



Learning about latent dynamic trading demand *

Xiao Chen¹ · Jin Hyuk Choi² · Kasper Larsen¹ · Duane J. Seppi³

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Abstract

We present an equilibrium model of dynamic trading, learning, and pricing by strategic investors with trading targets and price impact. Since trading targets are private, investors filter the child order flow dynamically over time to estimate the latent underlying parent trading demand imbalance and to forecast its impact on subsequent price-pressure dynamics. We prove existence of an equilibrium and solve for equilibrium trading strategies and prices as the solution to a system of coupled ODEs. Trading strategies are combinations of trading towards investor targets, liquidity provision for other investors' demands, and speculation based on learning about latent underlying trading-demand imbalances.

Keywords Order-splitting · Optimal order execution · Subgame perfect nash equilibrium · Dynamic learning · Trading targets · Speculation

JEL codes G11 · G12

1 Introduction

The price formation process in financial markets involves equating supply and demand for securities over time for arriving investors with heterogeneous trading preferences. In present day markets, large investors act on their underlying trading preferences, sometimes called

✉ Kasper Larsen
KL756@math.rutgers.edu

Xiao Chen
xc206@scarletmail.rutgers.edu

Jin Hyuk Choi
jchoi@unist.ac.kr

Duane J. Seppi
ds64@andrew.cmu.edu

¹ Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

² Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), UNIST-gil 50, Ulsan 689-798, Republic of Korea

³ Tepper School of Business, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213, USA

parent demands, by splitting their trading into dynamic sequences of smaller orders, called *child orders* (see O'Hara [32]), to minimize their price impact. Since the parent demands driving child-order trading are private information, investors use information from arriving child orders to form inferences over time about the dynamically evolving fundamental state of the market. In particular, investors learn about imbalances in the underlying aggregate parent demands and the associated pressure on future market-clearing prices and incorporate this information in their current child orders. Given the widespread prevalence of optimized order-splitting of parent orders into flows of child orders, dynamic learning about aggregate parent demands is a critical part of market dynamics.¹

This paper is the first to provide an analytically tractable equilibrium model of dynamic learning, trading, and pricing with parent trading demands. We consider a continuous-time model with high-frequency trading at times $t \in [0, 1]$ over short time-horizons with $[0, 1]$ being a day or an hour. Trading occurs between price-sensitive optimizing traders with two different types of parent trading targets: One group has fixed individual targets, and the other group wants to track a stochastically evolving target over time. Since parent targets are initially not public, information about parent demand imbalances is partially revealed through market-clearing stock prices. Our analysis models the equilibrium dynamic learning process, stock holdings, and stock-price processes.

Our main results are:

- We construct and solve two different equilibrium models: A simpler price-friction equilibrium and a subgame perfect Nash financial-market equilibrium. In the price-friction equilibrium, price impact is due to an exogenous trading friction, but in the subgame Nash equilibrium, price impact includes both exogenous frictions and an endogenous price impact due to market clearing with constrained market asset-holding capacity. We find that these two equilibria are numerically similar.
- Intraday price drifts due to price pressure change over the trading day and are path-dependent. This leads to time-varying incentives for investors to provide liquidity to the child orders of other investors.
- A practical application of our model is that we can compute total trading costs for investors given the effects of dynamic learning and optimal trading by other investors. We show these costs are quadratic in the rebalancers' trading targets.
- Trading in our model reflects a combination of liquidity provision and speculation but not predatory trading. We conjecture that the absence of predatory trading is because our model replaces the exogenous price-elastic residual supply used in both Brunnermeier and Pedersen [9] and Carlin, Lobo, and Viswanathan [10] with endogenous demands coming from rational profit-maximizing investors.

Our paper advances several strands of research on market microstructure. First, dynamic learning and trading have been extensively studied in the context of markets with strategic investors with long-lived asymmetric information as in Kyle [29]. However, equilibrium trading, learning, and pricing with optimal dynamic order-splitting by large uninformed investors are less understood. Thus, we model price pressure to equate supply and demand rather than adverse selection. Second, Grossman and Miller [21] model pricing and liquidity provision with impatient traders who submit single orders equal to their parent demands and with symmetric payoff information. In contrast, we model liquidity provision with optimal order-splitting of parent demands into child order flows. Third, Choi, Larsen, and Seppi [12] construct an equilibrium with optimal dynamic trading and learning in a market with a

¹ See van Kervel and Menkveld [26], Korajczyk and Murphy [28], and van Kervel, Kwan, and Westerholm [25].

strategic rebalancer with an end-of-day trading target and an informed investor who trades on private long-lived asset-payoff information. By filtering the order flow over time, the rebalancer learns about the underlying asset payoff, the informed investor learns about the rebalancer's trading target, and market makers learn about both when setting prices. That earlier paper provides a characterization result for equilibrium and gives numerical examples but does not have an existence proof or analytic solutions. In contrast, our model is solved analytically and gives the equilibrium in closed form. Fourth, Brunnermeier and Pedersen [9] and Carlin, Lobo, and Viswanathan [10] show how dynamic rebalancing by a large investor can lead to predatory trading. However, these papers abstract from the learning problem by assuming the parent trading needs are publicly observable. They also make an ad hoc assumption about the price sensitivity of a residual market-maker trading demand due to exogenous price-elastic noise traders. In contrast, our model assumes the underlying parent trading demands are private information, which leads to a learning problem. In addition, our prices are rationally set with no ad hoc residual demand. Fifth, a large body of research models optimal order-splitting strategies for a single strategic investor given an exogenous pricing rule with no learning about latent trading demands of other investors (see, e.g., Almgren and Chriss [3, 4], Almgren [2], and Schied and Schöneborn [34]). In contrast, we solve for optimal trades, learning, and pricing jointly. van Kervel, Kwan, and Westerholm [25] solve for optimal trading strategies for two dynamic rebalancers with learning over time about each other's latent trading demands. This leads to predictions about the effect of aggregate parent demand on individual investor child orders, which are then verified empirically. However, they assume an ad hoc linear pricing rule, and there are no existence proofs or analytic solutions. In contrast, price pressure in our Nash model is partly endogenously determined in equilibrium, and we solve our model analytically. As in van Kervel, Kwan, and Westerholm [25], trading in our model is a combination of speculation on expected future price changes and trading-demand accommodation.

The mathematics of our model is tractable because we use a modeling approach from the asset-pricing literature for non-dividend paying stocks. The simplification involves finding equilibrium price drifts that clear the market without determining the levels of market-clearing prices as discounted future cash flows. Karatzas and Shreve [27, Chap. 4] use this approach in complete market settings, and Cuoco and He [14] consider an extension to incomplete markets. Atmaz and Basak [1] show that non-dividend paying stocks are relevant for asset pricing. However, the non-dividend paying stock approach is new in the mainstream microstructure literature. Gârleanu and Pedersen [20], Bouchard, Fukasawa, Herdegen, and Muhle-Karbe [7], and Noh and Weston [31] use the zero-dividend stock approach to model prices given exogenous transaction costs. We extend this approach to include learning and endogenous price impact.

2 Model

We model equilibrium trading, learning, and pricing in a market with a risky stock and a riskless bank account over a short time horizon $[0, 1]$ (e.g., a trading day). For simplicity, the net supply of both the stock and bank account are set to zero. Since the time horizon is short, the risk-free interest rate on the bank account is set to zero. Stock differs from the bank account in two ways: First, investors have individual parent demands for the stock. Second, stock prices are stochastic over time. Stock valuation can be viewed as the sum of two components: One component is a fundamental valuation of future dividends absent price

pressure from trading targets. The other component is incremental price pressure for markets to clear given parent trading demand imbalances. It is the price pressure component that is the focus of our analysis. Our analysis treats these two components as being orthogonal and, for simplicity, normalizes the dividend valuation component to zero. Thus, hereafter, when we refer to the “stock price”, this is shorthand for the “price pressure valuation component of stock prices.” Our prices are random due to random trading demand imbalances. In a more complicated model, a separate fundamental dividend valuation component could be added to our stock-price pressure valuation to get the full stock price.

Two different groups of investors trade in our equilibrium model.

- (i) Price-sensitive rebalancers. Rebalancer $i \in \{1, \dots, M\}$ maximizes her expected profit subject to a parent trading target \tilde{a}_i where \tilde{a}_i is private information for i . The targets $(\tilde{a}_1, \dots, \tilde{a}_M)$ are assumed independent and homogeneously distributed $\tilde{a}_i \sim \mathcal{N}(0, \sigma_{\tilde{a}}^2)$ for all rebalancers $i \in \{1, \dots, M\}$ with identical zero means and standard deviations $\sigma_{\tilde{a}}$. The aggregate target is

$$\tilde{a}_\Sigma := \sum_{i=1}^M \tilde{a}_i. \tag{2.1}$$

Rebalancer i 's control is her stock holdings, which are denoted by $(\theta_{i,t})_{t \in [0,1]}$ for $i \in \{1, \dots, M\}$. For simplicity, the initial endowed holdings of both the bank account and the stock are normalized to zero for all rebalancers.

When \tilde{a}_i is close to zero ($\tilde{a}_i \approx 0$), rebalancer i is a “high-frequency” liquidity provider with inventory penalties. Because \tilde{a}_i is private information for i , other traders $k, k \neq i$, do not know whether rebalancer i has an active latent trading demand ($|\tilde{a}_i| \gg 0$) or is a liquidity provider ($\tilde{a}_i \approx 0$).

- (ii) Price-sensitive trackers. Trackers $j \in \{M + 1, \dots, M + \bar{M}\}$ all track a dynamic target given by a common exogenous Brownian motion process w_t over time $t \in [0, 1]$

$$w_t := w_0 + w_t^\circ, \quad t \in (0, 1], \tag{2.2}$$

where the initial target is $w_0 \sim \mathcal{N}(0, \sigma_{w_0}^2)$, and w_t° is a standard Brownian motion that starts at zero, has a zero drift, and a unit volatility.² While trackers observe the same w_t at time $t \in [0, 1]$, rebalancers do not and instead filter w_t over time $t \in [0, 1]$. Tracker j 's control is her stock holdings, which are denoted by $(\theta_{j,t})_{t \in [0,1]}$ for $j \in \{M + 1, \dots, M + \bar{M}\}$. Their initial stock and money market holdings are also normalized to zero.

We assume the random variables $(\tilde{a}_1, \dots, \tilde{a}_M)$, w_0 , and $(w_t^\circ)_{t \in [0,1]}$ are all independent.

van Kerval, Kwan, and Westerholm [25] show that interactions between multiple heterogeneous investors are an empirically important part of the trading process. Our model with $M \geq 1$ and $\bar{M} \geq 1$ lets us analyze such trading interactions. In the following, index $k \in \{1, \dots, M + \bar{M}\}$ denotes any generic trader, index $i \in \{1, \dots, M\}$ denotes a rebalancer, and index $j \in \{M + 1, \dots, M + \bar{M}\}$ denotes a tracker. This allows us to express the stock-market clearing condition as

² Adding a volatility coefficient σ_w in front of w_t° in (2.2) does not increase model flexibility because — as we shall see — the stock volatility γ is a free model parameter and γ and σ_w would play identical roles. Moreover, our model can be extended to include a drift term $\mu_w t$ for a constant μ_w in (2.2).

$$0 = \sum_{k=1}^{M+\bar{M}} \theta_{k,t} = \underbrace{\sum_{i=1}^M \theta_{i,t}}_{\text{rebalancer demand}} + \underbrace{\sum_{j=M+1}^{M+\bar{M}} \theta_{j,t}}_{\text{tracker demand}}, \quad t \in [0, 1]. \tag{2.3}$$

Investor stock demands change over time due to stochastic shocks to the tracker target w_t and due to randomness in imperfect learning about the rebalancer targets. As a result, the stock-price process that clears the market as in (2.3) changes randomly over time. Thus, stock randomness in our model — given that the fundamental dividend valuation is normalized to zero — comes from learning about traders’ parent targets (which are initially private information of the individual rebalancers and the trackers) and from random changes over time in the trackers’ target w_t .³

Investor information is represented as generic filtrations $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$ for rebalancers and trackers. These filtrations are constructed explicitly in the equilibria considered below. In the price-friction equilibrium in Sect. 3, the filtrations $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$ are

$$\begin{aligned} \sigma(\tilde{a}_i, S_{i,u})_{u \in [0,t]}, \quad t \in [0, 1], \quad i \in \{1, \dots, M\}, \\ \sigma(w_u, S_{j,u})_{u \in [0,t]}, \quad t \in [0, 1], \quad j \in \{M + 1, \dots, M + \bar{M}\}, \end{aligned} \tag{2.4}$$

where $S_{i,t}$ and $S_{j,t}$ denote perceived stock-price processes for a rebalancer i and a tracker j . In the Nash equilibrium in Sect. 4, more complicated filtrations are needed to derive traders’ optimal off-equilibrium response functions.

Our model is a model of dynamic learning. As we shall see, trackers infer the aggregate target \tilde{a}_Σ in (2.1) from the initial stock price, and so trackers have no need to filter the rebalancers’ individual targets $(\tilde{a}_1, \dots, \tilde{a}_M)$. The situation is different for each rebalancer $i \in \{1, \dots, M\}$, who only observes her own target \tilde{a}_i and past and current stock prices. When $\sigma_{w_0} > 0$, these observations are insufficient to infer \tilde{a}_Σ and w_t separately, so rebalancer i filters based on \tilde{a}_i and on past and current stock-price observations to learn about the underlying latent parent demands \tilde{a}_Σ and w_t . In contrast, when $\sigma_{w_0} := 0$, the model only has static learning about \tilde{a}_Σ at time $t = 0$ from the initial stock price. At later times $t \in (0, 1]$, the rebalancers can infer w_t from their stock-price observations. The static learning model with $\sigma_{w_0} := 0$ was developed in Choi, Larsen, and Seppi [13].

2.1 Individual maximization problems

This section introduces the individual maximization problems. A generic trader k ’s optimal stock holdings are determined in terms of a trade-off between expected terminal wealth $X_{k,1}$ and a penalty for deviations of their holdings $\theta_{k,t}$ over time from their parent target \tilde{a}_i (rebalancers) or Brownian motion w_t (trackers). An investor’s terminal wealth $X_{k,1}$ depends on the stock prices $S_{k,t}$ associated with k ’s holdings $\theta_{k,t}$ over time. An exogenous continuous (deterministic) function $\kappa : [0, 1] \rightarrow [0, \infty]$ models the severity of the target penalty over time.⁴ For example, more severe target penalties later in the day would be associated with a penalty severity function κ that is increasing with time t . The rebalancer and tracker objectives are

³ Our model features asymmetric information and learning about parent demands. However, because there are no stock dividends, there can be no asymmetric information related to future dividends.

⁴ Our analysis can be extended to allow for different penalty functions for the two groups of traders.

$$\begin{aligned} & \sup_{\theta_{i,t} \in \mathcal{F}_{i,t}} \mathbb{E} \left[X_{i,1} - \int_0^1 \kappa(t) (\tilde{a}_i - \theta_{i,t})^2 dt \middle| \mathcal{F}_{i,0} \right], \quad i \in \{1, \dots, M\}, \\ & \sup_{\theta_{j,t} \in \mathcal{F}_{j,t}} \mathbb{E} \left[X_{j,1} - \int_0^1 \kappa(t) (w_t - \theta_{j,t})^2 dt \middle| \mathcal{F}_{j,0} \right], \quad j \in \{M + 1, \dots, M + \bar{M}\}, \end{aligned} \tag{2.5}$$

where \tilde{a}_i is the ideal holdings for rebalancer i and w_t is the ideal holdings for tracker j at time $t \in [0, 1]$. However, stock-market clearing prevents $\theta_{i,t}$ and $\theta_{j,t}$ from being \tilde{a}_i and w_t . The suprema in (2.5) are taken over progressively measurable holding processes $\theta_{i,t}$ and $\theta_{j,t}$ with respect to traders’ filtrations $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$. As we shall see in Sects. 3 and 4 below, our traders optimally use controls given as smooth functions evaluated at a finite set of state processes (i.e., Markov controls). The next section constructs such a set of Markovian state processes. To rule out doubling strategies, we require square integrability

$$\mathbb{E} \left[\int_0^1 \theta_{k,t}^2 dt \right] < \infty, \quad k \in \{1, \dots, M + \bar{M}\}. \tag{2.6}$$

Terminal wealth $X_{k,1}$ in (2.5) is generated by trader k ’s perceived wealth process

$$dX_{k,t} := \theta_{k,t} dS_{k,t}, \quad X_{k,0} := 0, \quad k \in \{1, \dots, M + \bar{M}\}, \tag{2.7}$$

which is affected by k ’s holdings $\theta_{k,t}$ both directly and also indirectly via the impact of k ’s holdings on an associated perceived stock-price process $S_{k,t}$. Trader k ’s holdings $\theta_{k,t}$ are price sensitive because market-clearing price pressure affects price drifts and, thus, investor wealth. In (2.7), the zero initial wealth $X_{k,0} = 0$ is because trader k ’s initial endowed money market and stock holdings are normalized to zero. Thus, \tilde{a}_i and w_t are ideal holding changes relative to investors’ normalized initial zero holdings. Given the objectives in (2.5), trading reflects a combination of motives: Investors seek to have stock holdings close to their own targets a_i and w_t , but they also seek to increase their expected terminal wealth by trading on price pressure from other investors trading on their targets. Thus, traders demand liquidity (to come close to their targets) and supply liquidity for markets to clear (by being willing to deviate from their targets so that other traders can trade towards their targets, given the appropriate price incentives), and speculate on future predictable price pressure.

Our remaining model construction involves specifying investor stock-price perceptions $S_{i,t}$ and $S_{j,t}$ and the associated investor filtrations $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$. We then state conditions that these perceptions and filtrations must satisfy in equilibrium. Finally, we give theoretical results that ensure equilibria exist.

2.2 State processes

The fundamental underlying state of the market in our model depends on the aggregate parent demand imbalances \tilde{a}_Σ and $\bar{M} w_t$. As already noted, there is a significant informational difference between trackers and rebalancers. Each tracker directly observes w_t in (2.2) and — as we shall see — can therefore infer the aggregate rebalancer target \tilde{a}_Σ in (2.1) from the initial stock price. In contrast, rebalancers learn about w_t and \tilde{a}_Σ using dynamic filtering. Thus, the rebalancer filtrations $\mathcal{F}_{i,t}, i \in \{1, \dots, M\}$, and tracker filtrations $\mathcal{F}_{j,t}, j \in \{M + 1, \dots, M + \bar{M}\}$, are not nested. Rebalancers know prices and their individual target \tilde{a}_i , whereas trackers know \tilde{a}_Σ, w_t , and prices.

Before considering specific stock-price perceptions in Sects. 3 and 4 below, we describe a set of conjectured state processes $(Y_t, \eta_t, q_{i,t}, w_{i,t})$ for rebalancer $i \in \{1, \dots, M\}$. These processes are all endogenous in the equilibria we construct. However, it is convenient to

describe the state processes’ informational properties first, before showing how they arise in equilibrium. The processes (Y_t, η_t) are public in that they are adapted to $\mathcal{F}_{k,t}$ for all traders $k \in \{1, \dots, M + \bar{M}\}$. Furthermore, η_t will be adapted to $\sigma(Y_u)_{u \in [0,t]}$. The state processes $(q_{i,t}, w_{i,t})$ are specific to individual rebalancers. They are adapted to i ’s filtration $\mathcal{F}_{i,t}$, but they are not adapted to other traders’ filtrations $\mathcal{F}_{k,t}$ for $k \neq i$.

Rebalancers learn by extracting information about aggregate demand imbalances from stock prices. In the equilibria we construct, the information extracted from stock prices over time t is a state process Y_t , which has the form

$$Y_t := w_t - B(t)\tilde{a}_\Sigma, \quad t \in [0, 1], \tag{2.8}$$

where $B : [0, 1] \rightarrow \mathbb{R}$ is a smooth deterministic function of time that is endogenously determined in equilibrium. The function $B(t)$ controls how \tilde{a}_Σ and w_t are mixed in stock prices. The process Y_t is not directly observable for the rebalancers, but Lemma 3.1 below shows that Y_t can be inferred from stock prices. Because rebalancer $i \in \{1, \dots, M\}$ also knows her own target \tilde{a}_i , by knowing Y_t over time $t \in [0, 1]$, she equivalently knows

$$\begin{aligned} Y_{i,t} &:= Y_t + B(t)\tilde{a}_i \\ &= w_t - B(t)(\tilde{a}_\Sigma - \tilde{a}_i). \end{aligned} \tag{2.9}$$

Unlike Y_t in (2.8), the process $Y_{i,t}$ is independent of rebalancer i ’s private trading target \tilde{a}_i and satisfies

$$\sigma(\tilde{a}_i, Y_u)_{u \in [0,t]} = \sigma(\tilde{a}_i, Y_{i,u})_{u \in [0,t]}, \quad t \in [0, 1]. \tag{2.10}$$

Rebalancers use knowledge of Y_t to estimate \tilde{a}_Σ and w_t from stock prices at time t . For a continuously differentiable function $B : [0, 1] \rightarrow \mathbb{R}$, we define two processes

$$\begin{aligned} q_{i,t} &:= \mathbb{E} \left[\tilde{a}_\Sigma - \tilde{a}_i \mid \sigma(Y_{i,u})_{u \in [0,t]} \right], \\ dw_{i,t} &:= dw_t - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})dt, \quad w_{i,0} := Y_{i,0}, \end{aligned} \tag{2.11}$$

for each rebalancer $i \in \{1, \dots, M\}$ and $t \in [0, 1]$. The expectation $q_{i,t}$ describes what rebalancer i has learned up through time t about the aggregate target $\tilde{a}_\Sigma - \tilde{a}_i$ of the other rebalancers.⁵ In particular, $q_{i,t}$ is a path-dependent process because it depends on the path of $Y_{i,s}$ over time $s \in [0, t]$.

Let the function $\Sigma(t)$ denote the remaining variance

$$\Sigma(t) := \mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}] = \mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})^2], \quad t \in [0, 1], \tag{2.12}$$

where the second equality follows from the zero-mean assumptions for $(\tilde{a}_1, \dots, \tilde{a}_M)$ and w_0 . Because the targets $(\tilde{a}_1, \dots, \tilde{a}_M)$ are assumed independent and homogeneously distributed $\mathcal{N}(0, \sigma_a^2)$, the initial variance $\Sigma(0) = \mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0})^2]$ is identical across all rebalancers $i \in \{1, \dots, M\}$. This property and the formula for $\Sigma(t)$ in (2.15) below imply that $\Sigma(t)$ is also identical for all index $i \in \{1, \dots, M\}$ for all $t \in [0, 1]$.

Now consider the $w_{i,t}$ processes. Eq. (2.11) gives the dynamics of $Y_{i,t}$ as

$$\begin{aligned} dY_{i,t} &= dw_t - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i)dt \\ &= dw_{i,t} - B'(t)q_{i,t}dt. \end{aligned} \tag{2.13}$$

The following result is a special case of the Kalman-Bucy result from filtering theory (See Appendix B for details).

⁵ The process $Y_{i,t}$ is also informative about the current value of the trackers’ target w_t . Using (2.9) and (2.11), we have $\mathbb{E}[w_t | \sigma(Y_{i,u})_{u \in [0,t]}] = \mathbb{E}[Y_{i,t} + B(t)(\tilde{a}_\Sigma - \tilde{a}_i) | \sigma(Y_{i,u})_{u \in [0,t]}] = Y_{i,t} + B(t)q_{i,t}$.

Lemma 2.1 (Kalman-Bucy) *For a continuously differentiable function $B : [0, 1] \rightarrow \mathbb{R}$, the process $w_{i,t}$ is independent of \tilde{a}_i , is a Brownian motion, and satisfies (modulo \mathbb{P} null sets)*

$$\sigma(\tilde{a}_i, Y_{i,u})_{u \in [0,t]} = \sigma(\tilde{a}_i, w_{i,u})_{u \in [0,t]}, \quad t \in [0, 1]. \tag{2.14}$$

Furthermore, the remaining variance at time t is given by

$$\Sigma(t) = \frac{1}{\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0}] + \int_0^t (B'(u))^2 du}, \quad t \in [0, 1]. \tag{2.15}$$

◇

Because the process $w_{i,t}$ is independent of \tilde{a}_i , $w_{i,t}$ is also a Brownian motion with respect to the filtration $\sigma(\tilde{a}_i, w_{i,u})_{u \in [0,t]}$. Furthermore, Lemma 2.1 shows that $(\tilde{a}_i, w_{i,t})$ are informationally equivalent with $(\tilde{a}_i, Y_{i,t})$ in the sense that (2.14) holds. However, while $w_{i,t}$ on the left in (2.11) is observable by rebalancers, the individual terms w_t and \tilde{a}_Σ in $w_{i,t}$'s decomposition on the right of (2.11) are not.

The stock-market clearing condition (2.3) lets us relate prices to the state processes driving investor demands. The sum $\sum_{i=1}^M q_{i,t}$ is an important term in this relation, so the following decomposition results are useful:

Lemma 2.2 *Let $B : [0, 1] \rightarrow \mathbb{R}$ be a continuously differentiable function.*

1. *The decomposition*

$$\sum_{i=1}^M q_{i,t} = \eta_t + A(t)\tilde{a}_\Sigma, \quad t \in [0, 1], \tag{2.16}$$

holds with the process η_t being adapted to $\sigma(Y_u)_{u \in [0,t]}$ with Y_t in (2.8) and

$$\begin{aligned} A'(t) &= -(B'(t))^2 \Sigma(t)(A(t) + 1), \quad A(0) = -\frac{(M-1)B(0)^2 \sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2 \sigma_a^2}, \\ d\eta_t &= -(B'(t))^2 \Sigma(t)\eta_t dt - MB'(t)\Sigma(t)dY_t, \quad \eta_0 = -\frac{M(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2 \sigma_a^2} Y_0. \end{aligned} \tag{2.17}$$

2. *The inverse relation*

$$q_{i,t} = \frac{\eta_t}{M} - F_1(t) \left(\frac{(M-1)B(0)^2 \sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2 \sigma_a^2} + F_2(t) \right) \tilde{a}_i \tag{2.18}$$

holds with deterministic functions $F_1(t)$ and $F_2(t)$ given by the ODEs

$$\begin{aligned} F_1'(t) &= -B'(t)^2 \Sigma(t)F_1(t), \quad F_1(0) = 1, \\ F_2'(t) &= \frac{B'(t)^2 \Sigma(t)}{F_1(t)}, \quad F_2(0) = 0. \end{aligned} \tag{2.19}$$

◇

There are two key points: First, no investor knows $\sum_{i=1}^M q_{i,t}$, but it can be decomposed into a public term η_t and a term $A(t)\tilde{a}_\Sigma$ that trackers know but not the rebalancers. Second, from (2.17), the process η_t depends on the path of Y_s over time $s \in [0, t]$. Thus, the state process η_t reflects common path dependence due to w_t . The expression (2.18) shows that the individual rebalancer expectation $q_{i,t}$ includes a common learning component $\frac{\eta_t}{M}$ and then the effect of i 's private information \tilde{a}_i . In particular, it follows from (2.19), that $F_1(t)$ and $F_2(t)$ are both positive so that, consistent with intuition, the loading on \tilde{a}_i is negative in (2.18).

3 Price-friction equilibrium

Investor perceptions of the impact of their trading on stock prices are a key part of the optimizations in (2.5) and the resulting market equilibrium. We consider two specifications of investor stock-price perceptions. This section presents a simplified model in which perceived price impact is a fully exogenous trading friction. This approach is analogous to the exogenous price impact used in van Kervel, Kwan, and Westerholm [25]. We then solve for the endogenous stock-price process that clears the market (and also satisfies some weak consistency conditions) and the associated optimized investor-holding processes. Sect. 4 presents a richer model of price impact in which investor stock-price perceptions are partially endogenized in a subgame perfect Nash financial-market equilibrium.

Our equilibrium construction is a conjecture-and-verify analysis. Section 3.1 conjectures functional forms for investor perceptions of stock-price dynamics. Section 3.2 defines equilibrium and then solves for equilibrium price-perception coefficients and the associated price dynamics and holdings that satisfy the definition of equilibrium.

3.1 Stock-price perceptions

Recall that price pressure is different from the value of future dividends. It is a valuation adjustment needed to clear the stock market given trading demand imbalances. This allows us to model price pressure as zero-dividend asset prices as in, e.g., Karatzas and Shreve [27, Chap. 4].

Rebalancers optimize (2.5) with respect to perceived stock-price processes of the form

$$\begin{aligned}
 dS_{i,t}^f &:= \left\{ f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \right\} dt + \gamma dw_{i,t}, \\
 S_{i,0}^f &:= Y_0, \quad i \in \{1, \dots, M\},
 \end{aligned}
 \tag{3.1}$$

where $f_0, f_1, f_2, f_3 : [0, 1] \rightarrow \mathbb{R}$ are continuous (deterministic) functions of time $t \in [0, 1]$ and (α, γ) are constants. The “ f ” superscript indicates that the perceived price $S_{i,t}^f$ is defined with respect to a particular set of coefficient functions f in (3.1). The stock-price drift in (3.1) is perceived by rebalancer i to be affine in a set of state processes. Consistent with intuition, we will see that in equilibrium the loadings $f_0(t)$ and $f_3(t)$ on Y_t and η_t are negative. In particular, Y_t with $B(t) < 0$ measures a mix of aggregate demand from rebalancers and trackers, and η_t reflects public expectations of aggregate private rebalancer expectations about other rebalancers’ parent-demand imbalances, both of which depress price change expectations. The other coefficients describe the perceived impact of rebalancer i on the stock-price drift.

Theorem 3.5 below endogenously determines (f_0, f_1, f_2, f_3) in equilibrium. The exogenous parameters (α, γ) can be found by calibrating model output to empirical data. The term $\alpha\theta_{i,t}$ allows for ad hoc trading frictions. The price-friction parameter α is an exogenous model input. Price taking is a special case with $\alpha := 0$, whereas the empirically relevant case is $\alpha < 0$ such that buy (sell) orders decrease (increase) the future stock-price drifts.

The innovations in the rebalancers’ perceived stock prices $dw_{i,t}$ come from new information rebalancer i learns over time about the underlying parent-demand state variable Y_t , which has both a direct effect on the future stock-price drift and an additional indirect effect via its impact on η_t since η_t is adapted to $\sigma(Y_u)_{u \in [0,t]}$ from Lemma 2.2.

The zero-dividend stock valuation approach (see, e.g., Chapter 4 in Karatzas and Shreve, [27]) has several consequences: First, we model perceived and equilibrium stock-price drifts

rather than price levels. Second, in (3.1), the stock’s volatility and initial value are not determined in equilibrium but rather are model inputs. For simplicity, we set the volatility to be a constant $\gamma > 0$ (i.e., positive demand innovations $dw_{i,t}$ increase prices), and the initial price is set to be Y_0 in (3.1). However, other choices of S_0 would work equally well as long as S_0 satisfies $\sigma(S_0) = \sigma(Y_0)$.

The next result shows that $w_{i,t}$ is rebalancer i ’s innovations process in the sense that $w_{i,t}$ is a Brownian motion relative to i ’s filtration defined with perceived stock prices $S_{i,t}^f$ in (3.1) and such that $S_{i,t}^f$ and $w_{i,t}$ generate the same information.

Lemma 3.1 *Let $f_0, f_1, f_2, f_3 : [0, 1] \rightarrow \mathbb{R}$ be continuous functions and let $B : [0, 1] \rightarrow \mathbb{R}$ be a continuously differentiable function. For a rebalancer $i \in \{1, \dots, M\}$, let $\theta_{i,t}$ satisfy (2.6) and be progressively measurable with respect to $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}$ with $S_{i,t}^f$ defined in (3.1) and Y_t defined in (2.8). Then, modulo \mathbb{P} -null sets, we have*

$$\sigma(\tilde{a}_i, w_{i,u})_{u \in [0,t]} = \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}, \quad t \in [0, 1], \quad i \in \{1, \dots, M\}. \tag{3.2}$$

◇

Thus, given a path of perceived prices generated by a price process $S_{i,t}^f$ of the form in (3.1) and her personal target \tilde{a}_i , rebalancer i can infer the path of $w_{i,t}$. Furthermore, given the path $w_{i,t}$, rebalancer i can infer $Y_{i,t}$ using (2.14) and, thus, can infer Y_t from (2.10). Consequently, rebalancer i can infer $(q_{i,t}, \eta_t)$ where we recall from Lemma 2.2 that η_t is adapted to $\sigma(Y_t)_{t \in [0,1]}$.

Trackers optimize (2.5) with respect to a perceived stock-price process of the form

$$\begin{aligned} dS_{j,t}^{\bar{f}} &:= \left\{ \bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\theta_{j,t} \right\} dt + \gamma dw_t, \\ S_{j,0}^{\bar{f}} &:= Y_0, \quad j \in \{M + 1, \dots, M + \bar{M}\}, \end{aligned} \tag{3.3}$$

where $\bar{f}_3, \bar{f}_4, \bar{f}_5 : [0, 1] \rightarrow \mathbb{R}$ are continuous (deterministic) functions, and the α is a constant.⁶ Trackers have different information in that they observe w_t directly and can infer \tilde{a}_Σ from the initial stock price Y_0 using (2.8) and their knowledge of w_0 . Therefore, their perceived stock prices differ from those of the rebalancers. Theorem 3.5 below endogenously determines $(\bar{f}_3, \bar{f}_4, \bar{f}_5)$ in equilibrium, and (α, γ) are exogenous model inputs. Again, $\alpha := 0$ is the special case of price-taking.

The motivation for these price perceptions for the trackers is as follows. First, the perceptions in (3.3) allow trackers to condition their perceived price drift to take into account price pressure from target imbalances \tilde{a}_Σ and w_t that depress expected price changes. Since trackers and rebalancers trade differently on their targets, the price-drift impacts \bar{f}_4 and \bar{f}_5 are in general different. Second, the trackers understand that the state process Y_t affects the rebalancer demand and, thus, the stock-price drift. However, Y_t does not need to be included explicitly in the tracker perceived price drift in (3.3) since Y_t is given by a linear combination of \tilde{a}_Σ and w_t , which are already included in the drift. Third, trackers know that rebalancers’ can infer η_t and that this potentially affects their price perceptions in (3.1), and, thus, is likely to affect their trading, and, thus, is likely to affect pricing. Thus, trackers allow for the pricing effect of η_t in their perceptions in (3.3). Fourth, as already noted, α allows for possible exogenous trading frictions, if any.

An important difference between rebalancer and tracker perceived prices in (3.1) and (3.3) is that rebalancer price dynamics are based on the informational innovations $dw_{i,t}$,

⁶ Our model can be extended to allow for a different price-friction coefficient $\bar{\alpha}$ with $\bar{\alpha} \neq \alpha$ for the trackers.

whereas tracker price dynamics are based on the tracker target changes dw_t . Reconciling the price perceptions of rebalancers and trackers will impose restrictions on equilibrium price perceptions and holdings and will rely on the relation between $dw_{i,t}$ and dw_t in (2.11).

Given the price perceptions in (3.1) and (3.3), we solve (2.5) for optimal rebalancer and tracker holdings.

Lemma 3.2 *Let $f_0, f_1, f_2, f_3, \bar{f}_3, \bar{f}_4, \bar{f}_5 : [0, 1] \rightarrow \mathbb{R}$ and $\kappa : [0, 1] \rightarrow (0, \infty]$ be continuous functions, let $B : [0, 1] \rightarrow \mathbb{R}$ be continuously differentiable, let $\alpha \leq 0$, and let the perceived stock-price process in the wealth dynamics (2.7) be as in (3.1) and (3.3). Then, for $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}$ and $\mathcal{F}_{j,t} := \sigma(w_u, S_{j,u}^{\bar{f}})_{u \in [0,t]}$, and, provided the holding processes*

$$\begin{aligned} \hat{\theta}_{i,t} &:= \frac{f_0(t)}{2(\kappa(t) - \alpha)} Y_t + \frac{f_1(t) + 2\kappa(t)}{2(\kappa(t) - \alpha)} \tilde{a}_i + \frac{f_2(t)}{2(\kappa(t) - \alpha)} q_{i,t} + \frac{f_3(t)}{2(\kappa(t) - \alpha)} \eta_t, \\ \hat{\theta}_{j,t} &:= \frac{\bar{f}_3(t)}{2(\kappa(t) - \alpha)} \eta_t + \frac{\bar{f}_5(t) + 2\kappa(t)}{2(\kappa(t) - \alpha)} w_t + \frac{\bar{f}_4(t)}{2(\kappa(t) - \alpha)} \tilde{a}_\Sigma, \end{aligned} \tag{3.4}$$

satisfy (2.6), the traders’ maximizers for (2.5) are $\hat{\theta}_{i,t}$ for rebalancer $i \in \{1, \dots, M\}$ and $\hat{\theta}_{j,t}$ for tracker $j \in \{M + 1, \dots, M + \bar{M}\}$. \diamond

The proof of Lemma 3.2 shows that pointwise quadratic maximization gives the maximizers for (2.5) for rebalancers and trackers for arbitrary f and \bar{f} functions.

Stock-price perceptions play two interconnected roles in our model. First, rebalancers and trackers solve their optimization problems in (2.5) based on their perceptions in (3.1) and (3.3) for how hypothetical holdings $\theta_{i,t}$ and $\theta_{j,t}$ affect price dynamics. Second, investor stock-price perceptions affect how they learn from observed prices. In particular, Lemma 3.1 shows that rebalancers use their stock-price perceptions (3.1) to infer the aggregate demand state variable Y_t based on past and current stock prices. In other words, dynamic learning by rebalancers depends critically on their stock-price perceptions. Similarly, trackers also use their stock-price perception of Y_0 in (3.3) to infer the aggregate parent demand \tilde{a}_Σ from the initial price at time $t = 0$. However, thereafter, there is no additional learning from prices by the trackers at $t > 0$ since they directly observe their target w_t .

3.2 Equilibrium

This section defines our first of two equilibrium concepts and then derives price-perception coefficients for the conjectured functional form in Sect. 3.1 that satisfy the equilibrium definition along with the associated equilibrium price dynamics and holdings. The notion of equilibrium in our first construction is relatively simple, being based just on market clearing and consistency of investor price perceptions.

Definition 3.3 Deterministic functions of time $f_0, f_1, f_2, f_3, \bar{f}_3, \bar{f}_4, \bar{f}_5, B : [0, 1] \rightarrow \mathbb{R}$ constitute a *price-friction equilibrium* if:

- (i) Maximizers $\hat{\theta}_{k,t}$ for (2.5) exist for traders $k \in \{1, \dots, M + \bar{M}\}$ given the stock-price perceptions (3.1) and (3.3) for filtrations $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}$ and $\mathcal{F}_{j,t} := \sigma(w_u, S_{j,u}^{\bar{f}})_{u \in [0,t]}$.
- (ii) Inserting trader k ’s maximizer $\hat{\theta}_{k,t}$ into the perceived stock-price processes (3.1) and (3.3) produces identical stock-price processes across all traders $k \in \{1, \dots, M + \bar{M}\}$. This common equilibrium stock-price process is denoted by \hat{S}_t .

(iii) The money and stock markets clear. ◇

Definition 3.3 places only minimal restrictions on the perceived stock-price coefficient functions in (3.1) and (3.3): Markets must clear and result in consistent perceived stock-price processes when all investors use their equilibrium strategies. Section 4 below considers a subgame perfect Nash extension of our basic model that imposes more restrictions on allowable off-equilibrium stock-price perceptions such as off-equilibrium market clearing and various consistency requirements.

Definition 3.3(ii) requires that in equilibrium rebalancers and trackers perceive identical stock-price dynamics when using their equilibrium holdings. However, rebalancers and trackers have different information (i.e., rebalancers form imperfect inferences about w_t and \tilde{a}_Σ , whereas trackers observe w_t directly and infer \tilde{a}_Σ at time 0). The resolution of this apparent paradox is investors’ different information sets: Trackers and rebalancers all agree on $d\hat{S}_t$, but they disagree on how to decompose $d\hat{S}_t$ into drift and volatility components. Because the trackers observe w_t , they can use dw_t in their decomposition of $d\hat{S}_t$. However, w_t is not adapted to the rebalancers’ filtrations and can therefore not be used in their $d\hat{S}_t$ decompositions. Instead, rebalancers use their innovations processes $dw_{i,t}$ when decomposing $d\hat{S}_t$ into drift and volatility. By replacing $dw_{i,t}$ in $dS_{i,t}^f$ in (3.1) with the decomposition of $dw_{i,t}$ in terms of dw_t from (2.11), we can rewrite $dS_{i,t}^f$ in (3.1) as

$$dS_{i,t}^f = \left\{ f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})\gamma \right\} dt + \gamma dw_t, \quad i \in \{1, \dots, M\}. \tag{3.5}$$

Therefore, to ensure identical equilibrium stock-price perceptions for all rebalancers and trackers, it suffices to match the drift of $dS_{j,t}^f$ in (3.3) for the equilibrium holdings $\theta_{j,t} = \hat{\theta}_{j,t}$, $j \in \{M+1, \dots, M+\bar{M}\}$, with the drift of $dS_{i,t}^f$ in (3.5) for the equilibrium holdings $\theta_{i,t} := \hat{\theta}_{i,t}$, $i \in \{1, \dots, M\}$. This produces the following equilibrium requirement:

$$f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\hat{\theta}_{i,t} - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})\gamma = \bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\hat{\theta}_{j,t}, \tag{3.6}$$

for all rebalancers $i \in \{1, \dots, M\}$ and all trackers $j \in \{M+1, \dots, M+\bar{M}\}$. We note that the right-hand side of (3.6) does not depend on the rebalancer index i . Matching up coefficients in front of $(\tilde{a}_i, \tilde{a}_\Sigma, q_{i,t}, \eta_t, w_t)$ in (3.6) using $\hat{\theta}_{i,t}$ and $\hat{\theta}_{j,t}$ in (3.4) and Y_t in (2.8) produces five equations. In addition, inserting $\hat{\theta}_{i,t}$ and $\hat{\theta}_{j,t}$ in (3.4) into the market-clearing condition (2.3) and using (2.16) produce three more equations from matching $(\tilde{a}_\Sigma, \eta_t, w_t)$ coefficients. All in all, we have eight equilibrium restrictions for $(f_0, f_1, f_2, f_3, \bar{f}_3, \bar{f}_4, \bar{f}_5)$ and B' , which give the equilibrium coefficient functions (A.1) in Appendix A and the ODE for $B(t)$ in (3.7) below.

Our equilibrium existence result is based on the following technical lemma. It guarantees the existence of a solution to an autonomous system of coupled ODEs. In particular, given rebalancer stock-price perceptions of the form in (3.1) with an aggregate demand state variable Y_t process of the form in (2.8) (and the associated η_t process), we must construct a deterministic function $B(t)$ that gives an equilibrium.

Lemma 3.4 *Let $\kappa : [0, 1] \rightarrow [0, \infty]$ be a continuous and integrable function (i.e., $\int_0^1 \kappa(t)dt < \infty$). For an initial constant $B(0) \in \mathbb{R}$, the coupled ODEs*

$$\begin{aligned}
 B'(t) &= \frac{2\kappa(t)(\bar{M}B(t) + 1)}{\gamma(A(t) + \bar{M} + 1)}, \\
 A'(t) &= -(B'(t))^2 \Sigma(t)(A(t) + 1), \quad A(0) = -\frac{(M-1)B(0)^2 \sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2 \sigma_a^2}, \\
 \Sigma'(t) &= -(B'(t))^2 \Sigma(t)^2, \quad \Sigma(0) = \frac{(M-1)\sigma_a^2 \sigma_{w_0}^2}{(M-1)B(0)^2 \sigma_a^2 + \sigma_{w_0}^2},
 \end{aligned}
 \tag{3.7}$$

have unique solutions with $\Sigma(t) \geq 0$, $\Sigma(t)$ decreasing, $A(t) \in [-1, 0]$, $A(t)$ decreasing for $t \in [0, 1]$, and $B(t), B'(t) < 0$ when $\bar{M}B(0) + 1 < 0$. ◇

The solutions to the ODEs for $A(t)$ and $\Sigma(t)$ in (3.7) agree with the expressions in (2.15) and (2.17). The exogenous price-friction coefficient α does not appear in the ODEs (3.7). It is possible to restate the ODE system (3.7) using a single path-dependent ODE. The special case $B(0) := -\frac{1}{\bar{M}}$ produces a model with no dynamic learning because $B'(t) = 0$ implies $\Sigma'(t) = 0$ and so $dq_{i,t} = 0$.

The following theorem gives the price-friction equilibrium in terms of the ODEs (3.7). In this theorem, the price-friction parameter α , volatility γ , and initial value $B(0) \in \mathbb{R}$ are free parameters. The intuition for $B(0)$ being free is discussed after our equilibrium construction in Theorem 3.5.

Theorem 3.5 *Let $\kappa : [0, 1] \rightarrow (0, \infty)$ be continuous, let the functions (B, A, Σ) be as in Lemma 3.4, and let $\alpha \leq 0$. Then, we have:*

- (i) *A price-friction equilibrium exists and is given by the price-perception functions (A.1) in Appendix A.*
- (ii) *The equilibrium in (i) has holdings $\hat{\theta}_{i,t}$ for rebalancer i and $\hat{\theta}_{j,t}$ for tracker j given by*

$$\begin{aligned}
 \hat{\theta}_{i,t} &= -\frac{\gamma B'(t) - 2\kappa(t)}{2\kappa(t) - \alpha} \tilde{a}_i - \frac{\gamma B'(t)}{2\kappa(t) - \alpha} q_{i,t} \\
 &\quad + \frac{\gamma B'(t)}{(M+\bar{M})(2\kappa(t) - \alpha)} \eta_t - \frac{2\bar{M}\kappa(t)}{(M+\bar{M})(2\kappa(t) - \alpha)} Y_t, \quad i \in \{1, \dots, M\}, \\
 \hat{\theta}_{j,t} &= \frac{\gamma B'(t)}{(M+\bar{M})(2\kappa(t) - \alpha)} \eta_t + \frac{2M\kappa(t)}{(M+\bar{M})(2\kappa(t) - \alpha)} w_t \\
 &\quad + \frac{\gamma(A(t) - M + 1)B'(t) - 2\kappa(t)}{(M+\bar{M})(2\kappa(t) - \alpha)} \tilde{a}_\Sigma, \quad j \in \{M + 1, \dots, M + \bar{M}\}.
 \end{aligned}
 \tag{3.8}$$

- (iii) *The equilibrium in (i) has the equilibrium stock-price process \hat{S}_t given by $\hat{S}_0 := w_0 - B(0)\tilde{a}_\Sigma$ and dynamics with respect to the trackers' filtrations $\mathcal{F}_{j,t} := \sigma(w_u, S_{j,u}^{\bar{f}})_{u \in [0,t]}$ given by*

$$d\hat{S}_t = \left\{ \frac{\gamma B'(t)}{M+\bar{M}} \eta_t - \frac{2\bar{M}\kappa(t)}{M+\bar{M}} w_t + \frac{\gamma(A(t) - M + 1)B'(t) - 2\kappa(t)}{M+\bar{M}} \tilde{a}_\Sigma \right\} dt + \gamma dw_t,
 \tag{3.9}$$

and dynamics with respect to the rebalancers' filtrations $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}$ given by

$$d\hat{S}_t = \left\{ \frac{\gamma B'(t)}{M+\bar{M}} \eta_t - \frac{2\bar{M}\kappa(t)}{M+\bar{M}} Y_t - \gamma B'(t)(\tilde{a}_i + q_{i,t}) \right\} dt + \gamma dw_{i,t}.
 \tag{3.10}$$

◇

Several observations follow from Theorem 3.5:

1. Lemma 3.1 ensures that rebalancer i can infer her innovations process $w_{i,t}$ from perceived prices $S_{i,t}^f$ and \tilde{a}_i , but rebalancer i cannot infer the trackers' target w_t from the equilibrium prices \hat{S}_t in (3.9). This is because the aggregate target \tilde{a}_Σ also appears in the drift of $d\hat{S}_t$ and \tilde{a}_Σ is not observed by individual rebalancers.
2. The equilibrium holdings (3.8) follow from inserting the equilibrium f and \bar{f} functions in (A.1) in Appendix A into (3.4). Thus, the holdings in (3.8) are expressed in terms of the investors' state processes, which, in particular, are adapted to the investors' filtrations. However, these state processes are not mutually independent and so we give such representations of (3.8) in (A.2) and (A.3) in Appendix A. First, the price-friction equilibrium rebalancer holdings $\hat{\theta}_{i,t}$ in (3.8) can be written in terms of the independent variables $(\tilde{a}_i, \tilde{a}_\Sigma - \tilde{a}_i, w_0)$ and a residual orthogonal term given as a stochastic integral with respect to w_t° of a deterministic function of time. Likewise, the price-friction equilibrium tracker holdings $\hat{\theta}_{j,t}$ can be written in terms of the independent variables (\tilde{a}_Σ, w_0) and a residual orthogonal term given in terms of a stochastic integral with respect to w_t° of a deterministic function of time. Both these residual terms are Gaussian. Section 3.4 illustrates the loading coefficients on these independent state processes.
3. Because the exogenous price-friction coefficient $\alpha \leq 0$ does not appear in the ODEs (3.7), α is irrelevant for the equilibrium stock-price dynamics (3.9). However, α does affect the equilibrium holdings in (3.8).
4. The stock-price volatility γ affects the stock-price drift and holdings via its impact on $B(t)$ in (3.7) and, thus, on (A.1).
5. It can seem paradoxical that trackers and rebalancers all perceive the same equilibrium stock-price process \hat{S}_t , but they decompose its dynamics $d\hat{S}_t$ into different perceived drifts and martingale terms (i.e., they have different Itô decompositions). The resolution lies in the rebalancers and trackers having different filtrations.⁷ The drift and martingale terms in (3.10) are not adapted to $\mathcal{F}_{j,t}$ and the drift and martingale terms in (3.9) are not adapted to $\mathcal{F}_{i,t}$. The dynamics (3.9) and (3.10) all produce the same process \hat{S}_t because the innovations process $w_{i,t}$ in (2.11) links dw_t with $dw_{i,t}$ and the drift term $B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})dt$.
6. Investors' off-equilibrium perceived stock-price drifts differ linearly from their equilibrium drifts due to the differences $\theta_{k,t} - \hat{\theta}_{k,t}$ between their off-equilibrium and equilibrium holdings.⁸ Rebalancer i 's perceived stock-price drift in (3.1) can be decomposed for arbitrary holdings $\theta_{i,t}$ as

$$\begin{aligned}
 & f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \\
 & = \frac{\gamma B'(t)}{M+M} \eta_t - \frac{2\bar{M}\kappa(t)}{M+M} Y_t - \gamma B'(t)(\tilde{a}_i + q_{i,t}) + \alpha(\theta_{i,t} - \hat{\theta}_{i,t}),
 \end{aligned}
 \tag{3.11}$$

⁷ We nickname this our “Rashomon Theorem” after the 1950 movie in which different characters perceive the same event differently given their different perspectives. Rebalancers and trackers both start with private information so their filtrations are not nested. However, in equilibrium, stock-price dynamics depend on w_t and \tilde{a}_Σ . Because the trackers know w_0 at time $t = 0$, they infer \tilde{a}_Σ from $S_{j,0} = w_0 - B(0)\tilde{a}_\Sigma$, and so they have no need to filter at later times. On the other hand, rebalancer i only has noisy dynamic predictions $\mathbb{E}[\tilde{a}_\Sigma | \mathcal{F}_{i,t}] = q_{i,t} + \tilde{a}_i$ of the aggregate parent imbalance \tilde{a}_Σ given her inferences based on the individual parent targets \tilde{a}_i and stock-price observations.

⁸ Eqs. (3.11) and (3.12) are similar to Eq. (3.14) in Choi, Larsen, and Seppi [13].

where we have used the formulas for (f_0, f_1, f_2, f_3) in (A.1) in Appendix A. Likewise, for arbitrary holdings $\theta_{j,t}$, tracker j 's perceived stock-price drift in (3.3) is

$$\begin{aligned} & \bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\theta_{j,t} \\ &= \frac{\gamma B'(t)}{M+M} \eta_t - \frac{2\bar{M}\kappa(t)}{M+M} w_t + \frac{\gamma(A(t)-M+1)B'(t)-2\kappa(t)}{M+M} \tilde{a}_\Sigma + \alpha(\theta_{j,t} - \hat{\theta}_{j,t}), \end{aligned} \tag{3.12}$$

where we have used the formulas for $(\bar{f}_3, \bar{f}_4, \bar{f}_5)$ in (A.1) in Appendix A.

Continuity between equilibrium and off-equilibrium is a reasonable property of investor stock-price perceptions. The representation of the perceived rebalancer drift in (3.11) relative to $\hat{\theta}_{i,t}$ from (3.8) also explains the presence of the rebalancer-specific terms $(\tilde{a}_i, q_{i,t})$ in the rebalancers' perceptions in (3.1).

7. Investors initially use block trades at time 0 to trade to positions $\theta_{i,0}$ and $\theta_{j,0}$ from (3.8) that are generically different from their initial normalized holdings of 0. Thereafter, investors trade continuously at times $t > 0$.
8. Theorem 3.5 verifies that price-perception coefficients in (3.1) and (3.3) can be constructed such that an equilibrium satisfying Definition 3.3 exists. However, as with many other rational expectation models, we do not have a proof of uniqueness. For example, there may be other public state variables in addition to η_t that could hypothetically be included in the perceived price drifts that might also be associated with other equilibria as defined in Definition 3.3.

The function $B(t)$ from (3.7) is key both in constructing the equilibrium and for interpreting the equilibrium price and holding processes. First, there is the issue that the initial value $B(0)$ is a free input in Theorem 3.5. The intuition is that our model determines equilibrium stock-price drifts but not price levels. As can be seen in Theorem 3.5(iii), $B(0)$ controls the initial price level in our model. Second, the relation between $B(t)$ and price levels allows us to impose additional structure on $B(t)$. In particular, w_t and \tilde{a}_Σ represent different types of demand imbalances. Thus, if $B(t) < 0$, then Y_t in (2.8) plays the role of an aggregate demand state variable. How the two component quantities w_t and \tilde{a}_Σ are mixed in the aggregate demand state variable Y_t is different given the two components' different informational dynamics (i.e., \tilde{a}_Σ is not time dependent while w_t changes randomly over time) and given their different impacts on investor demands (i.e., each rebalancer only knows their personal \tilde{a}_i component of \tilde{a}_Σ where other rebalancers' targets do not affect investor i 's parent demand whereas w_t affects both an individual tracker's parent demand and is also information about other trackers' parent demands). It seems reasonable that the sign of the impact of w_t and \tilde{a}_Σ on the price level should be the same, which imposes the additional restriction that $B(t) < 0$. From Lemma 3.4, a sufficient condition for $B(t) < 0$ for all $t \in [0, 1]$ is $\bar{M}B(0) + 1 < 0$, which implies $B'(t) < 0$.⁹

With the economically reasonable parametric restriction that $B'(t) < 0$ and given that $\alpha \leq 0$ so that $\alpha - 2\kappa(t) < 0$, we can sign the impact of various quantities in the model on holdings and prices, which leads to the following comparative statics:

1. In (3.8), the equilibrium holdings $\hat{\theta}_{i,t}$ of rebalancers are positively related to their parent targets \tilde{a}_i . This is intuitive because rebalancers want holdings close to \tilde{a}_i . Rebalancer holdings $\hat{\theta}_{i,t}$ are negatively related to the aggregate demand imbalance state variable Y_t . The fact that $\theta_{i,t}$ is decreasing in Y_t is consistent with the theoretical results and empirical evidence in van Kervel, Kwan, and Westerholm [25] that investors buy less when there is

⁹ This sufficient condition follows because the denominator in (3.7) is positive given that $A(t) \in [-1, 0]$ so that the numerator in (3.7) determines the sign of $B'(t)$.

- a positive parent-demand imbalance for other investors in the market. The same intuition applies to the negative impact of the common component η_t on $\hat{\theta}_{i,t}$. However, the impact of $q_{i,t}$ on $\hat{\theta}_{i,t}$ is positive. The intuition is that when rebalancer i expects the other rebalancers (given i 's ability to filter using her private target information \tilde{a}_i) to have a net positive parent-demand imbalance $\mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i | \mathcal{F}_{i,t}]$ from (2.11), she buys at time t to speculate on the resulting anticipated positive drift in future price pressure in (3.10).
- In (3.8), the equilibrium holdings $\hat{\theta}_{j,t}$ of trackers are increasing in w_t (which reflects both her own parent demand and also information about the parent demands of other trackers). Tracker holdings $\hat{\theta}_{j,t}$ are also decreasing in η_t , which is related to imbalances in rebalancers' aggregate parent demand expectations. The negative effect of η_t is consistent with the van Kervel, Kwan, and Westerholm [25] liquidity-provision result and empirical evidence. However, the impact of \tilde{a}_Σ is ambiguous in (3.8), and numerical calculations in Sect. 3.4 show that the sign is positive. This is again consistent with speculation on future predicted price pressure due to the tracker's superior information about aggregated latent parent-demand imbalances.
 - The equilibrium stock-price drift in (3.9) is decreasing in the tracker parent demand w_t . However, the impact of \tilde{a}_Σ in the price drift is again ambiguous, which is related to information about \tilde{a}_Σ being useful in forecasting future price pressure.

3.3 Tractability and model structure

This section discusses the key modeling features that make our model tractable. First, we assume all traders seek to maximize their individual objectives in (2.4). Linear-quadratic objectives have been used extensively in the literature because of their tractability. Such objectives have been used in, e.g., Sannikov and A. Skrzypacz [33], Gârleanu and Pedersen [20], and Bouchard, Fukasawa, Herdegen, and Muhle-Karbe [7]. The linear-quadratic objectives (2.5) allow us to solve for the optimal holdings in Lemma 3.2 using quadratic pointwise optimization. In the price-friction equilibrium, we could equivalently use dynamic programming to produce the same optimal holdings.

Second, our stock does not pay dividends, which means that only the stock drift can be endogenously determined in equilibrium. Models with non-dividend paying stocks have been used extensively in the literature. The monograph Karatzas and Shreve [27] gives an overview.¹⁰ In particular, non-dividend paying stock models have been used for short horizon models like ours where consumption only takes place at the terminal time.¹¹ The rebalancers' dynamic learning produces forward-running filtering equations and by considering a non-dividend paying stock, we circumvent having additional backward-running equations. Equilibrium models with both forward and backward-running equations include Kyle [29],

¹⁰ Similar to a money market account, a non-dividend paying stock is a *financial asset* in the sense that holding one stock at time $t = 1$, gives one unit of consumption at $t = 1$. Likewise, being short one stock at $t = 1$, means the trader provides one unit of consumption at $t = 1$. Both the bank account and the non-dividend paying stock have exogenous initial prices and volatilities. It is custom for the money market account's initial price to be one and its volatility to be zero. For the non-dividend paying stock, we set the initial price to be Y_0 , its volatility to be a positive constant $\gamma > 0$, and determine endogenously the drift.

¹¹ There are long-lived non-dividend paying stocks too as; see, for example, Atmaz and Basak [1] write: "For example, Hartzmark and Solomon [24] find that over the long-sample of 1927-2011, the average proportion of no-dividend stocks is around 35% and accounts for 21.3% of the aggregate US stock market capitalization. Similarly, by taking into account of rising share repurchase programs since the mid-1980ies, Boudoukh et al. [8] report that over the 1984-2003 period, the average proportion of no-dividend stocks is 64% and no-payout stocks, i.e., no dividends or no share repurchases, is 51% with the relative market capitalizations of 16.4% and 14.2%, respectively."

Foster and Viswanathan [18, 19], Back, Cao, and Willard [5], and Choi, Larsen, and Seppi [12].

Third, price impact is often modeled as the impact of investor holdings and orders on price levels (e.g., as in Almgren [2]) and as the impact of orders on price changes (e.g., Kyle [29]). However, for the sake of tractability, we follow Cuoco and Cvitanic [15] and model price impact in terms of the impact of investor holdings on the price drift. Price impact matters for the trading decisions of strategic investors because of its effect on future expected price changes (e.g., high holding demand raises prices which lowers expected future price appreciation). Our price-friction specification simply assumes directly that investor holdings affect expected future price changes. Thus, while our price impact specification is a simplification, it is a reasonable simplification that preserve the essential economics of price impact.

Fourth, instead of exogenous noise traders, we use optimizing trackers with a Brownian motion target w_t . Grossman and Stiglitz [22] and Kyle [29] are standard references with an exogenous Gaussian stock supply. Gaussian noise traders are also used in the predatory trading models in Brunnermeier and Pederson [9] and Carlin, Lobo, and Viswanathan [10]. In our setting, we could eliminate trackers by setting $\bar{M} := 0$ and replace the stock-market clearing condition (2.3) by using w_t to model the exogenous stock supply as in

$$w_t = \sum_{i=1}^M \theta_{i,t}, \quad t \in [0, 1]. \tag{3.13}$$

Including noise traders as in (3.13) in the model would be tractable in the price-friction equilibrium. However, surprisingly, exogenous noise-traders complicate constructing a Nash equilibrium with dynamic learning, whereas — as we show in Sect. 4 — optimizing trackers and market learning in (2.3) produce a subgame perfect Nash financial-market equilibrium in closed form. The models in Sannikov and Skrzypacz [33] and Choi, Larsen, and Seppi [13] have optimizing trackers but no dynamic learning.

3.4 Numerics

Our price-friction equilibrium is straightforward to compute numerically. This is because equilibrium stock prices and holdings are available in closed form given the solutions to the associated coupled ODEs in (3.7). We illustrate our models for several different parameterizations. In these parameterizations, there are $M := 5$ rebalancers and $\bar{M} := 10$ trackers. The penalty function is a constant over the trading day and set to $\kappa(t) := 1$. The rebalancer target volatility is normalized to $\sigma_{\bar{a}} := 1$ whereas we consider $\sigma_{w_0} \in \{\frac{1}{10}, 1\}$ to illustrate the impact of dynamic learning. Recall that $\sigma_{w_0} := 0$ gives the model with only initial learning of \bar{a}_Σ as developed in Choi, Larsen, and Seppi [13]. To be consistent with our negative $B(t)$ restriction, we consider an initial value $B(0) := -0.2$. We consider two stock-price volatility parameters $\gamma \in \{\frac{1}{2}, 1\}$ and a zero price-friction parameter $\alpha := 0$ (i.e., the competitive equilibrium). As noted above, α does not affect the endogenous price-drift coefficients, but α does affect investor holdings.

3.4.1 Equilibrium holdings

First, we consider equilibrium holdings. Fig. 1 shows the coefficient functions for the equilibrium stock holdings $\hat{\theta}_{k,t}$ in (3.8) for rebalancers and trackers using their orthogonal

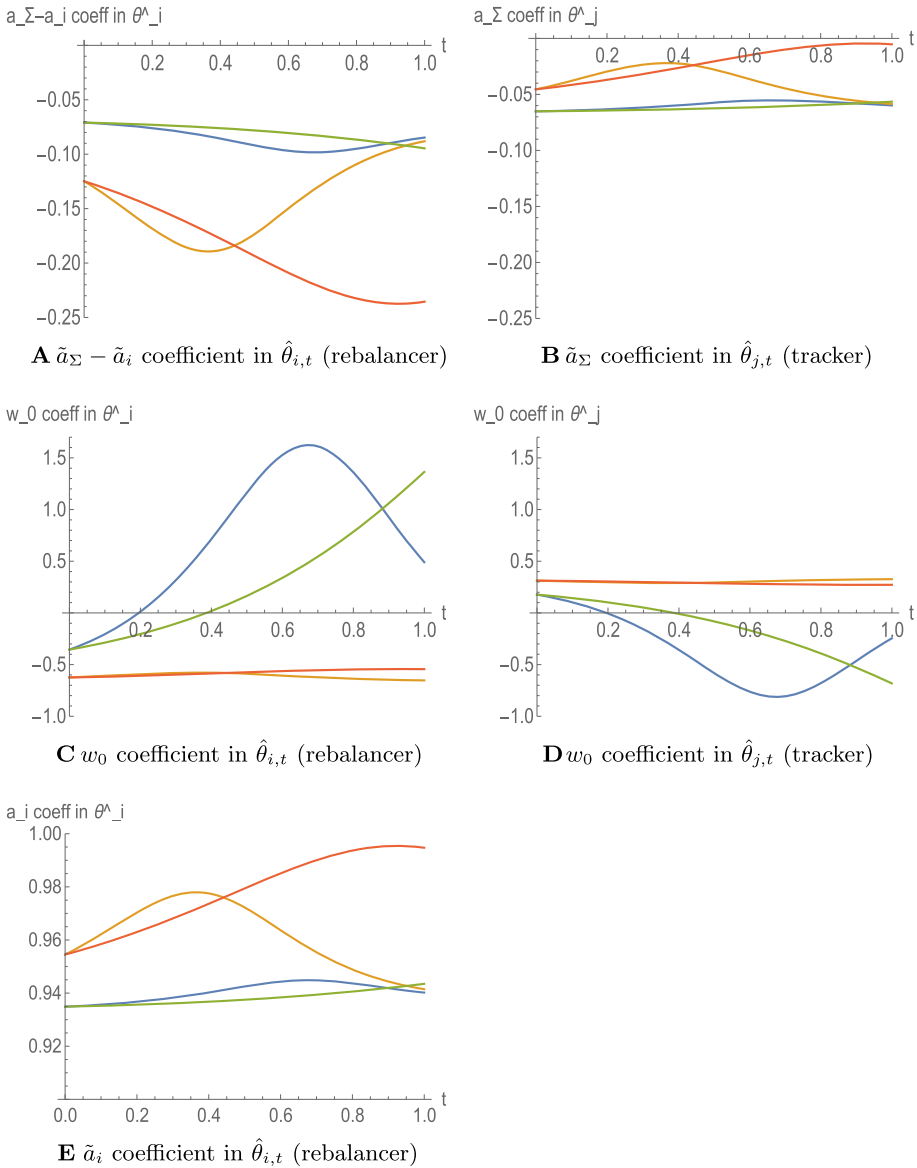


Fig. 1 Plots of coefficient loadings over time for holdings $\hat{\theta}_{k,t}$ using the orthogonal representations in (A.2) and (A.3) in Appendix A. The exogenous model parameters are $\sigma_{\bar{a}} := 1, M := 5, \bar{M} := 10, \alpha := 0, B(0) := -0.2, \kappa(t) := 1$ for $t \in [0, 1], (\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

representations in (A.2) and (A.3) in Appendix A. Alternatively, we could plot coefficient loadings on the state processes $(\bar{a}_i, q_{i,t}, \eta_t, Y_t)$ and $(\eta_t, w_t, \bar{a}_\Sigma)$ in (3.8). We prefer to illustrate orthogonal loadings to avoid cancelation effects in the different state processes.

Fig. 1E shows rebalancer i 's loadings over time on her own parent target a_i . As expected, these loadings are positive, but they are less than 1 because trading towards a positive target

depresses equilibrium price drifts in order for markets to clear. The rebalancer loading on \tilde{a}_i is over 0.9, which implies a large initial block trade at time $t = 0$. The negative coefficients on $\tilde{a}_\Sigma - \tilde{a}_i$ (for rebalancer i) in Fig. 1A and \tilde{a}_Σ (for tracker j) in Fig. 1B are demand accommodation. In particular, rebalancers and trackers reduce their holdings when other rebalancers want to buy. The loadings on w_0 in Fig. 1C and 1D are more subtle. When the initial tracker target w_0 has a high volatility (as in the red and amber trajectories), the tracker holdings load positively on w_0 over time in Fig. 1D, and the negative rebalancer loadings in Fig. 1C indicate demand accommodation by the rebalancers. However, when the initial tracker target has low volatility (as in the green and blue trajectories), the initial positive tracker loadings on w_0 eventually flip signs as do the initial negative rebalancer loadings. At first glance, this is puzzling. The explanation is that, as noted above, the trackers and rebalancers have different stock-price drift perceptions in (3.9) and (3.10) given their different filtrations. In particular, there is dynamic learning over time by the rebalancers based on the information Y_t inferred from prices, whereas the trackers are fully informed about \tilde{a}_Σ and w_t (trackers infer \tilde{a}_Σ at time 0). Fig. 3C and 3D below illustrate that the rebalancers' and trackers' stock-drift perceptions are quite different in these two low σ_{w_0} parameterizations.

In addition to the effects illustrated in Fig. 1, investor holdings are also affected by the realized path of $w_t = w_0 + w_t^\circ$ over time. This is because of fluctuations in the underlying tracker parent demand and also due to the effect of w_t° on dynamic learning by the rebalancers. Appendix A shows the exact specification of this term in the tracker holdings (given as a dw_u° integral of a deterministic function). Given the linearity of investor holdings and since the Brownian motion w_t° has zero expected increments, this random path effect disappears in ex ante expected investor holdings.

To summarize, Fig. 1 shows there are three main drivers of investor holdings: First, investors' holdings in most cases are drawn partially towards their own targets \tilde{a}_i and w_t . Second, investors provide partial accommodation to other investors' parent demands. Third, dynamic learning and speculation on the price drift affect demand accommodation. Interestingly, there is no evidence in Fig. 1 of predatory trading. Predatory trading differs from demand accommodation in that a predator first trades in the same direction as another investor and then subsequently unwinds her position. In this context, the hump-shape of the blue trajectories (for low σ_{w_0}) in Fig. 1C and 1D do not indicate predatory trading: Because w_0 is the trackers' own target, the blue hump in Fig. 1D cannot reflect predatory trading. Furthermore, the blue hump-shaped trajectory in Fig. 1C also differs from predatory trading because the tracker and rebalancer loadings have opposite signs as seen in Fig. 1D. This is due to market clearing. For example, when the rebalancers are buying given $w_0 > 0$, the trackers are actually selling. Instead of predatory trading, we shall see below, that the blue trajectories are explained by price perceptions and dynamic learning.

Fig. 2 plots the instantaneous intraday unconditional trading autocorrelations

$$\rho_k(t) := \lim_{h \downarrow 0} \frac{\text{corr}(\hat{\theta}_{k,t+h} - \hat{\theta}_{k,t}, \hat{\theta}_{k,t+2h} - \hat{\theta}_{k,t+h})}{h}, \quad k \in \{1, \dots, M + \bar{M}\}, \quad (3.14)$$

for the price-friction equilibrium holding processes for both the rebalancer and tracker in (3.8). These autocorrelations are scaled by the time step $h > 0$ (the unscaled versions converge to zero as $h \downarrow 0$).

Thus, consistent with empirical evidence, trading is autocorrelated due to order splitting. Fig. 2 shows that rebalancers' orders are positively autocorrelated (2A) whereas trackers' orders exhibit negative autocorrelation (2B).

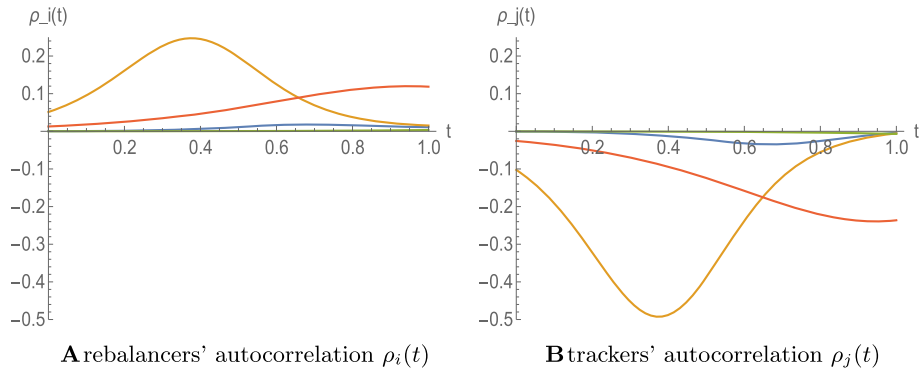


Fig. 2 Plots of unconditional autocorrelation (3.14) of trading over time. The exogenous model parameters are $\sigma_{\tilde{a}} := 1, M := 5, \bar{M} := 10, \alpha := 0, B(0) := -0.2, \kappa(t) := 1$ for $t \in [0, 1]$, and $(\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

Market clearing forces the intraday instantaneous unconditional cross correlation between rebalancers’ and trackers’ holdings to be negatively perfectly correlated

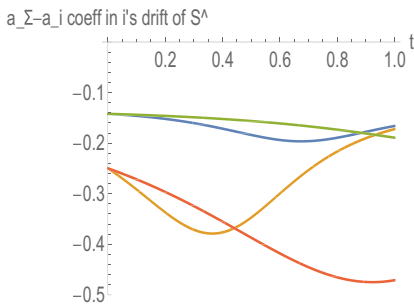
$$\lim_{h \downarrow 0} \text{corr}(\hat{\theta}_{i,t+h} - \hat{\theta}_{i,t}, \hat{\theta}_{j,t+h} - \hat{\theta}_{j,t}) = -1, \tag{3.15}$$

for all $i \in \{1, \dots, M\}$ and $j \in \{M + 1, \dots, M + \bar{M}\}$.

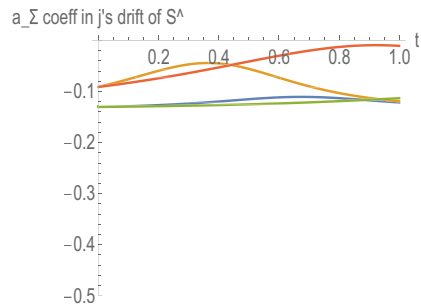
3.4.2 Equilibrium prices

Next, we consider the price-friction equilibrium stock-price dynamics in (3.9) and (3.10). For the trackers, we can rewrite the drift in (3.9) in terms of the independent random variables (\tilde{a}_Σ, w_0) and an residual orthogonal term given as a stochastic integral with respect to w_t° of a deterministic function of time. For the rebalancers, we can rewrite the perceived drift in (3.10) in terms of the independent random variables $(\tilde{a}_\Sigma - \tilde{a}_i, w_0, \tilde{a}_i)$ and an residual orthogonal term given as a stochastic integral with respect to w_t° of a deterministic function of time. These formulas are given in (A.4) and (A.5) in Appendix A and are illustrated in Fig. 3.

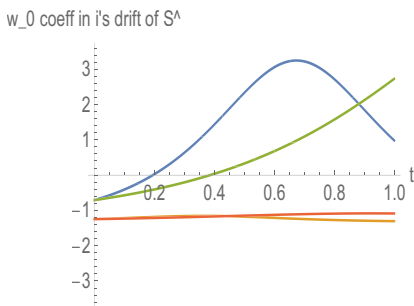
Fig. 3 shows that positive parent demands $\tilde{a}_i, \tilde{a}_\Sigma - \tilde{a}_i,$ and \tilde{a}_Σ all depress perceived stock-price drifts. The same is true for the tracker perceived stock-price drift loading on the initial tracker parent demand w_0 . However, the relation between the rebalancer perceived drift and w_0 is more nuanced. When the initial tracker demand volatility σ_{w_0} is high (red and amber lines in Fig. 3C and 3D), then rebalancers perceive that w_0 depresses the price drift. However, when σ_{w_0} is low, then the dynamic learning process — given the inability of rebalancers to observe w_0 directly — causes the rebalancer perceived stock-price drift loading on w_0 to change sign. The blue and green lines in Fig. 3C and 3D illustrate that low values of σ_{w_0} make the trackers use their superior knowledge of w_0 to manipulate stock-price perceptions to create gains from trade that outweigh their penalties. More specifically, the blue and green lines in Fig. 1C and 1D show that rebalancers have large positive stock holdings and trackers have large negative holdings based on a positive realization $w_0 > 0$. Such large negative holdings imply that trackers incur large inventory penalties because they deviate from the target trajectory $w_t = w_0 + w_t^\circ$. Trackers find this behavior optimal because their blue and green lines in Fig. 3D are negative (giving trackers large gains from trade) and rebalancers



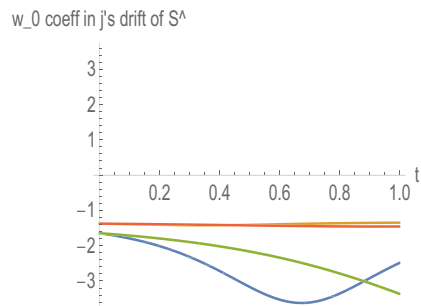
A: $\tilde{a}_\Sigma - \tilde{a}_i$ coefficient in rebalancers' stock drift



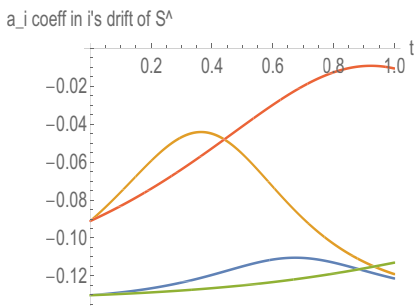
B: \tilde{a}_Σ coefficient in trackers' stock drift



C: w_0 coefficient in rebalancers' stock drift



D: w_0 coefficient in trackers' stock drift



E: \tilde{a}_i coefficient in rebalancers' stock drift

Fig. 3 Plots of coefficient loadings over time in stock-price drifts in (A.5) (rebalancer i) and (A.4) (tracker j). The exogenous model parameters are $\sigma_{\tilde{a}} := 1$, $M := 5$, $\bar{M} := 10$, $\alpha := 0$, $B(0) := -0.2$, $\kappa(\tilde{a}) := 1$ for $t \in [0, 1]$, and $(\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

are willing to hold these large positive stock positions because their blue and green lines in Fig. 3C are positive (giving also rebalancers large gains from trade).

Fig. 4A plots the instantaneous intraday unconditional stock-price autocorrelation, which is again scaled relative to h

$$\rho(t) := \lim_{h \downarrow 0} \frac{\text{corr}(\hat{S}_{t+h} - \hat{S}_t, \hat{S}_{t+2h} - \hat{S}_{t+h})}{h}, \quad t \in [0, 1), \tag{3.16}$$

for the equilibrium stock-price process \hat{S}_t .

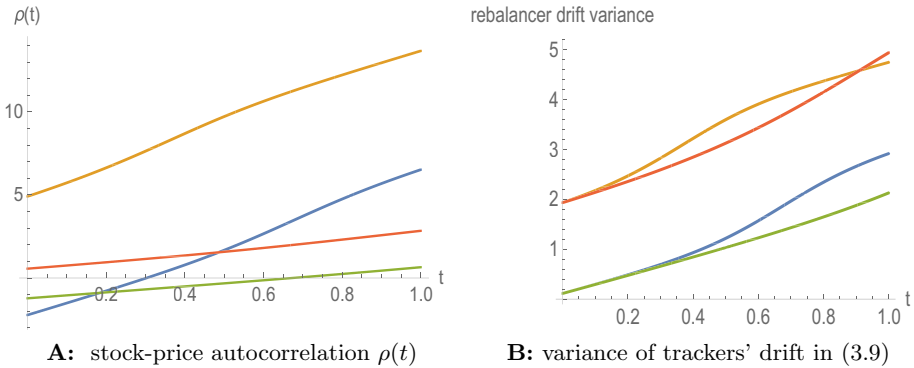


Fig. 4 Plots of the stock-price scaled autocorrelation (3.16) and of the variance of trackers' equilibrium stock-price drift over time for the equilibrium stock-price dynamics $d\hat{S}_t$ in (3.9). The exogenous model parameters are $\sigma_{\tilde{a}} := 1, M := 5, \bar{M} := 10, \alpha := 0, B(0) := -0.2, \kappa(t) := 1$ for $t \in [0, 1]$, and $(\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

Price pressure from persistent parent demands lead to rising intraday price autocorrelation over the trading day. Fig. 4B plots the time trajectory of the unconditional variance of intraday price drifts over the trading day based on the trackers' equilibrium perceptions in (3.9). Predictable price drifts are important in actual markets as incentives for intraday liquidity provision by HFT market makers (represented in our model by rebalancers with realizations $\tilde{a}_i \approx 0$). We see that price-drift variability due to price pressure increases over the trading day.

3.4.3 Equilibrium learning

Fig. 5A shows that $\sigma_{w_0} > 0$ controls the starting point $\Sigma(0) > 0$ whereas $\gamma > 0$ controls the speed of learning (i.e., how negative the slope of $\Sigma(t)$ is). For example, the green and red lines ($\gamma = 1$) illustrate a slower speed of learning relative to the amber and blue lines ($\gamma = 0.1$). These effects on $\Sigma'(t)$ come from Fig. 5B and the formula for $\Sigma(t)$ in terms $B'(t)$ in (2.15). The red line in Fig. 5A also shows that the remaining variance $\Sigma(1)$ at $t = 1$ can be substantial.

3.4.4 Equilibrium welfare

In this section, we study the impact of the exogenous model input $B(0) \in \mathbb{R}$ on equilibrium welfare. There are many ways to measure social welfare (see, e.g., Vayanos [35, Section 6]). We follow Du and Zhu [17, Eq. 42] and consider maximizing the expected aggregate certainty equivalent for the $M + \bar{M}$ investors. The certainty equivalent $CE_k \in \mathbb{R}$ for investor $k \in \{1, \dots, M + \bar{M}\}$ is defined by the expressions in (2.5). The aggregate expected welfare is given by

$$\sum_{k=1}^{M+\bar{M}} \mathbb{E}[CE_k], \tag{3.17}$$

where the expectation in (3.17) is ex ante in the sense that it is taken over the random variables $(\tilde{a}_1, \dots, \tilde{a}_M)$ and w_0 (Gaussian and independent).

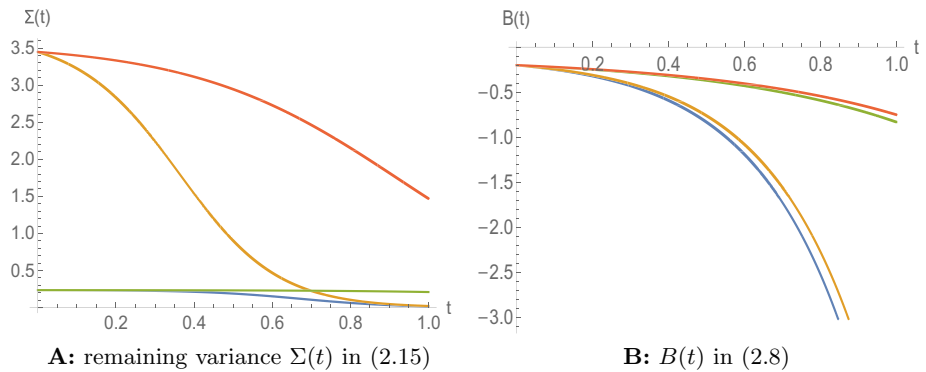


Fig. 5 Plots of solutions of ODEs in (3.7) over time. The exogenous model parameters are $\sigma_{\tilde{a}} := 1, M := 5, \bar{M} := 10, \alpha := 0, B(0) := -0.2, \kappa(t) := 1$ for $t \in [0, 1]$, and $(\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

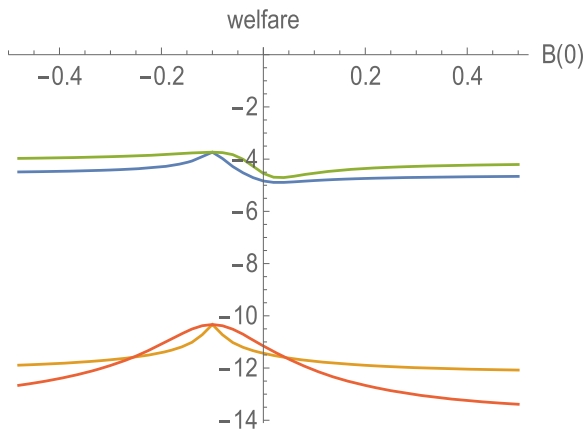


Fig. 6 Plots of aggregate expected welfare in (3.17) for varying $B(0)$. The exogenous model parameters are $\sigma_{\tilde{a}} := 1, M := 5, \bar{M} := 10, \kappa(t) := 1$ for $t \in [0, 1]$, $\alpha := 0$, and $(\gamma, \sigma_{w_0}) = (\frac{1}{2}, \frac{1}{10})$ (blue), $(\frac{1}{2}, 1)$ (amber), $(1, \frac{1}{10})$ (green), and $(1, 1)$ (red).

Fig. 6 shows that in the price-friction equilibrium with $\alpha := 0$, expected welfare is maximized at $B(0) = \frac{1}{M}$. This is not too surprising because $B(0) = \frac{1}{M}$ implies full revelation and no dynamic learning takes place for $t > 0$ (see the discussion after Lemma 3.4). Aggregate welfare is decreasing in the initial tracker parent standard deviation σ_{w_0} both because more demand accommodation is required and also because the rebalancer learning problem is more difficult. This effect can be seen by comparing the blue and green (low initial standard deviation) and amber and red (high initial standard deviation) cases in Fig. 6.

4 Subgame perfect Nash equilibrium

This section builds on the analysis in Section 3 by endogenizing stock-price perceptions and price impact. In particular, we partially endogenize the impact of an investor’s hypotheti-

cal off-equilibrium holdings on off-equilibrium market-clearing stock prices based on her perceptions of how other investors perceive prices and on other investors’ resulting optimal response functions to her off-equilibrium holdings. More specifically, a subgame perfect Nash equilibrium involves describing how each trader k_0 (who might be a rebalancer i_0 or a tracker j_0 with their different filtrations) perceives all other traders’ price perceptions.

The major difference between the price-friction equilibrium in Sect. 3 and our subgame perfect Nash equilibrium lies in the traders’ stock-price perceptions. For a subgame perfect Nash equilibrium, investor stock-price perceptions must be such that:

- (i) Trader k_0 ’s own stock-price perceptions must be consistent with market-clearing for any off-equilibrium holdings $\theta_{k_0,t}$ used by k_0 , when other traders’ holding responses are optimal given the stock-price dynamics k_0 perceives other traders $k \neq k_0$ to have. This off-equilibrium market-clearing requirement can be found in, e.g., Vayanos [35].
- (ii) Trader k_0 ’s equilibrium holdings are found by solving her optimization problem using her own market-clearing stock-price dynamics from (i).
- (iii) All optimizers from (i) must be consistent with traders’ equilibrium holdings in (ii).

Definition 4.3 below makes properties (i)–(iii) operational. We refer to the last property (iii) as a consistency requirement between off- and on-equilibrium holdings.

4.1 Optimal off-equilibrium responses

In our subgame perfect Nash model, a generic trader k_0 perceives that other rebalancers and trackers have stock-price perceptions of the form

$$\begin{aligned}
 dS_{i,t}^Z &:= \left\{ Z_t + \mu_1(t)\tilde{a}_i + \mu_2(t)q_{i,t} + \mu_3(t)\eta_t + \alpha\theta_{i,t} \right\} dt + \gamma dW_{i,t}, \\
 S_{i,0}^Z &:= Z_0, \quad i \in \{1, \dots, M\}, \\
 dS_{j,t}^Z &:= \left\{ Z_t + \bar{\mu}_4(t)\tilde{a}_\Sigma + \bar{\mu}_5(t)w_t + \alpha\theta_{j,t} \right\} dt + \gamma dW_{j,t}, \\
 S_{j,0}^Z &:= Z_0, \quad j \in \{M + 1, \dots, M + \bar{M}\},
 \end{aligned}
 \tag{4.1}$$

where $W_{i,t}$ and $W_{j,t}$ are Brownian motions and Z_t is an arbitrary Itô process (i.e., Z_t is a sum of drift and volatility). The “Z” superscript in (4.1) indicates that the perceived stock prices $S_{i,t}^Z$ and $S_{j,t}^Z$ are defined with respect to Z_t . We use the market-clearing condition (2.3) to construct two such Itô processes in (4.5) and (4.8) below. These Z_t processes differ from Y_t in (3.1) and (3.3) in that we use Z_t to capture the effect of arbitrary off-equilibrium stock holdings by trader k_0 on market-clearing prices given optimal responses by other investors $k, k \neq k_0$. We then go on to determine endogenously the deterministic functions $(\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5)$ in equilibrium in Theorem 4.5 below.

Lemma 4.1 gives traders’ optimal response to an arbitrary Itô process Z_t and is the Nash equilibrium analogue of Lemma 3.2.

Lemma 4.1 (Optimal responses to Z_t) *Let $\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5 : [0, 1] \rightarrow \mathbb{R}$ and $\kappa : [0, 1] \rightarrow (0, \infty)$ be continuous functions, let $\alpha \leq 0$, let $(Z_t)_{t \in [0,1]}$ be an Itô process, and let the perceived stock-price process in the wealth dynamics (2.7) be as in (4.1). Then, Z_t is adapted to both $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, Y_u, W_{i,u}, S_{i,u}^Z)_{u \in [0,t]}$ and $\mathcal{F}_{j,t} := \sigma(\tilde{a}_\Sigma, w_u, Y_u, W_{j,u}, S_{j,u}^Z)_{u \in [0,t]}$ and, provided*

$$\begin{aligned}
 \theta_{i,t}^Z &:= \frac{1}{2(\kappa(t)-\alpha)} Z_t + \frac{2\kappa(t)+\mu_1(t)}{2(\kappa(t)-\alpha)} \tilde{a}_i + \frac{\mu_2(t)}{2(\kappa(t)-\alpha)} q_{i,t} + \frac{\mu_3(t)}{2(\kappa(t)-\alpha)} \eta_t, \\
 \theta_{j,t}^Z &:= \frac{1}{2(\kappa(t)-\alpha)} Z_t + \frac{2\kappa(t)+\bar{\mu}_5(t)}{2(\kappa(t)-\alpha)} w_t + \frac{\bar{\mu}_4(t)}{2(\kappa(t)-\alpha)} \tilde{a}_\Sigma,
 \end{aligned}
 \tag{4.2}$$

satisfy (2.6), the maximizer for (2.5) is $\theta_{i,t}^Z$ for rebalancer $i \in \{1, \dots, M\}$ and $\theta_{j,t}^Z$ for tracker $j \in \{M + 1, \dots, M + \bar{M}\}$. \diamond

Similar to Lemma 3.2, Lemma 4.1 is proven using pointwise quadratic maximization. Unlike Y_t in Lemma 3.2, there is no Markov structure imposed on Z_t in Lemma 4.1, which makes dynamical programming inapplicable. Therefore, the simplicity of the linear-quadratic objectives in (2.5) is crucial for the proof of the optimality of $\theta_{i,t}^Z$ and $\theta_{j,t}^Z$ in (4.2).

4.2 Market-clearing stock-price perceptions

Investor k_0 's perceptions about other investors' stock-price perceptions ensure that the stock market clears for any choice of k_0 's holdings. Thus, when solving for trader k_0 's individual equilibrium holdings, we require k_0 's perceived stock-price process (denoted by $S_{k_0,t}^v$ below) clears the stock market for arbitrary hypothetical holdings $\theta_{k_0,t}$. We assume that a given trader $k_0 \in \{1, \dots, M + \bar{M}\}$ perceives that other traders $k \neq k_0$ perceive the stock-price processes in (4.1). Hence, trader k_0 perceives that other traders $k, k \neq k_0$, optimally hold $\theta_{k,t}^Z$ in (4.2) shares of stock. Given this, we then find market-clearing $Z_{k_0,t}$ processes associated with arbitrary hypothetical holdings $\theta_{k_0,t}$ for trader k_0 .

First, consider a rebalancer $i_0 \in \{1, \dots, M\}$. We construct a process $Z_{i_0,t}$ such that the stock market clears in the sense

$$0 = \underbrace{\theta_{i_0,t}}_{\text{rebalancer}} i_0 + \underbrace{\sum_{i=1, i \neq i_0}^M \theta_{i,t}^{Z_{i_0}}}_{\text{other rebalancers}} + \underbrace{\sum_{j=M+1}^{\bar{M}} \theta_{j,t}^{Z_{i_0}}}_{\text{trackers}}, \quad t \in [0, 1], \tag{4.3}$$

where $\theta_{i_0,t}$ denotes an arbitrary stock-holdings process for rebalancer i_0 and other investors' responses $\theta_{k,t}^{Z_{i_0}}$ are from (4.2) for $Z_t := Z_{i_0,t}$. Clearly, any solution $Z_{i_0,t}$ of (4.3) is specific for rebalancer i_0 . To describe one particular solution $Z_{i_0,t}$, we insert (4.2) into (4.3). This produces an affine equation in $(\theta_{i_0,t}, Z_{i_0,t}, \tilde{a}_{i_0}, q_{i_0,t}, \eta_t, w_t, \tilde{a}_\Sigma)$. Because rebalancer i cannot observe nor infer w_t and \tilde{a}_Σ separately, she has to filter based on observing a linear combination of w_t and \tilde{a}_Σ given by $Y_t := w_t - B(t)\tilde{a}_\Sigma$ where $B : [0, 1] \rightarrow \mathbb{R}$ is a continuously differentiable function satisfying

$$B(t) = - \frac{A(t)\mu_2(t) + \bar{M}\bar{\mu}_4(t) + 2\kappa(t) + \mu_1(t)}{2\bar{M}\kappa(t) + \bar{M}\bar{\mu}_5(t)}, \tag{4.4}$$

where $A(t)$ is as in (2.17). The specific form of (4.4) comes from rewriting (4.3) in terms of $(\theta_{i_0,t}, Z_{i_0,t}, \tilde{a}_{i_0}, q_{i_0,t}, \eta_t, Y_t)$ rather than $(\theta_{i_0,t}, Z_{i_0,t}, \tilde{a}_{i_0}, q_{i_0,t}, \eta_t, w_t, \tilde{a}_\Sigma)$. Because $A(t)$ in (2.17) depends on $B(t)$, Eq. (4.4) is a fixed point requirement for $B(t)$. Below, we show that the coupled ODEs in (4.19) characterize (A, B) in (4.4), and we give conditions ensuring that (4.19) has a solution. Given a solution $B(t)$ to (4.4), we use $Y_t := w_t - B(t)\tilde{a}_\Sigma$ from (2.8) to express a solution of (4.3) as¹²

$$Z_{i_0,t} := \frac{2(\alpha - \kappa(t))}{M + \bar{M} - 1} \theta_{i_0,t} + \frac{2\kappa(t) + \mu_1(t)}{M + \bar{M} - 1} \tilde{a}_{i_0} + \frac{\mu_2(t)}{M + \bar{M} - 1} q_{i_0,t} - \frac{(M - 1)\mu_3(t) + \mu_2(t)}{M + \bar{M} - 1} \eta_t - \frac{\bar{M}(2\kappa(t) + \bar{\mu}_5(t))}{M + \bar{M} - 1} Y_t, \quad t \in [0, 1]. \tag{4.5}$$

¹² The specific $B(t)$ function in (4.4) lets us combine w_t and \tilde{a}_Σ terms from (4.3) into the Y_t term in (4.5) using $Y_t = w_t - B(t)\tilde{a}_\Sigma$ from (2.8).

The process $Z_{i_0,t}$ in (4.5) captures the impact of arbitrary holdings $\theta_{i_0,t}$ by rebalancer i_0 on market-clearing stock prices given i_0 's perceptions of how other traders $k \neq i_0$ optimally respond using $\theta_{k,t}^{Z_{i_0}}$ from (4.2) with $Z_t := Z_{i_0,t}$.

Next, we describe rebalancer i_0 's stock-price perceptions for $i_0 \in \{1, \dots, M\}$. Rebalancer i_0 filters based on her own target \tilde{a}_i and on observations of past and current perceived market-clearing stock prices $S_{i_0,u}^v$ defined by

$$dS_{i_0,t}^v := \left\{ v_0(t)Z_{i_0,t} + v_1(t)\tilde{a}_{i_0} + v_2(t)q_{i_0,t} + v_3(t)\eta_t + \alpha\theta_{i_0,t} \right\} dt + \gamma dw_{i_0,t}, \tag{4.6}$$

$$S_{i_0,0}^v := Y_0, \quad i_0 \in \{1, \dots, M\},$$

where $(\tilde{a}_{i_0}, \theta_{i_0,t})$ are known and $(Z_{i_0,t}, q_{i_0,t}, \eta_{i_0,t})$ are inferred by rebalancer i_0 . The “ v ” superscript in (4.6) indicates that the perceived stock prices are defined with respect to a particular set of deterministic functions (v_0, v_1, v_2, v_3) , which we endogenously determine in Theorem 4.5 below. More specifically, by observing \tilde{a}_{i_0} and $(S_{i_0,u}^v)_{u \in [0,t]}$ defined in (4.6), rebalancer i_0 infers $Y_t := w_t - B(t)\tilde{a}_\Sigma$ from (2.8) using the Volterra argument behind Lemma 3.1. To see this, we insert (4.5) into (4.6) to produce rebalancer i_0 's perceived market-clearing stock-price dynamics

$$dS_{i_0,t}^v = \left\{ \left(\frac{v_0(t)(2\kappa(t)+\mu_1(t))}{M+\bar{M}-1} + v_1(t) \right) \tilde{a}_{i_0} + \left(\frac{\mu_2(t)v_0(t)}{M+\bar{M}-1} + v_2(t) \right) q_{i_0,t} \right. \\ \left. + \left(v_3(t) - \frac{v_0(t)((M-1)\mu_3(t)+\mu_2(t))}{M+\bar{M}-1} \right) \eta_t - \frac{\bar{M}v_0(t)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1} Y_t \right. \\ \left. + \left(\alpha - \frac{2v_0(t)(\kappa(t)-\alpha)}{M+\bar{M}-1} \right) \theta_{i_0,t} \right\} dt + \gamma dw_{i_0,t}. \tag{4.7}$$

Because the expressions multiplying $(\tilde{a}_{i_0}, q_{i_0,t}, \eta_t, Y_t, \theta_{i_0,t})$ in (4.7) are continuous (deterministic) functions of time $t \in [0, 1]$, Lemma 3.1 applies and shows that by observing \tilde{a}_{i_0} and $(S_{i_0,u}^v)_{u \in [0,t]}$ in (4.7) over time $t \in [0, 1]$, rebalancer i_0 can infer $w_{i_0,t}$. Subsequently, rebalancer i_0 can use (2.10) and (2.14) to also infer Y_t over time $t \in [0, 1]$.

Next, consider a tracker $j_0 \in \{M + 1, \dots, M + \bar{M}\}$. For arbitrary off-equilibrium holdings $\theta_{j_0,t}$, the market-clearing solution $Z_{j_0,t}$ from

$$0 = \underbrace{\theta_{j_0,t}}_{\text{tracker } j_0} + \underbrace{\sum_{j=M+1, j \neq j_0}^{\bar{M}} \theta_{j,t}^{Z_{j_0}}}_{\text{other trackers}} + \underbrace{\sum_{i=1}^M \theta_{i,t}^{Z_{j_0}}}_{\text{rebalancers}}, \quad t \in [0, 1], \tag{4.8}$$

is given by

$$Z_{j_0,t} := \frac{2(\alpha-\kappa(t))}{M+\bar{M}-1} \theta_{j_0,t} - \frac{M\mu_3(t)+\mu_2(t)}{M+\bar{M}-1} \eta_t - \frac{(\bar{M}-1)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1} w_t \\ - \frac{A(t)\mu_2(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{M+\bar{M}-1} \tilde{a}_\Sigma, \tag{4.9}$$

where $A(t)$ is as in (2.17). Once again, $Z_{j_0,t}$ captures tracker j_0 's perceptions of the impact of her holdings $\theta_{j_0,t}$ on market-clearing stock prices given j_0 's perceptions of other investors' $k \neq j_0$ responses $\theta_{k,t}^{Z_{j_0}}$ to $\theta_{j_0,t}$.

Tracker j_0 's perceived market-clearing stock-price process is defined as

$$dS_{j_0,t}^{\bar{v}} := \left\{ Z_{j_0,t} + \bar{v}_3(t)\eta_t + \bar{v}_4(t)\tilde{a}_\Sigma + \bar{v}_5(t)w_t + \alpha\theta_{j_0,t} \right\} dt + \gamma dw_t, \tag{4.10}$$

$$S_{j_0,0}^{\bar{v}} := Y_0, \quad j_0 \in \{M + 1, \dots, M + \bar{M}\},$$

where $\bar{v}_3, \bar{v}_4, \bar{v}_5 : [0, 1] \rightarrow \mathbb{R}$ are deterministic functions of time (endogenously determined Theorem 4.5 below). Inserting (4.9) into (4.10) gives tracker j_0 's perceived market-clearing stock-price dynamics

$$\begin{aligned}
 dS_{j_0,t}^{\bar{v}} = & \left\{ \left(\bar{v}_3(t) - \frac{M\mu_3(t)+\mu_2(t)}{M+\bar{M}-1} \right) \eta_t \right. \\
 & + \left(\bar{v}_5(t) - \frac{(\bar{M}-1)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1} \right) w_t \\
 & + \left(\bar{v}_4(t) - \frac{A(t)\mu_2(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{M+\bar{M}-1} \right) \tilde{a}_{\Sigma} \\
 & \left. + \frac{\alpha(M+\bar{M}+1)-2\kappa(t)}{M+\bar{M}-1} \theta_{j_0,t} \right\} dt + \gamma dw_t.
 \end{aligned}
 \tag{4.11}$$

We note that tracker j_0 's perceived market-clearing stock-price dynamics $dS_{j_0,t}^{\bar{v}}$ in (4.11) are driven by the exogenous Brownian motion w_t from (2.2) whereas rebalancer i_0 's stock prices $dS_{i_0,t}^v$ in (4.7) are driven by i_0 's innovations process $dw_{i_0,t}$ from (2.11). This is due to the different information sets of rebalancers and trackers.

Unlike the price-friction equilibrium in Theorem 3.5, we see from (4.7) and (4.11) that, even with no direct price impact in the sense $\alpha := 0$ in (4.6) and (4.10), the remaining net price impacts $-\frac{2v_0(t)\kappa(t)}{M+\bar{M}-1}$ and $-\frac{2\kappa(t)}{M+\bar{M}-1}$ of $\theta_{i,t}$ and $\theta_{j,t}$ are nonzero. This is because price pressure in (4.7) and (4.11) clears the stock market for arbitrary holdings $\theta_{i,t}$ and $\theta_{j,t}$.

The next result gives the optimal holdings $\theta_{k,t}^*$ for all traders $k_0 := k \in \{1, \dots, M + \bar{M}\}$ given their perceptions of market-clearing stock prices in (4.7) and (4.11). While both $\theta_{k,t}^*$ and the optimal response holdings $\theta_{k,t}^Z$ in (4.2) maximize (2.5), they differ because they are based on different perceived stock-price processes. On one hand, the optimal responses $\theta_{k,t}^Z$ in (4.2) are based on the stock-price perceptions in (4.1). On the other hand, the optimizer $\theta_{k,t}^*$ is based on the market-clearing stock-price perceptions in (4.7) and (4.11).

Lemma 4.2 [Trader k 's maximizer for market-clearing stock-price perceptions] *Let $v_0, v_1, v_2, v_3, \bar{v}_3, \bar{v}_4, \bar{v}_5 : [0, 1] \rightarrow \mathbb{R}$ and $\kappa : [0, 1] \rightarrow (0, \infty]$ be continuous functions with $v_0 > 0$ and assume $\alpha \leq 0$. Let the perceived market-clearing stock-price processes in the wealth dynamics (2.7) be given by (4.7) and (4.11) with corresponding filtrations $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^v)_{u \in [0,t]}$ and $\mathcal{F}_{j,t} := \sigma(w_u, S_{j,u}^{\bar{v}})_{u \in [0,t]}$ for $i \in \{1, \dots, M\}$ and $j \in \{M + 1, \dots, M + \bar{M}\}$. Then, provided the holding processes*

$$\begin{aligned}
 \theta_{i,t}^* := & \frac{2\kappa(t)(M+\bar{M}+v_0(t)-1)+(M+\bar{M}-1)v_1(t)+\mu_1(t)v_0(t)}{2(\kappa(t)-\alpha)(M+\bar{M}+2v_0(t)-1)} \tilde{a}_i \\
 & + \frac{(M+\bar{M}-1)v_2(t)+\mu_2(t)v_0(t)}{2(\kappa(t)-\alpha)(M+\bar{M}+2v_0(t)-1)} q_{i,t} \\
 & - \frac{v_0(t)((M-1)\mu_3(t)+\mu_2(t))-(M+\bar{M}-1)v_3(t)}{2(\kappa(t)-\alpha)(M+\bar{M}+2v_0(t)-1)} \eta_t \\
 & - \frac{\bar{M}v_0(t)(2\kappa(t)+\bar{\mu}_5(t))}{2(\kappa(t)-\alpha)(M+\bar{M}+2v_0(t)-1)} Y_t, \\
 \theta_{j,t}^* := & \frac{(M+\bar{M}-1)\bar{v}_3(t)-M\mu_3(t)-\mu_2(t)}{2(M+\bar{M}+1)(\kappa(t)-\alpha)} \eta_t \\
 & + \frac{(M+\bar{M}-1)\bar{v}_5(t)+2M\kappa(t)-(\bar{M}-1)\bar{\mu}_5(t)}{2(M+\bar{M}+1)(\kappa(t)-\alpha)} w_t \\
 & - \frac{A(t)\mu_2(t)-(M+\bar{M}-1)\bar{v}_4(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{2(M+\bar{M}+1)(\kappa(t)-\alpha)} \tilde{a}_{\Sigma},
 \end{aligned}
 \tag{4.12}$$

satisfy (2.6), the traders' maximizers for (2.5) are $\theta_{i,t}^*$ for rebalancer $i \in \{1, \dots, M\}$ and $\theta_{j,t}^*$ for tracker $j \in \{M + 1, \dots, M + \bar{M}\}$. ◇

From Lemma 4.2, we note that a generic rebalancer i_0 has filtration $\sigma(\tilde{a}_{i_0}, S_{i_0,u}^v)_{u \in [0,t]}$ whereas she perceives that other rebalancers $i \neq i_0$ have filtrations $\sigma(\tilde{a}_i, Y_u, W_{i,u}, S_{i,u}^Z)_{u \in [0,t]}$ as in Lemma 4.1. Because these are i_0 's off-equilibrium perceptions, this is allowable as long as they are consistent with i 's equilibrium holdings. We require this consistency in Definition 4.3(iii) below. We also note from Lemma 4.1 that rebalancer i can infer $Z_{i_0,t}$ in (4.5). In turn, this allows rebalancer $i, i \neq i_0$, to also know the process

$$\frac{2(\alpha - \kappa(t))}{M + \bar{M} - 1} \theta_{i_0,t} + \frac{2\kappa(t) + \mu_1(t)}{M + \bar{M} - 1} \tilde{a}_{i_0} + \frac{\mu_2(t)}{M + \bar{M} - 1} q_{i_0,t}. \tag{4.13}$$

However, knowing (4.13) is insufficient for rebalancer $i, i \neq i_0$, to infer rebalancer i_0 's private target \tilde{a}_{i_0} .

4.3 Equilibrium

Definition 4.3 Deterministic functions of time $\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5, \nu_0, \nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_4, \bar{\nu}_5 : [0, 1] \rightarrow \mathbb{R}$ constitute a *subgame perfect Nash financial-market equilibrium* if:

- (i) For $k \in \{1, \dots, M + \bar{M}\}$, trader k 's maximizer $\theta_{k,t}^*$ for (2.5) exists given the market-clearing stock-price perceptions (4.7) and (4.11).
- (ii) For $k \in \{1, \dots, M + \bar{M}\}$, inserting trader k 's maximizer $\theta_{k,t}^*$ into the perceived market-clearing stock-price processes (4.7) and (4.11) produces identical stock-price processes across all traders. This common equilibrium stock-price process is denoted by S_t^* .
- (iii) Optimizers and equilibrium holdings must be consistent in the sense that trader k 's perceived response to trader k_0 's maximizer $\theta_{k_0,t}^*$ is trader k 's maximizer $\theta_{k,t}^*$.
- (iv) The money and stock markets clear. ◇

The identical stock-price requirement in Definition 4.3(ii) is similar to the one in Definition 3.3(ii). We see from the rebalancers' perceptions (4.6) that both the drifts and the martingale terms have i dependence. Similar to (3.5), we replace $dw_{i,t}$ in $dS_{i,t}^v$ in (4.6) with the decomposition of $dw_{i,t}$ in terms of dw_t in (2.11) and rewrite $dS_{i,t}^v$ in (4.6) as

$$dS_{i,t}^v = \left\{ \nu_0(t)Z_{i,t} + \nu_1(t)\tilde{a}_i + \nu_2(t)q_{i,t} + \nu_3(t)\eta_t + \alpha\theta_{i,t} - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})\gamma \right\} dt + \gamma dw_t, \quad i \in \{1, \dots, M\}. \tag{4.14}$$

Therefore, to ensure identical equilibrium stock-price perceptions for all traders $k \in \{1, \dots, M + \bar{M}\}$, it suffices to match the drift of $dS_{j,t}^v$ in (4.10) for $j \in \{M + 1, \dots, M + \bar{M}\}$ with the drift of $dS_{i,t}^v$ in (4.14) for the optimal holdings $\theta_{i,t} := \theta_{i,t}^*$ for $i \in \{1, \dots, M\}$. This produces the requirement

$$\begin{aligned} & \nu_0(t)Z_{i,t}^* + \nu_1(t)\tilde{a}_i + \nu_2(t)q_{i,t} + \nu_3(t)\eta_t + \alpha\theta_{i,t}^* - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})\gamma \\ & = Z_{j,t}^* + \bar{\nu}_3(t)\eta_t + \bar{\nu}_4(t)\tilde{a}_\Sigma + \bar{\nu}_5(t)w_t + \alpha\theta_{j,t}^*, \end{aligned} \tag{4.15}$$

for all rebalancers $i \in \{1, \dots, M\}$ and all trackers $j \in \{M + 1, \dots, M + \bar{M}\}$. The right-hand side of (4.15) does not depend on the rebalancer index i . In (4.15), the process $Z_{i,t}^*$ is (4.5)

evaluated at $\theta_{i,t} := \theta_{i,t}^*$, and $Z_{j,t}^*$ is (4.9) evaluated at $\theta_{j,t} := \theta_{j,t}^*$ so that:

$$\begin{aligned} Z_{i,t}^* &:= \frac{2(\alpha - \kappa(t))}{M + \bar{M} - 1} \theta_{i,t}^* + \frac{2\kappa(t) + \mu_1(t)}{M + \bar{M} - 1} \tilde{a}_i + \frac{\mu_2(t)}{M + \bar{M} - 1} q_{i,t} \\ &\quad - \frac{(M - 1)\mu_3(t) + \mu_2(t)}{M + \bar{M} - 1} \eta_t - \frac{\bar{M}(2\kappa(t) + \bar{\mu}_5(t))}{M + \bar{M} - 1} Y_t, \\ Z_{j,t}^* &:= \frac{2(\alpha - \kappa(t))}{M + \bar{M} - 1} \theta_{j,t}^* - \frac{M\mu_3(t) + \mu_2(t)}{M + \bar{M} - 1} \eta_t \\ &\quad - \frac{(\bar{M} - 1)(2\kappa(t) + \bar{\mu}_5(t))}{M + \bar{M} - 1} w_t - \frac{A(t)\mu_2(t) + (\bar{M} - 1)\bar{\mu}_4(t) + 2\kappa(t) + \mu_1(t)}{M + \bar{M} - 1} \tilde{a}_\Sigma, \end{aligned} \tag{4.16}$$

for rebalancers $i \in \{1, \dots, M\}$ and trackers $j \in \{M + 1, \dots, M + \bar{M}\}$.

As for the consistency requirement in Definition 4.3(iii), we first fix a rebalancer $i_0 \in \{1, \dots, M\}$. We require that the response holdings in (4.2) are consistent with $\theta_{i_0,t}^*$ in the sense that

$$\begin{aligned} \theta_{i,t}^* &= \frac{1}{2(\kappa(t) - \alpha)} Z_{i_0,t}^* + \frac{2\kappa(t) + \mu_1(t)}{2(\kappa(t) - \alpha)} \tilde{a}_i + \frac{\mu_2(t)}{2(\kappa(t) - \alpha)} q_{i,t} + \frac{\mu_3(t)}{2(\kappa(t) - \alpha)} \eta_t, \\ \theta_{j,t}^* &= \frac{1}{2(\kappa(t) - \alpha)} Z_{i_0,t}^* + \frac{2\kappa(t) + \bar{\mu}_5(t)}{2(\kappa(t) - \alpha)} w_t + \frac{\bar{\mu}_4(t)}{2(\kappa(t) - \alpha)} \tilde{a}_\Sigma, \end{aligned} \tag{4.17}$$

for rebalancers $i \in \{1, \dots, M\} \setminus \{i_0\}$ and trackers $j \in \{M + 1, \dots, M + \bar{M}\}$. Second, we fix a tracker $j_0 \in \{M + 1, \dots, M + \bar{M}\}$ and require that the response holdings in (4.2) must be consistent with $\theta_{j_0,t}^*$ in the sense that

$$\begin{aligned} \theta_{i,t}^* &= \frac{1}{2(\kappa(t) - \alpha)} Z_{j_0,t}^* + \frac{2\kappa(t) + \mu_1(t)}{2\alpha - 2\kappa(t)} \tilde{a}_i + \frac{\mu_2(t)}{2(\kappa(t) - \alpha)} q_{i,t} + \frac{\mu_3(t)}{2(\kappa(t) - \alpha)} \eta_t, \\ \theta_{j,t}^* &= \frac{1}{2(\kappa(t) - \alpha)} Z_{j_0,t}^* + \frac{2\kappa(t) + \bar{\mu}_5(t)}{2(\kappa(t) - \alpha)} w_t + \frac{\bar{\mu}_4(t)}{2(\kappa(t) - \alpha)} \tilde{a}_\Sigma, \end{aligned} \tag{4.18}$$

for rebalancers $i \in \{1, \dots, M\}$ and trackers $j \in \{M + 1, \dots, M + \bar{M}\} \setminus \{j_0\}$.

Similar to the price-friction equilibrium, our Nash equilibrium existence result is based on a technical lemma, which guarantees the existence of a solution to an autonomous system of coupled ODEs.

Lemma 4.4 *Let $\kappa : [0, 1] \rightarrow (0, \infty]$ be a continuous and integrable function (i.e., $\int_0^1 \kappa(t) dt < \infty$), let $M + \bar{M} > 2$, and let $\alpha \leq 0$. For a constant $B(0) \in \mathbb{R}$, the coupled ODEs*

$$\begin{aligned} B'(t) &= \frac{\left\{ 2\kappa(t) \left(\bar{M} B(t) (M + \bar{M} - 1) (\alpha (M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)) \right. \right. \\ &\quad \left. \left. + (M + \bar{M} - 2) (\alpha (M + \bar{M} + 1) - 2(M + \bar{M})\kappa(t)) \right) \right\}}{\left\{ \gamma \left(A(t) (M + \bar{M} - 2) (\alpha (M + \bar{M} + 1) - 2(M + \bar{M})\kappa(t)) \right. \right. \\ &\quad \left. \left. + \alpha (M^2 + M - 1)\bar{M} + M^2 + 2M\bar{M}^2 - M + \bar{M}^3 - 2 \right) \right. \\ &\quad \left. - 2 \left((M^2 - 1)\bar{M} + (2M - 1)\bar{M}^2 + (M - 2)M + \bar{M}^3 \right) \kappa(t) \right\}}, \\ A'(t) &= -(B'(t))^2 \Sigma(t) (A(t) + 1), \quad A(0) = -\frac{(M - 1)B(0)^2 \sigma_a^2}{\sigma_{w_0}^2 + (M - 1)B(0)^2 \sigma_a^2}, \\ \Sigma'(t) &= -(B'(t))^2 \Sigma(t)^2, \quad \Sigma(0) = \frac{(M - 1)\sigma_a^2 \sigma_{w_0}^2}{(M - 1)B(0)^2 \sigma_a^2 + \sigma_{w_0}^2}, \end{aligned} \tag{4.19}$$

have unique solutions with $\Sigma(t) \geq 0$, $\Sigma(t)$ decreasing, $A(t) \in [-1, 0]$, and $A(t)$ decreasing for $t \in [0, 1]$. ◇

The affine ODE for $B(t)$ in (4.19) is more complicated than the corresponding affine ODE in (3.7) because the Nash equilibrium has the additional fixed point requirement in (4.4) that

is absent in the price-friction equilibrium. However, both ODEs for $B(t)$ are affine. It is possible to restate the ODE system (4.19) using a single path-dependent ODE. The special case $\alpha := 0$ and $B(0) := -\frac{1}{\bar{M}} + \frac{1}{\bar{M}(\bar{M}+1)^2}$ produces a Nash model with no dynamic learning because $B'(t) = 0$ implies $\Sigma'(t) = 0$ and so $d\eta_t = dq_{i,t} = 0$. The resulting Subgame perfect Nash equilibrium model only has learning at $t = 0$ and can be seen as a special case of Choi, Larsen, and Seppi [13].

Our main theoretical result gives a Nash equilibrium in terms of the ODEs (4.19). In this theorem, the price-friction parameter $\alpha \leq 0$, volatility $\gamma > 0$, and initial value $B(0) \in \mathbb{R}$ are free parameters.

Theorem 4.5 *Let $\kappa : [0, 1] \rightarrow (0, \infty)$ be continuous, let the functions (B, A, Σ) be as in Lemma 4.4, let $M + \bar{M} > 2$, and let $\alpha \leq 0$. Then, we have:*

- (i) *A subgame perfect Nash financial-market equilibrium exists and is given by the functions in (A.6) in Appendix A.*
- (ii) *The Nash equilibrium in (i) has holdings given by*

$$\begin{aligned} \theta_{i,t}^* := & -\frac{(M + \bar{M} - 2)(2\kappa(t) - \gamma B'(t))}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)} \tilde{a}_i \\ & + \frac{\gamma(M + \bar{M} - 2)B'(t)}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)} q_{i,t} \\ & - \frac{\left\{ \gamma(M + \bar{M} - 2)^2 B'(t)(\alpha(M + \bar{M} + 1) - 2(M + \bar{M})\kappa(t)) \right\}}{\left\{ (\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))(\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} \right. \\ & \left. + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)) \right\}} \eta_t \\ & + \frac{\left\{ 2\bar{M}(M + \bar{M} - 2)(M + \bar{M} - 1)\kappa(t) \right\}}{\left\{ \alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) \right. \\ & \left. - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t) \right\}} Y_t, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \theta_{j,t}^* := & -\frac{\gamma(M + \bar{M} - 2)(M + \bar{M} - 1)B'(t)}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} \eta_t \\ & - \frac{2M(M + \bar{M} - 2)(M + \bar{M} - 1)\kappa(t)}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} w_t \\ & + \frac{(M + \bar{M} - 2)(M + \bar{M} - 1)(\gamma(-A(t) + M - 1)B'(t) + 2\kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} \tilde{a}_\Sigma, \end{aligned}$$

for rebalancers $i \in \{1, \dots, M\}$ and trackers $j \in \{M + 1, \dots, M + \bar{M}\}$.

- (iii) *The Nash equilibrium in (i) has the stock-price process S_t^* given by $S_0^* := w_0 - B(0)\tilde{a}_\Sigma$ and dynamics with respect to the trackers' filtrations $\mathcal{F}_{j,t} := \sigma(w_u, S_{j,u}^v)_{u \in [0,t]}$ given by*

$$\begin{aligned} dS_t^* = & \left\{ \frac{\gamma(M + \bar{M} - 2)B'(t)(\alpha(M + \bar{M} + 1) - 2(M + \bar{M})\kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} \eta_t \right. \\ & - \frac{2\bar{M}(M + \bar{M} - 1)\kappa(t)(\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} w_t \\ & - \left. \frac{(M + \bar{M} - 2)(\alpha(M + \bar{M} + 1) - 2(M + \bar{M})\kappa(t))(\gamma(-A(t) + M - 1)B'(t) + 2\kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)} \tilde{a}_\Sigma \right\} dt \\ & + \gamma dw_t, \end{aligned} \tag{4.21}$$

and dynamics with respect to the rebalancers' filtrations $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^V)_{u \in [0,t]}$ given by

$$dS_t^* = \left\{ \frac{\gamma(M+\bar{M}-2)B'(t)(\alpha(M+\bar{M}+1)-2(M+\bar{M})\kappa(t))}{\alpha((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3)-2((M+\bar{M}-2)(M+\bar{M})^2+\bar{M})\kappa(t)} \eta_t - \frac{2\bar{M}(M+\bar{M}-1)\kappa(t)(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t))}{\alpha((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3)-2((M+\bar{M}-2)(M+\bar{M})^2+\bar{M})\kappa(t)} Y_t - \gamma B'(t)(\tilde{a}_i + q_{i,t}) \right\} dt + \gamma dw_{i,t}. \tag{4.22}$$

◇

The following observations follow from Theorem 4.5:

1. The logic for the initial value $B(0)$ being a free input parameter is the same as in the price-friction equilibrium.
2. The price-friction parameter α and stock-price volatility γ affect the stock-price drift and holdings via its impact on $B(t)$ in (4.19). The dependence on α is different from the price-friction equilibrium where the corresponding $B(t)$ in (3.7) is independent of α . The reason is that α affects the perceived optimal responses in (4.2).
3. Similar to (3.11) and (3.12), for an arbitrary trader $k_0 \in \{1, \dots, M + \bar{M}\}$ and her arbitrary holdings $\theta_{k_0,t}$, the optimal responses in (4.2) can be decomposed as

$$\begin{aligned} \theta_{i,t}^{Z_{k_0}} &= \theta_{i,t}^* - \frac{1}{M + \bar{M} - 1} (\theta_{k_0,t} - \theta_{k_0,t}^*), \quad i \in \{1, \dots, M\}, \\ \theta_{j,t}^{Z_{k_0}} &= \theta_{j,t}^* - \frac{1}{M + \bar{M} - 1} (\theta_{k_0,t} - \theta_{k_0,t}^*), \quad j \in \{M + 1, \dots, M + \bar{M}\}, \end{aligned} \tag{4.23}$$

where the equilibrium holdings $(\theta_{i,t}^*, \theta_{j,t}^*, \theta_{k_0,t}^*)$ are in (4.20).¹³

4. The subgame perfect Nash financial-market equilibrium is attractive because of its reasonable off-equilibrium market-clearing perceptions. However, although much of the mathematic structure is similar, the expressions for the equilibrium stock price and holding coefficients are algebraically more complex. Nonetheless, our numerical results in Sect. 3.4 below show that the differences between the price-friction and the subgame perfect Nash financial-market equilibria are quantitatively small. This, in turn, suggests that the economic logic from the price-friction equilibrium carries over to the Nash equilibrium.

4.4 Numerics

We have experimented extensively with the subgame perfect Nash model's numerics, and its numerics are very similar to the numerics of the price-friction equilibrium in Sect. 3. The numerical similarity of the two equilibria suggests that the intuitions for the signs of the various coefficients in the price-friction equilibrium carry over to the subgame perfect Nash financial-market equilibrium. Because the two equilibria produce similar numerics, it appears that the in-equilibrium market-clearing requirement (common in both equilibria) has a much larger effect on equilibrium prices relative to the off-equilibrium market-clearing requirement (only present in the subgame perfect Nash equilibrium).

¹³ This is similar to Eq. (2.16) in Chen, Choi, Larsen, and Seppi [11].

5 Empirical predictions

The primary contribution of our analysis is theoretical. The Kyle model has provided a tractable framework for a large body of theoretical research on price discovery and dynamic order splitting given long-lived asymmetric information about stock cash flows. However, no corresponding tractable framework exists for modeling price discovery and dynamic order splitting with private trading targets (e.g., by large index funds). Our model provides such a framework. While our zero-dividend modeling approach precludes statements about the impact of order on price levels, our analysis does have empirical implications for intraday price drifts:

First, intraday price predictability is an important empirical driver of high-frequency liquidity provision. Our model’s equilibrium price dynamics in (3.9) and (4.21) suggest that intraday price drifts are path dependent (via the η_t term) and also that learning about parent-demand imbalances early in the trading day is associated with predictable price drifts later in the day.

Second, our analysis provides insights about the determinants of price impact as it relates to imbalance-related parent trading demands and toxic cumulative order flow. In particular, the holdings $\theta_{k,t}$ are cumulative trading up through time t , and large parent targets \tilde{a}_i lead to toxic streams of orders. Our subgame perfect Nash model endogenizes the price-drift impact of investor holdings (i.e., cumulative trading). The Nash model’s price-friction coefficient in the rebalancer’s perceived stock-price dynamics (4.7) is given by

$$-\frac{2v_0(t)(\kappa(t) - \alpha)}{M + \bar{M} - 1} = -2\frac{\kappa(t) - \alpha}{M + \bar{M} - 2}, \tag{5.1}$$

where we have inserted $v_0(t)$ from (A.6). An implication of (5.1) is that if, as is widely believed, investor target penalties become stronger as time passes (i.e., if $\kappa(t)$ increases with time), then our Nash model predicts that the total price impact in (5.1) should increase. On its face, this is contrary to evidence in Barardehi and Bernhardt [6] that price impact declines over the trading day. We conjecture, however, that a richer model can be reconciled with these stylized facts if the number of investors (and, thus, the available inventory bearing capacity to absorb aggregate parent demand imbalances) is also allowed to grow as the market approaches the end of the trading day. Increased investor participation toward the end of the trading day is also empirically common.

6 Measuring execution costs

As an application, this section gives a measure of a rebalancer’s costs of rebalancing from zero endowed shares at time $t = 0$ to a given target \tilde{a}_i . We present the measure in the price-friction equilibrium in Sect. 3 (the Nash analogue is logically similar and produces similar numerics). In the price-friction equilibrium, rebalancer i ’s value function is

$$J(\tilde{a}_i, 0, \eta_0, Y_0, q_{i,0}) := \mathbb{E}\left[\int_0^1 \hat{\theta}_{i,t} d\hat{S}_t - \int_0^1 \kappa(t)(\tilde{a}_i - \hat{\theta}_{i,t})^2 dt \mid \mathcal{F}_{i,0}\right], \tag{6.1}$$

where $\hat{\theta}_{i,t}$ denotes rebalancer i ’s equilibrium stock holdings in (3.8) and $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S_{i,u}^f)_{u \in [0,t]}$ where the f coefficient functions are as in (A.1) in Appendix A for

$i \in \{1, \dots, M\}$. We seek a value function $J = J(\tilde{a}_i, s, q, Y, q_i)$ such that the process

$$J(\tilde{a}_i, s, \eta_s, Y_s, q_{i,s}) + \int_0^s \left\{ \hat{\theta}_{i,t} \left(f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\hat{\theta}_{i,t} \right) - \kappa(t)(\tilde{a}_i - \hat{\theta}_{i,t})^2 \right\} dt, \quad s \in [0, 1], \tag{6.2}$$

is a martingale with respect to $\mathcal{F}_{i,t}$. Because rebalancer i 's objective in (2.5) is linear-quadratic, the value function J is again linear-quadratic in the state processes. Thus, J can be written as

$$J(\tilde{a}_i, s, \eta, Y, q_i) = J_0(s) + J_\eta(s)\eta + J_Y(s)Y + J_{q_i}(s)q_i + J_{\eta\eta}(s)\eta^2 + J_{\eta Y}(s)\eta Y + J_{Y Y}(s)Y^2 + J_{q_i q_i}(s)q_i^2 + J_{q_i \eta}(s)q_i \eta + J_{q_i Y}(s)q_i Y, \tag{6.3}$$

for deterministic functions of time ($J_0, J_\eta, J_Y, J_{q_i}, J_{\eta\eta}, J_{\eta Y}, J_{Y Y}, J_{q_i q_i}, J_{q_i \eta}, J_{q_i Y}$). These functions are given by a coupled set of ODEs with zero terminal conditions (we omit the ODEs for brevity). In (6.3), the dummy variables (η, Y, q_i) are real numbers and $s \in [0, 1]$.

To quantify the costs associated with rebalancer i 's trading target \tilde{a}_i , the quadratic mapping RC (Rebalancing Costs) defined by

$$RC(\tilde{a}_i) := J(0, 0, \eta, Y, q_i) - J(\tilde{a}_i, 0, \eta, Y, q_i), \tag{6.4}$$

measures the dependence the change in profit (i.e., change in value function) associated with a non-zero target \tilde{a}_i . The rebalancing cost RC in (6.4) for a target \tilde{a}_i is computed as the difference between the value function evaluated at \tilde{a}_i and the function evaluated at $\tilde{a}_i = 0$. Since the value function J is highest at $\tilde{a}_i = 0$, the measure RC is positive.

Figure 7 plots the rebalancer's value function J for different target values \tilde{a}_i for different model parameterizations. When the target \tilde{a}_i is close to zero, the rebalancers become high-frequency liquidity providers. Their value function is positive due expected profit from liquidity provision and price-pressure speculation. As the target moves away from zero, the rebalancer starts to have larger stock-holding penalties that eventually drive the rebalancer's value function negative. Interestingly, the impact of the stock-price volatility parameter γ on the rebalancer's value function can be positive or negative. Liquidity providing rebalancers

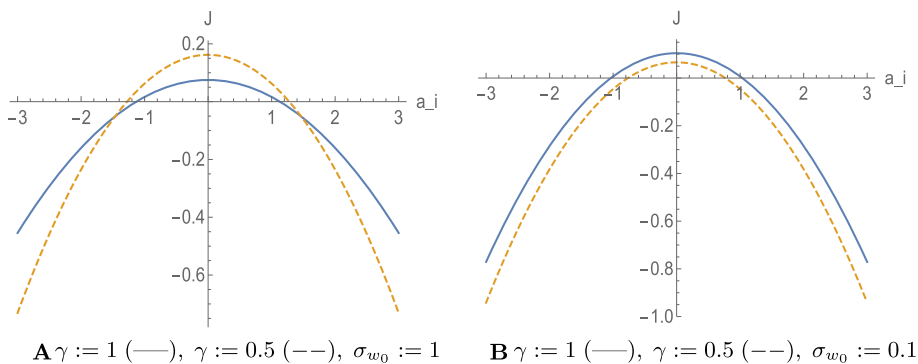


Fig. 7 Plots of the rebalancers' value function J for various values of (γ, σ_{w_0}) . The exogenous model parameters are $\sigma_{\tilde{a}_i} := 1, M := \bar{M} := 10, \alpha := -0.1, B(0) := -1, \kappa(t) := 1$ for $t \in [0, 1]$, and $w_0 := B(0)(\tilde{a}_\Sigma - \tilde{a}_i)$

are better off with a small γ whereas rebalancers with large rebalancing targets are better off when γ is large.

7 Conclusion

This paper presents the first analytically tractable model of dynamic learning about parent trading-demand imbalances with optimized order-splitting. In particular, we provide closed-form expressions prices and stock holdings in terms of solutions to systems of coupled ODEs in both the price-friction and Subgame perfect Nash equilibria. Trading in our models reflects a combination of reaching investor's own trading targets, liquidity provision so that markets can clear, and speculation based on predictions of future price pressure.

There are many interesting directions for future research based on our analysis. First, replacing the zero-dividend stock approach with valuation based on a terminal payoff would be a significant technical step. Second, the model could be enriched by allowing for investor heterogeneity in the form of different penalty functions $\kappa(t)$ and by having multiple tracker targets (which would weaken the trackers' informational advantage). Third, it would be interesting to investigate if other off-equilibrium refinements have larger equilibrium effects. Fourth, incorporating risk-aversion into the investors' objectives would be interesting too. For example, how can Lemma 4.1 be extended if the objectives in (2.5) are changed to exponential utilities?

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A Formulas

A.1 Price-perception coefficients for the price-friction equilibrium

$$\begin{aligned}
 \bar{f}_0(t) &:= \frac{4\bar{M}\kappa(t)(\kappa(t) - \alpha)}{(M + \bar{M})(\alpha - 2\kappa(t))}, \quad \bar{f}_1(t) := \frac{2\gamma B'(t)(\kappa(t) - \alpha) + 2\alpha\kappa(t)}{\alpha - 2\kappa(t)}, \\
 \bar{f}_2(t) &:= \frac{2\gamma B'(t)(\kappa(t) - \alpha)}{\alpha - 2\kappa(t)}, \quad \bar{f}_3(t) := \frac{2\gamma B'(t)(\alpha - \kappa(t))}{(M + \bar{M})(\alpha - 2\kappa(t))}, \\
 \bar{f}_3(t) &:= \frac{2\gamma B'(t)(\alpha - \kappa(t))}{(M + \bar{M})(\alpha - 2\kappa(t))}, \\
 \bar{f}_4(t) &:= \frac{2(\alpha - \kappa(t))(\gamma(A(t) - M + 1)B'(t) - 2\kappa(t))}{(M + \bar{M})(\alpha - 2\kappa(t))}, \\
 \bar{f}_5(t) &:= \frac{2\kappa(t)(\alpha(M - \bar{M}) + 2\bar{M}\kappa(t))}{(M + \bar{M})(\alpha - 2\kappa(t))}.
 \end{aligned} \tag{A.1}$$

A.2 Orthogonal representations for the price-friction equilibrium

Let the deterministic functions $F_1(t)$ and $F_2(t)$ be as in (2.19).

A.2.1 Price-friction equilibrium holdings

The price-friction equilibrium holdings $\hat{\theta}_{i,t}$ in (3.8) for rebalancer $i \in \{1, \dots, M\}$ has an orthogonal representation given by

$$\begin{aligned}
 \hat{\theta}_{i,t} = & - \frac{\left\{ 2\kappa(t)\left(A(t)(\bar{M}B(t) + M + \bar{M})\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) \right. \right. \\
 & + MF_1(t)F_2(t)(\bar{M}B(t) + 1)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) \\
 & + (M - 1)\left(B(0)^2M\sigma_a^2F_1(t)(\bar{M}B(t) + 1) - \bar{M}B(t)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right)\right) \\
 & \left. \left. + \bar{M}(M + \bar{M})\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right)\right\}}{\left\{(M + \bar{M})(A(t) + \bar{M} + 1)(\alpha - 2\kappa(t))\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right)\right\}} \tilde{a}_i \\
 & + \frac{\left\{ 2\bar{M}\kappa(t)\left(B(t)(-A(t)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) \right. \right. \\
 & + \bar{M}F_1(t)\left(F_2(t)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) + (M - 1)B(0)^2\sigma_a^2\right) \\
 & - (\bar{M} + 1)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) \\
 & \left. \left. + F_1(t)\left(F_2(t)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right) + (M - 1)B(0)^2\sigma_a^2\right)\right\}}{\left\{(M + \bar{M})(A(t) + \bar{M} + 1)(\alpha - 2\kappa(t))\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right)\right\}} (\tilde{a}_\Sigma - \tilde{a}_i) \\
 & + \frac{2\bar{M}\kappa(t)\left(1 - \frac{B(0)(M - 1)\sigma_a^2F_1(t)(\bar{M}B(t) + 1)}{(A(t) + \bar{M} + 1)\left((M - 1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2\right)}\right)}{(M + \bar{M})(\alpha - 2\kappa(t))} w_0 \\
 & + \frac{2\bar{M}\kappa(t)}{(M + \bar{M})(\alpha - 2\kappa(t))} w_t^\circ - \frac{2\bar{M}F_1(t)\kappa(t)(\bar{M}B(t) + 1)}{(M + \bar{M})(A(t) + \bar{M} + 1)(\alpha - 2\kappa(t))} \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ.
 \end{aligned} \tag{A.2}$$

The price-friction equilibrium holdings $\hat{\theta}_{j,t}$ in (3.8) for tracker $j \in \{M + 1, \dots, M + \bar{M}\}$ has an orthogonal representation given by

$$\begin{aligned} \hat{\theta}_{j,t} = & - \frac{\left\{ 2\kappa(t) \left(\bar{M} B(t) \left(A(t) \left((M-1) B(0)^2 \sigma_a^2 + \sigma_{w_0}^2 \right) \right. \right. \right. \\ & + M F_1(t) F_2(t) \left((M-1) B(0)^2 \sigma_a^2 + \sigma_{w_0}^2 \right) \\ & + (M-1) \left(B(0)^2 M \sigma_a^2 F_1(t) - (M-1) B(0)^2 \sigma_a^2 - \sigma_{w_0}^2 \right) \\ & + M F_1(t) \left(F_2(t) \left((M-1) B(0)^2 \sigma_a^2 + \sigma_{w_0}^2 \right) \right. \\ & \left. \left. \left. + (M-1) B(0)^2 \sigma_a^2 - (M + \bar{M}) \left((M-1) B(0)^2 \sigma_a^2 + \sigma_{w_0}^2 \right) \right) \right\}}{\left\{ (M + \bar{M}) (A(t) + \bar{M} + 1) (\alpha - 2\kappa(t)) \left((M-1) B(0)^2 \sigma_a^2 + \sigma_{w_0}^2 \right) \right\}} \tilde{a}_\Sigma \\ & + \frac{2M\kappa(t) \left(\frac{B(0)(M-1)\sigma_a^2 F_1(t)(\bar{M}B(t)+1)}{(A(t)+M+1)\left((M-1)B(0)^2\sigma_a^2+\sigma_{w_0}^2\right)} - 1 \right)}{(M + \bar{M})(\alpha - 2\kappa(t))} w_0 \\ & - \frac{2M\kappa(t)}{(M + \bar{M})(\alpha - 2\kappa(t))} w_t^\circ + \frac{2M F_1(t)\kappa(t)(\bar{M}B(t) + 1)}{(M + \bar{M})(A(t) + \bar{M} + 1)(\alpha - 2\kappa(t))} \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ. \end{aligned} \tag{A.3}$$

A.2.2 Price-friction equilibrium stock dynamics

For the trackers, we can rewrite the drift in (3.9) in terms of (\tilde{a}_Σ, w_0) and an residual orthogonal term as

$$\begin{aligned} & \frac{\gamma B'(t)}{M + \bar{M}} \eta_t - \frac{2\bar{M}\kappa(t)}{M + \bar{M}} w_t + \frac{\gamma(A(t) - M + 1)B'(t) - 2\kappa(t)}{M + \bar{M}} \tilde{a}_\Sigma \\ & = - \frac{\frac{B(0)\gamma(M-1)M\sigma_a^2 F_1(t)B'(t)}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2} + 2\bar{M}\kappa(t)}{M + \bar{M}} w_0 \\ & + \frac{\gamma B'(t) \left(A(t) + M F_1(t) \left(\frac{(M-1)B(0)^2\sigma_a^2}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2} + F_2(t) \right) - M + 1 \right) - 2\kappa(t)}{M + \bar{M}} \tilde{a}_\Sigma \\ & - \frac{2\bar{M}\kappa(t)}{M + \bar{M}} w_t^\circ - \frac{\gamma M F_1(t)B'(t)}{M + \bar{M}} \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ. \end{aligned} \tag{A.4}$$

For the rebalancers, we can rewrite the drift in (3.10) in terms of $(\tilde{a}_\Sigma - \tilde{a}_i, w_0, \tilde{a}_i)$ and an residual orthogonal term as

$$\begin{aligned} & - \gamma B'(t) (\tilde{a}_i + q_{i,t}) + \frac{\gamma B'(t)}{M + \bar{M}} \eta_t - \frac{2\bar{M}\kappa(t)}{M + \bar{M}} Y_t \\ & = \frac{\bar{M} \left(\frac{B(0)\gamma(M-1)\sigma_a^2 F_1(t)B'(t)}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2} - 2\kappa(t) \right)}{M + \bar{M}} w_0 \\ & + \frac{\gamma B'(t) \left(M F_1(t) \left(\frac{(M-1)B(0)^2\sigma_a^2}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2} + F_2(t) \right) - M - \bar{M} \right) + 2\bar{M}B(t)\kappa(t)}{M + \bar{M}} \tilde{a}_i \\ & + \frac{\bar{M} \left(\gamma F_1(t)B'(t) \left(- \frac{(M-1)B(0)^2\sigma_a^2}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2} - F_2(t) \right) + 2B(t)\kappa(t) \right)}{M + \bar{M}} (\tilde{a}_\Sigma - \tilde{a}_i) \\ & - \frac{2\bar{M}\kappa(t)}{M + \bar{M}} w_t^\circ + \frac{\gamma \bar{M} F_1(t)B'(t)}{M + \bar{M}} \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ. \end{aligned} \tag{A.5}$$

A.3 Price-perception coefficients for the Nash equilibrium

$$\begin{aligned}
 \mu_1(t) &:= \frac{2\gamma(M + \bar{M} - 2)B'(t)(\kappa(t) - \alpha) + 2\kappa(t)(\alpha(M + \bar{M} - 4) + 2\kappa(t))}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)}, \\
 \mu_2(t) &:= -\frac{2\gamma(M + \bar{M} - 2)B'(t)(\alpha - \kappa(t))}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)}, \\
 \mu_3(t) &:= \frac{\left\{ - (4\gamma(M + \bar{M} - 2)B'(t)(\alpha - \kappa(t))^2) \right\}}{\left\{ ((\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))(\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} \right. \\
 &\quad \left. + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)) \right\}}, \\
 \bar{\mu}_4(t) &:= \frac{\left\{ - (2(M + \bar{M} - 2)(\alpha - \kappa(t))(\gamma B'(t)(\alpha(-2A(t) + (M + \bar{M} - 1)(M + \bar{M})^2 - 2) - 2\kappa(t) \right. \\
 &\quad \left. (-A(t) + (3M - 2)\bar{M}^2 + M(3M - 4)\bar{M} + (M - 1)^2M + \bar{M}^3 + \bar{M} - 1)) + 4\kappa(t)(\alpha - \kappa(t))) \right\}}{\left\{ (\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))(\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} \right. \\
 &\quad \left. + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)) \right\}}, \\
 \bar{\mu}_5(t) &:= \frac{\left\{ (2\kappa(t)(\alpha(M^2(3\bar{M} - 5) + M^3 + M(\bar{M}(3\bar{M} - 10) + 6) \right. \\
 &\quad \left. + (\bar{M} - 4)(\bar{M} - 1)\bar{M})) + 2(M^2 + 2M(\bar{M} - 1) + (\bar{M} - 1)\bar{M})\kappa(t)) \right\}}{\left\{ \alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) \right. \\
 &\quad \left. - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t) \right\}}, \\
 v_0(t) &:= \frac{1}{M + \bar{M} - 2} + 1, \\
 v_1(t) &:= \frac{2\alpha(M + \bar{M} - 2)\kappa(t) - 2\gamma(M + \bar{M} - 1)B'(t)(\alpha - \kappa(t))}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)}, \\
 v_2(t) &:= -\frac{2\gamma(M + \bar{M} - 1)B'(t)(\alpha - \kappa(t))}{\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t)}, \\
 v_3(t) &:= \frac{\left\{ - (M + \bar{M} - 1)4\gamma B'(t)(\alpha - \kappa(t))^2 \right\}}{\left\{ ((\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))(\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} \right. \\
 &\quad \left. + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)) \right\}}, \\
 \bar{v}_3(t) &:= \frac{2\gamma(M + \bar{M} - 2)B'(t)(\alpha - \kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)}, \\
 \bar{v}_4(t) &:= \frac{\left\{ - (2(M + \bar{M} - 2)(\alpha - \kappa(t))(\gamma B'(t)(\alpha(-A(t)(M + \bar{M} + 2) + (M + \bar{M} - 1)(M^2 + 2M\bar{M} + M + \bar{M}^2) \right. \\
 &\quad \left. - \bar{M} - 2) + 2\kappa(t)(A(t)(M + \bar{M}) + M^2(1 - 3\bar{M}) - M^3 - 3M(\bar{M} - 1)\bar{M} + M - (\bar{M} - 2)\bar{M}^2)) + 2\kappa(t) \right. \\
 &\quad \left. (\alpha(M + \bar{M} + 2) - 2(M + \bar{M})\kappa(t))) \right\}}{\left\{ (\alpha(M + \bar{M}) - 2(M + \bar{M} - 1)\kappa(t))(\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2) \right. \\
 &\quad \left. M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)) \right\}}, \\
 \bar{v}_5(t) &:= \frac{2(M + \bar{M} - 1)\kappa(t)(\alpha(M^2 + 2M(\bar{M} - 1) + (\bar{M} - 4)\bar{M}) + 2\bar{M}\kappa(t))}{\alpha((3M - 1)\bar{M}^2 + M(3M - 2)\bar{M} + (M - 2)M(M + 1) + \bar{M}^3) - 2((M + \bar{M} - 2)(M + \bar{M})^2 + \bar{M})\kappa(t)}.
 \end{aligned}
 \tag{A.6}$$

B Kalman–Bucy filtering

The proof of Lemma 2.1 follows from the well-known Kalman-Bucy result in filtering theory and can be found in, e.g., Lipster and Shiryaev [30, Chapter 8]. We note that the solution to the Riccati equation (B.3) below is given by (2.15).

Theorem B.1 (Kalman-Bucy) *Let $B : [0, 1] \rightarrow \mathbb{R}$ be a continuously differentiable function and consider the Gaussian observation process $Y_{i,t} := w_t - B(t)(\tilde{a}_\Sigma - \tilde{a}_i)$ from (2.9) with dynamics*

$$dY_{i,t} = dw_t - B'(t)(\tilde{a}_\Sigma - \tilde{a}_i)dt, \quad Y_{i,0} = w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i) \tag{B.1}$$

and corresponding innovations process $w_{i,t}$ in (2.11). Then, (2.14) holds and the filtering property in (2.11) holds if $q_{i,t}$ has dynamics given by

$$\begin{aligned} dq_{i,t} &= -B'(t)\Sigma(t)dY_{i,t} - (B'(t))^2\Sigma(t)q_{i,t}dt \\ &= -B'(t)\Sigma(t)dw_{i,t}, \\ q_{i,0} &= \mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i | \sigma(Y_{i,0})] \\ &= \mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i | \sigma(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i))] \\ &= -\frac{B(0)\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i]}{\mathbb{V}[w_0] + B(0)^2\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i]}(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)) \\ &= -\frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2}(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)), \end{aligned} \tag{B.2}$$

and the remaining variance is given by

$$\Sigma'(t) = -(B'(t))^2\Sigma(t)^2, \tag{B.3}$$

with initial value

$$\begin{aligned} \Sigma(0) &= \mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0}] \\ &= \mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0})^2] \\ &= \mathbb{E}\left[\left(\tilde{a}_\Sigma - \tilde{a}_i + \frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2}(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i))\right)^2\right] \\ &= \left(\frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2}\right)^2\sigma_{w_0}^2 + \left(1 - B(0)\frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2}\right)^2(M-1)\sigma_a^2 \\ &= \frac{(M-1)\sigma_a^2\sigma_{w_0}^2}{(M-1)B(0)^2\sigma_a^2 + \sigma_{w_0}^2}. \end{aligned} \tag{B.4}$$

C Remaining proofs

Proof of Lemma 2.2 To see that (2.16) holds, we use the Kalman-Bucy filter (B.2) to write

$$q_{i,t} = q_{i,0} - \int_0^t B'(u)\Sigma(u)dw_{i,u}, \quad t \in [0, 1]. \tag{C.1}$$

Then,

$$\begin{aligned} \sum_{i=1}^M q_{i,0} &= -\frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2} (Mw_0 - B(0)(M\tilde{a}_\Sigma - \tilde{a}_\Sigma)) \\ &= -\frac{(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2} (MY_0 + B(0)\tilde{a}_\Sigma), \\ \sum_{i=1}^M B'(t)\Sigma(t)dw_{i,t} &= B'(t)\Sigma(t) \left(Mdw_t + B'(t) \left\{ \tilde{a}_\Sigma + \sum_{i=1}^M q_{i,t} - M\tilde{a}_\Sigma \right\} dt \right) \\ &= B'(t)\Sigma(t) \left(MdY_t + B'(t) \left\{ \tilde{a}_\Sigma + \sum_{i=1}^M q_{i,t} \right\} dt \right). \end{aligned} \tag{C.2}$$

To explicitly solve for $\sum_{i=1}^M q_{i,t}$, we note

$$\begin{aligned} &de^{\int_0^t (B'(u))^2 \Sigma(u) du} \sum_{i=1}^M q_{i,t} \\ &= e^{\int_0^t (B'(u))^2 \Sigma(u) du} \left\{ (B'(t))^2 \Sigma(t) \sum_{i=1}^M q_{i,t} dt - \sum_{i=1}^M B'(t)\Sigma(t)dw_{i,t} \right\} \\ &= -e^{\int_0^t (B'(u))^2 \Sigma(u) du} B'(t)\Sigma(t) \left(MdY_t + B'(t)\tilde{a}_\Sigma dt \right). \end{aligned} \tag{C.3}$$

We get the solution $\sum_{i=1}^M q_{i,t}$ by integrating

$$\begin{aligned} \sum_{i=1}^M q_{i,t} &= e^{-\int_0^t (B'(u))^2 \Sigma(u) du} \sum_{i=1}^M q_{i,0} \\ &\quad - \int_0^t e^{-\int_s^t (B'(u))^2 \Sigma(u) du} B'(s)\Sigma(s) \left(MdY_s + B'(s)\tilde{a}_\Sigma ds \right). \end{aligned} \tag{C.4}$$

Thus, the decomposition (2.16) holds with

$$\begin{aligned} A(t) &:= -e^{-\int_0^t (B'(u))^2 \Sigma(u) du} \frac{(M-1)B(0)^2\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2} - \int_0^t e^{-\int_s^t (B'(u))^2 \Sigma(u) du} (B'(s))^2 \Sigma(s) ds, \\ \eta_t &:= -e^{-\int_0^t (B'(u))^2 \Sigma(u) du} \frac{M(M-1)B(0)\sigma_a^2}{\sigma_{w_0}^2 + (M-1)B(0)^2\sigma_a^2} Y_0 - M \int_0^t e^{-\int_s^t (B'(u))^2 \Sigma(u) du} B'(s)\Sigma(s)dY_s. \end{aligned} \tag{C.5}$$

For the second part, we write the solution to the Ornstein-Uhlenbeck SDE for $d\eta_t$ in (2.17) as

$$\eta_t = F_1(t) \left(\eta_0 + MF_2(t)\tilde{a}_\Sigma - M \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ \right), \tag{C.6}$$

where the deterministic functions $F_1(t)$ and $F_2(t)$ are given by the ODEs in (2.19). Similarly, the the Ornstein-Uhlenbeck SDE for $dq_{i,t}$ in (B.2) has solution

$$q_{i,t} = F_1(t) \left(q_{i,0} + F_2(t)(\tilde{a}_\Sigma - \tilde{a}_i) - \int_0^t \frac{B'(s)\Sigma(s)}{F_1(s)} dw_s^\circ \right). \tag{C.7}$$

By comparing (C.6) and (C.7), we get (2.18). ◇

Proof of Lemma 3.1 The inclusion “ \supseteq ” in (3.2) follows from (2.4), (2.10), and (2.14). To see the inclusion “ \subseteq ”, we use Y_t in (2.8), η_t in (C.5), and $q_{i,t}$ in (B.2) to find deterministic functions h_0, h , and H such that

$$\begin{aligned} dS_{i,t}^f - \alpha\theta_{i,t}dt &= \left\{ f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t \right\} dt + \gamma dw_{i,t} \\ &= \left\{ h_0(t)Y_0 + h(t)\tilde{a}_i + \int_0^t H(u,t)dw_{i,u} \right\} dt + \gamma dw_{i,t}. \end{aligned} \tag{C.8}$$

We define

$$dZ_{i,t} := dS_{i,t}^f - \left\{ \alpha\theta_{i,t} + h_0(t)Y_0 + h(t)\tilde{a}_i \right\} dt, \quad Z_{i,0} := w_{i,0}, \tag{C.9}$$

The inclusion “ \subseteq ” in (3.2) will follow from the inclusion

$$\sigma(w_{i,u})_{u \in [0,t]} \subseteq \sigma(Z_{i,u})_{u \in [0,t]}. \tag{C.10}$$

To see (C.10), let $t_0 \in [0, t]$ be arbitrary and let $f(s), s \in [0, t]$, solve the following Volterra integral equation of the second kind (such f exists by Lemma 4.3.3 in Davis [16] because $\gamma \neq 0$):

$$\int_r^t f(s)H(r,s)ds + f(r)\gamma = 1_{[0,t_0]}(r), \quad r \in [0, t]. \tag{C.11}$$

This gives us

$$\begin{aligned} \int_0^t f(s)dZ_{i,s} &= \int_0^t f(s) \int_0^s H(r,s)dw_{i,r}ds + \int_0^t f(s)\gamma dw_{i,s} \\ &= \int_0^t \int_r^t f(s)H(r,s)dsdw_{i,r} + \int_0^t f(s)\gamma dw_{i,s} \\ &= \int_0^t \left(\int_r^t f(s)H(r,s)ds + f(r)\gamma \right) dw_{i,r} \\ &= \int_0^t 1_{[0,t_0]}(r)dw_{i,r} \\ &= w_{i,t_0} - w_{i,0}. \end{aligned} \tag{C.12}$$

◇

Proof of Lemma 3.2 Consider a rebalancer $i \in \{1, \dots, M\}$. For arbitrary holdings $\theta_{i,t}$, the expectation in the i 'th objective in (2.5) is

$$\begin{aligned} &\mathbb{E} \left[\int_0^1 \theta_{i,t} dS_{i,t}^f - \int_0^1 \kappa(t)(\tilde{a}_i - \theta_{i,t})^2 dt \right] \\ &= \mathbb{E} \left[\int_0^1 \theta_{i,t} \left\{ f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \right\} dt - \int_0^1 \kappa(t)(\tilde{a}_i - \theta_{i,t})^2 dt \right]. \end{aligned} \tag{C.13}$$

The equality in (C.13) follows from the square integrability condition (2.6), which ensures that the stochastic integral $\int_0^s \theta_{i,t} dw_{i,t}$ is a martingale with zero expectation. We can maximize the integrand in (C.13) pointwise because the second-order condition $\alpha < \kappa(t)$ holds. This gives the first formula in (3.4).

The second formula for a tracker j in (3.4) is proved similarly.

◇

Proof of Lemma 3.4 The local Lipschitz property of the ODEs (3.7) ensures that there exists a maximal interval of existence $[0, \tau)$ with $\tau \in (0, \infty]$ by the Picard-Lindelöf theorem (see, e.g., Theorem II.1.1 in Hartman [23]). We assume that $\tau < 1$ and construct a contradiction. To this end, we set

$$K := \int_0^1 \kappa(s) ds < \infty. \tag{C.14}$$

First, the Riccati ODE for $\Sigma(t)$ has the explicit solution in (2.15), which cannot explode as $t \uparrow \tau$ (even if $B(t)$ should explode as $t \uparrow \tau$).

Second, the initial value $A(0)$ in (3.7) ensures $A(0) \geq -1$ and to see that implies $A(t) \geq -1$ for all $t \in [0, \tau)$, we note

$$\frac{\partial}{\partial t} (A(t) + 1) = -(B'(t))^2 \Sigma(t) (A(t) + 1), \tag{C.15}$$

which implies

$$A(t) + 1 = (A(0) + 1) e^{-\int_0^t (B'(s))^2 \Sigma(s) ds} \geq 0. \tag{C.16}$$

This shows that $A(t)$ cannot explode as $t \uparrow \tau$ (even if $B(t)$ should explode as $t \uparrow \tau$).

Third, we show $B(t)$ is uniformly bounded for $t \in [0, \tau)$; hence, also $B(t)$ cannot explode as $t \uparrow \tau$. This then gives the desired contradiction because of Theorem II.3.1 in Hartman [23]. The affine ODE for $B(t)$ in (3.7) has the explicit solution

$$B(t) = e^{\int_0^t \frac{2\bar{M}\kappa(s)}{\gamma(A(s)+1+\bar{M})} ds} \left(B(0) + \int_0^t \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M})} e^{-\int_0^s \frac{2\bar{M}\kappa(u)}{\gamma(A(u)+1+\bar{M})} du} ds \right). \tag{C.17}$$

We can use K in (C.14) to produce the upper bound

$$\int_0^t \frac{2\bar{M}\kappa(s)}{\gamma(A(s)+1+\bar{M})} ds \leq \int_0^t \frac{2\bar{M}\kappa(s)}{\gamma\bar{M}} ds \leq \frac{2K}{\gamma}, \quad t \in [0, \tau). \tag{C.18}$$

In turn, the bound (C.18) and (C.17) imply

$$\begin{aligned} |B(t)| &\leq e^{\frac{2K}{\gamma}} \left(|B(0)| + \int_0^t \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M})} ds \right) \\ &\leq e^{\frac{2K}{\gamma}} \left(|B(0)| + \frac{2K}{\gamma\bar{M}} \right), \end{aligned} \tag{C.19}$$

for $t \in [0, \tau)$. Because the upper bound in (C.19) is uniform over $t \in [0, \tau)$, $B(t)$ cannot explode as $t \uparrow \tau$. \diamond

Proof of Theorem 3.5 To see that the holdings in (3.8) satisfy the square integrability condition (2.6), we insert $B'(t)$ from (3.7) to get

$$\begin{aligned} \hat{\theta}_{i,t} &= -\frac{2\kappa(t)(A(t)+\bar{M}(1-B(t)))}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} \tilde{a}_i + \frac{2\kappa(t)(\bar{M}B(t)+1)}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} q_{i,t} \\ &\quad - \frac{2\kappa(t)(\bar{M}B(t)+1)}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} \eta_t + \frac{2\bar{M}\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))} Y_t, \\ \hat{\theta}_{j,t} &= -\frac{2\kappa(t)(\bar{M}B(t)+1)}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} \eta_t - \frac{2\bar{M}\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))} w_t \\ &\quad + \frac{2\kappa(t)(\bar{M}B(t)-A(t)+M-1)+M+\bar{M}}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} \tilde{a}_{\Sigma}. \end{aligned} \tag{C.20}$$

Because $\kappa : [0, 1] \rightarrow (0, \infty)$ is continuous, $\kappa(t)$ is uniformly bounded. This gives us that $B'(t)$ in (3.7) is also uniformly bounded. As a consequence, the variances $\mathbb{V}[q_{i,t}]$, $\mathbb{V}[\eta_t]$, and $\mathbb{V}[Y_t]$ are also uniformly bounded functions of $t \in [0, 1]$. Therefore, the holding processes in (C.20) satisfy (2.6) if the coefficient functions for $(\tilde{a}_i, q_{i,t}, \eta_t, Y_t, w_t, \tilde{a}_\Sigma)$ are square integrable over $t \in [0, 1]$. For example, the coefficient function for \tilde{a}_i in $\hat{\theta}_{i,t}$ is bounded because

$$\left| \frac{2\kappa(t)(A(t)+\bar{M}(1-B(t)))}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))} \right| \leq \frac{2|A(t)+\bar{M}(1-B(t))|}{\bar{M}}, \tag{C.21}$$

which is continuous for $t \in [0, 1]$. Similarly, the remaining coefficients functions can be seen to be bounded too. The optimality in Definition 3.3(i) then follows from Lemma 3.2 and the fact that the holdings (3.8) are those in (3.4) with the f functions in (A.1) inserted.

Definition 3.3(ii)+(iii) are ensured by the specific f functions in (A.1). ◇

Proof of Lemma 4.1 Lemma A.1 in Choi, Larsen, and Seppi [13] and the continuity of Z_t 's paths imply that Z_t is adapted to both $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$. The rest of this proof is similar to the proof of Lemma 3.2 given above and is therefore omitted. ◇

Proof of Lemma 4.2 The rebalancers' second-order condition is

$$\frac{(\alpha - \kappa(t))(M + \bar{M} + 2v_0(t) - 1)}{M + \bar{M} - 1} < 0, \tag{C.22}$$

whereas the trackers' second-order condition is $\alpha < \kappa(t)$. Inequality (C.22) holds because $v_0(t) \geq 0$ and $\alpha < \kappa(t)$. The rest of this proof is similar to the proof of Lemma 3.2 given above and is therefore omitted. ◇

Proof of Lemma 4.4 The proof only requires minor changes to the proof of Lemma 3.4. As before, we let $[0, \tau)$ be the maximal interval of existence with $\tau \in (0, \infty]$ and assume that $\tau < 1$ to construct a contradiction. As in the proof of Lemma 3.4, $\Sigma(t) = \frac{1}{\Sigma(0)} + \int_0^t (B'(t))^2 dt$ and $A(t) \geq -1$. Next, to show $B(t)$ is bounded on $[0, \tau)$, we rewrite the ODE for $B(t)$ in (4.19) as

$$B'(t) = \frac{2\kappa(t)(B(t)(c(t) + \bar{M}) + 1)}{\gamma(A(t) + 1 + \bar{M} + c(t))}. \tag{C.23}$$

where the deterministic function $c(t)$ is defined as

$$c(t) := \frac{2\bar{M}(\kappa(t) - \alpha)}{(M + \bar{M} - 2)(2(M + \bar{M})\kappa(t) - \alpha(M + \bar{M} + 1))}, \quad t \in [0, 1]. \tag{C.24}$$

Because $\alpha \leq 0$ and $\kappa(t) > 0$, we have $c(t) > 0$. Furthermore, $c(t)$ is bounded because

$$\begin{aligned} c(t) &\leq \frac{2\bar{M}(\kappa(t) - \alpha)}{(M + \bar{M} - 2)(M + \bar{M} + 1)(\kappa(t) - \alpha)} \\ &= \frac{2\bar{M}}{(M + \bar{M} - 2)(M + \bar{M} + 1)} \\ &=: c_0, \end{aligned} \tag{C.25}$$

where the inequality follows from $2(M + \bar{M}) > (M + \bar{M} + 1)$ and the positivity of $\kappa(t)$. Because $A(t) + 1 \geq 0$ and $c(t) > 0$ we get the two estimates

$$\int_0^t \frac{2\kappa(s)(c(s) + \bar{M})}{\gamma(A(s) + 1 + \bar{M} + c(s))} ds \leq \frac{2(c_0 + \bar{M})}{\gamma\bar{M}} K, \tag{C.26}$$

$$\int_0^t \frac{2\kappa(s)}{\gamma(A(s) + 1 + \bar{M} + c(s))} ds \leq \frac{2}{\gamma\bar{M}} K,$$

where K is as in (C.14). Similar to (C.17), the explicit solution of (C.23) is

$$B(t) = e^{\int_0^t \frac{2\kappa(s)(c(s)+\bar{M})}{\gamma(A(s)+1+\bar{M}+c(s))} ds} B(0) + \int_0^t e^{\int_s^t \frac{2\kappa(u)(c(u)+\bar{M})}{\gamma(A(u)+1+\bar{M}+c(u))} du} \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M}+c(s))} ds. \tag{C.27}$$

Combing this expression for $B(t)$ with the bounds (C.26) produces

$$B(t) \leq e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}} |B(0)| + \int_0^t e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}} \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M}+c(s))} ds \tag{C.28}$$

$$\leq e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}} (|B(0)| + \frac{2K}{\gamma\bar{M}}).$$

◇

Proof of Theorem 4.5 From (C.23) we see that

$$|B'(t)| = \frac{2\kappa(t)(|B(t)|(c(t) + \bar{M}) + 1)}{\gamma(A(t) + 1 + \bar{M} + c(t))} \tag{C.29}$$

$$\leq \frac{2\kappa(t)(|B(t)|(c_0 + \bar{M}) + 1)}{\gamma\bar{M}},$$

where c_0 is defined in (C.25). Because $\kappa(t)$ is continuous on $t \in [0, 1]$, $\kappa(t)$ is bounded and from (C.19) we know that $B(t)$ is bounded too. Therefore, from (C.29), we see that $B'(t)$ is also uniformly bounded. Consequently, the variances $\mathbb{V}[q_{i,t}]$, $\mathbb{V}[\eta_t]$, and $\mathbb{V}[Y_t]$ are also uniformly bounded functions of $t \in [0, 1]$.

As before, the coefficient functions for $(\tilde{a}_i, q_{i,t}, \eta_t, Y_t, w_t, \tilde{a}_\Sigma)$ in (4.20) are all uniformly bounded for $t \in [0, 1]$. Therefore, the square-integrability condition (2.6) holds.

The requirements in Definition 4.3 follow from the definition of the functions in (A.6). ◇

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