Journal for Research in Mathematics Education
1992, Vol. 23, No. 3, 194–222

LEARNING TO TEACH HARD MATHEMATICS: DO NOVICE TEACHERS AND THEIR INSTRUCTORS GIVE UP TOO EASILY?

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This article analyzes from several vantage points a classroom lesson in which a student teacher was unsuccessful in providing a conceptually based justification for the standard division-of-fractions algorithm. We attempt to understand why the lesson failed, what it reveals about learning to teach, and what the implications are for mathematics teacher education. We focus on (a) the student teacher's beliefs about good mathematics teaching, her knowledge related to division of fractions, and her beliefs about learning to teach; and (b) the treatment of division of fractions in the mathematics methods course she took. The student teacher's conception of good mathematics teaching included components compatible with current views of effective mathematics teaching. However, these beliefs are difficult to achieve without a stronger conceptual knowledge base and a greater commitment to use available resources and to engage in hard thinking than she possessed. Further, the mathematics methods course did not require the student teacher to reconsider her knowledge base, to confront the contradictions between her knowledge base and at least some of her beliefs, or to reassess her beliefs about how she would learn to teach. These findings suggest that mathematics teacher education programs should reconsider how they provide subject matter knowledge and opportunities to teach it, and whether and how they challenge student teachers' existing beliefs.

The assertion that knowledge related to subject matter is an essential component of teachers' professional knowledge is neither new nor controversial (Ball & McDiarmid, 1990b). Shulman and colleagues (e.g., Shulman & Grossman, 1988) have proposed that knowledge of subject matter for teaching consists of two overlapping knowledge domains: subject matter knowledge and pedagogical content knowledge. Their conceptualization has served as a framework for much of the current research on teacher knowledge (Ball & McDiarmid, 1990b), including our own.

This research was supported in part by the National Science Foundation under Grant No. MDR 8652476. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

We would like to thank the numerous school division and university personnel who participated in this research project. We extend special thanks to the eight novice teachers who opened their lives to us and spent countless hours talking with us about their teaching. Without their support and commitment, the project would not have been possible.

We also wish to acknowledge the graduate students who assisted with data analysis: Susan Hagen at Virginia Tech and Althea Pearlman at the University of Colorado.

Despite this generally accepted conceptual framework, researchers do not agree on the elements of knowledge that are essential for effective subject matter teaching. Further, we do not know what impact limitations in teachers' subject matter knowledge and pedagogical content knowledge have on their ability to teach effectively. Nor do we understand well the role that teacher preparation programs play in the development of teachers' knowledge in these domains (Ball & McDiarmid, 1990b). These questions were central concerns in the study that we drew on in this article. Primary goals of our study were to describe novice teachers' emergent knowledge, beliefs, thinking, and actions related to the teaching of mathematics; to understand the interdependence and mutual influence of these components of teaching and learning to teach; and to examine the impact of teacher education experiences on the process of learning to teach.

In this article, we examine one student teacher's knowledge related to a single topic in the elementary and middle school mathematics curriculum—division of fractions. Our analysis focuses on a classroom lesson in which Ms. Daniels (all names used for study participants are pseudonyms) was unsuccessful in responding to a student's request for a conceptually based justification for the standard division-of-fractions algorithm. We attempt to understand what occurred in that teaching episode and its implications for Ms. Daniels's development as a mathematics teacher by looking closely at two factors: (a) her own system of knowledge and beliefs related to division of fractions, and (b) the treatment of division of fractions in the mathematics methods course she took. We begin the article with a discussion of subject matter knowledge and pedagogical content knowledge for teaching mathematics. We then present the teaching episode and our analyses.

Subject Matter Knowledge and Pedagogical Content Knowledge

In recent years, much progress has been made toward identifying and describing components of the two knowledge domains most central to knowledge of subject matter for teaching: subject matter knowledge and pedagogical content knowledge. Subject matter knowledge includes knowledge of the key facts, concepts, principles, and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field (Grossman, Wilson & Shulman, 1989). In the area of mathematics, Ball (1988, 1991) suggests that the subject matter knowledge needed for teaching includes both knowledge *of* mathematics and knowledge *about* mathematics. She argues that to teach mathematics effectively, individuals must have knowledge of mathematics characterized by an explicit conceptual understanding of the principles and meaning underlying mathematical procedures and by connectedness—rather than compartmentalization—of mathematical topics, rules, and definitions. A person must also have knowledge about the nature and discourse of mathematics and an understanding of what it means to know and do mathematics.

Grossman et al. (1989) include beliefs about the subject matter as another component of subject matter knowledge. They suggest that "...teachers' beliefs about the subject matter, including an orientation toward the subject matter, contribute to the ways in which teachers think about their subject matter and the choices they make in their teaching" (p. 27). Pedagogical content knowledge, or subject-specific pedagogical knowledge, consists of an understanding of how to represent specific topics and issues in ways that are appropriate to the diverse abilities and interests of learners. Two critical components are a knowledge of representations and a subject-specific knowledge of learners. Regarding these two components, Shulman (1986) notes that pedagogical content knowledge includes

> ...for the most regularly taught topics in one's subject area, the most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing the subject that make it comprehensible to others....[It] also includes an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to learning. (p. 9)

McDiarmid, Ball, and Anderson (1989), like Shulman, see instructional representations as central to the task of teaching subject matter. These authors also point out the dependence of pedagogical content knowledge on subject matter knowledge. "[T]o develop, select, and use appropriate representations, teachers must understand the content they are representing, the ways of thinking and knowing associated with this content, and the pupils they are teaching" (p. 198).

The National Council of Teachers of Mathematics (NCTM) acknowledged the role of subject matter knowledge and pedagogical content knowledge in effective mathematics teaching and targeted both knowledge domains as important elements in the preparation of mathematics teachers. For example, both domains are explicitly addressed in the set of standards for the professional development of teachers of mathematics presented in the *Professional Standards for Teaching Mathematics* (NCTM, 1991). Standard 2, "Knowing Mathematics and School Mathematics," states that "The education of teachers of mathematics should develop their knowledge of the content and discourse of mathematics, including mathematical concepts, procedures, and the connections among them...." (p. 132). Standard 3, "Knowing Students as Learners of Mathematics," suggests that "the preservice and continuing education of teachers of mathematics should provide multiple perspectives on students as learners of mathematics...." (p. 144). Standard 4, "Knowing Mathematics Pedagogy," indicates that "the preservice and continuing education of teachers of mathematics should develop teachers' knowledge of and ability to use and evaluate...ways to represent mathematics concepts and procedures...." (p. 151).

Subject matter knowledge and pedagogical content knowledge were also major foci in our Learning to Teach Mathematics research project. In keeping with Ball's arguments and the recommendations in the teaching standards, and in light of recent evidence regarding limitations in preservice teachers' knowledge of mathematical concepts (e.g., Ball, 1990a, 1990b), we were particularly interested in our participants' knowledge of mathematical concepts and their understanding of the connections among concepts and procedures. We conducted interviews to assess student teachers' subject matter knowledge and pedagogical content knowledge at several points in time during their final year of teacher preparation. We examined how

knowledge in these domains was addressed in the mathematics methods course. And we analyzed the student teachers' classroom teaching for evidence of their own conceptual and procedural knowledge of the topics they taught, their knowledge of student understanding, and their ability to generate appropriate representations for teaching procedures and concepts.

The Teaching Episode

The lesson that serves as the focal point of this article occurred on 20 April 1989 in the sixth grade classroom where Ms. Daniels was student teaching. It took place during "morning math," a time set aside by Mr. Blake (the cooperating teacher) for reviewing mathematics skills learned during the year, in preparation for the Survey of Basic Skills tests that were administered throughout the school division in early May. Morning math sessions typically began at 8:30 a.m. and were 20 to 30 minutes in length.

Ms. Daniels planned to focus on review and practice of the division-of-fractions algorithm. As she explained to the researcher after the lesson, "I hadn't planned to have to give an explanation because I figured since it was a review, that, you know, we'd just have to review the process of it." At the beginning of the session, pupils worked independently for approximately 15 minutes on eight practice problems on subtraction and multiplication of fractions that Ms. Daniels had written on the board. She then reviewed the division-of-fractions algorithm using the problem 3/4 divided by 1/2 as an example. She wrote on the board as she explained:

> "OK, you keep your first term the same. OK, 3/4 remains 3/4. When you divide fractions, it says well, to divide fractions we have to change the operation to multiplication and then flip or invert the second number. Not the first one, the second one. You look at the sign. It says change it to multiplication, and the number after the sign is the number that you invert. Does that make sense? OK, then it is just a matter of multiplication. Does anybody have problems with that part? Multiplication is very simple. You just multiply your two numerators together, 3 times 2 gives me what?"[1]

Ms. Daniels answered several procedural questions. Then Elise asked, "I was just wondering why, up there when you go and divide it and down there you multiply it, why do you change over?" Ms. Daniels immediately recognized Elise's question as calling for a conceptual explanation, and she attempted to respond by providing a concrete example and accompanying diagram:

> "Well, as you learned before, when you divide a fraction into a fraction, the process is to flip the second one and then multiply. And say we have a wall, OK, and we divide it into fourths. 1/4 of it is already painted, OK. So we have 3/4 of it left to paint. Right? You agree with me?" Ms. Daniels drew a rectangle on the front board, drew three vertical lines to divide it into four congruent parts, and shaded one part.

[1]In accord with traditional anthropological usage, we are using the following conventions throughout the article: (a) Extended excerpts from field notes are presented in block format, (b) verbatim statements by participants in the field notes are enclosed in quotation marks, (c) researchers' explanations and clarifications added to field notes when writing the paper are enclosed in brackets, and (d) extended quotes from interviews are presented in block format without quotation marks.

"But we only have enough paint to paint half of these three fourths. So half of 3/4 would be between about right there. Right, do you agree with that?" Ms. Daniels drew a line down the middle of the unshaded portion to divide it in half. Elise replied, "Yes." Ms. Daniels continued: "There is 1/4 on each side plus half of a fourth. So now if we look at this, this fourth was divided in half, so we divide this fourth in half and this fourth in half. We are left with 1, 2, 3, 4, 5, 6." [She drew vertical lines to divide each of the remaining unshaded fourths in half.] "And if we had this fourth divided in half, it would be what kind of unit?" [She drew a vertical line to divide the shaded fourth in half.] "How many units is my wall divided into now? 1, 2, 3, 4, 5, 6, 7, 8. But 2/8 is already covered. We see right here that we have enough paint to cover this many more eighths. Right? When we divide it into eighths, leaving us with how many eighths, 1, 2, 3. OK, oh wait. I did something wrong here."

Ms. Daniels realized that she had made an error. She paused for about 2 minutes, studying the board. She then decided to abandon the attempt to provide a concrete example, saying:

"Well, I am just trying to show you so you can visualize what happens when you divide fractions, but it is kind of hard to see. We'll just use our rule for right now and let me see if I can think of a different way of explaining it to you. OK? But for right now, just invert the second number and then multiply."

Ms. Daniels stood at the board, working on the division problem. The students were working independently, apparently on the problems they had been given at the beginning of the lesson. After a few minutes, Ms. Daniels walked over to Mr. Blake's desk and looked at the presentation of division of fractions in the teacher's manual. She said to the researcher (who was sitting at Mr. Blake's desk), "I just did multiplication." She did not indicate to the students that the example illustrated multiplication. Further, she did not attempt a correct representation on the following day.

For the remainder of the lesson, Ms. Daniels focused on computational procedures for division of fractions and related topics such as converting a mixed number to an improper fraction and visa versa. She demonstrated use of the algorithms and provided guided and independent practice. The morning math session lasted over one hour.

The interview with Ms. Daniels conducted prior to the lesson reveals that she did not think about representations to use in demonstrating division of fractions when planning the lesson. In fact, she did not plan to provide a conceptual explanation at all. As she explained, "I knew they had already had it [division of fractions] before, so I just figured the main thing was to make sure they remembered to invert and multiply when dividing." Her planning, which was done the morning of the lesson, consisted of selecting a few problems from the appropriate chapter reviews and chapter tests in the text to give to the students to solve.

In discussing the episode with the researcher later that day, Ms. Daniels explained that when faced with Elise's question, "I attempted to do something I had learned about...in the methods course, but it didn't work because I did the wrong thing.... The example I had given was multiplication." However, despite her realization that "the explanation...wasn't very good," she was basically pleased with the lesson. As

she explained, "I think by the end of the time, that they had picked up on it." Her major concern was that "I just spent too much time on it. I mean, as a result, I had to cut short my other lessons...." She elaborated, "As far as time-wise, you just don't have time to reteach every single thing."

For us, this episode raised many questions about what Ms. Daniels knew and believed about teaching division of fractions. How thorough was her own knowledge of division of fractions? What did she really believe about how the topic should be taught? Why did Ms. Daniels not later investigate the topic so she would be able to provide an explanation the next time someone asked Elise's question? How, if at all, did the mathematics methods course affect her subject matter knowledge and pedagogical content knowledge of division of fractions? And how can we explain what she did and did not learn about division of fractions in that course?

These questions are the focus of the analyses presented in this paper. In the sections that follow, we briefly describe the Learning to Teach Mathematics study that provided the episode and the data for our analysis of it. We then present our analyses of the various factors that seem to have contributed to what Ms. Daniels did and did not learn about division of fractions during her teacher preparation program. We end the paper with a discussion of numerous factors that would have to be overcome, in order for this teacher education program (and others that are probably similar to it) to accomplish the goals for preservice teachers' knowledge development specified in the *Professional Standards for Teaching Mathematics* (NCTM, 1991).

THE LEARNING TO TEACH MATHEMATICS STUDY

Research Design and Conceptual Framework

The Learning to Teach Mathematics study was designed to examine the process of becoming a middle school mathematics teacher by following a small number of novice teachers throughout their final year of teacher preparation and first year of teaching. As Figure 1 depicts, our primary goal was to describe and understand the novice teachers' knowledge, beliefs, thinking, and actions related to the teaching of mathematics over the 2-year course of the study. We drew heavily on Shulman's theoretical model of domains of teachers' professional knowledge (e.g., Shulman & Grossman, 1988) to develop the knowledge and beliefs component of the framework. Our conceptualization was modified by an initial review of a sample of the early data. We decided to investigate knowledge and beliefs related to the following domains: mathematics, general pedagogy, mathematics-specific pedagogy, mathematics curriculum, learners and learning, elementary school, middle school, learning to teach, teachers as professionals, and self as teacher. We examined the nature of participants' thinking during preactive, interactive, and postactive teaching (Jackson, 1968). We analyzed their teaching actions for patterns in lesson structure and in characteristics of lesson components such as explanations and representations. The double arrow between knowledge and beliefs (Box 1) and classroom thinking and actions (Box 2) in Figure 1 reflects our interest in exploring the interdependence and mutual influence of these components of teaching.

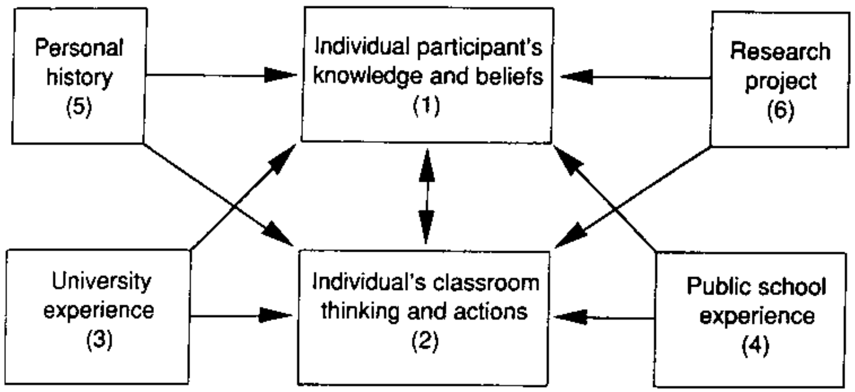

Figure 1. Becoming a middle school mathematics teacher.

Additional goals of the project were to describe and explain the contexts for learning to teach created by the novice teachers' university teacher education experiences (Box 3) and their experiences in the public schools (Box 4) where they student taught (Year 1) and held their first teaching jobs (Year 2). Because the university and the public school are the two major contexts for learning the culture and social organization related to learning to teach in most teacher preparation programs, we hypothesized that these contexts would be the major sources of external influence on the process of learning to teach. Secondary sources of influence were expected to be the novice teachers' personal histories (Box 5) and the research project itself (Box 6). (For a more in-depth discussion of the conceptual framework for the study, see Borko et al., 1990; Brown et al., in preparation; Jones et al., 1989.)

In this article, we focus on only a portion of the conceptual framework. We examine a single episode in one participant's student teaching experiences. We attempt to understand her thinking and actions (Box 2) in that episode by analyzing the episode in terms of her knowledge and beliefs (Box 1) and her experiences in the university teacher education program (Box 3). We focus on this portion of our framework because our analysis suggests that in Ms. Daniels's case, her existing knowledge and beliefs and the mathematics methods course had the most direct influence on how she taught mathematics and how she was learning to teach it.

Participants and Setting

Eight seniors in an elementary teacher education program at a large southern university participated in the first year of the project. All eight were members of a cohort of 38 students in a year-long senior year experience or model that included professional course work and student teaching. The model was specifically intended for preservice teachers interested in middle school teaching.

All eight participants had selected mathematics as an area of concentration (consisting of approximately 20 semester hours of course work in mathematics, statistics, and computer science) and indicated an intention to teach middle school mathematics on graduation. They were selected purposefully to represent diverse educational backgrounds and a range of competencies in mathematics. All eight were average or above average in their academic performance, compared to other students in the model. We also attempted to select participants consistent with the ethnic and gender makeup of the cohort. The participants included seven white females and one black female.

Ms. Daniels, the participant whose teaching is analyzed in this article, had the most extensive mathematics background of any of the student teachers in the program, having completed her first three years at the university as a mathematics major. Ms. Daniels maintained a C average through 2 years of calculus, an introductory course in mathematical proof, a first course in modern algebra, and four computer science courses. Like many mathematics majors, she hit the wall in a second modern algebra course and an advanced calculus course, receiving very low grades in these courses. Ms. Daniels reported "I got to my junior-level classes and ended up hating it. I thought about something else I would enjoy doing because I knew I would never enjoy math. So teaching is what I came up with." She was turned down by the secondary mathematics teacher education program because of her grades in mathematics and so decided to major in elementary education with a mathematics concentration. She expressed a preference to teach "something that is a little higher than beginning math—something like algebra." Although Ms. Daniels had completed more courses in advanced mathematics than other participants, she had the fewest courses related to elementary mathematics. Most other participants had completed a three-course sequence in Concepts in Mathematics that was developed specifically for elementary education majors. She had studied the content of the first two Concepts in Mathematics courses, those that dealt with number topics, and earned credit for them by examination. The route allowed her to miss opportunities provided by the courses to explore elementary number concepts and operations. She enrolled in the third course, which dealt with elementary geometric topics, and received a B in that course.

The design of the teacher education program called for each cohort member to have four different student teaching placements (7 weeks each; two each semester) in a city unified school district of approximately 15,000 students. During the first three placements, the cohort taught for half of the school day and took courses taught by university faculty; during the final placement, they taught the full school day. During the first 12 weeks of the academic year, mathematics, language arts, and reading methods courses were taught; during the second 12 weeks, courses in science and social studies methods and diagnosis were taught.

Ms. Daniels's first student teaching placement was in a self-contained sixth-grade classroom in an elementary school. Her second placement was in a second-grade classroom. Her third assignment was with a mathematics teacher in a junior high school. For her fourth placement, she returned to the sixth grade, but this time to another classroom in a different elementary school. The teaching episode occurred in her fourth placement.

Data Collection

Our conceptual framework guided the design of the data collection instruments and procedures. We relied primarily on interviews and observations to gather information pertinent to each component in Figure 1. These data were supplemented by questionnaires and written documents. All interviews were semistructured, based on protocols that we developed and piloted, audiotaped, and transcribed. During the observations, written notes were taken by the observer to record nonverbal communication as well as writing on the chalkboards and overhead projectors. Observations were also audiotaped. The audiotapes and written notes were used to prepare detailed field notes of the observations. Below, we identify primary data sources for the three components of the framework addressed in this paper.

Participants' knowledge and beliefs. The primary source of information about participants' knowledge and beliefs was a baseline interview[2] administered at the beginning, middle, and end of the school year. Open-ended questions, many of which were based on vignettes describing hypothetical classroom situations involving mathematics, were intended to elicit participants' knowledge and beliefs about mathematics, pedagogy, mathematics pedagogy, learning to teach, and other domains of teachers' professional knowledge and beliefs (e.g., Shulman & Grossman, 1988).

Division of fractions was one topic that received special attention in the interview. Participants were asked to compare the presentation of this topic in two sixth-grade textbook sections (provided by the interviewer), to describe how they would teach the topic to a sixth-grade class and how they would evaluate student learning, and to react to a hypothetical student's homework assignment on the topic. We probed with questions such as "What kinds of problems do pupils have with this material?" and "How could you tell if your students were getting it?". We asked participants how they would respond to a pupil who says:

> I know that when I'm supposed to divide two fractions, I have to turn one of the numbers upside down and multiply, but I don't know why all of a sudden it gets changed to multiplication, so I forget which one to turn upside down and I get a bunch of the problems wrong.

Interview data were supplemented by responses to a questionnaire[3] administered several days prior to each interview. One item on that questionnaire requested participants to select the appropriate story problem(s) to illustrate what 1 1/4 divided by 1/2 means.

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[2]The baseline interview is a modification of an interview developed by the National Center for Research on Teacher Education (NCRTE) at Michigan State University and was used with the Center's permission. See Ball and McDiarmid (1990a) for information about the original interview protocol and NCRTE (1988) for a description of the research program for which it was developed.

[3]The questionnaire is also a modification of a questionnaire developed by NCRTE and was also used with the Center's permission.

Another source of data was participants' written work for the mathematics methods course. A number of assignments and assessments required the participants to answer questions related to rational number concepts and operations.

University experience. To gather information about the university experience, we observed each session of the mathematics methods course, interviewed the instructor after each class session about his goals and objectives for the session and his reactions to it, and interviewed participants about their reactions to the course. We also interviewed the participants, their methods instructors, the university supervisors, and the teacher education program director about their overall impressions of the university's teacher education program. To supplement these data, we collected documents pertaining to the teacher education program and to the students' progress in it.

Classroom thinking and actions. To gather information about the novice teachers' thinking and actions in the classroom, we conducted week-long visits to each participant's class near the end of her first, third, and fourth student teaching placements. Primary data sources were daily observations of the participants' mathematics instruction, interviews about their planning for that instruction, and interviews asking for their reactions to the lessons and to specific lesson components (e.g., selected explanations, demonstrations, examples, and student activities). These data were supplemented by copies of written lesson plans, worksheets, and other handouts.

Data Analysis

We organized the data analysis into six strands, representing the six components of the conceptual framework, for the first stage of data analysis. Individual researchers or teams of researchers each assumed responsibility for coding and analyzing data related to a particular strand, in order to develop a description of that component of the conceptual framework.

For this paper, analysis for the classroom thinking and actions strand consisted of coding and examining all data related to the teaching episode in order to construct a description of Ms. Daniels's actions in that episode, as well as her planning, interactive thinking, and reflections about the lesson. That analysis resulted in the description presented in the section of the paper titled "The Teaching Episode."

For the knowledge and beliefs strand, we sorted the coded baseline data and methods course artifacts to identify data specific to division of fractions and closely related topics. We then analyzed these data from each of the three data collection cycles to build a picture of Ms. Daniels's knowledge and beliefs about division of fractions, how it is taught and learned, and how one learns to teach it. Analysis of the university experience for this paper consisted of coding all the university data, identifying major themes in these data (see Eisenhart, Behm, & Romagnano, 1991, for a summary of themes), and then reconsidering all occasions when multiplication or division of fractions came up in the methods class. The evidence pertaining to division of fractions was then placed in the context of the students' overall university experience of teacher education. Results of these analyses are presented in the following sections of the paper.

MS. DANIELS'S KNOWLEDGE AND BELIEFS ABOUT MATHEMATICS, PEDAGOGY, MATHEMATICS PEDAGOGY, AND LEARNING MATHEMATICS

We begin by examining Ms. Daniels's beliefs about what constitutes good mathematics teaching, her knowledge related to division of fractions, and her beliefs about learning to teach division of fractions, as revealed by our analysis of her responses to questions in the baseline interview and questionnaire and methods course artifacts. We find that her conception of good mathematics teaching included several components that are compatible with current views of effective mathematics teaching (e.g., NCTM, 1991) but are difficult to achieve when teaching division of fractions and probably impossible to achieve without stronger subject matter and pedagogical content knowledge than she possessed. In addition, Ms. Daniels's beliefs included certain ideas about her own subject matter knowledge and about how she would learn to teach topics that she had not yet mastered—ideas that seemed to block her access to what she needed to know in order to provide a conceptually based justification for the division of fractions algorithm.

Ms. Daniels's Beliefs About Good Teaching of Mathematics

Ms. Daniels, like many novice teachers, drew on her own experiences as a student in mathematics classrooms, her methods course experiences, and her experiences as a student teacher to develop her personal beliefs about the characteristics of good mathematics teaching (Britzman, 1986). Almost all Ms. Daniels's beliefs about good mathematics teaching were verbalized in the interviews we conducted before or shortly after the beginning of the year. These beliefs seemed to have their origins in her own experiences of school mathematics. As the year progressed, Ms. Daniels drew more heavily on her experiences in the mathematics methods course and her own teaching when discussing her beliefs. However, as we shall see, her later experiences did not cause her to change her beliefs nor to resolve the contradictions among them.

In general, Ms. Daniels believed that good mathematics teaching included primarily (a) making mathematics relevant for students and (b) making mathematics meaningful to students. Making mathematics relevant for students required teachers to incorporate into their lessons (a) applications of mathematics that students can use in their everyday lives, (b) applications of mathematics that students believe might be useful someday or to someone, and (c) mathematics-related activities that students enjoy. Making mathematics meaningful meant that students should be encouraged to "understand the math, not just know the process, but to understand the reasoning behind it and the logic of it." This was accomplished, in Ms. Daniels's opinion, primarily by (a) giving good explanations and (b) providing explanations "that are on the students' level."

Making mathematics relevant. In her earliest interviews with us, Ms. Daniels expressed the belief that good teachers make mathematics relevant through applications. Real-life applications had piqued her own interest in learning mathematics. She explained that she had always liked mathematics in the lower and middle grades because "...it makes sense,...and it has a practical use in real life. Look at all the things we use math for." She cited money and time as specific examples and stressed that almost everything in life relates to mathematics.

Ms. Daniels's experiences in the methods course seemed to reinforce her belief that good mathematics teachers use applications. She recalled a class discussion about teaching division of fractions and used it to support her belief:

> Like we talked about in the methods course, you want to make it relevant to them so you [give them] problems they can relate to much better than just cutting up a bunch of paper plates or something. I would try to present it to them in a way that they will know that the skill they're learning in this lesson will be of some use to them later.

Throughout the year, even after the methods course, Ms. Daniels was quite global in her discussion of applications and rarely volunteered specific examples. When pressed, she talked about general categories of applications, like shopping or science, but not specific problem situations. Further, in both the second and third baseline interviews, in seeming contradiction to her earlier assertions that mathematics is found in almost every activity in life, Ms. Daniels began to talk about how difficult it is in the mathematics classroom to relate mathematics to students' lives. For example, in response to a question about how she might relate mathematics to other school subjects, Ms. Daniels said:

> Math is the hardest subject to relate to other things....I haven't learned much about how you can make math a hands-on experience....It takes a lot of time and thinking to come up with activities that the kids will enjoy. It doesn't mean there aren't a lot of things you can do with it, because there are. I just don't know what they are yet.

Although the methods course did not, in Ms. Daniels's view, provide her with a repertoire of application-oriented, enjoyable activities, it did confirm her belief that making mathematics fun is a component of good and relevant teaching.

> In our math [methods] course we talked about how math can be really boring unless you make it—you have to make it fun because it's not just naturally going to be everyone's enjoyment.

Ms. Daniels, like most novice teachers, expressed a high level of anxiety about classroom management. The notion that students did not naturally enjoy mathematics served to strengthen her beliefs about the importance of integrating applications and other fun activities into her teaching. She believed that if she could maintain students' interest by making mathematics relevant and fun for them, she could win the management battle and encourage them to learn the mathematics.

Ms. Daniels's belief that good teachers should employ relevant applications—both to heighten students' interest in mathematics and to keep their attention focused on school work—compelled her to want to use applications in her teaching. But over time it appeared that she was not capable of providing, nor was she being provided, the applications she wished to have. A similar discrepancy between Ms. Daniels's beliefs and her knowledge base was evident in her attempts to make mathematics meaningful.

Making mathematics meaningful. Ms. Daniels's earliest statements about making mathematics meaningful were based on her own experiences as a student. In particular, she described former teachers who had provided good explanations—explanations that she could understand. These good teachers had also used games and other activities to keep students interested; they had made mathematics fun. But Ms. Daniels was quick to assert that, although it was important to have fun in mathematics class, fun was not sufficient. Teachers must also be able to explain math "in a way that everyone could relate to," and they must be able to tailor explanations to the level of different students.

Later in her student teaching year, Ms. Daniels began to articulate a belief that mathematics could be made meaningful by helping students to visualize mathematics, by providing "something they can actually see or touch to see why the process is the way it is." Although this belief did not show up strongly in the data until after the methods course, Ms. Daniels did not discuss any ways in which the methods course contributed to it. However, she never articulated what she really meant by visualizing; she simply asserted that visualization plays a prominent role in learning mathematics, especially in learning about fractions. In January she provided this example:

> Like if you divide 1/4 by 1/2, it will be 1/2. How would a child understand it? You have 1/4 and you divide it by 1/2 which is less than 1. It is kind of hard to visualize....I think the biggest problem is being able to visualize it, the process of what actually happens. Because if you can visualize it, then you can estimate. When you get your final answer, you'll know if that's in the right range or not.

Despite Ms. Daniels's belief in the importance of making mathematics meaningful, we found consistently that she could not provide an illustration of an explanation that students "could relate to" or one that would be on a particular student's level. As will be described in more detail in the next sections, Ms. Daniels's knowledge of what to do to make mathematics relevant and meaningful was superficial and fragmented, at least in the case of division of fractions.

Ms. Daniels's Knowledge and Beliefs Related to Learning and Teaching Division of Fractions

Implementation of Ms. Daniels's beliefs about good mathematics teaching would seem to require that the teacher have considerable knowledge of the mathematical content to be taught and an extensive repertoire of pedagogical tools. Over the course of her student teaching year, Ms. Daniels's knowledge and beliefs related to division of fractions changed somewhat. However, there is considerable evidence that her understanding of division of fractions remained quite superficial and that her repertoire of ways to apply or represent division of fractions (pedagogical content knowledge) was limited, even by the end of the student teaching year.

Ms. Daniels's entering knowledge and beliefs. In interviews with Ms. Daniels at the beginning of her student teaching year, we were unable to get her to talk about division of fractions in a meaningful way. When asked how she might try to explain division of fractions to a sixth-grade student, she responded after a long pause, "Every time I've had trouble with fractions, I make the little pie thing. I mean pictures are about the only way you can show someone how fractions work." She began to describe how she would use a picture to demonstrate division of fractions. However, she stopped before completing the description, saying she was not sure what she would do. When asked in the same interview how she might respond to a student confused about the invert-and-multiply algorithm for division of fractions, she responded:

> A good way to remember this problem would be the number you are dividing by is the number you want to invert. The number to the right of the sign is the one you flip and when you invert the number that means you invert the sign or change the sign to the opposite of division which is multiplication.

There is little evidence of conceptual understanding in either of these attempts at explanation. And, in fact, Ms. Daniels admitted, "I don't know why you invert and multiply, I just know that's the rule." However, she did express a desire to know more about this algorithm. We asked her why it might be important to understand the algorithm as well as be able to use it. Her response anticipates Elise's question.

> Because someone could very well ask me that question and I couldn't tell them why. I should know that. I mean, if they're trying to picture visually, that's really hard. I mean that's about the only way I've been able to see fractions until I've worked with them a lot. So...to understand how they're divided, I would have to think about it visually and that's really hard...to divide a part that's already been divided is very difficult to visualize.

Our analysis suggests that in spite of her extensive work in advanced mathematics, Ms. Daniels entered her student teaching year with only a rote understanding of division of fractions and no knowledge of representations that might enable her to teach the topic except by demonstration of the algorithm. These limitations would clearly make it difficult for her to implement her ideas about good mathematics teaching.

Ms. Daniels's knowledge and beliefs after the mathematics methods course. There is some evidence that during the student teaching year, Ms. Daniels's knowledge of division of fractions developed beyond simply knowing how to invert and multiply. However, although she certainly had more knowledge about division as a concept and some pedagogical tools for representing division of fractions, she continued to be unable to draw on this knowledge to construct coherent explanations or powerful representations, even away from the pressure of the classroom. Further, she seemed to be confused about the role that applications and representations could play in developing an understanding of the invert-and-multiply algorithm. The following examples illustrate both the strengths and the weaknesses of her developing knowledge system.

Ms. Daniels did seem to have an understanding of the measurement or quotative interpretation of division and could draw on that understanding when working with fractions. Frequently, when asked to talk about a problem like 1/2 divided by 1/4, she would restate it as "How many 1/4's are there in 1/2?" On an exam for the methods course she was asked to "solve the problem 1 divided by 5/8 using semiconcrete representations and show what would appear on the chalkboard." She solved the problem by first restating it as "How many 5/8's are in one?" and then drawing a number line representing 0 to 1. She divided the segment into eighths, marked off one section of 5/8 and then indicated that 3 eighths were left, representing 3/5 of what was needed for another 5/8.

Ms. Daniels also seemed to understand that multiplication and division are inverse operations. She drew on this knowledge when responding to the item in the baseline interviews about a child who expresses confusion about inverting and multiplying. However, on both occasions her responses seemed to turn into suggestions for remembering the algorithm, rather than coherent explanations of the algorithm. In January she responded:

> I would point out that the reason we turn one of the numbers upside down and multiply is because a reciprocal can also be called an inverse fraction. And division is the inverse operation of multiplication. And so you look at the first number and when you see that dividing sign and you know it's fractions, you turn that into a multiplication problem and since multiplication is the inverse operation of division, then you have to take that second number you see or your divisor and turn it over because you're doing the inverse to it as you would with the division sign.

We found that after the completion of the methods course Ms. Daniels described explanations of division of fractions that depended on applications and visual representations. However, these descriptions were quite global and, when pressed for details, she was often unable to respond. Further, when she did respond, her illustrations provided additional evidence of the limits of her knowledge of division of fractions. They consistently contained applications requiring, or at least suggesting, multiplication of fractions. For example, when asked how she might teach division of fractions to sixth-grade students, Ms. Daniels replied that she would present situations that would make it relevant to the students, rather than using paper plates like the textbook did. After providing a few examples of division situations, she would ask the students to demonstrate division for her, to "think of other ways we use dividing fractions in everyday life." When pressed to give an example she could use, Ms. Daniels presented a situation that suggested a multiplication problem.

> If you and 2 other people were running a relay and the total distance of the relay was a mile or 2/3 of a mile. And each of you had to run 1/4 of that. [How far would each person have to run?]

Later in the interview Ms. Daniels mentioned that students should be able to draw a diagram to indicate their understanding of division. When asked to provide a diagram for a problem she had generated, 1/4 divided by 1/2, she began to describe a diagram for multiplication based on a Cartesian product interpretation of multiplication. This diagram had been demonstrated in the methods course. She faltered in the middle of her description, and the interviewer asked her to diagram a simpler case of 2 divided by 1/2. She said, "I don't know…I mean, we didn't really do that [in our methods class]."

Ms. Daniels appeared to be drawing on explanations modeled for her in the mathematics methods class. However, her recollection of these explanations was only partial, and she seemed unable to use her own understanding of the content to construct appropriate or complete explanations.

Ms. Daniels also was confused about the role that applications and visualizations play in developing an understanding of the invert-and-multiply algorithm. At times, she asserted that applications enable students to understand why the division-of-fractions algorithm involves multiplication by the reciprocal of the divisor. That is, she suggested that visual representations can be used to derive the symbolic algorithm. In late April, she explained:

> Well, maybe if you were willing to feed the whole class you'd have to have probably about 2 or 3 pizzas. So if you said, "Well, how many 12ths are in three?" And each pizza was cut into 12 sections each. Just count how many pieces you have.…I mean they'd see that the pizza is divided into sections and they would see why you flip the second number.

In the same interview, however, she talked about how the visual and concrete methods of finding solutions to division-of-fraction problems support the division algorithm by showing students that the algorithm gives a correct solution. For example, she described how she might evaluate whether her students were "getting it":

> Maybe ask them to explain: "Tell me, tell it back to me why you flip the second number or how you visualize it." Maybe give them a problem, a division problem and have them come up with a story behind it or how you could use that in real life.…Just that using something visual they could show me how they got that answer. That would be to support when they flip the second number. Because if they didn't, then it's not going to turn out right and they're going to realize what they're doing.

This confusion may have been associated with an unrealistic expectation on Ms. Daniels's part that applications and visual tools can be used to derive or justify the standard algorithm for division of fractions. These visual tools are a valid means of verifying solutions obtained through use of the symbolic algorithm. However, no nonsymbolic representation suggested by Ms. Daniels leads to a derivation of that algorithm.

The examples given above, from interviews with Ms. Daniels after the classroom episode, suggest that having completed the mathematics methods course, having taught division of fractions, and approaching the end of her student teaching, she was still unable to provide a clear explanation of division of fractions. Although she used language that indicated she was drawing on her knowledge of division as the inverse of multiplication and that implied a measurement interpretation of division, her explanations continued to be thinly veiled suggestions for remembering the algorithm. Further, her recall of problem situations from the methods class suggests that her understanding of both multiplication and division of fractions was not strong.

Learning to Teach Division of Fractions

Ms. Daniels seemed to realize at the beginning of the year that she had a lot to learn about instructional representations for division of fractions. She also verbalized a desire to develop a knowledge of mathematics that went beyond simply knowing how and included knowing why. She expressed a belief that she could figure out explanations for some questions such as "why invert and multiply" for herself, by thinking hard and using visual representations to assist her thinking.

> You just have to take like 3/5 divided by 3/4 and look at it. Draw a picture of it, see how it turns out and then try to work it the opposite way. I mean, the normal way you would do with whole numbers and compare the visual example to the number you get and that should tell you right there.

However, she also looked to the mathematics methods course as a potential source for that knowledge. "Hopefully [it will] help me to have a deeper understanding myself of mathematics."

After completing the methods course, Ms. Daniels continued to express an interest in strengthening her repertoire of instructional representations. When asked in each baseline interview if there was anything she wished she knew more about in order to teach division of fractions, she consistently answered "More about coming up with examples of like 2 divided by 1/2. I don't know how to do that."

Over time Ms. Daniels spoke more and more about the role of practice and of sources such as the textbook in learning mathematical explanations and representations. Correspondingly, she spoke less about the role of the methods course or her own ability to generate explanations through hard thinking. As time passed, she seemed much less inclined to try and figure things out for herself through hard thinking. In January when asked how she might learn to give better explanations, she asserted:

> I just need more practical experience of being in front of the class and being put on the spot and just thinking on my feet....It's just going to have to come with a lot of practical experience. Maybe I could tutor one-on-one.

In April, she restated her belief in the role of practice in learning to teach. She commented:

> I'm not too adequate with fractions. I mean I know how to manipulate them but I don't know why. I guess I kind of know why after the math [methods] class, but I'm sure the more practice I have at teaching my weaker areas, that the more I'll understand them.

Regarding her beliefs about the role of the textbook, it is revealing to trace Ms. Daniels's reactions to the inclusion of information about typical student errors in the teachers' editions of the textbook sections on division of fractions. Before the student teaching year began Ms. Daniels's review of these sections included an almost angry response to the student error information. "If I'm going to teach math, I'm going to have to know it to the point that I can look at a student's paper and see what's wrong—not read in a book what errors could be!" By January she was positive about the student error section. "You'll know what to expect from your students, and you can think about how you would explain it in a different way rather than being put on the spot. It would make it a lot easier." In April, she recalled her failed explanation of why invert and multiply and commented,

> They prepare you for the errors that students might make and the different thoughts they might bring into it that would confuse them. This book does prepare you for that. Like I was talking about my explanation [of division of fractions in the episode highlighted earlier]. A lot of times I don't know how to explain it any other way because I'm not really sure why they get things wrong or what they're thinking. This book is a lot more geared towards that to prepare the teacher.

Thus the teachers' edition of the textbook had become a welcome source of information for Ms. Daniels.

We can conjecture that Ms. Daniels was disappointed by the mathematics methods course. She did not appear to have learned the examples and explanations she had hoped to. Her one course in methods of teaching mathematics behind her, she turned to her own practice and the textbook. Unfortunately, these sources typically offer little that might assist Ms. Daniels in developing her own knowledge base or in implementing her own beliefs about good teaching.

Good Teaching of the Division of Fractions: The Teaching Episode

Many elements of Ms. Daniels's knowledge and belief system seem to have come into play in the teaching episode. Elise's question provided Ms. Daniels with an opportunity to implement several of her beliefs about good mathematics teaching. In line with these beliefs, she constructed a problem situation intended to be an application of division of fractions with some degree of relevance to her students' lives. She explicitly said in the lesson that she was doing this so Elise "can visualize what happens when you divide fractions." However, limitations in Ms. Daniels's knowledge of division of fractions and of pedagogy related to division of fractions apparently hindered her in this attempt at good teaching. As was the case in her responses to baseline interview questions throughout the year, she was unable to come up with an appropriate representation and, instead, devised a representation for multiplication of fractions.

This experience might have led Ms. Daniels to go back to her notes from the methods course or to consult other sources in an attempt to learn an appropriate representation, in case the question ever arose again. However, in the baseline interview conducted several days after the classroom episode, we see that she did not, in fact, try to find or learn an appropriate representation. For the third time in the student teaching year, we asked Ms. Daniels how she might explain division of fractions to sixth-grade students. Her response was, again, a very vague and global description, one that used pizzas as a concrete example. She then tried to recall an activity from her mathematics methods course: "Maybe do like [the instructor] showed us how to fold the paper. I think that was—I never had looked at my notes yet. I think it was dividing fractions that he did that with."

In summary, then, we find that Ms. Daniels's beliefs about good mathematics teaching cannot be implemented in the case of division of fractions because she does not have a conceptual understanding of the topic. Further she cannot translate what is presented about division of fractions in the math methods course into her own knowledge base. Finally, her increasing belief in the value of practice and the textbook—to help her learn to teach the hard mathematics that she has not yet mastered—seems to be an obstacle that prevents her from realizing that she could and should figure out, using resources such as those provided by the methods course, a conceptually based justification for the division-of-fractions algorithm. In the next section, which analyzes the teacher education program, we suggest why that program served to reinforce, rather than disrupt, Ms. Daniels's existing beliefs and knowledge base.

THE INFLUENCE OF THE TEACHER EDUCATION PROGRAM

On the basis of the data gathered about the university program, we think that Ms. Daniels's inability to provide an appropriate response to Elise's question reflects a confluence of subtle linguistic and conceptual difficulties that occurred at the point of trying to teach and learn the invert-and-multiply rule. Although we think these difficulties occurred, at least in part, because of the nature of this algorithm (in comparison to the conceptually easier and previously taught algorithms for addition, subtraction, and multiplication), we expect that similar difficulties arise in the teaching and learning of other, particularly complex, mathematical algorithms.

One linguistic difficulty surrounded the meaning of "to provide an explanation for," that is, the meaning embedded in and the nature of the answer foreshadowed by the question, "Why do you invert and multiply?" On the one hand, there is a "usual everyday meaning" (Orton, 1987, p. 126) for "to provide an explanation for": to supply a clear statement of the reason(s) for. On the other hand, within the "mathematical register" (Pimm, 1987, Chapter 4), providing an explanation for algorithms can take the form of (a) supplying applications to real-world situations, (b) employing concrete or semi-concrete representations, or (c) manipulating symbols. Thus, depending on the meaning given to the question (why invert and multiply?), the answer might take one of four different forms. This instance of linguistic ambiguity was combined, in the case of division of fractions, with a shift in the way the mathematics methods instructor used manipulatives in his class. Further, at least some students, including Ms. Daniels (as revealed earlier in the discussion of her mathematics knowledge), did not have a good conceptual knowledge of division of fractions.

It is our contention that these overlapping difficulties went largely undetected in the mathematics methods course for three major reasons. First, the instructor felt considerable pressure to move quickly through material in order to cover everything necessary for teaching elementary and middle school mathematics in one 12-week course. Second, because of the student teachers' previous course work in mathematics, the instructor anticipated they would have already mastered the mathematics knowledge necessary to understand the linguistic and conceptual conventions and shifts he was using in the class. And third, many of the student teachers apparently did not have the command of verbal expressions or conceptual knowledge necessary to articulate their difficulties, nor did they think it was particularly important for them to develop such skills during their teacher education course work. This situation, which we describe in more detail below, made it possible for Ms. Daniels to complete her teacher education course work without gaining a better conceptual knowledge of division of fractions and without knowing how to explain the invert-and-multiply rule and thereby set her up for what occurred when she faced Elise's question.

The Mathematics Methods Course

Ms. Daniels recalled thinking back to the methods course when Elise asked her question. The same question had come up and been covered in the methods course during a portion of three hour-long lab periods devoted to multiplication and

division of fractions. But as Ms. Daniels later correctly remembered, the example she tried to use to help Elise had been used in the methods course to illustrate multiplication of fractions.

The activity Ms. Daniels remembered and tried to transfer to the unpainted fence problem—folding and shading a piece of paper—was introduced first thing in a Monday lab focused on a new topic, multiplication of fractions. This lab proceeded in what was by then (November) standard practice in the labs: First the problem (in this case, "1/4 of 1/2") was introduced; then the use of the manipulative was demonstrated by the instructor; next the students asked a few questions; then the instructor asked them to provide stories to go with the problem; finally he asked them to form groups to tell each other their stories and to demonstrate the manipulative themselves. This sequence was consistent with the instructor's stated intention to organize his presentation of mathematics content by initially modeling the target activity (both verbally and using a manipulative) and then converting the class to peer groups in which each student teacher would use words and the manipulative to teach the content to other group members.

From the beginning of the labs pertaining to multiplication and division of fractions, there were indications that at least some of the student teachers had difficulty finding a way to talk about the paper-folding activity so it could be used to provide an explanation of the multiplication-of-fractions algorithm. After the instructor told them to begin by folding the paper in half, a student asked, "How do you explain that you always do the second number first?" The response was, "If I want one fourth of one half, that says I have to have the half first." The student wondered aloud, "Do you think they [my students] will understand?" The instructor repeated his verbal description, but the students' difficulties persisted. They could translate the problem into a story, but they faltered when trying to use everyday language to explain (by which they seemed to mean: offering plain reasons for) why certain procedures (the algorithm) were appropriate to the problem and what the conceptual link was between the paper folding and the algorithm. The instructor's assistance did not seem to help them, and in their groups, they concentrated instead on the mechanics of folding the paper. At one point a student said, "I'd be terrified if something like this came up in my class." The students' problems seemed to increase with the move to division of fractions.

The instructor believed that the ground work for division of fractions had been laid during his earlier instruction of addition, subtraction, and especially multiplication. For addition and subtraction of whole numbers and fractions, he had talked through and illustrated the 1-1 correspondence between use of the manipulative and steps in the algorithm. This approach was extended for multiplication, as in talking about 3×4 as 3 sets of 4, $3 \times 1/2$ as 3 sets of 1/2, and $1/4 \times 1/2$ as 1/4 of 1/2, and then using manipulatives, such as paper folding, to demonstrate each example. For division of fractions, the instructor drew on the measurement interpretation of division to relate it to whole numbers. He continued by saying that 1 divided by 2/3 is the same as asking, "How many 2/3's are there in 1?" He demonstrated the problem by folding a strip of paper into thirds, tearing off two of the thirds, and showing that half of a two-thirds segment is left. Therefore, 1 divided by 2/3 is 1 1/2.

The instructor then began a discussion of the invert-and-multiply rule by presenting the complex fraction derivation of the rule for 1 divided by 3/8. He switched to symbols and formal mathematical language to explain that the derivation demonstrated that the algorithm produced a correct answer to any division problem. He suggested, but did not make explicit at the time, that there is no direct or concrete way to demonstrate, using a manipulative, the derivation of this algorithm. The instructor then presented several additional examples using the complex fraction derivation. When working the final example, 7/8 divided by 3/4, he pointed out that, having written the division problem as a complex fraction, he would—

> ...multiply both numerator and denominator by 4/3 because that would give him a denominator of 1. "The answer is in the numerator.... You can see that the 3/4 has been inverted." He then remarked that it seemed that some of the students were following this, while others were not. The instructor stressed that he was simply multiplying the original complex fraction by 1, and then renaming 1 in such a way that the denominators multiplied to get 1. "The final numerator looks like you inverted and multiplied." The instructor then noted that sixth graders don't follow this very well.
>
> "So when I show division of fractions, here's what I do. I get the basic concept down using subtraction. So that asking this question, 'How many 2/3's in 1?,' I can get them so that they see it's 1 1/2. And asking, 'How many 3/4's in 7/8?,' I can get kids to visualize and see that the answer is going to be between 1 and 2. Now once they have done some of that, then I have to go through this routine (complex fractions and multiplying to get 1 in the denominator). And some of the kids in the class, the bright ones who are good in math, will have a pretty good understanding of what's happening here. The rest of them, I just have to take it on faith. Because after I go through this kind of an exercise, what I end up doing is showing that this (7/8 divided by 3/4) is the same thing as this ($7/8 \times 4/3$)...."
>
> "And for those who understand it, it's great. For those who don't understand this (complex fraction manipulations), they're still left with two things. One is they're left with the conceptualization of what division means. And then they're also given a way to get an answer. And for some of them, these two things will be very poorly connected. They'll understand what this means (invert and multiply) and they'll know this gets the answer, but they really won't understand why this gets the answer."

From the instructor's standpoint, the student teachers should have been able to recognize the conceptual shift that was made when he moved away from the use of manipulatives to obtain a solution to the division-of-fractions problem and toward the complex fraction derivation of the standard algorithm. That is, they should have realized that in division of fractions, (a) there is no direct relationship between stories or concrete and semiconcrete representations of the measurement interpretation of division of fractions and the standard algorithm; (b) representations can be used to verify a solution obtained through use of the algorithm, but not to derive the algorithm; and (c) the derivation of the algorithm demonstrated by the instructor was not within the knowledge constraints of many young learners.

The student teachers' responses to the course material suggested that they did not recognize the conceptual distinction and may have been confused by the linguistic distinction implied in the instructor's approach to explanation. Their confusion was exemplified in their repeated requests for an "explanation" of the invert-and-multiply rule and their disorientation when the response took the form of verification. It seemed that the student teachers were indicating that they expected or wanted the stories and the concrete or semiconcrete representations to be the vehicles for providing explanations (reasons) for the correct algorithm. This seems to have been the same route that Ms. Daniels embarked on as she tried to answer Elise's question during the division-of-fractions review lesson.

The Teacher Education Program

Turning to the teacher education program as a whole, we found that Ms. Daniels and the other student teachers came to place more importance on learning activities that could be directly imported to their student teaching classrooms than on verbal, theoretical, or conceptual information covered in their university course work. We think that this preference for what the student teachers called "ideas that will work" was created by multiple demands placed on the student teachers by the design of the teacher education program. (A detailed description of the teacher education program and its demands is beyond the scope of this paper; however, interested readers may want to consult Eisenhart et al., 1991, for more information.)

In brief, the objectives and purposes of the teacher education program were many. From the university's standpoint, its responsibility was to give student teachers the cognitive and pedagogical skills they would need to be successful teachers. The public schools were expected to enculturate the student teachers into the norms of the school. And the student teachers were to take responsibility first for acquiring what was presented to them both by the university and the public schools and then for fashioning their own style and command of teaching. In this tangle of competing expectations, the student teachers had to find their own way. They had to take responsibility for the organization and use of their time; they had to solve their own problems of competing priorities and demands; they had to make their own decisions about what to do when and where; they had to plan and conduct their own lessons.

Given the multiple demands that they faced in the program and the fact that they had to student teach during a portion of every school day, the students simply did not have the time to construct carefully and reflect on their own set of classroom activities, routines, and strategies for each lesson they were called on to teach. Instead, they had to piece together classroom activities—activities they could use, often the very next day—from the ideas they gathered from their university professors, cooperating teachers, and peers. In other words, they needed "ideas that will work"—ideas or activities that could be imported, with little modification and almost immediately, into their own classrooms—in order to meet the requirements they and others set for them. Facing these situations of practice, the student teachers were most interested in techniques that would hold their students' attention and interest while simultaneously providing the necessary subject matter content. Further, they selectively focused on practically relevant ideas, ignoring other aspects (for example, the theoretical aspects) of their university course work. The following statement from Ms. Daniels illustrates the student teachers' frustration with the teacher education program as a whole.

> A lot of curriculum and instruction classes have been so much theory...and being in the [student teaching] classroom you really want to start applying what you're learning, but you can't because [the theory's] so cut and dry. It's only been in one or two of our classes that...the professor...gives us any ideas to use in the classroom.

One of the language arts instructors noted the same tendency among the student teachers.

> I don't like to teach...: "This is one method. This is another method. This is another method." I try to give them a thread of theoretical understanding....If they understand the theoretical...they can go beyond it. But it's real hard because the theory isn't nearly as compelling [to them] as the methods.... I don't know how well [I] did. I know that [the student teachers] love the strategies.

The Teaching Episode

In the teaching episode, Ms. Daniels needed just such an idea that would work. She apparently did not think she had the time to wait and think through an example to make sure it was correct before she started to answer Elise's question. She began in the way she had learned to begin in the mathematics methods course, that is, with an application or a concrete or semiconcrete representation. By the time she realized her example would not work, she had already devoted more time than she had anticipated or scheduled for the topic. Further, she apparently had no readily available alternate representation that she knew would work.

In hindsight it appears that what happened to Ms. Daniels in her classroom was constituted, at least in part, by her experiences in the methods course and the teacher education program. At the least, the university program and the student teachers' response to it did not create the conditions in which Ms. Daniels overcame the limitations of her knowledge about division of fractions. She did learn (or was encouraged to sustain her prior belief about) how she should begin an explanation (i.e., with an application or representation), but she did not learn the conceptual information she needed to complete the explanation or to answer Elise's question. She apparently did not have and did not acquire the words she needed, the mental picture she needed, or the conceptual knowledge she needed to produce quickly, and in front of a class, an adequate explanation. Given this situation, it does not seem surprising that when Ms. Daniels ran into problems with the example and out of time, she abandoned her efforts to provide a conceptual explanation for Elise's question and drilled her students on the algorithm instead.

CONCLUSIONS

In the introduction to this article, we raised a number of questions about Ms. Daniels's knowledge and beliefs related to teaching division of fractions. We now return to those questions, as we consider further the issue of why she was unsuccessful in responding to Elise's question and suggest possible implications of our explanations for teacher education reform.

Any explanation for why Ms. Daniels was unsuccessful at teaching the conceptual underpinnings of the division-of-fractions algorithm in the teaching episode must take into account the fact that division of fractions is a difficult topic to learn and to teach (Fendel, 1987). It is certainly not surprising that Ms. Daniels entered her final year of teacher preparation with a limited repertoire of instructional representations for division of fractions and limited knowledge of what students understand about the topic. Data from the Teacher Education and Learning to Teach (TELT) Study conducted by the National Center for Research on Teacher Education suggest that the majority of students entering elementary and secondary preservice teacher education programs are not able to select or generate appropriate representations for division of fractions (Ball, 1990a, 1990b).

From our perspective, an explanation that stops here is inadequate. At the time of the teaching episode, Ms. Daniels was different from the preservice elementary teachers in the TELT study in several important respects. Her background in mathematics was stronger, since she had successfully completed over 2 years of course work as a mathematics major. She had also successfully completed her mathematics methods course, in which the topic of division of fractions was explicitly addressed. Why, then, was she unable to provide a successful response to Elise's question? How can we explain what she did and did not learn about division of fractions in the mathematics methods course, or what knowledge she was and was not able to draw on when constructing an explanation? Our separate analyses of Ms. Daniels's knowledge and beliefs and of the influence of the teacher education program provide partial answers to this question. Here, we consider the two sets of analyses together, in order to provide a more comprehensive explanation.

Given Ms. Daniels's extensive course work in mathematics, why did she begin her final year of teacher preparation with a weak understanding of the meaning underlying at least some mathematical procedures that are a regular part of the sixth-grade curriculum? One answer to this question seems to lie in the nature of university mathematics courses. The mathematics courses taken by mathematics majors during their first 2 years of university study typically do not stress meaningful learning of mathematics. Rather, they emphasize rote learning of numerous computational techniques. Courses that do stress conceptual topics typically treat these topics at high levels of abstraction, encouraging rigorous proof. They assume some level of knowledge of rational number concepts and procedures, but do not address rational number topics that are central to the middle school mathematics curriculum (Committee on Mathematical Education of Teachers, 1991; National Research Council, 1991).

By taking advanced mathematics courses rather than courses designed specifically for preservice elementary teachers, Ms. Daniels probably had less opportunity than other novice teachers in her program to explore elementary number concepts and operations. It is likely that in her university studies of mathematics, her knowledge of rational number concepts was not challenged, nor was she given the opportunity to evaluate her own understanding of elementary mathematics or encouraged to construct explanations or representations of that understanding. In hindsight, it appears that by allowing Ms. Daniels to earn credit for the "Concepts in Mathematics" courses by examination, the university teacher education program gave up an opportunity to foster her acquisition of the subject matter knowledge necessary for teaching middle school mathematics, and it reinforced her belief that her knowledge of mathematics was sufficient for teaching mathematics in elementary and middle schools.

Perhaps a more perplexing question is why Ms. Daniels did not learn the conceptual information and representations that she needed to produce an adequate explanation of division of fractions during the mathematics methods course. Our analyses suggest several reasons. First, Ms. Daniels's prior belief in her own strong, or at least adequate, knowledge base probably interfered with her ability to recognize that she did not have the subject matter knowledge necessary to implement her beliefs about good teaching for at least some topics in the sixth-grade curriculum. The fact that Ms. Daniels tested out of some courses by examination and that the methods course instructor could (or had to) assume the student teachers' mastery of subject matter knowledge would further obscure her limitations.

This explanation is confirmed and extended by what the baseline data reveal about Ms. Daniels's knowledge of mathematics. Although Ms. Daniels believed that students should learn to understand (rather than just do) mathematics and that she should learn explanations and visual representations that she could use to foster meaningful learning, she herself seemed to know and have learned these explanations and representations by rote. Possibly, Ms. Daniels did not understand what it might mean to know mathematics, at least the mathematics of the division-of-fractions algorithm, any differently than she did. Her own success in K–12 and early university mathematics appears to have been the result of her success in rote learning of fairly complex mathematical procedures and her ability to apply these procedures in a variety of problem situations. If this was the case, then her own previous success may have contributed to a sense that rote knowledge is, at least in cases like division of fractions, enough to know. If so, the conceptual emphasis on the topic in the methods course would have seemed irrelevant.

Another reason why the methods course may have failed to facilitate Ms. Daniels's learning of the conceptual underpinnings of the division-of-fractions algorithm is that it did not clear up her apparent confusion about the role that applications and visualizations can play in developing an understanding of division of fractions. Although the topic was directly addressed, the course presented conflicting understandings and expectations regarding the nature of an explanation, especially in the case of division of fractions. In addition, the student teachers' selective attention to ideas from the methods course that could be applied directly to their own classrooms contributed to a situation in which Ms. Daniels was not likely to work very hard to develop her own conceptual understanding of the algorithm.

There is yet another perplexing question in this puzzle of Ms. Daniels's knowledge, beliefs, thinking, and actions. Why did she not investigate the topic of division of fractions, once her explanation failed, so that she might be better prepared to provide an explanation the next time Elise's question was asked? She did, after all, express concern about just such a question in her responses to the first baseline interview. And, she now had clear evidence that her fear that "someone could very well ask me that question and I couldn't tell them why" was well-founded. One

answer to this question seems to lie in Ms. Daniels's emerging belief that a person learns to teach mathematics through practice, a belief that is shared by many prospective teachers (McDiarmid, 1990).

This belief is troublesome for at least two reasons. First, it suggests that Ms. Daniels was not aware of the limitations of unexamined experience (i.e., simply doing without reflecting) as a tool for learning (Dewey, 1938), limitations revealed in her own case by the fact that she was not able to come up with a correct representation of division of fractions in the baseline interview that followed the teaching episode. Further, at least in part because of her belief in the power of practice, Ms. Daniels did not seem to feel that it was her responsibility to actively seek to improve her understanding of the mathematics she was teaching, either by consulting resources or by engaging in hard thinking of her own. The fact that she had not gone back to her notes from the math methods course or to any other source by the time of the final baseline interview, to look for an adequate instructional representation, confirms our fears regarding the negative impact that Ms. Daniels's belief about practice seemed to have on her learning to teach. Other reasons Ms. Daniels may not have pursued a conceptual explanation for division of fractions after the episode are (a) that she did not really understand the importance of such an explanation for fostering students' meaningful learning of division of fractions, and (b) that the demands she faced in the teacher education program compelled her to think ahead to her next lesson rather than to reflect on the one that was already over (Eisenhart et al., 1991).

Implications for Teacher Education

What are the implications of these findings for teacher education and mathematics education reform? What have we learned from analyzing, in-depth and from several vantage points, a single episode in one student teacher's learning-to-teach experience?

First, prospective teachers must be given the opportunity in their university course work to strengthen their subject matter knowledge. One recommendation offered in the teacher education reform literature is that the number of academic courses required for certification—elementary as well as secondary—be increased (e.g., Carnegie Forum, 1986; Holmes Group, 1986). Ms. Daniels's experience provides reason to question that recommendation. In fact, it provides support for the position voiced by Anderson (1989) that simply increasing the number of academic courses required for certification will not guarantee that prospective teachers acquire the subject matter knowledge they need for teaching. Academic courses, as they currently are taught, do not do a particularly good job of fostering such knowledge. In fact, evidence is mounting that students in mathematics courses can meet the expectations for satisfactory work without developing a conceptual understanding of the subject matter (Ball & McDiarmid, 1990b). What is needed, then, is improvement in the curriculum and instruction offered in university mathematics departments. Courses that focus on the conceptual development of important topics in elementary mathematics, such as rational number concepts and procedures, and that challenge prospective teachers' knowledge of these topics are needed. As the Committee on the Mathematical Education of Teachers (1991) suggests:

> In order for teachers to implement the curriculum envisioned by the NCTM Standards, they must have opportunities in their collegiate courses to do mathematics: explore, analyze, construct models, collect and represent data, present arguments, and solve problems. The content of collegiate level courses must reflect the changes in emphases and content of the emerging school curriculum and the rapidly broadening scope of mathematics itself. (preface, no page number)

Second, university course work must provide prospective teachers with the opportunity to strengthen their pedagogical content knowledge. One area of particular difficulty seems to be developing the concepts and language to draw connections between representations and applications on the one hand and algorithms and procedures on the other. Prospective teachers must have time and incentives in their teacher education programs to permit and encourage the kind of practice and reflection necessary for the development of these components of their professional knowledge base. Instructors must find ways to permit students to talk about and talk through their reasoning and their breakdowns with others who are more proficient and thus can model and assist them. University mathematics courses, as described above, could partially meet this need. However, methods courses, in conjunction with field experiences, should bear primary responsibility for supporting the learning of pedagogical content knowledge. In large classes, such as the 38-person methods class described in this article, even small groups of students working cooperatively may not have the opportunity to speak and to demonstrate their mathematical concepts to others who are more proficient and thus in a position to truly help them.

However, our analyses suggest that it may not be sufficient to simply provide these opportunities for prospective teachers in their mathematics methods classes. Given the competing demands and pressures they feel, prospective teachers selectively attend to some elements of what they are taught and ignore others. More specifically, "...prospective teachers do not see the relevance of much that they are taught. Without immediate need for the knowledge, they do not attend to it closely" (McDiarmid, 1990, p. 12).

For prospective teachers to take advantage of the opportunities we provide, we must find ways to challenge their fundamental beliefs about learning, teaching, and learning to teach. In the case of Ms. Daniels, key factors seem to be her beliefs about learning to teach and about her own knowledge of mathematics. Although several of her beliefs about mathematics learning (e.g., the importance of learning mathematics with understanding) and teaching (e.g., the role of applications and visualizations) were compatible with the spirit of mathematics education reform (e.g., NCTM, 1989), they could not be supported by her knowledge of mathematics and mathematics pedagogy. Further, Ms. Daniels's beliefs about the central role of practice in learning to teach may have inhibited any efforts to improve her knowledge except by practice. Ms. Daniels's belief in the adequacy of her own knowledge of mathematics was also a problem. In this case, the university's implicit

support of that belief, as evidenced by the teacher education program's permission for her to test out of the Concepts of Mathematics course and the mathematics methods instructor's assumptions about mathematical knowledge the students had mastered prior to the course, exacerbated the problem. A systematic effort to force Ms. Daniels to rethink her understanding of learning to teach, to reconsider her assessment of her own knowledge of mathematics, and to confront the contradictions between her beliefs and knowledge might have made her more receptive to the opportunities for learning provided in the mathematics methods course.

It will never be possible, within the constraints of a single mathematics methods course or even an entire preservice teacher preparation program, to enable prospective teachers to learn all that they need to know and believe about mathematics and mathematics pedagogy in order to teach effectively. However, by providing the kinds of experiences described above, teacher educators might be able to prepare teachers who can identify the constraints of their beliefs, the limits of their knowledge, and the restrictive demands of their situations, and who are equipped with the tools and attitudes that support the independent development of beliefs, knowledge, thinking, and actions long after the conclusion of their teacher preparation programs.

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