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Least Squares Methods for $2m$th Order Elliptic Boundary-Value Problems

By J. H. Bramble and A. H. Schatz

Abstract. In this paper we consider a general class of boundary-value problems for $2m$th order elliptic equations including nonhomogeneous essential boundary conditions and nonselfadjoint problems. Approximation methods involving least squares approximation of the data are presented and corresponding error estimates are proved. These methods can be considered in the category of Rayleigh-Ritz-Galerkin methods and have the special feature that the trial functions need not satisfy the boundary conditions. A special case of the trial functions which is studied are spline functions defined on a uniform mesh of width h (or more generally piecewise polynomial functions). For a given "well set" boundary-value problem for a $2m$th order operator the theory presented will provide a method with any prescribed order of accuracy r which is optimal in the sense that the best approximation in the underlying subspace is of order of accuracy r.

1. Introduction. In this paper, we present, and give error estimates for a class of least squares methods for the approximation of solutions of general boundary-value problems for $2m$th order elliptic partial differential equations. The estimates are based on an abstract approximation theorem which we prove. This theorem is also applied to obtain some purely approximation theoretic results.

The results on boundary-value problems are a generalization of those of [14], where we treat only the Dirichlet problem for second order operators. A description of some of them is given in [15].

As is well known, the ordinary Rayleigh-Ritz method for the treatment of boundary-value problems with essential boundary conditions requires that the trial functions satisfy boundary conditions. In the least squares method this difficulty is not present. For other approaches where the requirement of satisfication of boundary conditions has been avoided, the reader is referred to the works of Aubin [4], [5], Babuška [6] and Nitsche [26]. Although most of this work has been aimed at second-order problems, it could be generalized in some directions to higher order problems.

Little has been done, however, of a general nature on the treatment of higher order equations with general boundary conditions by any method. We present here, for the first time, a general theory for the approximation of solutions of such problems. For a given "well set" boundary-value problem for a $2m$th order operator, our theory will provide a method with any prescribed order of accuracy r which is optimal in the sense that best approximation in the underlying subspace is of the order of accuracy r. Some special features of the method presented in this paper are as follows:

(1) The trial functions are not required to satisfy any boundary conditions.

(2) Selfadjoint problems and nonselfadjoint problems are treated with equal ease.

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(3) Problems whose associated quadratic form is not positive definite are treated.

(4) The data are used in an L_2 sense.

(5) The matrix of the resulting linear system is always symmetric and positive definite.

An outline of the paper follows. The first part contains the above-mentioned general approximation theorem. We begin Part II with a description of the function spaces and subspaces to be used later. The remainder of Part II is devoted to some approximation theoretic applications of the general theorem; e.g. we study some approximation theoretic properties on $\partial\Omega$ of functions which are assumed to have certain approximation theoretic properties on Ω. The main part of the paper is in Part III. Here we present and study least squares approximations for general boundary-value problems for $2m$th order elliptic operators. Many error estimates are proved; e.g. L_2 estimates are given as well as interior estimates for derivatives. A specific example of a boundary problem for the biharmonic operator is presented. Finally, two methods for the Dirichlet problem for Poisson's equation, which are not contained in the general theory, are studied. The proofs of the estimates in these methods are again an application of the abstract theorem of Part I.

PART I

2. Preliminaries. For each $j = 1, \cdots, n$, $n \geqq 1$, let $X(j, \theta_i)$, $0 \leqq \theta_i \leqq 1$, be a one-parameter family of Banach spaces (real or complex), with corresponding norms $\|\cdot\|_{X(j,\theta_i)}$ satisfying the following condition:

Condition I. (a) For each $j = 1, \cdots, n$, $X(j, \theta_i') \subset X(j, \theta_i)$ for any $0 \leqq \theta_i < \theta_i' \leqq 1$ with a continuous injection, i.e.

$$\|u\|_{X(j,\theta_i)} \leqq C_1(\theta_i, \theta_i')\, \|u\|_{X(j,\theta_i')} \tag{2.1}$$

for all $u \in X(j, \theta_i')$ where $C_1(\theta_i, \theta_i')$ is a constant which may depend on the choices of θ_i and θ_i'.

(b) Let Y be any Banach space with norm $\|\cdot\|_Y$. For fixed j, let $T \in \mathcal{L}(X(j, 0); Y) \cap \mathcal{L}(X(j, 1); Y)$ (where for X and Y Banach spaces, $\mathcal{L}(X; Y)$ is used to denote the space of bounded linear mappings of X into Y) with

$$\|Tu\|_Y \leqq m_0\|u\|_{X(j,0)}, \qquad \|Tu\|_Y \leqq m_1\|u\|_{X(j,1)}. \tag{2.2}$$

Then, for each $0 < \theta_i < 1$, $T \in \mathcal{L}(X(j, \theta_i); Y)$ with

$$\|Tu\|_Y \leqq C_2(\theta_i) m_0^{1-\theta_i} m_1^{\theta_i} \|u\|_{X(j,\theta_i)}, \tag{2.3}$$

where $C_2(\theta_i)$ is a constant which depends at most on θ_i and is independent of the choice of Y and T.

We note that Condition I(b) states that for each j the spaces $X(j, \theta_i)$, $0 < \theta_i < 1$, are interpolation spaces between the spaces $X(j, 0)$ and $X(j, 1)$ (cf. [17]).

Let $\theta \in \mathbf{R}^n$, $\theta = (\theta_1, \cdots, \theta_n)$. We shall say that $a \leqq \theta \leqq b$ for two real numbers a and b if $a \leqq \theta_i \leqq b$, $j = 1, \cdots, n$. For each $\theta \in \mathbf{R}^n$, $0 \leqq \theta \leqq 1$, denote by $X(\theta)$ the product space $X(\theta) = \prod_{j=1}^n X(j, \theta_i)$ with the norm

$$\|u\|_{X(\theta)} = \sum_{j=1}^n \|u_j\|_{X(j,\theta_i)}, \qquad u = (u_1, \cdots, u_n). \tag{2.4}$$

It will be convenient for us to renorm $X(\theta)$ in the following way: For each $\alpha \in \mathbf{R}^n$ and $h \in \mathbf{R}^1$ with $h > 0$, $X(\theta; h; \alpha)$ will denote the Banach space whose elements are those of $X(\theta)$, but furnished with the equivalent norm

$$||u||_{X(\theta;h;\alpha)} = \sum_{j=1}^{n} h^{-\alpha_j}||u_j||_{X(j,\theta_j)}. \tag{2.5}$$

We note that the constants of equivalence depend on h and α and when $\alpha = (0, \cdots, 0)$ the norms (2.4) and (2.5) are equal.

3. Approximation-Theoretic Properties of Certain Subspaces of $X(\theta)$. In order to motivate the approximation-theoretic result we wish to prove, let us consider the simplest case in which $X(\theta) = X(1, \theta_1)$, $0 \leqq \theta_1 \leqq 1$, a single one-parameter family of Banach spaces satisfying Condition I. Suppose that there exists a one-parameter family S^h, $0 < h \leqq 1$, of closed subspaces of $X(1, \beta_1)$ for some $0 \leqq \beta_1 < 1$, having the approximation property that for some $\sigma_1 > 0$

$$\inf_{\chi_1 \in S^h} ||u_1 - \chi_1||_{X(1,\beta_1)} \leqq Ch^{\sigma_1}||u_1||_{X(1,1)},$$

where C is a constant which is independent of h and $u_1 \in X(1, 1)$.

Suppose that the subspaces S^h are also closed in $X(1, 0)$. We now ask what properties with respect to best approximation in $X(1, 0)$ do the subspaces S^h have which follow from their properties on $X(1, \beta_1)$? Since $X(1, 0) \subset X(1, \beta_1)$ with a continuous injection, we trivially have

$$\inf_{\chi_1 \in S^h} ||u_1 - \chi_1||_{X(1,0)} \leqq Ch^{\sigma_1}||u||_{X(1,1)}.$$

We shall show in this case that since the spaces $X(1, \theta)$ satisfy Condition I the stronger inequality

$$\inf_{\chi_1 \in S^h} ||u_1 - \chi_1||_{X(1,0)} \leqq Ch^{\sigma_1\theta_1/(1-\beta_1)}||u_1||_{X(1,\theta_1)}$$

holds for each $0 \leqq \theta_1 \leqq 1$. Here C is some constant which is independent of h and $u_1 \in X(1, \theta_1)$. In the nontrivial case, in which $\theta_1 = 1$ and $0 < \beta_1 < 1$, we have $\sigma_1\theta_1/(1 - \beta_1) = \sigma_1/(1 - \beta_1) > \sigma_1$.

In this section, we shall generalize this situation. Briefly, we shall consider product spaces $X(\theta)$ as described previously. We shall assume that for some $\beta \in \mathbf{R}^n$, there exists a one-parameter family of closed subspaces S^h of $X(\beta)$ having certain properties with respect to best approximation in $X(\beta)$ (see below) and then show that this implies that they have certain not so evident approximation properties in the weighted spaces $X(0; h; \alpha)$, for some appropriate choice of α.

In Parts II and III of this paper, we shall consider some specific applications where the assumed properties of the approximating subspaces are easily verifiable.

Let $\beta, \sigma \in \mathbf{R}^n$ with $0 \leqq \beta < 1$ and $0 < \sigma$ and let $h \in \mathbf{R}^1$ with $0 < h \leqq 1$. For fixed values of β and σ, let S^h denote a family of subspaces of $X(\beta)$ which are closed in both $X(\beta)$ and $X(0)$ having the following property:

Condition II. For all $u \in X(1)$ there exists a constant C_3, independent of u and h, such that

$$\inf_{\chi \in S^h} ||u - \chi||_{X(\beta)} = \inf_{\chi \in S^h} \sum_{j=1}^{n} ||u_j - \chi_j||_{X(j,\beta_j)} \leqq C_3 \sum_{j=1}^{n} h^{\sigma_j}||u_j||_{X(j,1)}. \tag{3.1}$$

Remark. In all our applications, the spaces S^h will be finite-dimensional and therefore automatically closed in $X(\beta)$ and $X(0)$.

We shall now state the main result of this section.

THEOREM 3.1. *Suppose that Condition* I *is satisfied. For given* S^h, *satisfying Condition* II, *let* $\alpha_j = \beta_j\sigma_j(1 - \beta_j)$ *and* $0 \leqq \mu_j \leqq \alpha_j$ *for* $j = 1, \cdots, n$. *Then*

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\mu_j} ||u_j - \chi_j||_{X(j,0)} \leqq C(\theta) \sum_{j=1}^{n} h^{-\mu_j + (\sigma_j + \mu_j)\theta_j} ||u_j||_{X(j,\theta_j)} \tag{3.2}$$

for all $u = (u_1, \cdots, u_n) \in \prod_{j=1}^{n} X(j, \theta_j)$, *where* $\theta = (\theta_1, \cdots, \theta_n)$ *with* $0 \leqq \theta_j \leqq 1$ *and* $C(\theta)$ *is independent of* u *and* h *but may depend on the choice of* θ.

The proof of Theorem 3.1 is lengthy. We shall first need some lemmas.

LEMMA 3.1. *Suppose that the conditions of Theorem* 3.1 *are satisfied and that there exist* η_j, $0 \leqq \eta_j \leqq \alpha_j$, $j = 1, \cdots, n$, *such that for all* $u \in \prod_{j=1}^{n} X(j, \beta_j)$

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} ||u_j - \chi_j||_{X(j,0)} \leqq C_4 \sum_{j=1}^{n} h^{-\eta_j} ||u_j||_{X(j,\beta_j)}, \tag{3.3}$$

where C_4 *is a constant which is independent of* u *and* h. *Let* k *be an integer* $1 \leqq k \leqq n$ *for which* $\eta_k = \max[\eta_1, \cdots, \eta_n]$. *Then*

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} ||u_j - \chi_j||_{X(j,0)} \leqq C_4' \left(h^{-\eta_k \beta_k} ||u_k||_{X(k,\beta_k)} + \sum_{j \neq k} h^{-\eta_j} ||u_j||_{X(j,\beta_j)} \right), \tag{3.4}$$

where

$$C_4' = \max \{ C_2(\beta_k)(C_3 C_4)^{\beta_k}, C_4 \}.$$

The following remarks will be useful in the proof.

Remark 3.1. If X is a Banach space and N a closed subspace of X, then the quotient space X/N is a Banach space with norm $||\{u\}||_{X/N} = \inf_{v \in N} ||u - v||_X$ where $\{u\}$ denotes the equivalence class to which u belongs. The triangle inequality then states that

$$\inf_{v \in N} ||u_1 + u_2 - v||_X \leqq \inf_{v_1 \in N} ||u_1 - v_1||_X + \inf_{v_2 \in N} ||u_2 - v_2||_X.$$

Remark 3.2. Let X and Y be Banach spaces $X \subset Y$ and let N be a closed subspace of both X and Y. Suppose that there exists a constant C such that for all $u \in X$

$$\inf_{v \in N} ||u - v||_Y \leqq C ||u||_X.$$

Then, for all $u \in X$ and for the same constant C

$$\inf_{v \in N} ||u - v||_Y \leqq C \inf_{w \in N} ||u - w||_X.$$

Proof. We may assume without loss of generality that $\eta_k = \max[\eta_1, \cdots, \eta_n] > 0$ and therefore $\beta_k > 0$, for otherwise (3.4) is trivial. Let δ_{ij} denote the Kronecker delta. Then in view of Remark 3.1

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} ||u_j - \chi_j||_{X(j,0)} \leqq \sum_{l=1}^{n} \left(\inf_{\psi^l \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} ||u_j \delta_{jl} - \psi_j^l||_{X(j,0)} \right). \tag{3.5}$$

Now for $l = k$ we trivially have

$$\inf_{\psi^k \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} ||u_j \delta_{jk} - \psi_j^k||_{X(j,0)} \leqq h^{-\alpha_k} ||u_k||_{X(k,0)}. \tag{3.6}$$

Suppose now that $u_k \in X(k, 1)$, then in view of our assumption (3.3) and Remark 3.2

$$\inf_{\psi^k \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j \delta_{jk} - \psi_j^k \|_{X(j,0)} \leqq C_4 \inf_{\psi \in S^h} h^{-\eta_k} \sum_{j=1}^{n} \| u_j \delta_{jk} - \psi_j \|_{X(j,\beta_j)}. \tag{3.7}$$

Since S^h satisfies (3.1), we have from (3.7)

$$\inf_{\psi^k \in S^h} \sum_{j=1}^{h} h^{-\alpha_j} \| u_j \delta_{jk} - \psi_j^k \|_{X(j,0)} \leqq C_3 C_4 h^{-\eta_k + \sigma_k} \| u_k \|_{X(k,1)}. \tag{3.8}$$

We can now "interpolate" the inequalities (3.6) and (3.8) using our assumption (b) of Condition I. Let Y be the quotient space of $X(0; h; \alpha)$ modulo the subspace S^h and T be the mapping of $X(k, 0)$ into Y defined as follows: $T = T_2 \circ T_1$ where T_1 is the injection mapping of $X(k, 0)$ into $X(0; h; \alpha)$ and T_2 is the canonical mapping of $X(0; h; \alpha)$ into Y which associates with each element of $X(0; h; \alpha)$ its equivalence class in Y. Now, in view of the inequalities (3.6) and (3.8), $T \in \mathfrak{L}(X(k, 0); Y) \cap \mathfrak{L}(X(k, 1); Y)$ where in (2.2) we may take $m_0 = h^{-\alpha_k}$, $m_1 = C_3 C_4 h^{-\eta_k + \sigma_k}$. Hence, in view of (2.3), we have

$$\inf_{\psi^k \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j \delta_{jk} - \psi_j^k \|_{X(j,0)} \leqq C_2(\beta_k)(C_3 C_4)^{\beta_k} h^{-\eta_k \beta_k} \| u_k \|_{X(k,\beta_k)} \tag{3.9}$$

for all $u_k \in X(k, \beta_k)$.

From (3.3) we also have that for $l \neq k$

$$\inf_{\psi^l \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j \delta_{jl} - \psi_j^l \|_{X(j,0)} \leqq C_4 h^{-\eta_l} \| u_l \|_{X(l,\beta_l)}. \tag{3.10}$$

The proof of the lemma now follows from the inequalities (3.5), (3.9) and (3.10).

Lemma 3.2. *Suppose that the conditions of Theorem* 1 *are satisfied. Then, for all* $u \in X(\beta)$, *we have*

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j - \chi_j \|_{X(j,0)} \leqq C_5 \sum_{j=1}^{n} \| u_j \|_{X(j,\beta_j)}, \tag{3.11}$$

where C_5 *is a constant which is independent of* h *and* u.

Proof. Let $M = \max \{C_1(0, \beta_1), \cdots, C_1(0, \beta_n), C_2(\beta_1), \cdots, C_2(\beta_n), C_3, 1\}$. Now we have

$$\begin{aligned} \inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j - \chi_j \|_{X(j,0)} &\leqq \sum_{j=1}^{n} h^{-\alpha_j} \| u_j \|_{X(j,0)} \\ &\leqq M \sum_{j=1}^{n} h^{-\alpha_j} \| u_j \|_{X(j,\beta_j)}, \end{aligned} \tag{3.12}$$

where we may assume that max $\alpha_j > 0$, $j = 1, \cdots, n$, for otherwise (3.12) is just (3.11). In general then, (3.12) is a poor estimate. We shall now systematically improve (3.12) by using Lemma 3.1 in order to obtain the estimate (3.11). We proceed as follows. Let $\alpha_k = \max\{\alpha_1, \cdots, \alpha_n\}$. Now apply Lemma 3.1 to the inequality (3.12) where we take $C_4 = M$ and $\alpha_j = \eta_j$, $j = 1, \cdots, n$. It is easy to see from (3.4) that

$$\inf_{\chi \in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \| u_j - \chi_j \|_{X(j,0)} \leqq M^{1+2\rho} \sum_{j=1}^{n} h^{-\eta_j^{(1)}} \| u \|_{X(j,\beta_j)} \tag{3.13}$$

where $\rho = \max\{\beta_1, \cdots, \beta_n\}$ and $\eta_j^{(1)} = \alpha_j$ if $j \neq k$ and $\eta_k^{(1)} = \alpha_k \beta_k$. We repeat this process using Lemma 3.1, now using the inequality (3.13) instead of (3.12). Iterating after s steps we arrive at

$$\inf_{\chi\in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \|u_j - \chi_j\|_{X(j,0)} \leqq M_s \sum_{j=1}^{n} h^{-\eta_j^{(s)}} \|u\|_{X(j,\beta_j)} \tag{3.14}$$

where

$$\begin{aligned} \eta_j^{(s)} &= \alpha_j(\beta_j)^{s_j}, \qquad \alpha_j \neq 0, \\ &= 0, \qquad\qquad \alpha_j = 0, \end{aligned}$$

$j = 1, \cdots, n$, $s = \sum_{j=1}^{n} s_j$ and $M_s = M^{[2\sum_{l=0}^{s} \rho^l - 1]}$. Since $M_s \leqq M^{2/(1-\rho)}$ and $\alpha_j(\beta_j)^{s_j}$ can be made arbitrarily small for s_j sufficiently large, it follows that each $\eta_j^{(s)}$ can be made arbitrarily small after a sufficient number of iterations which proves the lemma.

Proof of Theorem 3.1. We have, noting Remark 3.1,

$$\inf_{\chi\in S^h} \sum_{j=1}^{n} h^{-\mu_j} \|u - \chi_j\|_{X(j,0)} \leqq \sum_{l=1}^{n} \left(\inf_{\psi^l\in S^h} \sum_{j=1}^{n} h^{-\mu_j} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,0)} \right). \tag{3.15}$$

Now, for each $l = 1, \cdots, n$, trivially

$$\inf_{\psi^l\in S^h} \sum_{j=1}^{n} h^{-\mu_j} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,0)} \leqq h^{-\mu_l} \|u_l\|_{X(l,0)}. \tag{3.16}$$

In view of Lemma 3.2, Remark 3.2 and (3.1), we have

$$\begin{aligned} \inf_{\psi^l\in S^h} \sum_{j=1}^{n} h^{-\mu_j} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,0)} &\leqq \inf_{\psi^l\in S^h} \sum_{j=1}^{n} h^{-\alpha_j} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,0)} \\ &\leqq C_5 \inf_{\psi^l\in S^h} \sum_{j=1}^{n} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,\beta_j)} \\ &\leqq C_5C_3 h^{\sigma_l} \|u_l\|_{X(l,1)} \end{aligned} \tag{3.17}$$

for all $u_l \in X(l, 1)$.

We can interpolate the inequalities (3.16) and (3.17). In (b) of Condition I, we take Y to be the quotient space of $X(0; h; \mu)$ modulo the subspace S^h, T to be the same mapping as in the proof of Lemma 3.1, with k there replaced by any $l = 1, \cdots, n$, $m_0 = h^{-\mu_l}$ and $m_1 = C_3C_5h^{\sigma_l}$. Then, for any $0 < \theta_l < 1$, we have

$$\inf_{\psi^l\in S^h} \sum_{j=1}^{n} h^{-\mu_j} \|u_j\delta_{jl} - \psi_j^l\|_{X(j,0)} \leqq C_1(\theta_l)(C_3C_5)^{\theta_l} h^{-\mu_l + (\sigma_l+\mu_l)\theta_l} \|u_l\|_{X(l,\theta_l)}. \tag{3.18}$$

The inequality (3.2) now follows easily from (3.15) and (3.18).

As a consequence of Theorem 3.1 we have the following:

COROLLARY 3.1. *Suppose that the conditions of Theorem* 3.1 *are satisfied and let* $T \in \mathfrak{L}(X(0; h; \mu); X(0; h; \mu))$ *such that* $T\chi = \chi$ *for all* $\chi \in S^h$. *Then*

$$\|u - Tu\|_{X(0;h;\mu)} \leqq C(\theta) \sum_{j=1}^{n} h^{-\mu_j + (\sigma_j+\mu_j)\theta_j} \|u_j\|_{X(j,\theta_j)}, \tag{3.19}$$

where u, μ, σ *and* θ *are as in Theorem* 3.1.

Proof. By hypothesis

$$||u - Tu||_{X(0;h;\mu)} \leqq C\,||u||_{X(0;h;\mu)}.$$

Hence, for any $\chi \in S^h$,

$$\begin{aligned} ||u - Tu||_{X(0;h;\mu)} &= ||u - \chi - T(u - \chi)||_{X(0;h;\mu)} \\ &\leqq C\,||u - \chi||_{X(0;h;\mu)} = C\left(\sum_{j=1}^{n} h^{-\mu_j}\,||u_j - \chi_j||_{X(j,0)}\right). \end{aligned}$$

The proof now follows from Theorem 3.1 by choosing χ to be the best approximation in S^h to u in $X(0; h; \mu)$.

Part II

4. Some Function Spaces, Subspaces and Approximation Theoretic Properties. In this part, we introduce some particular function spaces and certain classes of finite-dimensional subspaces. We shall then apply Theorem 3.1 to obtain various approximation theoretic results concerning these function spaces.

A. *Some Particular Spaces and Their Properties.* We shall first briefly state some notions concerning the theory of interpolation of Banach spaces (see [17], [24] for further details).

Suppose $\mathcal{B}_0$ and $\mathcal{B}_1$ are Banach spaces which are continuously imbedded in a topological vector space $\mathcal{B}$. Their sum

$$\mathcal{B}_0 + \mathcal{B}_1 = \{u; u = u_0 + u, u_i \in \mathcal{B}_i, i = 0, 1\}$$

is a subspace of $\mathcal{B}$. For $u \in \mathcal{B}_0 + \mathcal{B}_1$ and $0 < t < \infty$ form the functional

$$K(t, u) = \inf_{u=u_0+u_1} (||u||_{\mathcal{B}_0} + t||u||_{\mathcal{B}_1}).$$

If $0 < \theta < 1$, $1 \leqq q \leqq \infty$, we denote by $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ the Banach space with the norm

$$(4.1)\qquad \begin{aligned} ||u||_{(\mathcal{B}_0,\mathcal{B}_1)_{\theta,q}} &= \left(\int_0^\infty (t^{-\theta} K(t, u))^q \frac{dt}{t}\right)^{1/q}, \quad &&\text{if } 1 \leqq q < \infty, \\ &= \sup_{t>0} t^{-\theta} K(t, u), &&\text{if } q = \infty. \end{aligned}$$

We note that if $\mathcal{B}_0 = \mathcal{B}_1$ then $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q} = \mathcal{B}_0$ with

$$(4.2)\qquad ||u||_{(\mathcal{B}_0,\mathcal{B}_1)_{\theta,q}} = C(q, \theta)||u||_{\mathcal{B}_0}$$

where

$$(4.3)\qquad \begin{aligned} C(q, \theta) &= \left(\frac{1}{q\theta(1 - \theta)}\right)^{1/q}, \quad &&\text{if } 1 \leqq q < \infty, \\ &= 1, &&\text{if } q = \infty. \end{aligned}$$

The next lemma says that the spaces $(\mathcal{B}_0, \mathcal{B}_1)^{\theta,q}$ satisfy Condition I(b).

Lemma 4.1. *Suppose that Y is a Banach space and $\mathcal{B}_0$, $\mathcal{B}_1$ are as above. Let $T \in \mathcal{L}(\mathcal{B}_i; Y)$, $i = 0, 1$, with*

$$(4.4)\qquad ||Tu||_Y \leqq M_i||u||_{\mathcal{B}_i}, \qquad i = 0, 1.$$

Then for $0 < \theta < 1$ *and* $1 \leqq q \leqq \infty$, $T \in \mathcal{L}((\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}, Y)$ *and*

$$||Tu||_Y \leqq (C(q, \theta))^{-1} M_0^{1-\theta} M_1^{\theta} ||u||_{(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}}, \tag{4.5}$$

where $C(q, \theta)$ *is given by* (4.3).

We shall now specialize to some particular function spaces. Suppose that Ω is an open set in N-dimensional Euclidian space $\mathbf{R}^N$ such that either $\Omega = \mathbf{R}^N$ or else Ω is bounded with a C^∞ boundary $\partial\Omega$ which lies on one side of Ω (see [23]). We shall consider the following spaces of real (or complex valued) functions defined on Ω.

(i) *The Sobolev Spaces* $W_p^m(\Omega)$. (a) Let $V = C^\infty(\bar{\Omega})$ if Ω is bounded and $V = C_0^\infty(\mathbf{R}^N)$ if $\Omega = \mathbf{R}^N$. If $1 \leqq p < \infty$ and m is a nonnegative integer, then $W_p^m(\Omega)$ is the completion of V under the norm

$$||u||_{W_p{}^m} = \left(\sum_{|\alpha| \leqq m} ||D^\alpha u||_{W_p{}^0}^p \right)^{1/p}$$

where α denotes a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N)$, $D^\alpha u = \partial^{|\alpha|} u / (\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N})$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $W_p^0 = L_p(\Omega)$ with

$$||u||_{W_p{}^0}^p = \int_\Omega |u|^p \, dx.$$

Note. We have written W_p^m instead of $W_p^m(\Omega)$ when the domain considered is evident.

(b) If $m > 0$ and not an integer, set $m = [m] + s$ where $[m]$ is the integral part of m. Then, $W_p^s(\Omega) = W_p^s$ is the subspace of $W_p^{[m]}$ of all elements u such that

$$||u||_{W_p{}^m} = \left(||u||_{W_p{}^{[m]}}^p + \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p}$$

is finite.

(c) Again, let $V = C^\infty(\bar{\Omega})$ if Ω is bounded and $V = C_0^\infty(\mathbf{R}^N)$ if $\Omega = \mathbf{R}^N$. If $p = 2$ and m is any real number $m < 0$, then $W_2^m(\Omega) = W_2^m$ is the completion of V under the norm

$$||u||_{W_2{}^m} = \sup_{v \in V} \frac{(u, v)}{||v||_{W_2{}^{-m}}},$$

where

$$(u, v) = \int_\Omega u \bar{v} \, dx.$$

(ii) *The Spaces* $C^m(\bar{\Omega}) = W_\infty^m(\Omega)$. If m is a nonnegative integer and Ω is bounded, we set $W_\infty^m(\Omega) = W_\infty^m = C^m(\bar{\Omega})$, the set of functions having continuous derivatives up to order m on $\bar{\Omega}$ with the usual norm

$$||u||_{W_\infty{}^m} = \sum_{|\alpha| \leqq m} \max_{x \in \bar{\Omega}} |u(x)|.$$

If $m > 0$ and not an integer, set $m = [m] + s$. Then $W_\infty^m(\Omega)$ is the subset of $W_\infty^{[m]}(\Omega)$ such that the norm

$$||u||_{W_\infty{}^m} = ||u||_{W_\infty{}^{[m]}} + \sum_{|\alpha| = [m]} \sup_{x, y \in \bar{\Omega}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^s},$$

is finite.

(iii) *Besov Spaces* $B_p^{m,q}(\Omega)$. For any $0 < m < l$, l an integer, $1 \leqq q \leqq \infty$ and $1 \leqq p \leqq \infty$, $B_p^{m,q}(\Omega)$ is the Besov space defined by interpolation

$$B_p^{m,q}(\Omega) = (W_p^0(\Omega), W_p^l(\Omega))_{\theta,q}, \qquad \text{where } \theta = m/l.$$

The definition can be shown to be independent of the choice of l (up to equivalence of norms).

(iv) *The Boundary Spaces* $W_p^m(\partial\Omega)$ *and* $B_p^{m,q}(\partial\Omega)$. Suppose that Ω is bounded and either $m > 0$ and $1 \leqq p \leqq \infty$ or $-\infty < m < \infty$ and $p = 2$. Since $\partial\Omega$ is of class C^∞, $\bar{\Omega}$ can be covered by an interior subdomain and a finite number of boundary patches $\{0_j\}$ with the following property: For each j there is a C^∞ homeomorphism T_j which maps $\bar{0}_j$ onto the ball $|x| \leqq 1$ and $0_j \cap \partial\Omega$ into the hyperplane $x_N = 0$.

The set $\{0_j \cap \partial\Omega\}$ forms an $(N-1)$-dimensional open covering of $\partial\Omega$. Let $\sum \xi_j^2 = 1$ be a partition of unity subordinate to this covering which may be assumed to be such that $\xi_j \in C_0^\infty(0_j)$. Let $\mathcal{S}^K$ denote the unit ball in $\mathbf{R}^K$, then $T_j(0_j) = \mathcal{S}^N$ and $T_j(0_j \cap \partial\Omega) = \mathcal{S}^{N-1}$. For $g \in C^\infty(\partial\Omega)$ set $\tau_j g(x) = g(T_j^{-1}x)$, $x \in \mathcal{S}^{N-1}$ and

$$(4.6) \qquad |g|_{W_p^m} = \left(\sum (\|\tau_j \xi_j g\|_{W_p^m}^{\mathcal{S}^{N-1}})^p\right)^{1/p},$$

$$(4.7) \qquad |g|_{B_p^{m,q}} = \left(\sum (\|\tau_j \xi_j g\|_{B_p^{m,q}}^{\mathcal{S}^{N-1}})^p\right)^{1/p},$$

where the norms on the right of (4.6) and (4.7) denote the norm in $W_p^m(\mathcal{S}_1^{N-1})$ and $B_p^{m,q}(\mathcal{S}_1^{N-1})$, respectively. The spaces $W_p^m(\partial\Omega)$ and $B_p^{m,q}(\partial\Omega)$, respectively, are defined to be the completions of $C^\infty(\partial\Omega)$ with respect to the norms defined in (4.6) and (4.7). It is not difficult to show that all possible choices of the 0_j and ξ_j give equivalent spaces.

We shall now list some known results concerning the above-mentioned spaces (see [21]) which we shall need in the applications of Theorem 3.1.

Lemma 4.2. (a) *If* $p = q = 2$ *then* $B_2^{m,2}(\Omega) = W_2^m(\Omega)$ *for all* $m > 0$.

(b) *If* $1 \leqq p \leqq \infty$ *and* $m > 0$ *is not an integer then* $B_p^{m,p}(\Omega) = W_p^m(\Omega)$.

(c) *If* $1 \leqq p \leqq \infty$ *and* $m > 0$ *is any real number then* $W_p^m(\Omega) \subset B_p^{m,\infty}(\Omega)$.

(d) *If* $0 < m_0 < m_1$, $0 < \theta < 1$, $1 \leqq q_0, q_1, q \leqq \infty$ *and* $1 \leqq p \leqq \infty$, *then*

$$(B_p^{m_0,q_0}(\Omega), B_p^{m_1,q_1}(\Omega))_{\theta,q} = B_p^{m,q}(\Omega), \qquad m = (1-\theta)m_0 + \theta m_1,$$

where if either m_0 *or* m_1 *are integers then* $B_p^{m_i,q_i}(\Omega)$ *may be replaced by* $W_p^{m_i}(\Omega)$; *i.e. if* m_0 *and* m_1 *are integers* $(W_p^{m_0}(\Omega), W_p^{m_1}(\Omega))_{\theta,q} = B_p^{m,q}(\Omega)$.

(e) *If* $-\infty < m_0 \leqq m_1 < \infty$ *and* $0 < \theta < 1$ *then*

$$(W_2^{m_0}(\Omega), W_2^{m_1}(\Omega))_{\theta,2} = W_2^m(\Omega), \qquad m = (1-\theta)m_0 + \theta m_1.$$

(f) *In the case that* Ω *is bounded and* $\partial\Omega$ *is as described previously then the results of* (a), (b), (c), (d) *and* (e) *hold with* Ω *replaced by* $\partial\Omega$. *In the above, equality is to be understood with equivalence of norms and containment is to be understood in the sense of a continuous injection.*

B. *Some Finite-Dimensional Subspaces of* $W_p^k(\Omega)$. One of the purposes of this paper is to investigate some approximation properties of certain classes of finite-dimensional subspaces of the spaces defined in 4.A. Many different subspaces which have been described recently in the literature have certain approximation theoretic properties in common. We shall consider a class of subspaces defined by a single property common to most of these specific subspaces. More precisely, we define the subspaces $S^h_{k,r,p}(\Omega)$ as follows: Let h, $0 < h < 1$, be a parameter. For any two given nonnegative integers k and r, with $k < r$ and $1 \leqq p \leqq \infty$, $S^h_{k,r,p}(\Omega)$ is any one parameter family of subspaces of $W^k_p(\Omega)$ such that: (*) For any $u \in W^r_p(\Omega)$, there exists a $\bar{u} \in S^h_{k,r,p}(\Omega)$ and a constant C independent of h and u such that

$$||u - \bar{u}||_{W_p^k} \leqq Ch^{r-k}||u||_{W_p^r}. \tag{4.8}$$

This is obviously equivalent to the condition that

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} ||u - \chi||_{W_p^k} \leqq Ch^{r-k}||u||_{W_p^r}. \tag{4.9}$$

In the literature, $S^h_{k,r,p}(\Omega)$ (there is no standard notation) sometimes denotes subspaces satisfying the seemingly stronger condition (**): For any $u \in W^j_p(\Omega)$, there exists a constant C independent of h and u such that

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} ||u - \chi||_{W_p^l} \leqq Ch^{j-l}||u||_{W_p^j} \tag{4.10}$$

for all nonnegative integers j and l with $l \leqq k$ and $l \leqq j \leqq r$.

The construction of such spaces has attracted much attention recently. For example, S. Hilbert [22] constructs spaces of splines on uniform meshes in $\mathbf{R}^N$ which satisfy condition (*) for certain choices of k and r. Schultz [29] has studied many finite-dimensional subspaces on rectilinear domains in $\mathbf{R}^N$ which satisfy (*). The papers of Aubin [3], Bramble and Zlámal [16] and Di Gugliemo [18] also contain examples of subspaces satisfying (*). The work of Babuška [7] and Fix and Strang [19] are also important in this regard.

C. *Some Other Properties of the Subspaces.* As the first application of Theorem 3.1, we shall show that (4.9) implies (4.10) and in the case $p = 2$, (4.9) implies (4.10) where l and j also may be taken to be negative. In the case $p = 2$, this says that a space of the type $S^h_{k,r,2}(\Omega)$ is automatically a space of type $S^h_{l,j,2}(\Omega)$ where l and j are any real numbers satisfying $l \leqq k$ and $l \leqq j \leqq r$.

THEOREM 4.1. *Let $S^h_{k,r,p}(\Omega)$ be given satisfying* (4.9), *then*

(i) *If l and j are any two real numbers satisfying* $0 \leqq l \leqq k$ *and* $l \leqq j \leqq r$, *there exists for all $u \in W^j_p(\Omega)$ a constant $C = C(l, j)$ which is independent of h and u such that*

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} ||u - \chi||_{W_p^l} \leqq Ch^{j-l}||u||_{W_p^j}. \tag{4.11}$$

(ii) *If* $p = 2$, *then* (4.11) *holds where we may take l and j to be any two real numbers* (*positive or negative*) *satisfying* $l \leqq k$ *and* $l \leqq j \leqq r$.

Proof. We first prove (i). In Theorem 3.1 let us take the case where $n = 1$ and $X(1, \theta) = W_p^{l(1-\theta_1)+r\theta_1}(\Omega)$, $0 \leqq \theta_1 \leqq 1$, and $\beta_1 = (k - l)/(r - l) > 0$; that is $X(1, \beta_1) = W^k_p(\Omega)$. Hence (2.1) is satisfied. In order to show that Condition I is satisfied, it remains to show that (2.3) holds. Take Y to be any Banach space and in Lemma 4.1 $\mathcal{B}_0 = W^l_p(\Omega)$, $\mathcal{B}_1 = W^r_p(\Omega)$ and $q = \infty$. Then, for any $T \in \mathcal{L}(\mathcal{B}_i; Y)$, $i = 1, 2$, satisfying (4.4), we have

$$||Tu||_Y \leqq M_0^{1-\theta_1}M_1^{\theta_1}||u||_{B_p^{\delta,\infty}}, \qquad 0 < \theta_1 < 1,$$

where $\delta = l(1 - \theta_1) + r\theta_1$. Now, since $W^\delta_p(R) \subseteq B_p^{\delta,\infty}(R)$ with a continuous injection, we find that

$$||Tu||_Y \leqq C(\theta_1)M_0^{1-\theta_1}M_1^{\theta_1}||u||_{W_p^\delta}, \qquad 0 < \theta_1 < 1.$$

Therefore, (2.3) is satisfied and hence Condition I is.

We now take $S^h = S^h_{k,r,p}(\Omega)$ in Theorem 3.1, where $S^h_{k,r,p}(\Omega)$ satisfies (4.9) and therefore $\sigma_1 = r - k$. We therefore have from (3.2) taking $\mu_1 = \alpha_1 = k - l$ and $\theta_1 = (j - l)/(r - l)$ that

$$\inf_{\chi \in S_{k,r,p}{}^h(\Omega)} h^{-(k-l)} \|u - \chi\|_{W_p{}^l} \leqq C(l, j) h^{-k+j} \|u\|_{W_p{}^j}$$

and (4.11) follows, which completes the proof of (i).

The proof of (ii) is simpler. Here we take $X(1, \theta_1) = W_2^{l(1-\theta_1)+r\theta_1}(\Omega)$, $0 \leqq \theta_1 \leqq 1$, where l is any real number satisfying the conditions of (ii) of this theorem. Clearly, (2.1) is satisfied. To show that (b) of Condition I is satisfied, we take $\mathfrak{B}_0 = W_2^l(\Omega)$ and $\mathfrak{B}_1 = W_2^r(\Omega)$ with $q = 2$ in Lemma 4.1. Then,

$$\|Tu\|_Y \leqq 2\sqrt{2}\, m_0^{1-\theta_1} m_1^{\theta_1} \|u\|_{(W_2{}^l, W_2{}^r)_{\theta_1,2}}.$$

But $(W_2^l(\Omega), W_2^r(\Omega))_{\theta,2} = W_2^{l(1-\theta)+\theta r}(\Omega)$ with equivalence of norms which proves (2.3) and hence Condition I is satisfied. The remainder of the proof of (ii) now proceeds as in the proof of (i) and will not be given.

D. *Approximation on* $\partial\Omega$. We shall now consider some questions which arise naturally. Suppose that Ω is bounded and we are given a subspace of type $S^h_{k,r,p}(\Omega)$. Let us look at the restrictions to $\partial\Omega$ of elements of $S^h_{k,r,p}(\Omega)$ (assuming now that these are well defined in the sense of a trace and belong to some Sobolev space on $\partial\Omega$).

In Theorem 4.2, we shall consider the following question: What properties, relative to best approximation, do the restrictions to $\partial\Omega$ of elements of $S^h_{k,r,p}(\Omega)$ have relative to functions in Sobolev spaces on $\partial\Omega$? We shall show that in fact they have some very nice approximation properties on $\partial\Omega$ which are analogous to their properties on Ω. Other questions of a related nature are also treated. Before proceeding, we shall need some preliminaries.

We shall use B_j, $j = 0, \cdots, l$, to denote boundary differential operators of the form

$$B_j u = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x)\, D^\alpha u(x), \qquad x \in \partial\Omega, \tag{4.12}$$

where, for simplicity, we shall assume that the coefficients $b_{j\alpha}(x) \in C^\infty(\partial\Omega)$ and $0 \leqq m_j$ denotes the order of the highest order derivatives occurring.

Definition. A system of boundary operators $\{B_j\}$, $j = 0, \cdots, l$, of the form (4.12) is said to be normal if $m_i \neq m_j$ for $i \neq j$ and if for each $j = 0, \cdots, l$

$$\sum_{|\alpha| = m_j} b_{j\alpha}(x) \nu^\alpha \neq 0$$

at each point $x \in \partial\Omega$. Here, $\nu = \nu(x)$ is a unit normal vector to $\partial\Omega$ at x and $\nu^\alpha = \nu_1^{\alpha_1} \cdots \nu_N^{\alpha_N}$. The system $\{B_j\}$ is said to be a Dirichlet system of order $l + 1$ if it is normal and $m_j = j$ for $j = 0, \cdots, l$.

We note that given a normal system $\{B_j\}$, we can always add other boundary operators to the system so that the resulting system is a Dirichlet system of order $m_l + 1$. We shall also need the following result concerning traces on $\partial\Omega$.

LEMMA 4.3 (CF. [24]). *Let B_j be an operator of the form* (4.12). *The mapping $u \to B_j u$ of $C^\infty(\bar\Omega)$ into $C^\infty(\partial\Omega)$, completed by continuity, is continuous from $W_p^s(\Omega) \to B_p^{s-m_j-1/p,p}(\partial\Omega)$ for any $m_j + 1/p < s$ and $1 < p < \infty$. Furthermore, let $\{B_j\}$ be a Dirichlet system of order $l + 1$. Then, there exists a continuous linear extension operator*

E mapping $\prod_{j=0}^{l} B_p^{s-j-1/p,p}(\partial\Omega) \to W_p^s(\Omega)$ *such that* $B_j(Eg) = g_j$, $j = 0, \cdots, l$, *for all* $g \in \prod_{j=0}^{l} B_p^{s-j-1/p,p}(\partial\Omega)$ *with* $l + 1/p < s$ *and* $1 < p < \infty$.

We are now in a position to prove our first result concerning the approximation-theoretic properties on $\partial\Omega$ of the subspaces of the type $S_{k,r,p}^h(\Omega)$.

THEOREM 4.2. *Let* $\{B_j\}$, $j = 0, \cdots, l$, *be normal with* $m_0 < \cdots < m_l$ *and suppose that* $S_{k,r,p}^h(\Omega)$, $1 < p < \infty$, *is given satisfying* (4.9) *with* $m_l + 1/p < k$.

(i) *Let* s_j *and* λ_j *be given real numbers satisfying* $0 \leqq s_j \leqq k - m_j - 1/p$ *and* $0 \leqq \lambda_j \leqq r - m_j - 1/p - s_j$, $j = 0, \cdots, l$. *Then, for all* $g = (g_0, \cdots, g_l) \in \prod_{j=0}^{l} W^{s_j+\lambda_j}(\partial R)$

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} \left(\sum_{j=0}^{l} h^{m_j+s_j} |g_j - B_j\chi|_{W_p^{s_j}} \right) \leqq C\left(\sum_{j=0}^{l} h^{m_j+s_j+\lambda_j} |g_j|_{W_p^{s_j+\lambda_j}} \right) \tag{4.13}$$

where C *is a constant which is independent of* h *and* g.

(ii) *If* $p = 2$, *then* (4.13) *holds with* λ_j *as above and* s_j *any real number satisfying* $s_j \leqq k - m_j - 1/p$.

Before proceeding with the proof, let us note one consequence of Theorem 4.2. Suppose we denote by $S_{k,r,p}^h(\partial\Omega)$ any one-parameter family of finite-dimensional subspaces of $W_p^k(\partial\Omega)$ having the property that for any $g \in W_p^r(\partial\Omega)$

$$\inf_{\chi \in S_{k,r,p}^h(\partial\Omega)} |g - \chi|_{W_p^k} \leqq Ch^{r-k} |g|_{W_p^r}, \tag{4.14}$$

where C is a constant which is independent of h and g. Suppose we take $l = 0$ in Theorem 4.2 and $B_0 u = u|_{\partial\Omega}$. In this case, we have the following:

COROLLARY 4.1. *The restrictions to the boundary of the elements of a space of type* $S_{k,r,p}^h(\Omega)$, $1 < p < \infty$, *is a space of type* $S_{\rho,\sigma,p}^h(\partial\Omega)$ *for all real numbers* $0 \leqq \rho \leqq k - 1/p$ *and* $\rho \leqq \sigma \leqq r - 1/p$. *Furthermore, if* $p = 2$, *then this is also true for all real numbers* $\rho \leqq k - 1/p$ *and* $\rho \leqq \sigma \leqq r - 1/p$.

Proof of Theorem 4.2. Without loss of generality, we may assume that the $\{B_j\}$ is a Dirichlet system of order $l + 1$. For if it is not then, as remarked previously, we can always augment it so that it is one. Let us denote by $\{B_j'\}$, $j = 0, \cdots, m_l$, the augmented system where $B_{m_j}' = B_j$. If (4.13) holds for the augmented system for given $g = (g_0', \cdots, g_{m_l}')$, it certainly holds for the original system by taking $g_{m_j}' = g_j$ for $j = 0, \cdots, l$, and $g_j' = 0$ otherwise. Now taking $\{B_j\}$ to be a Dirichlet system of order $l + 1 = m_l + 1$, let E be the extension operator discussed in Lemma 4.3. Since $1 < p < \infty$, $\rho - j - 1/p$ is not an integer for any integer ρ, and hence $B_p^{\rho-j-1/p,p}(\partial\Omega) = W_p^{\rho-j-1/p}(\partial\Omega)$ for $j + 1/p < \rho$, $j = 0, \cdots, l$. Set $Eg = u$ for $g \in \prod_{j=0}^{l} B_p^{r-j-1/p,p}(\partial\Omega)$. Then, from Lemma 4.3, we have that for any $\chi \in S_{k,r,p}^h(\Omega)$

$$\sum_{j=0}^{l} |g_j - B_j\chi|_{W_p^{k-j-1/p}} \leqq C||u - \chi||_{W_p^k}. \tag{4.15}$$

In (4.15), let χ be the best approximation in $S_{k,r,p}^h(\Omega)$ to u in the norm of $W_p^k(\Omega)$. Then from (4.15), (4.9) and Lemma 4.3, it follows that

$$\begin{aligned} \inf_{\psi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} |g_j - B_j\psi|_{W_p^{k-j-1/p}} &\leqq Ch^{r-k}||u||_{W_k^r} \\ &\leqq Ch^{r-k} \sum_{j=0}^{l} |g_j|_{W_p^{r-j-1/p}}. \end{aligned} \tag{4.16}$$

In order to prove part (i) of Theorem 4.2, we shall apply Theorem 3.1 in the following manner: Let us take $X(j, \theta_i) = W_p^{\rho_i}(\partial\Omega)$ where $\rho_i = \theta_i(r - j + 1 - 1/p) + (1 - \theta_i)s_i$ for $0 < \theta_i < 1$ and $j = 1, \cdots, l + 1$. Obviously Condition I(a) is satisfied and it is easily seen that Condition I(b) is satisfied using the fact that $W_p^{\rho_i}(\partial\Omega) \subset (W_p^{s_i}(\partial\Omega), W_p^{r-j+1-1/p}(\partial\Omega))_{\theta_i,\infty} = B_p^{\rho_i,\infty}(\partial\Omega)$ for $0 < \theta_i < 1$, and s_i satisfying, $0 \leqq s_i < r - j + 1 - 1/p$, $j = 1, \cdots, l + 1$. Now, taking

$$\prod_{j=1}^{l+1} X(j, \beta_i) = \prod_{j=1}^{l+1} W_p^{k-j+1-1/p}(\partial\Omega),$$

it follows that $\beta_i = (k - j + 1 - 1/p - s_i)/(r - j + 1 - 1/p - s_i)$ for $j=1, \cdots, l + 1$. We now identify S^h with the finite-dimensional subspace of $\prod_{j=1}^{l+1} W_p^{k-j+1-1/p}(\partial\Omega)$ of elements of the form $(B_0\chi, \cdots, B_l\chi)$ for $\chi \in S^h_{k,r,p}(\Omega)$. Hence we have from (4.16) that Condition II is satisfied with $\sigma_i = r - k$, $j = 1, \cdots, l + 1$. Now $\alpha_i = \beta_i\sigma_i/(1 - \beta_i) = k - j + 1 - 1/p - s_i$, $j = 1, \cdots, l + 1$, and we obtain from (3.2) that

$$\text{(4.17)} \quad \inf_{\chi \in S_{k,r,p}^h(\Omega)} \left(\sum_{j=0}^{l} h^{-(k-j-1/p-s_j)} |g_i - B_i\chi|_{W_p^{s_i}} \right) \leqq C \sum_{j=0}^{l} h^{-(k-j-1/p-s_j)+\lambda_i} |g_i|_{W_p^{s_i+\lambda_i}}$$

for any $1 < p < \infty$ and s_i and λ_i satisfying the conditions of the theorem. The inequality (4.13) now follows from (4.17) by multiplying both sides of (4.17) by $k - 1/p$ and hence part (i) is proved. The proof of part (ii) is simpler and will be left to the reader.

Remark 4.1. It can be easily seen from the proof of Theorem 4.2 that (i) of Theorem 4.2 also holds if the norms $|\cdot|_{W_p^{s_i+\lambda_i}}$ are replaced by $|\cdot|_{B_p^{s_i+\lambda_i,p}}$ on the right-hand side of (4.13) provided we take $0 < \lambda_i \leqq r - m_i - 1/p - s_i$ whenever s_i is an integer.

THEOREM 4.3. *Let us assume that*

(i) $\{B_j\}$, $j = 0, \cdots, l$, *is a normal system with* $m_0 < \cdots < m_l$.

(ii) $S^h_{k,r,p}(\Omega)$, $1 < p < \infty$, *is given with* $m_{l+1/p} < k < r$.

(iii) β, *and* s_i, $j = 0, \cdots, l$, *are real numbers satisfying* $m_l + 1/p < \beta < r$ *and* $0 \leqq s_i \leqq k - m_i - 1/p$, $j = 0, \cdots, l$.

(iv) $s_i \leqq \beta - m_i - 1/p$ *if* $\beta - m_i - 1/p \neq$ *integer*, $j = 0, \cdots, l$, *or* $s_i < \beta - m_i - 1/p$ *if* $\beta - m_i - 1/p =$ *integer*, $j = 0, \cdots, l$.

Then, for all $u \in W_p^\beta(R)$,

$$\text{(4.18)} \quad \inf_{\chi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} h^{m_i+s_i} |B_iu - B_i\chi|_{W_p^{s_i}} \leqq Ch^{\beta-1/p}||u||_{W_p^\beta},$$

where C is a constant which is independent of h and u.

Proof. Let us first consider the case where $\beta - m_i - 1/p$ is not an integer $j = 0, \cdots, l$. Then, a straightforward application of Theorem 4.2, with $\lambda_i = \beta - m_i - 1/p - s_i$, $j = 0, \cdots, l$, yields

$$\text{(4.19)} \quad \inf_{\chi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} h^{m_i+s_i} |B_iu - B_i\chi|_{W_p^{s_i}} \leqq Ch^{\beta-1/p} \sum_{j=0}^{l} |B_iu|_{W_p^{\beta-m_i-1/p}}.$$

But since $\beta - m_j - 1/p$ is not an integer

$$\sum_{j=0}^{l} |B_j u|_{W_p^{\beta - m_j - 1/p}} = \sum_{j=0}^{l} |B_j u|_{B_p^{\beta - m_j - 1/p, p}} \leqq C \|u\|_{W_p^\beta},$$

which together with (4.19) completes the proof for this case.

If the $\beta - m_j - 1/p$ are integers, for $j = 0, \cdots, l$, then we have, in view of Remark 4.1 and the assumption that $s_j < \beta - m_j - 1/p$ for $j = 1, \cdots, l$, that

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} h^{m_j + s_j} |B_j u - B_j \chi|_{W_p^{s_j}} \leqq C h^{\beta - 1/p} \sum_{j=0}^{l} |B_j u|_{B_p^{\beta - m_j - 1/p, p}},$$

from which the desired result easily follows.

For our next result, we shall prove an analog of Theorem 4.3, except in this case we shall not require that the system of operators $\{B_j\}$ be normal.

THEOREM 4.4. *Suppose that*

(i) $\{B_j\}$, $j = 0, \cdots, l$, *is a system of boundary operators of the form* (4.12) *of orders* $m_0 \leqq \cdots \leqq m_l$ ($\{B_j\}$ *need not be normal*).

(ii) $S_{k,r,p}^h(\Omega)$, $1 < p < \infty$, *is given with* $m_l + 1/p < k < r$.

(iii) s *and* β *are real numbers satisfying* $m_l + 1/p < \beta \leqq r$,

$$0 \leqq s \leqq \min(k - m_l - 1/p, \beta - m_l - 1/p),$$

if $\beta - m_l - 1/p$ *is not an integer and* $0 \leqq s < \min(k - m_l - 1/p, \beta - m_l - 1/p)$, *if* $\beta - m_l - 1/p$ *is an integer.*

Then, for all $u \in W_p^\beta(\Omega)$

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} |B_j(u - \chi)|_{W_p^{s + (m_l - m_j)}} \leqq C h^{\beta - 1/p - s - m_l} \|u\|_{W_p^\beta} \tag{4.20}$$

where C *is a constant which is independent of* h *and* u.

Proof. Using the smoothness properties of the coefficients $b_{j\alpha}$ and of $\partial\Omega$ it is not hard to see that for any $\chi \in S_{k,r,p}^h(\Omega)$

$$\sum_{j=0}^{l} |B_j u - \chi|_{W_p^{s + m_l - m_j}} \leqq C \sum_{j=0}^{m_l} \left| \frac{\partial^j (u - \chi)}{\partial \nu^j} \right|_{W_p^{s + m_l - j}}$$

where $\partial^j / \partial \nu^j$ denotes the jth order inward normal derivative to $\partial\Omega$ and C is a constant which is independent of u, χ and h. Hence, it follows that

$$\inf_{\chi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{l} |B_j(u - \chi)|_{W_p^{s + m_l - m_j}} \leqq \inf_{\psi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{m_l} \left| \frac{\partial^j (u - \psi)}{\partial \nu^j} \right|_{W_p^{s + m_l - j}}. \tag{4.21}$$

We can now apply Theorem 4.3 in the case that $B_j = \partial^j / \partial \nu^j$ and $s_j = s + m_l - j$ for $j = 0, \cdots, m_l$. We obtain from (4.18) that

$$\inf_{\psi \in S_{k,r,p}^h(\Omega)} \sum_{j=0}^{m_l} h^{s + m_l} \left| \frac{\partial^j (u - \psi)}{\partial \nu^j} \right|_{W_p^{s + m_l - j}} \leqq C h^{\beta - 1/p} \|u\|_{W_p^\beta} \tag{4.22}$$

for s and β satisfying the conditions of the theorem. The inequality (4.20) now follows from (4.21) and (4.22) which completes the proof.

PART III

In this part, we consider some general classes of boundary-value problems for $2m$th order elliptic operators. We shall present here a general theory on the approximation of solutions of such problems.

5. Preliminaries.

A. *Boundary-Value Problems.* Let Ω be a bounded domain in $\mathbf{R}^N$ with boundary $\partial\Omega$. We shall assume (for convenience) that $\partial\Omega$ is of class C^∞ and shall consider in Ω the operator A of order $2m$ with infinitely differentiable real coefficients:

$$(5.1) \qquad Au = A(x, D)u = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x) D^\beta u),$$

where, as usual, $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $\beta = (\beta_1, \cdots, \beta_N)$ are multi-indices, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. Set

$$A_0(x, \xi) = \sum_{|\alpha|, |\beta| = m} (-1)^m a_{\alpha\beta}(x) \xi^{\alpha+\beta}$$

where $\xi^{\alpha+\beta} = \xi_1^{\alpha_1+\beta_1} \cdots \xi_N^{\alpha_N+\beta_N}$. Note that A is not necessarily formally selfadjoint. Its formal adjoint A^* is given by

$$A^* u = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\beta\alpha}(x) D^\beta u).$$

It is assumed that A is uniformly elliptic; i.e., there exists a constant $a > 0$ independent of x such that

$$a^{-1} |\xi|^{2m} \leqq |A_0(x, \xi)| \leqq a |\xi|^{2m}$$

for all $x \in \bar{\Omega}$ and all $\xi \in \mathbf{R}^N$.

We shall consider the boundary problem

$$(5.2) \qquad \begin{aligned} Au &= f \quad \text{in } \Omega, \\ B_j u &= g_j, \qquad j = 0, \cdots, m-1, \quad \text{on } \partial\Omega, \end{aligned}$$

where the B_j's are given boundary differential operators of order m_j, $0 \leqq m_j < 2m - 1$ and f and g_j are given. The operators B_j are defined by (4.12).

All functions considered in remaining sections will be real valued. The conditions we shall place on our problem are as follows:

Condition III. (i) A is uniformly elliptic with coefficients in $C^\infty(\bar{\Omega})$.

(ii) The boundary system $\{B_j\}$ is normal, covers the operator A (see e.g. [23]) and has coefficients in $C^\infty(\partial\Omega)$. The order of B_j is m_j where we assume them ordered so that $0 \leqq m_0 < \cdots < m_{m-1} \leqq 2m - 1$.

(iii) The only solution of (5.2) in $C^\infty(\bar{\Omega})$ with zero data is the zero solution.

Remark 5.1. The smoothness requirements on the coefficients of the differential operators and the domain can be weakened considerably (cf. [12]).

B. *Sobolev Spaces.* In the remaining sections, we shall be concerned primarily with an L_2 theory for least squares methods. For simplicity, the inner products on the spaces $W_2^s(\Omega)$ and $W_2^s(\partial\Omega)$, introduced in Part II, will be denoted by $(\cdot\,, \cdot)_s$ and $\langle\cdot\,, \cdot\rangle_s$, respectively, and $\|\cdot\|_s$ and $|\cdot|_s$ will be used for their corresponding norms. We shall occasionally be interested in the Sobolev spaces $W_2^s(G)$ where G is an open set in $\mathbf{R}^N$, $G \neq \Omega$. In this case, the inner product and norm on $W_2^s(G)$ will be denoted by $(\cdot\ ,\ \cdot)_s^G$ and $||\cdot||_s^G$, respectively. For precise definitions, the reader is referred to Part II, Section 4A.

Remark 5.2. The spaces $W_2^s(G)$ have the following property (cf. [23]) which shall be useful later on. Let $s_1 < s_2$ be any two real numbers, then for any $0 < \theta < 1$ and all $u \in W_2^{s_2}(G)$

$$||u||_s^G \leqq C(\theta, s_1, s_2)(||u||_{s_1}^G)^{1-\theta}(||u||_{s_2}^G)^\theta \tag{5.3}$$

where $s = (1 - \theta)s_1 + \theta s_2$.

C. *Definition of a Solution and Some Further Preliminaries.* For any set of real numbers l and s_j, $j = 0, \cdots, m-1$, $W^{(l,s_j)}$ will denote the product space $W^{(l,s_j)} = W_2^l(\Omega) \times \prod_{j=0}^{m-1} W_2^{s_j}(\partial\Omega)$ with the inner product

$$(\cdot\ ,\ \cdot)_{(l,s_j)} = (\cdot\ ,\ \cdot)_l + \sum_{j=0}^{m-1} \langle \cdot\ ,\ \cdot \rangle_{s_j}. \tag{5.4}$$

It will be essential for later purposes to consider different inner-product structures on $W^{(l,s_j)}$. Let $0 < h < \infty$ and γ_j, $j = 0, \cdots, m-1$, be given real numbers. Then $W_{(h,\gamma_j)}^{(l,s_j)}$ will denote the Hilbert space whose elements are those of $W^{(l,s_j)}$ but equipped with the inner product

$$(\cdot\ ,\ \cdot)_{(l,s_j,h,\gamma_j)} = (\cdot\ ,\ \cdot)_l + \sum_{j=0}^{m-1} h^{-2\gamma_j}\langle \cdot\ ,\ \cdot \rangle_{s_j}. \tag{5.5}$$

We note that the norm induced by (5.5) is equivalent to the norm induced by (5.4).

We shall next give definitions of weak solutions of (5.2) under various assumptions on the regularity of the data. We shall use the following result which is basic in what follows. In this theorem and throughout the paper we shall use C to denote a generic constant not necessarily the same in any two places.

THEOREM 5.1 (CF. [12], [28]). *Under Condition* III, *we have that for any real number p*

$$||u||_p \leqq C\left(||Au||_{p-2m} + \sum_{j=0}^{m-1} |B_j u|_{p-m_j-1/2}\right) \tag{5.6}$$

for all $u \in C^\infty(\bar{\Omega})$. The constant C is independent of u but in general depends on p.

Definition. Let $F = (f, g_0, \cdots, g_{m-1}) \in W_2^{(p-2m, p-m_j-1/2)}$ and $F_n = (f^n, g_0^n, \cdots, g_{m-1}^n) \in C^\infty(\bar{\Omega}) \times \prod_{j=0}^{m-1} C^\infty(\partial\Omega)$ converge to F in $W_2^{(p-2m, p-m_j-1/2)}$ as $n \to \infty$. Let $u_n \in C^\infty(\bar{\Omega})$ be the solution of (5.2) with data F_n (it is well known that such u_n exists and is unique). Then in view of (5.6), we define the weak solution of (5.2) with data F to be the unique limit in $W_2^p(\Omega)$ of the sequence $\{u_n\}$.

In the case that $p \geqq 2m$, we have the following:

THEOREM 5.2 (CF. [23]). *Under Condition* III, *we have that if p is a real number $p \geqq 2m$, the mapping $\mathcal{P}u = (Au, B_0u, \cdots, B_{m-1}u)$ is a homeomorphism of $W_2^p(\Omega)$ onto $W_2^{(p-2m, p-m_j-1/2)}$.*

It will be convenient for later purposes to give an alternative form for (5.6) in the case that $p \leqq m_0 + \frac{1}{2}$.

LEMMA 5.1. *Suppose that Condition* III *is satisfied and let $h > 0$, $\gamma_j \geqq 0$, $j = 0, \cdots, m-1$, and $p \leqq m_0 + \frac{1}{2}$ be given real numbers. Then, for all $u \in W_2^p(\Omega)$ such that $Au \in W_2^{p-2m}(\Omega)$ and $B_ju \in W_2^{p-m_j-1/2}(\partial\Omega)$ (in the sense defined above), we have*

$$(5.7) \qquad \|u\|_p \leqq C \sup_{v \in C^\infty(\bar\Omega)} \left\{ \frac{(Au, Av)_0 + \sum_{j=0}^{m-1} h^{-2\gamma_j} \langle B_j u, B_j v \rangle_0}{\|Av\|_{2m-p} + \sum_{j=0}^{m-1} h^{-2\gamma_j} |B_j v|_{m_j+1/2-p}} \right\}$$

where C is a constant which is independent of u, h and γ_j, $j = 0, \cdots, m-1$.

Proof. In view of (5.6) and the definitions of Sobolev norms of negative order, we have

$$(5.8) \qquad \|u\|_p \leqq C \left\{ \sup_{\psi \in C^\infty(\bar\Omega)} \frac{(Au, \psi)_0}{\|\psi\|_{2m-p}} + \sum_{j=0}^{m-1} \sup_{\varphi_j \in C^\infty(\partial\Omega)} \frac{\langle B_j u, \varphi_j \rangle_0}{|\varphi_j|_{m_j+1/2-p}} \right\}$$

where we may restrict ourselves to those functions ψ and φ_j for which $\|\psi\|_{2m-p} = 1$ and $|\varphi_j|_{m_j+1/2-p} = 1$, $j = 0, \cdots, m-1$. Now, let $\{\psi^n\}$ and $\{\varphi_j^n\}$, $j = 0, \cdots, m-1$, be maximizing sequences (subject to the above conditions) for each of the respective terms on the right-hand side of (5.8). For each n let $v_n \in C^\infty(\bar\Omega)$ be the unique solution of $Av_n = \psi^n$ in Ω, $B_j v_n = h^{2\gamma_j}\varphi_j^n$ on $\partial\Omega$, $j = 0, \cdots, m-1$. Then

$$\|u\|_p \leqq C \left\{ \lim_{n\to\infty} \frac{(Au, Av_n)_0 + \sum_{j=0}^{m-1} h^{-2\gamma_j} \langle B_j u, B_j v_n \rangle_0}{\|Av_n\|_{2m-p} + \sum_{j=0}^{m-1} h^{-2\gamma_j} |B_j v_n|_{m_j+1/2-p}} \right\}$$

for which the lemma easily follows.

6. Finite Dimensional Subspaces of $W_2^k(\Omega)$. In this section, we shall discuss some approximation-theoretic properties of subspaces of the type $S_{k,r,2}^h(\Omega) = S_{k,r}^h$, discussed previously in Part II, Section 4. Our aim is to show that they have certain approximation-theoretic properties relative to the "data spaces" of the differential operators considered here. We refer the reader to Part II, Section 4B for a discussion of their basic properties.

THEOREM 6.1. *Let $S_{k,r,2}^h(\Omega) = S_{k,r}^h$ satisfy (4.9) with $2m = k < r$. Then, for all $F = (f, g_0, \cdots, g_{m-1}) \in W^{(\lambda,\lambda_j)}$, where $0 \leqq \lambda \leqq r - 2m$, $0 \leqq \lambda_j \leqq r - m_j - \frac{1}{2}$, $j = 0, \cdots, m-1$, there exists a constant C independent of F and h such that for $\gamma_j = 2m - m_j - \frac{1}{2}$ (where $0 \leqq m_j \leqq 2m-1$ is the order of B_j), $j = 0, \cdots, m-1$,*

$$(6.1) \qquad \inf_{\chi \in S_{k,r}^h} \left(\|f - A\chi\|_0 + \sum_{j=0}^{m-1} h^{-\gamma_j} |g_j - B_j\chi|_0 \right) \leqq C\left(h^\lambda |f|_\lambda + \sum_{j=0}^{m-1} h^{-\gamma_j+\lambda_j} |g|_{\lambda_j} \right).$$

Proof. The proof of Theorem 6.1 will follow from Theorem 5.1 and Theorem 3.1. In Theorem 3.1, let us identify the spaces $X(j, \theta_j)$, $j = 1, \cdots, m+1$, $0 \leqq \theta_j \leqq 1$, as follows: $X(1, \theta_1) = W_2^{\theta_1(r-2m)}(\Omega)$ and $W(j, \theta_j) = W_2^{\theta_j(r-m_j-s-1/2)}(\partial\Omega)$, $j = 2, \cdots, m+1$. Certainly, Condition I is satisfied. We now take

$$\beta_1 = 0, \qquad \beta_j = (2m - m_{j-2} - \tfrac{1}{2})/(r - m_{j-2} - \tfrac{1}{2}), \qquad j = 2, \cdots, m+1.$$

Then

$$\prod_{j=0}^{m+1} X(j, \beta_j) = L_2(\Omega) \times \prod_{l=0}^{m-1} W_2^{2m-m_l-1/2}(\partial\Omega) = W^{(0,2m-m_l-1/2)}.$$

Now, consider $\mathcal{P}(S_{k,r}^h)$ the image of $S_{k,r}^h$ under the mapping $\mathcal{P}(\chi) = (A\chi, B_0\chi, \cdots, B_{m-1}\chi)$. Since $k = 2m$ and $m_l \leqq 2m-1$, it follows that $\mathcal{P}(S_{k,r}^h)$ is a finite-dimensional subspace of $\prod_{j=0}^{m+1} X(j, \beta_j)$ and hence of

$$\prod_{j=0}^{m+1} X(j, 0) = L_2(\Omega) \times \prod_{l=1}^{m-1} L_2(\partial\Omega) = W^{(0,0)}.$$

We shall now show that the space $\mathcal{P}(S^h_{k,r})$ satisfies Condition II of Theorem 3.1 with $\sigma_j = r - 2m$. In fact, let $F = (f, g_0, \cdots, g_{m-1}) \in W^{(r-2m, r-m_j-1/2)}$. By Theorem 5.2, there exists a unique solution $u \in W_2^r(R)$ of (5.2) with data F, and from (4.9)

$$\inf_{\chi \in S_{k,r}^h} ||u - \chi||_{2m} \leqq h^{r-2m} ||u||_r. \tag{6.2}$$

But by Theorem 5.2, the norms $||u - \chi||_{2m}$ and $||u||_r$ are equivalent to the norms

$$||A(u - \chi)||_0 + \sum_{j=0}^{m-1} |B_j(u - \chi)|_{2m-m_j-1/2}$$

and

$$||Au||_{r-2m} + \sum_{j=0}^{m-1} |B_j u|_{r-m_j-1/2},$$

respectively. It then easily follows from (6.2) that

$$\inf_{\chi \in S_{k,r}^h} \left(||f - A\chi||_0 + \sum_{j=0}^{m-1} |\varphi_j - B_j\chi|_{2m-m_j-1/2} \right) \leqq Ch^{r-2m} \left(||f||_{r-2m} + \sum_{j=0}^{m-1} |g_j|_{r-m_j-1/2} \right) \tag{6.3}$$

which was to be shown.

Now, in Theorem 3.1, we take

$$S^h = \mathcal{P}(S^h_{2m,r}), \beta_1 = 0, \qquad \beta_j = (2m - m_{j-2} - \tfrac{1}{2})/(r - m_{j-2} - \tfrac{1}{2}), \qquad j = 2, \cdots, m + 1,$$

and $\sigma_j = r - 2m, j = 1, \cdots, m + 1$. We have that $\alpha_1 = 0$ and $\alpha_j = 2m - m_{j-2} - \frac{1}{2} = \gamma_{j-2}$, $j = 2, \cdots, m + 1$. Hence, from (3.2) we obtain

$$\begin{aligned} &\inf_{\chi = S_{k,r}^h} ||f - A\chi||_0 + \sum_{j=0}^{m-1} h^{-\gamma_j} |g_j - B_j\chi|_0 \\ &\leqq C(\theta_1, \cdots, \theta_{m+1}) \Big\{ h^{\theta_1(r-2m)} ||f||_{\theta_1(r-2m)} \\ &\qquad + \sum_{j=0}^{m-1} h^{-\gamma_j+\theta_{j+2}(r-m_j-1/2)} |g_j|_{\theta_{j+2}(r-m_j-1/2)} \Big\} \end{aligned} \tag{6.4}$$

where $0 \leqq \theta_l \leqq 1$ for $l = 1, \cdots, m + 1$. Now, setting $\theta_{j+2}(r - m_j - \frac{1}{2}) = \lambda_j$, $j = 0, \cdots, m - 1$, in (6.4), we obtain the desired inequality (6.1).

7. Least Squares Methods for $2m$th order Boundary-Value Problems. In this section, we shall consider a least squares method for the approximation of the solution of (5.2) using the subspaces $S^h_{k,r}$ as approximating functions. The scheme we shall consider is a generalization of the scheme considered in [14].

Let u be the solution of (5.2), where for the present we shall assume that $F = (f, g_0, \cdots, g_{m-1}) \in W^{(0,0)}$. The first approximation scheme we shall consider is as follows:

Let $S^h_{k,r}$ be given with $2m = k < r$. Find $w \in S^h_{k,r}$ such that

$$(7.1) \qquad (f - Aw, A\varphi)_0 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \langle g_j - B_j w, B_j\varphi\rangle_0 = 0,$$

that is

$$\int_\Omega (f - Aw) A\varphi \, dx + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \int_{\partial\Omega} (g_j - B_j w) B_j \varphi \, d\sigma = 0$$

for all $\varphi \in S^h_{k,r}$.

Since $\mathcal{P}(S^h_{k,r})$ (the image of $S^h_{k,r}$ under the mapping $\mathcal{P}\varphi = (A\varphi, B_0\varphi, \cdots, B_{m-1}\varphi)$) is a finite-dimensional subspace of $W^{(0,0)}_{(h,2m-m_j-1/2)}$, by (iii) of Condition III (uniqueness), w exists and is unique. It is determined by solving a linear system of algebraic equations whose coefficients depend only on f, g_j and h. An alternative way of stating (7.1) is the following: Among all $\chi \in S^h_{k,r}$, find the one which minimizes the functional

$$(7.2) \qquad \Phi(\chi) = \|f - A\chi\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \, |g - B_j\chi|_0^2.$$

In the scheme given in (7.1) and (7.2) above we could have chosen $k_j h^{-2(2m-m_j-1/2)}$ instead of the coefficients $h^{-2(2m-m_j-1/2)}$ where the k_j's are any fixed constants which are independent of h. All the results which follow remain valid for that case.

Let us note some features of this scheme: (i) For each given $S^h_{k,r}$, the weighting factors $h^{-2(2m-m_j-1/2)}$, $j = 0, \cdots, m-1$, are constants. (ii) The trial functions are not required to satisfy the boundary conditions. (iii) Only L_2 inner-products are used in the computation of the solution. (iv) The operator A need not be selfadjoint.

Let $e = u - w$. The following theorems give error estimates for the approximation scheme discussed above.

THEOREM 7.1. *Suppose that Condition* III *is satisfied and u is the solution of* (5.2) *with $F = (f, g_0, \cdots, g_{m-1}) \in W^{(0,0)}$. For given $S^h_{k,r}$, with $2m \leqq k < r$, let w be the solution of the approximate problem* (7.1) *and set $e = u - w$.*

Case 1. *Suppose that $4m \leqq r$ and l satisfies $4m - r \leqq l \leqq m_0 + \frac{1}{2}$. Then*

$$(7.3) \qquad \begin{aligned} \|e\|_l &\leqq Ch^{2m-l}\Big(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \, |B_j e|_0^2\Big)^{1/2} \\ &\leqq Ch^{2m-l}\Big(\|f\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} \, |g_j|_0\Big). \end{aligned}$$

Case 2. (i) *If $2m < r < 4m$ and $4m - r \leqq m_0 + \frac{1}{2}$, then for $4m - r \leqq l \leqq m_0 + \frac{1}{2}$*

$$(7.4) \qquad \begin{aligned} \|e\|_l &\leqq Ch^{2m-l}\Big(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \, |B_j e|_0^2\Big)^{1/2} \\ &\leqq Ch^{2m-l}\Big(\|f\|_0 + \sum h^{-(2m-m_j-1/2)} \, |g_j|_0\Big). \end{aligned}$$

(ii) *If* $2m < r < 4m$ *and* $m_0 + \frac{1}{2} < 4m - r$, *then*

$$(7.5)\quad \begin{aligned} \|e\|_{m_0+1/2} &\leqq Ch^{r-2m}\left(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_je|_0^2\right)^{1/2} \\ &\leqq Ch^{r-2m}\left(\|f\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} |g_j|_0\right). \end{aligned}$$

In (7.3), (7.4) *and* (7.5), *C is a constant which is independent of h and F.*

If the data are smoother, we have:

COROLLARY 7.1. *Suppose that the conditions of Theorem* 7.1 *are satisfied and in addition* $F = (f, g_0, \cdots, g_{m-1}) \in W^{(\lambda,\lambda_j)}$, *where* $0 \leqq \lambda \leqq r - 2m$ *and* $0 \leqq \lambda_j \leqq r - m_j - \frac{1}{2}$, $j = 0, \cdots, m-1$. *Then,*

(a) *Case* 1. *If* $4m \leqq r$ *and* $4m - r \leqq l \leqq m_0 + \frac{1}{2}$,

$$(7.6)\quad \|e\|_l \leqq Ch^{2m-l}\left(h^\lambda \|f\|_\lambda + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j} |g|_{\lambda_j}\right).$$

Case 2. (i) *If* $2m < r < 4m$ *and* $4m - r \leqq m_0 + \frac{1}{2}$, *then, for* $4m - r \leqq l \leqq m_0 + \frac{1}{2}$,

$$(7.7)\quad \|e\|_l \leqq Ch^{2m-l}\left(h^\lambda \|f\|_\lambda + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j} |g_j|_{\lambda_j}\right).$$

(ii) *If* $2m < r < 4m$ *and* $4m - r > m_0 + \frac{1}{2}$,

$$(7.8)\quad \|e\|_{m_0+1/2} \leqq Ch^{r-2m}\left(h^\lambda \|f\|_\lambda + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j} |g_j|_{\lambda_j}\right).$$

(b) *If* $2m < r$, *then, for each* $t = 0, \cdots, m-1$,

$$(7.9)\quad |g_t - B_t w|_0 \leqq Ch^{2m-m_t-1/2}\left(h^\lambda \|f\|_\lambda + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j} |g_j|_{\lambda_j}\right).$$

In (7.6), (7.7), (7.8) *and* (7.9), *C is a constant which is independent of h and F.*

Proof of Corollary 7.1. Since

$$(7.10)\quad \begin{aligned} &\left(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_je|_0^2\right)^{1/2} \\ &\qquad \leqq C \inf_{\chi \in S_{k,r}^h} \left(\|f - A\chi\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} |g_j - B_j\chi|_0\right), \end{aligned}$$

the inequalities (7.6), (7.7) and (7.8) follow immediately from (7.3), (7.4) and (7.5), respectively, after applying Theorem 6.1. The inequality (7.9) is a special case of (7.10).

Proof of Theorem 7.1. From Lemma 5.1, we have

$$(7.11)\quad \begin{aligned} \|e\|_l &\leqq C \sup_{\psi \in C^\infty(\bar\Omega)} \frac{(Ae, A\psi)_0 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} \langle B_je, B_j\psi\rangle_0}{\|A\psi\|_{2m-l} + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_j\psi|_{m_j+1/2-l}} \\ &\equiv C \sup_{\psi \in C^\infty(\bar\Omega)} \{Q(e, \psi)\}. \end{aligned}$$

Since e satisfies (7.1), we have that for each $\psi \in C^\infty(\bar\Omega)$ and all $\chi \in S_{k,r}^h$,

$$(7.12)\quad \begin{aligned}(Ae, A\psi)_0 &+ \sum_{j=0}^{m-1} h^{-2(m-m_j-1/2)} \langle B_j e, B_j\psi\rangle_0 \\ &= (Ae, A\psi - A\chi)_0 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)}\langle B_j e, B_j\psi - B_j\chi\rangle_0 \\ &\leqq C\left(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_j e|_0^2\right)^{1/2} \\ &\quad\cdot\left(\|A\psi - A\chi\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} |B_j\psi - B_j\chi|_0\right),\end{aligned}$$

where we have used Schwarz's inequality. Now, for each $\psi \in C^\infty(\bar{\Omega})$, we choose χ so that it minimizes the last term in parentheses. Hence, for fixed ψ, we obtain

$$(7.13)\qquad Q(e, \psi) \leqq \left(\|Ae\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_j e|_0^2\right)^{1/2} Q^*(\psi),$$

where

$$(7.14)\qquad Q^*(\psi) = \frac{\inf_{\varphi\in S_{k,r}^h} (\|A\psi - A\varphi\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} |B_j\psi - B_j\varphi|_0)}{\|A\psi\|_{2m-l} + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)} |B_j\psi|_{m_j+1/2-l}}.$$

We now appeal to Theorem 6.1 for the approximation-theoretic properties of the subspaces $S_{k,r}^h$ in order to estimate $Q^*(\psi)$. We obtain from (6.1) that

$$(7.15)\quad \begin{aligned}\inf_{S_{k,r}^h} &\left(\|A\psi - A\varphi\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)} |B_j\psi - B_j\varphi|_0\right) \\ &\leqq C\left(h^\lambda \|A\psi\|_\lambda + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j} |B_j\psi|_{\lambda_j}\right) \\ &\leqq Ch^\lambda\left(\|A\psi\|_\lambda + \sum_{j=0}^{m-1} h^{(2m-m_j-1/2)+\lambda_j-\lambda} |B_j\psi|_{\lambda_j}\right) \\ &\leqq Ch^\lambda\left(\|A\psi\|_{2m-l} + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2-\lambda_j+\lambda)} |B_j\psi|_{m_j+1/2-l}\right)\end{aligned}$$

where $\lambda = \min(r - 2m, 2m - l)$ and

$$\lambda_j = \min(r - m_j - \tfrac{1}{2}, m_j + \tfrac{1}{2} - l), \qquad j = 0, \cdots, m-1.$$

First, suppose that the conditions of either Case 1 or Case 2(i) of Theorem 7.1 are satisfied, i.e. $4m - r \leqq l \leqq m_0 + \frac{1}{2}$ and $2m < r$. Then, $2m - l \leqq r - 2m$ and hence $\lambda = 2m - l$. Therefore,

$$-\lambda_j + \lambda = \max(-l - r + 2m + m_j + \tfrac{1}{2}, 2m - m_j - \tfrac{1}{2}) \leqq 2m - m_j - \tfrac{1}{2}.$$

Hence,

$$2m - m_j - \tfrac{1}{2} - \lambda_j - \lambda \leqq 2(2m - m_j - \tfrac{1}{2})$$

and from (7.14) and (7.15), we obtain

$$(7.16)\qquad Q^*(\psi) \leqq Ch^{2m-l}$$

where C is a constant which is independent of ψ and h. Therefore, we immediately obtain the inequalities (7.3) and (7.4) from (7.16), (7.13) and (7.11).

For Case 2(ii) of Theorem 7.1, i.e. when $l = m_0 + \frac{1}{2} < 4m - r$ and $2m < r < 4m$, we have that the inequality (7.15) holds with $\lambda = r - 2m$. Here, we also have that $2m - m_i - \frac{1}{2} - \lambda_i - \lambda \leqq 2(2m - m_i - \frac{1}{2})$. Therefore, in this case, (7.14) yields

$$Q^*(\psi) \leqq Ch^{r-2m} \tag{7.17}$$

where C is a constant which is independent of ψ and h. The inequality (7.5) now follows easily from (7.17), (7.13) and (7.11), which completes the proof of the theorem.

A discussion of the results of Theorem 7.1 and Corollary 7.1 is in order. In the derivation of the error estimates, the only property of the subspaces $S^h_{k,r}$, which we assumed, was (4.9) (see Section 4B). One can show (cf. Theorem 5.1) that this implies that $S^h_{k,r}$ only has the following property regarding best approximation: For each $u \in W_2^\beta(\Omega)$

$$\inf_{\chi \in S_{k,r}^h} ||u - \chi||_l \leqq C_1 h^{\beta - l} ||u||_\beta \tag{7.18}$$

for each pair of real numbers l and β satisfying $l \leqq k$, $l \leqq \beta \leqq r$. C_1 is a constant which is independent of h and u.

The approximate solution w of (7.1) has, in many circumstances, the same properties as those of the best approximation. In order to show this, let us assume that $F = (f, g_0, \cdots, g_{m-1}) \in W^{(\beta-2m, \beta - m_j - 1/2)}$ for $\beta > 2m$, hence $u \in W_2^\beta(\Omega)$ and the norms $||u||_\beta$ and $||f||_{\beta-2m} + \sum_{j=0}^{m-1} |g_j|_{\beta - m_j - 1/2}$ are equivalent.

Now, suppose that w is the solution of (7.1) for a given $S^h_{k,r}$ with $r \geqq 4m$. Then, the inequality (7.6) yields

$$||u - w||_l \leqq Ch^{\beta - l} ||u||_\beta \tag{7.19}$$

for $4m - r \leqq l \leqq m_0 + \frac{1}{2}$ and $2m \leqq \beta \leqq r$, which essentially reproduces the property (7.18) in this range. We note that since in this case $0 \leqq m_0$ and $4m - r \geqq 0$, the estimate (7.19) always includes the case $l = 0$ (i.e. an estimate in $L_2(\Omega)$ where, we remind the reader, that m_0 is the order of lowest order boundary operator which was assumed to be operator B_0). We note that when $l = 0$ and $\beta = r$, we obtain

$$||u - w||_0 \leqq Ch^r ||u||_r.$$

If $2m < r < 4m$ and $4m - r \leqq m_0 + \frac{1}{2}$, then the estimate (7.19) also holds. However, in this case, $4m - r > 0$ which says that the approximate solution reproduces the property (4.16) when measured in an appropriately high norm (which in this case does not include the $W_2^0(\Omega) = L_2(\Omega)$ norm).

In the case that $2m < r < 4m$ and $m_0 + \frac{1}{2} < r - 4m$, (4.8) yields

$$||u - w||_{m_0 + 1/2} \leqq Ch^{r + \beta - 4m} ||u||_\beta$$

for $2m \leqq \beta \leqq r$. In this case, our results do not indicate that w has properties comparable to the best approximation measured in any norm up to order $m_0 + \frac{1}{2}$. This will be illustrated by specific examples in Section 9.

B. *Interior Estimates.*

THEOREM 7.2. *Suppose that Condition* III *is satisfied and u is the solution of* (5.2) *for given* $F = (f, g_0, \cdots, g_{m-1}) \in W^{(\lambda, \lambda_j)}$ *where* $0 \leqq \lambda \leqq r - 2m$ *and* $0 \leqq \lambda_j \leqq r - m_j - \frac{1}{2}$, $j = 0, \cdots, m-1$. *Let* w *be the solution of the approximate problem* (7.1) *and suppose that* Ω_1 *is any compact subdomain of* Ω. *Then, for each* $4m - r \leqq l \leqq 2m$

$$(7.20) \qquad ||e||_l^{\Omega_1} \leqq Ch^{2m-l}\left(h^{\lambda}\,||f||_{\lambda} + \sum_{j=0}^{m-1} h^{-(2m-m_j-1/2)+\lambda_j}\,|g_j|_{\lambda_j}\right),$$

where C *is a constant which is independent of* h *and* F *but may depend on* Ω_1.

Remark 7.1. When $\lambda = \beta - 2m$ and $\lambda_j = \beta - m_j - \frac{1}{2}$, $j = 0, 1, \cdots, m-1$, for $2m \leqq \beta \leqq r$, the inequality (4.18) may be written as

$$(7.21) \qquad ||e||_l^{\Omega_1} \leqq Ch^{\beta-l}\,||u||_{\beta}$$

for $4m - r \leqq l \leqq 2m$ and $2m \leqq \beta \leqq r$. Thus, for example, if $r \geqq 4m$, this says that the error can be estimated in all norms from $L_2(\Omega_1)$ to $W_2^{2m}(\Omega_1)$ with the best possible power of h. If $2m < r < 4m$, then this can also be done provided the error is measured in a sufficiently high norm $W_2^l(\Omega_1)$ with $4m - r \leqq l \leqq 2m$.

Remark 7.2. For some values of r and l, Theorem 7.2 may be improved by replacing $||e||_l^{\Omega_1}$ with $||e||_l$ on the left-hand side of (7.20). These cases are covered by the inequalities (7.6) and (7.7) of Corollary 7.1. In Theorem 7.2, we are mainly interested in obtaining estimates in norms $W_2^l(\Omega_1)$ for $l > m_0 + \frac{1}{2}$.

Proof. As remarked above, we shall only need a proof of (7.20) in the case that $m_0 + \frac{1}{2} < l \leqq 2m$. However, it is just as easy to provide a proof for the case in which $0 \leqq l \leqq 2m$. We start with the well-known interior estimate (cf. [1])

$$(7.22) \qquad ||v||_{2m}^{\Omega_1} \leqq C(||Av||_0 + ||v||_0)$$

and the estimate

$$(7.23) \qquad ||v||_0 \leqq C\left(||Av||_{-2m} + \sum_{j=0}^{m-1} |B_jv|_{-m_j-1/2}\right).$$

Hence, from (7.22) and (7.23), we have

$$(7.24) \qquad ||v||_{2m}^{\Omega_1} \leqq C\left(||Av||_0 + \sum_{j=0}^{m-1} |B_jv|_{-m_j-1/2}\right).$$

Interpolating the two inequalities (7.23) and (7.24) (with $||v||_0$ replaced with $||v||_0^{\Omega_1}$ in (7.23)), we easily obtain, taking $v = e$,

$$(7.25) \qquad ||e||_l^{\Omega_1} \leqq C\left(||Ae||_{l-2m} + \sum_{j=0}^{m-1} |B_je|_{-m_j-1/2}\right), \qquad 0 \leqq l \leqq 2m.$$

Using the procedure given in Lemma 5.1, (7.25) may be rewritten in the form

$$(7.26) \qquad ||e||_l^{\Omega_1} \leqq C \sup_{\psi \in C^\infty(\bar{\Omega})} \frac{(Ae, A\psi)_0 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)}\langle B_je, B_j\psi\rangle_0}{||A\psi||_{2m-l} + \sum_{j=0}^{m-1} h^{-2(2m-m_j-1/2)}\,|B_j\psi|_{m_j+1/2}}$$

for $0 \leqq l \leqq 2m$. The proof of Theorem 7.2 now proceeds in the same manner as the proof of Theorem 7.1 and Corollary 7.1 and will be left to the reader.

C. *Interior Estimates for Higher Derivatives.* In Theorem 7.2, we were able to obtain estimates for the $2m$th order derivatives of the error on compact subsets of Ω, using only the property (4.9) of the subspaces $S_{k,r}^h$. The question naturally arises as to whether derivatives of order higher than $2m$ can be estimated, in the case that $k > 2m$, with the "correct" rate of convergence. This can be done provided we make some additional reasonable assumptions as to the properties of the subspaces $S^h_{k,r}$. Our first assumption is commonly called an "inverse" assumption and is shared by many of the spaces $S^h_{k,r}$ which are used in practice. More precisely, we shall assume that the subspaces $S^h_{k,r}$ have the following property:

(***) Let Ω_1 be any open subset of Ω, $\bar{\Omega}_1 \subset \Omega$. Corresponding to Ω_1, there exists a Lipschitz domain Ω_2 with $\Omega_1 \subset \Omega_2$ and $\bar{\Omega}_2 \subset \Omega$ such that

$$\|\chi\|_k^{\Omega_2} \leqq Ch^{-1} \|\chi\|_{k-1}^{\Omega_2} \tag{7.27}$$

for all $\chi \in S^h_{k,r}$, where C is a constant which is independent of χ and h.

Remark 7.3. These assumptions can be verified if for example we take $S^h_{k,r}$ to be the restrictions to Ω of splines defined on a uniform mesh of width h, provided h is taken sufficiently small (cf. for example Babuška [9]). Some further remarks concerning the assumption (***) will be made immediately after Theorem 7.4.

Before proceeding with the interior estimates for the derivatives of the error of up to order k, we shall show that the assumption (7.27) implies that the subspaces $S^h_{k,r}$ have some further properties.

THEOREM 7.3. *Suppose that $S^h_{k,r}$ satisfies* (7.27). *Then, for each pair of real numbers α and β satisfying $\beta \leqq \alpha \leqq k$ and $\beta \leqq k - 1$,*

$$\|\chi\|_\alpha^{\Omega_2} \leqq Ch^{\beta-\alpha} \|\chi\|_\beta^{\Omega_2} \tag{7.28}$$

for all $\chi \in S^h_{k,r}$, where C is a constant which is independent of χ and h.

For simplicity we shall prove (7.28) in the case that α and β are integers. The proof in the general case follows using similar arguments. Using (5.3) (which is also valid if G is a Lipschitz domain), we have that if s is any integer then for any $\chi \in S^h_{k,r}$,

$$(\|\chi\|_{s-1}^{\Omega_2})^2 \leqq C \|\chi\|_s^{\Omega_2} \|\chi\|_{s-2}^{\Omega_2} \tag{7.29}$$

where C is independent of χ. Now suppose that (7.28) holds for $k = s$, then

$$(\|\chi\|_{s-1}^{\Omega_2})^2 \leqq Ch^{-1} \|\chi\|_{s-2}^{\Omega_2} \|\chi\|_{s-1}^{\Omega_2}$$

or

$$\|\chi\|_{s-1}^{\Omega_2} \leqq Ch^{-1} \|\chi\|_{s-2}^{\Omega_2}. \tag{7.30}$$

A simple induction argument using (7.29) now gives us that (7.30) holds for all integers $s \leqq k - 1$.

Now, let α and β be any two integers satisfying the conditions of this theorem. We obtain, after a finite number of steps,

$$\|\chi\|_\alpha^{\Omega_2} \leqq Ch^{-1} \|\chi\|_{\alpha-1}^{\Omega_2} \leqq \cdots \leqq Ch^{\beta-\alpha} \|\chi\|_\beta^{\Omega_2},$$

which was to be shown.

Let us denote by $S^h_{k,r}(\mathbf{R}^N)$, for $0 < h < 1$, any one-parameter family of finite-dimensional subspaces of $W_2^k(\mathbf{R}^N)$ having the following property: For all $u \in W_2^k(\mathbf{R}^N)$

$$\inf_{S_{k,r}{}^h(\mathbf{R}^N)} \|u - \psi\|_k^{\mathbf{R}^N} \leqq Ch^{r-k} \|u\|_r^{\mathbf{R}^N} \tag{7.31}$$

where C is a constant which is independent of h and u.

As discussed previously (see Section 4), such subspaces have been constructed by several authors.

Remark 7.4. It follows from an easy application of the Calderón extension theorem that the restrictions to Ω of elements of a subspace $S^h_{k,r}(\mathbf{R}^N)$ form a subspace satisfying (4.9).

In view of Remark 7.4, we shall make the following assumption:

(****) $S^h_{k,r}$ is the restriction to Ω of elements of a space of type $S^h_{k,r}(\mathbf{R}^N)$.

We shall now prove

THEOREM 7.4. *Assume that the conditions of Theorem* 7.2 *hold with* $F = (f, g_0, \cdots, g_{m-1}) \in W^{(\beta-2m,\beta-m_j-1/2)}$, *where now* $S^h_{k,r}$ *satisfies* (7.27), (***) *and* (****) *with* $2m + 1 \leqq k < r$. *Then,*

$$\|e\|_l^{\Omega_1} \leqq Ch^{\beta-l} \|u\|_\beta \tag{7.32}$$

for each l *and* β *satisfying* $4m - r \leqq l \leqq k$ *and* $l \leqq \beta \leqq r$. *The constant* C *is independent of* h *and* F, *but, in general, depends on* Ω_1.

Remark 7.5. In (***), we have assumed that $\bar{\Omega}_2 \subset \Omega$. If we had assumed that the inequality (7.27) held on Ω, then we can also obtain estimates for the derivatives of the error up to order k on Ω which are analogous to those obtained in Theorem 7.4. However, subspaces whose elements satisfy inequalities of the type (7.27) for domains Ω of general shape are not easy to construct.

Proof. The inequality (7.32), in the case that $4m - r \leqq l \leqq 2m$, is just the inequality (7.20), where then our assumption on F implies that $u \in W_2^\beta(\Omega)$. Hence, we need only consider the case in which $2m + 1 \leqq l \leqq k$. Obviously,

$$\|u - w\|_l^{\Omega_1} \leqq \|u - w\|_l^{\Omega_2}$$

and hence it is sufficient to prove the inequality (7.32) with Ω_1 replaced by Ω_2.

Let $2m + 1 \leqq l \leqq k$ and suppose that $l \leqq \beta \leqq r$. By a theorem of Calderón [1], we can extend u to all of $\mathbf{R}^N$ so that

$$\|u\|_\beta^{\mathbf{R}^N} \leqq C \|u\|_\beta, \tag{7.33}$$

where we have again denoted the extended u by u. Now, let u_l denote the best approximation in $S^h_{k,r}(\mathbf{R}^N)$ to u in the norm of $W_2^l(\mathbf{R}^N)$. Then,

$$\begin{aligned} \|u - w\|_l^{\Omega_2} &\leqq \|u - u_l\|_l^{\mathbf{R}^N} + \|u_l - w\|_l^{\Omega_2} \\ &\leqq \|u - u_l\|_l^{\mathbf{R}^N} + h^{-1} \|u_l - w\|_{l-1}^{\Omega_2} \\ &\leqq \|u - u_l\|_l^{\mathbf{R}^N} + h^{-1} \|u - u_l\|_{l-1}^{\mathbf{R}^N} + h^{-1} \|u - w\|_{l-1}^{\Omega_2}, \end{aligned} \tag{7.34}$$

where, since $u_l \in S^h_{k,r}(\mathbf{R}^N)$, the restriction of u_l to Ω is in $S^h_{k,r}$ and hence $u_l - w$ satisfies (7.27).

Now, it follows from (7.31) that

$$\|u - u_l\|_l^{\mathbf{R}^N} \leqq Ch^{\beta-l} \|u\|_\beta^{\mathbf{R}^N} \leqq Ch^{\beta-l} \|u\|_\beta, \qquad l \leqq \beta \leqq r. \tag{7.35}$$

We claim that for any integer l

$$h^{-1} \|u - u_l\|_{l-1}^{\mathbf{R}^N} \leqq Ch^{\beta-l} \|u\|_\beta^{\mathbf{R}^N} \quad \text{for} \quad l \leqq \beta \leqq r. \tag{7.36}$$

Now for any integer l we have

$$\|u - u_l\|_{l-1}^{\mathbf{R}^N} = \sup_{\psi \in W_2^{l+1}(\mathbf{R}^N)} \frac{(u - u_l, \psi)_l^{\mathbf{R}^N}}{\|\psi\|_{l+1}^{\mathbf{R}^N}}$$

where $(\cdot\,,\,\cdot)_l^{\mathbf{R}^N}$ denotes the inner product on $W_2^l(\mathbf{R}^N)$. From the definition of u_l it follows that

$$||u - u_l||_{l-1}^{\mathbf{R}^N} = \sup_{\psi \in W_2^{l+1}(\mathbf{R}^N)} \frac{(u - u_l,\, \psi - \psi_l)_l^{\mathbf{R}^N}}{||\psi||_{l+1}^{\mathbf{R}^N}},$$

where for each ψ, ψ_l denotes the best approximation in $S_{k,r}^h(\mathbf{R}^N)$ to ψ in the norm of $W_2^l(\mathbf{R}^N)$. Then, using Schwarz's inequality and the property (7.31), we have

$$||u - u_l||_{l-1}^{\mathbf{R}^N} \leqq ||u - u_l||_l^{\mathbf{R}^N}\left(\sup_{\psi \in W_2^{l+1}(\mathbf{R}^N)} \frac{||\psi - \psi_l||_l^{\mathbf{R}^N}}{||\psi||_{l+1}^{\mathbf{R}^N}}\right)$$

$$\leqq Ch\, ||u - u_l||_l^{\mathbf{R}^N} \leqq Ch^{\beta - l + 1}\, ||u||_\beta^{\mathbf{R}^N}, \qquad l \leqq \beta \leqq r,$$

from which (7.36) easily follows.

From (7.34), (7.35) and (7.36), we then have

$$(7.37) \qquad ||u - w||_l^{\Omega_2} \leqq C(h^{\beta - l}\, ||u||_\beta + h^{-1}\, ||u - w||_{l-1}^{\Omega_2})$$

for any integer $2m \leqq l \leqq k$ and $l \leqq \beta \leqq r$. Hence, the inequality

$$(7.38) \qquad ||u - w||_l^{\Omega_2} \leqq Ch^{\beta - l}\, ||u||_\beta$$

would follow immediately from (7.37) if we could show that

$$(7.39) \qquad ||u - w||_{l-1}^{\Omega_2} \leqq Ch^{\beta - l + 1}\, ||u||_\beta .$$

In view of Theorem 7.2 or in this case (7.21), the inequality (7.39) holds when $l = 2m + 1$ and $2m \leqq \beta \leqq r$. But then, by induction, it follows that (7.39) holds for all integers $2m \leqq l \leqq k$ and real numbers β, $l \leqq \beta \leqq r$. The proof of (7.32) now follows by interpolation.

D. *An Example*. We shall now consider a specific example of the theory presented in this section. In [14], several examples were given which illustrated the theory for Dirichlet's problem for second-order elliptic operators. Since the purpose of this paper was to extend the theory to higher-order operators and general boundary conditions, our example will illustrate this. In particular, we shall consider a biharmonic problem whose associated boundary conditions are neither Dirichlet nor natural boundary conditions. In our example, the operator is selfadjoint and the usual associated bilinear form is positive definite, however, we should note that our method works equally well if these properties of the problem are not present, provided Condition III is satisfied.

Let us consider the problem of finding approximate solutions of the second boundary-value problem for elastic plates. Here we take Ω to be a two-dimensional region (with smooth boundary) and u to be the deflection (i.e. displacement) of the plate $\bar{\Omega}$. The symbol σ is used to denote Poisson's ratio and D will signify the plate rigidity. The boundary-value problem to be considered can be stated as

$$(7.40) \qquad \Delta^2 u = f \quad \text{in } \Omega$$

and

$$(7.41) \qquad u = g_0, \qquad M(u) = g_1 \quad \text{on } \partial\Omega.$$

Here, f, g_0, and g_1 are prescribed data and $M(u)$ is the bending moment; i.e.

$$M(u) = -D\left[\Delta u - (1-\sigma)\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{\rho}\frac{\partial u}{\partial \nu}\right)\right]. \tag{7.42}$$

In (7.42), $\partial^2 u/\partial s^2$ denotes the second tangential derivative of u, $\partial u/\partial \nu$ is the exterior normal derivative and ρ is the radius of curvature on $\partial\Omega$.

In this example, the approximate problem (7.1) takes the following form: For given $S^h_{k,r}$ with $4 \leqq k < r$, find $w \in S^h_{k,r}$ such that

$$\int_\Omega (f - \Delta^2 w)\,\Delta^2\varphi\,dx + h^{-7}\int_{\partial\Omega}(g_0 - w)\varphi\,ds + h^{-3}\int_{\partial\Omega}(g - M(w))M(\varphi)\,ds = 0 \tag{7.43}$$

for all $\varphi \in S^h_{k,r}$.

If we take w to be the solution of (7.43), then (7.6), (7.8) and (7.9) of Corollary 7.1 yield the following error estimates over Ω and on $\partial\Omega$.

If $8 \leqq r$, then

$$\|u - w\|_0 \leqq Ch^4(h^\lambda \|f\|_\lambda + h^{-7/2+\lambda_0}|g_0|_{\lambda_0} + h^{-3/2+\lambda_1}|g_1|_{\lambda_1}). \tag{7.44}$$

If $4 < r < 8$, then

$$\|u - w\|_{1/2} \leqq Ch^{r-4}(h^\lambda \|f\|_\lambda + h^{-7/2+\lambda_0}|g_0|_{\lambda_0} + h^{-3/2+\lambda_1}|g_1|_{\lambda_1}). \tag{7.45}$$

For any $4 < r$

$$|u - w|_0 = |g - w|_0 \leqq C(h^{7/2+\lambda}\|f\|_\lambda + h^{\lambda_0}|g_0|_\lambda + h^{2+\lambda_1}|g_1|_{\lambda_1}) \tag{7.46}$$

and

$$|M(u) - M(w)|_0 = |g_1 - M(w)|_0 \leqq C(h^{3/2+\lambda}\|f\|_\lambda + h^{-2+\lambda_0}|g_0|_{\lambda_0} + h^{\lambda_1}|f|_{\lambda_1}). \tag{7.47}$$

In (7.44), (7.45), (7.46) and (7.47), λ, λ_0 and λ_1 are restricted to $0 \leqq \lambda \leqq r - 4$, $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$ and $0 \leqq \lambda_1 \leqq r - \frac{5}{2}$.

If we take for example $k = 5$, $r = 6$ and $S^h_{k,r}$ to be quintic splines on a uniform mesh of width h, then we obtain from (7.45), (7.46) and (7.47)

$$\|u - w\|_{1/2} \leqq Ch^4(\|f\|_2 + |g_0|_{11/2} + |g_1|_{7/2}) \leqq Ch^4\|u\|_6,$$

$$|g_0 - w|_0 \leqq Ch^{11/2}(\|f\|_2 + |g_0|_{11/2} + |g_1|_{7/2}) \leqq Ch^{11/2}\|u\|_6$$

and

$$|g_1 - M(w)|_0 \leqq Ch^{7/2}(\|f\|_2 + |g_0|_{11/2} + |g_1|_{7/2}) \leqq Ch^{7/2}\|u\|_6.$$

If we take $k = 7$, $r = 8$ and $S^h_{k,r}$ to be splines of order eight, then the two extreme cases of (7.44) yield

$$\|u - w\|_0 \leqq Ch^4(\|f\|_0 + h^{1/2}|g_0|_0 + h^{5/2}|g_1|_0)$$

and

$$\|u - w\|_0 \leqq Ch^8(\|f\|_4 + |g_0|_{15/2} + |g_1|_{11/2}) \leqq Ch^8\|u\|_8.$$

From (7.46) and (7.47) we obtain

$$|g_0 - w|_0 \leqq Ch^{15/2}(\|f\|_4 + |g_0|_{15/2} + |g_1|_{11/2} \leqq Ch^{15/2}\|u\|_8$$

and

$$|g_1 - M(w)|_0 \leqq Ch^{11/2}(||f||_4 + |g_0|_{15/2} + |g_1|_{11/2}) \leqq Ch^{11/2}\,||u||_8.$$

Interior estimates are easily obtained from Theorems 7.3 and 7.4. If we take $k = 5$, $r = 6$, $S^h_{k,r}$ to be say quintic splines and Ω_1 to be an open subset of Ω with $\bar{\Omega}_1 \subset \Omega$, we obtain from (7.32) that

$$||u - w||_l^{\Omega_1} \leqq Ch^{6-l}\,||u||_6$$

for any $2 \leqq l \leqq 5$. In particular, in the two extreme cases we have

$$||u - w||_2^{\Omega_1} \leqq Ch^4\,||u||_6$$

and

$$||u - w||_5^{\Omega_1} \leqq Ch\,||u||_6.$$

If we take $k = 7$, $r = 8$ and $S^h_{k,r}$ to be splines of order eight, then

$$(7.48) \qquad ||u - w||_l^{\Omega_1} \leqq Ch^{8-l}\,||u||_8$$

for any $0 \leqq l \leqq 7$.

We note that for $0 \leqq l \leqq \frac{1}{2}$, Corollary 7.1 says that the inequality (7.48) also holds with $||u - w||_l$, replacing $||u - w||_l^{\Omega_1}$ on the left-hand side.

8. Other Approximation Schemes for Dirichlet's Problem.

A. We shall first briefly consider some approximation schemes which are closely related to schemes previously discussed. For simplicity, we shall restrict the discussion to Dirichlet's problem

$$(8.1) \qquad \begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

In the scheme (7.1), the approximate solution in this case is determined as the minimum of the functional

$$(8.2) \qquad \Phi(\varphi) = ||f - \Delta\varphi||_0^2 + h^{-3}\,|g - \varphi|_0^2, \qquad \varphi \in S^h_{k,r}.$$

We could, if we wished, determine an approximate solution as the minimum of the functional

$$(8.3) \qquad \Phi_1(\varphi) = ||f - \Delta\varphi||_s^2 + h^{-3+2(s-s_0)}\,|g - \varphi|_{s_0}^2$$

for a given pair of real numbers s and s_0. Estimates for the error in these schemes could be obtained using the methods presented previously in this paper.

Let us discuss the results one can obtain for the particular choice of $s = 0$ and $s_0 = 1$. This scheme has certain advantages over the scheme (7.1). This will be discussed immediately after Theorem 8.1. An equivalent formulation of this scheme is as follows: For given $S^h_{k,r}$ with $2 \leqq k < r$, find $v \in S^h_{k,r}$ such that

$$(8.4) \qquad (f - \Delta v, \Delta\varphi)_0 + h^{-1}\langle g - v, \varphi\rangle_1 = 0 \quad \text{for all } \varphi \in S^h_{k,r}.$$

It is easy to see that there exists a unique solution of this equation and that it can be determined by solving a linear system of algebraic equations.

If we assume for simplicity that g is defined in a neighborhood $\mathfrak{N}$ of $\partial\Omega$ and $g \in W_2^{3/2}(\mathfrak{N})$, then we may take as a definition

$$\langle g, w\rangle_1 = \int_{\partial\Omega} gw \, d\sigma + \int_{\partial\Omega} \left(\sum_{i=1}^{N} \frac{\partial g}{\partial x_i} \frac{\partial w}{\partial x_i} - \frac{\partial g}{\partial \nu} \frac{\partial w}{\partial \nu} \right) d\sigma,$$

where $\partial v/\partial \nu$ denotes the interior normal derivative of v to $\partial\Omega$.

We shall not give any details of the proof of the error estimates, except to say that we use a generalization of Theorem 3.1 which also follows easily from Theorem 3.1.

THEOREM 8.1. *Suppose $S_{k,r}^h$ is given satisfying* (*) *with* $2 \leqq k \leqq r$ *and let u be the solution of* (2.2) *for given data $F = (f, g) \in W^{(\lambda, \lambda_0)}$ where $0 \leqq \lambda \leqq r - 2$ and $1 \leqq \lambda_0 \leqq r - \frac{1}{2}$. Let v be the solution of the approximate problem* (8.4). *Then, for any $4 - r \leqq l \leqq \frac{3}{2}$*

$$||u - v||_l \leqq Ch^{2-l}(h^\lambda \, ||f||_\lambda + h^{-3/2+\lambda_0} \, |g|_{\lambda_0}), \tag{8.5}$$

where C is a constant which is independent of h and F.

Let us compare the estimates that have been obtained for the schemes (7.1) and (8.4). Suppose that $F = (f, g) \in W^{(\beta, \beta-1/2)}$ for $2 \leqq \beta \leqq r$. Then, as previously discussed if w is the approximate solution of (8.1) obtained from (8.2), the following error estimates hold: If $4 \leqq r$ and $4 - r \leqq l \leqq \frac{1}{2}$, then

$$||u - w||_l \leqq Ch^{\beta-l} \, ||u||_\beta. \tag{8.6}$$

If $r = 3$, then

$$||u - w||_{1/2} \leqq Ch^{\beta-1} \, ||u||_\beta. \tag{8.7}$$

On the other hand, if we take v to be the approximate solution obtained from (8.4), then (8.5) yields the error estimate

$$||u - v||_l \leqq Ch^{\beta-l} \, ||u||_\beta \tag{8.8}$$

for any $4 - r \leqq l \leqq \frac{3}{2}$.

The advantages of the scheme (8.4) are now easy to see. Namely, one can obtain estimates for the error in the $W_2^1(\Omega)$ norm with the correct order of convergence. This is an improvement over the previous schemes for any $3 \leqq r$. In the case that $k = 2$ and $r = 3$, one obtains from (8.8), using for example quadratic splines on a uniform mesh of size h, the estimate

$$||u - v||_1 \leqq Ch^2 \, ||u||_3.$$

In this case, the scheme (7.1) yields

$$||u - w||_{1/2} \leqq Ch^2 \, ||u||_3.$$

B. Let us again consider the problem of approximating the solution of Dirichlet's problem (8.1). Besides the approximation scheme analyzed in [6], Babuška (private communication) has proposed the following scheme for the approximation of the solution of (8.1): Find $v \in S_{k,r}^h$ which minimizes the functional

$$\Phi_2(\psi) = ||u - \Delta\varphi||_0^2 + \rho^{-1}(h) \left| \rho(h) \frac{\partial u}{\partial \nu} - \varphi \right|_0^2 \tag{8.9}$$

among all $\varphi \in S^h_{k,r}$. Here, $\rho(h)$ is a given function of h which is to be chosen appropriately. It is easily seen that if $v \in S^h_{k,r}$ is the function which minimizes (8.9), then

$$\int_R \Delta v \, \Delta\varphi \, dx + \rho(h)^{-1} \int_{\partial R} v\varphi \, d\sigma - \int_R f\varphi \, dx \int_{\partial R} g \frac{\partial \varphi}{\partial \nu} \, d\sigma = 0$$

for all $\varphi \in S^h_{k,r,2}(\Omega)$. Hence, v can be determined from a system of linear algebraic equations in terms of the data f and g.

We shall prove some error estimates for this scheme with $\rho(h)$ chosen as $\rho(h) = h^{-3}$. We believe this to be the best possible choice. We shall make use of a result which was proved in [14]. It is also a special case of Theorem 6.1. For convenience we shall restate that result in this special case.

LEMMA 8.1. *Let $S^h_{k,r}$ be given satisfying* (4.9) *with* $2 \leqq k < r$. *Then, for any* $F = (f, g) \in W^{(\lambda,\lambda_0)}$, *where* $0 \leqq \lambda \leqq r - 2$ *and* $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$,

$$\inf_{\varphi \in S_{k,r}{}^h} (\|f_1 - \Delta\varphi\|_0^2 + h^{-3} |g - \varphi|_0^2)^{1/2} \leqq C(h^\lambda \|f\|_\lambda + h^{-3/2+\lambda_0} |g|_{\lambda_0})$$

where C is a constant which is independent of F and h.

The following error estimates for the scheme (8.9) easily follow from Lemma 8.1.

THEOREM 8.2. *Let* $u \in W_2^{2+\lambda}(\Omega)$ *be the solution of* (8.1) *for given data* $F = (f, g) \in W^{(\lambda,\lambda+3/2)}$ *where* $\frac{3}{2} < \lambda \leqq r - 2$. *Suppose that* $S^h_{k,r}$ *is given,* $2 \leqq k < r$, $r \geqq 4$ *and let v be the solution of the approximate problem* (8.9). *Then*

$$\|u - \Delta v\|_0 \leqq Ch^\lambda \|u\|_\lambda \tag{8.10}$$

where C is a constant which is independent of h and u.

Remark 8.1. If we denote by w the solution of the approximation scheme (8.2) (or equivalently (7.1)), then in the case that $2 \leqq k < r$, $r \geqq 4$, we can obtain from Corollary 7.1 that

$$\|u - w\|_0 \leqq Ch^\lambda \|u\|_\lambda \tag{8.11}$$

for any $2 \leqq \lambda \leqq r$. A comparison between (8.10) and (8.11) shows that the maximum rate of convergence in (8.10) is of the order h^{r-2} while (8.11) gives us the higher maximum rate of order h^r. Note also that the matrices in the two schemes (8.2) and (8.9) are identical.

Proof. We certainly have (with $\rho(h) = h^3$) that

$$\|u - \Delta v\|_0^2 \leqq \inf_{\psi \in S_{k,r}{}^h} \left(\|u - \Delta\psi\|_0^2 + h^{-3} \left|h^3 \frac{\partial u}{\partial \nu} - \psi\right|_0^2\right)^{1/2}.$$

Applying Lemma 8.1, it follows that for $0 \leqq \lambda \leqq r - 2$, and $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$,

$$\|u - \Delta v\|_0^2 \leqq C\left(h^\lambda \|u\|_\lambda + h^{3/2+\lambda_0} \left|\frac{\partial u}{\partial \nu}\right|_{\lambda_0}\right). \tag{8.12}$$

Now, from Lemma 4.3, we have that if $\frac{3}{2} < \lambda$ then

$$\left|\frac{\partial u}{\partial \nu}\right|_{\lambda-3/2} \leqq C \|u\|_\lambda \tag{8.13}$$

where C is independent of u. Choosing $\lambda_0 = \lambda - \frac{3}{2}$ in (8.13) and using (8.12), we easily obtain the desired inequality (8.11) which completes the proof.

Cornell University
Ithaca, New York 14850

1. S. AGMON, *Lectures on Elliptic Boundary Value Problems,* Van Nostrand Math. Studies, no. 2, Van Nostrand, Princeton, N. J., 1965. MR **31** #2504.

2. J. P. AUBIN, "Behavior of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods," *Ann. Scuola Norm. Sup. Pisa,* Ser. 3, v. 21, 1967, pp. 599–637. MR **38** #1391.

3. J. P. AUBIN, "Interpolation et approximation optimales et 'spline functions'," *J. Math. Anal. Appl.,* v. 24, 1968, pp. 1–24. MR **37** #6651.

4. J. P. AUBIN, "Approximation des problèms aux limites non homogènes et régularite de la convergence," *Calcolo,* v. 6, 1969, pp. 117–139.

5. J. P. AUBIN, *Variational Methods for Non-Homogeneous Boundary Value Problems,* SYNSPADE, 1970 (B. E. Hubbard, editor). (To appear.)

6. I. BABUŠKA, *Numerical Solution of Boundary Value Problems by the Perturbed Variational Principle,* University of Maryland, Technical Note BN-624, College Park, Md., 1969.

7. I. BABUŠKA, *Approximation by Hill Functions,* University of Maryland, Technical Note BN-648, College Park, Md.

8. I. BABUŠKA, *The Finite Element Method for Elliptic Equations With Discontinuous Coefficients,* University of Maryland, Technical Note BN-631, College Park, Md.

9. I. BABUŠKA, *Error-Bounds for Finite Element Method,* University of Maryland, Technical Note BN-630, College Park, Md.

10. I. BABUŠKA, *Computation of Derivatives in the Finite Element Method,* University of Maryland, Technical Note BN-650, College Park, Md.

11. I. BABUŠKA, *Finite Element Method for Domains With Corners,* University of Maryland, Technical Note BN-636, College Park, Md.

12. JU. M. BEREZANSKII, *Expansion in Eigenfunctions of Self Adjoint Operators,* Naukova Dumka, Kiev, 1965; English transl., Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, R.I., 1968. MR **36** #5768; 5769.

13. J. H. BRAMBLE & S. HILBERT, "Bounds for a class of linear functionals with application to Hermite interpolation," *Numer. Math.* (To appear.)

14. J. H. BRAMBLE & A. H. SCHATZ, "Rayleigh-Ritz-Galerkin methods for Dirichlet's problem using subspaces without boundary conditions," *Comm. Pure Appl. Math.,* v. 23, 1970, pp. 653–675.

15. J. H. BRAMBLE & A. H. SCHATZ, *On the Numerical Solution of Elliptic Boundary Value Problems by Least Squares Approximation of the Data,* SYNSPADE, 1970 (B. E. Hubbard, editor). (To appear.)

16. J. H. BRAMBLE & M. ZLÁMAL, "Triangular elements in the finite element method," *Math. Comp.,* v. 24, 1970, pp. 809–820.

17. P. L. BUTZER & H. BERENS, *Semi-Groups of Operators and Approximation,* Die Grundlehren der math. Wissenschaften, Band 145, Springer-Verlag, New York, 1967. MR **37** #5588.

18. F. DI GUGLIELMO, "Construction d'approximations des espaces de Sobolev sur des reseaux en simplexes," *Calcolo,* v. 6, 1969, pp. 279–331.

19. G. FIX & G. STRANG, "Fourier analysis of the finite element method in Ritz-Galerkin theory," *Studies in Appl. Math.,* v. 48, 1969, pp. 265–273.

20. K. O. FRIEDRICHS & H. B. KELLER, *A Finite Difference Scheme for Generalized Neumann Problems,* Proc. Sympos. Numerical Solution of Partial Differential Equations (Univ. Maryland, 1965) Academic Press, New York, 1966, pp. 1–19. MR **34** #3803.

21. P. GRISVARD, "Commutativité de deux functeurs d'interpolation," *J. Math. Pures Appl.,* Ser. 9, v. 45, 1966, pp. 143–290. MR **36** #4361; 4362.

22. S. HILBERT, *Numerical Methods for Elliptic Boundary Problems,* Ph.D. Thesis, University of Maryland, College Park, Md., 1969.

23. J. L. LIONS & E. MAGENES, *Problèmes aux Limites non Homogènes et Applications.* Vol. I, Travaux et Recherches Mathématiques, no. 17, Dunod, Paris, 1968. MR **40** #512.

24. E. MAGENES, *Spazi d' Interpolazione ed Equazioni a Derivate Parziali,* Atti del Settimo Congresso dell'Unione Matematica Italiana (Genova, 1963) Edizioni Cremonese, Rome, 1965, pp. 134–197. MR **35** #5925.

25. J. NITSCHE, *Lineare Spline-Funktionen und die Methode von Ritz für elliptische Randwertprobleme.* (Preprint.)

26. J. NITSCHE, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind.* (Preprint.)

27. Ja. A. Roitberg & Z. G. Šeftel, "A homeomorphism theorem for elliptic systems, and its applications," *Mat. Sb.*, v. 78 (120), 1969, pp. 446–472 = *Math. USSR Sb.*, v. 7, 1969, pp. 439–465. MR **40** #539.
28. M. Schechter, "On L^p estimates and regularity. II," *Math. Scand.* v. 13, 1963, pp. 47–69. MR **32** #6052.
29. M. Schultz, "Multivariate spline functions and elliptic problems," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 523–538.
30. M. Schultz, "Rayleigh-Ritz-Galerkin methods for multidimensional problems," *SIAM J. Numer. Anal.*, v. 6, 1969, pp. 570–582.
31. G. Strang, *The Finite Element Method and Approximation Theory,* SYNSPADE, 1970 (B. E. Hubbard, editor). (To appear.)
32. R. Varga, *Hermite Interpolation-Type Ritz Methods for Two-Point Boundary Value Problems,* Proc. Sympos. Numerical Solution of Partial Differential Equations (Univ. Maryland, 1965) Academic Press, New York, 1966, pp. 365–373. MR **34** #5302.
33. M. Zlámal, "On the finite element method," *Numer. Math.*, v. 12, 1968, pp. 394–409. MR **39** #5074.