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LEAST-SQUARES FINITE ELEMENT APPROXIMATIONS TO SOLUTIONS OF INTERFACE PROBLEMS*

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Abstract. A least-squares finite element method for second-order elliptic boundary value problems having interfaces due to discontinuous media properties is proposed and analyzed. Both Dirichlet and Neumann boundary data are treated. The boundary value problems are recast into a first-order formulation to which a suitable least-squares principle is applied. Among the advantages of the method are that nonconforming, with respect to the interface, approximating subspaces may be used. Moreover, the grids used on each side of an interface need not coincide along the interface. Error estimates are derived that improve on other treatments of interface problems and a numerical example is provided to illustrate the method and the analyses.

Key words. least-squares finite element methods, interface problems

AMS subject classification. 65N30

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1. Introduction.

Least-squares finite element methods are the subject of much current interest; a small sample of the recent literature is given by [1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. The obvious advantages of this class of methods is that the discrete problems one must solve are symmetric and positive definite. However, the practicality of these methods is still not fully documented due to a lack of study of the behavior of the methods in the presence of "difficulties" arising from, for example, the use of low-order piecewise polynomial spaces, the application of mixed Dirichlet–Neumann boundary conditions, the discretization of nonconvex polygonal domains, and the need to conserve some global quantity such as mass. Some of these issues were addressed from a computational point of view in [12]. The purpose of this paper is to address another difficulty by defining and analyzing a least-squares finite element method for second-order elliptic equations with discontinuous coefficients; more specifically, we consider interface problems.

One of the first finite element methods (not of least-squares type) treating interface problems was proposed in [2], and a survey of finite element methods for such problems can be found in [3]. In [1], a least-squares method for the interface problem of Poisson equations is introduced after a general theory of the least-squares method has been developed. The authors of [1] were well aware that proving error estimates for the method is difficult, and therefore the weights they use in the terms related to the interface conditions cannot be rigorously justified. In this paper, following the approach of [8] and [13], we formulate the problem as a first-order system and then apply least-squares principles to this system. The theory of [8] can also be applied to the interface problem. However, the error estimate there requires that solutions be sufficiently smooth, which may not be true for interface problems. Least-squares finite element methods for interface problems are also considered in [15].

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To avoid global regularity requirements, we introduce two terms in the least-squares functional that are related to the conditions on the interface. Our error analysis shows that the method has nearly optimal order of accuracy with respect to an appropriately defined norm. The weights used for these terms are justified by the error estimate and are supported by our numerical experiments.

The paper is organized as follows. In the next section we introduce the problem and some necessary notations. An existence and uniqueness theorem is stated. Then, in section 3 we define and analyze the least-squares finite element method for the case of Dirichlet boundary conditions. A coercive property for the least-squares functional is proved and error estimates are obtained. In section 4 we extend the analyses to problems with inhomogeneous Neumann boundary conditions. Finally, in section 5, a computational example is presented.

In order to keep the exposition simple, our discussion is in the context of a single interface separating two subdomains in each of which the coefficients of the partial differential equations are "smooth." However, our algorithms and results extend in an obvious manner to problems with multiple interfaces and domains, so long as the assumed regularity results within the subdomains separated by the interfaces remain valid. In particular, we will assume that each of the subdomains has a "smooth" boundary or, in the very special situations for which this can be arranged, each has a convex boundary.

2. Statement of the problem. Assume that Ω is an open bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with smooth boundary. Ω_1 and Ω_2 are two open subsets of Ω such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. Let $\Gamma_0 = \partial\Omega$, $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$, $\Gamma_1 = \partial\Omega \cap \overline{\Omega}_1$, and $\Gamma_2 = \partial\Omega \cap \overline{\Omega}_2$. Here, Γ is referred to as the *interface*. Throughout, we assume that the subdomains Ω_1 and Ω_2 both have smooth (or in very special situtations, convex) boundaries. Smooth boundaries can occur, for example, if $\overline{\Omega}_1 \subset \Omega$, $\Omega_2 = \Omega - \overline{\Omega}_1$, and Ω_1 has a smooth boundary. Convex subdomains result, for example, if one subdivides a rectangle into smaller rectangles.

Consider the following elliptic boundary value problem on Ω:

$$-\mathrm{div}\,(A_i \nabla u_i) + c_i u_i = f_i \quad \text{in } \Omega_i,\; i = 1, 2\,, \tag{2.1}$$

$$u_i|_{\Gamma_i} = 0\,, \quad i = 1, 2\,, \tag{2.2}$$

$$u_1|_\Gamma = u_2|_\Gamma\,, \qquad \text{and} \qquad \big(A_1 \nabla u_1 \cdot \mathbf{n}_1\big)|_\Gamma + \big(A_2 \nabla u_2 \cdot \mathbf{n}_2\big)|_\Gamma = 0\,, \tag{2.3}$$

where $c_i \geq c > 0$ and $A_i = (a^i_{lk})$, $i = 1$, 2, and $l, k = 1, \ldots, n$, are $n \times n$ positive definite matrices so that, if λ^i_j, $j = 1, \ldots, n$, denote the eigenvalues of A_i, then there exist two constants C_a and C_b such that

$$0 < C_a \leq \lambda^i_j \leq C_b, \quad i = 1,\, 2\,,\; j = 1, \ldots, n\,.$$

The cases for which $c_1 = 0$ and/or $c_2 = 0$ may also be treated at the expense of greatly complicating the analyses. The constants appearing in our estimates will, in general, depend on C_a and C_b and, in particular, on the ratio C_b/C_a. In (2.3), $\mathbf{n}_i$ denotes the unit outer normal vector on Ω_i, $i = 1, 2$.

For $k \geq 0$, we denote by $H^k(\mathcal{D})$ the standard Sobolev space consisting of functions defined over the domain $\mathcal{D}$ and having square integrable derivatives of order up to k. For negative values of k, these spaces are also defined in the usual manner as appropriate dual spaces. In particular, $H^{-1}(\mathcal{D})$ is the dual space of $H^1_0(\mathcal{D})$, where the latter is the space of functions having one square integrable derivative with respect to $\mathcal{D}$ and that vanish on the boundary of that domain. Also, $\mathbf{H}^k(\mathcal{D}) = (H^k(\mathcal{D}))^n$ denotes the space of vector-valued functions, each of whose components belongs to $H^k(\mathcal{D})$. The standard Sobolev norm for functions belonging to $H^k(\mathcal{D})$ and $\mathbf{H}^k(\mathcal{D}) = (H^k(\mathcal{D}))^n$ is denoted by $\|\cdot\|_{k,\mathcal{D}}$.

For $k \geq 0$, define the Banach spaces

$$\dot{H}^k(\Omega) = \{u = (u_1, u_2) \quad | \quad u_i = u|_{\Omega_i} \in H^k(\Omega_i)\,, \;\; i = 1,2\}$$

with norm $|||u|||_k = \|u_1\|_{k,\Omega_1} + \|u_2\|_{k,\Omega_2}$ and

$$\dot{\mathbf{V}}^k(\Omega) = \{\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \quad | \quad \mathbf{v}_i = \mathbf{v}|_{\Omega_i} \in \mathbf{H}^k(\Omega_i)\,, \;\; i = 1,2\}$$

with norm $|||\mathbf{v}|||_k = \|\mathbf{v}_1\|_{k,\Omega_1} + \|\mathbf{v}_2\|_{k,\Omega_2}$. We may extend these definitions to $k = -1$; for example,

$$\dot{H}^{-1}(\Omega) = \{u = (u_1, u_2) \quad | \quad u_i = u|_{\Omega_i} \in H^{-1}_{\Gamma_i}(\Omega_i)\,, \;\; i = 1,2\}\,,$$

where $H^{-1}_{\Gamma_i}(\Omega_i)$ denotes the dual space of $H^1_{\Gamma_i}(\Omega_i) = \{u \in H^1(\Omega_i) \,|\, u = 0 \text{ on } \Gamma_i\}$, $i = 1, 2$. Note that, generally, $\dot{H}^k(\Omega) \not\subset H^k(\Omega)$ and $\dot{\mathbf{V}}^k(\Omega) \not\subset \mathbf{V}^k(\Omega)$. In particular, we will work with the space $\dot{H}^1(\Omega)$, which is generally not a subspace of $H^1(\Omega)$, so that approximations for $\{u \mid u|_{\Omega_i} = u_i\,,\; i = 1,2\}$ will be nonconforming in the sense that these approximations need not belong to $H^1(\Omega)$.

Let

$$H = \dot{H}^1(\Omega)\,, \quad \mathbf{V} = \dot{\mathbf{V}}^1(\Omega)\,, \quad \text{and} \quad H_0 = \{u \in H \;\mid\; u|_{\Gamma_0} = 0\}\,. \tag{2.4}$$

Along the interface Γ, let $[u]_\Gamma = u_1 - u_2$ and $[\mathbf{v}\cdot\mathbf{n}]_\Gamma = \mathbf{v}_1\cdot\mathbf{n}_1 + \mathbf{v}_2\cdot\mathbf{n}_2$. Let

$$H(\Gamma,\Omega) = \{u \in H \quad | \quad [u]_\Gamma = 0\}\,, \qquad H_0(\Gamma,\Omega) = H(\Gamma,\Omega) \cap H_0\,, \tag{2.5}$$

and

$$\mathbf{V}(\Gamma,\Omega) = \{\mathbf{v} \in \mathbf{V} \quad | \quad [\mathbf{v}\cdot\mathbf{n}]_\Gamma = 0\}\,. \tag{2.6}$$

Note that $H(\Gamma,\Omega) \subset H^1(\Omega)$. Also, define the Hilbert spaces

$$\mathbf{V}(\mathrm{div},\Omega_i) = \{\mathbf{v} \in \mathbf{L}_2(\Omega_i) = (L_2(\Omega_i))^n \quad | \quad \mathrm{div}\,\mathbf{v} \in L_2(\Omega_i)\}$$

with norm $\|\mathbf{v}\|_{\mathrm{div},\Omega_i} = \|\mathbf{v}\|_{0,\Omega_i} + \|\mathrm{div}\,\mathbf{v}\|_{0,\Omega_i}$ and

$$\mathbf{V}(\mathrm{div}) = \{\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}) \quad | \quad \mathbf{v}_i \in \mathbf{V}(\mathrm{div},\Omega_i)\}$$

with norm $|||\mathbf{v}|||_{\mathrm{div}} = \|\mathbf{v}_1\|_{\mathrm{div},\Omega_1} + \|\mathbf{v}_2\|_{\mathrm{div},\Omega_2}$.

Concerning the problem (2.1)–(2.3), we have the following result.

THEOREM 2.1. *Assume, for* $k \geq 1$, *that* $f \in \dot{H}^{k-2}(\Omega)$, $a^i_{lj} \in H^k(\overline{\Omega}_i)$, $i = 1, 2$, $l, j = 1, \ldots, n$, *and* $c_i \in H^k(\overline{\Omega}_i)$ *for* $i = 1, 2$. *Then, there exists a unique solution* $u \in \dot{H}^k(\Omega)$ *for* (2.1)–(2.3).

Proof. See [16]. ∎

3. Least-squares finite element approximations. We rewrite (2.1)–(2.3) as a system of first-order differential equations:

$$-\mathrm{div}\,(\mathbf{v}_i) + c_i u_i = f \quad \text{in } \Omega_i\,, \;\; i = 1, 2\,, \tag{3.1}$$

$$A_i \nabla u_i - \mathbf{v}_i = 0 \quad \text{in } \Omega_i\,, \;\; i = 1, 2\,, \tag{3.2}$$

$$u|_{\Gamma_0} = 0\,, \qquad [u]_\Gamma = 0\,, \qquad \text{and} \qquad [\mathbf{v}\cdot\mathbf{n}]_\Gamma = 0\,. \tag{3.3}$$

We introduce subspaces $H^h \subset H_0$ and $\mathbf{V}^h \subset \mathbf{V}$ parameterized by h, usually chosen to be some measure of the grid size such as the largest diameter of the triangles in a triangulation of Ω. Note that H^h need not be a subset of $H^1(\Omega)$ so that in this sense our method is nonconforming.

We assume that the subspaces H^h and $\mathbf{V}^h$ possess the approximation properties

$$\inf_{u_h \in H^h} \|u - u_h\|_{k,\Omega_i} \le C h^{s-k} \|u\|_{s,\Omega_i} \quad \forall\, u \in \dot{H}^s(\Omega)\,,\; u^h \in H^h\,,\; i = 1, 2$$

and

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{v} - \mathbf{v}^h\|_{k,\Omega_i} \le h^{s-k} \|\mathbf{v}\|_{s,\Omega_i} \quad \forall\, \mathbf{v} \in \dot{\mathbf{V}}^s(\Omega)\,,\; \mathbf{v}^h \in \mathbf{V}^h\,,\; i = 1, 2\,,$$

where $0 < k < s$. As a result, we have that

$$\inf_{u_h \in H^h} |||u - u_h|||_k \le C h^{s-k} |||u|||_s \quad \forall\, u \in \dot{H}^s(\Omega)\,,\; u^h \in H^h \tag{3.4}$$

and

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} |||\mathbf{v} - \mathbf{v}^h|||_k \le h^{s-k} |||\mathbf{v}|||_s \quad \forall\, \mathbf{v} \in \dot{\mathbf{V}}^s(\Omega)\,,\; \mathbf{v}^h \in \mathbf{V}^h\,, \tag{3.5}$$

where $0 < k < s$. We also assume that the following inverse inequality holds in H^h: there exists a constant C such that for $u^h = (u_1^h, u_2^h) \in H^h$,

$$\|u_1^h - u_2^h\|_{1/2,\Gamma} \le \frac{C}{h^{1/2}} \|u_1^h - u_2^h\|_{0,\Gamma}\,. \tag{3.6}$$

Note that if the restrictions to Γ of the approximating spaces in Ω_1 and Ω_2 coincide, then the inverse property (3.6) is simply the inverse property in the usual sense.

3.1. The least-squares functional. We define a functional on $H_0 \times \mathbf{V}$ as follows. For $u \in H_0$ and $\mathbf{v} \in \mathbf{V}$, let

$$\begin{aligned} \mathcal{J}(u, \mathbf{v}; f) = &\sum_{i=1}^{2} (\| - \mathrm{div}\,\mathbf{v}_i + c_i u_i - f\|_0^2 + \|A_i \nabla u_i - \mathbf{v}_i\|_0^2) \\ &+ \frac{1}{h^{1+\epsilon_0}} \int_\Gamma [u]_\Gamma^2 \, d\Gamma + \frac{1}{h^{\epsilon_1}} \int_\Gamma [\mathbf{v}\cdot\mathbf{n}]_\Gamma^2 \, d\Gamma\,, \end{aligned} \tag{3.7}$$

where $f \in \dot{H}^0(\Omega)$ and $\epsilon_0, \epsilon_1 > 0$. Note that if $u \in H_0(\Gamma, \Omega)$ and $\mathbf{v} \in \mathbf{V}(\Gamma, \Omega)$, then the last two terms in (3.7) vanish. Also, note that if u and $\mathbf{v}$ satisfy (3.1)–(3.3), then $\mathcal{J}(u, \mathbf{v}; f) = 0$.

The functional $\mathcal{J}(\cdot, \cdot; \cdot)$ satisfies the following coercivity property.

PROPOSITION 3.1. *Let* $u_h = (u_1^h, u_2^h) \in H^h$, $\mathbf{v}^h = (\mathbf{v}_1^h, \mathbf{v}_2^h) \in \mathbf{V}^h$, $u = (u_1, u_2) \in H_0(\Gamma, \Omega)$, *and* $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}(\Gamma, \Omega)$. *Then, for* h *sufficiently small, there exists a constant* $C > 0$ *independent of* h *such that*

$$\mathcal{J}(u - u^h, \mathbf{v} - \mathbf{v}^h; 0) \geq C \left(|||u - u^h|||_1^2 + |||\mathbf{v} - \mathbf{v}^h|||_{\text{div}}^2 \right) . \tag{3.8}$$

Proof. Let

$$\widehat{\mathcal{J}}(u, \mathbf{v}) = \sum_{i=1}^{2} (\| - c_i^{-1/2} \text{div}\, \mathbf{v}_i + c_i^{1/2} u_i \|_0^2 + \| A_i^{\frac{1}{2}} \nabla u_i - A_i^{-\frac{1}{2}} \mathbf{v}_i \|_0^2) + \frac{1}{h^{1+\epsilon_0}} \int_\Gamma [u]_\Gamma^2 \, d\Gamma + \frac{1}{h^{\epsilon_1}} \int_\Gamma [\mathbf{v} \cdot \mathbf{n}]_\Gamma^2 \, d\Gamma .$$

It is easy to see that $\mathcal{J}(\cdot, \cdot; 0)$ and $\widehat{\mathcal{J}}(\cdot, \cdot)$ are equivalent, i.e., that there exist two positive constants C_1 and C_2 such that

$$C_1 \mathcal{J}(u, \mathbf{v}; 0) \leq \widehat{\mathcal{J}}(u, \mathbf{v}) \leq C_2 \mathcal{J}(u, \mathbf{v}; 0)$$

for all $u \in H$ and $\mathbf{v} \in \mathbf{V}$. Thus, it suffices to prove that

$$\widehat{\mathcal{J}}(u - u^h \,,\, \mathbf{v} - \mathbf{v}^h) \geq C \left(|||u - u^h|||_1^2 + |||\mathbf{v} - \mathbf{v}^h|||_{\text{div}}^2 \right)$$

for some constant $C > 0$. Now, since $u \in H_0(\Gamma, \Omega)$ and $\mathbf{v} \in \mathbf{V}(\Gamma, \Omega)$, we have that

$$\begin{aligned} &\widehat{\mathcal{J}}(u - u^h \,,\, \mathbf{v} - \mathbf{v}^h) \\ &\quad = \sum_{i=1}^{2} \Big(\| - c_i^{-1/2} \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h) + c_i^{1/2} (u_i - u_i^h) \|_0^2 \\ &\qquad + \| A_i^{1/2} \nabla (u_i - u_i^h) - A_i^{-1/2} (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 \Big) \\ &\qquad + \frac{1}{h^{1+\epsilon_0}} \int_\Gamma |u_1^h - u_2^h|^2 \, d\Gamma + \frac{1}{h^{\epsilon_1}} \int_\Gamma ((\mathbf{v}_1^h - \mathbf{v}_2^h) \cdot \mathbf{n})^2 \, d\Gamma. \end{aligned}$$

Integrating by parts, one obtains, for $i = 1, 2$,

$$\begin{aligned} &\| - c_i^{-1/2} \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h) + c_i^{1/2} (u_i - u_i^h) \|_0^2 + \| A_i^{1/2} \nabla (u_i - u_i^h) - A_i^{-1/2} (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 \\ &\quad = \| c_i^{-1/2} \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 + \| c_i^{1/2} (u_i - u_i^h) \|_0^2 - 2 \int_{\Omega_i} \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h)(u_i - u_i^h) \, d\Omega \\ &\qquad + \| A_i^{1/2} \nabla (u_i - u_i^h) \|_0^2 + \| A_i^{-1/2} (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 - 2 \int_{\Omega_i} \nabla (u_i - u_i^h) \cdot (\mathbf{v}_i - \mathbf{v}_i^h) \, d\Omega \\ &\quad = \| c_i^{-1/2} \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 + \| c_i^{1/2} (u_i - u_i^h) \|_0^2 + \| A_i^{1/2} \nabla (u_i - u_i^h) \|_0^2 \\ &\qquad + \| A_i^{-1/2} (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 - 2 \int_\Gamma (u_i - u_i^h)(\mathbf{v}_i - \mathbf{v}_i^h) \cdot \mathbf{n} \, d\Gamma . \end{aligned}$$

Hence, for some constant $C_3 > 0$,

$$\begin{aligned} &\widehat{\mathcal{J}}(u - u^h, \mathbf{v} - \mathbf{v}^h) \\ &\quad \geq C_3 \sum_{i=1}^{2} \left(\| \text{div}\, (\mathbf{v}_i - \mathbf{v}_i^h) \|_0^2 + \| u_i - u_i^h \|_0^2 + \| \nabla (u_i - u_i^h) \|_0^2 + \| \mathbf{v}_i - \mathbf{v}_i^h \|_0^2 \right) \\ &\qquad + 2 \int_\Gamma (u_1 - u_1^h)(\mathbf{v}_1 - \mathbf{v}_1^h) \cdot \mathbf{n} \, d\Gamma - 2 \int_\Gamma (u_2 - u_2^h)(\mathbf{v}_2 - \mathbf{v}_2^h) \cdot \mathbf{n} \, d\Gamma \\ &\qquad + \frac{1}{h^{1+\epsilon_0}} \int_\Gamma (u_1^h - u_2^h)^2 \, d\Gamma + \frac{1}{h^{\epsilon_1}} \int_\Gamma \left((\mathbf{v}_1^h - \mathbf{v}_2^h) \cdot \mathbf{n} \right)^2 \, d\Gamma . \end{aligned}$$

By the definition of $H_0(\Gamma,\Omega)$ and $\mathbf{V}(\Gamma,\Omega)$, trace theorems, and the inverse property (3.6) on H^h, we have, for some constant $C_4>0$,

$$\begin{aligned}&\left|\int_\Gamma (u_1-u_1^h)(\mathbf{v}_1-\mathbf{v}_1^h)\cdot\mathbf{n}\,d\Gamma-\int_\Gamma (u_2-u_2^h)(\mathbf{v}_2-\mathbf{v}_2^h)\cdot\mathbf{n}\,d\Gamma\right|\\ &\qquad=\left|\int_\Gamma (u_1^h-u_2^h)(\mathbf{v}_1-\mathbf{v}_1^h)\cdot\mathbf{n}\,d\Gamma+\int_\Gamma (u_2-u_2^h)(\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n}\,d\Gamma\right|\\ &\qquad\le \|u_1^h-u_2^h\|_{1/2,\Gamma}\|(\mathbf{v}_1-\mathbf{v}_1^h)\cdot\mathbf{n}\|_{-1/2,\Gamma}+\|u_2-u_2^h\|_{0,\Gamma}\|(\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n}\|_{0,\Gamma}\\ &\qquad\le C_4\left(\epsilon|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2+\frac{1}{h\epsilon}\|u_1^h-u_2^h\|_{0,\Gamma}^2+\frac{1}{\epsilon}\|\mathbf{v}_1^h-\mathbf{v}_2^h\|_{0,\Gamma}^2+\epsilon|||u-u^h|||_1^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}\widehat{\mathcal{J}}(u-u^h,\mathbf{v}-\mathbf{v}^h)&\ge (C_3-C_4\epsilon)\left(|||u-u^h|||_1^2+|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2\right)\\ &\quad+\left(\frac{1}{h^{1+\epsilon_0}}-\frac{2C_4}{h\epsilon}\right)\int_\Gamma |u_1^h-u_2^h|^2\,d\Gamma+\left(\frac{1}{h^{\epsilon_1}}-\frac{2C_4}{\epsilon}\right)\int_\Gamma \left((\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n}\right)^2\,d\Gamma\,.\end{aligned}$$

We first choose ϵ small enough so that $C_3-C_4\epsilon>0$. Then, for this fixed ϵ, we choose h_0 sufficiently small so that $2C_4h_0^{\epsilon_0}\le\epsilon$ and $2C_4h_0^{\epsilon_1}\le\epsilon$. Thus, for $0<h<h_0$, we have that

$$\widehat{\mathcal{J}}(u-u^h,\mathbf{v}-\mathbf{v}^h)\ge C(|||u-u^h|||_1^2+|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2)\,.\qquad\square$$

Proposition 3.1 shows a certain coercive property about the functional $\mathcal{J}$. If we choose $u=0$ and $\mathbf{v}=\mathbf{0}$ in (3.8), then we see that the coercive property is true on the finite-dimensional subspace $H^h\times\mathbf{V}^h$ of $H\times\mathbf{V}$, i.e., $\mathcal{J}(u^h,\mathbf{v}^h;0)\ge C(\|u^h\|_1^2+\|\mathbf{v}^h\|_{H(\mathrm{div})}^2)$ for $(u^h,\mathbf{v}^h)\in H^h\times\mathbf{V}^h$. However, this does not hold for all elements of $H\times\mathbf{V}$. Neverthless, Proposition 3.1 suffices for us to obtain an error estimate for the least-squares finite element approximations of the solution of (3.1)–(3.3).

3.2. Finite element approximations. We define $(u_*^h,\ \mathbf{v}_*^h)$ to be the solution of the following problem:

$$J(u_*^h,\mathbf{v}_*^h;f)=\min_{u_h\in H^h,\mathbf{v}^h\in\mathbf{V}^h} J(u^h,\mathbf{v}^h;f)\,. \tag{3.9}$$

We then have the following error estimate.

THEOREM 3.2. *Let $s>0$. Assume that the solution $(u,\mathbf{v})$ of (3.1)–(3.3) satisfies $u\in\dot{H}^{s+1}(\Omega)\cap H_0(\Gamma,\Omega)$ and $\mathbf{v}\in\dot{V}^{s+1}(\Omega)\cap\mathbf{V}(\Gamma,\Omega)$. Then, for h sufficiently small and for $0<\epsilon_1\le\ 1$ and any $\delta>\epsilon_0>0$, there exists a constant $C>0$ such that*

$$|||u-u_*^h|||_1+|||\mathbf{v}-\mathbf{v}_*^h|||_{\mathrm{div}}\le Ch^{s-\delta}(|||u|||_{s+1}+|||\mathbf{v}|||_{s+1})\,. \tag{3.10}$$

Proof. By the approximation properties (3.4) and (3.5), there exist $\widehat{u}^h\in H^h$ and $\widehat{\mathbf{v}}^h\in\mathbf{V}^h$ such that

$$|||u-\widehat{u}^h|||_1\le Ch^s|||u|||_{s+1} \tag{3.11}$$

and

$$|||\mathbf{v}-\widehat{\mathbf{v}}^h|||_1\le Ch^s|||\mathbf{v}|||_{s+1}\,. \tag{3.12}$$

By Proposition 3.1 and the definition of u_*^h and $\mathbf{v}_*^h$,

$$
\begin{aligned}
\|\!|u-u_*^h\|\!|_1^2 + \|\!|\mathbf{v}-\mathbf{v}_*^h\|\!|_{\text{div}}^2 &\le C\mathcal{J}(u-u_*^h, \mathbf{v}-\mathbf{v}_*^h; 0) \\
&= C\mathcal{J}(u_*^h, \mathbf{v}_*^h; f) \le C\mathcal{J}(\widehat{u}^h, \widehat{\mathbf{v}}^h; f) = C\mathcal{J}(u-\widehat{u}^h, \mathbf{v}-\widehat{\mathbf{v}}^h; 0) \\
&\le \|\!|u-\widehat{u}^h\|\!|_1^2 + \|\!|\mathbf{v}-\widehat{\mathbf{v}}^h\|\!|_{\text{div}}^2 \\
&\qquad + \frac{1}{h^{1+\epsilon_0}}\int_\Gamma (\widehat{u}_1^h - \widehat{u}_2^h)^2\, d\Gamma + \frac{1}{h^{\epsilon_1}}\int_\Gamma \left((\widehat{\mathbf{v}}_1^h - \widehat{\mathbf{v}}_2^h)\mathbf{n}\right)^2\, d\Gamma\,.
\end{aligned}
\tag{3.13}
$$

Using trace theorems, we have that

$$
\begin{aligned}
&\frac{1}{h^{1+\epsilon_0}}\int_\Gamma (\widehat{u}_1^h - \widehat{u}_2^h)^2\, d\Gamma + \frac{1}{h^{\epsilon_1}}\int_\Gamma ((\widehat{\mathbf{v}}_1^h - \widehat{\mathbf{v}}_2^h)\cdot\mathbf{n})^2\, d\Gamma \\
&\quad = \frac{1}{h^{1+\epsilon_0}}\int_\Gamma (\widehat{u}_1^h - u_1 + u_2 - \widehat{u}_2^h)^2\, d\Gamma \\
&\qquad + \frac{1}{h^{\epsilon_1}}\int_\Gamma \left((\widehat{\mathbf{v}}_1^h - \mathbf{v}_1 + \mathbf{v}_2 - \widehat{\mathbf{v}}_2^h)\cdot\mathbf{n}\right)^2\, d\Gamma \\
&\quad \le \frac{2}{h^{1+\epsilon_0}}\left(\int_\Gamma (\widehat{u}_1^h - u)^2\, d\Gamma + \int_\Gamma (\widehat{u}_2^h - u)^2\, d\Gamma\right) \\
&\qquad + \frac{2}{h^{\epsilon_1}}\left(\int_\Gamma \left((\widehat{\mathbf{v}}_1^h - \mathbf{v})\cdot\mathbf{n}\right)^2\, d\Gamma + \int_\Gamma \left((\widehat{\mathbf{v}}_2^h - \mathbf{v})\cdot\mathbf{n}\right)^2\, d\Gamma\right) \\
&\quad \le \frac{1}{h^{1+\epsilon_0}}\|\!|u-\widehat{u}^h\|\!|_{(1/2)(1+2\delta-\epsilon_0)}^2 + \frac{1}{h^{\epsilon_1}}\|\!|\mathbf{v}-\widehat{\mathbf{v}}^h\|\!|_{1-\frac{\epsilon_1}{2}}^2 \\
&\quad \le h^{2s-2\delta}\|\!|u\|\!|_{s+1}^2 + \frac{1}{h^{\epsilon_1}}h^{2(s+\frac{\epsilon_1}{2})}\|\!|\mathbf{v}\|\!|_{s+1}^2 \le Ch^{2(s-\delta)}(\|\!|u\|\!|_{s+1}^2 + \|\!|\mathbf{v}\|\!|_{s+1}^2)
\end{aligned}
\tag{3.14}
$$

for h sufficiently small. Combining (3.11)–(3.14) yields (3.10). ∎

Remark. The conclusion of Theorem 3.2 is also valid for problems with homogeneous Newmann boundary conditions and mixed homogeneous boundary conditions.

Remark. Theorem 3.2 is a generalization of Theorem 5.1 of [8]. We merely require that $u \in \dot{H}^{s+1}(\Omega)$ and $\mathbf{v} \in \dot{\mathbf{V}}^{s+1}(\Omega)$, i.e., regularity within each subdomain and not across interfaces. Furthermore, we allow for the use of nonconforming elements in the sense that the finite element functions $u^h \in H^h$ need not belong to $H^1(\Omega)$.

4. Inhomogeneous Neumann boundary conditions. We now consider problem (3.1) with the homogeneous Dirchlet boundary condition replaced by an inhomogeneous Neumann boundary condition; i.e., we consider the problem

$$-\text{div}\,(\mathbf{v}_i) + c_i u_i = f \quad \text{in } \Omega_i\,,\ \ i = 1, 2\,, \tag{4.1}$$

$$A_i \nabla u_i - \mathbf{v}_i = 0 \quad \text{in } \Omega_i\,,\ \ i = 1, 2\,, \tag{4.2}$$

$$\mathbf{v}\cdot\mathbf{n}|_{\Gamma_0} = g\,, \qquad [u]_\Gamma = 0\,, \qquad \text{and} \qquad [\mathbf{v}\cdot\mathbf{n}]_\Gamma = 0\,. \tag{4.3}$$

Define the functional $\mathcal{K}(u, \mathbf{v}; f, g)$ on $H \times V$ as follows. For $u \in H$ and $\mathbf{v} \in V$,

$$
\begin{aligned}
\mathcal{K}(u, \mathbf{v}; f, g) &= \sum_{i=1}^{2}(\|-\text{div}(\mathbf{v}_i) + c_i u_i - f\|_0^2 + \|A_i\nabla u_i - \mathbf{v}_i\|_0^2) \\
&\quad + \frac{1}{h^{1+\epsilon_0}}\int_\Gamma [u]_\Gamma^2\, d\Gamma + \frac{1}{h^{\epsilon_1}}\int_\Gamma [\mathbf{v}\cdot\mathbf{n}]_\Gamma^2\, d\Gamma. + \frac{1}{h^{\epsilon_2}}\int_{\Gamma_0}(\mathbf{v}\cdot\mathbf{n} - g)^2\, d\Gamma.
\end{aligned}
\tag{4.4}
$$

First we prove a coercivity property for $\mathcal{K}$; the result and its proof are similar to that of Proposition 3.1. Let $H^h \subset H$ and $\mathbf{V}^h \subset \mathbf{V}$ be finite-dimensional subspaces satisfying the approximation properties (3.4) and (3.5).

PROPOSITION 4.1. *Let* $(u^h, \mathbf{v}^h) \in H^h \times \mathbf{V^h}$ *with* $u^h = (u_1^h, u_2^h)$ *and* $\mathbf{v}^h = (\mathbf{v}_1^h, \mathbf{v}_2^h)$ *and* $(u, \mathbf{v}) \in H(\Gamma, \Omega) \times \mathbf{V}(\Gamma, \Omega)$ *with* $u = (u_1, u_2)$ *and* $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$*. Then, for* h *sufficiently small, there is a constant* $C > 0$ *independent of* h *such that*

$$\mathcal{K}(u - u^h, \mathbf{v} - \mathbf{v}^h; 0, 0) \geq C(|||u - u^h|||_1^2 + |||\mathbf{v} - \mathbf{v}^h|||_{\mathrm{div}}^2)\,.$$

Proof. Define

$$\begin{aligned}\widehat{\mathcal{K}}(u, \mathbf{v}) = &\sum_{i=1}^{2}(\| - c_i^{-1/2}\mathrm{div}\,(\mathbf{v}_i) + c_i^{1/2}u_i\|_0^2 + \|A_i^{1/2}\nabla u_i - A_i^{-1/2}\mathrm{v}_i\|_0^2 \\ &+ \frac{1}{h^{1+\epsilon_0}}\int_\Gamma [u]_\Gamma^2\, d\Gamma + \frac{1}{h^{\epsilon_1}}\int_\Gamma [\mathbf{v}\cdot\mathbf{n}]_\Gamma^2\, d\Gamma + \frac{1}{h^{\epsilon_2}}\int_{\Gamma_0} (\mathbf{v}\cdot\mathbf{n})^2\, d\Gamma\,.\end{aligned} \tag{4.5}$$

It is easy to see that $\mathcal{K}(\cdot; 0, 0)$ and $\widehat{\mathcal{K}}(\cdot)$ are equivalent, i.e., that there exist two positive constants C_1 and C_2 such that

$$C_1\mathcal{K}(u, \mathbf{v}; 0, 0) \leq \widehat{\mathcal{K}}(u, \mathbf{v}) \leq C_2\mathcal{K}(u, \mathbf{v}; 0, 0)$$

for all $u \in H$ and $\mathbf{v} \in \mathbf{V}$. Thus, it suffices to prove that

$$\widehat{\mathcal{K}}(u - u^h, \mathbf{v} - \mathbf{v}^h) \geq C(|||u - u^h|||_1^2 + |||\mathbf{v} - \mathbf{v}^h|||_{\mathrm{div}}^2)$$

for some constant $C > 0$ and all $u \in H(\Gamma, \Omega)$ and $\mathbf{v} \in V(\Gamma, \Omega)$. Now, by the definition of $H(\Gamma, \Omega)$ and $V(\Gamma, \Omega)$,

$$\begin{aligned}&\widehat{\mathcal{K}}(u - u^h, \mathbf{v} - \mathbf{v}^h) \\ &= \sum_{i=1}^{2}(\| - c_i^{-1/2}\mathrm{div}\,(\mathbf{v}_i - \mathbf{v}_i^h) + c_i^{1/2}(u_i - u_i^h)\|_0^2 + \|A_i^{1/2}\nabla(u_i - u_i^h) - A_i^{-1/2}(\mathbf{v}_i - \mathbf{v}_i^h)\|_0^2) \\ &+ \frac{1}{h^{1+\epsilon_0}}\int_\Gamma |u_1^h - u_2^h|^2\, d\Gamma + \frac{1}{h^{\epsilon_1}}\int_\Gamma ((\mathbf{v}_1^h - \mathbf{v}_2^h)\cdot\mathbf{n})^2\, d\Gamma + \frac{1}{h^{\epsilon_2}}\int_{\Gamma_0} (\mathbf{v}\cdot\mathbf{n} - \mathbf{v}^h\cdot\mathbf{n})^2\, d\Gamma\,.\end{aligned}$$

Integrating by parts, one has

$$\begin{aligned}&\|-c_i^{-1/2}\mathrm{div}\,(\mathbf{v}_i - \mathbf{v}_i^h) + c_i^{1/2}(u_i - u_i^h)\|_0^2 + \|A_i^{1/2}\nabla(u_i - u_i^h) - A_i^{-1/2}(\mathbf{v}_i - \mathbf{v}_i^h)\|_0^2 \\ &\quad = \|c_i^{-1/2}\mathrm{div}\,(\mathbf{v}_i - \mathbf{v}_i^h)\|_0^2 + \|c_i^{1/2}(u_i - u_i^h)\|_0^2 - 2\int_{\Omega_i} \mathrm{div}(\mathbf{v}_i - \mathbf{v}_i^h)(u_i - u_i^h)\, d\Omega \\ &\qquad + \|A_i^{1/2}\nabla(u_i - u_i^h)\|_0^2 + \|A_i^{-1/2}(\mathbf{v}_i - \mathbf{v}_i^h)\|_0^2 - 2\int_{\Omega_i} \nabla(u_i - u_i^h)\cdot(\mathbf{v}_i - \mathbf{v}_i^h)\, d\Omega \\ &\quad = \|\mathrm{div}\,(\mathbf{v}_i - \mathbf{v}_i^h)\|_0^2 + \|u_i - u_i^h\|_0^2 + \|A_i^{1/2}\nabla(u_i - u_i^h)\|_0^2 \\ &\qquad + \|A_i^{-1/2}(\mathbf{v_i} - \mathbf{v_i^h})\|_0^2 - 2\int_\Gamma (u_i - u_i^h)(\mathbf{v}_i - \mathbf{v}_i^h)\cdot\mathbf{n}\, d\Gamma \\ &\qquad + 2\int_{\Gamma_0} (u - u^h)(\mathbf{v} - \mathbf{v}^h)\cdot\mathbf{n}\, d\Gamma\,.\end{aligned}$$

Hence,

$$
\begin{aligned}
&\widehat{\mathcal{K}}(u-u^h,\mathbf{v}-\mathbf{v}^h)\\
\geq &C\sum_{i=1}^{2}\big(\|\mathrm{div}\,(\mathbf{v}_i-\mathbf{v}_i^h)\|_0^2+\|u_i-u_i^h\|_0^2+\|\nabla(u_i-u_i^h)\|_0^2+\|\mathbf{v}_i-\mathbf{v}_i^h\|_0^2\big)\\
&+2\int_\Gamma(u_1-u_1^h)(\mathbf{v}_1-\mathbf{v}_1^h)\cdot\mathbf{n}\,d\Gamma-2\int_\Gamma(u_2-u_2^h)(\mathbf{v}_2-\mathbf{v}_2^h)\cdot\mathbf{n}\,d\Gamma\\
&+2\int_{\Gamma_0}(u-u^h)(\mathbf{v}-\mathbf{v}^h)\cdot\mathbf{n}\,d\Gamma+\frac{1}{h^{1+\epsilon_0}}\int_\Gamma(u_1^h-u_2^h)^2\,d\Gamma\\
&+\frac{1}{h^{\epsilon_1}}\int_\Gamma((\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n})^2\,d\Gamma+\frac{1}{h^{\epsilon_2}}\int_{\Gamma_0}(\mathbf{v}\cdot\mathbf{n}-\mathbf{v}^h\cdot\mathbf{n})^2\,d\Gamma\\
=&I+II+III+IV,
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
I=&C\sum_{i=1}^{2}(\|\mathrm{div}\,(\mathbf{v}_i-\mathbf{v}_i^h)\|_0^2+\|u_i-u_i^h\|_0^2+\|\nabla(u-u_h)\|_0^2+\|\mathbf{v}_i-\mathbf{v}_i^h\|_0^2),\\
II=&2\left(\int_\Gamma(u_1-u_1^h)(\mathbf{v}_1-\mathbf{v}_1^h)\cdot\mathbf{n}\,d\Gamma-\int_\Gamma(u_2-u_2^h)(\mathbf{v}_2-\mathbf{v}_2^h)\cdot\mathbf{n}\,d\Gamma\right),\\
III=&2\int_{\Gamma_0}(u-u^h)(\mathbf{v}-\mathbf{v}^h)\cdot\mathbf{n}\,d\Gamma,\\
IV=&\frac{1}{h^{1+\epsilon_0}}\int_\Gamma(u_1^h-u_2^h)^2\,d\Gamma\\
&\qquad+\frac{1}{h^{\epsilon_1}}\int_\Gamma((\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n})^2\,d\Gamma+\frac{1}{h^{\epsilon_2}}\int_{\Gamma_0}(\mathbf{v}\cdot\mathbf{n}-\mathbf{v}^h\cdot\mathbf{n})^2\,d\Gamma\,.
\end{aligned}
$$

By the proof of Theorem 3.1 we have that for $\epsilon>0$,

$$
|II|\leq C\epsilon|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2+\frac{1}{h\epsilon}\|u_1^h-u_2^h\|_{0,\Gamma}^2+\frac{1}{\epsilon}\|\mathbf{v}_1^h-\mathbf{v}_2^h\|_{0,\Gamma}^2+\epsilon|||u-u_h|||_1^2\,.
$$

Using the Schwartz inequality we have that

$$
|III|=2\left|\int_{\Gamma_0}(u-u^h)(\mathbf{v}\cdot\mathbf{n}-\mathbf{v}^h\cdot\mathbf{n})\,d\Gamma\right|\leq\epsilon_3\int_{\Gamma_0}(u-u^h)^2\,d\Gamma+\frac{1}{\epsilon_3}\int_{\Gamma_0}(\mathbf{v}\cdot\mathbf{n}-\mathbf{v}^h\cdot\mathbf{n})^2\,d\Gamma
$$

for $\epsilon_3>0$. Hence

$$
\begin{aligned}
\widehat{\mathcal{K}}(u-u^h,\mathbf{v}-\mathbf{v}^h)\geq&(C_1-\epsilon-\epsilon_3)|||u-u^h|||_1^2+(C_2-\epsilon)|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2\\
&+\left(\frac{1}{h^{1+\epsilon_0}}-\frac{1}{h\epsilon}\right)\int_\Gamma|u_1^h-u_2^h|^2\,d\Gamma+\left(\frac{1}{h^{\epsilon_1}}-\frac{1}{\epsilon}\right)\int_\Gamma((\mathbf{v}_1^h-\mathbf{v}_2^h)\cdot\mathbf{n})^2\,d\Gamma\\
&+\left(\frac{1}{h^{\epsilon_2}}-\frac{1}{\epsilon_3}\right)\int_{\Gamma_0}(\mathbf{v}\cdot\mathbf{n}-\mathbf{v}^h\cdot\mathbf{n})^2\,d\Gamma\,.
\end{aligned}
$$

We first choose ϵ and ϵ_3 small enough so that $C_i-\epsilon-\epsilon_3>0$. Then, for this fixed ϵ, we let h_0 be sufficiently small so that $h_0^{1+\epsilon_0}\leq h\epsilon$, $h_0^{\epsilon_1}<\epsilon$, and $h_0^{\epsilon_2}<\epsilon_3$. Thus, for $0>h>h_0$, we obtain

$$
\widehat{\mathcal{K}}(u-u^h,\mathbf{v}-\mathbf{v}^h)\geq C\left(|||u-u^h|||_1^2+|||\mathbf{v}-\mathbf{v}^h|||_{\mathrm{div}}^2\right)\,.
$$

The proof is complete. ∎

Assume that $(u^h_*, \mathbf{v}^h_*)$ is the solution of the following problem:

$$\mathcal{K}(u^h_*, \mathbf{v}^h_*; f, g) = \min_{u^h \in H^h, \mathbf{v}^h \in \mathbf{V}^h} \mathcal{K}(u^h, \mathbf{v}^h; f, g)\,.$$

We have the following error estimate.

THEOREM 4.2. *Let $s > 0$. Assume that the solutions u and $\mathbf{v}$ of (4.1)–(4.3) satisfy $u \in \dot{H}^{s+1}(\Omega)$ and $\mathbf{v} \in \dot{V}^{s+1}(\Omega)$. Then, for h sufficiently small and for $0 < \epsilon_1, \epsilon_2 \le 1$, and any $\delta > \epsilon_0 > 0$, there exists a constant $C > 0$ such that*

$$|||u - u^h_*|||_1 + |||\mathbf{v} - \mathbf{v}^h_*|||_{\mathrm{div}} \le Ch^{s-\delta}(|||u|||_{s+1} + |||\mathbf{v}|||_{s+1})\,. \tag{4.7}$$

Proof. By the approximation properties (3.4) and (3.5), there exist $\widehat{u}^h \in H^h$ and $\widehat{\mathbf{v}}^h \in \mathbf{V}^h$ such that

$$|||u - \widehat{u}^h|||_1 \le Ch^s |||u|||_{s+1}$$

and

$$|||\mathbf{v} - \widehat{\mathbf{v}}^h|||_1 \le Ch^s |||\mathbf{v}|||_{s+1}\,.$$

By Proposition 4.1 and the definition of u^h_* and $\mathbf{v}^h_*$ we have that

$$\begin{aligned}
&|||u - u^h_*|||_1^2 + |||\mathbf{v} - \mathbf{v}^h_*|||_{\mathrm{div}}^2 \\
&\le C\mathcal{K}(u - u^h_*, \mathbf{v} - \mathbf{v}^h_*; 0, 0) = C\mathcal{K}(u^h_*, \mathbf{v}^h_*; f, g) \\
&\le C\mathcal{K}(\widehat{u}^h, \widehat{\mathbf{v}}^h; f, g) = C\mathcal{K}(u - \widehat{u}^h, \mathbf{v} - \widehat{\mathbf{v}}^h; 0, 0) \\
&\le |||u - \widehat{u}^h|||_1^2 + |||\mathbf{v} - \widehat{\mathbf{v}}^h|||_{\mathrm{div}}^2 + \frac{1}{h^{1+\epsilon_0}} \int_\Gamma (\widehat{u}^h_1 - \widehat{u}^h_2)^2 \, d\Gamma \\
&\quad + \frac{1}{h^{\epsilon_1}} \int_\Gamma ((\widehat{\mathbf{v}}^h_1 - \widehat{\mathbf{v}}^h_2) \cdot \mathbf{n})^2 \, d\Gamma + \frac{1}{h^{\epsilon_2}} \int_{\Gamma_0} ((\widehat{\mathbf{v}}^h - \mathbf{v}) \cdot \mathbf{n})^2 \, d\Gamma\,.
\end{aligned} \tag{4.8}$$

From the proof of Theorem 3.2, we have that

$$\begin{aligned}
&\frac{1}{h^{1+\epsilon_0}} \int_\Gamma (\widehat{u}^h_1 - \widehat{u}^h_2)^2 \, d\Gamma + \frac{1}{h^{\epsilon_1}} \int_\Gamma ((\widehat{\mathbf{v}}^h_1 - \widehat{\mathbf{v}}^h_2) \cdot \mathbf{n})^2 \, d\Gamma \\
&\quad \le h^{2s-2\delta} |||u|||_{s+1}^2 + \frac{1}{h^{\epsilon_1}} h^{2(s+\frac{\epsilon_1}{2})} |||\mathbf{v}|||_{s+1}^2 \le Ch^{2(s-\delta)}(|||u|||_{s+1}^2 + |||\mathbf{v}|||_{s+1}^2)\,.
\end{aligned} \tag{4.9}$$

Using trace theorems we have that

$$\begin{aligned}
\frac{1}{h^{\epsilon_2}} \int_{\Gamma_0} (\mathbf{v} \cdot \mathbf{n} - \widehat{\mathbf{v}}^h \cdot \mathbf{n})^2 \, d\Gamma &\le \frac{1}{h^{\epsilon_2}} |||\mathbf{v} - \widehat{\mathbf{v}}^h|||_{1-\frac{\epsilon_2}{2}}^2 \\
&\le \frac{1}{h^{\epsilon_2}} h^{2(s+\frac{\epsilon_2}{2})} |||\mathbf{v}|||_{s+1}^2 = Ch^{2s} |||\mathbf{v}|||_{s+1}^2
\end{aligned} \tag{4.10}$$

for h sufficiently small. Substituting (4.9) and (4.10) into (4.8), we obtain (4.7). ◻

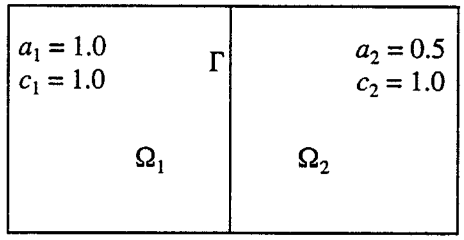

FIG. 1. *Interface problem used in computational example.*

5. Numerical results. In this section we report the results of computations which illustrate our method and error analysis. We take for the domain the rectangle $\Omega = (0, 2) \times (0, 1)$. The interface occurs at $x = 1$ so that $\Omega_1 = (0, 1) \times (0, 1)$ and $\Omega_2 = (1, 2) \times (0, 1)$. In (2.1)–(2.3), $A_i = \text{diag}\,(a_i, a_i)$ with $a_1 = 1$ and $a_2 = 1/2$ and $c_1 = c_2 = 1$; see Figure 1.

For the exact solution, we choose

$$u_1(x, y) = \sin(\pi x)\sin(\pi y), \qquad (x, y) \in \Omega_1 = (0, 1) \times (0, 1)$$

and

$$u_2(x, y) = -\sin(2\pi x)\sin(\pi y), \qquad (x, y) \in \Omega_2 = (1, 2) \times (0, 1)\,.$$

The right-hand sides f_1 and f_2 in (2.1) are then determined from this choice for (A_1, c_1, u_1) and (A_2, c_2, u_2), respectively. Note that the global solution merely belongs to $H^1(\Omega)$. We choose $\epsilon_0 = 2/3$ and $\epsilon_1 = 3/4$ in the functional (3.4).

Standard techniques of the calculus of variations may be used to deduce that any solution $(u_*^h, \mathbf{v}_*^h)$ of (3.8) necessarily satisfies the variational problem: find $(u_*^h, \mathbf{v}_*^h) \in H^h \times V^h$ such that

$$B\big((u_*^h, \mathbf{v}_*^h), (\widetilde{u}^h, \widetilde{\mathbf{v}}^h)\big) = F\big((\widetilde{u}^h, \widetilde{\mathbf{v}}^h)\big) \qquad \forall\ (\widetilde{u}^h, \widetilde{\mathbf{v}}^h) \in H^h \times V^h\,, \tag{5.1}$$

where, for $u^h = (u_1^h, u_2^h)$, $\mathbf{v}^h = (\mathbf{v}_1^h, \mathbf{v}_2^h)$, $\widetilde{u}^h = (\widetilde{u}_1^h, \widetilde{u}_2^h)$, $\mathbf{v}^h = (\widetilde{\mathbf{v}}_1^h, \widetilde{\mathbf{v}}_2^h)$, we have

$$\begin{aligned} B\big((u^h, \mathbf{v}^h),&(\widetilde{u}^h, \widetilde{\mathbf{v}}^h)\big) \\ &= \sum_{i=1}^{2} \Big(\big(\text{div}\,(\mathbf{v}_i^h) + c_i u_i\,,\ \text{div}\,(\widetilde{\mathbf{v}}_i^h) + c_i \widetilde{u}_i^h\big)_{\Omega_i} + \big(A_i \nabla u_i^h - \mathbf{v}_i^h\,,\ A_i \nabla \widetilde{u}_i^h - \widetilde{\mathbf{v}}_i^h\big)_{\Omega_i} \Big) \\ &\quad + \frac{1}{h^{1+\epsilon_0}} \big(u_1^h - u_2^h\,,\ \widetilde{u}_1^h - \widetilde{u}_2^h\big)_\Gamma + \frac{1}{h^{\epsilon_1}} \big((\mathbf{v}_1^h - \mathbf{v}_2^h) \cdot \mathbf{n}\,,\ (\widetilde{\mathbf{v}}_1^h - \widetilde{\mathbf{v}}_2^h) \cdot \mathbf{n}\big)_\Gamma \end{aligned}$$

and

$$F\big((\widetilde{u}^h, \widetilde{\mathbf{v}}^h)\big) = \sum_{i=1}^{2} (f_i\,,\ -\text{div}\,(\widetilde{\mathbf{v}}_i^h) + \widetilde{u}_i^h)_{\Omega_i}\,.$$

Here, $(\cdot, \cdot)_{\Omega_i}$ and $(\cdot, \cdot)_\Gamma$ denote the $L^2(\Omega_i)$ and $L^2(\Gamma)$ inner products, respectively.

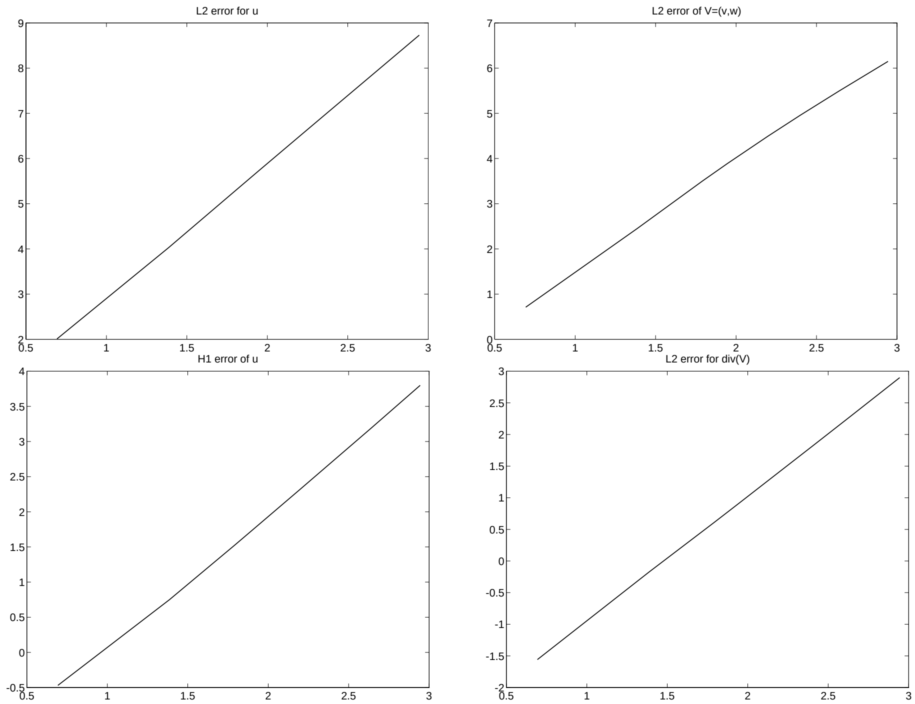

FIG. 2. *Negative of logarithm of error vs.* $-\log(h)$.

TABLE 1
Rates of convergence.

Function	L^2 error	H^1 error
u	3.315	1.837
v	2.321	1.214
w	2.138	1.038
$\operatorname{div} \mathbf{v}$	1.898	—

For our numerical results, globally continuous piecewise quadratic finite element functions based on uniform triangulations of Ω_i, $i = 1, 2$, were used for all unknowns, i.e., u_i^h and the components of $\mathbf{v}_i^h$, $i = 1, 2$. The nodes of the triangulations of Ω_1 and Ω_2 coincide on the interface Γ. Hence, we expect that convergence rates will be determined according to (3.9) with $s = 3$.

Figure 2 displays the L_2 error of the approximate solutions for $u = (u_1, u_2)$, $\mathbf{v} = (A_1\nabla u_1, A_2\nabla u_2)$, the error of u in the H^1 seminorm, and the L_2 norm error of $\operatorname{div} \mathbf{v}$. In Table 1, we list the rates of convergence estimated by linear regression. These convergence rates match our error estimates in section 3.

Remark. From Table 1, we see that the L^2 error in the approximation to u is one order higher than that for its derivative.

Remark. As shown in [9] and [13], if $\|\operatorname{curl} \mathbf{v}\|^2$ is added to the standard least-squares functional, then the optimal error estimate in the H^1-norm for $\mathbf{v}$ may be achieved. Also, the error for $\mathbf{v}$ in the L^2 norm of is one order higher. Our numerical experience (see Figure 3) indicates that if we add $\|\operatorname{curl} \mathbf{v}\|^2$ to the functional (3.4), then the H^1 error in the approximation $\mathbf{v}$ is seemingly better, but the L^2 error is not improved.

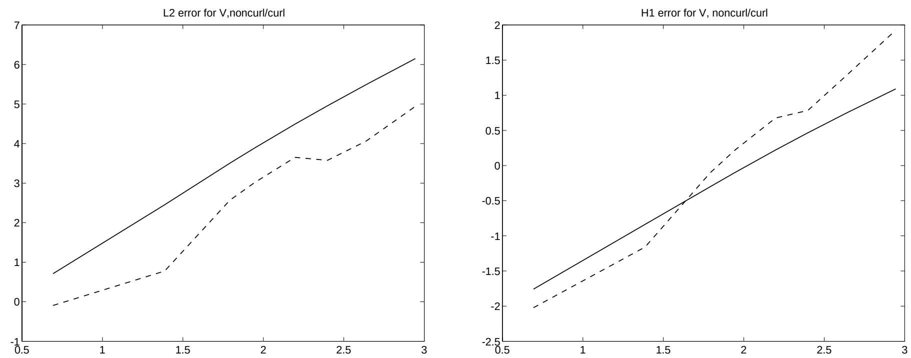

FIG. 3. *Negative of logrithm of error vs.* $-\log(h)$. *Solid line: without curl term in functional; dashed line: with curl term in functional.*

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