i,$$

where we used also (2.1).

Let $d_{01} : \Omega^{p,q}(TM^0) \rightarrow \Omega^{p,q+1}(TM^0)$ be the foliated derivative on (TM^0, G, F_V) . It is well-known, [10], that $d_{01}^2 = 0$. If $f \in \Omega^0(TM^0)$ and $\theta \in \Omega^{0,1}(TM^0)$, then we locally have

$$d_{01}f = \frac{\partial f}{\partial y^i} \delta y^i, \quad d_{01}\theta = \left(\frac{\partial \theta_i}{\partial y^j} - \frac{\partial \theta_j}{\partial y^i} \right) \delta y^j \wedge \delta y^i.$$

The foliated derivative define the leafwise cohomology of a foliated manifold. In this paper we present only the 1-dimensional leafwise cohomology group of (TM^0, G, F_V) , which is interesting for our research.

We recall that a $(0,1)$ -form θ is called d_{01} -exact if there exists $f \in \Omega^0(TM^0)$ such that $\theta = d_{01}f$ and it is called d_{01} -closed if $d_{01}\theta = 0$. The Poincare lemma assures that every d_{01} -closed form is d_{01} -locally exact.

Let $Z^{0,1}(TM^0)$ and $B^{0,1}(TM^0)$ be the groups of $(0,1)$ -forms which are, respectively, d_{01} -closed and d_{01} -exact. Obviously, $B^{0,1}(TM^0)$ is a subgroup of $Z^{0,1}(TM^0)$. The quotient group

$$(2.11) \quad H^1(F_V) = \frac{Z^{0,1}(TM^0)}{B^{0,1}(TM^0)},$$

is called the 1-dimensional *leafwise cohomology* group of the foliated manifold (TM^0, G, F_V) .

We recall that $f \in \Omega^0(TM^0)$ is called *basic* if $d_{01}f = 0$, or, equivalent, if it is constant on the leaves, so it doesn't depend on x . Let Φ be the sheaf of germs of basic functions on (TM^0, G, F_V) . An important result is the de Rham theorem:

Theorem 2.4. *The cohomology group $H^1(F_V)$ is isomorphic with the 1-dimensional Čech cohomology group of TM^0 with coefficients in the sheaf Φ :*

$$H^1(F_V) \approx \mathbf{H}^1(TM^0, \Phi).$$

3 Leafwise 2-jets on (TM^0, G, F_V)

If M is a differential real manifold, we know, [1], [8], that two functions $f, g \in \Omega^0(M)$ determine the same *2-jet* at a fixed point $x \in M$ if $f(x) = g(x) = 0$ and their first and second derivative are equal at x .

On a foliated manifold two differentiable real functions determine the same *leafwise 2-jet* at a point if they determine the same 2-jet in the leaf through that point ([5]). So, for (TM^0, G, F_V) we have the following definition:

Definition 3.1. We say that $f, g \in \Omega^0(TM^0)$ determine the same *leafwise 2-jet*, or *l,2-jet* at $(x, y) \in TM^0$ if they determine the same 2-jet at (x, y) in $T_x M^0$.

The relation "to determine the same l,2-jet at (x, y) " is an equivalence on $\Omega^0(TM^0)$. The equivalence class containing f is denoted by $j_{(x,y)}^{l,2}f$ and it is called the

$l,2$ -jet of f at x . By the above definition we have that the equality $j_{(x,y)}^{l,2}f = j_{(x,y)}^{l,2}\tilde{f}$ is locally expressed by:

$$f(x, y) = \tilde{f}(x, y) = 0, \quad \frac{\partial f}{\partial y^i}(x, y) = \frac{\partial \tilde{f}}{\partial y^i}(x, y), \quad \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y) = \frac{\partial^2 \tilde{f}}{\partial y^i \partial y^j}(x, y),$$

for every $i, j = \overline{1, n}$. These conditions have geometrical meaning and they can be considered as definition for " f, g determine the same $l,2$ -jet at (x, y) ".

For every $f, g \in \Omega^0(TM^0)$ and $\alpha \in \mathbb{R}$, we define the following operations:

$$\begin{aligned} j_{(x,y)}^{l,2}f + j_{(x,y)}^{l,2}g &= j_{(x,y)}^{l,2}(f + g), \\ \alpha \cdot j_{(x,y)}^{l,2}f &= j_{(x,y)}^{l,2}(\alpha \cdot f), \\ j_{(x,y)}^{l,2}f \cdot j_{(x,y)}^{l,2}g &= j_{(x,y)}^{l,2}(f \cdot g). \end{aligned}$$

It is easy to see that the above operations are well-defined. For example, we shall prove that the product doesn't depend on the representatives. Let $\tilde{f} \in j_{(x,y)}^{l,2}f$ and $\tilde{g} \in j_{(x,y)}^{l,2}g$. By the definition 3.1, it results $f(x, y) = g(x, y) = \tilde{f}(x, y) = \tilde{g}(x, y) = 0$, and

$$\begin{aligned} \frac{\partial f}{\partial y^i}(x, y) &= \frac{\partial \tilde{f}}{\partial y^i}(x, y), & \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y) &= \frac{\partial^2 \tilde{f}}{\partial y^i \partial y^j}(x, y), \\ \frac{\partial g}{\partial y^i}(x, y) &= \frac{\partial \tilde{g}}{\partial y^i}(x, y), & \frac{\partial^2 g}{\partial y^i \partial y^j}(x, y) &= \frac{\partial^2 \tilde{g}}{\partial y^i \partial y^j}(x, y). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial(f \cdot g)}{\partial y^i}(x, y) &= \frac{\partial f}{\partial y^i}(x, y) \cdot g(x, y) + \frac{\partial g}{\partial y^i}(x, y) \cdot f(x, y) = \frac{\partial(\tilde{f} \cdot \tilde{g})}{\partial y^i}(x, y) = 0 \\ \frac{\partial^2(f \cdot g)}{\partial y^i \partial y^j}(x, y) &= 2 \frac{\partial f}{\partial y^i}(x, y) \cdot \frac{\partial g}{\partial y^j}(x, y) = 2 \frac{\partial \tilde{f}}{\partial y^i}(x, y) \cdot \frac{\partial \tilde{g}}{\partial y^j}(x, y) = \frac{\partial^2(\tilde{f} \cdot \tilde{g})}{\partial y^i \partial y^j}(x, y), \end{aligned}$$

which shows that $j_{(x,y)}^{l,2}(f \cdot g) = j_{(x,y)}^{l,2}(\tilde{f} \cdot \tilde{g})$, q.e.d.

Now let $f \in \Omega^0(TM^0)$ such that $f(x, y) = 0$ for a fixed point $(x, y) \in TM^0$. Let $(U, (x^i, y^i))$ be a local chart at (x, y) , such that $y^i(x, y) = 0, i = \overline{1, n}$. In U , both the functions f and

$$y^i \frac{\partial f}{\partial y^i}(x, y) + \frac{1}{2} y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y),$$

determine the same $l,2$ -jet at (x, y) , so we obtain the local expression of $j_{(x,y)}^{l,2}f$:

$$j_{(x,y)}^{l,2}f = \frac{\partial f}{\partial y^i}(x, y) j_{(x,y)}^{l,2}y^i + \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y) j_{(x,y)}^{l,2}y^i \cdot j_{(x,y)}^{l,2}y^j.$$

Moreover, we say that it is a $l,2$ -jet at (x, y) every expression locally given in $(U, (x^i, y^i))$ by

$$(3.1) \quad \omega_{(x,y)}^2 = \omega_i(x, y) \cdot j_{(x,y)}^{l,2}y^i + \frac{1}{2} \omega_{ij}(x, y) j_{(x,y)}^{l,2}y^i \cdot j_{(x,y)}^{l,2}y^j,$$

where $\omega_i, \omega_{ij} \in \Omega^0(U)$ are satisfying $\omega_{ij} = \omega_{ji}$ and if \tilde{U} is another domain of local chart at (x, y) , then in $U \cap \tilde{U}$ these functions satisfy

$$\tilde{\omega}_{i_1} = \omega_i \frac{\partial y^i}{\partial \tilde{y}^{i_1}}, \quad \tilde{\omega}_{i_1 j_1} = \omega_i \frac{\partial^2 y^i}{\partial \tilde{y}^{i_1} \partial \tilde{y}^{j_1}} + \omega_{ij} \frac{\partial y^i}{\partial \tilde{y}^{i_1}} \frac{\partial y^j}{\partial \tilde{y}^{j_1}}.$$

Taking into account (2.1), the transformation of the component functions of a 1,2-jet at (x, y) are:

$$(3.2) \quad \tilde{\omega}_{i_1} = \omega_i \frac{\partial x^i}{\partial \tilde{x}^{i_1}}, \quad \tilde{\omega}_{i_1 j_1} = \omega_{ij} \frac{\partial x^i}{\partial \tilde{x}^{i_1}} \frac{\partial x^j}{\partial \tilde{x}^{j_1}}.$$

We denote by $J_{(x,y)}^{l,2}(TM^0)$ the space of 1,2-jets at (x, y) and we can see that this is a real algebra. The space $J^{l,2}(TM^0) = \cup_{(x,y) \in TM^0} J_{(x,y)}^{l,2}(TM^0)$ is a fiber bundle over TM^0 , where the fiber coordinates are $(\omega_i, \omega_{ij})_{1 \leq i < j \leq n}$.

Definition 3.2. An 1,2-jet at (x, y) locally given by (3.1) is called *homogeneous* if $\omega_i = 0$ for all $i = \overline{1, n}$.

The relations (3.2) show that the notion of homogeneous 1,2-jet has geometrical meaning. We denote by $\Theta_{(x,y)}^{l,2}(TM^0)$ the space of homogenous 1,2-jets at (x, y) . The space $\Theta^{l,2}(TM^0) = \cup_{(x,y) \in TM^0} \Theta_{(x,y)}^{l,2}(TM^0)$ is a fiber bundle over TM^0 , where the fiber coordinates are $(\omega_{ij})_{1 \leq i < j \leq n}$.

The smooth sections of $J^{l,2}(TM^0)$, $\Theta^{l,2}(TM^0)$ are called the *fields of l,2-jets*, and the *fields of homogeneous l,2-jets* (or h,l,2-jets) on TM^0 , respectively. The spaces of these sections are denoted by $J^2(F_V)$, $\Theta^2(F_V)$, respectively. We have the inclusion $\Theta^2(F_V) \subset J^2(F_V)$.

Let $j^{l,2}$ be the map which assigns to every $f \in \Omega^0(TM^0)$ its field of 1,2-jets

$$(3.3) \quad j^{l,2} : \Omega^0(TM^0) \rightarrow J^2(F_V).$$

The local expression of this map is

$$(3.4) \quad j^{l,2} f = \frac{\partial f}{\partial y^i} j^{l,2} y^i + \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j.$$

The changing rules for the fields of 1,2-jets of the coordinates are:

$$j^{l,2} \tilde{y}^{i_1} = \frac{\partial \tilde{y}^{i_1}}{\partial y^i} j^{l,2} y^i + \frac{1}{2} \frac{\partial^2 \tilde{y}^{i_1}}{\partial y^i \partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j,$$

$$j^{l,2} \tilde{y}^{i_1} \cdot j^{l,2} \tilde{y}^{j_1} = \frac{\partial \tilde{y}^{i_1}}{\partial y^i} \frac{\partial \tilde{y}^{j_1}}{\partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j.$$

Taking into account relations (2.1), the above transformations become:

$$(3.5) \quad j^{l,2} \tilde{y}^{i_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} j^{l,2} y^i, \quad j^{l,2} \tilde{y}^{i_1} \cdot j^{l,2} \tilde{y}^{j_1} = \frac{\partial \tilde{x}^{i_1}}{\partial x^i} \frac{\partial \tilde{x}^{j_1}}{\partial x^j} j^{l,2} y^i \cdot j^{l,2} y^j.$$

Let ω^2 be an arbitrary field of 1,2-jets on TM^0 , locally given by

$$\omega^2 = \omega_i \cdot j^{l,2}y^i + \frac{1}{2}\omega_{ij}j^{l,2}y^i \cdot j^{l,2}y^j,$$

where the relations (3.2) hold. The relations (3.2) and (3.5) show that the expressions

$$\omega^1 = \omega_i \cdot j^{l,2}y^i, \quad \omega_o^2 = \frac{1}{2}\omega_{ij}j^{l,2}y^i \cdot j^{l,2}y^j,$$

have geometrical meaning. Moreover, ω_o^2 is a field of h,1,2-jets. We call ω^1 the *1-type part* of ω^2 and ω_o^2 the *homogeneous part* of ω^2 . It results that every field of 1,2-jets on TM^0 is a sum of its 1-type and homogeneous parts:

$$(3.6) \quad \omega^2 = \omega^1 + \omega_o^2.$$

Let $J_{(x,y)}^{l,2,1}(TM^0)$ the set of all the expressions $\omega_i(x,y) \cdot j_{(x,y)}^{l,2}y^i$ satisfying (3.2) and (3.5). The space

$$J^{l,2,1}(TM^0) = \cup_{(x,y) \in TM^0} J_{(x,y)}^{l,2,1}(TM^0)$$

is a fiber bundle over TM^0 , and a subbundle of $J^{l,2}(TM^0)$. The space of its sections is denoted by $J^{2,1}(F_V)$.

Using the relation (3.6) it is easy to verify that

Proposition 3.1. *The bundle of 1,2-jets on the foliated manifold TM^0, F_V is the direct sum of the bundles $J^{l,2,1}(TM^0)$ and $\Theta^{l,2}(TM^0)$.*

An immediate consequence is the decomposition

$$J^2(F_V) = J^{2,1}(F_V) \oplus \Theta^2(F_V).$$

Proposition 3.2. *The space $J^{2,1}(F_V)$ is a $\Omega^0(TM^0)$ -module, isomorphic with the module $\Omega^{0,1}(TM^0)$ of (0,1)-forms on TM^0 .*

Proof. Let $\xi : \Omega^{0,1}(TM^0) \rightarrow J^{2,1}(F_V)$ be the map which assigns to every (0,1)-form θ locally given by (2.9) the field of 1,2-jets ω^2 locally given by $\theta_i \cdot j^{l,2}y^i$. From (2.10), (3.2) and (3.5), it results that ξ is well-defined. Also, it is an isomorphism of modules.

Remark 3.1. One could say that every field of 1,2-jets on TM^0 is congruent modulo $\Omega^{0,1}(TM^0)$ to every field of h,1,2-jets on the same manifold.

We can see that the map ξ from the proof of Proposition 3.2 satisfies

$$(\xi \circ d_{01})f = (j^{l,2}f)^1, \quad \forall f \in \Omega^0(TM^0),$$

where the second member is the 1-type part of the field of 1,2-jets of the function f . Let $\delta^{l,2}$ be the map who assigns to every $f \in \Omega^0(TM^0)$ the homogeneous part of $j^{l,2}f$:

$$(3.7) \quad \delta^{l,2} : \Omega^{0,1}(TM^0) \rightarrow \Theta^2(F_V), \quad \delta^{l,2}f = (j^{l,2}f)_o.$$

By the relations (3.3), (3.4), (3.6)-(3.7), we have

Proposition 3.3. *The map $j^{l,2}$ satisfies*

$$j^{l,2} = \xi \circ d_{01} + \delta^{l,2}.$$

Now, we define the following map

$$D : \Omega^{0,1}(TM^0) \rightarrow \Theta^2(F_V),$$

given locally by

$$D(\theta_i \delta y^i) = \frac{1}{4} \left(\frac{\partial \theta_i}{\partial y^j} + \frac{\partial \theta_j}{\partial y^i} \right) \cdot j^{l,2} y^i \cdot j^{l,2} y^j.$$

By a direct calculation, D has geometrical meaning. Moreover, for an arbitrary $f \in \Omega^0(TM^0)$, we can compute in every local chart

$$(D \circ d_{01})f = D \left(\frac{\partial f}{\partial y^i} \delta y^i \right) = \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j} j^{l,2} y^i \cdot j^{l,2} y^j = (j^{l,2} f)_o = \delta^{l,2} f.$$

If we denote by ζ the map $\xi + D$, then $\zeta : \Omega^{0,1}(TM^0) \rightarrow J^2(F_V)$ is satisfying by proposition 3.3 the following relation

$$(3.8) \quad j^{l,2} = \zeta \circ d_{01}.$$

4 Mastrogiamo leafwise cohomology on (TM^0, G, F_V)

In this section we use the fields of leafwise 2-jets from the previous section for defining two cohomology groups of the foliated manifold (TM^0, G, F_V) , one of them being essentially dependent by a given connection on TM^0 .

Definition 4.1. We say that a field of 1,2-jets $\omega^2 \in J^2(F_V)$ is $j^{l,2}$ -exact if $\omega^2 = j^{l,2} f$ for some $f \in \Omega^0(TM^0)$, and it is $j^{l,2}$ -closed if it is locally $j^{l,2}$ -exact.

We denote by $E^2(F_V)$, $C^2(F_V)$ the spaces of fields of 1,2-jets on TM^0 which are $j^{l,2}$ -exact, respectively $j^{l,2}$ -closed. Obviously, $E^2(F_V) \subset C^2(F_V)$. We call the *Mastrogiamo leafwise cohomology* group of TM^0 the quotient group

$$H^{l,2}(F_V) = \frac{C^2(F_V)}{E^2(F_V)}.$$

Proposition 4.1. *If ζ is the map from the relation (3.8), then we have*

$$\zeta(Z^{0,1}(TM^0)) = C^2(F_V), \quad \zeta(B^{0,1}(TM^0)) = E^2(F_V),$$

where the spaces $Z^{0,1}(TM^0)$, $B^{0,1}(TM^0)$ are from the relation (2.11).

Proof: Let $\theta \in Z^{0,1}(TM^0)$, which means that it is a $(0,1)$ -form satisfying $d_{01}\theta = 0$. The Poincaré lemma assures that in every domain U from TM^0 there exists $f_U \in \Omega^0(U)$ such that in U we have $\theta = d_{01}f_U$. We compute in U

$$\zeta(\theta) = (\zeta \circ d_{01})(f_U) = j^{l,2} f_U$$

using relation (3.8). It results that $\zeta(\theta)$ is locally $j^{l,2}$ -exact, so it belongs to $C^2(F_V)$.

Considering now an arbitrary $\omega^2 \in C^2(F_V)$, we have that in every domain U from TM^0 there is a differentiable function f_U such that $\omega^2 = j^{l,2}f_U$. Let θ be the $(0,1)$ -form locally given in U by $\theta = d_{01}f_U$. This form is globally defined because the field of 1,2-jets ω^2 is globally defined. Moreover, $d_{01}\theta = 0$ and $\zeta(\theta) = \omega^2$, so $C^2(F_V) \subset \zeta(Z^{0,1}(TM^0))$, which entails the first equality. The second equality follows by an analogous argument, q.e.d..

An immediate consequence is:

Theorem 4.2. *The map ζ induces an isomorphism between the 1-leafwise cohomology $H^1(F_V)$ and the Mastrogiacomio leafwise cohomology group $H^{l,2}(F_V)$.*

Proof. The map

$$\zeta^* : H^1(F_V) \rightarrow H^{l,2}(F_V), \quad \zeta^*([\theta]) = [\zeta(\theta)]^{l,2},$$

is well-defined by Proposition 4.1 and it is an isomorphism of groups.

From Theorems 2.1 and 4.1, it results:

Theorem 4.3. *The Mastrogiacomio leafwise cohomology group of the foliated manifold (TM^0, G, F_V) and the 1-dimensional Cech cohomology group of TM^0 with coefficients in the sheaf Φ of basic functions are isomorphic.*

In the following, let $\tilde{\nabla}$ be the Levi-Civita connection of the metric G on TM^0 and ∇ the connection induced by $\tilde{\nabla}$ in the structural bundle, from the section 2. The following map is well-defined:

$$\delta_{\tilde{\nabla}}^{l,2} : \Omega^{0,1}(TM^0) \rightarrow \Theta^{2,1}(F_V),$$

locally given by

$$(4.1) \quad \delta_{\tilde{\nabla}}^{l,2} f = \frac{1}{2} \left(\frac{\partial^2 f}{\partial y^i \partial y^j} - C_{ij}^k \frac{\partial f}{\partial y^k} \right) j^{l,2} y^i \cdot j^{l,2} y^j,$$

where C_{ij}^k are the coefficients of the restriction of ∇ to the leaves, given by (2.5). Indeed, if U and \tilde{U} are two domains of local charts which overlap, then we have in $U \cap \tilde{U}$, [4]:

$$\tilde{C}_{i_1 j_1}^{k_1} \cdot \frac{\partial x^k}{\partial \tilde{x}^{k_1}} = C_{ij}^k \cdot \frac{\partial x^i}{\partial \tilde{x}^{i_1}} \cdot \frac{\partial x^j}{\partial \tilde{x}^{j_1}}.$$

Using the above transformation, by a direct calculation results that the second member of (4.1) satisfies the conditions (3.2) (here $\omega_i = 0$), so $\delta_{\tilde{\nabla}}^{l,2} f$ is a field of h,1,2-jets on TM^0 .

Definition 4.2. We say that a field of h,1,2-jets $\omega^2 \in \Theta^2(F_V)$ is $\delta_{\tilde{\nabla}}^{l,2}$ -exact if $\omega^2 = \delta_{\tilde{\nabla}}^{l,2} f$ for some $f \in \Omega^0(TM^0)$, and it is $\delta_{\tilde{\nabla}}^{l,2}$ -closed if it is locally $\delta_{\tilde{\nabla}}^{l,2}$ -exact.

We denote by $E_{\tilde{\nabla}}^2(F_V)$, $C_{\tilde{\nabla}}^2(F_V)$ the spaces of fields of h,1,2-jets on TM^0 which are $\delta_{\tilde{\nabla}}^{l,2}$ -exact, $\delta_{\tilde{\nabla}}^{l,2}$ -closed, respectively. Obviously, $E_{\tilde{\nabla}}^2(F_V) \subset C_{\tilde{\nabla}}^2(F_V)$. We call the

Mastrogiacomo leafwise cohomology group with respect to ∇ of TM^0 the quotient group

$$H_{\nabla}^{l,2}(F_V) = \frac{C_{\nabla}^2(F_V)}{E_{\nabla}^2(F_V)}.$$

Let Ω_{∇} be the sheaf of germs of differentiable functions on TM^0 which satisfy $\nabla df = 0$. Locally, this condition is equivalent with the following system:

$$(4.2) \quad \frac{\partial^2 f}{\partial y^i \partial y^j} - C_{ij}^k \frac{\partial f}{\partial y^k} = 0, \quad (\forall) i, j = \overline{1, n}.$$

The integrability conditions $\frac{\partial^3 f}{\partial y^i \partial y^j \partial y^h} = \frac{\partial^3 f}{\partial y^h \partial y^i \partial y^j}$ yield that the solutions of (4.2) have to satisfy

$$S_{ijh}^l \frac{\partial f}{\partial y^l} = 0, \quad (\forall) i, j, h = \overline{1, n},$$

where

$$S_{ijh}^l = \frac{\partial C_{ij}^l}{\partial y^h} - \frac{\partial C_{jh}^l}{\partial y^i} + C_{ij}^k C_{kh}^l - C_{jh}^k C_{ki}^l,$$

are the curvature coefficients of the restriction of ∇ along the leaves. On the other hand, taking into account the well-known relations, [4],

$$y^i C_{ij}^k = 0,$$

from the system (4.2) it follows

$$y^i \frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) j = \overline{1, n}.$$

which is equivalent to

$$Lf - f \in \Phi(TM^0),$$

where L is the vertical Liouville vector field on TM^0 ,

$$L = y^i \frac{\partial}{\partial y^i}.$$

For an arbitrary foliated manifold we proved in [7] that the Mastrogiacomo leafwise cohomology with respect to a connection ∇ and the 1-dimensional Čech cohomology group with coefficients in the sheaf Ω_{∇} of that manifold, are isomorphic ([7], theorem 4.1, p.83). So, we have

Theorem 4.4. *If (M, F) is a Finsler manifold, for the foliated manifold (TM^0, G, F_V) the following isomorphism holds:*

$$H_{\nabla}^{l,2}(F_V) \approx \mathbf{H}^1(TM^0, \Omega_{\nabla}).$$

In the following we determine the Mastrogiacomo leafwise cohomology group of TM^0 with respect to ∇ in two particular cases.

(I) If the Finsler manifold (M, F) is Riemannian, which is equivalent from remark (2.1) to relation $C_{ij}^k = 0$ for all indices i, j, k , and ∇ is the induced connection on the

structural bundle by the Levi-Civita connection on TM^0 , then the sheaf Ω_{∇} contains the germs of functions which are satisfying

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) i, j = \overline{1, n}.$$

The solutions of the above system are $f = \alpha_i y^i + \alpha$, with α_i, α basic functions. Hence, we have the isomorphism of shaves $\Omega_{\nabla} \approx \Phi^{n+1}$, where Φ is the sheaf of germs of basic functions on TM^0 . So, by Theorem 4.3, it results

$$H_{\nabla}^{l,2}(F_V) \approx \mathbf{H}^1(TM^0, \Phi^{n+1}).$$

Taking into account theorem 2.1, we obtain that the 1-leafwise cohomology group of (TM^0, G, F_V) is a proper subgroup of $H_{\nabla}^{l,2}(F_V)$.

(II) Let (M, F) be a 4-dimensional 4-roots space, where the fundamental functions F is locally given by $F = \sqrt[4]{y^1 y^2 y^3 y^4}$. We suppose that F is well-defined on TM . By a straightforward calculation we obtain $g_{ii} = -\frac{F^2}{8} \frac{1}{(y^i)^2}$, $g_{ij} = \frac{F^2}{8} \frac{1}{y^i y^j}$, and further

$$C_{ii}^i = -\frac{3}{8(y^i)^2}, \quad C_{ii}^k = \frac{y^k}{8(y^i)^2}, \quad C_{ik}^i = \frac{1}{8y^k}, \quad C_{ij}^k = -\frac{y^k}{8y^i y^j},$$

for all indices $i, j, k \in \{1, 2, 3, 4\}$ such that $i \neq j \neq k \neq i$.

Then the system (4.2) becomes

$$\begin{cases} 8(y^i)^2 \frac{\partial^2 f}{\partial (y^i)^2} + 4y^i \frac{\partial f}{\partial y^i} - Lf = 0 \\ 8y^i y^j \frac{\partial^2 f}{\partial y^i \partial y^j} + 2y^i \frac{\partial f}{\partial y^i} + 2y^j \frac{\partial f}{\partial y^j} - Lf = 0 \end{cases}$$

for all $i, j \in \{1, 2, 3, 4\}$, where the repetition of the same index does not means summation. So, this system has 10 equations. The integrability conditions give that a solution f has to satisfy

$$(4.3) \quad y^1 \frac{\partial f}{\partial y^1} = y^2 \frac{\partial f}{\partial y^2} = y^3 \frac{\partial f}{\partial y^3} = y^4 \frac{\partial f}{\partial y^4},$$

hence the system becomes

$$\frac{\partial^2 f}{\partial (y^i)^2} = 0, \quad \frac{\partial^2 f}{\partial y^i \partial y^j} = 0, \quad (\forall) i, j \in \{1, 2, 3, 4\}, i \neq j.$$

The solutions are $f = \alpha_1 y^1 + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4 + \alpha$, where α_i, α are basics functions, so they do not depend on y . These solutions have to satisfy as well the relations (4.3), hence

$$\alpha_1(x)y^1 = \alpha_2(x)y^2 = \alpha_3(x)y^3 = \alpha_4(x)y^4, \quad (\forall) x \in M.$$

It results that $\alpha_i = 0$ for all $i \in \{1, 2, 3, 4\}$. We have then $\Omega_{\nabla} \approx \Phi$.

For this particular Finsler manifold we obtain

$$H_{\nabla}^{l,2}(F_V) \approx \mathbf{H}^l(TM^0, \Phi).$$

Taking into account theorem 2.1, it follows that the Mastrogiacomio leafwise cohomology group with respect to ∇ is isomorphic with the 1-leafwise cohomology group of the foliated manifold (TM^0, G, F_V) .

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