Learning to coordinate: A recursion theoretic perspective*

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Abstract

We consider two players each of whom attempts to predict the behavior of the other, using no more than the history of earlier predictions. Behaviors are limited to a pair of options, conventionally denoted by $0, 1$. Such players face the problem of *learning to coordinate choices.* The present paper formulates their situation recursion theoretically, and investigates the prospects for success. A pair of players build up a matrix with two rows and infinitely many columns, and are said to "learn" each other if cofinitely many of the columns show the same number in both rows (either 0 or 1). Among other results we prove that there are two collections of players that force all other players to choose their camp. Each collection is composed of players that learn everyone else in the same collection, but no player that learns all members of one collection learns any member of the other.

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1 Introduction

Sam and Sally like to meet daily in the park, pretending each time that it's yet another chance encounter, walking side by side in shy silence. Each shows up punctually at either noon or 6:00 p.m. hoping the other will have made the same choice. The shifting constraints on their schedules, however, make it hard to predict who will select which time of arrival, and both suffer disappointment when there is mismatch. So both Sam and Sally set about trying to predict the other's choices, desiring to act in concert. Their predictions are based on no more than the history of earlier events. For example, on the sixth morning, each might contemplate the matrix:

Sam:	noon	6:00	noon	noon	6:00
Sally:	noon	noon	6:00	6:00	6:00

Their separate decisions extend the matrix to a sixth column, which helps determine the choices made on the seventh morning, and so on without end. We can think of Sam's policy as a function that maps each such matrix (of any finite size) into the set $\{\text{noon}, 6{:}00 \text{ p.m.}\}$, and similarly for Sally. It is said that Sam "learns" Sally's policy — for short, "Sam learns Sally" — just in case he eventually begins to select arrival times that match Sally's forever after. (In such circumstances, we may also say that Sally learns Sam.) If Sam is clever enough, Sally could embody any of a wide range of policies without compromising ultimate success, and in this case we say that Sam learns the entire set of potential policies, even though only one of them will be embodied by Sally's dispositions.

Sam and Sally face the problem of *learning to coordinate choices*. The present paper formulates their situation abstractly and investigates the prospects for success. To keep matters simple, we consider only two players facing the same two options on each trial; the options are denoted 0 and 1. A player will thus be identified with a function from the set of all finite binary sequences into $\{0, 1\}$, where any such sequence is conceived as the history of moves of an opposing player. From a sequence of length n, a player can reconstruct the $2 \times n$ binary matrix that includes his own responses through move n. So it is not necessary to represent both rows of the matrix explicitly in players' inputs; just the opposing player's moves suffice. In the obvious way, a pair of players build up a matrix with two rows and infinitely many columns. The players are said to "learn" each other if the rows in cofinitely many of the columns agree (that is, both are 0 or both are 1). This conception of players and coordination is threadbare, but highlights the cognitive problem raised by repeated games between the same participants; each must discover a strategy that fits the other's play. A similar paradigm of learned coordination is raised in [Kelly, 1996, p. 267-8].

The foregoing paradigm will be cast in a recursion-theoretical framework, similar to the development of *Formal Learning Theory* [Jain *et al.*, 1999]. Within the latter tradition (and also in the paradigm to be developed here), the hypotheses of the learner are generated by a computational process that meets various constraints but need not be justified by recourse to probability. Discussion of the contrast between Formal Learning Theory and probabilistic approaches to induction is available in [Earman, 1992, Kelly, 1996, Martin & Osherson, 1998] and in references cited there.

After presentation of the coordination paradigm in the next section, we discuss some of its properties, including its relation to Formal Learning Theory. One of our theorem concerns the existence of two collections of players that force all other players to choose their camp. Each collection is composed of players that learn everyone else in the same collection. But no player that learns all members of one collection learns any member of the other. Other results concern cooperation by special classes of players, for example, those with limited memory in the sense that their present moves depend on a fixed number of immediately preceding stages of the game.

2 The paradigm

2.1 Notation

We use N to denote the natural numbers, 0, 1, 2, ... Let s be an infinite sequence over a set S (by which we mean, an ω-sequence over S). Then for $i \in N$, $s(i)$ is the value of s in the ith position (counting from 0), and $s[i]$ is the initial finite sequence in s of length i. The length of a finite sequence σ is denoted by $length(\sigma)$. The finite sequence of length zero is denoted by $\emptyset$. Given finite sequence σ and $i < length(\sigma)$, we denote by $\sigma(i)$ the element in σ's ith position (counting from 0). Thus, $(1, 0, 1, 1, 0)(0) = 1$ and $(1, 0, 1, 1, 0)(1) = 0$.

The set of all finite sequences over $\{0, 1\}$ is denoted by *BISEQ*. We do not distinguish between finite sequences of length one and their sole member. Concatenation among finite sequences is denoted by $*$.

2.2 Players and learnability

We now record the official definition of a "player," followed by a definition of the game played between two of them.

(1) DEFINITION: Any function from *BISEQ* into $\{0, 1\}$ is a *player.* (Players can thus be partial or total, computable or uncomputable.) A total function from *BISEQ* into $\{0, 1\}$ is called a *total player.*

Let players a, b be given. We define by induction two finite or infinite sequences over $\{0, 1\}$ to be denoted $R\langle a, b\rangle$ and $R\langle b, a\rangle$. The first may be called "a's response to b," and the second "b's response to a."

(2) DEFINITION: Suppose that for a given $n \in N$ both $R\langle a, b\rangle[n]$ and $R\langle b, a\rangle[n]$ are defined. Then $R\langle a, b\rangle(n) = a(R\langle b, a\rangle[n])$ and $R\langle b, a\rangle(n) = b(R\langle a, b\rangle[n])$.

Thus, for $n = 0$, the definition implies that $R\langle a, b\rangle(0) = a(\emptyset)$ and $R\langle b, a\rangle(0) = b(\emptyset)$. Observe that $R\langle a, b\rangle$ is an ω-sequence if and only if $R\langle b, a\rangle$ is too. They are both finite if at any stage one of them is undefined on the sequence produced at the prior stage by the other. Successful encounters between players give rise to "learning," as defined next.

(3) DEFINITION: Let player a and set A of players be given. We say that *a learns A* just in case for all $b \in A$, $R\langle a, b\rangle$ and $R\langle b, a\rangle$ are both ω-sequences and are almost everywhere identical. If some player learns A, then A is said to be *learnable,* otherwise *unlearnable.*

We say that "a learns b" in place of "a learns $\{b\}$." Observe:

(4) LEMMA: For all total players a, b,

(a) a learns a;

(b) If a learns b then b learns a.

However, "learns" is not an equivalence relation since it is not transitive. For example, define player a such that for all $\sigma \in$ *BISEQ*,

$$a(\sigma) = \begin{cases} 1 & \text{if } \mathit{length}(\sigma) > 0 \text{ and } \sigma(0) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that the constant 1 player learns a, and that a learns the constant 0 function. But the constant 1 function does not learn the constant 0 function.

2.3 Computable players

Let φ_i, $i \in N$ be an acceptable ordering of all partial recursive functions from N to N. (See [Machtey & Young, 1978] for background on acceptable orderings.) We fix a recursive isomorphism between N and *BISEQ*. Via the isomorphism we can conceive each φ_i as a partial recursive function from *BISEQ* to N. Going a step further, we conceive every positive number as a code for 1; the range of φ_i then collapses to a subset of $\{0, 1\}$. Under these conventions, the φ_i's list all of the *partial recursive players*, that is, all computable, possibly partial functions from *BISEQ* to $\{0, 1\}$. Since the remainder of our discussion concerns such players, we henceforth drop the qualifier "partial recursive," and abide by the following convention.

(5) CONVENTION: By "player" in what follows is meant "partial recursive player," in the sense given by coding *BISEQ* and $\{0, 1\}$, as described above. By "total player" is meant "total recursive player," i.e., a total computable map from *BISEQ* to $\{0, 1\}$.

3 Elementary facts about learnability

We now present a few elementary facts about learnability that will help to fix intuitions, and set the stage for later developments. For the first proposition, we recall that a collection $\mathcal{C}$ of learners is *uniformly recursive* just in case there is a total computable function f with $\mathcal{C} = \{\varphi_{f(i)} \,|\, i \in N\}$.

(6) PROPOSITION: Every uniformly recursive collection of total players is learnable. Moreover, the learner can be taken to be total.

Before proving the proposition, we provide a definition, that will also be useful elsewhere.

(7) DEFINITION: Let $\sigma \in$ *BISEQ* and player f be given. We define $\overline{f(\sigma)} \in$ *BISEQ* by induction on the length of σ. $\overline{f(\emptyset)} = f(\emptyset)$. Suppose that $\overline{f(\tau)}$ is defined for $\tau \in$ *BISEQ*, and that $x \in \{0, 1\}$ is given. Then $\overline{f(\tau * x)} = \overline{f(\tau)} * f(\tau * x)$.

Thus, if f is defined on all the initial segments of σ (which will be the case if f is total), then $\overline{f(\sigma)}$ is the finite sequence of 0's and 1's that f produces in response to the sequence σ. Notice that when $\overline{f(\sigma)}$ is defined, its length is positive. We denote by $\overline{f(\sigma)}-$ the result of removing the last (rightmost) digit from $\overline{f(\sigma)}$.

Proof of Proposition (6): Let total recursive $e : N \to N$ be such that for all $i \in N$, $\varphi_{e(i)}$ is total. We define by induction a player f that learns $C = \{\varphi_{e(i)} \,|\, i \in N\}$. Let $f(\emptyset) = \varphi_{e(0)}(\emptyset)$. Suppose that f is defined on all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \leq n$. Given $x \in \{0, 1\}$ and $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) = n$, let $i(\sigma) \in N$ be greatest with $i(\sigma) \leq n$, and $\overline{\varphi_{e(j)}(\overline{f(\sigma)-})-} \neq \sigma$ for all $j < i(\sigma)$. Set $f(\sigma * x) = \varphi_{e(i(\sigma))}(\overline{f(\sigma)})$. Clearly, f is total recursive.

To see that f learns C, let $g \in C$ be given. Suppose that $g = \varphi_{e(n)}$. Then it is easy to verify that for some $m \leq n$, $R\langle \varphi_{e(n)}, f\rangle(p) = R\langle \varphi_{e(m)}, f\rangle(p) = R\langle f, \varphi_{e(m)}\rangle(p)$ for cofinitely many $p \in N$. ■

The converse to Proposition (6) is false. Indeed:

(8) PROPOSITION: There is a learnable collection of total players that is contained in no learnable, uniformly recursive collection of total players.

To prove the proposition we rely on other useful definitions along with an obvious lemma.

(9) DEFINITION: Let $\sigma \in \mathit{BISEQ}$ and player f be given. We say that f *starts with* σ just in case for all $\tau \in \mathit{BISEQ}$ with $\mathit{length}(\tau) = m < \mathit{length}(\sigma)$, $f(\tau) = \sigma(m)$.

(10) DEFINITION: Let $\sigma \in \mathit{BISEQ}$ and total player g be given. Let player f be such that:

(a) f starts with σ;

(b) for all $\tau \in \mathit{BISEQ}$ with $\mathit{length}(\tau) \geq \mathit{length}(\sigma)$, $f(\tau) = 1 - g(\overline{f(\tau-)})$.

Then f is said to *disagree with g starting at σ*.

Of course, if player g is total, and f disagrees with g starting at $\sigma \in \mathit{BISEQ}$, then f is total also. The following lemma is obvious.

(11) LEMMA: Let $\sigma \in \mathit{BISEQ}$ and total player g be given. Suppose that player f disagrees with g starting at σ. Then g does not learn f.

Proof of Proposition (8): Given $n \in N$ with φ_n total, let a_n denote the (total) player that disagrees with φ_n starting at 1^n0. We claim that $\mathcal{C} = \{a_n \,|\, \varphi_n \text{ total}\}$ witnesses the proposition.

To see that $\mathcal{C}$ is learnable, let player p operate as follows. For as long as the input is an unbroken string of 1's, p puts out 1. If the first zero is encountered after receiving n 1's, then for all $\tau \in \mathit{BISEQ}$ with $\tau \supseteq 1^n0$, and for all $x \in \{0,1\}$, $p(\tau * x) = 1 - \varphi_n(\overline{p(\tau)})$. It is clear that if φ_n is total, then p learns a_n. (But p is not total: if n is an index for $\emptyset$, then p is defined on no sequence that extends 1^n0.) It is also clear that no total player learns $\mathcal{C}$ since each total φ_n fails to learn a_n. Hence $\mathcal{C}$ is contained in no learnable, uniformly recursive collection of total players since Proposition (6) implies that any such collection (and all of its subsets) is learnable by a total player. ■

As a corollary to the preceding proof we see that restricting attention to just the total players reduces the possibilities for cooperation. Indeed:

(12) Corollary: There is a learnable collection of total players that cannot be learned by any total player.

Proof: Let $\mathcal{C}$ be as defined in the proof of Proposition (8). As shown above, $\mathcal{C}$ is a learnable collection of total players but every total player fails to learn at least one $f \in \mathcal{C}$. ■

3.1 No strict improvement of the competence of total players

It is natural to consider player f to be more cooperative than player g if the set of players that f learns strictly includes the set that g learns. We shall now see that such a relation never obtains if f and g are total. To proceed, we rely on the following counterpart to Definition (10).

(13) Definition: Let $\sigma \in \mathit{BISEQ}$ and player g be given. Let player f be such that:

- (a) f starts with σ;
- (b) for all $\tau \in \mathit{BISEQ}$ with $\mathit{length}(\tau) \geq \mathit{length}(\sigma)$, $f(\tau) = g(\overline{f(\tau-)})$.

Then f is said to *agree with g starting at σ*.

Obviously, if player g is total and player f agrees with g starting at $\sigma \in \mathit{BISEQ}$, then f is total also. Equally obviously:

(14) Lemma: Let $\sigma \in \mathit{BISEQ}$ and total player g be given. Suppose that player f agrees with g starting at σ. Then g learns f.

(15) Proposition: For all distinct total players f, g, there is a total player h that f learns and that g does not. (Two total players are distinct if they have different values on a common binary sequence.)

Proof: Recall that for $\sigma \in BISEQ$ of positive length, $\sigma-$ is the result of removing the last (rightmost) digit from σ. Also recall that for all players f and $\sigma \in BISEQ$, $length(\overline{f(\sigma)}) > 0$. Let total players f, g be distinct. Then there is $\sigma \in BISEQ$ such that $f(\sigma) \neq g(\sigma)$. Choose (total) player h_0 that agrees with f starting at $\overline{f(\sigma)}$, and (total) player h_1 that disagrees with g starting at $\overline{g(\sigma)}$. Define player h such that:

(a) for all $\tau \in BISEQ$ with $length(\tau) = n < length(\sigma)$, $h(\tau) = \sigma(n)$;

(b) for all $\tau \in BISEQ$ such that $\tau \supseteq \overline{f(\sigma)}$, $h(\tau) = h_0(\tau)$.

(c) for all $\tau \in BISEQ$ such that $\tau \supseteq \overline{g(\sigma)}$, $h(\tau) = h_1(\tau)$.

(d) for all other $\tau \in BISEQ$, $h(\tau) = 0$.

Then h agrees with f starting at σ, so by Lemma (14) f learns h. On the other hand, h disagrees with g starting at σ, so by Lemma (11), g does not learn h. ■

(16) COROLLARY: Given player f, let $scope(f)$ be the class of players that f learns. There are no total players g, f such that $scope(g) \subset scope(f)$.

That is, the competence of a total player cannot be strictly improved. On the other hand, it is obvious that the competence of some partial players (e.g., the empty player) can be strictly improved (e.g., by the uniform 1-player).

4 Uncooperativeness

The following result shows that there can be two societies such that a child can learn to cooperate with either of them but not both. Indeed, any player who cooperates with one of the societies will be incapable of cooperating with *any* member of the other.

(17) PROPOSITION: There are two infinite sets A, B of total players with the following properties.

- (a) Every pair of players in A learn each other, and every pair of players in B learn each other. (That is, for all $a, b \in A$, a learns b, and for all $a, b \in B$, a learns b.)
- (b) No player that learns A learns any member of B, and no player that learns B learns any member of A. (That is, suppose that player c learns A. Then c learns no $b \in B$. Similarly, suppose that player c learns B. Then c learns no $a \in A$.)

Proof: We say that player a is "1-tempted" just in case there is a player c such that for cofinitely many $m \in N$, $R\langle a, c\rangle(m) = 1$. Similarly, we say that a is "0-tempted" just in case there is a player c such that for cofinitely many $m \in N$, $R\langle a, c\rangle(m) = 0$. (The sets of 1-tempted and 0-tempted players intersect, but are distinct.)

Let a_i, $i \in N$ enumerate all the 1-tempted players. (No assumption is made about the effective calculability of this enumeration.) We define by induction a sequence Z_i of players. It will be the case that:

(18) For all $i \in N$,

- (a) Z_i is not 1-tempted;
- (b) for all $j < i$, $R\langle Z_j, Z_i\rangle \neq R\langle a_i, Z_i\rangle$ (that is, Z_j does not appear to Z_i to be a_i);
- (c) for all $j < i$, $R\langle Z_i, Z_j\rangle \neq R\langle a_j, Z_j\rangle$ (that is, Z_i does not appear to Z_j to be a_j);
- (d) a_i does not learn Z_i;
- (e) for all players c, if c learns Z_i then c is 0-tempted.

Let $i \in N$ be given, and suppose that for all $j < i$, Z_j has been defined, and verifies (18) for j. We define Z_i and show that it also verifies (18).

The first part of the definition is designed to satisfy (18)b. Since a_i is 1-tempted, there is a player c such that $R\langle a_i, c\rangle(m) = 1$ for cofinitely many $m \in N$. By our induction hypothesis and (18)a, for all $j < i$, Z_j is not 1-tempted. So, for all $j < i$ there is $m(j) \in N$ with $R\langle Z_j, c\rangle[m(j)] \neq R\langle a_i, c\rangle[m(j)]$. It follows that there is $n_0 \in N$ such that for all $j < i$, $R\langle Z_j, c\rangle[n_0] \neq R\langle a_i, c\rangle[n_0]$. For all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \leq n_0$ we define $Z_i(\sigma) = c(\sigma)$. Hence, for all $j < i$, $R\langle Z_j, Z_i\rangle[n_0] = R\langle Z_j, c\rangle[n_0] \neq R\langle a_i, c\rangle[n_0] = R\langle a_i, Z_i\rangle[n_0]$. It follows immediately that for all $j < i$, $R\langle Z_j, Z_i\rangle \neq R\langle a_i, Z_i\rangle$, verifying (18)b.

The next part of the definition of Z_i is designed to satisfy (18)c. For all $j < i$ and all $\tau \in \mathit{BISEQ}$ of length $n_0 + j$, $Z_i(\tau) = 1 - a_j(\tau)$. It follows that for all $j < i$, $R\langle Z_i, Z_j\rangle[n_0 + j + 1] \neq R\langle a_j, Z_j\rangle[n_0 + j + 1]$, which directly implies (18)c.

To complete the definition of Z_i, let $\gamma \in \mathit{BISEQ}$ be given with $\mathit{length}(\gamma) = q_0 \geq n_0 + i$. Proceeding by induction we suppose that for all $\delta \in \mathit{BISEQ}$ with $\mathit{length}(\delta) < q_0$, $Z_i(\delta)$ is defined. So in particular [by Definition (2)], both $R\langle a_i, Z_i\rangle[q_0]$ and $R\langle Z_i, a_i\rangle[q_0]$ are defined. We distinguish two cases.

Case 1: $\gamma \neq R\langle a_i, Z_i\rangle[q_0]$. Then $Z_i(\gamma) = 0$.

Case 2: $\gamma = R\langle a_i, Z_i\rangle[q_0]$. Then if q_0 is even, $Z_i(\gamma) = 0$, and if q_0 is odd, $Z_i(\gamma) = 1 - a_i(R\langle Z_i, a_i\rangle[q_0])$.

Note that if $i = 0$, there is no $j < i$, hence $n_0 = i = 0$. Thus $Z_0(\emptyset)$ can be defined arbitrarily, and for the remaining binary sequences the construction proceeds according to Cases 1 and 2.

Let us verify (18)a,d,e. Cases 1 and 2 ensure that for all players b, $R\langle Z_i, b\rangle[2n] = 0$ for cofinitely many $n \in N$, hence Z_i is not 1-tempted, verifying (18)a. It is clear from case 2 that for infinitely many $n \in N$, $R\langle a_i, Z_i\rangle(n) \neq R\langle Z_i, a_i\rangle(n)$, which implies (18)d. For (18)e, let player c be given. If $R\langle c, Z_i\rangle = R\langle a_i, Z_i\rangle$ then case 2 implies that c does not learn Z_i. So, assume that $R\langle c, Z_i\rangle \neq R\langle a_i, Z_i\rangle$. It follows from case 1 that $R\langle Z_i, c\rangle(n) = 0$ for cofinitely many n. Hence if c learns Z_i, then $R\langle c, Z_i\rangle(n) = 0$ for cofinitely many n, in which case c is 0-tempted. This verifies (18)e. Finally, we observe that Z_i is total computable because for all $j \leq i$, a_j is total computable.

We take A of the proposition to be $\{Z_i \,|\, i \in N\}$. To verify (17)a with respect to A, let $i, j \in N$ be given with $i < j$. By (18)b, $R\langle Z_j, Z_i\rangle \neq R\langle a_i, Z_i\rangle$. Hence, case 1 implies that $R\langle Z_i, Z_j\rangle(n) = 0$ for cofinitely many $n \in N$. Similarly, by (18)c, $R\langle Z_i, Z_j\rangle \neq R\langle a_j, Z_j\rangle$. Hence, case 1 implies that $R\langle Z_j, Z_i\rangle(n) = 0$ for cofinitely many $n \in N$. It follows that Z_i learns Z_j. Relying on Lemma (4), we have thus demonstrated:

(19) For all $i, j \in N$, Z_i learns Z_j.

Now let b_i, $i \in N$ enumerate all the 0-tempted players. In fashion parallel to the preceding discussion we may define a sequence Y_i of players such that:

(20) For all $i \in N$,

- (a) Y_i is not 0-tempted;
- (b) for all $j < i$, $R\langle Y_j, Y_i\rangle \neq R\langle b_i, Y_i\rangle$ (that is, Y_j does not appear to Y_i to be b_i);
- (c) for all $j < i$, $R\langle Y_i, Y_j\rangle \neq R\langle b_j, Y_j\rangle$ (that is, Y_i does not appear to Y_j to be b_j);
- (d) b_i does not learn Y_i;
- (e) for all players c, if c learns Y_i then c is 1-tempted.

We take B of the proposition to be $\{Y_i \,|\, i \in N\}$. By the same reasoning as before, we have:

(21) For all $i, j \in N$, Y_i learns Y_j.

The first clause of Proposition (17) follows immediately from (19) and (21). For the second clause, let player c and $i \in N$ be given, and suppose that c learns $Y_i \in B$. We show that c does not learn $A = \{Z_i \mid i \in N\}$. By (20)e, c is 1-tempted. Hence there is $j \in N$ with $c = a_j$. So by (18)d, c does not learn Z_j, and thus does not learn A. The complementary case where c learns $Z_i \in A$ is parallel. ■

5 Players with special properties.

5.1 Three kinds of players

Further insight into coordination can be achieved by studying subsets of players who embody characteristics that might be seen in human players. In the present section we consider three such subsets. Intuitively, "forgiving" players are not deterred from coordination by the early moves of their partners, "blind" players pay no attention to their opponents, and "memory limited" players base their choices only on the recent history of the game. Formally, these ideas are defined as follows.

(22) DEFINITION: Two total players are *finite variants* (of each other) iff the symmetric difference of their graphs is finite. Let total player f be given.

- (a) f is *forgiving* just in case for all pairs g, h of total players that are finite variants of each other, f learns g iff f learns h.
- (b) f is *blind* just in case for all $\sigma, \tau \in \mathit{BISEQ}$ of the same length, $f(\sigma) = f(\tau)$.
- (c) Let $n \in N$ be given. f is n *memory-limited* just in case for all $\sigma, \tau \in \mathit{BISEQ}$ with length at least n, if the latest ("rightmost") finite sequence of length n in σ and τ are identical, then $f(\sigma) = f(\tau)$.

We now explore various properties of these kinds of players. To begin, it is shown that the blind and the forgiving form but one set of players.

5.2 All and only the blind are forgiving

(23) PROPOSITION: Every blind player is forgiving.

Proof: Suppose that player f is blind, that f learns total g, and that total g' is a finite variant of g. Because f learns g we may choose $m \in N$ such that:

(24) For all $p \geq m$, $R\langle f, g\rangle(p) = R\langle g, f\rangle(p)$.

Because g and g' are finite variants, we may choose $n > m$ such that for all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \geq n$, $g'(\sigma) = g(\sigma)$. Let $p \geq n \geq m$ be given. Then $R\langle f, g'\rangle[p] = R\langle f, g\rangle[p]$ by f's blindness. By the choice of p it follows that $R\langle g', f\rangle(p) = R\langle g, f\rangle(p)$. From the last two equations and (24) it follows that $R\langle f, g'\rangle(p) = R\langle g', f\rangle(p)$. ■

(25) Proposition: Every forgiving player is blind.

Proof: Suppose that player g is not blind. We will show that g is not forgiving. If g is not total then by Definition (22)a g is not forgiving and we are done. So assume that g is total. By g's non-blindness, there are $\sigma_0, \sigma_1 \in \mathit{BISEQ}$ such that $\mathit{length}(\sigma_0) = \mathit{length}(\sigma_1) = n$ and $g(\sigma_0) \neq g(\sigma_1)$. Choose player f_0 that agrees with g starting at σ_0, and player f_1 that disagrees with g starting at σ_1. Define total player h such that:

(a) for all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \leq n$, $h(\sigma) = \sigma_0(\mathit{length}(\sigma))$,

(b) for all $\sigma \in \mathit{BISEQ}$ with $\sigma \supseteq \overline{g(\sigma_0)}$, $h(\sigma) = f_0(\sigma)$,

(c) for all $\sigma \in \mathit{BISEQ}$ with $\sigma \supseteq \overline{g(\sigma_1)}$, $h(\sigma) = f_1(\sigma)$,

(d) for all other $\sigma \in \mathit{BISEQ}$, $h(\sigma) = 0$.

Then h agrees with g starting at σ_0, so by Lemma (14) g learns h. Now define total player h' such that:

(a) for all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \leq n$, $h'(\sigma) = \sigma_1(\mathit{length}(\sigma))$,

(b) for all $\sigma \in \mathit{BISEQ}$ with $\sigma \supseteq \overline{g(\sigma_0)}$, $h'(\sigma) = f_0(\sigma)$,

(c) for all $\sigma \in \mathit{BISEQ}$ with $\sigma \supseteq \overline{g(\sigma_1)}$, $h'(\sigma) = f_1(\sigma)$,

(d) for all other $\sigma \in \mathit{BISEQ}$, $h(\sigma) = 0$.

Then h' disagrees with g starting at σ_1, so by Lemma (11) g does not learn h' Since h and h' are finite variants, g is not forgiving. ■

Putting Propositions (23) and (25) together yields the identity of the blind and forgiving subsets of players.

Let f be the constant 1 function, let g be the constant 0 function. Then both f and g are blind. Plainly, no blind player learns $\{f, g\}$. We thus obtain as a corollary to Propositions (23) and (25):

(26) Corollary: There is a set of two forgiving players that no forgiving player learns.

5.3 Unlearnability of blind and forgiving players

Blindness (and hence, forgivingness) is such a debilitating property that it might be thought possible to learn the entire class of blind players (albeit, not necessarily blindly). This is not the case, however, as will be seen in the present subsection. We rely on the following fact, proved via a technique introduced in [Blum & Blum, 1975].

(27) Proposition: There are two learnable collections of blind players whose union is not learnable.

We proceed via a definition and some lemmas.

(28) Definition: Call a player f *blindly zero* just in case f is blind, and for all but finitely many $\sigma \in \mathit{BISEQ}$, $f(\sigma) = 0$. Let $\mathcal{Z}$ be the collection of all blindly zero players.

Clearly:

(29) Lemma:

(a) $\mathcal{Z}$ is learnable, and is composed of blind players.

(b) Every player that learns $\mathcal{Z}$ is total.

Now recall that the *standard enumeration* of an r.e. set W_i is the sequence of numbers that results from the kind of dovetailing construction described in [Rogers, 1987]. In particular, repetitions are allowed, hence some standard enumerations are infinite binary sequences. Let us demonstrate:

(30) Lemma: Let total player f be given. Then there is an infinite binary sequence s with the following properties:

(a) s starts with a sequence of form $1^m 0$,

(b) the standard enumeration of W_m is s; and

(c) for all $n \geq m + 1$, $s(n) = 1 - f(s[n])$.

Proof: Let total player f be given. Via a simple inductive definition it is easy to verify the existence of a total recursive $h : N \to N$ such that for all $k \in N$,

(a) the standard enumeration s of $W_{h(k)}$ starts with a sequence of form $1^k 0$;

(b) for all $n \geq k+1$, $s(n) = 1 - f(s[n])$.

By the Recursion Theorem there is $m \in N$ such that the standard enumerations of W_m and $W_{h(m)}$ are the same. Hence, the standard enumeration s of W_m starts with a sequence of form $1^m 0$, and for all $n \geq m+1$, $s(n) = 1 - f(s[n])$. ■

Now let total player f be given. Let s be the binary sequence guaranteed by Lemma (30). Define blind player b_f as follows. For all $\sigma \in \mathit{BISEQ}$, $b_f(\sigma) = s(\mathit{length}(\sigma))$. The following fact is clear.

(31) LEMMA:

(a) Let total player f be given. Then f does not learn b_f.

(b) Let F be any collection of total players. Then $\{b_f \mid f \in F\}$ is a learnable collection of blind players.

The foregoing lemmas in hand, we can finally prove Proposition (27).

Proof of Proposition (27): Let $\mathcal{C} = \{b_f \mid f \text{ learns } \mathcal{Z}\}$. By Lemmas (29)a and (31)b, each of $\mathcal{Z}$, $\mathcal{C}$ is a learnable collection of blind players. Let player f learn $\mathcal{Z}$. We must show that f does not learn $\mathcal{C}$. By Lemma (29)b, f is total. So by Lemma (31)a, f does not learn b_f. However, $b_f \in \mathcal{C}$ since f learns $\mathcal{Z}$. ■

Of course, it follows immediately that:

(32) COROLLARY: Neither the class of blind players nor the class of forgiving players is learnable.

5.4 Improving the blind competence of players

We return to the topic of Section 3.1, and consider improving the competence of players. Corollary (16) shows that no total player can be supplanted by another that learns a more inclusive set. If attention is limited to learning blind players, however, improvement is always possible. The matter is described in the following definition and proposition.

(33) DEFINITION: Given player f, let *b-scope*(f) be the class of blind players that f learns.

(34) PROPOSITION: For every player g there is a player f with *b-scope*$(g) \subset$ *b-scope*(f).

Proof: Let player f be given. Then by Corollary (32) there is blind $b \notin$ *b-scope*(f). Since b is blind, there is total recursive $b^* : N \to N$ such that for all $\sigma \in$ *BISEQ*, $b(\sigma) = b^*(\mathit{length}(\sigma))$. Define player g as follows. For all $\sigma \in$ *BISEQ*, $g(\sigma) = b^*(\mathit{length}(\sigma))$ if for all $k < \mathit{length}(\sigma)$, $\sigma(k) = b^*(k+1)$; otherwise, $g(\sigma) = f(\sigma)$. Relying on b's blindness, it is easy to see that *b-scope*$(g) =$ *b-scope*$(f) \cup \{b\} \supset$ *b-scope*(f). ■

Hence, competence for learning blind players can always be improved. However, this cannot be achieved uniformly recursively.

(35) PROPOSITION: There is no total recursive $h : N \to N$ with *b-scope*$(\varphi_i) \subset$ *b-scope*$(\varphi_{h(i)})$ for all $i \in N$.

Proof: By the Recursion Theorem, if there were such an h, then for some $m \in N$, $\varphi_m = \varphi_{h(m)}$. But then *b-scope*$(\varphi_m) =$ *b-scope*$(\varphi_{h(m)})$, contradiction. ■

The foregoing proposition shows there to be no algorithm $\mathcal{A}$ that accepts the program of an arbitrary player, and returns a program with enhanced ability to learn blind players. The nonexistence of such an $\mathcal{A}$, however, results from the requirement that it improve the competence not only of total players, but of strictly partial ones as well. If we are satisfied to increase the *b-scope* just of total players, then such improvement can be obtained algorithmically. This is revealed by the following contrast to Proposition (35).

(36) PROPOSITION: There is a total recursive $h : N \to N$ such that for all $i \in N$ with φ_i total, *b-scope*$(\varphi_i) \subset$ *b-scope*$(\varphi_{h(i)})$.

Proof: Suppose we are given $i \in N$ with φ_i total. Let b_i be the blind player that disagrees with f starting at $\emptyset$. It is easy to see that an index for b_i can be generated uniformly-recursively from i. Of course, $b_i \notin$ *b-scope*(φ_i). Given $\sigma \in$ *BISEQ*, define:

$$f(\sigma) = \begin{cases} b_i(\sigma) & \text{if} \quad \overline{b_i(\sigma)} = \sigma \\ \varphi_i(\sigma) & \text{otherwise} \end{cases}$$

Intuitively, f matches b_i's behavior for as long as the data appear to be generated by b_i. As soon as this strategy produces a mismatch, f reverts to using φ_i. Relying on the fact that b_i is blind, and that *b-scope*(φ_i) contains only blind players, it can easily be verified that *b-scope*$(f) =$ *b-scope*$(\varphi_i) \cup \{b_i\} \supset$ *b-scope*(φ_i). It is equally clear that a program for f can be generated uniformly recursively in i. ■

5.5 Memory-limitation

There remains part (c) of Definition (22), concerning players whose current choice depends on no more than the previous n moves (for some fixed n). Since an n memory-limited player is uniquely determined by its values on the finite set $\{\sigma \in \mathit{BISEQ} \,|\, \mathit{length}(\sigma) \leq n\}$, it is clear that there are only finitely many n memory-limited players. Let us now ask whether one member of this set learns all the others. The following proposition provides a negative answer.

(37) Proposition: Let $n \in N$ be given. No n memory limited player learns the (finite) set of all n memory limited players.

Proof: Let player f be n memory limited. Define player g inductively as follows. $g(\emptyset) = 1 - f(\emptyset)$. Suppose that g is defined for all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) \leq m$. Given $x \in \{0, 1\}$ and $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) = m$, define $g(\sigma{*}x) = 1 - f(\overline{g(\sigma)})$. Plainly, g is n memory limited since f is. It is equally clear that for all $p \in N$, $R\langle g, f\rangle(p) \neq R\langle f, g\rangle(p)$, so f does not learn g. ■

6 Learnability versus identifiability

We now consider the relation between learning to coordinate with another player versus inferring the program that drives her choices. The latter topic is studied within Formal Learning Theory, where program-discovery is called "identification." After presenting the identification paradigm, we prove some propositions that separate program discovery from learning to coordinate. For simplicity in what follows, we restrict attention to total players.

6.1 Definition of identifiability

Recall our fixed, computable isomorphism between BISEQ and N. It induces a computable enumeration β_i, $i \in N$, of BISEQ. Given $j \in N$ and total player g we denote by $g[j]$ the finite sequence $\langle \beta_0, g(\beta_0)\rangle \ldots \langle \beta_{j-1}, g(\beta_{j-1})\rangle$ (so $g[0] = \emptyset$). Thus, $g[j]$ provides partial information about the graph of g, hence about the input-output behavior of any program that implements g. Let $\mathit{GSEQ} = \{g[j] \,|\, j \in N \text{ and } g \text{ is a total player }\}$. So GSEQ includes all potential data about the input-output behavior of any player.

(38) Definition: Let total player g and computable $\psi : \mathit{GSEQ} \to N$ be given. We say that ψ *identifies* g just in case there is $n \in N$ such that:

(a) $g = \varphi_n$;

(b) for cofinitely many $j \in N$, $\psi(g[j]) = n$.

We say ψ *identifies* a collection P of total players just in case ψ identifies every $g \in P$. In this case, P is said to be *identifiable*. If no computable $\psi : GSEQ \to N$ identifies P, then P is said to be *unidentifiable*.

The definition is based on [Gold, 1967]. See [Jain *et al.*, 1999] for its analysis and elaboration within Formal Learning Theory.

Notice how different identification is from cooperation. In the identification paradigm the learner is shown the entire graph of the function to be identified, whereas in cooperation only that part of the graph responding to the learner is made manifest. The requirements for identification are commensurably higher than for cooperation. In the former, the learner must stabilize to a program that computes the entire graph of the presented function; for cooperation it suffices to predict the function's reaction to the learner's own behavior. These differences are connected to the symmetrical character of cooperation compared to the potential asymmetry of identification. Thus, one learner might identify the graph of another without the converse obtaining, whereas Lemma (4)b guarantees symmetry in cooperation. The fundamental distinction is that neither player is passive in the cooperation paradigm (unlike identification); each reacts to the moves of the other.

6.2 Identifiability does not imply learnability

If a class C of players is identifiable, then it is possible for a single learner to discover a program for an arbitrary $f \in C$ by examining the choices f makes in different situations. Armed with such a program, f's successive moves can be predicted, and coordination thereby assured. Such a strategy cannot be fully implemented in a coordination game, however, since players only see a proper subset of the behavior of their partners. (This is because the game is never started over from the beginning.) As a consequence, identifiability does not guarantee learnability. Indeed:

(39) Proposition: There is an identifiable collection of total players that is not learnable.

Proof: Let player ψ be given. We specify a total computable function $h : N \to \{0, 1\}$ with the following properties.

(40) For all $i \in N$,

(a) $\varphi_{h(i)}$ is a total player;

(b) for every $\sigma \in BISEQ$ of length less than i, $\varphi_{h(i)}(\sigma) = 0$;

(c) for every $\sigma \in BISEQ$ of length i, $\varphi_{h(i)}(\sigma) = 1$;

(d) $\varphi_{h(i)}$ does not learn ψ.

To define h, let i be given. Then h constructs a program $h(i)$ such that:

(41) For all $\sigma \in BISEQ$,

(a) if $length(\sigma) < i$ then $\varphi_{h(i)}(\sigma) = 0$;

(b) if $length(\sigma) = i$ then $\varphi_{h(i)}(\sigma) = 1$;

(c) if $length(\sigma) \geq i$ then for all $x \in \{0, 1\}$, $\varphi_{h(i)}(\sigma{*}x) = 1 - \psi(\overline{\varphi_{h(i)}(\sigma)})$.

Since ψ is total computable, the existence of an h satisfying (41) is easy to verify. Moreover, it is clear that any such h meets the conditions in (40). By the Recursion Theorem, there is $n \in N$ such that $\varphi_{h(n)} = \varphi_n$. We have thus established:

(42) For every player ψ there is a total player f_ψ with the following properties.

(a) $f_\psi = \varphi_n$, where n is least such that for some $\sigma \in BISEQ$ with $length(\sigma) = n$, $f_\psi(\sigma) = 1$.

(b) ψ does not learn f_ψ.

Let $C = \{f_\psi \mid \psi$ is a player$\}$. It follows immediately from (42)b that no player learns C. On the other hand, (42)a renders trivial the identification of C. ■

One of the reviewers suggests an alternative proof to the foregoing proposition. Learning a class of blind players can be represented as extrapolating a class of binary functions (in the sense of predicting the next value cofinitely often). It is observed in [Blum & Blum, 1975, p. 129, Footnote 1] and [?] that there are identifiable classes of recursive binary functions that cannot be extrapolated. So there are identifiable classes of (blind) players that cannot be learned.

6.3 Learnability does not imply identifiability

We now demonstrate that being able to coordinate with a class of players does not presuppose the ability to discover their programs. This is the converse to Proposition (39).

(43) PROPOSITION: There is a learnable collection of total players that is not identifiable.

Our proof rests on yet another paradigm of identification, now defined. We rely on the usual notation: given $n \in N$ and total $h : N \to \{0,1\}$, $h[n]$ denotes the finite initial segment of length n in h.

(44) DEFINITION: Let total computable $h : N \to \{0,1\}$ and computable $\psi : \mathit{BISEQ} \to N$ be given. We say that ψ *discovers* h just in case there is $n \in N$ such that:

(a) $h = \varphi_n$;

(b) for cofinitely many $j \in N$, $\psi(h[j]) = n$.

We say that ψ *discovers* a collection $\mathcal{C}$ of total computable $h : N \to \{0,1\}$ just in case ψ discovers every $h \in \mathcal{C}$.

Thus, discovery is distinguished from identification [Definition (38)] by the domains of the target functions. In Definition (38), the domain is *BISEQ*, whereas in Definition (44) the domain is N.

(45) LEMMA: [Gold, 1967]: No computable $\psi : \mathit{BISEQ} \to N$ discovers the class of all computable $h : N \to \{0,1\}$.

Proof of Proposition (43): Let computable $h : N \to \{0,1\}$ be given. We define player χ_h as follows.

(46) (a) $\chi_h(\emptyset) = h(0)$.

(b) For all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) > 0$ and $\sigma(0) = 1$, $\chi_h(\sigma) = 1$.

(c) For all $\sigma \in \mathit{BISEQ}$ with $\mathit{length}(\sigma) > 0$ and $\sigma(0) = 0$, $\chi_h(\sigma) = h(\mathit{length}(\sigma))$.

In other words, χ_h responds with h if faced with a player who issues 0 at the start; otherwise, χ_h responds with 1's [except maybe at the first move, which is always $h(0)$.] Because h is total computable, it is clear that (46) succeeds in defining a total (computable) player. Let $\mathcal{C} = \{\chi_h \mid h : N \to \{0,1\} \text{ is total computable}\}$. It follows immediately from (46)b that the player which maps *BISEQ* uniformly to 1 learns $\mathcal{C}$. So it remains to show that $\mathcal{C}$ is not identifiable.

From (46)a,c it is clear how to recover an index of h from one for χ_h, and also how to generate the graph of χ_h from the graph of h. These facts imply that if $\psi' : \mathit{GSEQ} \to N$ identified $\mathcal{C}$, then ψ' could be converted into $\psi : \mathit{BISEQ} \to N$ that discovers the class of all computable $h : N \to \{0,1\}$, contradicting Lemma (45). ■

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