LEARNING ABOUT REALITY FROM OBSERVATION

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ABSTRACT. Takens, Ruelle, Eckmann, Sano and Sawada launched an investigation of images of attractors of dynamical systems. Let A be a compact invariant set for a map f on $\mathbb{R}^n$ and let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$ be a "typical" smooth map. When can we say that A and $\phi(A)$ are similar, based only on knowledge of the images in $\mathbb{R}^m$ of trajectories in A? For example, under what conditions on $\phi(A)$ (and the induced dynamics thereon) are A and $\phi(A)$ homeomorphic? Are their Lyapunov exponents the same? Or, more precisely, which of their Lyapunov exponents are the same? This paper addresses these questions with respect to both the general class of smooth mappings ϕ and the subclass of delay coordinate mappings.

In answering these questions, a fundamental problem arises about an arbitrary compact set A in $\mathbb{R}^n$. For $x \in A$, what is the smallest integer d such that there is a C^1 manifold of dimension d that contains all points of A that lie in some neighborhood of x? We define a tangent space T_xA in a natural way and show that the answer is $d = \dim(T_xA)$. As a consequence we obtain a Platonic version of the Whitney embedding theorem.

1. Introduction

In *The Republic*, Plato writes of people who are chained in a cave for all of their lives, unable to observe life directly. Behind these people a fire burns and real objects cast shadows on the cave wall for them to see. Forced to base their knowledge of reality on inferences made from the shadows, they equate the shadows with reality. While philosophers may vigorously debate epistemological theory, it is certainly true that experimentalists are limited to observations that may not encode the full complexity of their systems.

As Ruelle and Takens have observed, it is very difficult to directly observe all aspects of the evolution of a high dimensional dynamical system such as a turbulent flow. Out of necessity, it is frequently the case that experimentalists study such systems by measuring a relatively low number of different quantities. We assume that all measurements have infinite precision in what follows. A central experimental question is the following.

Question 1.1. Is the measured data sufficient for us to understand the evolution of the dynamical system? In particular, does the measured data contain enough

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information to reconstruct dynamical objects of interest and recover coordinate independent dynamical properties such as attractor dimension and Lyapunov exponents? How many exponents can be determined?

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map and suppose $A \subset \mathbb{R}^n$ is a compact invariant set. Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. We always assume $m > 0$. We think of ϕ as a measurement function measuring m physical quantities, and for each point x in the state space $\mathbb{R}^n$ we say that $\phi(x)$ is the **measurement** associated with x. Motivated by an experimental point of view, we say that **observations are deterministic** if there exists an induced map $\bar{f}$ on $\phi(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \phi \downarrow & & \downarrow \phi \\ \phi(A) & \xrightarrow{\bar{f}} & \phi(A) \end{array}$$

The dynamics generated by $\bar{f}$ may be thought of as the shadows that traverse Plato's hypothetical cave wall. The global goal is to infer as much as possible about the dynamical system f from knowledge of the induced dynamics. In the absence of induced dynamics, experimenters increase m by either making more measurements or using delay coordinate maps. Assuming $\bar{f}$ exists, there is a considerable literature on how to compute the Lyapunov exponents associated with the induced system. Do these values correspond to those of the full system? What do we need to check to see this? We would like to state theorems of the following type.

Prototypical Theorem. *For a typical measurement map ϕ, if the induced map $\bar{f}$ exists and has certain properties, then the measurement map ϕ preserves dynamical objects of interest and dynamical invariants of the full system may be computed from the induced dynamics.*

Under what conditions do our observations allow us to make predictions? James Clerk Maxwell wrote of the fundamental importance of continuous dependence on initial data [2, 9]:

> "It is a metaphysical doctrine that from the same antecedents follow the same consequents. No one can gainsay this. But it is not of much use in a world like this, in which the same antecedents never again concur, and nothing ever happens twice.... The physical axiom which has a somewhat similar aspect is 'That from like antecedents follow like consequents'."

We ask what we can conclude if observations are deterministic and if the induced map $\bar{f}$ is continuous. Using a translation invariant concept of "almost every" on infinite dimensional vector spaces described in Section 2, we obtain the main C^0 conclusion.

Notation 1.2. For a map ψ we denote the restriction of ψ to a subset S of the domain of ψ by $\psi[S]$. Notice that this notation is not standard.

Let $\mathrm{Fix}(\bar{f})$ and $\mathrm{Per}_2(\bar{f})$ denote the collection of fixed points and period two points, respectively, of $\bar{f}$.

C^0 **Theorem.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map and let A be a compact invariant set. For almost every map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, there is an induced map $\bar{f}$ satisfying*

(1) $\bar{f}$ *is continuous and invertible, and*
(2) $\mathrm{Fix}(\bar{f})$ *and* $\mathrm{Per}_2(\bar{f})$ *are countable*

if and only if the following hold.

(1) *The measurement map* ϕ *is one to one on* A*.*
(2) *The sets* $\mathrm{Fix}(f[A])$ *and* $\mathrm{Per}_2(f[A])$ *are countable.*
(3) *The map* $f[A]$ *is continuous and invertible.*

Remark 1.3. If one can infer a property of A from a corresponding property of $\phi(A)$, we say that the property is **observable**. The boundedness of A is observable in the sense that if A is unbounded, then $\phi(A)$ is unbounded for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$. This applies to each of the embedding theorems in this paper.

Remark 1.4. Our goal is to obtain results with few or preferably no assumptions on f and A. Hypotheses should instead be placed on the observed objects, $\phi(A)$ and $\bar{f}$. This point of view motivates the definition of a Platonic result.

Definition 1.5. A result is said to be **Platonic** if it contains no hypotheses on the dynamical system f aside from the assumption of a finite-dimensional Euclidean phase space.

Does a typical measurement function preserve differential structure? If f is a diffeomorphism, A is a smooth submanifold of $\mathbb{R}^n$ and $\dim(A)$ is known a priori, one may appeal to the Whitney embedding theorem [6]. This theorem states that if A is a compact C^r k-dimensional manifold, where $r \geq 1$, then there is a C^r embedding of A into $\mathbb{R}^m$ where $m \geq 2k+1$. This situation is generic in the sense that the set of embeddings of A is open and dense in $C^r(A, \mathbb{R}^m)$. However, the experimentalist lacking a priori knowledge of the structure of A cannot rely on embedding theorems of Whitney type.

In Section 3 we define a notion of tangent space, denoted T_xA, suitable for a general compact subset A of $\mathbb{R}^n$ and we prove a manifold extension theorem. This result allows us to prove a Platonic version of the Whitney embedding theorem and to formulate a notion of diffeomorphism on A equivalent to the notion of injective immersion on A. We formulate our C^1 embedding theorems using this notion of diffeomorphism. Our Platonic C^1 theorem states that for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, the existence of an invertible quasidifferentiable (see Section 6) induced map $\bar{f}$ on $\phi(A)$ satisfying mild assumptions implies that ϕ is a diffeomorphism on A.

Platonic C^1 Theorem. *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a map. For almost every* $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$*, if there exists an invertible quasidifferentiable (see Section 6) induced map* $\bar{f}$ *on* $\phi(A)$ *satisfying*

(1) $\mathrm{Fix}(\bar{f})$ *and* $\mathrm{Per}_2(\bar{f})$ *are countable,*
(2) $\dim T_y(\phi(A)) < m \; \forall y \in \phi(A)$*, and*
(3) $D\bar{f}(y)[T_y\phi(A)]$ *is invertible* $\forall y \in \phi(A)$*,*

then the measurement mapping ϕ *is a diffeomorphism on* A*.*

It is difficult for a scientist to measure a large number of independent quantities simultaneously. For this reason one introduces the class of delay coordinate mappings. This mapping class was introduced into the literature by Takens [23].

Definition 1.6. Let $g \in C^1(\mathbb{R}^n, \mathbb{R})$. The **delay coordinate map** $\phi(f, g) : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$\phi(f, g)(x) = (g(x), g(f(x)), \ldots, g(f^{m-1}(x)))^{\mathrm{T}}$$

Analogs of several of our embedding results hold for the class of delay coordinate mappings. Since the delay coordinate mappings form a subspace of $C^1(\mathbb{R}^n, \mathbb{R}^m)$, it should be stressed that the delay coordinate results do not follow from the corresponding results about almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$. The following result addresses the observation of differentiable dynamics.

Delay Coordinate Map Theorem. *Let f be a diffeomorphism on $\mathbb{R}^n$ and let A be a compact invariant set. For almost every $g \in C^1(\mathbb{R}^n, \mathbb{R})$, if there is a quasidifferentiable induced map $\bar{f}$ satisfying*

1. $\bigcup_{i=1}^{2m} \mathrm{Per}_i(\bar{f})$ *is countable and*
2. *for each $p \in \{1, \ldots, m\}$ and $y \in \mathrm{Per}_p(\bar{f})$ we have*

$$D\bar{f}^p(y)[T_y\phi(f, g)(A)] \neq \gamma \cdot I \text{ for every } \gamma \in \mathbb{R},$$

then the delay map $\phi(f, g)$ is a diffeomorphism on A.

Assume that f and $\bar{f}$ are quasidifferentiable and invertible on A and $\phi(A)$, respectively, with invertible quasiderivatives at each point $x \in A$ and $y \in \phi(A)$. Suppose that ϕ is a diffeomorphism on A. We say that a Lyapunov exponent $\lambda(y, v)$ of $\bar{f}$ at $y \in \phi(A)$ is **true** if it does not depend on the choice of quasiderivative $D\bar{f}$ and if it is also a Lyapunov exponent of f at $\phi^{-1}(y) \in A$. The works of Eckmann, Ruelle, Sano and Sawada provide heuristic computational procedures for obtaining m Lyapunov exponents for a trajectory (y_k) of $\bar{f}$. They use the subset of measurement mappings generated by so-called delay coordinate mappings, the mapping class considered in the famous, fundamental paper of Takens [23]. In particular, the Eckmann and Ruelle algorithm (ERA) [3] uses a linear fitting of the tangent map and has proven to be computationally efficient in giving the complete Lyapunov spectrum of many dynamical systems. Mera and Morán [14] find conditions ensuring the convergence of this algorithm for a smooth dynamical system on a $C^{1+\alpha}$ submanifold supporting an ergodic invariant Borel probability measure. Our exponent characterization theorem establishes a rigorous connection between the observed Lyapunov exponents and the Lyapunov exponents of $f[A]$. Under our assumptions, an observed Lyapunov exponent $\lambda(y, v)$ is a true Lyapunov exponent if and only if $v \in T_y\phi(A)$.

Suppose A is a manifold of dimension d. Implementation of the full Eckmann and Ruelle algorithm yields m observed Lyapunov exponents, d of which are true. The remaining $m-d$ exponents are spurious, artifacts of the embedding process. In order to identify the d true exponents, one must either devise a method to identify the spurious exponents a fortiori or modify ERA to completely avoid the computation of spurious exponents. Several authors propose a modified ERA in which the tangent maps are computed only on the tangent spaces and not on the ambient space $\mathbb{R}^m$. Mera and Morán [15] discuss the convergence of the modified ERA. This technique eliminates the computation of spurious exponents but requires that tangent spaces be computed along orbits. We propose a new technique based on the exponent characterization theorem that allows for the a fortiori determination of the spurious exponents without requiring the computation of tangent spaces along orbits. We describe this algorithm in Section 7 following the statement of the exponent characterization theorem.

1.1. **The case of linear f and ϕ.** We illustrate our ansatz with the case where f and ϕ are linear.

Proposition 1.7. *Let f be linear on $\mathbb{R}^n$ and let A be an invariant subspace on which f is an isomorphism. If the restriction of f to A is not a scalar multiple of the identity, then for almost every $\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ in the sense of Lebesgue measure, there is an induced map on $\phi(A)$ if and only if ϕ is an isomorphism on A.*

Key issues are raised by this proposition. Notice that if there exists $c \in \mathbb{R}$ for which $f(x) = cx$ for all $x \in A$, then $y \mapsto cy$ is the induced map on $\phi(A)$ even if ϕ is not one to one on A. Since this is a theory of observation, when possible the assumptions should be verifiable from observation. The following alternative version of the proposition transfers the assumption onto the induced dynamics in a manner that will be followed throughout this paper.

Proposition 1.8. *Let f be linear on $\mathbb{R}^n$ and let A be an invariant subspace on which f is an isomorphism. For almost every $\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$, there is an induced map on $\phi(A)$ and this induced map is not identically a scalar multiple of the identity if and only if ϕ is an isomorphism on A and the restriction of f to A is not a scalar multiple of the identity.*

Remark 1.9. The hypothesis that f is an isomorphism on A is observable in the sense mentioned earlier. The key point is that if $f[A]$ is not one-to-one, then for almost every $\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ there does not exist an injective induced map $\bar{f}$ on $\phi(A)$.

1.2. **What does "typical" mean?** The conclusions of the linear propositions hold for almost every linear ϕ with respect to Lebesgue measure. In the general situation we will consider the space of C^1 measurement mappings. In order to prove versions of our prototypical theorem, we must first clarify what we mean by a "typical" measurement mapping ϕ. The notion of typicality may be cast in topological terms. In this setting, "typical" would be used to refer to an open and dense subset or a residual subset of mappings. For example, consider the topological Kupka-Smale theorem.

Definition 1.10. Let M be a smooth, compact manifold. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is said to be **Kupka-Smale** if

(1) The periodic points of f are hyperbolic.
(2) If p and q are periodic points of f, then $W^s(p)$ is transverse to $W^u(q)$.

Theorem 1.11 (Kupka-Smale [17])**.** *The set of Kupka-Smale diffeomorphisms is residual in* $\mathrm{Diff}^r(M)$.

The topological notion of typicality is not the appropriate conceptualization for the experimentalist interested in a probabilistic result on the likelihood of a given property in a function space. Any Cantor set of positive measure illustrates the difference between the topological and measure theoretic notions of a small set. The discord between topological typicality and probabilistic typicality is also evident in the following dynamical examples.

Example 1.12. Arnold [1] studied the family of circle diffeomorphisms

$$f_{\omega,\epsilon}(x) = x + \omega + \epsilon \sin(x) \pmod{2\pi},$$

where $0 \leq \omega \leq 2\pi$ and $0 \leq \epsilon < 1$ are parameters. For each ϵ we define the set

$$S_\epsilon = \{\omega \in [0, 2\pi] : f_{\omega,\epsilon} \text{ has a stable periodic orbit}\}.$$

For $0 < \epsilon < 1$, the set S_ϵ is a countable union of disjoint open intervals (one for each rational rotation number) and is an open dense subset of $[0, 2\pi]$. However, the Lebesgue measure of S_ϵ converges to 0 as $\epsilon \to 0$.

There are even more striking examples where the Baire categorical and measure theoretic notions of typicality yield diametrically opposite conclusions about the size of a set.

Example 1.13. Misiurewicz [16] proved that the mapping $z \mapsto e^z$ on the complex plane is topologically transitive, implying that a residual set of initial points yield dense trajectories. On the other hand, Lyubich [13] and Rees [18] proved that Lebesgue almost every initial point has a trajectory whose limit set is a subset of the real axis.

Finally, we consider Lyapunov exponents. This example is particularly relevant because the work of Eckmann, Ruelle, Sano, and Sawada on the computation of these exponents motivated this paper.

Example 1.14 (Lyapunov Exponents)**.** Let $f : M \to M$ be a C^1 diffeomorphism on a compact finite-dimensional Riemannian manifold M. For $(x, v) \in TM$, $\|v\| \neq 0$, the number

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v\|$$

should the limit exist is called the Lyapunov exponent of f at (x, v), denoted $\lambda(x, v)$. We say that $x \in M$ is a **regular point** for f if there are Lyapunov exponents

$$\lambda_1(x) > \cdots > \lambda_l(x)$$

and a splitting

$$T_x M = \bigoplus_{i=1}^{l} E_i(x)$$

of the tangent space to M at x such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)u\| = \lambda_j(x) \quad (u \in E_j(x) \setminus \{0\} \text{ and } 1 \leq j \leq l).$$

While the periodic points of f are always regular points, frequently the set of regular points is a topologically small subset of M. Quite often this set is Baire first category and it may even be finite [24]. From a measure theoretic point of view the situation is completely different.

Theorem 1.15 (Oseledec Multiplicative Ergodic Theorem [24, 11])**.** *The set of regular points for f has full measure with respect to any f-invariant Borel probability measure on M.*

The Oseledec theorem holds in the more general context of measurable cocycles over invertible measure-preserving transformations of a Lebesgue space (X, μ) [11].

Let $f : X \to X$ be an invertible measure preserving transformation and let $L : X \to GL(n, \mathbb{R})$ be a measurable cocycle over X. If

$$\log^+ \|L^{\pm 1}(x)\| \in L^1(X, \mu)$$

then almost every $x \in X$ is a regular point for (f, L).

The following example illustrates that Lyapunov exponents may not exist for a residual set of points. Let $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $p \neq q$. Consider the Markov map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} px, & \text{if } 0 \leqslant x < \frac{1}{p}; \\ qx - \frac{q}{p}, & \text{if } \frac{1}{p} \leqslant x \leqslant 1. \end{cases}$$

This transformation represents the full shift on two symbols with probabilities $1/p$ and $1/q$. Lebesgue measure is invariant under f and ergodic, thus the Lyapunov exponent at Lebesgue almost every $x \in [0,1]$ exists and is equal to

$$\frac{\log(p)}{p} + \frac{\log(q)}{q}$$

by virtue of the Birkhoff ergodic theorem. On the other hand, we claim that no Lyapunov exponent exists for a residual set of points. For $n \in \mathbb{N}$, set

$$V_{p,n}(x) = \frac{1}{n}(|\{0 \leqslant i \leqslant n-1 : f^i(x) \in [0, 1/p)\}|).$$

Fix $\alpha > 1/p$ and $\beta < 1/p$. Define for each $N \in \mathbb{N}$ the sets $C_N = \{x : \exists\, n \geqslant N$ for which $V_{p,n}(x) \geqslant \alpha\}$ and $D_N = \{x : \exists\, n \geqslant N$ for which $V_{p,n}(x) \leqslant \beta\}$. The set C_N contains an open interval to the right of each preimage of $1/p$, and thus C_N contains an open and dense subset of $[0,1]$. Similarly, D_N contains an open interval to the left of each preimage of $1/p$, and thus D_N also contains an open and dense subset of $[0,1]$. No Lyapunov exponent exists for points in the residual set

$$\bigcap_{N=1}^{\infty} C_N \cap D_N$$

because $V_{p,n}(x)$ does not converge for such points.

Motivated by the probabilistic interpretation of typicality, we will use the notion of prevalence developed in [7, 8]. See the references given in [8] for closely related concepts. The notion of prevalence generalizes the translation invariant concept of Lebesgue full measure to infinite-dimensional Banach spaces.

1.3. **Overview of this paper.** Section 2 develops the relevant prevalence theory and demonstrates that cardinality and boundedness are observable properties. In §3 we define a notion of tangent space suitable for general compact subsets of $\mathbb{R}^n$ and we prove the manifold extension theorem. The manifold extension theorem is used in §4 to derive a Platonic version of the Whitney embedding theorem. We present our embedding theorems in §5 and §6 and our results on delay coordinate mappings and Lyapunov exponents in §7.

1.4. **The Transference Method.** Schematically our embedding theorems are developed in the following way. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a dynamical system and let A be a compact invariant set. We want to require no regularity assumptions about f nor do we wish to assume that f is invertible. For a map g, a subset D of the domain of g and any property L, write $(g, L; D)$ to indicate that the restriction of g to D has property L. Let $\mathcal{S}$ denote a collection of properties of a dynamical system. Let $\mathcal{Q}$ denote a collection of properties of maps in the measurement function space $C^1(\mathbb{R}^n, \mathbb{R}^m)$. For example, $\mathcal{Q}$ might consist of the assertion that $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is a homeomorphism on A. We are interested in the ability of the observer to make inferences; that is, in results of the form

$$(\bar{f}, \mathcal{L}; \phi(A)) \Rightarrow (\phi, \mathcal{Q}) \text{ for almost every } \phi, \tag{1.1}$$

where $\mathcal{L}$ is a collection of properties of $\bar{f}$. In other words, the existence of an induced map $\bar{f}$ satisfying properties $\mathcal{L}$ implies that ϕ satisfies properties $\mathcal{Q}$. We first prove

$$(f, \mathcal{S}; A) \Rightarrow ((\bar{f}, \mathcal{L}_1; \phi(A)) \Leftrightarrow (\phi, \mathcal{Q})) \text{ for a.e. } \phi.$$

The Platonic version of the theorem is obtained by replacing each assumption on f with one on $\bar{f}$. For $P \in \mathcal{S}$, we replace the assumption

$$(f, P; A)$$

with one on $\bar{f}$, giving

$$(\bar{f}, \mathcal{L}_1 \cup \mathcal{S}; \phi(A)) \Leftrightarrow ((\phi, \mathcal{Q}) \text{ and } (f, \mathcal{S}; A)) \text{ for a.e. } \phi.$$

In particular, (1.1) holds with $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{S}$. In essence the Platonic version has been obtained by transferring the hypotheses $(f, P; A)$ for $P \in \mathcal{S}$ onto the induced dynamics. Prevalence statements allow for these transfers. Properties for which this program may be implemented are said to be **observable**.

2. Prevalence (Measure-Theoretic Transversality)

Let V be a complete metric linear space.

Definition 2.1. A Borel measure μ on V is said to be **transverse** to a Borel set $S \subset V$ if the following holds:

1. There exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$, and
2. for every $v \in V$ we have $\mu(S + v) = 0$.

For example, μ might be Lebesgue measure supported on a finite-dimensional subspace of V.

Definition 2.2. A Borel set $S \subset V$ is called **shy** if there exists a measure transverse to S. More generally, a subset of V is called shy if it is contained in a shy Borel set. The complement of a shy set is called a **prevalent** set.

A subset of $\mathbb{R}^n$ is shy if and only if it has Lebesgue measure zero. For a map ϕ contained in a prevalent subset S of a linear function space V, we say that ϕ is typical. Employing the language of the finite dimensional case, we say that **almost every** element of V lies in S (in the sense of prevalence).

Using the notion of prevalence, researchers have reformulated several topological and dynamical theorems. Sauer, Yorke, and Casdagli prove in [21] a prevalence version of the Whitney embedding theorem.

Theorem 2.3 (Prevalence Whitney Embedding Theorem [21]). *Let A be a compact subset of $\mathbb{R}^n$ of box dimension d and let m be an integer greater than $2d$. For almost every smooth map $\phi : \mathbb{R}^n \to \mathbb{R}^m$,*

(1) *ϕ is one to one on A and*
(2) *ϕ is an immersion on each compact subset C of a smooth manifold contained in A.*

This theorem is not Platonic because the dimension assumption is on A. In Section 4 we prove a Platonic Whitney embedding theorem as a corollary of the manifold extension theorem.

The reformulation of a genericity theorem of Kupka-Smale type requires a notion of prevalence for nonlinear function spaces such as the space of diffeomorphisms of a compact smooth manifold. Kaloshin in [10] develops such a notion and proves a prevalence version of the Kupka-Smale theorem for diffeomorphisms.

2.1. **Cardinality Preservation.** In Sections 5, 6 and 7 we will need to know how a typical smooth projection affects the cardinality of a set. We show that for a set $A \subset \mathbb{R}^n$, A and $\phi(A)$ have the same cardinality for a.e. $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$. We begin by assuming that A is a countable set.

Proposition 2.4. *Let $A \subset \mathbb{R}^n$ be a countable set. Almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is one to one on A. In particular, if A is countably infinite, then $\phi(A)$ is also countably infinite for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. We write $A = \{x_i : i \in \mathbb{N}\}$. For $i \neq j$ let $C_{ij} = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m) : \phi(x_i) = \phi(x_j)\}$. We first show that C_{ij} is shy. Let $B(x_i, r_i)$ be a metric ball such that $x_j \notin B(x_i, r_i)$. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map such that

(1) $\beta > 0$ on $B(x_i, r_i)$ and
(2) $\mathrm{supp}(\beta) = \overline{B(x_i, r_i)}$.

Let $v \in \mathbb{R}^m$ be a nonzero vector and let μ be the Lebesgue measure supported on the one dimensional subspace

$$\{tv\beta : t \in \mathbb{R}\}.$$

For any $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, it is evident that $\phi + tv\beta \in C_{ij}$ for at most one $t \in \mathbb{R}$. Thus C_{ij} is a shy subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$ because μ is transverse to it. The set

$$\bigcap_{\substack{i,j \in \mathbb{N} \\ i \neq j}} C^1(\mathbb{R}^n, \mathbb{R}^m) \setminus C_{ij}$$

consists of functions that map A injectively into $\mathbb{R}^m$. This set is prevalent because the countable intersection of prevalent sets is prevalent (see [7]). □

Plato would have us consider the prisoner's question where the cardinality of A is not known a priori. For a typical ϕ, does the countability of $\phi(A)$ imply the countability of A? The next proposition answers this question affirmatively with the help of the following lemma.

Lemma 2.5. *Let $A_0 \subset \mathbb{R}^n$ be an uncountable set. Lebesgue almost every function $\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ maps A_0 to an uncountable set.*

Proof. It suffices to consider the scalar case $m = 1$. For each $\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ there exists a unique vector $v \in \mathbb{R}^n$ such that $\phi(x) = (x, v)$ for all $x \in \mathbb{R}^n$. Suppose by way of contradiction that the set

$$\{\phi \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}) : \phi(A_0) \text{ is countable}\}$$

has positive measure. This implies that there exist n linearly independent vectors $\{v_i : i = 1, \ldots, n\}$ such that the functions ϕ_{v_i} given by $x \mapsto (x, v_i)$ map A_0 to a countable set. Let A_1 be an uncountable subset of A_0 such that $\phi_{v_1}(A_1) = \{y_1\}$. Inductively construct a collection of sets $\{A_i : i = 1, \ldots, n\}$ satisfying

(1) A_i is uncountable for each i,
(2) $A_i \subset A_{i-1}$ for each i, and
(3) $\phi_{v_i}(A_i) = \{y_i\}$.

We have $\phi_{v_i}(A_n) = \{y_i\}$ for each i, so A_n consists of one point. This contradiction establishes the lemma. □

Proposition 2.6. *Let A_0 be an uncountable set. For almost every*

$$\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m),$$

$\phi(A_0)$ is uncountable.

Proof. Once again it suffices to consider the scalar case $m = 1$. We show that the set

$$S = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}) : \phi(A_0) \text{ is countable}\}$$

is shy. Let $\{\phi_{e_i}\}$ be a basis for $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R})$ and let μ be the Lebesgue measure on $\mathbb{R}^n$. Write $\alpha = (\alpha_i)$ for a vector in $\mathbb{R}^n$ and for $\phi \in C^1(\mathbb{R}^n, \mathbb{R})$ set

$$\phi_\alpha := \phi + \sum_{i=1}^{n} \alpha_i \phi_{e_i}.$$

If S is not shy, there exists some $g \in S$ such that

$$\mu\{\alpha : g_\alpha(A_0) \text{ is countable}\} > 0$$

where μ denotes n dimensional Lebesgue measure. Without loss of generality assume that $g(A_0)$ is countable. There is at least one point y such that $g^{-1}(y) \cap A_0$ is uncountable. Shrinking A_0 if necessary, without loss of generality we may assume that g maps A_0 to a single point; that is, g is constant on A_0. There exist n linearly independent vectors $\{v_i\}$ such that the functions $\phi_{v_i} + g$ map A_0 to a countable set. As in the proof of (2.5) we inductively construct a collection of sets $\{A_i : i = 1, \ldots, n\}$ satisfying

(1) A_i is uncountable for each i,
(2) $A_i \subset A_{i-1}$ for each i, and
(3) $(\phi_{v_i} + g)(A_i) = \{y_i\}$.

We have $(\phi_{v_i} + g)(A_n) = \{y_i\}$ for each i, so A_n consists of one point. This contradiction establishes the proposition. □

2.2. **Preservation of Unboundedness.** We now consider the question of how a typical smooth projection affects the boundedness of a set. For a typical ϕ, does the boundedness of $\phi(A)$ imply that A is bounded?

Proposition 2.7 (Unboundedness Preservation)**.** *Assume $A \subset \mathbb{R}^n$ is unbounded. Then $\phi(A)$ is unbounded for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. It suffices to assume $m = 1$. We show that the set

$$V = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}) : \phi(A) \text{ is bounded}\}$$

is shy. As above, let μ be the Lebesgue measure on $\mathbb{R}^n$ and for $\phi \in C^1(\mathbb{R}^n, \mathbb{R})$ and $(\alpha_i) \in \mathbb{R}^n$ write

$$\phi_\alpha := \phi + \sum_{i=1}^{n} \alpha_i \phi_{e_i}.$$

If V is not shy, there exists some $g \in V$ such that

$$\mu\{\alpha : g_\alpha(A) \text{ is bounded}\} > 0.$$

Without loss of generality assume that $g(A) \subset [-d, d]$ for some $d > 0$. There exist n linearly independent vectors $\{v_i\}$ and scalars $c_i > 0$ such that the functions $g + \phi_{v_i}$ map A into $[-c_i, c_i]$. Thus A is contained in the set

$$\bigcap_{i=1}^{n} \phi_{v_i}^{-1}([-c_i - d, c_i + d]),$$

a bounded solid polygon. This contradiction establishes the proposition. □

Remark 2.8. We conclude that for a typical $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, the boundedness of $\phi(A)$ implies that A is bounded. That is, the boundedness of A is an observable property.

3. Enveloping Manifolds

Let A be a compact subset of $\mathbb{R}^n$ and let $x \in A$. We say that a C^1 manifold M is an enveloping manifold for A at x if there exists a neighborhood $N(x)$ of x such that $M \supset N(x) \cap A$ and if the dimension of M is minimal with respect to this property. We demonstrate the existence of a C^1 enveloping manifold M for each $x \in A$.

Definition 3.1. Let $D_x A$ be the set of all directions v for which there exist sequences (y_i) and (z_i) in A such that $y_i \to x$, $z_i \to x$, and $\frac{z_i - y_i}{\|z_i - y_i\|} \to v$. The tangent space at x relative to A, denoted $T_x A$, is the smallest linear space containing $D_x A$.

We note that this is one of the two obvious ways to define the tangent space at a point in an arbitrary compact subset of $\mathbb{R}^n$. The other would be to fix $y_i = x$ in the above definition, but the resulting tangent space would be too small for our purposes. In general neither the tangent space itself nor its dimension will vary continuously with $x \in A$. Nevertheless, the tangent space varies upper semicontinuously with $x \in A$. More precisely, we have

Lemma 3.2. *The function $x \mapsto \dim(T_x A)$ is upper semicontinuous on A. In fact, $T_x A$ depends upper semicontinuously on $x \in A$ in the sense that if $x_i \to x$ where $x_i \in A$ and $v_i \to v$ where $v_i \in T_{x_i} A$ then $v \in T_x A$. In other words, $\{(x, v) : x \in A$, $v \in T_x A\}$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$. If $T_x A$ has constant dimension on a set $A_0 \subset A$, then $T_x A$ is continuous on A_0 in the same sense.*

Definition 3.3. The **tangent dimension** of A, denoted $\dim_T(A)$, is given by

$$\dim_T(A) = \max_{x \in A}(\dim T_x A).$$

Example 3.4. In Figure 1 the tangent space $T_p A$ is two-dimensional while $T_x A$ is one-dimensional for all other points $x \in A$. Choosing $(y_i) \subset A$ and $(z_i) \subset A$ such that $y_i \to p$, $z_i \to p$, and y_i and z_i lie on a vertical line for each i, we obtain the tangent vector $v \in T_p A$. Thus $\dim_T(A) = 2$.

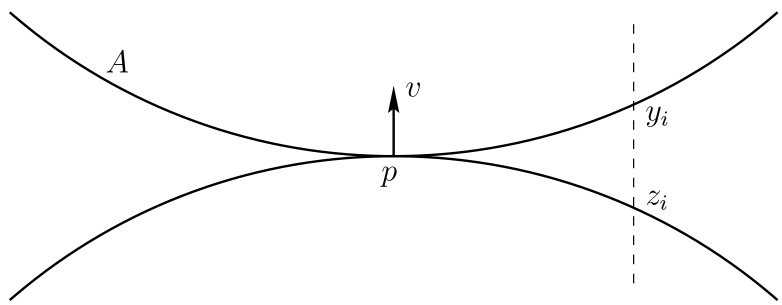

FIGURE 1. A Cusp

We are now in position to state a surprising theorem.

Theorem 3.5 (Manifold Extension Theorem)**.** *For each $x \in A$ there exists an enveloping manifold M for A at x with $T_x M = T_x A$.*

Conjecture 3.6. We believe that integrability is an intrinsic feature of the definition of the tangent space. We therefore conjecture that a global version of the manifold extension theorem holds. Namely, there exists a manifold M such that $\dim(M) = \dim_T(A)$ and $A \subset M$.

Proof. Recall that for a map ψ we denote the restriction of ψ to a subset S of the domain of ψ by $\psi[S]$. Let $m = \dim(T_x A)$. There exists a compact neighborhood N of x such that $\dim(T_y A) \leq m$ for all $y \in N \cap A$. Let π denote the orthogonal projection of $\mathbb{R}^n$ onto $T_x A$. The projection map π induces the splitting $\mathbb{R}^n = T_x A \oplus E_x$. Using this splitting write (p, q) for points in $\mathbb{R}^n$. If $((p_i, q_i))$ is a sequence such that $(p_i, q_i) \in N \cap A$ for each i and $(p_i, q_i) \to x$ then $\frac{\|q_{i+1} - q_i\|}{\|p_{i+1} - p_i\|} \to 0$. We may assume N has been chosen sufficiently small so that π maps $T_y A$ injectively into $T_x A$ for each $y \in N \cap A$ and that $\pi[N \cap A]$ is one to one. Hence we may define ψ on $\pi(N \cap A)$ by $\psi(p) := q$ for $(p, q) \in N \cap A$. Repeated use of our main technical tool, the Whitney extension theorem, will allow us to extend ψ to a C^1 function defined on a neighborhood in $T_x A$ of $\pi(A \cap N)$. We first state a C^1 version of the Whitney extension theorem for compact domains.

Definition 3.7. Let $Q \subset \mathbb{R}^m$ be a compact set and assume $f : Q \to \mathbb{R}^k$ and $L : Q \to \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^k)$ are given functions.

Notation 3.8.

(1) $R(y, z) := \frac{f(z) - f(y) - L(y) \cdot (z - y)}{\|z - y\|}$ (for all $y, z \in Q$, $y \neq z$).

(2) For $\delta > 0$, set
$$\rho(\delta) := \sup_{\substack{y,z\in Q \\ 0<\|z-y\|\le\delta}} \|R(y,z)\|.$$

The pair (f, L) is said to be a **Whitney $\mathbf{C^1}$ function pair** on Q if f and L are continuous and if ρ satisfies

$$\rho(\delta) \to 0 \text{ as } \delta \to 0. \tag{3.1}$$

Notice that (3.1) is equivalent to the following uniformity condition stated by Whitney in [25]: Given any $w \in Q$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in Q$ and $z \in Q$ satisfy $\|y - w\| < \delta$ and $\|z - w\| < \delta$, then $\|R(y, z)\| \le \epsilon$.

Theorem 3.9 (Whitney Extension Theorem [5, 12, 25])**.** *Given a Whitney C^1 function pair (f, L) defined on a compact subset Q of $\mathbb{R}^m$, there exists a C^1 function $\tilde{f} : \mathbb{R}^m \to \mathbb{R}^k$ such that $\tilde{f} = f$ and $D\tilde{f} = L$ on Q.*

We now continue the proof of our manifold extension theorem. Let
$$d(y) = \dim(T_yA)$$
for $y \in A\cap N$. For $k \le m$ let $X_k = \{y \in N \cap A : d(y) = k\}$. We first find a function whose graph is a C^1 manifold which envelops X_m. For each $y \in N \cap A$, the tangent space T_yA may be viewed as a subspace of $T_xA \oplus E_x = \mathbb{R}^n$. For $y \in X_m$ define the linear operator $L_m(y) : T_xA \to E_x$ as follows. For $(v, w) \in D_yA$ let $L_m(y)v = w$. By linearity $L_m(y)$ is determined on T_yA. The linear operator $L_m(y)$ depends continuously on $y \in X_m$ since T_yA depends continuously on $y \in X_m$ by (3.2). The function pair (ψ, L_m) is Whitney C^1 on $\pi(X_m)$ because the uniformity condition of Whitney is implied by (3.1). Notice that the Whitney extension theorem can now only be used to extend $\psi[\pi(X_m)]$ because no obvious candidate exists for $L(y)$ for $y \notin X_m$. By applying the Whitney extension theorem, extend ψ to a function $\tilde{\psi}_1$ defined on $\pi(N)$. Notice that if $X_m = N \cap A$, the result is proved since the graph of $\tilde{\psi}_1$ constitutes an enveloping manifold for A at x.

The general case is handled inductively. Construct $\tilde{\psi}_1$ as above and make the nonlinear change of variable $(p, q) \to (p, q - \tilde{\psi}_1(p)) := (p, \psi_2(p))$. Consider the map $\psi_2[\pi(X_m) \cup \pi(X_{m-1})]$ and let $y \in \text{graph}(\psi_2[\pi(X_m) \cup \pi(X_{m-1})])$. The tangent space $T_y(\text{graph}(\psi_2[\pi(A)]))$ may be viewed as a subspace of $T_xA \oplus E_x = \mathbb{R}^n$. Define the linear map $L_{m-1}(y) : T_xA \to E_x$ as follows. If $y \in \text{graph}(\psi_2[\pi(X_m)])$, set $L_{m-1}(y) \equiv 0$. If $y \in \text{graph}(\psi_2[\pi(X_{m-1})])$, enlarge $T_y(\text{graph}(\psi_2[\pi(A)]))$ to a linear space $\tilde{T}_y$ of dimension m by adjoining one vector in T_xA orthogonal to $T_y(\text{graph}(\psi_2[\pi(A)]))$. For $(v, w) \in \tilde{T}_y$ let $L_{m-1}(y)v = w$. The linear operator $L_{m-1}(y)$ depends continuously on $y \in \text{graph}(\psi_2[\pi(X_m) \cup \pi(X_{m-1})])$ by (3.2). The function pair (ψ_2, L_{m-1}) is Whitney C^1 on $\pi(X_m) \cup \pi(X_{m-1})$ because the uniformity condition of Whitney is implied by (3.1). By applying the Whitney extension theorem, extend $\psi_2[\pi(X_m) \cup \pi(X_{m-1})]$ to a function $\tilde{\psi}_2$ defined on $\pi(N)$. Make the nonlinear change of variables $(p, q) \to (p, q - \tilde{\psi}_2(p)) = (p, \psi_3(p))$.

Assume now that the functions $\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_{k-1}$ and ψ_k have been constructed. Consider the map
$$\psi_k\left[\bigcup_{i=m-k+1}^{m} \pi(X_i)\right].$$

For each point y in the set

$$\text{graph}\left(\psi_k\left[\bigcup_{i=m-k+1}^{m}\pi(X_i)\right]\right)$$

the tangent space $T_y(\text{graph}(\psi_k[\pi(A)]))$ may be viewed as a subspace of $T_xA \oplus E_x = \mathbb{R}^n$. Define the linear map $L_{m-k+1}(y) : T_xA \to E_x$ as follows. If $y \in \text{graph}(\psi_k[\pi(X_m) \cup \cdots \cup \pi(X_{m-k+2})])$, set $L_{m-k+1}(y) \equiv 0$. On the other hand, if $y \in \text{graph}(\psi_k[\pi(X_{m-k+1})])$, enlarge $T_y(\text{graph}(\psi_k[\pi(A)]))$ to a linear space $\tilde{T}_y$ of dimension m by adjoining $k-1$ vectors in T_xA orthogonal to

$$T_y(\text{graph}(\psi_k[\pi(A)])).$$

For $(v,w) \in \tilde{T}_y$ let $L_{m-k+1}(y)v = w$. By (3.1) and (3.2) the function pair

$$(\psi_k, L_{m-k+1})$$

is Whitney C^1 on the set

$$\bigcup_{i=m-k+1}^{m}\pi(X_i).$$

By applying the Whitney extension theorem, extend the function

$$\psi_k\left[\bigcup_{i=m-k+1}^{m}\pi(X_i)\right]$$

to a function $\tilde{\psi}_k$ defined on $\pi(N)$. Make the change of variables $(p,q) \to (p, q - \tilde{\psi}_k(p)) := (p, \psi_{k+1}(p))$. After $m+1$ steps we obtain a map

$$\Psi := \sum_{i=1}^{m+1}\tilde{\psi}_i$$

defined on $\pi(N)$. The graph of Ψ constitutes an enveloping manifold M for A at x. □

Remark 3.10. Although our inductive procedure is canonical, observe that the Whitney extension theorem makes no claim of uniqueness. Assume that (f, L_1) and (f, L_2) are Whitney C^1 function pairs defined on a compact subset Q of $\mathbb{R}^m$ as in (3.9). Let $y \in \text{graph}(f)$ and let π denote the orthogonal projection of $\mathbb{R}^m \times \mathbb{R}^k$ onto $\mathbb{R}^m$. The tangent space $T_y(\text{graph}(f))$ may be viewed as a subspace of $\mathbb{R}^m \times \mathbb{R}^k$. The linear operators $L_1(y)$ and $L_2(y)$ must satisfy $L_1(y)v = L_2(y)v = w$ for all $(v,w) \in T_y(\text{graph}(f))$. However, $L_1(y)$ and $L_2(y)$ are determined only for $(v,w) \in T_y(\text{graph}(f))$. If $v \notin \pi(T_y(\text{graph}(f)))$, then $L_1(y)$ and $L_2(y)$ may be such that $L_1(y)v \neq L_2(y)v$.

4. Platonic Embedology

Recall the prevalence version of the Whitney embedding theorem.

Theorem 4.1 (Prevalence Whitney Embedding Theorem [21])**.** *Let A be a compact subset of $\mathbb{R}^n$ of box dimension d and let m be an integer greater than $2d$. For almost every smooth map $\phi : \mathbb{R}^n \to \mathbb{R}^m$,*

(1) *ϕ is one to one on A and*
(2) *ϕ is an immersion on each compact subset C of a smooth manifold contained in A.*

The manifold extension theorem implies a Platonic version of this result. We need a notion of diffeomorphism appropriate for a general compact subset A of $\mathbb{R}^n$.

Definition 4.2. We say that a measurement map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is a **diffeomorphism** on A if ϕ is injective on A and if for each $x \in A$ there exists an enveloping manifold M for A at x that is mapped diffeomorphically onto an enveloping manifold for $\phi(A)$ at $\phi(x)$.

We are now in position to formulate the Platonic Whitney embedding theorem.

Theorem 4.3 (Platonic Whitney Embedding Theorem)**.** *Let $A \subset \mathbb{R}^n$ be compact. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if $\phi(A)$ satisfies $\dim_T \phi(A) < \frac{m}{2}$, then ϕ is a diffeomorphism on A.*

Conjecture 4.4. The Platonic Whitney embedding theorem remains valid under the weaker assumption that $\dim_T \phi(A) < m$.

The proof of this result requires an understanding of the relationship between the box dimension of A and the dimension of the tangent spaces T_xA for $x \in A$. Working only with the definitions, the relationship is unclear. Illumination is provided by the manifold extension theorem.

Lemma 4.5. *Let $A \subset \mathbb{R}^n$ be compact. For each $x \in A$, there exists a neighborhood N of x such that $\dim(T_xA) \geqslant \dim_B(A \cap N)$.*

Proof. Fix $x \in A$. By the manifold extension theorem, there exists an enveloping manifold M for A at x and a neighborhood N of x such that $M \supset N \cap A$. The set $N \cap A$ is contained in a C^1 manifold of dimension $\dim(T_xA)$ and therefore $\dim(T_xA) \geqslant \dim_B(A \cap N)$. □

We now commence with the proof of the Platonic Whitney embedding theorem. Suppose there exists $x \in A$ such that $\dim(T_xA) \geqslant \frac{m}{2}$. In this case we would have that $\dim(T_{\phi(x)}\phi(A)) \geqslant \frac{m}{2}$ for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ as a consequence of the fact that almost every linear transformation has full rank. Therefore we may assume that $\dim(T_xA) < \frac{m}{2}\ \forall x \in A$. By the manifold extension theorem and the compactness of A, A is contained in a finite union $\bigcup_{i=1}^k M_i$ of enveloping manifolds such that $\dim(M_i) < \frac{m}{2}$ for each i. Box dimension is finitely stable, so one has

$$\dim_B(A) \leqslant \dim_B\left(\bigcup_{i=1}^k M_i\right) = \max_i \dim_B(M_i) < \frac{m}{2}.$$

The prevalence version of the Whitney embedding theorem (2.3) implies that almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is a diffeomorphism on A.

Remark 4.6. Suppose one only knows that $\dim_B(\phi(A)) < \frac{m}{2}$ for a typical ϕ. It is difficult to draw any conclusions in this case. Sauer and Yorke [20] exhibit a compact subset A of $\mathbb{R}^{10}$ with $\dim_B(A) = 3.5$ such that $\dim_B(\phi(A)) < 3$ for every $\phi \in C^1(\mathbb{R}^{10}, \mathbb{R}^6)$.

5. Observing A Continuous Dynamical System

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a dynamical system and let A be a compact invariant set. We make no a priori regularity assumptions about f. Let $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ and let

$B \subset \mathbb{R}^n$ be an open metric ball. Recall that if there exists a map $\bar{f} : \phi(A) \to \phi(A)$ such that for $x \in A$ the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \phi \downarrow & & \downarrow \phi \\ \phi(A) & \xrightarrow{\bar{f}} & \phi(A) \end{array}$$

commutes, then we say that $\bar{f}$ is the **induced map** associated with f.

Remark 5.1. If f is continuous, then the existence of $\bar{f}$ implies the continuity of $\bar{f}$.

Definition 5.2. The pair $(x_1, x_2) \in A \times A$ is **coincident** if $\phi(x_1) = \phi(x_2)$. The pair $(x_1, x_2) \in A \times A$ is said to be **dynamically separated** by B if

(1) (x_1, x_2) is coincident and
(2) $x_1 \notin B$, $x_2 \notin B$, $f(x_1) \in B$ and $f(x_2) \notin B$.

Definition 5.3. Let S_B be the set of maps ϕ in $C^1(\mathbb{R}^n, \mathbb{R}^m)$ for which the following hold:

(1) There exists some pair (x_1, x_2) dynamically separated by B, and
(2) for all such pairs we have $\phi(f(x_1)) = \phi(f(x_2))$.

Lemma 5.4. *The set S_B is a shy subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$.*

Proof. We construct a measure transverse to S_B. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map such that $\beta > 0$ on B and $\text{supp}(\beta) = \bar{B}$. Let $v \in \mathbb{R}^m$ be a nonzero vector. Let μ be the Lebesgue measure supported on the one dimensional subspace

$$\{tv\beta : t \in \mathbb{R}\}.$$

For any $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ it is evident that $\phi + tv\beta \in S_B$ for at most one $t \in \mathbb{R}$. Thus S_B is shy because μ is transverse to it. □

Definition 5.5. Let $\text{Fix}(f)$ denote the set of fixed points of f. Let $\text{Per}_2(f)$ denote the set of periodic points of f of period 2.

Proposition 5.6. *Suppose $f[A]$ is continuous and invertible. Assume that the sets $\text{Fix}(f[A])$ and $\text{Per}_2(f[A])$ are countable. For almost every map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ the following are equivalent:*

(1) *The map ϕ is one to one on A.*
(2) *The induced map $\bar{f}$ exists (and is therefore continuous).*

Proof.
((1) $\Rightarrow$ (2)) Define $\bar{f} := \phi \circ f \circ \phi^{-1}$.
((2) $\Rightarrow$ (1)) Let $\{B_i\}$ be a countable collection of open metric balls such that if $x, y \in A$ satisfy $x \neq y$ then there exists some B_i such that $x \in B_i$ and $y \notin B_i$. Consider the following three sets:

$$\begin{cases} G_1 = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m) : \phi \text{ is one to one on } \text{Fix}(f[A])\} \\ G_2 = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m) : \phi \text{ is one to one on } \text{Per}_2(f[A])\} \\ G_3 = \displaystyle\bigcap_{i=1}^{\infty} (S_{B_i})^C \end{cases}$$

The set G_1 is a prevalent subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$ by Proposition 2.4 because the fixed points of $f[A]$ are countable. Similarly, G_2 is prevalent. The set G_3 is a prevalent subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$ because $(S_{B_i})^C$ is prevalent for each i by (5.4) and because the countable intersection of prevalent sets is prevalent (see [7]). Thus $G_1 \cap G_2 \cap G_3$ is a prevalent subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$. Let $\phi \in G_1 \cap G_2 \cap G_3$ and assume that ϕ is not one to one on A. It follows that no induced map $\bar{f}$ exists. Since $\phi \notin S_{B_i}$ for all i, there exists a metric ball B_i and a coincident pair (x_1, x_2) dynamically separated by B_i such that $\phi(f(x_1)) \neq \phi(f(x_2))$. □

Proposition 2.6 allows us to improve this result by transferring the dynamical hypotheses onto the induced dynamics. We need a lemma indicating that the existence of a point of discontinuity of $f[A]$ precludes the existence of a continuous induced map for a typical measurement function.

Lemma 5.7. *Suppose $f[A]$ is discontinuous at some point $x \in A$. Then for a.e. $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, no continuous induced map exists.*

Theorem 5.8. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map. For almost every map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, there is an induced map $\bar{f}$ satisfying*

(1) *$\bar{f}$ is continuous and invertible, and*
(2) *$\mathrm{Fix}(\bar{f})$ and $\mathrm{Per}_2(\bar{f})$ are countable*

if and only if the following hold.

(1) *The measurement map ϕ is one to one on A.*
(2) *The sets $\mathrm{Fix}(f[A])$ and $\mathrm{Per}_2(f[A])$ are countable.*
(3) *The map $f[A]$ is continuous and invertible.*

Proof. We employ the transference method. If $f[A]$ is continuous and invertible and $\mathrm{Fix}(f[A])$ and $\mathrm{Per}_2(f[A])$ are countable sets, then (5.6) implies the result. If $\mathrm{Fix}(f[A])$ or $\mathrm{Per}_2(f[A])$ is uncountable then Proposition 2.6 implies that the statement of the theorem holds for almost every ϕ. Lemma 5.7 implies the result if $f[A]$ is discontinuous at some point. If $f[A]$ is not invertible, then for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ no invertible induced map exists. □

We now consider the possibility of recovering differential information.

6. Observing Differentiable Dynamics

Assume that f is a diffeomorphism on $\mathbb{R}^n$. The concept of a measurement function ϕ being an immersion on A usually requires A to be a manifold, but there is now an obvious extension.

Definition 6.1. We say the map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is an **immersion** on A if $D\phi(x)[T_x A] : T_x A \to T_{\phi(x)}\phi(A)$ is one to one for each $x \in A$.

Motivated by the theory of infinite-dimensional dynamical systems, we formulate our C^1 results using the notion of quasidifferentiability.

Definition 6.2. The function f is said to be quasidifferentiable on the set A if $f[A]$ is continuous and if for each $x \in A$ there exists a linear map $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$, the quasiderivative of f at x, such that

$$\frac{f(x_i) - f(y_i) - Df(x)(x_i - y_i)}{\|x_i - y_i\|} \to 0$$

for all sequences $(x_i) \subset A$ and $(y_i) \subset A$ such that $x_i \to x$ and $y_i \to x$.

Remark 6.3. The function f is Whitney C^1 if and only if f is quasidifferentiable and the quasiderivative varies continuously. Since continuity is observable, the C^1 embedding results to follow may be formulated with 'Whitney C^1' in place of 'quasidifferentiable.'

We would like to prove under the assumptions of (5.6) that for almost every ϕ, the existence of a quasidifferentiable induced map $\bar{f}$ implies that ϕ is an injective immersion on A. However, one extra hypothesis on f is needed; namely, that for each $x \in \mathrm{Fix}(f[A])$ we have

$$Df(x)[T_xA] \neq \gamma \cdot I \text{ for every } \gamma \in \mathbb{R}.$$

To see the need for this hypothesis, suppose that f is the identity map, A is countable, and there exists $x \in A$ such that $\dim(T_xA) = n > m$. In this case, the identity map on $\phi(A)$ is the induced map for every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, yet every ϕ fails to be immersive at x.

Consider a countable set $\{B_i = B(y_i, r_i)$ of open metric balls in $\mathbb{R}^n$ that separates points. Let $T(A) = \{(x, v) : x \in A,\ v \in T_xA\}$.

Definition 6.4. Let W_{B_i} be the set of measurement maps in $C^1(\mathbb{R}^n, \mathbb{R}^m)$ with the following properties:

(1) There exists some point $(x, v) \in T(A)$ such that $v \neq 0$, $x \notin B(y_i, 2r_i)$, $f(x) \in B(y_i, r_i)$, $D\phi(x)v = 0$, and
(2) for all such points we have $D\phi(f(x)) \circ Df(x)v = 0$.

Lemma 6.5. *The set W_{B_i} is shy.*

Proof. Let $F_1, \ldots, F_t$ be a basis for the nm dimensional space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $\beta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map with the following properties:

$$\begin{cases} (1) \quad \beta(x) = 1 \text{ for } x \in B(y_i, \frac{5}{4} r_i) \\ (2) \quad \mathrm{supp}(\beta) = \overline{B(y_i, \frac{3}{2} r_i)} \\ (3) \quad 0 < \beta \leq 1 \text{ on } B(y_i, \frac{3}{2} r_i) \end{cases}$$

Let P be the subspace of $C^1(\mathbb{R}^n, \mathbb{R}^m)$ spanned by the collection $\{\beta F_i : i = 1, \ldots, t\}$ and endow P with Lebesgue measure. For any ϕ, the set of vectors (α_i) for which

$$\phi + \beta \sum_{i=1}^{t} \alpha_i F_i \in W_{B_i}$$

is a subset of P of measure zero. $\square$

Lemma 6.6. *Let $x \in \mathrm{Fix}(f[A])$ and assume that $Df(x)[T_xA] \neq \gamma \cdot I$ for all $\gamma \in \mathbb{R}$. The set Z_x of measurement mappings satisfying*

(1) $\ker(D\phi(x)) \cap T_xA \neq \{0\}$ *and*
(2) $Df(x)(\ker(D\phi(x)) \cap T_xA) \subset \ker(D\phi(x))$

is a shy subset of $C^1(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. Consider the orthogonal decomposition $\mathbb{R}^n = T_xA \oplus E_x$. Let L be the subset of $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ consisting of maps that vanish on E_x and have norm at most one. Endow L with the normalized Lebesgue probability measure μ. For any $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, we claim that

$$\mu(\{F \in L : \phi + F \in Z_x\}) = 0. \tag{6.1}$$

If $\dim T_x A \leq m$ then (6.1) follows from the fact that almost every linear transformation has full rank. If $\dim T_x A > m$, then it suffices to consider the scalar case $m = 1$. Let $d = \dim(T_x A)$ and let $\{\phi_{e_i}\}$ be an orthonormal basis for $\mathrm{Lin}(T_x A, \mathbb{R})$, the unit ball of which we identify with L. Let ϕ_w represent $D\phi(x)[T_x A]$ with respect to the basis $\{\phi_{e_i}\}$. For a map $\phi_v \in \mathrm{Lin}(T_x A, \mathbb{R})$ such that $v + w \neq 0$, it is necessary that $v + w$ be an eigenvector of $Df(x)[T_x A]^T$ in order to have

$$Df(x)(\ker(\phi_{v+w}) \cap T_x A) \subset \ker(\phi_{v+w}).$$

If $Df(x)[T_x A]^T$ does not have an eigenvalue of multiplicity d, then (6.1) holds. Finally, notice that $Df(x)[T_x A]^T$ has an eigenvalue of multiplicity d if and only if $Df(x)[T_x A]$ is a scalar multiple of the identity. □

Proposition 6.7. *Suppose f is a diffeomorphism on $\mathbb{R}^n$. Assume that*

$$\mathrm{Fix}(f[A]) \text{ and } \mathrm{Per}_2(f[A])$$

are countable sets. Assume that for each $x \in \mathrm{Fix}(f[A])$ we have

$$Df(x)[T_x A] \neq \gamma \cdot I \text{ for every } \gamma \in \mathbb{R}.$$

Then for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if there is a quasidifferentiable induced map $\bar{f}$ then the measurement map ϕ is an injective immersion on A.

Proof. Consider the following sets:

$$\begin{cases} G_4 = \bigcap_{i=1}^{\infty} (W_{B_i})^C \\ G_5 = \bigcap_{x \in \mathrm{Fix}(f[A])} (Z_x)^C \end{cases}$$

The sets G_4 and G_5 are prevalent by (6.5) and (6.6) respectively. For ϕ in the prevalent set

$$\bigcap_{j=1}^{5} G_j,$$

the existence of a quasidifferentiable induced map $\bar{f}$ implies that ϕ is an injective immersion on A. □

Once again Proposition 2.6 allows us to transfer some of the hypotheses of this theorem onto the induced dynamics.

Theorem 6.8. *Suppose f is a diffeomorphism on $\mathbb{R}^n$. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if there is a quasidifferentiable induced map satisfying*

(1) $\mathrm{Fix}(\bar{f})$ *and* $\mathrm{Per}_2(\bar{f})$ *are countable and*
(2) *For each $y \in \mathrm{Fix}(\bar{f})$, $D\bar{f}(y)[T_y \phi(A)] \neq \gamma \cdot I$ for every $\gamma \in \mathbb{R}$*

then the following hold.

(1) *The measurement map ϕ is an injective immersion on A.*
(2) $\mathrm{Fix}(f[A])$ *and* $\mathrm{Per}_2(f[A])$ *are countable.*
(3) *For each $x \in \mathrm{Fix}(f[A])$, $Df(x)[T_x A] \neq \gamma \cdot I$ for every $\gamma \in \mathbb{R}$.*

Proof. It suffices to consider the cases in which the hypotheses of Proposition 6.7 fail to hold. If $\mathrm{Fix}(f[A]) \cup \mathrm{Per}_2(f[A])$ is uncountable, then for almost every ϕ there cannot exist an induced map satisfying $\mathrm{Fix}(\bar{f})$ and $\mathrm{Per}_2(\bar{f})$ are countable by Proposition 2.6. Suppose there exist $x \in \mathrm{Fix}(f[A])$ and $\gamma \in \mathbb{R}$ such that

$$Df(x)[T_x A] = \gamma \cdot I.$$

For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $D\phi(x)[T_x A]$ has full rank. If $\dim(T_x A) \geqslant m$ then the full rank of $D\phi(x)[T_x A]$ implies that $D\phi(x)$ maps $T_x A$ onto $T_{\phi(x)}\phi(A)$ and therefore the existence of a quasidifferentiable induced map would imply

$$D\bar{f}(\phi(x))[T_{\phi(x)}\phi(A)] = \gamma \cdot I.$$

If $\dim(T_x A) < m$ then the full rank of $D\phi(x)[T_x A]$ implies that $D\phi(x)$ maps $T_x A$ injectively into $T_{\phi(x)}\phi(A)$ and therefore surjectively onto $T_{\phi(x)}\phi(A)$. In this case, the existence of a quasidifferentiable induced map would imply

$$D\bar{f}(\phi(x))[T_{\phi(x)}\phi(A)] = \gamma \cdot I.$$

□

Using the manifold extension theorem we strengthen this theorem by utilizing the previously introduced notion of a diffeomorphism on A. We recall that definition here.

Definition 6.9. We say that a measurement map $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ is a **diffeomorphism** on A if ϕ is injective on A and if for each $x \in A$ there exists an enveloping manifold M for A at x that is mapped diffeomorphically onto an enveloping manifold for $\phi(A)$ at $\phi(x)$.

Theorem 6.10. *Suppose f is a diffeomorphism on $\mathbb{R}^n$. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if there is a quasidifferentiable induced map $\bar{f}$ satisfying*

1. $\mathrm{Fix}(\bar{f})$ *and* $\mathrm{Per}_2(\bar{f})$ *are countable and*
2. *For each $y \in \mathrm{Fix}(\bar{f})$, $D\bar{f}(y)[T_y\phi(A)] \neq \gamma \cdot I$ for every $\gamma \in \mathbb{R}$*

then the following hold.

1. *The measurement map ϕ is a diffeomorphism on A.*
2. $\mathrm{Fix}(f[A])$ *and* $\mathrm{Per}_2(f[A])$ *are countable.*
3. *For each $x \in \mathrm{Fix}(f[A])$, $Df(x)[T_x A] \neq \gamma \cdot I$ for every $\gamma \in \mathbb{R}$.*

Remark 6.11. Mera and Morán [14] provide a test for determining whether or not observed trajectories of $\bar{f}$ are consistent with the assumption that $\bar{f}$ belongs to a certain regularity class.

The C^1 Theorem (6.10) is not Platonic because we assume that f is a diffeomorphism on $\mathbb{R}^n$. We formulate a Platonic version of the C^1 Theorem by selecting new hypotheses on the induced map $\bar{f}$. The key modification is the replacement of the dynamical assumption on the nature of $D\bar{f}(y)[T_y\phi(A)]$ for $y \in \mathrm{Fix}(\bar{f})$ with the structural assumption that $\dim T_y(\phi(A)) < m \; \forall y \in \phi(A)$. The smoothness of f becomes an observable in this new setting. After presenting several technical preliminaries, we state and prove the main result. We assume only that f is a map throughout this section.

Lemma 6.12. *If $\dim T_x(A) \geqslant m$ for some $x \in A$, then for almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ one has $\dim T_{\phi(x)}\phi(A) \geqslant m$.*

Proof. The result follows from the fact that almost every linear transformation from one finite-dimensional vector space to another has full rank. ☐

Lemma 6.13. *Suppose there exist sequences $(x_i) \subset A$, $(y_i) \subset A$, and $x \in A$ such that $x_i \to x$, $y_i \to x$ and $\frac{x_i - y_i}{\|x_i - y_i\|} \to v \in T_x A$, but*

$$\left(\frac{f(x_i) - f(y_i)}{\|x_i - y_i\|} \right)$$

does not converge to a vector in $\mathbb{R}^n$. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, there does not exist a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ with $\dim T_y \phi(A) < m \ \forall y \in \phi(A)$.

Proof. We need to consider two cases. Assume that the sequence

$$\left(\frac{f(x_i) - f(y_i)}{\|x_i - y_i\|} \right) \tag{6.2}$$

has two limit points, v_1 and v_2. There cannot exist a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ if $v \notin \ker(D\phi(x)[T_x A])$ and $v_1 - v_2 \notin \ker(D\phi(f(x))[T_{f(x)} A])$. This condition is prevalent and therefore the lemma holds in the first case. Now suppose that the sequence (6.2) tends to infinity. If either $\dim(T_x A) \geqslant m$ or $\dim(T_{f(x)} A) \geqslant m$, then Lemma 6.12 implies that for almost every ϕ one does not have $\dim T_y \phi(A) < m \ \forall y \in \phi(A)$. If both $\dim(T_x A) < m$ and $\dim(T_{f(x)} A) < m$, then for almost every ϕ it follows that $D\phi(x)[T_x A]$ and $D\phi(f(x))[T_{f(x)} A]$ are injective. For such a ϕ, the existence of a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ would imply

$$\frac{\bar{f} \circ \phi(x_i) - \bar{f} \circ \phi(y_i)}{\|\phi(x_i) - \phi(y_i)\|} = \frac{\phi \circ f(x_i) - \phi \circ f(y_i)}{\|\phi(x_i) - \phi(y_i)\|} \to \infty,$$

a contradiction. ☐

Theorem 6.14 (Platonic C^1 Theorem)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map. For almost every $\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, if there exists an invertible quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ satisfying*

1. $\mathrm{Fix}(\bar{f})$ *and* $\mathrm{Per}_2(\bar{f})$ *are countable,*
2. $\dim T_y(\phi(A)) < m \ \forall y \in \phi(A)$, *and*
3. $D\bar{f}(y)[T_y \phi(A)]$ *is invertible* $\forall y \in \phi(A)$,

then the following hold.

1. *The measurement mapping ϕ is a diffeomorphism on A.*
2. *The mapping $f[A]$ is invertible.*
3. *The sets* $\mathrm{Fix}(f[A])$ *and* $\mathrm{Per}_2(f[A])$ *are countable.*
4. *The dynamical system f is quasidifferentiable on A and $Df(x)[T_x A]$ is invertible for all $x \in A$.*
5. *For each $x \in A$,* $\dim(T_x A) < m$.

Proof. See Sections 5 and 6 for the definitions of the sets G_1, G_2, G_3, and G_4. Let

$$G_6 = \{\phi \in C^1(\mathbb{R}^n, \mathbb{R}^m) : D\phi(x)[T_x A] \text{ is injective for each } x \in \mathrm{Fix}(f[A])\}.$$

If $\mathrm{Fix}(f[A])$ is countable and $\dim(T_x A) < m$ for each $x \in A$, then G_6 is prevalent. We employ the transference method to prove the Platonic C^1 Theorem.

If f satisfies conclusions (2), (3), (4), and (5), then for ϕ in the prevalent set

$$\left(\bigcap_{j=1}^{4} G_j\right) \bigcap G_6,$$

the existence of a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ implies that ϕ is an injective immersion on A. If $f[A]$ is not invertible, then for almost every ϕ, no invertible induced map exists. If $\mathrm{Fix}(f[A]) \cup \mathrm{Per}_2(f[A])$ is uncountable, then Proposition 2.6 implies that no induced map satisfying hypothesis (1) exists for almost every ϕ. If there exists $x \in A$ for which $\dim(T_x A) \geqslant m$, then Lemma 6.12 implies that $\dim T_{\phi(x)}\phi(A) \geqslant m$ for almost every ϕ and for such ϕ hypothesis (2) is not satisfied.

Suppose f is not quasidifferentiable on A. If $f[A]$ is not continuous, then Lemma 5.7 implies that for almost every ϕ there does not exist a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$. If f fails to be quasidifferentiable on A because the hypotheses of Lemma 6.13 are satisfied, then this lemma implies that for a.e. ϕ there does not exist a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ with $\dim T_y\phi(A) < m \ \forall y \in \phi(A)$. The remaining possibility is that for some $x \in A$ there exists a nonlinear map taking $T_x A$ into $T_{f(x)}A$. For a.e. ϕ, this precludes the existence of a quasidifferentiable induced map $\bar{f}$. Finally, suppose f is quasidifferentiable on A but $Df(x)[T_x A]$ is not invertible for some $x \in A$. In this case for a.e. ϕ there does not exist a quasidifferentiable induced map $\bar{f}$ on $\phi(A)$ satisfying hypothesis (3). □

We finish with theorems concerning delay coordinate mappings and Lyapunov exponents.

7. Delay Coordinate Embeddings and Lyapunov Exponents

We state delay coordinate embedding versions of our results and prove the exponent characterization theorem.

7.1. **Delay Coordinate Maps.** The following theorems do not follow from the previously established corresponding theorems for the general class of smooth measurement mappings because the delay coordinate mappings form a subspace of $C^1(\mathbb{R}^n, \mathbb{R}^m)$. Nevertheless, their veracity is established using essentially the same reasoning.

Theorem 7.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map. For almost every $g \in C^1(\mathbb{R}^n, \mathbb{R})$, there is an induced map $\bar{f}$ satisfying*

(1) *$\bar{f}$ is continuous and invertible, and*
(2) *$\bigcup_{i=1}^{2m} \mathrm{Per}_i(\bar{f})$ is countable*

if and only if the following hold.

(1) *The delay coordinate map $\phi(f, g)$ is one to one on A.*
(2) *The set $\bigcup_{i=1}^{2m} \mathrm{Per}_i(f[A])$ is countable.*
(3) *The map $f[A]$ is continuous and invertible.*

Theorem 7.2. *Let f be a diffeomorphism on $\mathbb{R}^n$. For a.e. $g \in C^1(\mathbb{R}^n, \mathbb{R})$, if there is a quasidifferentiable induced map $\bar{f}$ satisfying*

(1) *$\bigcup_{i=1}^{2m} \mathrm{Per}_i(\bar{f})$ is countable and*

(2) *for each* $p \in \{1, \dots, m\}$ *and* $y \in \mathrm{Per}_p(\bar{f})$ *we have*

$$D\bar{f}^p(y)[T_y\phi(f,g)(A)] \neq \gamma \cdot I \text{ for every } \gamma \in \mathbb{R}$$

then the following hold.

(1) *The delay coordinate map* $\phi(f,g)$ *is a diffeomorphism on* A.
(2) *The set* $\bigcup_{i=1}^{2m} \mathrm{Per}_i(f[A])$ *is countable.*
(3) *For each* $p \in \{1, \dots, m\}$ *and each* $x \in \mathrm{Per}_i(f[A])$, *we have*

$$Df^p(x)[T_xA] \neq \gamma \cdot I \text{ for every } \gamma \in \mathbb{R}.$$

7.2. **Lyapunov Exponents.** We conclude Section 7 with a discussion of Lyapunov exponents. Assume f and $\bar{f}$ are quasidifferentiable and invertible on A and $\phi(A)$, respectively, with invertible quasiderivatives at each point $x \in A$ and $y \in \phi(A)$. Suppose ϕ is a diffeomorphism on A. Assume $y \in \phi(A)$ is a regular point for $\bar{f}$ and recall that this implies the existence of a decomposition

$$\mathbb{R}^m = \bigoplus_{i=1}^{l} E_i(y)$$

such that

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \|D\bar{f}^k(y)v\| = \lambda_j(y) \quad (v \in E_j(y) \setminus \{0\} \text{ and } 1 \leq j \leq l).$$

Since the set of regular points $R(\bar{f})$ is invariant in the sense that

(1) $y \in R(\bar{f}) \Rightarrow \bar{f}^k(y) \in R(\bar{f})$ for all $k \in \mathbb{Z}$ and
(2) $D\bar{f}^{\pm 1}(E_i(y)) = E_i(\bar{f}^{\pm 1}(y))$ for $i = 1, \dots, l$,

we associate the Lyapunov exponents $\lambda_1 > \cdots > \lambda_l$ with the trajectory (y_k). Counting multiplicities, there are m Lyapunov exponents associated with (y_k) and we label them $\chi_1, \dots, \chi_m$ such that

$$\chi_1 \geqslant \chi_2 \geqslant \cdots \geqslant \chi_m.$$

In light of Remark 3.10 following the manifold extension theorem, we make the following definitions.

Definition 7.3. We say that a Lyapunov exponent $\lambda(y,v)$ of $\bar{f}$ is a **tangent** Lyapunov exponent if $v \in T_y\phi(A)$. A Lyapunov exponent $\lambda(y,v)$ of $\bar{f}$ is said to be a **transverse** Lyapunov exponent if it is not a tangent exponent.

Definition 7.4. A Lyapunov exponent $\lambda(y,v)$ of $\bar{f}$ is said to be a **true** Lyapunov exponent if it does not depend on the choice of quasiderivative $D\bar{f}$ and if it is also a Lyapunov exponent of f at $\phi^{-1}(y)$. We say that a Lyapunov exponent $\lambda(y,v)$ of $\bar{f}$ is **spurious** if there exists a quasiderivative $D\bar{f}$ for which

$$\lim_{k \to \pm\infty} \frac{1}{k} \log \|D\bar{f}^k(y)v\|$$

either does not exist or is not a Lyapunov exponent of f at $\phi^{-1}(y)$.

Theorem 7.5 (Exponent Characterization Theorem)**.** *Assume f and $\bar{f}$ are quasidifferentiable and invertible on A and $\phi(A)$, respectively, with invertible quasiderivatives at each point $x \in A$ and $y \in \phi(A)$. Suppose ϕ is a diffeomorphism on A. Assume that $y \in \phi(A)$ is a regular point for $\bar{f}$ such that* $\dim T_z\phi(A) = \dim T_y\phi(A)$ *for all $z \in \overline{(y_k)}$. The following characterizations hold for a Lyapunov exponent $\lambda(y,v)$ of $\bar{f}$.*

(1) *If the exponent $\lambda(y, v)$ is tangent then it is a true exponent.*
(2) *If the exponent $\lambda(y, v)$ is transverse then it is a spurious exponent.*

The tangent exponents of $\bar{f}$ correspond to the tangent exponents of f.

Remark 7.6. The tangent space $T_y\phi(A)$ admits the decomposition

$$T_y\phi(A) = \bigoplus_{i=1}^{l} V_i(y)$$

where $V_i(y)$ is a subspace of $E_i(y)$ for $i = 1, \ldots, l$.

Remark 7.7. From a computational point of view, one is interested in constructing algorithms to efficiently and accurately compute the Lyapunov spectrum and identify the true exponents. The existing technique ([3, 19, 15]) requires that one modify the Eckmann and Ruelle algorithm by computing the tangent maps only on the tangent spaces and not on the ambient space $\mathbb{R}^m$. Assuming A is a smooth submanifold, Mera and Morán [15] state conditions under which this modified ERA converges. Clearly this technique eliminates the computation of spurious exponents. However, one has to compute the tangent spaces along the entire orbit. In light of the exponent characterization theorem, we propose a new algorithm that eliminates the need to compute these tangent spaces.

Definition 7.8. A **forward filtration** of $\mathbb{R}^m$ is a nested collection of subspaces

$$\emptyset = F_0(y) \subset F_1(y) \subset F_2(y) \subset \cdots \subset F_m(y) = \mathbb{R}^m$$

such that

$$\lim_{k\to+\infty} \frac{1}{k} \log \|D\bar{f}^k(y)v\| = \chi_{m-j+1}$$

for $v \in F_j(y) \setminus F_{j-1}(y)$.

Definition 7.9. A **backward filtration** of $\mathbb{R}^m$ is a nested collection of subspaces

$$\emptyset = B_0(y) \subset B_1(y) \subset B_2(y) \subset \cdots \subset B_m(y) = \mathbb{R}^m$$

such that

$$\lim_{k\to-\infty} \frac{1}{k} \log \|D\bar{f}^k(y)v\| = \chi_j$$

for $v \in B_j(y) \setminus B_{j-1}(y)$.

Suppose that forward and backward filtrations have been computed. Assume that one may determine computationally if a given $(m-1)$-dimensional subspace of $\mathbb{R}^m$ contains $T_y\phi(A)$. For $j = 1, \ldots, m$, compute the Lyapunov vector

$$v_j \in B_j \cap F_{m-j+1}.$$

We now fix j and determine if $v_j \in T_y\phi(A)$. If $\mathrm{Span}\{v_i : i \neq j\} \supset T_y\phi(A)$ then $v_j \notin T_y\phi(A)$. If $\mathrm{Span}\{v_i : i \neq j\} \not\supset T_y\phi(A)$ then $v_j \in T_y\phi(A)$ and χ_j is a true Lyapunov exponent. The true Lyapunov exponents and $T_y\phi(A)$ have been determined. It would be interesting to compare the performance of this algorithm to that of existing ERA techniques.

Proof. Statement (1) follows from the fact that ϕ is a diffeomorphism on A. We establish (2) with a perturbation argument. Let $\alpha > 1$ and let $d = \dim T_y\phi(A)$.

For each $z \in \overline{(y_k)}$ there exists an enveloping manifold M_z for $\phi(A)$ at z with $T_z M_z = T_z \phi(A)$ and $\dim(M_z) = d$. Let

$$\{B(z, r_z) : z \in \overline{(y_k)}\}$$

be a collection of metric balls such that

$$B(z, r_z) \cap \phi(A) \subset \mathrm{Int}(M_z).$$

By compactness there exists a finite subcover

$$\{B(z_i, \frac{r_{z_i}}{2}) : i = 1, \ldots, N\}$$

of $\overline{(y_k)}$. We inductively construct a sequence $\{D\bar{f}_k : k = 1, \ldots, N\}$ of perturbations of $D\bar{f}$. Let $\beta : \mathbb{R}^m \to \mathbb{R}$ be a C^∞ map such that

$$\begin{cases} (1) & 1 \leq \beta \leq \alpha, \\ (2) & \beta(z) = \alpha \text{ for } z \in B(z_1, \frac{r_{z_1}}{2}), \text{ and} \\ (3) & \beta(z) = 1 \text{ on } \mathbb{R}^m \setminus B(z_1, r_{z_1}). \end{cases}$$

For each $z \in B(z_1, r_{z_1}) \cap M_{z_1}$, $\mathbb{R}^m$ admits the orthogonal decomposition

$$\mathbb{R}^m = T_z(M_{z_1}) \oplus E_z.$$

Using this decomposition we define $D\bar{f}_1$ as follows.

(1) $D\bar{f}_1[\phi(A) \cap \mathbb{R}^m \setminus B(z_1, r_{z_1})] = D\bar{f}[\phi(A) \cap \mathbb{R}^m \setminus B(z_1, r_{z_1})]$
(2) For $z \in \phi(A) \cap B(z_1, r_{z_1})$, define $D\bar{f}_1(z)$ by

$$D\bar{f}_1(z)v = \begin{cases} D\bar{f}(z)v, & \text{if } v \in T_z(M_{z_1}); \\ \beta(z) D\bar{f}(z)v, & \text{if } v \in E_z. \end{cases}$$

In this fashion we inductively construct the family of perturbations $\{D\bar{f}_k : k = 1, \ldots, N\}$. For $v \in (T_y \phi(A))^\perp$ we have

$$\varliminf_{k \to \infty} \frac{1}{k} \log \|D\bar{f}_N^k(y)v\| \geq \lambda(y, v) + \log(\alpha).$$

Since $\alpha > 1$ was arbitrary, it follows that if $\lambda(y, v)$ is transverse then it is spurious. □

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