Leavitt Path Algebras

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Abstract

The central concept of this thesis is that of Leavitt path algebras, a notion introduced by both Abrams and Aranda Pino in [AA1] and Ara, Moreno and Pardo in [AMP] in 2004. The idea of using a field K and row-finite graph E to generate an algebra $L_K(E)$ provides an algebraic analogue to Cuntz and Krieger's work with C^*-algebras of the form $C^*(E)$ (which, despite the name, are analytic concepts). At the same time, Leavitt path algebras also generalise the algebras constructed by W. G. Leavitt in [Le1] and [Le2], and it is from this connection that the Leavitt path algebras get their name.

Although the concept of a Leavitt path algebra is relatively new, in the years since the publication of [AA1] there has been a flurry of activity on the subject. Many results were initially shown for row-finite graphs, then extended to countable (but not necessarily row-finite) graphs (as in [AA3]) and then finally shown for completely arbitrary graphs (see, for example, [AR]). Most of the research has focused on the connections between ring-theoretic properties of $L_K(E)$ and graph-theoretic properties of E (for example [AA2], [AR] and [ARM2]), the socle and socle series of a Leavitt path algebra ([AMMS1], [AMMS2] and [ARM1]) and analogues between $L_K(E)$ and their C^*-algebraic equivalents $C^*(E)$ (for example [To]). Some papers have classified certain sets of Leavitt path algebras, such as [AAMMS], which classifies the Leavitt path algebras of graphs with up to three vertices (and without parallel edges).

In Chapter 1 we will cover the ring-, module- and graph-theoretic background necessary to examine these algebras in depth, as well as taking a brief look at Morita equivalence, a concept that will prove useful at various points in this thesis.

We introduce Leavitt path algebras formally in Chapter 2 and look at various results that arise from the definition. We also examine simple and purely infinite simple Leavitt path algebras, as well as the 'desingularisation' process, which allows us to construct row-finite graphs from graphs containing infinite emitters in such a way that their corresponding Leavitt path algebras are Morita equivalent. In Chapter 3 we examine the socle and socle series of a Leavitt path algebra, while in Chapter 4 we examine Leavitt path algebras that are von Neumann regular, π-regular and weakly regular, as well as Leavitt path algebras that are self-injective. Finally, in Appendix A we give a detailed definition of a direct limit, a concept that recurs throughout this thesis.

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Contents

Chapter 1

Preliminaries

1.1 Ring Theory

In many texts, a ring R is required to be a monoid under multiplication; that is, R must contain a multiplicative identity. Indeed, many well-known ring theoretic results are based on the assumption that such an element exists. However, some authors omit this requirement, resulting in a more general definition of a ring. As we will see in Chapter 2, a Leavitt path algebra may not necessarily have an identity, so throughout this thesis we will assume the more general definition of a ring that does not require the existence of a multiplicative identity. In the case that R does have a multiplicative identity, we say that R **has identity** or that R **is unital**, and denote this identity by 1 (or 1_R) as usual.

The following definition gives a very useful generalisation of the concept of a multiplicative identity.

Definition 1.1.1. A ring R has **local units** if there exists a set of idempotents E in R such that, for every finite subset $X = \{x_1, \dots, x_n\} \subseteq R$, there exists an $e \in E$ such that $X \subseteq eRe$. In this case, $ex_i = x_i = x_ie$ for each $i = 1, \dots, n$ and e is said to be a **local unit** for the subset X.

Note that if a ring R has identity 1, then $\{1\}$ is a set of local units for R. In Chapter 2 we will show that every Leavitt path algebra has local units (but is not necessarily unital), and so we will be particularly interested in looking at results for rings with local units. As we shall see, extending results for unital rings to this more general case is straightforward in some cases, while in other cases it can be difficult or even impossible.

When working with rings that do not necessarily have identity we have to take care with the way certain things are defined. For example, for an arbitrary element x in an arbitrary ring R we define the **two-sided ideal generated by** x, denoted $\langle x \rangle$, to be the set

$$\langle x \rangle := \left\{ \sum_i r_i x s_i + \sum_j r'_j x + \sum_k x s'_k + n \cdot x \right\}$$

where $r_i, s_i, r'_j, s'_k \in R$, $n \in \mathbb{Z}$ and the sums are finite. If R is unital, it is easy to see that this expression simplifies to the more familiar definition $\langle x \rangle = \{\sum_i r_i x s_i \colon r_i, s_i \in R\}$. Furthermore, in the more general case that R has local units this simplification still holds, since we can find a nonzero idempotent $e \in R$ for which $ex = x = xe$.

Similarly, for an arbitrary element $a \in R$ we define the **principal left ideal generated by** a, denoted Ra, to be the set

$$Ra := \{ra + n \cdot a : r \in R, n \in \mathbb{Z}\}.$$

Once again, in the case that R has local units this simplifies to the more familiar definition $Ra = \{ra : r \in R\}$, since $a = ea$ for some idempotent $e \in R$.

If R is a ring with local units, then for any element $a \in R$ there exists an idempotent $e \in R$ such that $a \in eRe$, by definition. It is easy to see that eRe is a subring of R. Furthermore, note that eRe is always unital (with identity e), even if R is not. The following result concerns the subring eRe. Recall that a ring R is **simple** if the only two-sided ideals contained in R are $\{0\}$ and R itself.

Proposition 1.1.2. *Let R be a ring with local units. Then R is simple if and only if the subring eRe is simple for every nonzero idempotent $e \in R$.*

Proof. Suppose that R is simple and let e by any nonzero idempotent in R. To show that eRe is simple it suffices to show that, for any nonzero element $exe \in eRe$, the two-sided ideal of eRe generated by exe is equal to all of eRe. Take an arbitrary nonzero element $ex_0e \in eRe$. Since ex_0e is an element of R and R is simple, the two-sided ideal of R generated by ex_0e is equal to R. Now take another arbitrary element $eye \in eRe$. Since $R = \langle ex_0e \rangle$ and R has local units we can write $y = \sum_i r_i(ex_0e)s_i$, where each $r_i, s_i \in R$. Thus

$$eye = \textstyle\sum_i er_i(ex_0e)s_ie = \sum_i (er_ie)(ex_0e)(es_ie)$$

since e is an idempotent. Thus eye is contained in the two-sided ideal of eRe generated by ex_0e, and since eye was an arbitrary element this shows that eRe is simple.

Conversely, suppose that fRf is simple for every nonzero idempotent $f \in R$. As above, it suffices to show that, for any nonzero element $x \in R$, the two-sided ideal of R generated by x is equal to all of R. Take arbitrary nonzero elements $x_0, y \in R$. Since R has local units, there exists an idempotent $e \in R$ such that $x_0, y \in eRe$. Since eRe is simple, the two-sided ideal of eRe generated by x_0 must be all of eRe. Thus $y = \sum_i (er_ie)x_0(es_ie)$, where each $r_i, s_i \in R$. However, this sum is clearly contained in the two-sided ideal of R generated by x_0, and so x_0 generates the whole ring R, as required. □

We now move on to another definition that will be important when examining Leavitt path algebras.

Definition 1.1.3. A ring R is said to be **$\mathbb{Z}$-graded** if there is a family $\{R_n : n \in \mathbb{Z}\}$ of subgroups of the additive group $(R, +)$ for which

(i) $R_m R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}$, and

(ii) $R = \bigoplus_{n \in \mathbb{Z}} R_n$ as an abelian group.

In this case, the family $\{R_n : n \in \mathbb{Z}\}$ is said to be a **$\mathbb{Z}$-grading** of R, and elements of each subgroup R_n are called **homogeneous elements of degree** n. Thus each element in R can be written uniquely as a sum of homogeneous components (from the definition of a direct sum – see Section 1.2).

A familiar example of a $\mathbb{Z}$-graded ring is the ring of Laurent polynomials $R = K[x, x^{-1}]$ over a field K. If we define $R_n = \{kx^n : k \in K\}$ for each $n \in \mathbb{Z}$ then it is easy to see that conditions (i) and (ii) of the above definition hold. In many cases a ring R lends itself naturally to a $\mathbb{Z}$-grading; in particular, we will show that any Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$-graded ring. This concept of grading extends to several other ring-theoretic concepts, as the following definition illustrates.

Definition 1.1.4. An ideal I of a $\mathbb{Z}$-graded ring $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is said to be a **graded ideal** if $x = \sum_{n\in\mathbb{Z}} x_n \in I$ (with each $x_n \in R_n$) implies that each $x_n \in I$. In other words, an ideal I is graded if the homogeneous components of any element in I are in I themselves. Equivalently, we can write $I = \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$. Note that if R is a $\mathbb{Z}$-graded ring and e is an idempotent in R then the subring eRe is also $\mathbb{Z}$-graded.

If $R = \bigoplus_{n\in\mathbb{Z}} R_n$ and $S = \bigoplus_{n\in\mathbb{Z}} S_n$ are two $\mathbb{Z}$-graded rings, then a homomorphism $\phi : R \to S$ is said to be a **graded homomorphism** if $\phi(R_n) \subseteq S_n$ for each $n \in \mathbb{Z}$; that is, if ϕ takes homogeneous elements of degree n in R to homogeneous elements of degree n in S.

It can be shown (see, for example, [NV, page 6]) that if R is a $\mathbb{Z}$-graded ring and I is a graded ideal of R, then the quotient ring R/I is also $\mathbb{Z}$-graded. Similarly, if R and R/I are both $\mathbb{Z}$-graded then I must also be graded.

Lemma 1.1.5. *If $\phi : R \to S$ is a graded homomorphism between two graded rings R and S, then* $\ker(\phi)$ *is a graded ideal of R.*

Proof. Let $x \in \ker(\phi)$ and write $x = x_{n_1} + \cdots + x_{n_t}$, where each $x_{n_i} \in R_{n_i}$. Thus $0 = \phi(x) = \sum_{i=1}^{t} \phi(x_{n_i})$. Since ϕ is a graded homomorphism, $\phi(x_{n_i}) \in S_{n_i}$ for each i. However, since S is a graded ring, the element 0 can only be expressed one way as a sum of homogeneous components from each S_{n_i}, namely $0 = 0 + \cdots + 0$. Thus for each $i \in \{1, \ldots, t\}$ we have $\phi(x_{n_i}) = 0$ and so $x_{n_i} \in \ker(\phi)$, as required. $\square$

Let L be a left ideal of a ring R. Then L is said to be a **minimal left ideal** of R if $L \neq 0$ and there exists no left ideal K of R such that $0 \subset K \subset L$. Similarly, L is said to be a **maximal left ideal** of R if $L \neq R$ and there exists no left ideal M of R such that $L \subset M \subset R$.

The following lemma provides a useful way to determine when a principal left ideal is a minimal left ideal.

Lemma 1.1.6. *Let R be a ring and let x be a nonzero element of R. If $x \in Ra$ for every nonzero $a \in Rx$, then Rx is a minimal left ideal.*

Proof. Suppose Rx contains a nonzero left ideal I and take an arbitrary nonzero $a \in I$. Since $a = bx + nx$ for some $b \in R$ and $n \in \mathbb{N}$, we have $Ra \subseteq Rx$. Similarly, since $x \in Ra$ then $Rx \subseteq Ra$ and so $Rx = Ra$. Since $Ra \subseteq I$, we must have $I = Rx$ and so Rx is minimal. $\square$

Let R be a ring. The **Jacobson radical of** R, denoted $J(R)$, is the intersection of the family of all maximal left ideals of R. It can be shown that $J(R)$ is a two-sided ideal of R. (Similarly, the socle of R is the sum of all *minimal* left ideals of R; we will define this concept formally in Section 3.1). We now give two useful results concerning the Jacobson radical.

Lemma 1.1.7. *Let R be a ring. Then $J(R)$ contains no nonzero idempotents.*

Proof. It is well-known that every element $x \in J(R)$ is quasiregular, that is, there exists a $y \in R$ such that $x + y = -xy = -yx$ (see, for example, [D, Chapter 4]). Suppose that $J(R)$ contains an idempotent e. Then $-e \in J(R)$, so there exists a $y \in R$ such that $y - e = ey$. Multiplying on the left by $-e$ gives $-ey + e = -ey$, and thus $e = 0$. $\square$

The following lemma is from [AA3, Lemma 6.2].

Lemma 1.1.8. *Let R be a $\mathbb{Z}$-graded ring. Suppose that R contains a set of local units E such that each element of E is homogeneous. Then $J(R)$ is a graded ideal.*

Proof. Let $x \in J(R)$ and decompose $x = x_{n_1} + \cdots + x_{n_t}$ into a sum of its homogeneous components. Let e be an element of E such that $exe = x$. Then $x = exe = ex_{n_1}e + \cdots + ex_{n_t}e$. Since e is a local unit it must be an idempotent, and therefore e has degree 0. Thus $ex_{n_i}e$ is homogeneous with the same degree as x_{n_i}. Since the decomposition of an element into homogeneous components is unique, we must have $ex_{n_i}e = x_{n_i}$ for each $i \in \{1, \ldots, t\}$.

By Jacobson [J2, Proposition 3.7.1] we have that $J(R) \cap eRe = eJ(R)e = J(eRe)$. Since $x = exe$, we have $x \in J(R) \cap eRe$ and so $x \in J(eRe)$. Since each $x_{n_i} = ex_{n_i}e$, $x = x_{n_1} + \cdots + x_{n_t}$ is in fact the decomposition of x into graded components in eRe. Now, since eRe is a $\mathbb{Z}$-graded unital subring of R (with e as identity), we can apply Bergman [Be, Corollary 2] to get that $J(eRe)$ is a graded ideal of eRe. Thus $x_{n_i} \in J(eRe)$ for each $i \in \{1, \ldots, t\}$. Since $J(R) \cap eRe = J(eRe)$, we have $J(eRe) \subseteq J(R)$ and thus $x_{n_i} \in J(R)$ for each $i \in \{1, \ldots, t\}$, completing the proof. $\square$

A ring R is said to be **von Neumann regular** if, for every $a \in R$, there exists an $x \in R$ for which $a = axa$. Furthermore, we say that x is a **von Neumann regular inverse** or **quasi-inverse** for a. Note that any division ring is von Neumann regular, since we can simply choose $x = a^{-1}$ if a is nonzero. The question of which Leavitt path algebras are von Neumann regular (as well as other definitions of 'regular') will be visited in Section 4.2. The following lemma concerning von Neumann regular rings is from [G1, Lemma 1.3].

Lemma 1.1.9. *Let R be a ring and let J and K be two two-sided ideals in R with $J \subseteq K$. Then K is von Neumann regular if and only if J and K/J are von Neumann regular.*

Proof. If K is von Neumann regular, then clearly K/J is von Neumann regular. Now consider $a \in J$. Since $J \subseteq K$, there exists $x \in K$ such that $a = axa$. Now $y = xax \in J$ (since J is a two-sided ideal) and $aya = axaxa = axa = a$. Thus J is von Neumann regular.

Now suppose that K/J and J are both von Neumann regular and consider $a \in K$. Since K/J is von Neumann regular there exists $x \in K$ for which $a + J = axa + J$, so that $a - axa \in J$. Since J is von Neumann regular, there exists $y \in J$ for which $a - axa = (a - axa)y(a - axa)$. Thus $a = axa + (a - axa)y(a - axa) = a(x + y - xay - yax + xayax)a$, and so K is von Neumann regular. $\square$

We conclude this section with a useful result regarding the matrix ring $\mathbb{M}_n(K)$, where K is a field.

Lemma 1.1.10. *Let K be a field. Then $\mathbb{M}_n(K)$, the ring of $n \times n$ matrices over K, is simple for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$, let J be a nonzero two-sided ideal of $\mathbb{M}_n(K)$ and let $A = (a_{ij}) \in J$. If we can show that the $n \times n$ identity matrix I_n is in $\langle A \rangle$ (the two-sided ideal generated by A) then we have $\mathbb{M}_n(K) = \langle A \rangle = J$, proving that $\mathbb{M}_n(K)$ is simple. Let E_{ij} be the matrix unit with 1 in the (i, j) position and zeros elsewhere. Choose $i, j \in \{1, \dots, n\}$ such that $a_{ij} \neq 0$. Then $a_{ij}\, E_{11} = E_{i1} A\, E_{1j} \in \langle A \rangle$. Since K is a field, we have $(a_{ij}^{-1} E_{11})(a_{ij} E_{11}) = E_{11} \in \langle A \rangle$. By similar arguments, we have $E_{22}, \dots, E_{nn} \in \langle A \rangle$ and thus $I_n = E_{11} + E_{22} + \cdots + E_{nn} \in \langle A \rangle$, as required. $\square$

1.2 Module Theory

Let R be a ring. Recall that an abelian group $(M, +)$ is called a **left R-module** or a **left module over** R if there is a mapping $R \times M \to M$, given by $(r, m) \mapsto rm$, such that, for all $r, r_1, r_2 \in R$ and all $m, m_1, m_2 \in M$, we have

(i) $r(m_1 + m_2) = rm_1 + rm_2$,

(ii) $(r_1 + r_2)m = r_1 m + r_2 m$,

(iii) $r_1(r_2 m) = (r_1 r_2)m$, and

(iv) $1_R m = m$ (if R has identity).

If M is a left R-module, we sometimes denote M by ${}_R M$. Furthermore, we say that N is a **submodule** of M if N is a subgroup of M and $rn \in N$ for all $r \in R$

and all $n \in N$. Note that if we view a ring R as the additive abelian group $(R, +)$, then R can be seen as a left module over itself, with module multiplication given by multiplication in the ring R. Furthermore, the submodules of ${}_RR$ are simply the left ideals of R.

We can define a right R-module M similarly, with a mapping $M \times R \to M$ given by $(m, r) \mapsto mr$, and we sometimes denote this by M_R. Again, any ring R can be seen as a right module over itself, and the right submodules of R_R are the right ideals of R. In this section we will concern ourselves primarily with left R-modules, though analogous results and definitions exist for right R-modules in most cases.

Now let R be a ring and let M, N be left R-modules. A function $f : M \to N$ is called an **R-homomorphism** if f is a group homomorphism for which $f(rx) = rf(x)$ for all $x \in M$ and all $r \in R$. We denote by $\mathrm{Hom}_R(M, N)$ the additive abelian group of all R-homomorphisms from M to N. (This is easily seen to be a group if we define addition by $(f + g)(m) = f(m) + g(m)$ for all $f, g \in \mathrm{Hom}_R(M, N)$ and all $m \in M$.) Furthermore, in the case that $M = N$, $\mathrm{Hom}_R(M, N)$ is denoted by $\mathrm{End}_R(M)$ and is called the **endomorphism ring of** M. (Again, this is easily seen to be a ring if we define multiplication by $(f \cdot g)(m) = f(g(m))$ for all $f, g \in \mathrm{End}_R(M)$ and all $m \in M$.)

We denote by R-mod the category of all left R-modules together with all R-homomorphisms $f : M \to N$, where M and N are left R-modules. (See Section 1.3 for a formal definition of a category.) However, in this thesis we will concern ourselves with a slightly more restricted category of R-modules. We define an R-module M to be **unital** if

$$RM := \left\{ \sum_{i=1}^{n} r_i m_i : r_i \in R, m_i \in M \right\} = M,$$

and **nondegenerate** if $Rm = 0$ (for some $m \in M$) implies $m = 0$. We denote by R-Mod the subcategory of R-mod containing all unital and nondegenerate left R-modules and all R-homomorphisms between such modules (and define Mod-R to be the corresponding category of right R-modules.) Note that if R is a unital ring then every left R-module is unital and nondegenerate, and so R-Mod is the full category

R-mod.

Lemma 1.2.1. *Let R be a ring with local units and let M be a unital left R-module. Then*

(i) M is nondegenerate, and

(ii) for every $m \in M$ there is a local unit $e \in R$ such that $em = m$.

Proof. Let $m \in M$. Since M is unital, we can write $m = \sum_{i=1}^n r_i m_i$ for some $r_i \in R$ and $m_i \in M$. Since R has local units, there exists an idempotent $e \in R$ such that $r_i = er_i$ for each $i = 1, \dots, n$. Thus $m = \sum_{i=1}^n er_i m_i = e(\sum_{i=1}^n r_i m_i) = em$, proving (ii). Furthermore, if $Rm = 0$ then $m = em = 0$, showing that M is nondegenerate and thus proving (i). $\square$

Now let R and S be two rings. Suppose that M is a left R-module and a right S-module, with the property that $(rm)s = r(ms)$ for all $r \in R$, $s \in S$ and $m \in M$. Then we say that M is an **R-S-bimodule**, and we sometimes denote M by ${}_RM_S$. Furthermore, if M and N are two R-S-bimodules, then a map $f : M \to N$ is a **bimodule homomorphism** if it is both a homomorphism of left R-modules and right S-modules.

If A is a commutative ring then a ring R is called an **A-algebra** if R is an A-A-bimodule. For example, any ring R is a $\mathbb{Z}$-algebra, with the obvious module multiplications $nr = n \cdot r = rn$ for all $n \in \mathbb{Z}$ and all $r \in R$. In this thesis we will be primarily concerned with algebras over an arbitrary field K.

The following lemma gives a useful way of visualising the subring eRe of a ring R, where e is an idempotent. Recall that $\mathrm{End}_R(Re)$ is the ring of all R-homomorphisms from the left R-module Re to itself.

Lemma 1.2.2. *Let R be a ring and let e be an idempotent in R. Then $\mathrm{End}_R(Re) \cong (eRe)^{Op}$, where $(eRe)^{Op}$ is the opposite ring of eRe with multiplication $\cdot$ defined by $(er_1e) \cdot (er_2e) = (er_2e)(er_1e)$ for all $r_1, r_2 \in R$. Similarly, $\mathrm{End}_R(eR) \cong eRe$.*

Proof. Let f be an arbitrary R-homomorphism in $\mathrm{End}_R(Re)$ with $f(e) = r_f e$ for some $r_f \in R$. (Note that r_f defines the entire homomorphism f, since given any element $te \in Re$ we have $f(te) = t(f(e)) = tr_f e$.) Consider the map $\phi : \mathrm{End}_R(Re) \to (eRe)^{Op}$ with $\phi(f) = er_f e$. (To check this is a well-defined function, suppose that $r_f e = se$ for some $s \in R$ with $r_f \neq s$. Then $er_f e = ese$, and so ϕ is indeed well-defined.)

Now suppose that $f, g \in \mathrm{End}_R(Re)$ with $f(e) = r_f e$ and $g(e) = r_g e$ for some $r_f, r_g \in R$. Then $(f + g)(e) = f(e) + g(e) = (r_f + r_g)e$, and so

$$\phi(f + g) = e(r_f + r_g)e = er_f e + er_g e = \phi(f) + \phi(g)$$

as required. To check that ϕ is multiplicative, note that we must check that $\phi(fg) = \phi(f) \cdot \phi(g) = \phi(g)\phi(f)$. Now $(fg)(e) = f(r_g e) = f(r_g e^2) = r_g e f(e) = r_g e r_f e$, and so

$$\phi(fg) = e(r_g e r_f)e = (er_g e)(er_f e) = \phi(g)\phi(f).$$

Thus ϕ is a ring homomorphism. Now, given any $x \in (eRe)^{Op}$, say $x = ere$, let $f \in \mathrm{End}_R(Re)$ be the homomorphism defined by $f(e) = re$. Then $\phi(f) = ere = x$, and so ϕ is an epimorphism. Finally, suppose that $f \in \mathrm{End}_R(Re)$ with $\phi(f) = er_f e = 0$ (where r_f is an element of R for which $f(e) = r_f e$). Then $0 = er_f e = ef(e) = f(e^2) = f(e)$ and so, for any $se \in Re$, we have $f(se) = sf(e) = 0$ and thus $f = 0$. Therefore ϕ is a monomorphism, and so $\mathrm{End}_R(Re) \cong (eRe)^{Op}$.

Using a similar argument, if we define $\varphi : \mathrm{End}_R(eR) \to eRe$ to be the map $\varphi(f) = er_f e$, where r_f is an element of R for which $f(e) = er_f$, we obtain the isomorphism $\mathrm{End}_R(eR) \cong eRe$. $\square$

We now turn our attention to direct products and direct sums. Recall that if R is a ring and $\{A_i : i \in I\}$ is a family of left R-modules, we define the **direct product** of the family $\{A_i : i \in I\}$ to be the R-module formed by taking the cartesian product of the family and denote this by $\prod_{i \in I} A_i$. Furthermore, we define the **external direct sum** of the family to be

$$\bigoplus_{i \in I} A_i := \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i : a_i \neq 0_{A_i} \text{ for only a finite number of indices } i \in I \right\}.$$

Note that if I is a finite index set then we have $\prod_{i\in I} A_i = \bigoplus_{i\in I} A_i$.

Now if M is an R-module and $\{M_i : i \in I\}$ is a family of submodules of M, we say the sum $\sum_{i\in I} M_i$ is an **internal direct sum** if every element $m \in \sum_{i\in I} M_i$ has a unique representation in the form $\sum_{i\in I} m_i$, where each $m_i \in M_i$ and $m_i \neq 0$ for only a finite number of indices $i \in I$. We denote this by $\bigoplus_{i\in I} M_i$. If $M = \bigoplus_{i\in I} M_i$ then each M_i is said to be a **direct summand** of M.

It can be shown that a sum $\sum_{i\in I} M_i$ is direct if and only if

$$M_i \cap \left(\sum_{j\in I, j\neq i} M_j \right) = \{0\}$$

for all $i \in I$. We can also show that any internal direct sum can be regarded as an external direct sum, and vice versa, and hence there is no ambiguity in the notation.

The following result concerns left ideals generated by idempotents, and is useful when working with rings with local units (though it is valid for any ring).

Lemma 1.2.3. *Let R be an arbitrary ring. If $e \in R$ is an idempotent, then*

(i) Re is a direct summand of R;

(ii) any direct summand of Re is of the form Rf, where f is an idempotent; and

(iii) if $f \in Re$ is an idempotent then Rf is a direct summand of Re.

Proof. (i) Since $e = e^2$, we have $Re = \{re : r \in R\}$. Let $T = \{t - te : t \in R\}$. It is straightforward to see that T is a left ideal of R. For any $r \in R$, we have $r = re + (r - re)$, so clearly we have $R = Re + T$. Furthermore, suppose $x \in Re \cap T$. Then $x = se$ and $x = t - te$ for some $s, t \in R$. Using the fact that $e = e^2$, we therefore have $x = se = se^2 = (t - te)e = te - te = 0$. Thus $Re \cap T = \{0\}$ and so $R = Re \oplus T$, as required.

(ii) Suppose that $Re = B \oplus C$. Since $e = e^2 \in Re$ we have $e = f + g$, where $f \in B$ and $g \in C$. Furthermore, since $f \in Re$ we have $f = f'e$ for some $f' \in R$, and thus $f = f'e = f'e^2 = fe$. Therefore $f = fe = f(f+g) = f^2 + fg$ and so $f - f^2 = fg$. Now $f - f^2 \in B$ and $fg \in C$ (since C is a left ideal), and so $f^2 - f \in B \cap C = \{0\}$

and thus $f^2 = f$. We now show that $B = Rf$. Clearly $Rf \subseteq B$ since B is a left ideal. Now suppose that $x \in B$. Since $x \in Re$, we have $x = xe = x(f + g)$ and thus $x - xf = xg$. Once again this implies $x - xf \in B \cap C$ and so $x = xf \in Rf$, completing the proof.

(iii) We show that $Re = Rf \oplus R(e-ef)$. First, since $f \in Re$ we have $f = f'e$ for some $f' \in R$, and so $fe = f'e^2 = f'e = f$. Thus $Rf+R(e-ef) = Rfe+R(e-ef)e \subseteq Re$. Furthermore, if $re \in Re$, then $re = r(ef + e - ef) = (re)f + e(e - ef) \in Rf + R(e - ef)$, and so $Re = Rf + R(e - ef)$. To show the sum is direct, suppose that $x \in Rf \cap R(e - ef)$. Since f and $e - ef$ are idempotents, there exist $r, s \in R$ such that $x = rf = s(e - ef)$. Then $rf = rf^2 = s(e - ef)f = s(ef - ef) = 0$, as required. □

For a given left R-module M and index set I, we often write

$$M^I := \prod_{i\in I} M_i \quad \text{and} \quad M^{(I)} := \bigoplus_{i\in I} M_i$$

where $M_i = M$ for each $i \in I$. As noted above, if I is finite then we have $M^I = M^{(I)}$. Furthermore, for any $n \in \mathbb{N}$ we let M^n denote the direct sum (or product) of n copies of M.

Proposition 1.2.4. *Let R be a ring and let I be an index set. For any family of left R-modules $\{A_i : i \in I\}$ and any left R-module B, we have a group isomorphism*

$$\operatorname{Hom}\left(\bigoplus_{i\in I} A_i, B\right) \cong \prod_{i\in I} \operatorname{Hom}(A_i, B)$$

Proof. For each $j \in I$ we define $\pi_j : \bigoplus_{i\in I} A_i \to A_j$ to be the natural projection map $(a_i)_{i\in I} \mapsto a_j$, and $\phi_j : A_j \to \bigoplus_{i\in I} A_i$ to be the natural injection map $a_j \mapsto (b_i)_{i\in I}$, where $b_j = a_j$ and $b_i = 0$ for $i \neq j$. Let $f \in \operatorname{Hom}(\bigoplus_{i\in I} A_i, B)$. Then, for all $i \in I$ we have $f\phi_i \in \operatorname{Hom}(A_i, B)$. Define $\tau : \operatorname{Hom}(\bigoplus_{i\in I} A_i, B) \to \prod_{i\in I} \operatorname{Hom}(A_i, B)$ by $\tau(f) = (f\phi_i)_{i\in I}$. It is easy to show that τ is a group homomorphism.

Suppose that $\tau(f) = 0$ for some $f \in \operatorname{Hom}(\bigoplus_{i\in I} A_i, B)$, so that $f\phi_i = 0$ for all $i \in I$. Then, given any $(a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$, we have $f((a_i)_{i\in I}) = f(\sum_{i\in I} \phi_i(a_i)) = \sum_{i\in I} f\phi_i(a_i) = 0$ and so $f = 0$. Thus τ is a monomorphism. Now let $g = (g_i)_{i\in I} \in$

$\prod_{i\in I} \mathrm{Hom}(A_i, B)$, so that $g_i : A_i \to B$ is a homomorphism for each $i \in I$. Define $f \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, B)$ by $f((a_i)_{i\in I}) = \sum_{j\in I} g_j(\pi_j((a_i)_{i\in I}))$. Now, for each $j \in I$ we have $f\phi_j(a_j) = g_j(a_j)$ and so $f\phi_j = g_j$. Thus $\tau(f) = g$ and so τ is an epimorphism, completing the proof. □

Let R be a ring. A left R-module P is said to be **directly infinite** if P is isomorphic to a proper direct summand of itself. In other words, P is directly infinite if there exists a nonzero R-module Q such that $P \cong P \oplus Q$. Thus, for any $n \in \mathbb{N}$, we have

$$P \cong P \oplus B \cong (P \oplus B) \oplus B \cong P \oplus B^2 \cong \ldots \cong P \oplus B^n.$$

Furthermore, an idempotent e in a ring R is said to be **infinite** if the right ideal eR is directly infinite (as a right R-module). The ring R is said to be **purely infinite** if every right ideal of R contains an infinite idempotent. In other words, R is purely infinite if every right ideal of R contains a directly infinite right ideal of the form eR, where e is an idempotent.

The following result from [AGP, Theorem 1.6] gives a useful way of determining when a unital ring is purely infinite. We state it here without proof.

Theorem 1.2.5. *Let R be a simple unital ring. Then R is purely infinite if and only if the following conditions are satisfied:*

(i) R is not a division ring, and

(ii) for every nonzero element $x \in R$, there exist elements $s, t \in R$ such that $sxt = 1$.

In Section 2.3 we will be examining purely infinite simple Leavitt path algebras. As mentioned earlier, any Leavitt path algebra has local units but is not necessarily unital, and so we will need to adapt Theorem 1.2.5 for the more general case in which R has local units. This is not straightforward, however, and we will need to use Morita equivalence (introduced in Section 1.3) to do so.

The following proposition gives a useful way of determining when an idempotent is infinite.

Proposition 1.2.6. *Let R be a ring and let $e \in R$ be an idempotent. Then e is infinite if and only if there is an idempotent f in R and elements x, y in R such that*

$$e = xy, \quad f = yx, \quad \text{and} \quad fe = ef = f \neq e.$$

Proof. First suppose that e is infinite. Then $eR = B \oplus C$, where B, C are nonzero right ideals of R, and there is an R-isomorphism $\phi : eR \to B$. Since $e \in eR$ we have $e = f + g$, where $f \in B, g \in C$ and f, g are nonzero. Following the proof of Lemma 1.2.3 (ii) (but in the context of right ideals), we can conclude that f and g are idempotents and that $B = fR$ and $C = gR$, giving $eR = fR \oplus gR$. Now, since $f \in eR$ and e is an idempotent we have $f = ef$, as required. Similarly, $g = eg$ and so $g = eg = (f + g)g = fg + g^2$. Thus $g - g^2 = fg \in B \cap C = \{0\}$ and so $fg = 0$. Therefore we have $fe = f(f + g) = f^2 + fg = f$, as required. Furthermore, $f \neq e$ since g is nonzero.

Now, since $\phi : eR \to fR$ is an isomorphism, there exists $x \in eR$ such that $\phi(x) = f$ and there exists $y \in fR$ such that $\phi(e) = y$. Then

$$yx = \phi(e)x = \phi(ex) = \phi(x) = f.$$

Furthermore, we have

$$\phi(xy) = \phi(x)y = fy = y = \phi(e)$$

and so, since ϕ is a monomorphism, we also have $xy = e$, as required.

Conversely suppose that there exist elements $f, x, y \in R$ such that $f^2 = f$, $xy = e$, $yx = f$, and $ef = fe = f \neq e$. Let $g = e - f$, noting that $g \neq 0$ since $e \neq f$. Then $e = f + g$ and so $eR \subseteq fR + gR$. Moreover, $fR + gR = efR + (e - ef)R \subseteq eR$ and so we have $eR = fR + gR$. In fact, this last sum is direct since, if $fr_1 = gr_2$ for some $r_1, r_2 \in R$, then $fr_1 = f^2r_1 = fgr_2 = (fe - f^2)r_2 = 0$. Thus $eR = fR \oplus gR$.

We complete the proof by showing that there is an isomorphism $\phi : eR \to fR$. Recall that $eR = xyR$ and $fR = yxR$. Thus we can define an R-homomorphism $\phi : eR \to fR$ by setting $\phi(xyr) = yxyr$ for all $r \in R$. If $\phi(xyr) = 0$ then $yxyr = 0$ and so $xyr = er = e^2r = xyxyr = 0$, showing that ϕ is a monomorphism. Furthermore, given $yxr \in fR$ we have $\phi(xyxr) = yxyxr = f^2r = fr = yxr$, showing that ϕ is an epimorphism and thus completing the proof. □

Proposition 1.2.6 leads to the following useul corollary.

Corollary 1.2.7. *Let R be a ring and let S be a subring of R. If R has no infinite idempotents then S has no infinite idempotents.*

Proof. Suppose that R has no infinite idempotents but S has an infinite idempotent e. Then, by Proposition 1.2.6, there exists an idempotent $f \in S$ and elements $x, y \in S$ such that $e = xy, f = yx$ and $fe = ef = f \neq e$. Since these elements are also in R, Proposition 1.2.6 also gives that e is an infinite idempotent of R, a contradiction. $\square$

Definition 1.2.8. Let R be a ring, let M_1, M_2, …, M_n be left R-modules and for each $i = 1, \ldots, n-1$ let $f_i : M_i \to M_{i+1}$ be R-homomorphisms. We say that the sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \quad \cdots \quad \xrightarrow{f_{n-2}} M_{n-1} \xrightarrow{f_{n-1}} M_n$$

is **exact** if $\ker(f_{i+1}) = \operatorname{Im}(f_i)$ for each $i = 1, \ldots, n-1$. Furthermore, a **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Note that this implies that f is a monomorphism and g is an epimorphism.

We now define an important concept in module theory.

Definition 1.2.9. Let R be a ring. A left R-module P is said to be **projective** if, for any R-epimorphism $g : B \to C$, where B, C are left R-modules, and any R-homomorphism $f : P \to C$, there exists an R-homomorphism $h : P \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, "h"'] \arrow[d, "f"] & \\
B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

That is, $gh = f$.

This definition leads to the following useful lemma.

Lemma 1.2.10. *Let A and P be left R-modules, where P is projective, and let $f : A \to P$ be an R-homomorphism. If f is an epimorphism, then there exists a left R-module P' for which $A \cong P \oplus P'$.*

Proof. If f is an epimorphism, then by the projective nature of P there exists an R-homomorphism $h : P \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "1_P"] & \\ A \arrow[r, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

Thus we have $fh = 1_P$, the identity map on P. We begin by showing that $A = \mathrm{Im}(h) \oplus \ker(f)$. Let $x \in A$, and write $x = hf(x) + (x - hf(x))$. Now $hf(x) \in \mathrm{Im}(h)$, while $f(x - hf(x)) = f(x) - fhf(x) = 0$ (since $fh = 1_P$) and so $x - hf(x) \in \ker(f)$. Thus $A = \mathrm{Im}(h) + \ker(f)$. To show this sum is direct, suppose that $y \in \mathrm{Im}(h) \cap \ker(f)$. Then $y = h(z)$ for some $z \in P$. Furthermore, $0 = f(y) = f(h(z)) = 1_P(z) = z$, and so $y = h(z) = h(0) = 0$. Thus $A = \mathrm{Im}(h) \oplus \ker(f)$, as required. Since fh is a monomorphism, h must also be a monomorphism, and so $P \cong \mathrm{Im}(h)$. Letting $P' = \ker(f)$, we therefore have $A \cong P \oplus P'$, as required. □

A concept closely related to projective modules is that of injective modules.

Definition 1.2.11. Let R be a ring. A left R-module Q is said to be **injective** if, for any R-monomorphism $g : A \to B$, where A, B are left R-modules, and any R-homomorphism $f : A \to Q$, there exists an R-homomorphism $h : B \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, "h"] \\ & Q & \end{tikzcd}$$

That is, $hg = f$.

In the case that R is injective as a left module over itself, we say that R is **left self-injective**. We will examine self-injective Leavitt path algebras in Section 4.4.

Lemma 1.2.12. *Any direct summand of an injective R-module is injective.*

Proof. Suppose that $Q = M \oplus N$ is an injective R-module. Consider the following diagram of R-modules and R-homomorphisms, where g is a monomorphism:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \\ & M & \end{tikzcd}$$

Let $i : M \to Q$ and $\pi : Q \to M$ be the standard inclusion and projection maps, respectively. Now if is an R-homomorphism from A to Q, and so by the injectivity of Q there exists an R-homomorphism $\bar{h}$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[ddl, "\bar{h}"] \\ & M \arrow[d, "i"'] & \\ & Q & \end{tikzcd}$$

That is, $if = \bar{h}g$. Define $h : B \to M$ by $h = \pi\bar{h}$. Then $hg = \pi\bar{h}g = \pi i f = 1_M f = f$, and so M is injective. $\square$

A similar proof shows that the direct product of injective R-modules is also injective. Furthermore, we can use similar arguments to show that any direct summand of a projective R-module is projective, and that the direct sum of projective R-modules is projective.

We conclude this section with a series of results that generalise well-known results for R-modules, where R is unital, to the more general case that R has local units. This first result is from [ARM2, Proposition 2.2].

Proposition 1.2.13. *Let R be a ring with local units. Then for any idempotent $e \in R$, the left ideal Re is a projective module in the category R-Mod.*

Proof. Since R has local units, it is easy to see that Re is unital and nondegenerate and is therefore in the category R-Mod. Now consider the diagram

$$\begin{tikzcd} & Re \arrow[d, "f"] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

where B, C are in R-Mod and g is an epimorphism. Since $e = e^2 \in Re$, we can define $c = f(e)$. Furthermore, since g is an epimorphism there exists $b \in B$ for which $g(b) = c$. Now $ec = ef(e) = f(e^2) = f(e) = c$, and so $g(eb) = eg(b) = ec = c$. Define $h : Re \to B$ by $h(xe) = xeb$ for all $x \in R$. Since $xe = 0$ implies $xeb = 0$, h is a well-defined R-homomorphism. Thus, for any $x \in R$, $gh(xe) = g(xeb) = xg(eb) = xc = xf(e) = f(xe)$, and so $gh = f$ and Re is projective. □

Let R be a ring, let A be a right R-module, B a left R-module and let G be an abelian group. A function $f : A \times B \to G$ is called an R**-bilinear map** if, for all $a, a' \in A$, $b, b' \in B$ and $r \in R$, we have

(i) $f(a + a', b) = f(a, b) + f(a', b)$,

(ii) $f(a, b + b') = f(a, b) + f(a, b')$, and

(iii) $f(ar, b) = f(a, rb)$.

The **tensor product** of A and B over R is an abelian group $A \otimes_R B$ together with a bilinear map $\otimes : A \times B \to A \otimes_R B$ that is universal; that is, for every abelian group G and every bilinear map $f : A \times B \to G$, there exists a unique group homomorphism $\bar{f} : A \otimes_R B \to G$ for which the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "\otimes"] \arrow[dr, "f"'] & A \otimes_B N \arrow[d, "\bar{f}"] \\ & G \end{tikzcd}$$

That is, $\bar{f} \circ \otimes = f$. It can be shown that such an abelian group $A \otimes_R B$ will always exist (see, for example, [O, Section 2.2]). The group $A \otimes_R B$ is generated by elements of the form $a \otimes_R b$, where $a \in A$, $b \in B$ and $a \otimes_R b = \otimes((a, b))$.

Furthermore, if C is a right R-module, D a left R-module and $f : A \to C$ and $g : B \to D$ are R-homomorphisms, we can define a map $f \otimes_R g : A \otimes_R B \to C \otimes_R D$ by $f \otimes_R g((a, b)) = f(a) \otimes_R g(b)$ for all $a \in A$, $b \in B$. When it is clear we are taking the tensor product over R we may write $A \otimes_R B$ as simply $A \otimes B$. We use the tensor product to define the following important concept.

Definition 1.2.14. Let R be a ring. A module $M \in R$-Mod is said to be **flat** if the functor $- \otimes_R M$ is exact on the category R-Mod. That is, whenever

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence in R-Mod, then

$$0 \longrightarrow A \otimes M \xrightarrow{f \otimes 1_M} B \otimes M \xrightarrow{g \otimes 1_M} C \otimes M \longrightarrow 0$$

is also a short exact sequence.

It can be shown for any M that this sequence is always exact on the right-hand side, and so to show that M is flat it suffices to show that any monomorphism $f : A \to B$ gives rise to a monomorphism $f \otimes 1_M : A \otimes M \to B \otimes M$.

We now give several results concerning flat modules over rings with local units. We begin with the following lemma from [ARM2, Lemma 2.9].

Lemma 1.2.15. *Let R be a ring with local units. For any $M \in$ Mod-R, the map $\mu_M : M \otimes R \to M$ given by $\mu_M(\sum_{i=1}^n (m_i \otimes r_i)) = \sum_{i=1}^n m_i r_i$ is an isomorphism of right R-modules.*

Proof. First note that $M \otimes R$ is indeed a right R-module, with module multiplication given by $(\sum_{i=1}^n (m_i \otimes r_i))r = \sum_{i=1}^n (m_i \otimes (r_i r))$ for all $\sum_{i=1}^n (m_i \otimes r_i) \in M \otimes R$ and all $r \in R$. Since $M \in$ Mod-R, M is unital and so $MR = M$. Thus any $m \in M$ can be written $m = \sum_{i=1}^n m_i r_i$ for some $m_i \in M$ and $r_i \in R$, and so μ_M is an epimorphism. Now suppose $\sum_{i=1}^n m_i r_i = 0$. Since R has local units, there exists an idempotent $e \in R$ such that $r_i e = r_i$ for each $i = 1, \dots, n$. Then we have $\sum_{i=1}^n (m_i \otimes r_i) = \sum_{i=1}^n (m_i \otimes r_i e) = \sum_{i=1}^n (m_i r_i \otimes e) = (\sum_{i=1}^n m_i r_i) \otimes e = 0 \otimes e = 0$. Thus μ_M is a monomorphism, completing the proof. $\square$

This lemma leads to the following corollary.

Corollary 1.2.16. *Any ring R with local units is flat as a left R-module.*

Proof. Let $A, B \in \text{Mod-}R$ and let $f : A \to B$ be a monomorphism. Then, by Lemma 1.2.15, there exist isomorphisms $\mu_A : A \otimes R \to A$ and $\mu_B : B \otimes R \to B$. Let $a \otimes r$ be a generating element of $A \otimes R$. Then

$$f\mu_A(a \otimes r) = f(ar) = f(a)r = \mu_B(f(a) \otimes r) = \mu_B \circ (f \otimes 1_R)(a \otimes r)$$

and so $f\mu_A = \mu_B \circ (f \otimes 1_R)$; that is, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \\
0 \arrow[r] & A \otimes R \arrow[u, "\mu_A"] \arrow[r, "f\otimes 1_R"'] & B \otimes R \arrow[u, "\mu_B"']
\end{tikzcd}$$

Since f, μ_A and μ_B are monomorphisms, so too is $f \otimes 1_R$, and thus R is flat as a left R-module. $\square$

The following result regarding flat modules is from Rotman [Ro, Theorem 3.60]. Though Rotman's original result is for unital rings, the proof is the same for rings with local units and so we omit it.

Proposition 1.2.17. *Let R be a ring with local units, F a flat left R-module and K a submodule of F. Then F/K is a flat left R-module if and only if $K \cap IF = IK$ for every finitely generated right ideal I of R.*

Recall that, for a unital ring R, an R-module is said to be **free** if it has a basis; that is, a linearly independent generating set. The following definition, given in [ARM2, Definition 2.11], extends this notion to rings with local units.

Definition 1.2.18. Let R be a ring with local units and let F be a left R-module. Suppose there exists an index set I and sets $B = \{b_i\}_{i\in I} \subseteq F$ and $U = \{u_i\}_{i\in I} \subseteq R$, where each u_i is an idempotent and $b_i = u_i b_i$ for all $i \in I$. We say F is a **U-free left R-module** with **U-basis** B if, for all $x \in F$, there exists a unique family $\{r_i\}_{i\in I} \subseteq R$ (with only finitely many r_i nonzero) such that $r_i = r_i u_i$ for each $i \in I$ and

$$x = \sum_{i\in I} r_i b_i.$$

Note that, in particular, we have $F = \bigoplus_{i \in I} Rb_i$.

If R is a unital ring with identity 1, then taking $u_i = 1$ for each $i \in I$ reduces the definition of a U-free left R-module to the familiar definition of a free left R-module.

We can expand the following result from [Ro, Theorem 3.62] to the more general case involving local units and U-free modules, applying Proposition 1.2.17 in place of [Ro, Theorem 3.60]. Again, we will omit the proof.

Theorem 1.2.19. *Let R be a ring with local units and let F be a U-free left R-module. Then, for any submodule S of F, the following statements are equivalent:*

(i) F/S is a flat R-module.

(ii) For each element $x \in S$, there exists a homomorphism $f : F \to S$ such that $f(x) = x$.

(iii) For each finite set of elements $\{x_1, \ldots, x_n\}$ of S, there is a homomorphism $f : F \to S$ such that $f(x_i) = x_i$ for each $i = 1, \ldots, n$.

We conclude this section with the following proposition from [ARM2, Proposition 2.17], which generalises the well-known result for unital rings that any R-module is the epimorphic image of a free R-module.

Proposition 1.2.20. *If R is a ring with local units then every module $M \in R$-Mod is the epimorphic image of a U-free left R-module.*

Proof. Let $M \in R$-Mod. By Lemma 1.2.1, for every $m \in M$ there exists an idempotent $e_m \in R$ such that $m = e_m m$. Since M is unital, for any $x \in M$ we have $x = \sum_{m \in M} r_m m = \sum_{m \in M} r_m e_m m$, where only finitely many r_m are nonzero. Let $\phi : \bigoplus_{m \in M} Re_m \to M$ be the map $(r_m e_m)_{m \in M} \mapsto \sum_{m \in M} r_m e_m m$, which, by the above observation, is an epimorphism.

To show that $\bigoplus_{m \in M} Re_m$ is U-free, let $U = \{e_m\}_{m \in M}$ and $B = \{\mathbf{b}_m\}_{m \in M} \subseteq \bigoplus_{m \in M} Re_m$, where each $\mathbf{b}_m = (b_i)_{i \in M}$, with $b_i = e_m$ for $i = m$ and $b_i = 0$ otherwise. Then $\mathbf{b}_m = e_m \mathbf{b}_m$ for all $m \in M$. Furthermore, if we take an arbitrary $\mathbf{x} = (s_m e_m)_{m \in M} \in \bigoplus_{m \in M} Re_m$ (where each $s_m \in R$ and only finitely many s_m are nonzero), then taking $r_m = s_m e_m$ we have a unique family $\{r_m\}_{m\in M} \subseteq R$ such that $\mathbf{x} = \sum_{m\in M} r_m \mathbf{b}_m$ and $r_m e_m = r_m$. Thus $\bigoplus_{m\in M} Re_m$ is a U-free left R-module. $\square$

1.3 Morita Equivalence

In this section we examine the concept of 'Morita Equivalence', which was defined by Japanese mathematician Kiichi Morita in 1958. It is a powerful concept: if we can show that two rings are Morita equivalent, then these two rings will share various 'Morita invariant' ring-theoretic properties. We will appeal to Morita equivalence at several points in this thesis; in particular, we will show in this section that the property 'purely infinite' is Morita invariant (Theorem 1.3.17) and use this to expand Theorem 1.2.5 to rings with local units (Theorem 1.3.19).

Morita equivalence is a fairly deep and complex field of theory. Here we give enough background so that the basic concepts can be understood and we have sufficient tools to apply these concepts to relevant areas; however, some results will be stated without proof, as they require a large amount of background theory that would take us far outside the bounds of this thesis. We begin by looking at category theory.

Definition 1.3.1. A **category** $\mathcal{C}$ is made up of two sets: $\mathrm{Obj}(\mathcal{C})$, the set of **objects** in C, and $\mathrm{Mor}(\mathcal{C})$, the set of **morphisms** between objects in $\mathcal{C}$. If $A, B \in \mathrm{Obj}(\mathcal{C})$, we let $\mathrm{Mor}(A, B)$ denote the set of morphisms from A to B. Furthermore, if $f \in \mathrm{Mor}(A, B)$, we can denote this by the usual function notation $f : A \to B$.

Moreover, there exists an operation $\circ$ such that, for any $A, B, C \in \mathrm{Obj}(\mathcal{C})$, we have $\circ : \mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$. This operation is associative, so that for all $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and all $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$ and $h \in \mathrm{Mor}(C, D)$ we have $h \circ (g \circ f) = (h \circ g) \circ f$. Furthermore, for all $A \in \mathrm{Obj}(\mathcal{C})$ there exists a unique morphism $1_A \in \mathrm{Mor}(A, A)$ for which $f \circ 1_A = f$ and $1_A \circ g = g$, for all $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(C, A)$ and $B, C \in \mathrm{Obj}(\mathcal{C})$.

In Section 1.2 we introduced the category R-Mod. In light of the above definition, we can see that the objects of R-Mod are the unital, nondegenerate left R-modules, the morphisms are R-homomorphisms between such modules, and the operation $\circ$ is function composition.

Definition 1.3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **covariant functor** is a map $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$, denoted $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, that maps each object $A \in \mathrm{Obj}(\mathcal{C})$ to $\mathcal{F}(A) \in \mathrm{Obj}(\mathcal{D})$, and each morphism $f : A \to B$ to $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$, for all $A, B \in \mathrm{Obj}(\mathcal{C})$. Furthermore, this map must satisfy the following two conditions:

(i) for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$ and all $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$ we have $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$, and

(ii) for all $A \in \mathrm{Obj}(\mathcal{C})$ we have $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$.

A **contravariant functor** $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ is defined similarly, except that $\mathcal{G}$ takes each morphism $f : A \to B$ to $\mathcal{G}(f) : \mathcal{G}(B) \to \mathcal{G}(A)$, and thus condition (i) is modified to $\mathcal{G}(g \circ f) = \mathcal{G}(f) \circ \mathcal{G}(g)$, where f and g are defined as above.

We illustrate the concept of functors with the following example.

Example 1.3.3. Let R be a ring. Given $A, B \in R$-Mod, let $\mathrm{Hom}(A, B)$ denote the group of R-homomorphisms from A to B, as usual. Furthermore, let Ab denote the category of abelian groups. Now let M be a fixed module in R-Mod. We define the map $\mathcal{F} : \text{Mod-}R \to \mathrm{Ab}$ by setting $\mathcal{F}(A) = \mathrm{Hom}(M, A)$ for all $A \in R$-Mod. Furthermore, for all $A, B \in R$-Mod and all $f \in \mathrm{Hom}(A, B)$, we define $\mathcal{F}(f) : \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B)$ by $\mathcal{F}(f)(h) = fh$, for all $h \in \mathrm{Hom}(M, A)$. Then $\mathcal{F}(f)$ is a function from $\mathcal{F}(A)$ to $\mathcal{F}(B)$.

We show that $\mathcal{F}$ is a covariant functor. For all $A, B, C \in R$-Mod, and all $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, we have $\mathcal{F}(gf)(h) = gfh = \mathcal{F}(g)(fh) = \mathcal{F}(g)(\mathcal{F}(f)(h))$ for all $h \in \mathrm{Hom}(M, A)$, and thus $\mathcal{F}(gf) = \mathcal{F}(g) \circ \mathcal{F}(f)$, satisfying condition (i). Furthermore, $\mathcal{F}(1_A)(h) = h$ for all $h \in \mathrm{Hom}(M, A)$, and so $\mathcal{F}(1_A)$ is the identity on the group $\mathrm{Hom}(M, A) = \mathcal{F}(A)$, satisfying condition (ii). Thus $\mathcal{F}$ is a covariant functor.

We now move on to a concept that allows us to say when two functors are 'equivalent' in some way.

Definition 1.3.4. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $\mathcal{F}$ and $\mathcal{G}$ be two covariant functors from $\mathcal{C}$ to $\mathcal{D}$. A **natural transformation** η from $\mathcal{F}$ to $\mathcal{G}$ (denoted $\eta : \mathcal{F} \to \mathcal{G}$) associates to each $A \in \mathrm{Obj}(\mathcal{C})$ a morphism $\eta_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$, such that for every $f \in \mathrm{Mor}(A, B)$ (where $B \in \mathrm{Obj}(\mathcal{C})$) we have $\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A$. In other words,

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}$$

is a commutative diagram in the category $\mathcal{D}$. (Note that if $\mathcal{F}$ and $\mathcal{G}$ are contravariant functors then the horizontal arrows in the above diagram are reversed, so that $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(f) : \mathcal{G}(B) \to \mathcal{G}(A)$.)

Furthermore, if η_A is an isomorphism for all $A \in \mathrm{Obj}(\mathcal{C})$, then η is said to be a **natural isomorphism** or **natural equivalence**. In this case, we say that $\mathcal{F}$ and $\mathcal{G}$ are **naturally isomorphic** and write $\mathcal{F} \cong \mathcal{G}$.

The concept of natural equivalence now allows us to define Morita equivalence.

Definition 1.3.5. Let R and S be rings. We say that the categories R-Mod and S-Mod are **equivalent** if there exist functors $\mathcal{F} : R\text{-Mod} \to S\text{-Mod}$ and $\mathcal{G} : S\text{-Mod} \to R\text{-Mod}$ such that

$$\begin{aligned}
\mathcal{G} \circ \mathcal{F} &\cong \textit{the identity functor on R-Mod, and} \\
\mathcal{F} \circ \mathcal{G} &\cong \textit{the identity functor on S-Mod.}
\end{aligned}$$

Furthermore, if R-Mod and S-Mod are equivalent then we say that R is **Morita equivalent** to S.

A ring-theoretic property $\mathcal{P}$ is said to be **Morita invariant** if, whenever a ring R has property $\mathcal{P}$, so too does every ring S that is Morita equivalent to R. It can be shown that a property $\mathcal{P}$ is Morita invariant if it can be characterised purely in terms of R-Mod, without referencing elements of the modules or elements of R itself.

We now define a concept that allows us to give an alternative definition of Morita equivalence.

Definition 1.3.6. Let R and S be two rings, let ${}_RN_S$ and ${}_SM_R$ be two bimodules and let $(-,-) : N \times M \to R$ and $[-,-] : M \times N \to S$ be two maps. Furthermore, suppose we have two maps $\phi : N \otimes_S M \to R$ and $\varphi : M \otimes_R N \to S$ given by

$$\phi(n \otimes m) = (n, m) \quad \text{and} \quad \varphi(m \otimes n) = [m, n]$$

for which the following associativity conditions hold:

$$\phi(n \otimes m)n' = n\,\varphi(m \otimes n') \quad \text{and} \quad \varphi(m \otimes n)m' = m\,\phi(n \otimes m')$$

for all $m, m' \in M$ and $n, n' \in N$.

A **Morita context** is a sextuple $(R, S, M, N, \phi, \varphi)$ satisfying the above conditions. Furthermore, we say that this Morita context is **surjective** if both ϕ and φ are surjective. Note that in this case we have $R = NM$ and $S = MN$.

A ring R is said to be **idempotent** if $R^2 := \{\sum_{i=1}^n r_i s_i : r_i, s_i \in R\} = R$. Note that if R has local units, then for any $r \in R$ there exists an idempotent $e \in R$ such that $r = er \in R^2$, and so any ring with local units is idempotent. This definition allows us to give an equivalent condition for Morita equivalence, as we see in the following theorem from García and Simón [GS, Proposition 2.3], which we state without proof.

Theorem 1.3.7. *Let R and S be two idempotent rings. Then R and S are Morita equivalent if and only if there exists a surjective Morita context $(R, S, N, M, \phi, \varphi)$.*

We owe the following results to Ánh and Márki, whose research examining Morita equivalence for non-unital rings is invaluable, as we will require many of these results to be valid for rings that do not necessarily have identity. The first proposition is from [AM, Proposition 3.3].

Proposition 1.3.8. *Let R and S be two Morita equivalent rings with local units. Then the lattice of ideals of R is isomorphic to the lattice of ideals of S; in particular, R is simple if and only if S is simple.*

This second proposition is from [AM, Proposition 3.5].

Proposition 1.3.9. *Let R be a ring with local units. If there exists an idempotent $e \in R$ for which $R = ReR$, then R is Morita equivalent to the subring eRe.*

We now establish some definitions and results that are useful in the context of Morita equivalence.

Definition 1.3.10. A left R-module P is said to be a **generator** for R-Mod if every left R-module M is the epimorphic image of $P^{(I)}$ for some index set I. Furthermore, we say that P is a **progenerator** if P is a projective generator.

For any ring R, if we view R as a left R-module then R is a generator for R-Mod. To see this, let M be a module in R-Mod. Since $M = RM$ by the unital property of R-Mod, for any $m \in M$ we have $m = \sum_{n\in M} r_n n$ for some $r_n \in R$ (where only a finite number of r_n are nonzero). Let $I = M$ and define $\phi : R^{(I)} \to M$ by $\phi((r_m)_{m\in I}) = \sum_{m\in I} r_m m$. Then ϕ is an epimorphism and so R is a generator for R-Mod.

Definition 1.3.11. Let R be a ring and let P be a right R-module. The **trace** of P, denoted $\mathrm{tr}(P)$, is defined by

$$\mathrm{tr}(P) = \sum \{x \in R : x = g(p) \text{ for some } p \in R \text{ and some } g \in \mathrm{Hom}_R(P, R)\}.$$

It can be shown that $\mathrm{tr}(P)$ is a two-sided ideal of R (see, for example, [L2, Proposition 2.40]).

We now look at two results that allow us to determine when a right R-module P is a generator for Mod-R. The following result has been established for unital rings (see for example [L2, Theorem 18.8]). Here we extend it to rings with local units by adapting part of the proof of [AA2, Proposition 10], (i) $\Longleftrightarrow$ (ii).

Proposition 1.3.12. *Let R be a ring with local units and let P be a right R-module. If $\operatorname{tr}(P) = R$ then P is a generator for Mod-R.*

Proof. Let $E = \{e_i : i \in I\}$ be a set of local units for R. If $\operatorname{tr}(P) = R$, then for each $e_i \in E$ we can write $e_i = \sum_{t=i_1}^{i_{s(i)}} g_t(p_t)$ for some $p_t \in P$ and some $g_t \in \operatorname{Hom}_R(P, R)$. If we define $\lambda_{e_i} : R \to e_i R$ by $\lambda_{e_i}(r) = e_i r$, then letting $J_i = \{i_1, \dots, i_{s(i)}\}$ we have that $\lambda_{e_i} \circ \bigoplus_{t \in J_i} g_t : P^{(J_i)} \to R \to e_i R$ is an epimorphism. To see this, take an arbitrary $e_i r \in e_i R$. Then

$$e_i r = \lambda_{e_i}(e_i r) = (\lambda_{e_i}(e_i))r = \big(\lambda_{e_i}\big(\textstyle\sum_{t \in J_i} g_t(p_t)\big)\big)r = \big(\lambda_{e_i} \circ \bigoplus_{t \in J_i} g_t\big)(p_t r)_{t \in J_i}\,,$$

and so $\lambda_{e_i} \circ \bigoplus_{t \in J_i} g_t$ is indeed an epimorphism. Let J be the disjoint union of the sets J_i and define $\varphi : P^{(J)} \to R$ by $\varphi_{|P^{(J_i)}} = \lambda_{e_i} \circ \bigoplus_{t \in J_i} g_t$. Since any element $r \in R$ is contained in $e_i R$ for some local unit e_i, we have that φ is also an epimorphism.

Now take an arbitrary right R-module M. Since R is a generator for Mod-R, M is the epimorphic image of $R^{(\Lambda)}$ for some index set Λ. Thus M is the epimorphic image of $(P^{(J)})^{(\Lambda)}$ and so P is a generator for Mod-R. □

Proposition 1.3.12 leads to the following lemma, which has also been adapted from the proof of [AA2, Proposition 10], (i) $\Longleftrightarrow$ (ii).

Lemma 1.3.13. *Let R be a ring with local units and let P be a nonzero, finitely generated projective right R-module. If R is simple then P is a generator for Mod-R.*

Proof. Let P be a nonzero, finitely generated projective right R-module. Since P is finitely generated we can write $P = \sum_{i=1}^{n} x_i R$, where each $x_i \in P$. Define a homomorphism $\phi : R^n \to P$ by $\phi((a_1, \dots, a_n)) = \sum_{i=1}^{n} x_i a_i$. Since ϕ is an epimorphism and P is projective, there must exist $P' \in$ Mod-R for which $R^n \cong P \oplus P'$ (by Lemma 1.2.10).

Thus P is isomorphic to a direct summand of R^n (since, if $\theta : P \oplus P' \to R^n$ is an isomorphism, then $R^n = \theta(P) \oplus \theta(P')$) and so $\operatorname{Hom}_R(P, R^n) \neq 0$. However, since $\operatorname{Hom}_R(P, R^n) \cong (\operatorname{Hom}_R(P, R))^n$ (by the right R-module analogue of Proposition 1.2.4), we have $(\operatorname{Hom}_R(P, R))^n \neq 0$ and so $\operatorname{Hom}_R(P, R) \neq 0$. Thus $\operatorname{tr}(P)$ is nonzero and so, since $\operatorname{tr}(P)$ is a two-sided ideal of R and R is simple, we have $\operatorname{tr}(P) = R$. Thus, by Proposition 1.3.12, P is a generator for Mod-R. □

The significance of generators in the context of Morita equivalence is illustrated in the following two results, which we will state without proof. The first proposition is from Ánh and Márki [AM, Theorem 2.5] and generalises a well-known result for unital rings (see, for example, [L2, Proposition 18.33]) to the more general case of rings with local units. First, we need to define the concept of a locally projective module.

Definition 1.3.14. A module $P \in \text{Mod-}R$ is said to be **locally projective** if there exists a direct system $(P_i)_I$ of finitely generated projective summands of P for which $P = \varinjlim_{i \in I} P_i$. (See Appendix A for more information on direct limits.) Note that if P is a finitely generated projective module, then P is locally projective (taking $(P_i)_I$ to be simply P).

Proposition 1.3.15. *Let R and S be two rings with local units. Then R is Morita equivalent to S if and only if there is a locally projective generator $P_R = \varinjlim_{i \in I} P_i$ in Mod-R for which $S \cong \varinjlim_{i \in I} \text{End}(P_i)$.*

In the case that P_R is a progenerator in Mod-R, then Proposition 1.3.15 simplifies to 'R is Morita equivalent to S if and only if $S \cong \text{End}(P_R)$'. In particular, we have that R is Morita equivalent to $\text{End}(P_R)$.

This second proposition is from Lam [L2, Proposition 18.44]. While the result is given in a unital context, the proof is valid for any ring. Here we state it without proof.

Proposition 1.3.16. *Let R be a ring and let P_R be a progenerator in Mod-R. Then the lattice of right ideals in $\text{End}(P_R)$ is isomorphic to the lattice of submodules of P_R.*

Note that if R and S are two Morita equivalent rings and P_R and P_S are progenerators for R and S, respectively, then combining Propositions 1.3.8, 1.3.15 and 1.3.16 (and viewing R, S, $\text{End}(P_R)$ and $\text{End}(P_S)$ as right modules over themselves) we have that the lattices of submodules of R, S, $\text{End}(P_R)$, $\text{End}(P_S)$, P_R and P_S are all isomorphic.

We now come to the main result of this section, the proof of which has been expanded from [AA2, Proposition 10], (i) $\iff$ (ii).

Theorem 1.3.17. *Let R and S be two Morita equivalent rings with local units. Then R is purely infinite simple if and only if S is purely infinite simple; that is, the property 'purely infinite simple' is Morita invariant.*

Proof. Suppose that R is purely infinite simple and let P be a nonzero, finitely-generated projective right R-module. We know that P is a generator for Mod-R by Lemma 1.3.13. Since R is purely infinite, it must contain an infinite idempotent e such that the right ideal eR is directly infinite, so that there exists a nonzero submodule B of R such that

$$eR \cong B \oplus eR \cong \cdots \cong B^m \oplus eR$$

for all $m \in \mathbb{N}$. Now, since B is a direct summand of eR and eR is a projective right R-module (by the right R-module analogue of Proposition 1.2.13), B is also a projective right R-module. Furthermore, B is unital and nondegenerate since eR is unital and nondegenerate.

Since $B \subseteq eR \subseteq R$, we have $\mathrm{Hom}_R(B, R) \neq 0$ (since it contains the inclusion map from B to R) and so $\mathrm{tr}(B) \neq 0$. Thus, since R is simple, $\mathrm{tr}(B) = R$ and so B is a generator for Mod-R (by Proposition 1.3.12). Since P is finitely generated, there is an $n \in \mathbb{N}$ for which there exists an epimorphism $\alpha : B^n \to P$. Therefore, since P is projective, we have $B^n \cong P \oplus C$ for some submodule C of B^n (by Lemma 1.2.10). Thus, setting $Q = C \oplus eR$, we have

$$eR \cong B^n \oplus eR \cong P \oplus C \oplus eR = P \oplus Q.$$

Let $\eta : eR \to P \oplus Q$ be the above isomorphism. Let D be a nonzero submodule of P, so that $D' = \eta^{-1}(D)$ is a nonzero submodule – and therefore a nonzero right ideal – of eR. Since R is purely infinite, D' contains an infinite idempotent f. Thus $T' = fR$ is a directly infinite submodule of eR and so, since $f \in eR$, fR is a direct summand of eR (by Lemma 1.2.3 (iii)). Thus, letting $T = \eta(T')$, we have $T \subseteq D \subseteq P$. Futhermore, since $T' = fR$ is a direct summand of eR, T must be a direct summand of $P \oplus Q$ and therefore of P. Thus every submodule of P contains a nonzero direct summand of P that is directly infinite.

Conversely, suppose that for every nonzero, finitely generated projective right R-module P we have that every submodule of P contains a nonzero direct summand of P that is directly infinite. We show that R must be purely infinite. Let I be a nonzero right ideal of R and let $0 \neq x \in I$, so that $xR \subseteq I$. Since R has local units, $x = ex$ for some idempotent $e \in R$, and thus xR is a right ideal of eR. Now eR is a nonzero, finitely generated, projective (by Proposition 1.2.13) right R-module, and so xR contains a nonzero direct summand T of eR that is directly infinite. By Lemma 1.2.3 (ii) we have that $T = fR$, where f is an idempotent. Thus f is an infinite idempotent, and $f = f^2 \in fR \subseteq xR \subseteq I$. We can conclude that every nonzero right ideal of R contains an infinite idempotent and so R is purely infinite.

We already know that simplicity is a Morita invariant property (by Proposition 1.3.8). Furthermore, suppose that P_S is nonzero, finitely-generated projective right S-module. Then P_S is a generator for Mod-S (by Lemma 1.3.13) and so the lattice of submodules of P must be isomorphic to the lattice of submodules of P_S by our observation on page 28. Thus every submodule of P_S contains a nonzero direct summand of P_S that is directly infinite and so, as shown in the previous paragraph, S is purely infinite. Thus 'purely infinite' is a Morita invariant property between rings with local units, completing the proof. $\square$

Theorem 1.3.17 leads to the following useful result.

Proposition 1.3.18. *Let R be a ring with local units. Then R is purely infinite simple if and only if the subring eRe is purely infinite simple for every nonzero idempotent $e \in R$.*

Proof. Suppose that R is purely infinite simple. Then, for every nonzero idempotent $e \in R$ we have $R = ReR$ (by the simplicity of R) and so, by Proposition 1.3.9, R is Morita equivalent to eRe. Thus, since the property 'purely infinite simple' is a Morita invariant of R (by Theorem 1.3.17), eRe must be purely infinite simple. Conversely, if eRe is purely infinite simple for every nonzero idempotent $e \in R$ then

R is simple, by Proposition 1.1.2. Thus, Proposition 1.3.9 again gives that R is Morita equivalent to each nonzero ring eRe and so R is purely infinite simple. $\square$

We are now finally in a position to adapt Theorem 1.2.5 to the more general case in which R has local units. Note the subtle difference in condition (ii).

Theorem 1.3.19. *Let R be a simple ring with local units. Then R is purely infinite if and only if the following conditions are satisfied:*

(i) R is not a division ring, and

(ii) for every pair of nonzero elements $x, y \in R$, there exist elements $s, t \in R$ such that $sxt = y$.

Proof. Suppose R is purely infinite. Then R contains an idempotent e such that $eR = A \oplus B$, where $A \cong eR$ and B is nonzero. In particular, R contains a nonzero proper right ideal, and so R cannot be a division ring. Now choose a pair of nonzero elements $x, y \in R$. Since R has local units, there exists an idempotent e such that $x, y \in eRe$. By Proposition 1.3.18, we know that eRe must be purely infinite simple. Since e is the identity for eRe, by Theorem 1.2.5 there exist elements $s', t' \in eRe$ such that $s'xt' = e$. Thus, taking $s = s'$ and $t = t'y$, we have $sxt = y$ for $s, t \in R$, proving condition (ii).

Now suppose that conditions (i) and (ii) hold. Let I be a nonzero ideal of R and let x be a nonzero element of I. Then, by condition (ii), for any $y \in R$ there exist $a, b \in R$ such that $y = axb \in I$, and so R must be simple. Now let f be a nonzero idempotent of R (such an element must exist since R has local units). Since R is simple we have $RfR = R$, and so R is Morita equivalent to the subring fRf by Proposition 1.3.9. Thus the lattice of ideals of R is isomorphic to the lattice of ideals of fRf (by Proposition 1.3.8) and so it follows from condition (i) that fRf is not a division ring.

Now take an element $x \in fRf$. Applying condition (ii), we can find $s', t' \in R$ such that $s'xt' = f$. Let $s = fs'f$ and $t = ft'f$. Then, noting that $x = fxf$, we have $sxt = (fs'f)x(ft'f) = fs'(fxf)t'f = f(s'xt')f = f(f)f = f$. Since f is the identity for fRf and $s, t \in fRf$, Theorem 1.2.5 tells us that fRf is purely infinite. Furthermore, since R is simple, $R = RfR$ and so R and fRf are Morita equivalent (by Proposition 1.3.9). Thus Theorem 1.3.17 gives that R is purely infinite, completing the proof. □

The equivalence given in Theorem 1.3.19 will prove useful when we come to determine precisely which Leavitt path algebras are purely infinite simple in Section 2.3.

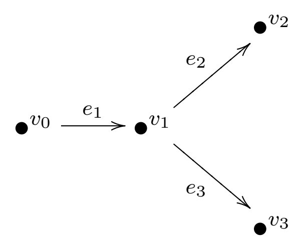
1.4 Graph Theory

As we will see, Leavitt path algebras are K-algebras that are generated, in a way, by directed graphs. In this section we will define a directed graph and introduce several important graph-theoretic concepts that will be useful when examining Leavitt path algebras.

Definition 1.4.1. A **directed graph** $E = (E^0, E^1, r, s)$ consists of two sets, E^0 and E^1, and two maps $r, s : E^1 \to E^0$. The elements of E^0 are called **vertices** and the elements of E^1 **edges**. For any edge e in E^1, $s(e)$ is the **source** of e and $r(e)$ is the **range** of e. If $s(e) = v$ and $r(e) = w$, then we say that v **emits** e and w **receives** e. Informally, we can think of e as having direction from v to w. If $r(e_1) = s(e_2)$ for some edges $e_1, e_2 \in E^1$, we say that e_1 and e_2 are **adjacent**.

Since we will be dealing exclusively with directed graphs in this thesis, we will henceforth refer to them as simply 'graphs'.

Example 1.4.2. Consider the graph E, where $E^0 = \{v_0, v_1, v_2, v_3\}$, $E^1 = \{e_1, e_2, e_3\}$ and $s(e_1) = v_0$, $r(e_1) = s(e_2) = s(e_3) = v_1$, $r(e_2) = v_2$ and $r(e_3) = v_3$. We can illustrate this with the following diagram:



From Definition 1.4.1 it follows that, for any vertex v in E^0, $s^{-1}(v)$ is the set of all edges emitted by v, while $r^{-1}(v)$ is the set of all edges received by v. If v does not emit any edges, so that $s^{-1}(v) = \emptyset$, then v is called a **sink**. If v does not receive any edges, it is called a **source**. Referring to the graph E in Example 1.4.2, we can see that v_0 is a source, while v_2, and v_3 are sinks.

If v is a vertex such that $|s^{-1}(v)| = \infty$ then v is called an **infinite emitter**. If v is either a sink or an infinite emitter, it is called a **singular vertex**. Otherwise, v is called a **regular vertex**. In other words, a vertex v is regular precisely when $0 < |s^{-1}(v)| < \infty$.

A graph E is said to be **finite** if E^0 is a finite set. If E contains no infinite emitters, then E is said to be **row-finite**. Furthermore, if all infinite emitters in a graph E emit a countably infinite number of edges then we say that E is **countable**. Note that a graph can be finite but not row-finite; for example, consider the graph

$$\bullet^u \xrightarrow{(\infty)} \bullet^v$$

where (∞) denotes an infinite number of edges from u to v (so that u is an infinite emitter). Many texts assume that a given graph E is row-finite, or even that E contains no singular vertices at all. However, in this thesis we will not be making any such assumptions unless stated otherwise.

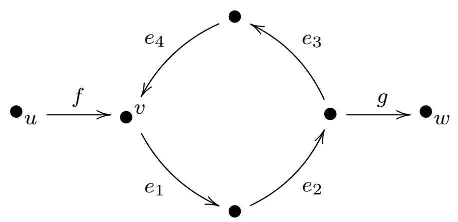
A **path** p in a graph E is a sequence of edges $e_1 e_2 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for all $i = 1, 2, \dots, n-1$. A path consisting of n edges is said to have **length** n, and we write $l(p) = n$. If a path p contains an infinite number of edges then we say that p has **infinite length**. The **source of** p, denoted $s(p)$, is the source of its initial edge, $s(e_1)$, while (if p has finite length) the **range of** p, denoted $r(p)$, is the range of its final edge, $r(e_n)$. It is also convenient to think of every vertex $v \in E^0$ as being a path of length 0, with $s(v) = v = r(v)$.

We denote the set of all paths in E by E^*. For a given path $p = e_1 \dots e_n \in E^*$, we define p^0 to be the set of all vertices in p; that is, $p^0 = \{s(e_1), r(e_i) : i = 1, 2, \dots\}$. Furthermore, if $q = e_1 \dots e_m$ for some $m \leq n$ then we say that q is an **initial subpath** of p.

A path p is said to be a **cycle** if $s(p) = r(p)$ and $s(e_i) \neq s(e_j)$ for all $i \neq j$. In other words, a cycle is a path that begins and ends on the same vertex and does not pass through any vertex more than once. If c is a cycle with $s(c) = r(c) = v$, then we say that c **is based at** v. If a graph E does not contain any cycles, it is said to be **acyclic**.

An edge $e \in E^1$ is said to be an **exit** to the path $p = e_1 \dots e_n$ if there exists an $i \in \{1, \dots, n\}$ such that $s(e) = s(e_i)$ but $e \neq e_i$. Note that an exit to a path p does not have to be external to the path itself. For example, if p contains two distinct edges e_i, e_j such that $s(e_i) = s(e_j)$, then both e_i and e_j are exits for p.

Example 1.4.3. Consider the following graph E:



Then $p = fe_1e_2g$ is a path in E^* with $s(p) = u$, $r(p) = w$ and $l(p) = 4$. If we let $q = fe_1$, then q is an initial subpath of p. Furthermore, $c = e_1e_2e_3e_4$ is a cycle in E based at v, and g is an exit for c.

Definition 1.4.4. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $p \in E^*$ such that $s(p) = v$ and $r(p) = w$. (Note that, because we consider a single vertex to be a path of length 0, it is possible that $v = w$.) In this case, we say that v **connects to** w.

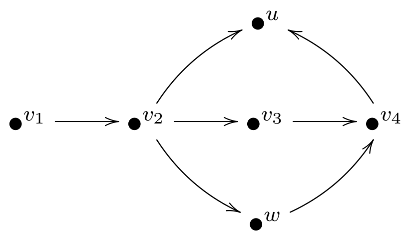
For a vertex $v \in E^0$, we define the **tree of** v, denoted $T(v)$, to be the set of all vertices in E^0 to which v connects; that is

$$T(v) = \{w \in E^0 : v \geq w\}.$$

Note that we always have $v \in T(v)$ since all vertices connect to themselves, by definition. We can extend the definition of a tree to an arbitrary subset X of E^0 by defining $T(X) = \bigcup_{v \in X} T(v)$. Since $v \in T(v)$ for each $v \in X$ we therefore have $X \subseteq T(X)$.

A vertex $v \in E^0$ is said to be a **bifurcation** (or **there is a bifurcation at** v) if v emits more than one edge; that is, $|s^{-1}(v)| > 1$. Furthermore, a vertex $u \in E^0$ is said to be a **line point** if there are no bifurcations or cycles based at any vertex $w \in T(u)$. Note that, by definition, any sink is a line point. We denote the set of all line points in E^0 by $P_l(E)$.

Example 1.4.5. Consider the following graph E:



Then, for example, we have $T(v_1) = E^0$, since there is a path from v_1 to every vertex in E, while $T(w) = \{w, v_4, u\}$. The only bifurcation in E is v_2. Furthermore, the line points in E are u, w, v_3 and v_4; that is, $P_l(E) = \{u, w, v_3, v_4\}$.

Definition 1.4.6. We denote by E^∞ the set of all paths of infinite length in E, and we denote by $E^{\leq\infty}$ the set E^∞ together with the set of all finite paths in E whose end vertex is a sink. A vertex v is **cofinal** if, for every path $p \in E^{\leq\infty}$, there exists a vertex w in p such that $v \geq w$. Furthermore, we say that a graph E is cofinal if all of its vertices are cofinal.

Now we define two concepts that will feature heavily in our study of Leavitt path algebras.

Definition 1.4.7. Let H be a subset of E^0. We say that

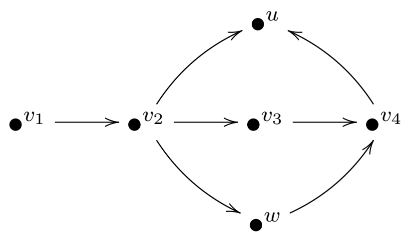
(i) H is **hereditary** if $v \in H$ implies $T(v) \subseteq H$, and

(ii) H is **saturated** if $\{r(e) : s(e) = v\} \subseteq H$ implies that $v \in H$, for every regular vertex $v \in E^0$.

In other words, a subset H is hereditary if, for each $v \in H$, every vertex that v connects to is also in H. Furthermore, a subset H is saturated if every regular vertex that feeds into H, and *only* into H, is also in H. In the study of Leavitt path algebras we will be particularly interested in subsets of E^0 that are both hereditary and saturated, which we call simply 'hereditary saturated subsets' of E^0. Note that if a vertex v is a line point then any vertex $w \in T(v)$ must be a line point, since $T(w) \subseteq T(v)$. Thus $P_l(E)$ is a hereditary subset of E^0 – however, it is not necessarily saturated.

Let X be an arbitrary subset of E^0. The **hereditary saturated closure of** X, denoted $\bar{X}$, is the smallest hereditary saturated subset containing X; that is, for any hereditary saturated subset H containing X, we have $\bar{X} \subseteq H$.

Example 1.4.8. Consider again the graph E from Example 1.4.5:



We can see that $S = \{v_1, v_2\}$ forms a saturated (but not hereditary) subset of E^0. Furthermore, $H = \{u, w, v_3, v_4\}$ forms a hereditary subset of E^0. Indeed, this is the set of line points of E, which is always hereditary, as noted above. However, H is not saturated, since $\{r(e) : s(e) = v_2\} = \{u, w, v_3\} \subset H$ but $v_2 \notin H$. It is easy to see that the hereditary saturated closure of H must contain v_2, and therefore must also contain v_1. Thus $\bar{H} = E^0$.

Since u is the only sink in E, and E is finite, $E^{\leq\infty}$ is the set of all paths in E^* that end in u. Since every vertex in E^0 connects to u, every vertex is cofinal and thus E is cofinal.

Lemma 1.4.9. *Let E be a graph and let X be a subset of E^0. Then the hereditary saturated closure of X is the set $\bigcup_{n=0}^{\infty} G_n(X)$, where*

$$G_0(X) = T(X), \text{ while for } n \geq 1,$$

$$G_n(X) = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \subseteq G_{n-1}(X)\} \cup G_{n-1}(X).$$

Proof. First, note that $G_m(X) \subseteq G_n(X)$ for each $m \leq n$. For ease of notation, we set $G(X) = \bigcup_{n=0}^{\infty} G_n(X)$. Now $X \subseteq T(X) = G_0(X) \subseteq G(X)$, and so $G(X)$ contains X. To show that $G(X)$ is hereditary, suppose that $v \in G(X)$ and let $w \in T(v)$. Furthermore, let $p = e_1 \dots e_l$ be a path with $s(p) = v$ and $r(p) = w$, and let n be the minimum integer for which $v \in G_n(X)$. If $n = 0$, then $v \in T(X)$ and so $w \in T(X) \subseteq G(X)$ and we are done. If $n \neq 0$, then by definition we have that $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq G_{n-1}(X)$. In particular, we have $r(e_1) \in G_{n-1}(X)$. Now let m be the minimum integer for which $r(e_1) \in G_m(X)$ (noting that $m \leq n-1$). If $m = 0$, then again $w \in T(X)$ and we are done; otherwise we have $r(e_2) \in G_{m-1}(X)$ by the same logic as above. Thus repeating this argument either yields that $w \in T(X)$ or $r(e_l) = w \in G_p(X)$ for some $p < n$. In either case, $w \in G(X)$ and so $G(X)$ is hereditary.

To show that $G(X)$ is saturated, suppose we have a regular vertex $v \in E^0$ such that $r(s^{-1}(v)) \subseteq G(X)$. Let n be the minimum integer for which $r(s^{-1}(v)) \subseteq G_n(X)$. Then by definition we have $v \in G_{n+1}(X) \subseteq G(X)$ and so $G(X)$ is saturated.

Finally, suppose that H is any hereditary saturated subset containing X. Since H is hereditary, it must contain $T(X)$, so that $T(X) = G_0(X) \subseteq H$. Furthermore, since H is saturated, H must contain the set $S_1 = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \subseteq T(X)\}$, and so $G_1(X) = S_1 \cup T(X) \subseteq H$. Continuing this argument, we see that $G_n(X) \subseteq H$ for each non-negative integer n, and so $G(X) = \bigcup_{n=0}^{\infty} G_n(X) \subseteq H$. Therefore $G(X)$ is the hereditary saturated closure of X, as required. $\square$

We end with a proposition that incorporates several of the concepts we have introduced in this section. This result generalises [APS, Lemma 2.8] from the row-finite case to an arbitrary graph E.

Proposition 1.4.10. *Let E be an arbitrary graph. The only hereditary saturated subsets of E^0 are $\emptyset$ and E^0 if and only if the following conditions are satisfied:*

(i) E is cofinal, and

(ii) for every singular vertex $u \in E^0$, we have $v \geq u$ for all $v \in E^0$.

Proof. Suppose that the only hereditary saturated subsets of E^0 are $\emptyset$ and E^0. Let $v \in E^0$ and $p \in E^{\leq\infty}$. To show that E is cofinal, it suffices to show that we can find a vertex $w \in p^0$ for which $w \in T(v)$. Let $X = \{v\}$. Then $\bar{X} \neq \emptyset$, and so $\bar{X} = E^0 = \bigcup_{n=0}^{\infty} G_n(X)$ (where each $G_n(X)$ is as defined in Lemma 1.4.9). Let $m \in \mathbb{N}$ be the minimum integer for which $G_m(v) \cap p^0 \neq \emptyset$ and let $w \in G_m(v) \cap p^0$. If $m > 0$, then the minimality of m implies that $w \notin G_{m-1}(v)$, and so w is a regular vertex and $r(s^{-1}(w)) \subseteq G_{m-1}(v)$. However, since $w \in p^0$ and w is not a sink, there must be some edge e in p for which $s(e) = w$. Thus $r(e) \in r(s^{-1}(w))$ and so $r(e) \in G_{m-1}(v) \cap p^0$, contradicting the minimality of m. Therefore $m = 0$ and so $w \in G_0(v) = T(v)$, as required. Now take a singular vertex $u \in E^0$ and let $v \in E^0$. Again, by our hypothesis there must exist a minimum integer $m \in \mathbb{N}$ for which $u \in G_m(v)$. Suppose that $m > 0$. Since u is singular, we must have $u \in G_{m-1}(v)$ (since only regular vertices are added with each iteration), a contradiction. Thus $m = 0$, and so $u \in T(v)$. Thus $v \geq u$, as required.

Now suppose that conditions (i) and (ii) hold and that there exists a hereditary saturated subset H of E^0 such that $\emptyset \subset H \subset E^0$. Choose a vertex v such that $v \in E^0 \backslash H$. Now v cannot be a singular vertex, because condition (ii) would imply that $w \geq v$ for any $w \in H$, and therefore that $v \in H$ by the hereditary nature of H. In particular, v is not a sink and so $s^{-1}(v) \neq \emptyset$. Furthermore, $r(s^{-1}(v)) \nsubseteq H$, for otherwise we would have $v \in H$ by the saturated property of H. Thus there exists an edge $e_1 \in s^{-1}(v)$ for which $r(e_1) \notin H$. Again, $r(e_1)$ cannot be a singular vertex, so we can repeat the above procedure to find an edge e_2 for which $s(e_2) = r(e_1)$ and $r(e_2) \notin H$, and so on. Thus we can form an infinite path $p = e_1 e_2 \dots$ with $p^0 \cap H = \emptyset$. We know that p is infinite since each vertex in p^0 is not in H and therefore cannot be a sink (by the argument used above). Thus $p \in E^\infty$. However, since E is cofinal, for any $w \in H$ there exists a vertex $u \in p^0$ such that $w \geq u$, and so $u \in H$, a contradiction. Thus the only hereditary saturated subsets of E^0 are $\emptyset$ and E^0, as required. $\square$

Chapter 2

Leavitt Path Algebras

2.1 Introduction to Leavitt Path Algebras

In this section we will define the central concept of this thesis, that of the Leavitt path algebra. This concept ties together many aspects of graph theory and ring theory, as we essentially construct a K-algebra from a given graph E by using its edges and vertices as generating elements, along with a new set of edges known as 'ghost edges'. As we shall see, there are many (often surprising) analogues between graph-theoretic properties of E and ring-theoretic properties of the associated Leavitt path algebra, $L_K(E)$. Furthermore, many well-known algebras, such as the matrix rings $\mathbb{M}_n(K)$ and the Leavitt algebras $L(1, n)$, are isomorphic to the Leavitt path algebra of some graph E. Thus a graph can often provide a simple visual representation of some of the more abstract properties of a particular ring.

We begin by defining the slightly simpler notion of a path algebra.

Definition 2.1.1. Let K be a field and E be an arbitrary graph. The **path K-algebra over** E, denoted $A(E)$, is defined to be the K-algebra generated by the sets E^0 and E^1, i.e. $K[E^0 \cup E^1]$, subject to the following relations:

(A1) $v_i v_j = \delta_{ij}\, v_i$ for all $v_i, v_j \in E^0$; and

(A2) $s(e)e = e = er(e)$ for all $e \in E^1$.

As we will see, the relations (A1) and (A2) defined on $A(E)$ essentially preserve the path structure of the associated graph E, hence the name 'path algebra'. In order to extend this concept to a Leavitt path algebra, we need to introduce the following concept.

Definition 2.1.2. For an arbitrary graph E, the **extended graph of** E is the graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined by

$$r'(e^*) = s(e), \quad s'(e^*) = r(e) \quad \text{and} \quad r'(e) = r(e), \quad s'(e) = s(e)$$

for all $e \in E^1$. For ease of notation, we usually denote the functions r' and s' as simply r and s.

Essentially, the extended graph introduces a new set of edges $(E^1)^*$, which is a copy of E^1 but with the direction of each edge reversed; that is, if $e \in E^1$ runs from u to v, then $e^* \in (E^1)^*$ runs from v to u. To distinguish between the two sets of edges, we refer to E^1 as the set of **real edges** and $(E^1)^*$ as the set of **ghost edges**. A path made up of only real edges is called a **real path**, while a path made up of only ghost edges is called a **ghost path**. For a real path $p = e_1 \dots e_n$, we denote the ghost path $e_n^* \dots e_1^*$ by p^*. When we refer to a 'path' in a graph E it is assumed that we are talking about a real path, unless stated otherwise. In particular, the notation E^* continues to refer to the set of real paths in E.

We are now able to define a Leavitt path algebra.

Definition 2.1.3. Let K be a field and let E be an arbitrary graph. The **Leavitt path algebra of** E **with coefficients in** K, denoted $L_K(E)$, is defined to be the K-algebra generated by the sets E^0, E^1 and $(E^1)^*$, i.e. $K[E^0 \cup E^1 \cup (E^1)^*]$, subject to the following relations:

(A1) $v_i v_j = \delta_{ij}\, v_i$ for all $v_i, v_j \in E^0$;

(A2) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$;

(CK1) $e_i^* e_j = \delta_{ij}\, r(e_j)$ for all $e_i, e_j \in E^1$; and

(CK2) $v = \sum_{\{e \in E^1 : s(e)=v\}} ee^*$ for every regular vertex $v \in E^0$.

In other words, the Leavitt path algebra of a graph E is the path K-algebra over the extended graph $\widehat{E}$, subject to the relations (CK1) and (CK2), which are known as the **Cuntz-Krieger relations**. Note that, by the (A1) relation, each $v \in E^0$ is an idempotent in $L_K(E)$ and the elements of E^0 are mutually orthogonal in $L_K(E)$. Thus the vertices of E form a set of orthogonal idempotents in $L_K(E)$.

We now give several examples of Leavitt path algebras. From this point we will always use K to denote an arbitrary field.

Example 2.1.4. The simplest possible example is the graph I_1 consisting of a single vertex v and no edges:

$$\bullet^v$$

In this case we have simply $L_K(I_1) = Kv$, which is isomorphic to the ring K. Similarly, if we add an extra vertex w to obtain the graph $I_1 \times I_1$:

$$\bullet^v \qquad \bullet^w$$

then we have $L_K(I_1 \times I_1) = Kv \oplus Kw \cong K^2$. (Note that $Kv + Kw = Kv \oplus Kw$ since v and w are mutually orthogonal.)

Things get more interesting if we add an edge e between v and w to form the graph M_2:

$$\bullet^v \xrightarrow{e} \bullet^w$$

In this case $L_K(M_2)$ is generated by the elements v, w, e, e^*, subject to the four Leavitt path algebra relations. We show that $L_K(M_2) \cong \mathbb{M}_2(K)$ by defining the map $\phi : L_K(M_2) \to \mathbb{M}_2(K)$ on the generators of $L_K(M_2)$ as follows:

$$\phi(v) = E_{11}, \quad \phi(w) = E_{22}, \quad \phi(e) = E_{12} \quad \text{and} \quad \phi(e^*) = E_{21},$$

where E_{ij} is the matrix unit with 1 in the (i, j) position and zeros elsewhere. We extend ϕ linearly and multiplicatively. Since any element in $\mathbb{M}_2(K)$ is a K-linear combination of the four matrix units listed above, ϕ is clearly an epimorphism. Furthermore, it is easy to see that ϕ is a monomorphism since these matrix units are linearly independent. However, we also need to check that ϕ is well-defined: specifically, that ϕ preserves the Leavitt path algebra relations on $L_K(M_2)$. This is often the most important step when defining a homomophism from a Leavitt path algebra to another ring, and often the most time-consuming. In this case, checking that ϕ preserves the relations is fairly straightforward since there are only a small number of generating elements; in larger graphs, this process can become quite messy and drawn-out.

Using the general matrix unit property that $E_{ij}E_{kl} = \delta_{jk}\, E_{il}$, it is easy to see that ϕ preserves the (A1), (A2) and (CK1) relations. For example, to check that the equality $e = er(e)$ is preserved by ϕ we must check that $\phi(e) = \phi(er(e))$ for all $e \in E^1$, which in this case reduces to showing

$$\phi(ew) = E_{12}E_{22} = E_{12} = \phi(e),$$

as required. To check that the (CK2) relation is preserved, recall that the relation is only defined at regular vertices. Thus we only need to check that the equality $v = ee^*$ is preserved by ϕ, which is easily seen since

$$\phi(v) = E_{11} = E_{12}E_{21} = \phi(e)\phi(e^*) = \phi(ee^*).$$

Thus ϕ is an isomorphism and so $L_K(M_2) \cong \mathbb{M}_2(K)$, as claimed.[1]

Example 2.1.5. We can generalise the above example by defining the **finite line graph with n vertices**, denoted M_n, to be the graph

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

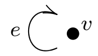
We show that $L_K(M_n) \cong \mathbb{M}_n(K)$. Similar to the case $n = 2$, we define ϕ on the generators of $L_K(M_n)$ by

$$\phi(v_i) = E_{ii}, \quad \phi(e_i) = E_{i(i+1)}, \quad \text{and} \quad \phi(e_i^*) = E_{(i+1)i}$$

[1]In this example, we can also show that ϕ is an isomorphism by showing that the generators v, w, e, e^* form a basis for $L_K(M_2)$, in which case there is no need to check that ϕ preserves the Leavitt path algebra relations on $L_K(M_2)$. However, we have chosen to use the latter method in order to emphasise the importance of this step.

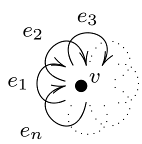
for all $i = 1, \dots n$. As in Example 2.1.4, it is straightforward to show that the Leavitt path algebra relations are preserved by ϕ. Furthermore, for any matrix unit $E_{ij} \in \mathbb{M}_n(K)$ with $i < j$ we have $E_{ij} = E_{i(i+1)}E_{(i+1)(i+2)} \dots E_{(j-1)j} = \phi(e_i e_{i+1} \dots e_{j-1})$. Similarly, if $i > j$ then $E_{ij} = \phi(e^*_{i-1} \dots e^*_{j+1} e^*_j)$. Since any element in $\mathbb{M}_n(K)$ is a K-linear combination of such matrix units, ϕ is an epimorphism. Again, it is easy to see that ϕ is a monomorphism, and so $L_K(M_n) \cong \mathbb{M}_n(K)$, as claimed.

Example 2.1.6. We define the **single loop graph**, denoted R_1, to be the graph



Consider $K[x, x^{-1}]$, the ring of Laurent polynomials with coefficients in K. By defining the map $\phi : L_K(R_1) \to K[x, x^{-1}]$ by $\phi(v) = 1$, $\phi(e) = x$ and $\phi(e^*) = x^{-1}$, it is straightforward to see that ϕ preserves the Leavitt path algebra relations and that $L_K(R_1) \cong K[x, x^{-1}]$.

We can extend the single loop graph to the **rose with n leaves graph**, denoted R_n:



For each $n \in \mathbb{N}$, we have that $L_K(R_n)$ is isomorphic to the Leavitt algebra $L(1, n)$, which is the unital K-algebra generated by elements $\{x_i, y_i : i = 1, \dots, n\}$ and subject to the following relations:

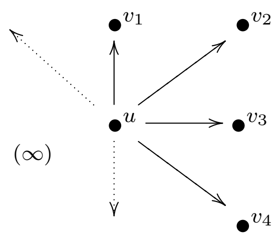
(i) $x_i y_j = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$; and

(ii) $\sum_{i=1}^n y_i x_i = 1$.

If we define $\phi : L_K(R_n) \to L(1, n)$ by $\phi(v) = 1$, $\phi(e_i) = y_i$ and $\phi(e^*_i) = x_i$, we can see that relations (i) and (ii) above correspond directly to the Leavitt path algebra relations (CK1) and (CK2) on $L_K(R_n)$. Furthermore, since $v^2 = v$ and $ve_i = e_i = e_i v$ (and $ve^*_i = e_i = e^*_i v$) for all $i = 1, \dots, n$, the relations (A1) and (A2) correspond directly to the unital properties of 1. From here the isomorphism is clear. Note that in the case $n = 1$ we have $K[x, x^{-1}] \cong L(1, 1)$, which is consistent with the single loop example above.

So far we have only looked at examples of Leavitt path algebras of row-finite graphs. The following graph contains an infinite emitter, which makes the situation slightly more complex.

Example 2.1.7. We define the **infinite clock graph**, denoted C_∞, to be the graph



where u emits a countably infinite number of edges. We show that $L_K(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$, where $\bigoplus_{i=1}^{\infty} \mathbb{M}_2(K)$ is the direct sum of a countably infinite number of copies of $\mathbb{M}_2(K)$ and $I_{22} = \prod_{i=1}^{\infty} E_{22}$. If we let e_i be the edge from u to v_i, then we can define a map $\phi : L_K(C_\infty) \to \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$ on the generators of $L_K(C_\infty)$ as follows:

$$\phi(u) = I_{22}, \quad \phi(v_i) = (E_{11})_i, \quad \phi(e_i) = (E_{21})_i, \quad \text{and} \quad \phi(e_i^*) = (E_{12})_i,$$

where $(A)_i$ denotes the element of $\bigoplus_{i=1}^{\infty} \mathbb{M}_2(K)$ with $A \in \mathbb{M}_2(K)$ in the i^{th} component and zeros elsewhere.

Note that the map here is similar to the mapping from $L_K(M_2) \to \mathbb{M}_2(K)$ in Example 2.1.4. Indeed, it is as if we have an infinite number of copies of the graph M_2 emanating from a single central vertex u. Thus, in a similar fashion to that example it is easy to see that the Leavitt path algebra relations are preserved by ϕ. As an example, we check that $\phi(ue_i) = \phi(e_i)$ for an arbitrary edge e_i:

$$\phi(ue_i) = \phi(u)\phi(e_i) = I_{22}(E_{21})_i = (E_{22})_i(E_{21})_i = (E_{21})_i = \phi(e_i),$$

as required. Note that we do not need to check the (CK2) relation as there are no regular vertices in C_∞. Finally, it is clear that ϕ is an isomorphism, as required.

From the four defining Leavitt path algebra relations we can deduce the product of two arbitrary generating elements in $L_K(E)$. For example, by applying relations

(A1) and (A2), we can deduce the product of two arbitrary edges $e_i, e_j \in E^1$:

$$e_i e_j = e_i r(e_i) s(e_j) e_j = \delta_{r(e_i),s(e_j)}\ e_i e_j.$$

Furthermore, for $e_i^*, e_j^* \in (E^1)^*$ we have:

$$e_i^* e_j^* = e_i^* s(e_i) r(e_j) e_j^* = \delta_{s(e_i),r(e_j)}\ e_i^* e_j^*.$$

Thus the product of two edges e_i and e_j is nonzero if and only if e_i and e_j are adjacent in the graph E. Extending this to an arbitrary number of edges $e_1, e_2, \ldots e_n \in E^1$, we can see that the product $e_1 e_2 \ldots e_n$ is nonzero if and only if $e_1 e_2 \ldots e_n$ is a path in E (and similarly the product $e_n^* \ldots e_2^* e_1^*$ is nonzero if and only if $e_n^* \ldots e_2^* e_1^*$ is a ghost path in E).

The relations (A1) and (A2) give similar results when multiplying an arbitrary vertex $v \in E^0$ with an arbitrary edge $e \in E^1$:

$$ve = \delta_{v,s(e)}\, e \quad \text{and} \quad ev = \delta_{v,r(e)}\, e.$$

And similarly, for an arbitrary $e^* \in (E_1)^*$ we have:

$$ve^* = \delta_{v,r(e)}\, e^* \quad \text{and} \quad e^* v = \delta_{v,s(e)}\, e^*.$$

Thus the product of a vertex by an edge is nonzero only when the vertex is the source of that edge, and the product of an edge by a vertex is nonzero only when the vertex is the range of that edge. Essentially the relations (A1) and (A2) can be seen as preserving the path structure of the graph E, as mentioned earlier. The following lemma from [AA1, Lemma 1.5] solidifies this concept. The proof here follows the same argument as the proof of [Rae, Corollary 1.15].

Lemma 2.1.8. *Let E be an arbitrary graph. Every monomial in $L_K(E)$ is of the form kpq^*, where $k \in K$ and $p, q \in E^*$. Specifically, every monomial can be expressed in one of two forms:*

(i) kv_i, *where* $k \in K$ *and* $v_i \in E^0$, *or*

(ii) $ke_{i_1} \ldots e_{i_m} e_{j_n}^* \ldots e_{j_1}^*$, *where* $k \in K, e_{i_1}, \ldots, e_{i_m}, e_{j_1}, \ldots, e_{j_n} \in E^1$ *and* $m, n \geq 0$, $m + n \geq 1$,

so that p and q are either paths of length 0 at the vertex v_i, or $p = e_{i_1} \dots e_{i_m}, q = e_{j_1} \dots e_{j_n}$ and at least one of p and q has length greater than 0.

Proof. We proceed by induction on the length of the monomial $kx_1 \dots x_t$, where each $x_i \in E^0 \cup E^1 \cup (E^1)^*$. For $t = 1$, it is clear that the monomial is either of type (i) or (ii) above. Now assume it is true that every monomial of length $t \geq 1$ can be written as a monomial of type (i) or (ii) and let $\beta = ky_1 \dots y_t y_{t+1}$, where each $y_i \in E^0 \cup E^1 \cup (E^1)^*$ and $k \in K$. Set $\alpha = ky_1 \dots y_t$, giving $\beta = \alpha y_{t+1}$. By our induction hypothesis on α, we have two cases:

Case 1: $\alpha = kv_i$ for some $v_i \in E^0$. If $y_{t+1} = v_j$ then $\beta = k\,\delta_{ij}\,v_i$ is of type (i). If $y_{t+1} = e_j$, where $e_j \in E^1$, then $\beta = kv_i s(e_j)e_j = k\,\delta_{v_i,s(e_j)}\,e_j$ is of type (ii). Similarly, if $y_{t+1} = e_j^*$ then β is again of type (ii).

Case 2: $\alpha = ke_{i_1} \dots e_{i_m} e_{j_n}^* \dots e_{j_1}^*$, with $m, n \geq 0$, $m + n \geq 1$ and each $e_i, e_j \in E^1$. We break this case into several subcases:

Case 2.1: $y_{t+1} = v_j$, for some $v_j \in E^0$ and $n > 0$. Then $e_{j_1}^* v_j = e_{j_1}^* s(e_{j_1}) v_j = \delta_{v_j,s(e_{j_1})}\,e_{j_1}^*$ and so $\beta = k\,\delta_{v_j,s(e_{j_1})}\,e_{i_1} \dots e_{i_m} e_{j_n}^* \dots e_{j_1}^*$ is of type (ii).

Case 2.2: $y_{t+1} = v_j$ for some $v_j \in E^0$ and $n = 0$. Then we must have $m > 0$ and so $\beta = k\,\delta_{v_j,r(e_{i_m})}\,e_{i_1} \dots e_{i_m}$ is again of type (ii).

Case 2.3: $y_{t+1} = e_j$ for some $e_j \in E^1$ and $n > 0$. By the (CK1) relation we have $e_{j_1}^* e_j = \delta_{j_1,j}\,r(e_j)$.

If $n > 1$, we have $\beta = k\,\delta_{j_1,j}\,\delta_{s(e_{j_2}),r(e_j)}\,e_{i_1} \dots e_{i_m} e_{j_n}^* \dots e_{j_2}^*$, which is of type (ii).

If $n = 1$ and $m > 0$, we have $\beta = k\,\delta_{j_1,j}\,\delta_{r(e_{i_m}),r(e_j)}\,e_{i_1} \dots e_{i_m}$, which is again of type (ii).

Finally, if $n = 1$ and $m = 0$, we have $\beta = k\,\delta_{j_1,j}\,r(e_j)$, which is of type (i).

Case 2.4: $y_{t+1} = e_j$ for some $e_j \in E^1$ and $n = 0$. Then we must have $m > 0$ and so $\beta = e_{i_1} \dots e_{i_m} e_j$ is of type (ii).

Case 2.5: $y_{t+1} = e_j^*$ for some $e_j \in E^1$ and $n > 0$. Then we have that $\beta = k\,\delta_{s(e_{j_1}),r(e_j)}\,e_{i_1} \dots e_{i_m} e_{j_n}^* \dots e_{j_1}^* e_j^*$ is of type (ii).

Case 2.6: $y_{t+1} = e_j^*$ for some $e_j \in E^1$ and $n = 0$. Then $m > 0$ and so $\beta = k\,\delta_{r(e_{i_m}),r(e_j)}\,e_{i_1} \dots e_{i_m} e_j^*$ is again of type (ii). □

In light of Lemma 2.1.8, we can now describe an arbitrary element of $L_K(E)$. Since any element in $L_K(E)$ is a K-linear combination of monomials in $L_K(E)$, an arbitrary element $\alpha \in L_K(E)$ is of the form

$$\alpha = \sum_{i=1}^{n} k_i p_i q_i^*,$$

where each $k_i \in K$ and each $p_i, q_i \in E^*$. In other words, $L_K(E)$ is spanned as a K-vector space by the set $\{pq^* : p, q \in E^*\}$. Note that a monomial pq^* is only nonzero if $r(p) = r(q)$.

Lemma 2.1.9. *Let E be an arbitrary graph. Then*

$$L_K(E) = \bigoplus_{v \in E^0} L_K(E)v$$

Proof. Let $x \in L_K(E)$. By Lemma 2.1.8, $x = k_1 p_1 q_1^* + \cdots + k_n p_n q_n^*$, where each $k_i \in K$ and each $p_i, q_i \in E^*$. Thus $x = k_1 p_1 q_1^* v_1 + \cdots + k_n p_n q_n^* v_n \in \sum_{v \in E^0} L_K(E)v$, where each $v_i = s(q_i)$, and so $L_K(E) = \sum_{v \in E^0} L_K(E)v$.

To show this sum is direct, suppose we have $y \in L_K(E)v \cap \sum_{w \in E^0, w \neq v} L_K(E)w$ for some $v \in E^0$. Then $y = av = \sum_{w \in E^0, w \neq v} a_w w$ for some $a, a_w \in L_K(E)$ (with only a finite number of a_w nonzero) and so $av = (av)v = (\sum_{w \in E^0, w \neq v} a_w w)v = 0$, since the vertices of E form a set of mutually orthogonal idempotents in $L_K(E)$ (by the (A1) relation). Thus $L_K(E) = \bigoplus_{v \in E^0} L_K(E)v$, as required. □

From Lemma 2.1.8 we know that every monomial in $L_K(E)$ is of the form kpq^*, where p and q are paths in E. But what happens when we form the product p^*q? The following lemma gives a useful result concerning such products.

Lemma 2.1.10. *Let E be an arbitrary graph and let p and q be two paths in E.*

(i) If p and q have the same length, then in $L_K(E)$ we have $p^*q = \delta_{p,q}\, r(p)$.

(ii) If p and q have different lengths, then in $L_K(E)$ we have

$$p^*q = \begin{cases} p_2^* & \text{if } q \text{ is an initial subpath of } p \text{ with } p = qp_2 \\ q_2 & \text{if } p \text{ is an initial subpath of } q \text{ with } q = pq_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. (i) Let $p = e_{i_1} \dots e_{i_n}$ and $q = e_{j_1} \dots e_{j_n}$, where each $e_{i_k}, e_{j_k} \in E^1$. By the (CK1) relation we have that $e_{i_k}^* e_{j_k} = \delta_{e_{i_k}, e_{j_k}} r(e_{i_k})$ for each $k \in \{1, \dots, n\}$. Also note that $r(e_{i_k}) = s(e_{i_{k+1}})$ and so the (A2) relation gives $r(e_{i_k}) e_{i_{k+1}} = e_{i_{k+1}}$. Thus we have

$$\begin{aligned} p^*q &= e_{i_n}^* \dots e_{i_1}^* e_{j_1} \dots e_{j_n} \\ &= (\delta_{e_{i_1}, e_{j_1}}) e_{i_n}^* \dots e_{i_2}^* e_{j_2} \dots e_{j_n} \\ &\quad \vdots \\ &= (\delta_{e_{i_1}, e_{j_1}} \dots \delta_{e_{i_{n-1}}, e_{j_{n-1}}}) e_{i_n}^* e_{j_n} \\ &= (\delta_{e_{i_1}, e_{j_1}} \dots \delta_{e_{i_{n-1}}, e_{j_{n-1}}} \delta_{e_{i_n}, e_{j_n}}) r(e_{j_n}) \end{aligned}$$

Thus, if $e_{i_k} \neq e_{j_k}$ for any $k \in \{1, \dots, n\}$, so that $p \neq q$, then the above equation gives $p^*q = 0$. Otherwise, if $p = q$ we have $p^*q = r(e_{j_n}) = r(p)$, as required.

(ii) If q is an initial subpath of p with $p = qp_2$, then applying (i) gives $p^*q = (qp_2)^*q = p_2^* q^*q = p_2^* r(q) = p_2^*$, since $r(q) = s(p_2)$. Similarly, if p is an initial subpath of q with $q = p\,q_2$, then $p^*q = p^*p\,q_2 = q_2$.

Now suppose that q is not an initial subpath of p and vice versa. Suppose that $l(p) > l(q)$ and write $p = p_1 p_2$, where $l(p_2) = l(q)$. Since $p_1 \neq q$, applying (i) gives $p^*q = p_2^* p_1^* q = 0$. If $l(q) > l(p)$, a similar argument completes the proof. □

Recall the definition of a $\mathbb{Z}$-graded ring from Section 1.1. If we equate degree in $L_K(E)$ with path length in E, it is natural to think of edges as elements of degree 1, ghost edges as elements of degree -1 and vertices as elements of zero degree. As it turns out, this intuitive grading does indeed fulfil the requirements for a $\mathbb{Z}$-grading on $L_K(E)$, as the following lemma from [AA1, Lemma 1.7] shows.

Lemma 2.1.11. *Let E be an arbitrary graph. Then $L_K(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by:*

$$\deg(v) = 0 \textit{ for all } v \in E^0; \quad \deg(e) = 1 \textit{ and } \deg(e^*) = -1 \textit{ for all } e \in E^1.$$

That is, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where for each $n \in \mathbb{N}$ we define

$$L_K(E)_n = \left\{ \textstyle\sum_i k_i p_i q_i^* : l(p_i) - l(q_i) = n \right\},$$

with each $k_i \in K$ and each $p_i, q_i \in E^*$.

Proof. From Lemma 2.1.8, it is clear that $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_K(E)_n$.

Now we want to show that $L_K(E)_m L_K(E)_n \subseteq L_K(E)_{m+n}$ for each $m, n \in \mathbb{Z}$. Consider nonzero monomials $x = p_1q_1^* \in L_K(E)_m$ and $y = p_2q_2^* \in L_K(E)_n$, where $p_1, q_1, p_2, q_2 \in E^*$. Note that we have $l(p_1) - l(q_1) = m$ and $l(p_2) - l(q_2) = n$. If $xy = p_1q_1^*p_2q_2^* = 0$ then we are done, so suppose that $xy \neq 0$. According to Lemma 2.1.10, we have three cases.

Case 1: $l(q_1) = l(p_2)$. Then $q_1^*p_2 = r(p_2)$ and so $xy = p_1q_2^*$. Since $l(p_1) - l(q_2) = (m + l(q_1)) - (l(p_2) - n) = m + n$, we have that $xy \in L_K(E)_{m+n}$.

Case 2: $l(q_1) > l(p_2)$. Then $q_1 = p_2q$ for some subpath q of q_1, and so $xy = p_1q^*q_2^*$. Since $l(p_1) - l(q_2q) = l(p_1) - (l(q_2) + l(q)) = l(p_1) - (l(q_2) + l(q_1) - l(p_2)) = (l(p_1) - l(q_1)) + (l(p_2) - l(q_2)) = m + n$, we again have that $xy \in L_K(E)_{m+n}$.

Case 3: $l(p_2) > l(q_1)$. Then a similar argument to Case 2 gives $xy \in L_K(E)_{m+n}$.

Finally, if $x = \sum_{i=1}^{r} p_{1_i}q_{1_i}^* \in L_K(E)_m$ and $y = \sum_{j=1}^{s} p_{2_j}q_{2_j}^* \in L_K(E)_n$, then from the argument above it is clear that $xy \in L_K(E)_{m+n}$. Thus $L_K(E)_m L_K(E)_n \subseteq L_K(E)_{m+n}$, as required. $\square$

We define the **degree** of an element $x \in L_K(E)$ to be the lowest number n for which $x \in \bigoplus_{m\leq n} L_K(E)_m$. Recall from Definition 1.1.3 that an element of $L_K(E)_n$ is said to be homogeneous of degree n, and so $\bigcup_{n\in\mathbb{Z}} L_K(E)_n$ is the set of homogeneous elements in $L_K(E)$.

Furthermore, if x is an arbitrary element of $L_K(E)$ and $d \in \mathbb{Z}^+$, we say that x is **representable as an element of degree d in real (or ghost) edges** if x can be written as a sum of monomials from the spanning set $\{pq^* : p, q \in E^*\}$ in such a way that d is the maximum length of a path p (or q) appearing in such monomials.

Note that an element $x \in L_K(E)$ can be representable as an element of different degrees in real edges. For example, the element $x = v$ (where $v \in E^0$ is a regular vertex) has degree 0 in real edges, but the (CK2) relation allows us to write $x = \sum_{s(e)=v} ee^*$, which has degree 1 in real edges.

Finally, it is natural to ask under what conditions $L_K(E)$ is unital and, more generally, under what conditions $L_K(E)$ has local units. We close this section with the following relevant lemma from [AA1, Lemma 1.6].

Lemma 2.1.12. *Let E be a graph.*

(i) If E^0 is finite, then $\sum_{v_i \in E^0} v_i$ is an identity for $L_K(E)$.

(ii) If E^0 is infinite, then E^0 generates a set of local units for $L_K(E)$.

Proof. (i) Suppose E^0 is finite and consider an arbitrary monomial $kpq^* \in L_K(E)$, where $p, q \in E^*$ and $k \in K$. Let $\alpha = \sum_{v_i \in E^0} v_i$. Then

$$\alpha(kpq^*) = \left(\sum_{v_i \in E^0} v_i\right) kpq^* = k\left(\sum_{v_i \in E^0} \delta_{v_i, s(p)} s(p)\right) pq^* = ks(p)pq^* = kpq^*.$$

Similarly, we can show that $(kpq^*)\alpha = kpq^*$. Since any element in $L_K(E)$ is a sum of such monomials, we must have that $\alpha x = x = x\alpha$ for all $x \in L_K(E)$. Thus α is an identity for $L_K(E)$.

(ii) Suppose E^0 is infinite. Consider a finite subset $X = \{a_i\}_{i=1}^t$ of $L_K(E)$. We can write each a_i as $a_i = \sum_{j=1}^{s(i)} k_j^i p_j^i (q_j^i)^*$, where each $p_j^i, q_j^i \in E^*$ and $k_j^i \in K$. Now define

$$V = \bigcup_{i=1}^{t} \{s(p_j^i), s(q_j^i) : j = 1, \dots, s(i)\}$$

and let $\beta = \sum_{v \in V} v$. Then, using the same arguments as in (i), it is easy to see that $\beta a_i = a_i = a_i \beta$ for each $a_i \in X$. Since β is a finite sum and an idempotent, it is a local unit for X. Thus E^0 generates a set of local units for $L_K(E)$. □

Lemma 2.1.12 tells us that, regardless of whether E^0 is finite or infinite, $L_K(E)$ will always have local units. This property will prove extremely useful when proving future results.

2.2 Results and Properties

In the previous section we defined a Leavitt path algebra for an arbitrary graph E and examined its basic structure. Now we continue to examine in detail some important properties of $L_K(E)$, including the extremely powerful result shown in Proposition 2.2.11. We begin by looking at the ideals of $L_K(E)$. This first lemma is from [AA1, Lemma 3.9].

Lemma 2.2.1. *Let E be an arbitrary graph and let J be an ideal of $L_K(E)$. The set of all vertices contained in J, i.e. $J \cap E^0$, forms a hereditary saturated subset of E^0.*

Proof. Let $H = J \cap E^0$ and let $v \in H$ and $w \in T(v)$. Then there exists a path $p \in E^*$ with $s(p) = v$ and $r(p) = w$. By Lemma 2.1.10, we have $w = p^*p = p^*vp \in J$ and so H is hereditary.

Now suppose that w is a regular vertex in E^0 such that for all $e \in E^1$ with $s(e) = w$ we have $r(e) \in H$. Then the (CK2) relation gives

$$w = \sum_{s(e)=w} ee^* = \sum_{s(e)=w} er(e)e^* \in J,$$

and so H is saturated. □

If G is a subset of E^0, then we denote by $I(G)$ the two-sided ideal in $L_K(E)$ generated by the elements of G. This definition gives us the following simple yet useful lemma from [APS, Lemma 2.1].

Lemma 2.2.2. *Let E be an arbitrary graph and let G be a subset of E^0. Then $I(G) = I(\bar{G})$, where $\bar{G}$ is the hereditary saturated closure of G.*

Proof. Let $H = I(G) \cap E^0$. By Lemma 2.2.1 we know that H is a hereditary saturated subset of E^0 containing G. By definition, $\bar{G}$ is the smallest such set, so $G \subseteq \bar{G} \subseteq H$ and, by extension, $I(G) \subseteq I(\bar{G}) \subseteq I(H)$. However, since $H \subseteq I(G)$ we have $I(H) \subseteq I(G)$ and so $I(G) = I(\bar{G}) = I(H)$, as required. □

The fact that any Leavitt path algebra has local units leads to the following useful lemma.

Lemma 2.2.3. *Let E be an arbitrary graph and let I be an ideal of $L_K(E)$. Then $I \cap E^0 = E^0$ if and only if $I = L_K(E)$.*

Proof. Suppose $I \cap E^0 = E^0$ and take an arbitrary element $x \in L_K(E)$. Since E^0 generates a set of local units for $L_K(E)$, there must be an $e \in I$ such that $ex = x = xe$. Since I is an ideal we must have $x \in I$, and so $I = L_K(E)$. The converse is obvious. □

Since $L_K(E)$ has local units for any graph E, we can apply many of the results in Section 1.2 to the category $L_K(E)$-Mod. We give a few examples of such applications here.

Lemma 2.2.4. *Let E be an arbitrary graph. The Leavitt path algebra $L_K(E)$ is a projective module in the category $L_K(E)$-Mod.*

Proof. By Lemma 2.1.9 we have $L_K(E) = \bigoplus_{v \in E^0} L_K(E)v$. Since each $v \in E^0$ is an idempotent and $L_K(E)$ has local units, we can apply Proposition 1.2.13 to obtain that each summand $L_K(E)v$ is projective in $L_K(E)$-Mod. Thus, since the direct sum of projective modules is also projective, $L_K(E)$ is projective. □

Lemma 2.2.4 tells us that every Leavitt path algebra is projective as a left module over itself (and we can show similarly that every Leavitt path algebra is projective as a right module over itself). However, the same is not true for injectivity; that is, not all Leavitt path algebras are left or right self-injective. In Section 4.4 we will examine self-injective Leavitt path algebras in detail.

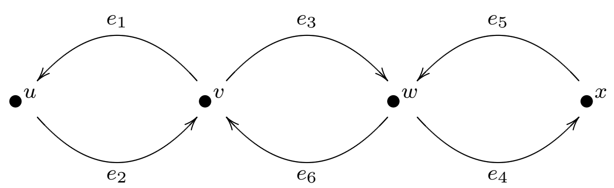
For an arbitrary graph E, Corollary 1.2.16 tells us that $L_K(E)$ is flat as a left $L_K(E)$-module. Furthermore, since $L_K(E) = \bigoplus_{v \in E^0} L_K(E)v$, then taking $B = E^0$ and $U = E^0$ in the definition of U-free module we can see that every Leavitt path algebra $L_K(E)$ is an E^0-free left $L_K(E)$-module with basis E^0.

Now we briefly return to graph theory to define the concept of a closed path.

Definition 2.2.5. A path $p = e_1 \dots e_n$ with $s(p) = v = r(p)$ is said to be a **closed path based at** v. Furthermore, if we have that $s(e_i) \neq v$ for all $i > 1$ we say that p is a **closed simple path based at** v. We denote the set of all closed paths based at v by $\mathrm{CP}(v)$, and the set of all closed simple paths based at v by $\mathrm{CSP}(v)$.

Note that any cycle is a closed simple path based at any of its vertices. However, a closed simple path based at v may not be a cycle as it may visit any of its vertices (other than v) more than once. Similarly, a closed path based at v may not be simple as it may visit v more than once. We illustrate this with the following example.

Example 2.2.6. Consider the following graph:



Now, e_1e_2 and e_3e_6 are both cycles based at v. Furthermore, the path $p = e_3e_4e_5e_6 \in \mathrm{CSP}(v)$ but is not a cycle, as p passes through w twice. Finally, the path $q = e_1e_2e_3e_4e_5e_6 \in \mathrm{CP}(v)$ but is not a closed simple path, as q passes through v twice.

We now use Lemma 2.1.10 to prove the following useful result from [AA1, Lemma 2.3] regarding closed paths.

Lemma 2.2.7. *Every closed path (of length greater than zero) can be decomposed into a unique series of closed simple paths (of length greater than zero); that is, for every $p \in \mathrm{CP}(v)$, there exist unique $c_1, \dots, c_m \in \mathrm{CSP}(v)$ (with $l(c_i) > 0$ for each $i \in \{1, \dots, m\}$) such that $p = c_1 \dots c_m$.*

Proof. Let $p = e_1 \dots e_n$ and let $e_{t_1}, \dots, e_{t_m}$ be the edges in p for which $r(e_{t_i}) = v$, where $t_1 < \cdots < t_m = n$. Let $c_1 = e_1 \dots e_{t_1}$ and $c_j = e_{t_{j-1}+1} \dots e_{t_j}$ for each $1 < j \leq m$. Thus $p = c_1 \dots c_m$, where each $c_j \in \mathrm{CSP}(v)$ and $l(c_j) > 0$.

To show that this decomposition is unique, suppose that $p = c_1 \dots c_r = d_1 \dots d_s$, with $c_i, d_j \in \mathrm{CSP}(v)$ and $l(c_i), l(d_j) > 0$. Furthermore, suppose that $r \geq s$. By Lemma 2.1.10 we have $c_1^*c_1 = v$, and so multiplication by c_1^* on the left gives $0 \neq vc_2 \dots c_m = c_1^*d_1 \dots d_s$. Since the right-hand side is nonzero, we must have $c_1 = d_1$, and so by Lemma 2.1.10 again we have $c_2 \dots c_r = d_2 \dots d_s$ (noting that $vc_2 = c_2$ and $vd_2 = d_2$). Repeating this process gives $c_i = d_i$ for each $i \in \{1, \dots, s\}$, and so $p = c_1 \dots c_s = d_1 \dots d_s$.

If $r > s$, we must have $pc_{s+1} \dots c_r = p$ and so $v = p^*p = p^*pc_{s+1} \dots c_r = c_{s+1} \dots c_r$, which is impossible since $l(c_i) > 0$ for each c_i. A similar argument shows that we cannot have $s > r$. Thus $r = s$ and so the decomposition is unique. $\square$

For a vertex v in an arbitrary graph E, the **range index of** v, denoted $n(v)$, is the cardinality of the set

$$R(v) := \{p \in E^* : r(p) = v\}.$$

Note that $n(v)$ is always nonzero, since $v \in R(v)$ for each $v \in E^0$. We apply this definition in the following lemma from [A, Lemma 4.4.3].

Lemma 2.2.8. *Let E be a finite and row-finite graph and let $v \in E^0$ be a sink. Then*

$$I_v := \operatorname{span}(\{\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta)\})$$

is an ideal of $L_K(E)$, and $I_v \cong \mathbb{M}_{n(v)}(K)$.

Proof. Take an arbitrary nonzero monomial $\alpha\beta^* \in I_v$, so that $r(\alpha) = v = r(\beta)$, and a nonzero monomial $\gamma\delta^* \in L_K(E)$ with $\gamma, \delta \in E^*$. Suppose that $\gamma\delta^*\alpha\beta^* \neq 0$. Then $\delta^*\alpha \neq 0$ and so (by Lemma 2.1.10) we have that either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$. In the latter case we must have that $l(q) = 0$, since $r(\alpha) = v$ is a sink, and so $\delta = \alpha$. Thus we can generalise to a single case in which $\alpha = \delta p$, where $l(p)$ may be zero. Then $\delta^*\alpha = p$ and so $\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$, since $r(p) = r(\alpha) = v$. This shows that I_v is a left ideal. Similarly, we can show that I_v is also a right ideal.

Now let $n = n(v)$, as defined above. Since E is both finite and row-finite, n must also be finite. Rename the elements in the set $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \dots, p_n\}$, giving $I_v = \operatorname{span}\{p_ip_j^* : i, j = 1, \dots, n\}$. Consider the expression $(p_ip_j^*)(p_kp_l^*)$ and suppose that $j \neq k$ and $(p_ip_j^*)(p_kp_l^*) \neq 0$. Then, as above, either $p_j = p_kp$ or $p_k = p_jq$ for some paths $p, q \in E^*$. In either case, $l(p) > 0$ or $l(q) > 0$ since $p_j \neq p_k$. However, this is impossible as v is a sink. Thus $j \neq k$ implies that $(p_ip_j^*)(p_kp_l^*) = 0$. Otherwise, we have $(p_ip_j^*)(p_jp_l^*) = p_ivp_l^* = p_ip_l^*$. Thus $\{p_ip_j^* : i, j = 1, \dots, n\}$ is a set of matrix units for I_v and so $I_v \cong \mathbb{M}_{n(v)}(K)$. $\square$

Lemma 2.2.8 leads to the following important result from [AAS, Proposition 3.5].

Lemma 2.2.9. *Let E be a finite, row-finite and acyclic graph, and let $\{v_1, \dots, v_t\}$ be the sinks of E. Then*

$$L_K(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n(v_i)}(K).$$

Proof. We begin by showing that $L_K(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where the I_{v_i} are the ideals defined in Lemma 2.2.8. Consider an arbitrary nonzero monomial $\alpha\beta^* \in L_K(E)$, where $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some $i \in \{1, \dots, t\}$, then $\alpha\beta^* \in I_{v_i}$. Otherwise, if $r(\alpha) \neq v_i$ for every i then $r(\alpha)$ is not a sink. Thus we can apply the (CK2) relation at $r(\alpha)$ (since E is row-finite), giving

$$\begin{aligned}\alpha\beta^* &= \alpha\Big(\sum\{e_1e_1^* : e_1 \in E^1,\, s(e_1) = r(\alpha)\}\Big)\beta^* \\ &= \sum\{\alpha e_1 e_1^* \beta^* : e_1 \in E^1,\, s(e_1) = r(\alpha)\}.\end{aligned}$$

Now consider a specific summand of the above expression, $\alpha e_1'(e_1')^*\beta^*$. Either $r(e_1') = v_i$ for some $i \in \{1 \dots, t\}$, in which case $\alpha e_1'(e_1')^*\beta^* \in I_{v_i}$, or $r(e_1)$ is not a sink, in which case we can expand the expression by again applying the (CK2) relation at $r(e_1')$, giving

$$\begin{aligned}\alpha e_1'(e_1')^*\beta^* &= \alpha e_1'\Big(\sum\{e_2e_2^* : e_2 \in E^1,\, s(e_2) = r(e_1')\}\Big)(e_1')^*\beta^* \\ &= \sum\{\alpha e_1' e_2 e_2^* (e_1')^* \beta^* : e_2 \in E^1,\, s(e_2) = r(e_1')\}.\end{aligned}$$

Suppose that repeating the above process yields a sequence of edges $e_1'e_2'\dots$ that never reaches a sink, and consider the infinite set of vertices $T = \{r(e_1'), r(e_2'), \dots\}$. Now this set of vertices must be distinct, since if $r(e_i') = r(e_j')$ for some $r(e_i), r(e_j) \in T$ then we would have a cycle in E, contradicting the fact that E is acyclic. However, we cannot have an infinite number of distinct vertices since E is finite. Thus, for each summand of $\alpha\beta^*$, we eventually reach an expression of the form $\alpha e_1'e_2' \dots e_n'(e_n')^* \dots (e_1')^*(e_2')^*\beta^*$, where $r(e_n')$ is a sink; that is, $r(e_n') = v_i$ for some $i \in \{1 \dots, t\}$. Thus each summand of $\alpha\beta^*$ is in I_{v_i} for some sink v_i, and since $\alpha\beta^*$ was an arbitrary monomial and these monomials generate $L_K(E)$, we have $L_K(E) = \sum_{i=1}^{t} I_{v_i}$.

To show that this sum is direct, consider two arbitrary monomials $\alpha_i\beta_i^* \in I_{v_i}$ and $\alpha_j\beta_j^* \in I_{v_j}$ for $i \neq j$. Suppose that $(\alpha_i\beta_i^*)(\alpha_j\beta_j^*) \neq 0$. As in the proof of Lemma 2.2.8, this implies that either $\alpha_j = \beta_i p$ or $\beta_i = \alpha_j q$ for some paths $p, q \in E^*$. Again, this is impossible as $\alpha_j \neq \beta_i$ (since $v_i \neq v_j$) and v_i, v_j are sinks. Thus $(\alpha_i\beta_i^*)(\alpha_j\beta_j^*) = 0$. Since such monomials generate I_{v_i} and I_{v_j}, we have $I_{v_i}I_{v_j} = \{0\}$ for all $i, j \in \{1, \dots, t\}$. Note also that since E is finite, $L_K(E)$ is unital (by Lemma 2.1.12). Since $L_K(E) = \sum_{i=1}^t I_{v_i}$, we have $1 = e_1 + \cdots + e_t$, where each $e_i \in I_{v_i}$. Now suppose there exists $x_1 \in I_{v_1}$ such that $x_1 = x_2 + \cdots + x_t$, where each $x_i \in I_{v_i}$. Now $x_1 = x_1(e_1 + \cdots + e_t) = x_1e_1$, since $I_{v_i}I_{v_j} = \{0\}$ for $i \neq j$, and so $x_1 = x_1e_1 = (x_2 + \cdots + x_t)e_1 = 0$. Repeating this argument, we have that $I_{v_i} \cap \sum_{j=1, j\neq i}^t I_{v_j} = \{0\}$ for each $i \in \{1, \dots, t\}$, and so the sum is direct. Finally, we apply Lemma 2.2.8 to complete the proof. $\square$

To illustrate Lemma 2.2.9, consider the finite line graph with t vertices, M_t:

$$\bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{t-1}} \xrightarrow{e_{t-1}} \bullet^{v_t}$$

Here M_t has a single sink v_t with $R(v_t) = \{v_t, p_{t-1}, \dots, p_2, p_1\}$, where we define $p_i = e_ie_{i+1}\dots e_{t-1}$ for each $i = 1, \dots, t-1$. Thus $n(v_t) = t$ and so Lemma 2.2.9 gives $L_K(M_t) \cong \mathbb{M}_t(K)$, which agrees with the formulation given in Example 2.1.5.

Definition 2.2.10. Let R be a ring with local units. The ring R is said to be **locally matricial** if $R = \varinjlim_{i\in I} R_i$, where $\{R_i : i \in I\}$ is an ascending chain of rings and each R_i is isomorphic to a finite direct sum of finite-dimensional matrix rings over K.

If E is a row-finite graph, we can use Lemma 2.2.9 to show that $L_K(E)$ is locally matricial if and only if E is acyclic (see, for example, [A], Proposition 4.4.6). However, to show this for an arbitrary graph E we will need some of the tools introduced in Section 4.1. This equivalence is finally proved in Theorem 4.2.3.

We now prove the following powerful result from [AMMS1, Proposition 3.1], which greatly simplifies the proof of several subsequent theorems. Though the original theorem was given in a row-finite context, the proof is still valid for arbitrary graphs. Here we have expanded the proof for ease of understanding.

Proposition 2.2.11. *Let E be an arbitrary graph. For every nonzero element $x \in L_K(E)$ there exist $y_1, \dots, y_r, z_1, \dots, z_s \in E^0 \cup E^1 \cup (E^1)^*$ such that*

(i) $y_1 \dots y_r x z_1 \dots z_s$ is a nonzero element in Kv for some $v \in E^0$, or

(ii) there exist a vertex w and a cycle without exits c based at w such that $y_1 \dots y_r x z_1 \dots z_s$ is a nonzero element in

$$wL_K(E)w = \left\{ \sum_{i=-m}^{n} k_i c^i \text{ for } m, n \in \mathbb{N}_0 \text{ and } k_i \in K \right\}.$$

These two cases are not mutually exclusive.

Proof. We first show that for a nonzero element x in $L_K(E)$, there is a path μ in E such that $x\mu$ is nonzero and in only real edges. Consider a vertex $v \in E^0$ such that $xv \neq 0$ (note that such a vertex will always exist, since if $x = \sum_{i=1}^{n} k_i p_i q_i^*$, where each $k_i \in K$ and $p_i, q_i \in E^*$, then choosing $v = s(q_1)$ ensures that $k_1 p_1 q_1^* v \neq 0$ and thus $xv \neq 0$). Write $xv = \sum_{i=1}^{m} \beta_i e_i^* + \beta$, where $\beta_i, \beta \in L_K(E), e_i \in E^1, e_i \neq e_j$ for $i \neq j$, β is a polynomial in real edges, and xv is represented as an element of minimal degree in ghost edges. We have two cases.

Case (1): $xve_i = 0$ for all $i \in \{1, \dots, m\}$. This gives, for each e_i, $xve_i = \beta_i + \beta e_i = 0$ and so $\beta_i = -\beta e_i$. Thus $xv = \sum_{i=1}^{m} -\beta e_i e_i^* + \beta = \beta(\sum_{i=1}^{m} -e_i e_i^* + v)$. Since $xv \neq 0$ we have $v - \sum_{i=1}^{m} e_i e_i^* \neq 0$. Since $s(e_i) = v$ for each e_i, by the (CK2) relation there must exist an $f \in E^1$ such that $s(f) = v$ but $f \neq e_i$ for each i. Thus $xvf = (\sum_{i=1}^{m} -\beta e_i e_i^* + \beta)f = 0 + \beta f \neq 0$ (since $r(\beta) = v$), and so, since β is in only real edges, we have a path $vf \in E^*$ such that xvf is nonzero and in only real edges.

Case (2): $xve_i \neq 0$ for some i, say for $i = 1$. Then $xve_1 = \beta_1 + \beta e_1$, with the degree in ghost edges of xve_1 strictly less than that of xv. If β_1 is a polynomial in only real edges, then we are done. Otherwise, we can repeat the above process, reducing the degree in ghost edges with each iteration until we are left with an element in only real edges (this must happen since the degree in ghost edges of xv must, of course, be finite).

Now we can assume that $x \in L_K(E)$ is a nonzero polynomial in only real edges. Write $x = \sum_{i=1}^{r} k_i \alpha_i$, where $0 \neq k_i \in K$ for each i and each α_i is a real path in

E with $\alpha_i \neq \alpha_j$ for $i \neq j$. Using induction on r, we will prove that multiplication on the left and/or right of x by elements from $E^0 \cup E^1 \cup (E^1)^*$ will produce either a nonzero scalar multiple of a vertex or a nonzero polynomial in a cycle with no exits. For $r = 1$, we have $x = k_1\alpha_1$. If α_1 is a vertex then we are done. Otherwise, $\alpha_1 = f_1 \dots f_n$ for some $f_i \in E^1$. Thus $f_n^* \dots f_1^* x = k_1 r(f_n)$, and so the proposition is true for $r = 1$.

Now assume that the property is true for any nonzero element that is the sum of less than r paths in the conditions above. Write $x = \sum_{i=1}^r k_i\alpha_i$ such that $\deg(\alpha_i) \leq \deg(\alpha_{i+1})$ and $k_i\alpha_i \neq 0$ for each i. Let $z = \alpha_1^* x$. Thus $0 \neq z = k_1 v + \sum_{i=2}^r k_i\beta_i$, where $v = r(\alpha_1)$ and $\beta_i = \alpha_1^*\alpha_i$. Note that $\deg(\beta_i) \leq \deg(\beta_{i+1})$ and $\beta_i \neq \beta_j$ for $i \neq j$.

If $\beta_i = 0$ for some i then we can apply our inductive hypothesis and we are done. Furthermore, if some β_i does not begin or end in v, then we can apply our inductive hypothesis to vz or zv (both nonzero since our β_i are distinct). Thus we can assume that each β_i is nonzero and begins and ends in v.

Now suppose that there exists some path τ such that $\tau^*\beta_i = 0$ for some, but not all, β_i. Then we can apply our inductive hypothesis to $\tau^* z \neq 0$ and we are done. Thus we can suppose that, for a given path τ, if $\tau^*\beta_i \neq 0$ for some i, then $\tau^*\beta_i \neq 0$ for all i. Let $\tau = \beta_j$ for some fixed j. Since $\tau^*\beta_j \neq 0$, we must have $\tau^*\beta_{j+1} \neq 0$. Since $\deg(\beta_j) \leq \deg(\beta_{j+1})$, by Lemma 2.1.10 we have that either $\beta_j = \beta_{j+1}$ or $\beta_{j+1} = \beta_j r_j$ for some path $r_j \in \mathrm{CP}(v)$. Since the β_i are distinct, we must have the latter case. Thus in general we have $\beta_{i+1} = \beta_i r_i$ for some path $r_i \in \mathrm{CP}(v)$, and so we can write $z = k_1 v + k_2\gamma_1 + k_3\gamma_1\gamma_2 + \dots + k_r\gamma_1\gamma_2 \dots \gamma_{r-1}$, where each γ_i is a closed path based at v.

Now write each γ_i as $\gamma_i = \gamma_{i_1} \dots \gamma_{i_{n(i)}}$, where each γ_{i_j} is a closed simple path based at v. If the paths γ_{i_j} are not identical, then we must have $\gamma_{1_1} \neq \gamma_{i_j}$ for some γ_{i_j}, and so $\gamma_{i_j}^*\gamma_{1_1} = 0$ (since one cannot be an initial subpath of the other). Thus we have $0 \neq \gamma_{i_j}^* z \gamma_{i_j} = k_1 v$, since γ_{1_1} appears in every term but the first.

Now assume that the paths are identical, so that $\gamma_{i_j} = \gamma$ (where $\gamma \in E^*$) for each i, j. If γ is not a cycle then γ must contain a cycle; that is, if $\gamma = e_1 \dots e_n$

(with each $e_i \in E^1$) then there exist $e_{i_1}, \dots, e_{i_k}$ with $i_1, \dots, i_k \in \{1, \dots, n\}$ such that $i_1 < \cdots < i_k$ and $d = e_{i_1} \dots e_{i_k}$ is a cycle based at v (noting that $k < n$). Thus we have that $d^*\gamma = 0$ (since d is clearly not an initial subpath of γ) and so $d^*zd = k_1 v$ and we are done.

Thus we can assume that z is a polynomial in the cycle $c = \gamma$. Suppose that f is an exit for c, so that $s(e) = s(f)$ for some edge e in c but $f \neq e$. Write $c = aeb$ (where $a, b \in E^*$) and let $\rho = af$, which is nonzero since $r(a) = s(e) = s(f)$. Then $\rho^* c = f^* a^* aec = f^* ec = 0$ and so $\rho^* z \rho = \rho^* k_1 v \rho = k_1 r(f)$ and we are done.

Finally, if c is a cycle with no exits based at v then $z \in \left\{ \sum_{i=-m}^{n} k_i c^i \text{ for } m, n \in \mathbb{N} \text{ and } k_i \in K \right\}$, where we understand $c^{-m} = (c^*)^m$ for $m \in \mathbb{N}$ and $c^0 = v$. Clearly this set is contained in $vL_K(E)v$ since each c^i begins and ends in v. To see the converse containment, first note that the elements of $vL_K(E)v$ must be linear combinations of monomials $\alpha\beta^*$, where $\alpha, \beta \in E^*$, $s(\alpha) = v = s(\beta)$ and $r(\alpha) = r(\beta)$. Now, since c has no exits, any path $p \in E^*$ with $s(p) = v$ must be of the form $c^n p'$, where $n \geq 0$ and p' is an initial subpath of c (for if p were to contain an edge distinct from any edge in c, that edge would constitute an exit for c). Thus $\alpha = c^m \alpha'$ and $\beta = c^n \beta'$ for some $m, n \geq 0$. Since α' and β' are initial subpaths of c and $r(\alpha') = r(\beta')$, we must have $\alpha' = \beta'$. Let $\alpha' = e_1 \dots e_k$. For any edge e in c, the vertex $s(e)$ emits only e (since c has no exits) and so applying the (CK2) relation at $s(e)$ yields $ee^* = s(e)$. Thus

$$\begin{aligned}
\alpha'(\beta')^* &= e_1 \dots e_{k-1} e_k e_k^* e_{k-1}^* \dots e_1^* \\
&= e_1 \dots e_{k-1} s(e_k) e_{k-1}^* \dots e_1^* \\
&= e_1 \dots e_{k-1} (e_{k-1})^* \dots e_1^* \\
&\ \ \vdots \\
&= e_1 e_1^* \\
&= v,
\end{aligned}$$

and so $\alpha\beta^* = c^m \alpha'(\beta')^* c^{-n} = c^m v c^{-n} = c^m (c^*)^n$. Again, using the fact that c has no exits we can apply the (CK2) relation to give $cc^* = v$ (letting $\alpha' = \beta' = c$ in the above equation). Thus $\alpha\beta^* = c^m (c^*)^n = c^{m-n}$, and so $vL_K(E)v$ is precisely the set of all polynomials in c.

To see that the two cases are not mutually exclusive, consider the graph E consisting of a single vertex v and a single loop e based at v, and take $x = e$. Thus $e^*xv = v$ (giving case (1)) and $vxv = e$, a cycle without exits (giving case (2)). $\square$

Proposition 2.2.11 leads to the following useful corollary from [AMMS2, Corollary 3.3].

Corollary 2.2.12. *Let E be an arbitrary graph. Then*

(i) every $\mathbb{Z}$-graded nonzero ideal of $L_K(E)$ contains a vertex, and

(ii) if E contains no cycles without exits, then every nonzero ideal of $L_K(E)$ contains a vertex.

Proof. (i) Let I be a $\mathbb{Z}$-graded nonzero ideal of $L_K(E)$ and let $0 \neq x \in I$. By Proposition 2.2.11, there exist $y, z \in L_K(E)$ such that $0 \neq yxz = \sum_{i=-m}^{n} k_i c^i$, where c is a cycle without exits based at a vertex w, each $k_i \in K$ and $m, n \in \mathbb{N}$. Since I is a graded ideal, each summand of yxz must also be in I (since each $k_i c^i$ is a homogeneous element of degree i in $L_K(E)$). Then, for $t \in \{-m, \dots, n\}$ such that $k_t c^t \neq 0$, we have $0 \neq (k_t^{-1}(c^*)^t)k_t c^t = w \in I$, as required.

(ii) Let J be a nonzero ideal of $L_K(E)$ and let $0 \neq x \in J$. Since E contains no cycles without exits, then again by Proposition 2.2.11 there must exist $y, z \in L_K(E)$ such that $0 \neq yxz = kv$ for some $v \in E^0$ and $k \in K$. Thus $0 \neq (k^{-1}v)kv = v \in I$, as required. $\square$

The following two 'Uniqueness theorems' are given by Tomforde as [To, Theorem 4.6] and [To, Theorem 6.8], respectively. In Tomforde's paper, the proofs are fairly involved. However, in light of Proposition 2.2.11 and its subsequent corollary, the results follow almost instantly.

Theorem 2.2.13 (Graded Uniqueness Theorem)**.** *Let E be an arbitrary graph and let A be a $\mathbb{Z}$-graded ring. If $\pi : L_K(E) \to A$ is a graded ring homomorphism for which $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is a monomorphism.*

Proof. By Lemma 1.1.5, $\ker(\pi)$ is a graded ideal of $L_K(E)$. So, by Corollary 2.2.12, if $\ker(\pi)$ is nonzero it must contain a vertex, contradicting the fact that $\pi(v) \neq 0$ for every vertex $v \in E^0$. Thus $\ker(\pi) = \{0\}$ and so π is a monomorphism. $\square$

Theorem 2.2.14 (Cuntz-Krieger Uniqueness Theorem)**.** *Let E be a graph in which every cycle has an exit and let A be a ring. If $\pi : L_K(E) \to A$ is a ring homomorphism for which $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is a monomorphism.*

Proof. Suppose that $\ker(\pi) \neq 0$. Since $\ker(\pi)$ is an ideal of $L_K(E)$ and E contains no cycles without exits, Corollary 2.2.12 tells us that $\ker(\pi)$ must contain a vertex. Thus contradicts the fact that $\pi(v) \neq 0$ for every vertex $v \in E^0$, and so $\ker(\pi) = \{0\}$ and thus π is a monomorphism. $\square$

In addition to the above two results, Proposition 2.2.11 also leads to the following useful theorem from [AMMS2, Theorem 3.7]. Recall that an element x in a ring R is said to be nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.

Theorem 2.2.15. *Let E be an arbitrary graph and let A be a graded K-algebra. If $\pi : L_K(E) \to A$ is a ring homomorphism for which $\pi(v) \neq 0$ for every vertex $v \in E^0$, and for which each cycle without exits in E is mapped to a non-nilpotent homogeneous element of nonzero degree, then π is a monomorphism.*

Proof. Suppose that $\ker(\pi)$ is nonzero. Since it is a nonzero ideal containing no vertices, by Proposition 2.2.11 $\ker(\phi)$ must contain a nonzero element of the form $x = \sum_{i=-m}^{n} k_i c^i$, where c is a cycle without exits based at a vertex w, each $k_i \in K$ and $m, n \in \mathbb{N}$. By hypothesis, $\pi(c) = h$, where h is a non-nilpotent homogeneous element of nonzero degree. Thus $\pi(x) = \sum_{i=-m}^{n} k_i \pi(c)^i = \sum_{i=-m}^{n} k_i h^i = 0$. Since h is not nilpotent, we must have $k_i = 0$ for each $i = -m, \dots, n$, and so $x = 0$, a contradiction. Thus $\ker(\phi) = \{0\}$ and so π is a monomorphism, as required. $\square$

2.3 Purely Infinite Simple Leavitt Path Algebras

We open this section with Theorem 2.3.1, which describes precisely which graphs yield simple Leavitt path algebras. From this result we then build toward Theorem 2.3.9, which describes precisely which graphs yield Leavitt path algebras that are both simple and purely infinite; that is, 'purely infinite simple'.

The following result was first shown for row-finite graphs in [AA1, Theorem 3.11] and then extended to arbitrary graphs in [AA3, Theorem 3.1]. In comparison to the published versions, the first part of the proof given here is much simpler, thanks to Proposition 2.2.11.

Theorem 2.3.1. *Let E be an arbitrary graph. Then the Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies the following conditions:*

(i) The only hereditary saturated subsets of E^0 are $\emptyset$ and E^0, and

(ii) every cycle in E has an exit.

Proof. Suppose statements (i) and (ii) are true and let J be a nonzero ideal of $L_K(E)$. Since E contains no cycles without exits, Proposition 2.2.11 tells us that J contains at least one vertex. Thus the vertices of J form a nonempty, hereditary saturated subset of E^0 (by Lemma 2.2.1) and so $J \cap E^0 = E^0$, by (i). Thus, by Lemma 2.2.3, we have that $J = L_K(E)$, proving $L_K(E)$ is simple.

Now suppose that there exists a hereditary saturated subset H of E^0 that is nonempty and is not equal to E^0. We will show that this implies that $L_K(E)$ cannot be simple. Define the graph $F = (F^0, F^1, r_F, s_F)$, where

$$F^0 = E^0 \backslash H, F^1 = r^{-1}(E^0 \backslash H), \quad r_F = r_{E^0 \backslash H}, s_F = s_{E^0 \backslash H}$$

In other words, F consists of all the vertices of E that are not in H, and all the edges whose range is not in H. To ensure that F is a well-defined graph, we must ensure that $s_F(F^1) \cup r_F(F^1) \subseteq F^0$. From the definition it is clear that $r_F(F^1) \subseteq F^0$. Furthermore, suppose that there exists an edge $e \in F^1$ with $s(e) \in H$. Then, by the hereditary nature of H, we have $r(e) \in H$, which contradicts the definition of F^1. Thus $s(e) \in F^0$, and so $s_F(F^1) \subseteq F^0$ and F is therefore well-defined.

Define a map $\phi : L_K(E) \to L_K(F)$ on the generators of $L_K(E)$ as follows:

$$\phi(v) = \begin{cases} v & \text{if } v \notin H \\ 0 & \text{if } v \in H, \end{cases} \quad \phi(e) = \begin{cases} e & \text{if } r(e) \notin H \\ 0 & \text{if } r(e) \in H \end{cases} \quad \text{and } \phi(e^*) = \begin{cases} e^* & \text{if } r(e) \notin H \\ 0 & \text{if } r(e) \in H. \end{cases}$$

Extend ϕ linearly and multiplicatively. To ensure that ϕ is a K-algebra homomorphism, we must check that it preserves the Leavitt path algebra relations on E. This is a relatively straightforward (though slightly tedious) process. We include it here for the sake of completeness.

First, we check that the (A1) relation holds, i.e. that $\phi(v_i)\phi(v_j) = \delta_{ij}\,\phi(v_i)$ for all $v_i, v_j \in E^0$. We must examine several different cases:

Case 1: $v_i, v_j \notin H$. Then $\phi(v_i)\phi(v_j) = v_iv_j = \delta_{ij}\,v_i = \delta_{ij}\,\phi(v_i)$.

Case 2: $v_i \notin H, v_j \in H$. Then $\delta_{ij}\,v_i = 0$ and so $\phi(v_i)\phi(v_j) = 0 = \delta_{ij}\,\phi(v_i)$. A similar argument holds for $v_i \in H, v_j \notin H$.

Case 3: $v_i, v_j \in H$. Then $\phi(v_i)\phi(v_j) = 0 = \delta_{ij}\,\phi(v_i)$.

Next, we check that the (A2) relations hold. First, we check that $\phi(s(e))\phi(e) = \phi(e)$ for all $e \in E^1$.

Case 1: $r(e) \notin H$. Then $s(e) \notin H$ and so $\phi(s(e))\phi(e) = s(e)e = e = \phi(e)$.

Case 2: $r(e) \in H$. Then $\phi(s(e))\phi(e) = 0 = \phi(e)$.

Similar arguments show that $\phi(e)\phi(r(e)) = \phi(e)$, $\phi(r(e))\phi(e^*) = \phi(e^*)$ and $\phi(e^*)\phi(s(e)) = \phi(e^*)$ for all $e \in E^1$.

Next we check that the (CK1) relation holds, i.e. that $\phi(e_i^*)\phi(e_j) = \delta_{ij}\,\phi(r(e_i))$ for all $e_i, e_j \in E^1$.

Case 1: $r(e_i), r(e_j) \notin H$. Then $\phi(e_i^*)\phi(e_j) = e_i^*e_j = \delta_{ij}\,r(e_i) = \delta_{ij}\,\phi(r(e_i))$.

Case 2: $r(e_i) \in H, r(e_j) \notin H$. Then $e_i \neq e_j$, so $\phi(e_i^*)\phi(e_j) = 0 = \delta_{ij}\,\phi(r(e_i))$. A similar argument holds for $r(e_i) \notin H, r(e_j) \in H$.

Case 3: $r(e_i), r(e_j) \in H$. Then again $\phi(e_i^*)\phi(e_j) = \phi(e_i^*)\phi(e_j) = 0 = \delta_{ij}\,\phi(r(e_i))$.

Finally, we check that the (CK2) relation holds, i.e. that $\phi\big(v - \sum_{s_E(e_i)=v} e_ie_i^*\big) = 0$ for all regular vertices $v \in E^0$.

Case 1: $v \in H$. By the hereditary nature of H, for every edge $e_i \in E^1$ with

$s(e_i) = v$, we have $r(e_i) \in H$. Thus

$$\phi\left(v - \sum_{s_E(e_i)=v} e_i e_i^*\right) = \phi(v) - \sum_{s_E(e_i)=v} \phi(e_i)\phi(e_i^*) = 0 - 0 = 0.$$

Case 2: $v \notin H$. Because H is saturated, there must exist at least one edge $e_i \in E^1$ such that $s(e_i) = v$ and $r(e_i) \notin H$ (for otherwise, if $r(e_i) \in H$ for all $e_i \in s^{-1}(v)$ then we must have $v \in H$, a contradiction). If $r(e_i) \notin H$, then $\phi(e_i)\phi(e_i^*) = e_i e_i^*$. Otherwise, $\phi(e_i)\phi(e_i^*) = 0$. Recalling that, in the graph F, v only emits edges e_i for which $r(e_i) \notin H$ in the original graph E (by definition), we have $\phi\big(\sum_{s_E(e_i)=v} e_i e_i^*\big) = \sum_{s_F(e_i)=v} \phi(e_i)\phi(e_i^*) = \sum_{s_F(e_i)=v} e_i e_i^*$. This gives

$$\phi\left(v - \sum_{s_E(e_i)=v} e_i e_i^*\right) = v - \sum_{s_F(e_i)=v} e_i e_i^* = 0.$$

Thus ϕ preserves the Leavitt path algebra relations on E, and so is a K-algebra homomorphism.

Now consider the ideal $\ker(\phi)$. Since H is nonempty, there must exist a vertex $v \in H$. Since $\phi(v) = 0$ and $v \neq 0$, we have $\ker(\phi) \neq \{0\}$. Furthermore, since $H \neq E^0$, there must exist a vertex $w \in E^0 \backslash H$. Since $\phi(w) = w \neq 0$, we have $\ker(\phi) \neq L_K(E)$. Thus $\ker(\phi)$ is a proper nontrivial ideal of $L_K(E)$, and so $L_K(E)$ is not simple, as required.

To complete the proof, we now suppose that E contains a cycle c without exits, and show again that this implies that $L_K(E)$ cannot be simple. Let v be the base of this cycle and consider the nonzero ideal $\langle v + c\rangle$. We show that $\langle v + c\rangle \neq L_K(E)$ by showing that $v \notin \langle v + c\rangle$. Let $c = e_{i_1} \dots e_{i_\sigma}$, where $s(e_{i_1}) = r(e_{i_\sigma}) = v$. Since c has no exits, we have that $cc^* = v$ (see the proof of Proposition 2.2.11, page 59). Furthermore, by Lemma 2.1.10 we know that $c^*c = v$. Furthermore, we must have $\mathrm{CSP}(v) = \{c\}$, since the existence of a closed simple path based at v that is distinct from c would imply that c has an exit.

We proceed by contradiction: suppose that $v \in \langle v + c\rangle$. Then there exist (nonzero) monic monomials $\alpha_t, \beta_t \in L_K(E)$ and scalars $k_t \in K$ such that

$$v = \sum_{t=1}^{n} k_t \alpha_t (v + c)\beta_t.$$

Each summand in the above expression must begin and end in v, for otherwise $v = v(v)v = \sum_{t=1}^{n} k_t v(\alpha_t(v+c)\beta_t)v = 0$, a contradiction. Furthermore, since c is based at v, we can write $v+c$ as $v(v+c)v$. Thus, since the right-hand side of the above expression is nonzero, each α_t and β_t must begin and end in v. Thus each α_t, β_t is a monomial in $vL_K(E)v$ and so, as shown in the proof of Proposition 2.2.11, we have that each α_t and β_t is equal to c^m or $(c^*)^n$ for some $m, n \in \mathbb{N}_0$.

Recalling that $cc^* = v = c^*c$, we have that c and c^* commute with $v+c$, and so $\alpha_t(v+c) = (v+c)\alpha_t$ for each $t = 1, \dots, n$. Thus

$$v = \sum_{t-1}^{n} k_t \alpha_t (v+c)\beta_t = \sum_{t-1}^{n} (v+c)(k_t\alpha_t\beta_t).$$

Now each $\alpha_t\beta_t$ term is a power of either c or c^*, and so we can write $v = (v+c)P(c, c^*)$, where P is a polynomial with coefficients in K, i.e.

$$P(c, c^*) = l_{-m}(c^*)^m + \cdots + l_0 v + \cdots + l_n c^n, \quad m, n \geq 0.$$

Suppose that $l_{-i} \neq 0$ for some index $i > 0$, and let m_0 be the maximum such index. Then

$$(v+c)P(c, c^*) = l_{-m_0}(c^*)^{m_0} + \text{ terms of higher degree } = v.$$

Thus we must have that $l_{-m_0} = 0$, a contradiction. Thus $l_{-i} = 0$ for all $i > 0$. Similarly, we can show that $l_i = 0$ for all $i > 0$. Thus $P(c, c^*) = l_0 v$, and so $v = (v+c)l_0 v = l_0(v+c)$, which is impossible since $\deg(v) = 0$ but $\deg(l_0(v+c)) = \deg(c) > 0$. Thus we have obtained our contradiction, proving that $L_K(E)$ is simple and completing the proof. □

Example 2.3.2. We now apply Theorem 2.3.1 to some of the Leavitt path algebras introduced in Section 2.1.

(i) *The finite line graph M_n.* For every $n \in \mathbb{N}$, M_n has no cycles, so trivially condition (ii) of Theorem 2.3.1 is satisfied. Furthermore, suppose that H is a nonempty hereditary saturated subset of E^0, so that $v_i \in H$ for some $i = 1, \dots, n$. Then, by the hereditary nature of H, we must have $v_{i+1}, \dots, v_n \in H$. Furthermore, by the saturated nature of H we must have $v_{i-1} \in H$, and thus inductively $v_{i-2}, \dots, v_1 \in H$.

Therefore $H = (M_n)^0$ and so condition (ii) is satisfied. Thus $L_K(M_n) \cong \mathbb{M}_n(K)$ is simple for all $n \in \mathbb{N}$, which agrees with the result given in Lemma 1.1.10.

(ii) *The single loop graph* R_1. The single loop in R_1 forms a cycle without an exit, so that condition (ii) is not satisfied and thus $L_K(R_1) \cong K[x, x^{-1}]$ is not simple.

(iii) *The rose with n leaves* R_n. Every edge $e_i \in (R_n)^1$ is a cycle, and if $n \geq 2$ then e_i has an exit, since any other edge is an exit. This satisfies condition (ii). Furthermore, condition (i) is trivially satisfied as R_n only has one vertex, so that the only nonempty subset of $(R_n)^0$ is $(R_n)^0$ itself. Thus $L_K(R_n) \cong L(1, n)$ is simple for all $n \geq 2$.

(iv) *The infinite clock graph* C_∞. In this case, for any radial vertex v_i we have that $\{v_i\}$ is a hereditary saturated subset of $(C_\infty)^0$, and so $L_K(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus K I_{22}$ is not simple.

This next corollary follows directly from Theorem 2.3.1 and Proposition 1.4.10, and offers an alternative set of conditions that are equivalent to $L_K(E)$ being simple.

Corollary 2.3.3. *Let E be an arbitrary graph. The Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies the following conditions:*

(i) every cycle in E has an exit,

(ii) E is cofinal, and

(iii) for every singular vertex $u \in E^0$, we have $v \geq u$ for all $v \in E^0$.

For a given graph E, we define the set

$$V_1 := \{v \in E^0 : |\operatorname{CSP}(v)| = 1\}.$$

We say that E satisfies **Condition (K)** if $V_1 = \emptyset$. In other words, E satisfies Condition (K) if no vertex in E^0 is the base of precisely one closed simple path.

The following lemma was first given for row-finite graphs in [AA2, Lemma 7] and then extended to arbitrary graphs in [AA3, Lemma 4.1].

Lemma 2.3.4. *Let E be an arbitrary graph. If $L_K(E)$ is simple, then E satisfies Condition (K).*

Proof. Suppose that $L_K(E)$ is simple, and suppose there exists a $v \in E^0$ such that $CSP(v) = \{p\}$. If p is not a cycle, it is easy to see that there exists a cycle based at v whose edges are a subset of the edges of p, contradicting the fact that $CSP(v) = \{p\}$. Thus p is a cycle and so, by condition (ii) of Theorem 2.3.1, there must exist an exit e for p.

Let A be the set of all vertices in p. Now $r(e) \notin A$, for otherwise we would have another closed simple path based at v distinct from p. Let $X = \{r(e)\}$ and let $\bar{X}$ be the hereditary saturated closure of X. Recall the definition of $G_n(X)$ from Lemma 1.4.9. Since $L_K(E)$ is simple, by condition (i) of Theorem 2.3.1 we have $\bar{X} = E^0$, and so we can find an $n \in \mathbb{N}$ such that

$$n = \min\{m : A \cap G_m(X) \neq \emptyset\}.$$

Let $w \in A \cap G_n(X)$ and suppose that $n > 0$. By the minimality of n, we have $w \notin G_{n-1}(X)$. Thus, by the definition of $G_n(X)$, w must be a regular vertex and $r(s^{-1}(w)) \subseteq G_{n-1}(X)$, i.e. w emits edges only into $G_{n-1}(X)$. Since w is a vertex in p, there must exist an edge f such that $s(f) = w$ and $r(f) \in A$. Thus $r(f) \in A \cap G_{n-1}(X)$, contradicting the minimality of n. So we must have $n = 0$, and therefore $w \in G_0(X) = T(r(e))$ (by definition). This means there is a path q from $r(e)$ to w. Since w is in the cycle p, and e is an exit for p, there must also be a path p' from w to $r(e)$, and so $p'q$ is a cycle based at w. However, this implies that $|CSP(v)| \geq 2$, a contradiction. □

The following useful result regarding infinite emitters in simple Leavitt path algebras is from [AA3, Lemma 4.2].

Lemma 2.3.5. *Let E be an arbitrary graph such that $L_K(E)$ is simple. If $z \in E^0$ is an infinite emitter, then* $\mathrm{CSP}(z) \neq \emptyset$*. In particular, if $L_K(E)$ is simple and E is acyclic, then E must be row-finite.*

Proof. Let $z \in E^0$ be an infinite emitter, and let $e \in s^{-1}(z)$. Since $L_K(E)$ is simple, by Corollary 2.3.3 (iii) we have that $r(e) \geq z$. Thus there is a closed simple path p

based at z and so $\mathrm{CSP}(z) \neq \emptyset$. Furthermore, it is easy to see that there is a cycle based at z made up of a subset of edges of p. Thus any graph E which is acyclic and for which $L_K(E)$ is simple cannot contain any infinite emitters. $\square$

The following result is from [AA2, Lemma 8].

Lemma 2.3.6. *If R is a directed union of a chain of finite-dimensional subalgebras, then R contains no infinite idempotents. In particular, R is not purely infinite.*

Proof. Suppose that R contains an infinite idempotent e. Then, by Proposition 1.2.6, there exists an idempotent $f \in R$ and elements $x, y \in R$ such that $e = xy$, $f = yx$ and $fe = ef = f \neq e$. Since R is the directed union of a chain of finite-dimensional subalgebras, the elements e, f, x, y must be contained in a finite-dimensional subalgebra S of R. Thus, applying Proposition 1.2.6 again we have that e is an infinite idempotent in S. Therefore $eS = A_1 \oplus B_1$, where $A_1 \neq \{0\}$, and there exists an isomorphism $\phi : eS \to B_1$. Define $\phi(A_1) = A_2$ and $\phi(B_1) = B_2$. Thus $B_1 = \phi(eS) = \phi(A \oplus B) = A_2 \oplus B_2$. Since $A_1 \neq \{0\}$ and ϕ is an isomorphism, $A_2 \neq \{0\}$ and so B_2 is properly contained in B_1. Once again, defining $\phi(A_2) = A_3$ and $\phi(B_2) = B_3$, we have $\phi(B_1) = B_2 = A_3 \oplus B_3$. By the same logic as above, B_3 is properly contained in B_2. Thus, repeating the process, we have an infinitely decreasing chain of right ideals

$$B_1 \supset B_2 \supset B_3 \supset \cdots$$

and so $eS = A_1 \oplus B_1 = A_1 \oplus A_2 \oplus B_2 = A_1 \oplus A_2 \oplus A_3 \oplus B_3 = \cdots$, contradicting the fact that S is finite-dimensional. $\square$

Recall that a ring R is locally matricial if $R = \varinjlim_{i \in I} R_i$, where $\{R_i : i \in I\}$ is an ascending chain of rings and each R_i is isomorphic to a finite direct sum of finite-dimensional matrix rings over K. Thus Lemma 2.3.6 leads immediately to the following corollary.

Corollary 2.3.7. *Let R be a ring. If R is locally matricial, then R is not purely infinite.*

The following proposition is from [AA2, Proposition 9]. Though the result is given there in a row-finite context, the proof still holds for arbitrary graphs.

Proposition 2.3.8. *Let E be an arbitrary graph. Suppose there exists a vertex $w \in E^0$ with the property that there are no closed simple paths based at any vertex $v \in T(w)$. Then the corner algebra $wL_K(E)w$ is not purely infinite.*

Proof. Define a new graph $H = (H^0, H^1, r, s)$, where $H^0 = T(w)$, $H^1 = s^{-1}(H^0)$ and r and s are the functions r_E and s_E restricted to the set H^1. To show this is a well-defined graph, it is enough to show that $r(s^{-1}(H^0)) \subseteq H^0$. Take a vertex $z \in H^0$ that is not a sink, and an edge e such that $s(e) = z$. Since $z \in T(w)$, we have $r(e) \in T(w) = H^0$, as required.

To show that $L_K(H)$ is a subalgebra of $L_K(E)$, we must show that the Leavitt path algebra relations hold in $L_K(H)$. It is clear that the first three relations hold; to show that the (CK2) relation holds, suppose that v is a regular vertex in H. Then v must be a regular vertex in E, and furthermore $s_H^{-1}(v) = s_E^{-1}(v) \subseteq H^1$, so the (CK2) relation holds in $L_K(H)$.

Since there are no closed simple paths based at any vertex $v \in T(w)$, H must be acyclic. Thus, by Theorem 4.2.3[2] $L_K(H)$ is locally matricial, and so by Corollary 2.3.7 $L_K(H)$ is not purely infinite. Since $wL_K(H)w$ is a subring of $L_K(H)$, and $L_K(H)$ does not contain any infinite idempotents, then by Corollary 1.2.7 $wL_K(H)w$ it cannot contain any infinite idempotents and is therefore not purely infinite.

Finally, we show that $wL_K(H)w = wL_K(E)w$. Let $\alpha = \sum_i k_i p_i q_i^*$ be an arbitrary element of $L_K(E)$, where $k_i \in K$ and $p_i, q_i \in E^*$. Then $w\alpha w = \sum_j k_{i_j} p_{i_j} q_{i_j}^*$, where $s(p_{i_j}) = w = s(q_{i_j})$. Thus $p_{i_j}, q_{i_j} \in L_K(H)$ and so $wL_K(E)w \subseteq wL_K(H)w$. Thus $wL_K(H)w = wL_K(E)w$ and so $wL_K(E)w$ is not purely infinite, as required. $\square$

We now come to the main proof of this section. This was first given for row-finite graphs in [AA2, Theorem 11] and then extended to arbitrary graphs in [AA3, Theorem 4.3]. It is here that we can apply Theorem 1.3.19, which we presented in Section 1.3.

[2]The proof of this theorem is independent to any results in this section.

Theorem 2.3.9. *Let E be an arbitrary graph. Then $L_K(E)$ is purely infinite simple if and only if E satisfies the following conditions:*

(i) The only hereditary saturated subsets of E^0 are $\emptyset$ and E^0,

(ii) every cycle in E has an exit, and

(iii) for every vertex $v \in E^0$, there is a vertex $u \in T(v)$ such that u is the base of a cycle.

Proof. Suppose that conditions (i), (ii) and (iii) hold. Theorem 2.3.1 tells us immediately that $L_K(E)$ is simple. Thus, to show that $L_K(E)$ is purely infinite, by Theorem 1.3.19 it suffices to show that $L_K(E)$ is not a division ring and that for any nonzero pair of elements $x, y \in L_K(E)$ there exist $s, t \in L_K(E)$ such that $sxt = y$. Together, conditions (ii) and (iii) show there exists at least one cycle with an exit in E, and thus there must exist two distinct edges e_1 and e_2 in E^1. Since $e_1^* e_2 = 0$, $L_K(E)$ has zero divisors and therefore cannot be a division ring.

Now let x, y be a pair of nonzero elements in $L_K(E)$. Since E contains no cycles without exits, by applying Proposition 2.2.11 we can find elements $a, b \in L_K(E)$ such that $axb = u$, where $u \in E^0$. By condition (iii), u connects to some vertex v at the base of a cycle c. Thus either $u = v$ or there is a path $p \in E^*$ with $s(p) = u$ and $r(p) = v$. By choosing $a' = b' = u$ in the former case, or $a' = p^*, b' = p$ in the latter, we have elements $a', b' \in L_K(E)$ such that $a'ub' = v$.

Since c is a closed simple path based at v and $L_K(E)$ is simple, Lemma 2.3.4 tells us there must be at least one other closed simple path q based at v with $q \neq c$. For each $m \in \mathbb{N}$, let $d_m = c^{m-1}q$. Since c cannot be a subpath of q, and vice versa, we have $c^*q = 0 = q^*c$. Using that $c^*c = v$ and assuming that $m > n$, we have $d_m^* d_n = (q^*(c^*)^{m-1})(c^{n-1}q) = q^*(c^*)^{m-n}q = 0$. Similarly, $d_m^* d_n = 0$ for $n > m$. For the case $m = n$, we have $d_m^* d_n = q^*vq = v$. Thus $d_m^* d_n = \delta_{m,n} v$ for all $m, n \in \mathbb{N}$.

Since $L_K(E)$ is simple, we have $\langle v \rangle = L_K(E)$, and so for an arbitrary $w \in E^0$ we can write $w = \sum_{i=1}^{t} a_i v b_i$ for some $a_i, b_i \in L_K(E)$. Let $a_w = \sum_{i=1}^{t} a_i d_i^*$ and

$b_w = \sum_{j=1}^t d_j b_j$. Using the fact that $d_i^* d_j = \delta_{i,j}\, v$, we have

$$a_w v b_w = \left(\sum_{i=1}^t a_i d_i^*\right) v \left(\sum_{j=1}^t d_j b_j\right) = \sum_{i=1}^t a_i v b_i = w.$$

In other words, for any vertex $w \in E^0$, we can find $a_w, b_w \in L_K(E)$ for which $a_w v b_w = w$.

By Lemma 2.1.12, we can find a finite subset of vertices $X = \{v_1, \ldots, v_s\}$ for which $e = \sum_{i=1}^s v_s$ is a local unit for y, so that $ey = e = ye$. Let a_{v_i}, b_{v_i} be elements for which $a_{v_i} v b_{v_i} = v_i$, for each $v_i \in X$. Let $s' = \sum_{i=1}^s a_{v_i} d_i^*$ and $t' = \sum_{j=1}^s d_j b_{v_j}$. This gives

$$s'vt' = \left(\sum_{i=1}^s a_{v_i} d_i^*\right) v \left(\sum_{j=1}^s d_j b_{v_j}\right) = \sum_{i=1}^s a_{v_i} v b_{v_i} = \sum_{i=1}^s v_s = e.$$

In summary, we have found elements $a, b, a', b', s', t' \in L_K(E)$ for which $axb = u$, $a'ub' = v$ and $s'vt' = e$. Let $s = s'a'a$ and $t = bb't'y$. Thus we have $sxt = (s'a'a)x(bb't'y) = s'a'(axb)b't'y = s'(a'ub')t'y = (s'vt')y = ey = y$, and so $L_K(E)$ is purely infinite.

Conversely, suppose that $L_K(E)$ is purely infinite simple. Again, conditions (i) and (ii) follow directly from the fact that $L_K(E)$ is simple (by Theorem 2.3.1). If condition (iii) does not hold, then there exists a vertex $w \in E^0$ such that no vertex $v \in T(w)$ is the base of a cycle. Since a cycle can be formed from a subset of edges of any closed path, there cannot be any closed simple path based at any vertex $v \in T(w)$ either. Thus, by Proposition 2.3.8, $wL_K(E)w$ is not purely infinite. Finally, Proposition 1.3.18 gives that $L_K(E)$ is not purely infinite, a contradiction. □

The following proposition from [AA3, Theorem 4.4] shows that, for any graph E for which $L_K(E)$ is simple, we have the following dichotomy.

Proposition 2.3.10. *Let E be an arbitrary graph. If E is simple, then either*

(i) $L_K(E)$ is purely infinite simple and E contains a cycle, or

(ii) $L_K(E)$ is locally matricial and E is acyclic.

Proof. If E is acyclic then Theorem 4.2.3 tells us that $L_K(E)$ is locally matricial. Otherwise, suppose E contains a cycle c. By Corollary 2.3.3 we have that E is cofinal, and so every vertex connects to the infinite path c^∞. Thus every vertex connects to a cycle, satisfying condition (iii) of Theorem 2.3.9. Since $L_K(E)$ is simple, conditions (i) and (ii) of Theorem 2.3.9 are satisfied (by Theorem 2.3.1), and thus $L_K(E)$ is purely infinite simple. □

Example 2.3.11. Of the Leavitt path algebras determined to be simple in Example 2.3.2, we now use Proposition 2.3.10 determine which of these are purely infinite simple.

(i) *The finite line graph M_n.* Since M_n is acyclic for all $n \in \mathbb{N}$, $L_K(M_n)$ must be locally matricial for all $n \in \mathbb{N}$. This is no surprise, considering that $L_K(M_n) \cong \mathbb{M}_n(K)$.

(ii) *The rose with n leaves R_n.* Since R_n contains n cycles for each $n \in \mathbb{N}$, $L_K(R_n) \cong L(1,n)$ must be purely infinite simple for all $n \geq 2$.

2.4 Desingularisation

Recall that a vertex $v \in E^0$ is said to be singular if v is either a sink or an infinite emitter. In this section we look at the process of 'desingularisation', in which we construct from a given graph E a new graph that contains no singular vertices; in other words, a graph that is row-finite and has no sinks. This concept was originally used in the C^*-algebra context in [BPRS]. The significance of the desingularisation process is illustrated in Theorem 2.4.5, in which we show that the Leavitt path algebra of a graph E is Morita equivalent to the Leavitt path algebra of its desingularisation.

Definition 2.4.1. Let E be a countable graph. A **desingularisation** of E is a graph F constructed from E that contains no singular vertices. We construct F by 'adding a tail' to each sink and infinite emitter in E^0. If v_0 is a sink in E, then we attach an infinite line graph at v_0 like so:

$$\bullet^{v_0} \longrightarrow \bullet^{v_1} \longrightarrow \bullet^{v_2} \longrightarrow \bullet^{v_3} \dashrightarrow$$

If v_0 is an infinite emitter in E, then we first list the edges $e_1, e_2, e_3, \ldots \in s^{-1}(v_0)$ (noting that the countable property of E allows us to list the edges in this way). Then we again attach an infinite line graph at v_0:

$$\bullet^{v_0} \xrightarrow{f_1} \bullet^{v_1} \xrightarrow{f_2} \bullet^{v_2} \xrightarrow{f_3} \bullet^{v_3} \dashrightarrow$$

We then remove the edges in $s^{-1}(v_0)$ and add an edge g_j from v_{j-1} (in the infinite line graph) to $r(e_j)$ for each $e_j \in s^{-1}(v_0)$. Effectively, we are removing each e_j and replacing it with the path $f_1 f_2 \ldots f_{j-1} g_j$ of length j. Note that both e_j and $f_1 f_2 \ldots f_{j-1} g_j$ have source v_0 and range $r(e_j)$.

Note also that the desingularisation of a graph may not necessarily be unique: differences may arise depending on the way in which we choose to order the edges in $s^{-1}(v_0)$ (in the case that v_0 is an infinite emitter).

We now give two examples of the desingularisation process. In these examples the desingularisation is in fact unique (up to isomorphism), due to the symmetry of the graphs.

Example 2.4.2. Consider the infinite edges graph

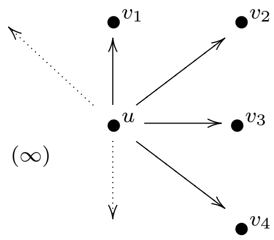
$$E_\infty : \qquad \bullet^{u} \xrightarrow{(\infty)} \bullet^{v}$$

Note that u is an infinite emitter and v is a sink, so we add a tail at both vertices in the desingularisation process. Furthermore, each edge emitted by u has range v, and so we obtain the desingularisation

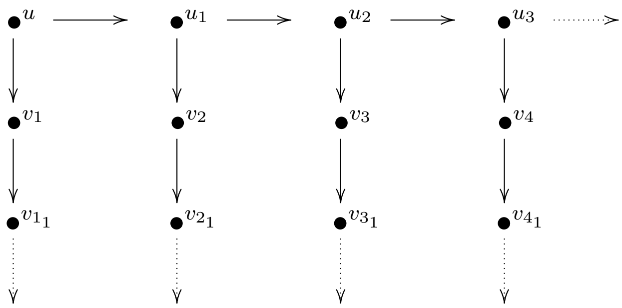
$$\begin{tikzcd}
\bullet^{u} \arrow[r] \arrow[d] & \bullet^{u_1} \arrow[r] \arrow[dl] & \bullet^{u_2} \arrow[r] \arrow[dll] & \bullet^{u_3} \arrow[r, dotted] \arrow[dlll] & {} \\
\bullet^{v} \arrow[r] & \bullet^{v_1} \arrow[r] & \bullet^{v_2} \arrow[r] & \bullet^{v_3} \arrow[r, dotted] & {}
\end{tikzcd}$$

Example 2.4.3. Recall the infinite clock graph

C_∞ :



Again, each vertex in this graph is a singularity, resulting in an infinite number of infinite tails. Thus the desingularisation of C_∞ looks like



The following proposition is from [AA3, Proposition 5.1].

Proposition 2.4.4. *Let E be a countable graph and let F be a desingularisation of E. Then there exists a monomorphism of K-algebras from $L_K(E)$ to $L_K(F)$.*

Proof. We define a map $\phi : L_K(E) \to L_K(F)$ on the generators of E as follows. First, we define $\phi(v) = v$ for all $v \in E^0$. Note that this is valid since no vertices are removed in the construction of F, only added. Next, if $s(e)$ is a regular vertex then we define $\phi(e) = e$ and $\phi(e^*) = e^*$. Furthermore, if $e = e_j \in s^{-1}(v_0)$, where v_0 is an infinite emitter, then we define $\phi(e_j) = f_1 f_2 \dots f_{j-1} g_j$ and $\phi(e_j^*) = g_j^* f_{j-1}^* \dots f_2^* f_1^*$, where $f_1, f_2, \dots, f_{j-1}$ and g_j are as in Definition 2.4.1.

Expand ϕ linearly and multiplicatively. In order to check that ϕ is a well-defined K-homomorphism, we must check that ϕ preserves the Leavitt path algebra relations on $L_K(E)$. Clearly the (A1) relation is preserved, since each vertex in $L_K(E)$ is mapped to itself in $L_K(F)$. Similarly, the (A2) relations are easily seen to be preserved, since $s(\phi(e)) = s(e)$ and $r(\phi(e)) = r(e)$ for all $e \in E^1$ (as noted in Definition 2.4.1). To check the (CK1) relation, note that the only nontrivial situation arises when $s(e_i) = s(e_j) = v_0$, where v_0 is an infinite emitter. In the case that $i = j$, we have $\phi(e_i^*)\phi(e_i) = (g_i^* f_{i-1}^* \dots f_2^* f_1^*)(f_1 f_2 \dots f_{i-1} g_i) = r(e_i) = \phi(r(e_i))$. On the other hand, if $i \neq j$ then $\phi(e_i^*)\phi(e_j) = (g_i^* f_{i-1}^* \dots f_2^* f_1^*)(f_1 f_2 \dots f_{j-1} g_j) = 0$, since $f_1 f_2 \dots f_{i-1} g_i$ and $f_1 f_2 \dots f_{j-1} g_j$ are not subpaths of each other. Thus the (CK1) relation is preserved. Finally, since regular vertices (and the edges they emit) are unchanged by ϕ, and we only evaluate the (CK2) relation at regular vertices, it is clear that the (CK2) relation is preserved. Thus ϕ is a well-defined K-homomorphism, as required.

Finally, we show that ϕ is a monomorphism. Suppose that $x \in \ker(\phi)$ and $x \neq 0$. By Proposition 2.2.11 there exist $y, z \in L_K(E)$ for which either $yxz = v \in E^0$ or $yxz = \sum_{i=-m}^{n} k_i c^i \neq 0$, where $m, n \in \mathbb{N}_0$, $k_i \in K$ and c is a cycle without exits in E. Since $\ker(\phi)$ is a two-sided ideal of $L_K(E)$, we have $yxz \in \ker(\phi)$. By definition, $\ker(\phi)$ contains no vertices (since $\phi(v) = v$ for all $v \in E^0$), and so we must have $\sum_{i=-m}^{n} k_i c^i \in \ker(\phi)$. Thus $\phi(\sum_{i=-m}^{n} k_i c^i) = \sum_{i=-m}^{n} k_i \phi(c)^i = 0$. Note that ϕ sends paths of length t to paths of length greater than or equal to t, and that c and $\phi(c)$ must have the same source and range. Furthermore, $\phi(c)$ cannot pass through any vertex more than once (from the definition of ϕ) and so $\phi(c)$ is a cycle in F. Since $L_K(F)$ is graded, this implies that each term $k_i \phi(c)^i = 0$, and thus each $k_i = 0$, which is impossible since $\sum_{i=-m}^{n} k_i c^i \neq 0$. Thus $\ker(\phi) = \{0\}$ and so ϕ is a monomorphism, as required. □

Proposition 2.4.4 leads to the following powerful result from [AA3, Theorem 5.2]. Here we have greatly expanded the proof to clarify the arguments and results used at each step.

Theorem 2.4.5. *Let E be a countable graph and let F be a desingularisation of E. Then the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are Morita equivalent.*

Proof. We begin by labelling the vertices of E as a sequence $\{v_l\}_{l=1}^{\infty}$. We can form idempotents $t_k := \sum_{l \leq k} v_l$ for each $k \in \mathbb{N}$. Note that for any subset $X \subseteq L_K(E)$ there exists a t_k such that $t_k x = x = x t_k$ for all $x \in X$ (see the proof of Lemma 2.1.12), and so $\{t_k : k \in \mathbb{N}\}$ forms a set of local units for $L_K(E)$. (Note that if E^0 is finite then t_k is simply the identity for $L_K(E)$ for all $k \geq |E^0|$, by Lemma 2.1.12.)

Let $t = t_k$ for an arbitrary $k \in \mathbb{N}$. We show that $tL_K(E)t \cong tL_K(F)t$. Recall the monomorphism $\phi : L_K(E) \to L_K(F)$ from Proposition 2.4.4, and consider the restriction map $\phi|_{tL_K(E)t} : tL_K(E)t \to L_K(F)$. Since $\phi(v) = v$ for each $v \in E^0$, we have $\phi(t) = t$, and thus for any $x = txt \in tL_K(E)t$ we have $\phi(txt) = t\phi(x)t$. Now if $t\phi(x)t = t\phi(y)t$ for some $y \in tL_K(E)t$, then clearly $\phi(x) = \phi(y)$ and so $x = y$, since ϕ is a monomorphism. Thus $\phi|_{tL_K(E)t}$ is a monomorphism from $tL_K(E)t$ to $tL_K(F)t$.

To show that $\phi|_{tL_K(E)t}$ is an epimorphism, consider an arbitrary element $x \in tL_K(F)t$. Then $x = \sum_{i=1}^{n} k_i p_i q_i^*$, where $k_i \in K$ and p_i, q_i are paths in F with $r(p_i) = r(q_i)$ and $s(p_i), s(q_i) \in \{v_l : l \leq k\}$ for each $i \in \{1, \ldots, n\}$. Suppose that p is a path in F with $s(p) \in \{v_l : l \leq k\}$. If $p = p_1 \ldots p_n$, where each p_i is an edge from the original graph E, then $p = \phi(p_1 \ldots p_n)$. If $p = f_1 f_2 \ldots f_{j-1} g_j$ (where the f_i and g_j are as defined in Definition 2.4.1), then $p = \phi(e_j)$, where $e_j \in s^{-1}(v_0)$ for some infinite emitter $v_0 \in E^0$. Furthermore, if p is a concatenation of two such paths, then clearly $p \in \mathrm{Im}(\phi)$. For all three cases above, clearly we also have $p^* \in \mathrm{Im}(\phi)$.

The final possible form for p is $p = p_1 \ldots p_n f_1 \ldots f_j$, (with $n \geq 0$), where the f_i form part of an infinite tail from either a sink or an infinite emitter, and each p_i is an edge from the original graph E. Let $s(f_1) = v_0$ and $r(f_j) = v_j$. By the desingularisation definition, any path q in F with $r(q) = v_j$ must be of the form $q_1 \ldots q_m f_1 \ldots f_j$ (with $m \geq 0$), where each q_i is an edge from the original graph E. Thus $pq^* = \phi(p_1 \ldots p_n)(f_1 \ldots f_j)(f_j^* \ldots f_1^*)\phi(q_m^* \ldots q_1^*)$ and so it suffices to show that $(f_1 \ldots f_j)(f_j^* \ldots f_1^*)$ is in the image of ϕ.

If v_0 is a sink, then each v_i along the infinite tail based at v_0 emits precisely one edge, namely f_{i+1}. Thus applying the (CK2) relation at v_i gives $v_i = f_{i+1} f_{i+1}^*$, and so

$$\begin{aligned}
(f_1 \dots f_{j-1} f_j)(f_j^* f_{j-1}^* \dots f_1^*) &= f_1 \dots f_{j-1} v_{j-1} f_{j-1}^* \dots f_1^* \\
&= f_1 \dots f_{j-1} (f_{j-1}^* \dots f_1^* \\
&= f_1 \dots) v_{j-2} \dots f_1^* \\
&\vdots \\
&= f_1 f_1^* \\
&= v_0,
\end{aligned}$$

and $v_0 = \phi(v_0)$, as required.

Now suppose that v_0 is an infinite emitter. Then each v_i along the infinite tail based at v_0 emits precisely two edges, namely f_{i+1} and g_{i+1}. Thus applying the (CK2) relation at v_i gives $v_i = f_{i+1} f_{i+1}^* + g_{i+1} g_{i+1}^*$, and so

$$\begin{aligned}
(f_1 \dots f_{j-1} f_j)(f_j^* f_{j-1}^* \dots f_1^*) &= f_1 \dots f_{j-1} (v_{j-1} - g_j g_j^*) f_{j-1}^* \dots f_1^* \\
&= (f_1 \dots f_{j-1})(f_{j-1}^* \dots f_1^*) \\
&\qquad - (f_1 \dots f_{j-1} g_j)(g_j^* f_{j-1}^* \dots f_1^*).
\end{aligned}$$

Repeating this expansion eventually gives

$$\begin{aligned}
(f_1 \dots f_{j-1} f_j)(f_j^* f_{j-1}^* \dots f_1^*) &= f_1 f_1^* - \sum_{i=2}^{j} (f_1 \dots f_{i-1} g_i)(g_i^* f_{i-1}^* \dots f_1^*) \\
&= v_0 - g_1 g_1^* - \sum_{i=2}^{j} (f_1 \dots f_{i-1} g_i)(g_i^* f_{i-1}^* \dots f_1^*) \\
&= \phi\left(v_0 - e_1 e_1^* - \sum_{i=2}^{j} e_i e_i^* \right)
\end{aligned}$$

and we are done. (Note that the inverse images of all of these paths also have source in the set $\{v_l : l \leq k\}$, and thus are indeed contained in $tL_K(E)t$.) Therefore $\phi|_{tL_K(E)t}$ is an isomorphism of K-algebras and so $tL_K(E)t \cong tL_K(F)t$.

From the definition of t_k, we can view $t_k L_K(E) t_k$ as the set of all elements in $L_K(E)$ generated by paths p with $s(p) \in \{v_l : l \leq k\}$. Thus we have $t_k L_K(E) t_k \subseteq t_{k+1} L_K(E) t_{k+1}$ (and $t_k L_K(F) t_k \subseteq t_{k+1} L_K(F) t_{k+1}$) for each $k \in \mathbb{N}$. For every pair

$i, j \in \mathbb{N}$ with $i \leq j$, let φ_{ij} be the inclusion map from $t_iL_K(E)t_i$ to $t_jL_K(E)t_j$ and let $\bar{\varphi}_{ij}$ be the inclusion map from $t_iL_K(F)t_i$ to $t_jL_K(F)t_j$. For such a pair i, j it is easy to see that $t_jt_i = t_i = t_it_j$, and so for any $x = t_ixt_i \in t_iL_K(E)t_i$ we have $t_jxt_j = t_j(t_ixt_i)t_j = t_ixt_i = x$. Thus we can view the inclusion map φ_{ij} as mapping $t_ixt_i \mapsto t_jxt_j$ (and similarly for $\bar{\varphi}_{ij}$). For ease of notation, let $\phi|_{t_kL_K(E)t_k} = \phi_k$ for all $k \in \mathbb{N}$. Thus for any $x = t_ixt_i \in t_iL_K(E)t_i$ we have

$$\bar{\varphi}_{ij}\phi_i(t_ixt_i) = \bar{\varphi}_{ij}(t_i\phi_i(x)t_i) = t_j\phi_i(x)t_j = \phi_j(t_jxt_j) = \phi_j\varphi_{ij}(t_ixt_i),$$

and so $\bar{\varphi}_{ij}\phi_i = \phi_j\varphi_{ij}$; that is, the following diagram commutes

$$\begin{tikzcd}
t_iL_K(E)t_i \arrow[r, "\phi_i"] \arrow[d, "\varphi_{ij}"'] & t_iL_K(F)t_i \arrow[d, "\bar{\varphi}_{ij}"] \\
t_jL_K(E)t_j \arrow[r, "\phi_j"'] & t_jL_K(E)t_j
\end{tikzcd}$$

for all $i, j \in \mathbb{N}$ with $i \leq j$. (Similarly, since ϕ_i is an isomorphism for all $i \in \mathbb{N}$, we also have $\varphi_{ij}\phi_i^{-1} = \phi_j^{-1}\bar{\varphi}_{ij}$.)

Clearly $(t_iL_K(E)t_i, \varphi_{ij})_{\mathbb{N}}$ and $(t_iL_K(F)t_i, \bar{\varphi}_{ij})_{\mathbb{N}}$ are direct systems of rings. Since these are both ascending chains of rings, the direct limits $\varinjlim_{i\in\mathbb{N}} t_iL_K(E)t_i$ and $\varinjlim_{i\in\mathbb{N}} t_iL_K(F)t_i$ exist (see Appendix A). For ease of notation, we set

$$R_E = \varinjlim_{i\in\mathbb{N}} t_iL_K(E)t_i \quad \text{and} \quad R_F = \varinjlim_{i\in\mathbb{N}} t_iL_K(F)t_i.$$

For each $i \in \mathbb{N}$, let φ_i be the map from $t_iL_K(E)t_i$ to R_E and let $\bar{\varphi}_i$ be the map from $t_iL_K(F)t_i$ to R_F as defined in Definition A.1.1.

Now, for each $i \in \mathbb{N}$ there exists a ring homomorphism $\bar{\varphi}_i\phi_i : t_iL_K(E)t_i \to R_F$, and furthermore, for $i \leq j$,

$$(\bar{\varphi}_j\phi_j)\varphi_{ij} = \bar{\varphi}_j(\phi_j\varphi_{ij}) = \bar{\varphi}_j(\bar{\varphi}_{ij}\phi_i) = (\bar{\varphi}_j\bar{\varphi}_{ij})\phi_i = \bar{\varphi}_i\phi_i.$$

Thus, by condition (ii) of Definition A.1.1, there exists a unique ring homomorphism $\mu : R_E \to R_F$ for which $\bar{\varphi}_i\phi_i = \mu\varphi_i$ for all $i \in \mathbb{N}$. By a similar argument, there exists a unique ring homomorphism $\mu' : R_F \to R_E$ for which $\varphi_i\phi_i^{-1} = \mu'\bar{\varphi}_i$ for all $i \in \mathbb{N}$. This situation is illustrated in the following commutative diagram, which holds for each pair $i, j \in \mathbb{N}$ with $i \leq j$:

$$\begin{tikzcd}
t_iL_K(E)t_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddd, bend right, "\phi_i"'] & & t_jL_K(E)t_j \arrow[dl, "\varphi_j"'] \arrow[ddd, bend left, "\phi_j"] \\
& R_E \arrow[d, bend right, "\mu"'] & \\
& R_F \arrow[u, bend right, "\mu'"'] & \\
t_iL_K(F)t_i \arrow[rr, "\bar{\varphi}_{ij}"'] \arrow[ur, "\bar{\varphi}_i"] \arrow[uuu, bend right, "\phi_i^{-1}"'] & & t_jL_K(F)t_j \arrow[ul, "\bar{\varphi}_j"'] \arrow[uuu, bend left, "\phi_j^{-1}"]
\end{tikzcd}$$

In summary, there exist unique ring homomorphisms $\mu : R_E \to R_F$ and $\mu' : R_F \to R_E$ that satisfy the following equations:

$$\bar{\varphi}_i\phi_i = \mu\varphi_i \quad \text{and} \quad \varphi_i\phi_i^{-1} = \mu'\bar{\varphi}_i.$$

From the second equation we have $\varphi_i = \mu'\bar{\varphi}_i\phi_i = \mu'\mu\varphi_i$ (substituting from the first equation) for all $i \in \mathbb{N}$, and so, by appealing to the uniqueness given in Definition A.1.1 (ii), we have $\mu'\mu = 1_{R_E}$. Similarly, the first equation gives $\bar{\varphi}_i = \mu\varphi_i\phi_i^{-1} = \mu\mu'\bar{\varphi}_i$, and so $\mu\mu' = 1_{R_F}$. Thus $\mu' = \mu^{-1}$ and we have $R_E \cong R_F$. However, as noted above, the set $\{t_k : k \in \mathbb{N}\}$ forms a set of local units for $L_K(E)$, and furthermore, for each pair $i, j \in \mathbb{N}$ with $i \leq j$ we have $t_i \in t_jL_K(E)t_j$. Thus, by Lemma A.1.2 we have $R_E = L_K(E)$, and so $R_F = \varinjlim_{k\in\mathbb{N}} t_kL_K(F)t_k \cong L_K(E)$.

Now suppose that w_0 is a singular vertex in E and let w_i be any vertex in F contained in the 'infinite tail' added at w_0 in the desingularisation process. Furthermore, let p_i denote the path $f_1 \dots f_i$ from w_0 to w_i in F^*. Define $\pi_i : L_K(F)w_0 \to L_K(F)w_i$ by $x \mapsto xp_i$. It is easy to see that π_i is a left $L_K(F)$-module homomorphism. Furthermore, $L_K(F)w_i$ is projective in $L_K(F)$-Mod, by Proposition 1.2.13. For an arbitrary $y \in L_K(F)w_i$, we have $y = yw_i = yp_i^*p_i = \pi_i(yp_i^*)$, and so π_i is an epimorphism. Thus, by Lemma 1.2.10, $L_K(F)w_i$ is isomorphic to a direct summand of $L_K(F)w_0$ as left $L_K(F)$-modules.

By Lemma 2.1.9, we have $L_K(F) = \bigoplus_{v \in F^0} L_K(F)v$. Furthermore, $L_K(F)$ is a generator for $L_K(F)$-Mod (see Definition 1.3.10). Now, from the above paragraph we have

$$\begin{aligned} L_K(F) &= \bigoplus_{v \in F^0} L_K(F)v \\ &= \left(\bigoplus_{v \in E^0} L_K(F)v \right) \oplus \left(\bigoplus_{w_i \in F^0} L_K(F)w_i \right) \\ &\cong \left(\bigoplus_{v \in E^0} L_K(F)v \right) \oplus \left(\bigoplus_{w_i \in F^0} A_{w_i} \right) \end{aligned}$$

where each $w_i \in F^0$ is contained in an infinite tail based at w_0 (for some singular vertex w_0) and A_{w_i} is a direct summand of $L_K(F)w_0$. For ease of notation, let $H = \bigoplus_{v \in E^0} L_K(F)v$. Then we have that $\bigoplus_{w_i \in F^0} A_{w_i}$ is a direct summand of H, since A_{w_i} is a direct summand of $L_K(F)w_0$ for each singular vertex $w_0 \in E^0$. From the above equation, we have an isomorphism between a subset of $H \oplus H$ and $L_K(F)$, which implies we have an epimorphism from $H \oplus H$ to $L_K(F)$. Since $L_K(F)$ is a generator for $L_K(F)$-Mod, for any $M \in L_K(F)$-Mod there exists an index set I and epimorphism $\tau : L_K(F)^{(I)} \to M$. This induces an epimorphism $\eta : H^{(2I)} = H^{(I)} \oplus H^{(I)} \to M$, and so H is a generator for $L_K(F)$-Mod.

Now, note that we have $L_K(F)t_k = L_K(F)(v_1 + \cdots + v_k) = \bigoplus_{\{v_i : i \leq k\}} L_K(F)v_i$ for each $k \in \mathbb{N}$. Thus is it easy to see that $\varinjlim_{k \in \mathbb{N}} L_K(F)t_k = \bigoplus_{v \in E^0} L_K(F)v = H$. Note that each $L_K(F)t_k$ is projective (by Proposition 1.2.13), is finitely generated (with generating set $\{t_k\}$) and is a direct summand of H. Thus H is a locally projective generator for $L_K(F)$-Mod (see Definition 1.3.14) and so by Proposition 1.3.15 any ring that is isomorphic to $\varinjlim_{k \in \mathbb{N}} \operatorname{End}(L_K(F)t_k)$ must be Morita equivalent to $L_K(F)$.

Finally, by Lemma 1.2.2 we have $\operatorname{End}(L_K(F)t_k) \cong t_k L_K(F) t_k$, and so

$$\varinjlim_{k \in \mathbb{N}} \operatorname{End}(L_K(F)t_k) \cong \varinjlim_{k \in \mathbb{N}} t_k L_K(F) t_k \cong L_K(E).$$

Thus $L_K(F)$ and $L_K(E)$ are Morita equivalent, completing the proof. □

Chapter 3

Socle Theory of Leavitt Path Algebras

In this chapter we define the notion of a socle and give a precise description of the socle of an arbitrary Leavitt path algebra in Section 3.2. Furthermore, we expand this definition to a socle series in Section 3.4, and again describe the socle series of a Leavitt path algebra, applying the concept of a quotient graph introduced in Section 3.3. To begin, we introduce some preliminary ring-theoretic definitions and results.

3.1 Preliminary Results

Definition 3.1.1. Let R be a ring. Recall that L is a minimal left ideal of R if $L \neq 0$ and there exists no left ideal K of R such that $0 \subset K \subset L$. The **left socle of** R, denoted $\mathrm{soc}_l(R)$, is defined to be the sum of the family of minimal left ideals of R (or the zero ideal, if R contains no minimal left ideals). We can define the right socle of R, denoted $\mathrm{soc}_r(R)$, similarly.

It is clear from the definition that $\mathrm{soc}_l(R)$ is a left ideal of R. However, what is slightly less obvious is that it is also a right ideal of R, as the following proposition shows.

Proposition 3.1.2. *For any ring R, $\mathrm{soc}_l(R)$ is a two-sided ideal of R.*

Proof. Since $\mathrm{soc}_l(R)$ is clearly a left ideal of R (since it is the sum of left ideals), it suffices to show that $\mathrm{soc}_l(R)$ is also a right ideal of R. Take an arbitrary nonzero element $s \in \mathrm{soc}_l(R)$ and an arbitrary nonzero $r \in R$. Since $s \in \mathrm{soc}_l(R)$, we can write $s = l_1 + \ldots + l_n$, where each $l_i \in L_i$ and L_i is a minimal left ideal of R. Thus $sr = l_1 r + \ldots + l_n r$, and so it suffices to show that $l_i r \in \mathrm{soc}_l(R)$ for each i.

Take an arbitrary minimal left ideal L_i of R and define $\phi : L_i \to R$ by $\phi(x) = xr$, for all $x \in L_i$. It is easy to see that ϕ is an R-module homomorphism: clearly ϕ is additive, and for any $r' \in R$ and $x \in L_i$ we have $\phi(r'x) = (r'x)r = r'(xr) = r'\phi(x)$.

Since $\ker(\phi)$ is a left ideal contained in L_i and L_i is minimal, then either $\ker(\phi) = L_i$ or $\ker(\phi) = \{0\}$. In the former case, this gives $\phi(L_i) = \{0\}$. In the latter case, ϕ is a monomorphism, and so $\phi : L_i \to \phi(L_i)$ is an isomorphism of left R-modules. Specifically, $\phi(L_i)$ is a minimal left ideal of R. In either case, $\phi(L_i) \subseteq \mathrm{soc}_l(R)$, and thus $xr \in \mathrm{soc}_l(R)$ for every $x \in L_i$. In particular, $l_i r \in \mathrm{soc}_l(R)$ and we are done. $\square$

A similar proof shows that $\mathrm{soc}_r(R)$ is also a two-sided ideal of R.

For a given ring R, a left R-module is **semisimple** if it is the direct sum of simple submodules. If we view R as a left module over itself, then R is semisimple if it is the direct sum of minimal left ideals. Thus we have that $\mathrm{soc}_l(R) = R$ if and only if R is semisimple.

An ideal I is said to be **nilpotent** if there exists a $k \in \mathbb{N}$ such that

$$I^k := \left\{ \sum_{i=1}^{n} x_{i_1} \ldots x_{i_k} : x_{i_j} \in I \text{ for all } i, j \text{ and } n \in \mathbb{Z} \right\} = 0.$$

A ring R is said to be **semiprime** if it contains no nonzero two-sided nilpotent ideals. Furthermore, a ring R is said to be **nondegenerate** if $aRa = 0$ for some $a \in R$ implies that $a = 0$. The following proposition shows that these two concepts are equivalent.

Proposition 3.1.3. *Let R be a ring. Then R is semiprime if and only if R is nondegenerate.*

Proof. Suppose that R is nondegenerate. Let I be a nonzero two-sided ideal of R such that $I^n = 0$ for some $n \in \mathbb{N}$. Let n_0 be the minimum such n and set $J = I^{n_0-1}$. Thus J is a nonzero two-sided ideal and $J^2 = 0$. Let a be an arbitrary element of J. Then, since $Ra \subseteq J$, we have $aRa \subseteq J^2 = 0$. Since R is nondegenerate, $a = 0$ and so $J = 0$, a contradiction. Thus R is semiprime.

Conversely, suppose that R is semiprime and that $aRa = 0$ for some $a \in R$. Recall that RaR is the two-sided ideal given by

$$RaR = \left\{ \sum_{i=1}^{n} r_i a s_i : r_i, s_i \in R, n \in \mathbb{Z} \right\}.$$

Then $(RaR)^2 = (RaR)(RaR) \subseteq R(aRa)R = 0$, and so $RaR = 0$, since R is semiprime. Now let J be the two-sided ideal generated by a, so that

$$J = \left\{ \sum_i r_i a s_i + \sum_j r'_j a + \sum_k a s'_k + ma : r_i, s_i, r'_j, s'_k \in R, m \in \mathbb{Z} \right\}.$$

Then any element of J^3 must be a sum of elements of the form xay, where $x, y \in R$, and so $J^3 \subseteq RaR$ and thus $J^3 = 0$. Since R is semiprime, we have that $J = 0$ and so $a = 0$, since $a \in J$. Thus R is nondegenerate. $\square$

The following proposition shows, somewhat surprisingly, that if R contains no nonzero two-sided nilpotent ideals then it cannot contain any nonzero left or right nilpotent ideals either.

Proposition 3.1.4. *Let R be a ring. Then R is semiprime if and only if R contains no nonzero left (or right) nilpotent ideals.*

Proof. Clearly if R contains no nonzero left (or right) nilpotent ideals then it contains no nonzero two-sided nilpotent ideals and must therefore be semiprime. To prove the converse, suppose that R is semiprime and let I be a nonzero left ideal of R such that $I^n = 0$ for some $n \in \mathbb{N}$. As in the proof of Proposition 3.1.3, we can find a left ideal J such that J is nonzero and $J^2 = 0$. Take an arbitrary element nonzero $x \in J$ and let L be the two-sided ideal generated by x, so that

$$L = \left\{ \sum_i r_i x s_i + \sum_j r'_j x + \sum_k x s'_k + mx : r_i, s_i, r'_j, s'_k \in R, m \in \mathbb{Z} \right\}.$$

Since $L^3 \subseteq RxR$ and $Rx \subseteq J$, we have

$$L^6 \subseteq (RxR)(RxR) \subseteq RxRxR \subseteq J^2R = 0.$$

Thus, since R is semiprime we have that $L = 0$, and so $x = 0$. Since x was an arbitrary element of J, we have that $J = 0$, a contradiction. Thus R contains no nonzero nilpotent left ideals. Similarly, we can show that R contains no nonzero nilpotent right ideals. □

We now move on to describing the general form of minimal left ideals. The following proposition is from [J2, Proposition 3.9.1].

Proposition 3.1.5. *Let D be a minimal left ideal of a ring R. Then either $D^2 = 0$ or D contains an idempotent e such that $D = Re = \{re : r \in R\}$.*

Proof. Suppose that $D^2 \neq 0$. Then there exists $b \in D$ such that $Db \neq 0$. Since Db is a nonzero left ideal contained in D and D is minimal, we have $Db = D$. Now let J be the left annihilator of b in R; that is, $J = \{r \in R : rb = 0\}$. It is clear that J is a left ideal of R and, furthermore, $J \cap D \neq D$, since otherwise we would have $Db = 0$. Since $J \cap D$ is a left ideal contained in D we must therefore have $J \cap D = 0$. Now, $Db = D$ implies that $eb = b$ for some $e \in D$. Thus $b = eb = e^2b$ and so $(e - e^2)b = 0$. Therefore $e - e^2 \in J \cap D = 0$ and so $e = e^2$. Since b is nonzero, e is nonzero, and so Re is a nonzero left ideal contained in D. Thus $Re = D$, as required. Finally, note that $Re = \{re : r \in R\}$ since e is an idempotent. □

Proposition 3.1.5 leads immediately to the following corollary.

Corollary 3.1.6. *Every minimal left ideal of a semiprime ring R is of the form Re, where e is an idempotent in R.*

We can show similarly that every minimal right ideal of a semiprime ring R is of the form eR, where e is an idempotent. Note that the converse is not necessarily true: for a given idempotent e in a semiprime ring R, Re and eR may not be minimal left or right ideals. However, the following proposition from [L1, Lemma 1.19] shows that if one of these is minimal then both are.

Proposition 3.1.7. *Let R be a semiprime ring and let e be an idempotent in R. Then Re is a minimal left ideal if and only if eR is a minimal right ideal.*

Proof. Suppose that Re is a minimal left ideal of R. To prove that eR is a minimal right ideal of R it suffices to show that $e \in aR$ for any nonzero $a \in eR$ (by Lemma 1.1.6). Now, if $a \in eR$ then $a = et = e^2t = ea$ (for some $t \in R$), and thus $aR = eaR$. Since R is semiprime it must be nondegenerate, and so $ea \neq 0$ implies that $eaRea \neq 0$. Thus $easea \neq 0$ for some $s \in R$. Let $\phi : Re \to Re$ be the R-homomorphism defined by $\phi(x) = xase$. Noting that $e = e^2 \in Re$, we have $\phi(e) = ease \neq 0$, and so $\mathrm{Im}(\phi) \neq 0$. Thus, since Re is a minimal left ideal we have $\mathrm{Im}(\phi) = Re$. Similarly, $\phi(e) \neq 0$ implies that $\ker(\phi) \neq Re$ and so $\ker(\phi) = 0$. Thus ϕ is an isomorphism of left R-modules. Therefore $e = \phi^{-1}\phi(e) = \phi^{-1}(ease) = ea\,\phi^{-1}(se) \in eaR = aR$, and so eR is a minimal right ideal of R. A similar argument shows the converse. □

Finally, we have this useful result from [J2, Theorem 4.3.1].

Proposition 3.1.8. *Let R be a ring. If R is semiprime, then* $\mathrm{soc}_l(R) = \mathrm{soc}_r(R)$.

Proof. Since R is semiprime, Corollary 3.1.6 tells us that that the left socle of R is the sum of minimal left ideals of the form Re, where e is an idempotent in R. Furthermore, by Proposition 3.1.7 we know that Re is a minimal left ideal if and only eR is a minimal right ideal. Thus, if $\mathrm{soc}_l(R) = \sum_i Re_i$, then $\sum_i e_iR \subseteq \mathrm{soc}_r(R)$. Therefore each $e_i \in e_iR \subseteq \mathrm{soc}_r(R)$ and so, since $\mathrm{soc}_r(R)$ is a two-sided ideal, $\mathrm{soc}_l(R) = \sum_i Re_i \subseteq \mathrm{soc}_r(R)$. Using a similar argument, we also have that $\mathrm{soc}_r(R) \subseteq \mathrm{soc}_l(R)$, and so $\mathrm{soc}_r(R) = \mathrm{soc}_l(R)$. □

3.2 The Socle of a Leavitt Path Algebra

In this section we show that the socle of a Leavitt path algebra $L_K(E)$ is closely related to the line points of the associated graph E. Indeed, in Theorem 3.2.11 we show that for any graph E we have $\mathrm{soc}(L_K(E)) = I(P_l(E))$, the ideal generated by the line points of E. We begin with the following proposition, shown in [AMMS2, Proposition 3.4].

Proposition 3.2.1. *For an arbitrary graph E, the Leavitt path algebra $L_K(E)$ is semiprime.*

Proof. Suppose that $L_K(E)$ is not semiprime, so that there exists a nonzero ideal I such that $I^2 = 0$. Take a nonzero $x \in I$. By Proposition 2.2.11, there exist $y, z \in L_K(E)$ such that either $yxz = kv$ for some nonzero $k \in K$ and some $v \in E^0$, or $yxz = \sum_{i=-m}^{n} k_i c^i$ for some $k_i \in K$ (not all zero) and some $c \in E^*$, where c is a cycle without exits in E. Now I cannot contain a vertex v, since $v = v^2 \in I^2 = 0$, a contradiction. So we must have $\sum_{i=-m}^{n} k_i c^i \in I$. Let $p = \sum_{i=-m}^{n} k_i c^i$ and let k be the (nonzero) coefficient of the term of maximum degree in p. Since $p^2 = 0$, we have $k^2 = 0$ and so $k = 0$, a contradiction. Thus $L_K(E)$ must be semiprime. □

Proposition 3.1.8 and Proposition 3.2.1 lead immediately to the following corollary.

Corollary 3.2.2. *Let E be an arbitrary graph. Then $\mathrm{soc}_l(L_K(E)) = \mathrm{soc}_r(L_K(E))$.*

In light of this result, we will drop the terms 'left' and 'right' and simply refer to the 'socle' of a Leavitt path algebra $L_K(E)$, which we denote by $\mathrm{soc}(L_K(E))$.

Recall that a vertex is a bifurcation if it emits two or more edges, and that a vertex v is a line point if there are no bifurcations or cycles based at any vertex $w \in T(v)$. We say that a path p **contains a bifurcation** if the set $p^0 \backslash \{r(p)\}$ contains a bifurcation. The following related lemma is from [AMMS1, Lemma 2.2], and though it is given there in a row-finite context, the proof remains valid for the arbitrary case.

Lemma 3.2.3. *Let E be an arbitrary graph and let u, v be in E^0, with $v \in T(u)$. If there is only one path joining u and v and it contains no bifurcations, then $L_K(E)u \cong L_K(E)v$ as left $L_K(E)$-modules.*

Proof. Let p be the unique path for which $s(p) = u$ and $r(p) = v$. By Lemma 2.1.10 we have that $p^*p = v$. Furthermore, since p contains no bifurcations, for each edge

e_i in p we have $s(e_i) = e_i e_i^*$ (by the (CK2) relation). Using the same logic as in the proof of Proposition 2.2.11, page 59, this gives $pp^* = u$.

Define a map $\phi_p : L_K(E)u \to L_K(E)v$ by $\phi_p(x) = xp$. Similarly, define a map $\phi_{p^*} : L_K(E)v \to L_K(E)u$ by $\phi_{p^*}(y) = yp^*$. These maps are easily seen to be left $L_K(E)$-module homomorphisms. Furthermore, we have $\phi_{p^*}\phi_p(x) = xpp^* = xu = x$ and $\phi_p\phi_{p^*}(y) = yp^*p = yv = y$. Thus ϕ_p and ϕ_{p^*} are mutual inverses, and so $L_K(E)u \cong L_K(E)v$ as left $L_K(E)$-modules, as required. □

We now embark on a series of results concerning left ideals and minimal left ideals of a Leavitt path algebra $L_K(E)$, building towards our main result in Theorem 3.2.11. The following proposition is from [AMMS1, Proposition 2.3], and though it is given in a row-finite context, it is easily adapted to the arbitrary case by requiring that u is a regular vertex rather than simply 'not a sink'.

Proposition 3.2.4. *Let E be an arbitrary graph and $u \in E^0$ be a regular vertex with $s^{-1}(u) = \{f_1, \dots, f_n\}$. Then $L_K(E)u = \bigoplus_{i=1}^n L_K(E)f_i f_i^*$. Furthermore, if $r(f_i) \neq r(f_j)$ for $i \neq j$ and we let $v_i = r(f_i)$, then $L_K(E)u \cong \bigoplus_{i=1}^n L_K(E)v_i$.*

Proof. By the (CK2) relation, we know that $u = \sum_{i=1}^n f_i f_i^*$, and so $L_K(E)u = \sum_{i=1}^n L_K(E)f_i f_i^*$. To show that this sum is direct, note that the $f_i f_i^*$ are orthogonal idempotents by the (CK1) relation: $(f_i f_i^*)(f_i f_i^*) = f_i(f_i^* f_i)f_i^* = f_i r(f_i) f_i^* = f_i f_i^*$, while $(f_i f_i^*)(f_j f_j^*) = f_i(f_i^* f_j)f_j^* = 0$ for $i \neq j$. Thus, if $x_i f_i f_i^* = \sum_{j=1, j\neq i}^n x_j f_j f_j^*$ for some $x_i, x_j \in L_K(E)$, multiplication on the right by $f_i f_i^*$ gives $x_i f_i f_i^* = 0$, and so the sum is direct.

To prove the second assertion, we define a map $\phi : L_K(E)u \to \bigoplus_{i=1}^n L_K(E)v_i$ by $\phi(x) = \sum_i x f_i$. It is clear that this map is a left $L_K(E)$-module homomorphism. Now suppose that $\phi(x) = \sum_{i=1}^n x f_i = 0$ for some $x \in L_K(E)u$. This gives $0 = \big(\sum_{i=1}^n x f_i\big) r(f_j) = x f_j$ for each $j \in \{1, \dots, n\}$ (since $r(f_i) \neq r(f_j)$ for $i \neq j$), and so $x = xu = \sum_{j=1}^n x f_j f_j^* = 0$. Thus $\ker(\phi) = \{0\}$ and so ϕ is a monomorphism. Now consider an arbitrary element $y = \sum_{i=1}^n y_i \in \bigoplus_{i=1}^n L_K(E)v_i$. Then $\sum_{i=1}^n y_i f_i^* \in L_K(E)u$ and

$$\phi\Big(\sum_{i=1}^n y_i f_i^*\Big) = \sum_{i=1}^n \big(\phi(y_i f_i^*)\big) = \sum_{i=1}^n \big(\sum_{j=1}^n y_i f_i^* f_j\big) = \sum_{i=1}^n (y_i f_i^* f_i) = y,$$

since $y_i f_i^* f_i = y_i v_i = y_i$ for each $i \in \{1, \ldots, n\}$. Thus ϕ is an epimorphism, completing the proof. $\square$

The following proposition from [AMMS2, Lemma 4.3] considers the case in which u is an infinite emitter.

Proposition 3.2.5. *Let E be an arbitrary graph and let $u \in E^0$ be an infinite emitter with $s^{-1}(u) = \{f_i\}_{i \in I}$ (where I is an infinite index set). Then $\bigoplus_{i \in I} L_K(E) f_i f_i^* \subset L_K(E)u$.*

Proof. First, note that the sum $\sum_{i \in I} L_K(E) f_i f_i^*$ is direct since $\{f_i f_i^*\}_{i \in I}$ is a set of mutually orthogonal idempotents (by the (CK1) relation). Now, since $r(f_i^*) = u$ for each i, we have the inclusion $\bigoplus_{i \in I} L_K(E) f_i f_i^* \subseteq L_K(E)u$. Suppose the converse containment holds, so that $u \in \bigoplus_{i \in I} L_K(E) f_i f_i^*$ (since $u = u^2 \in L_K(E)u$). Then $u = \sum_j x_j f_j f_j^*$, where $\{f_j\}$ is a finite subset of $s^{-1}(u)$ and each $x_j \in L_K(E)$. Since u is an infinite emitter, there exists a $g \in s^{-1}(u)$ such that $g \neq f_j$ for each j. Thus $g = ug = \sum_j x_j f_j f_j^* g = 0$ by the (CK1) relation, a contradiction. Thus $\bigoplus_{i \in I} L_K(E) f_i f_i^*$ is properly contained in $L_K(E)u$, as required. $\square$

The previous two results lead to the following corollary.

Corollary 3.2.6. *Let E be an arbitrary graph and let $u \in E^0$. If $T(u)$ contains a bifurcation then $L_K(E)u$ is not a minimal left ideal.*

Proof. Let $v \in T(u)$ be a bifurcation, and let p be a path from u to v. Let v_0 be the first bifurcation occuring in p, so that there are no bifurcations between u and v_0. Whether v_0 is a regular vertex or an infinite emitter, Proposition 3.2.4 and Proposition 3.2.5 give that $L_K(E)v_0$ is not a minimal left ideal, since for any $f_i \in s^{-1}(v_0)$ we have that $L_K(E) f_i f_i^*$ is a left ideal properly contained in $L_K(E)v_0$. By Lemma 3.2.3, we have $L_K(E)u \cong L_K(E)v_0$ as left $L_K(E)$-modules, and thus $L_K(E)u$ is not a minimal left ideal. $\square$

From Corollary 3.2.6 we can begin to see a relationship between minimal left ideals and line points forming. The following proposition from [AMMS1, Corollary 2.4] reinforces this notion. Though their proof is given in a row-finite setting, it holds for arbitrary graphs as well.

Proposition 3.2.7. *Let E be an arbitrary graph and let $u \in E^0$. If there is a closed path based at u, then $L_K(E)u$ is not a minimal left ideal.*

Proof. Let μ be a closed path based at u and suppose that $L_K(E)u$ is a minimal left ideal. By Corollary 3.2.6, there cannot be a bifurcation at any vertex in $T(u)$. In particular, μ cannot contain any bifurcations and so must be a cycle without exits. Consider the left ideal $L_K(E)(u+\mu)$. This ideal is nonempty, since $u+\mu = u(u+\mu) \in L_K(E)(u+\mu)$. Furthermore, it is contained in $L_K(E)u$ since $r(\mu) = u$. Thus, by the minimality of $L_K(E)u$, we have $L_K(E)(u+\mu) = L_K(E)u$. Specifically, we have $u \in L_K(E)(u+\mu)$.

Thus we can write $u = \sum_{i=1}^n k_i\alpha_i(u+\mu)$, where the α_i are monomials in $L_K(E)$ and $k_i \in K$. Using a similar argument to the one found in the proof of Proposition 2.2.11, each α_i must begin and end in u and is therefore either a power of μ or μ^* (since μ is a cycle without exits). Thus we can write $u = P(\mu,\mu^*)(u+\mu)$, where P is a polynomial with coefficients in K; that is,

$$P(\mu,\mu^*) = l_{-m}(\mu^*)^m + \cdots + l_0u + \cdots + l_n\mu^n,$$

where each $l_i \in K$ and $m,n \in \mathbb{N}$. Using the same argument found in the proof of Theorem 2.3.1, we can deduce that $l_{-i} = 0 = l_i$ for all $i > 0$. Thus $u = l_0u(u+\mu) = l_0(u+\mu)$, which is impossible, and so $L_K(E)$ cannot be minimal. ☐

The following proposition was first given in [AMMS1, Theorem 2.9] and then generalised to the arbitrary case in [AMMS2, Theorem 4.12]. However, a far simpler proof is given in [ARM1, Proposition 1.9], and it is this proof that we present below.

Proposition 3.2.8. *Let E be an arbitrary graph and let $v \in E^0$. Then $L_K(E)v$ is minimal if and only if $v \in P_l(E)$.*

Proof. Suppose that v is a line point in E. We begin by showing that every nonzero $L_K(E)$-endomorphism of $L_K(E)v$ is an automorphism. By Lemma 1.2.2 we have that $\mathrm{End}(L_K(E)v) \cong (vL_K(E)v)^{Op}$. Take an arbitrary element $x \in (vL_K(E)v)^{Op}$. Then $x = v(\sum_{i=1}^n k_ip_iq_i^*)v = \sum_{i=1}^n k_i(vp_iq_i^*v)$, where each $p_i, q_i \in E^*$ and $n \in \mathbb{N}$. If $vp_iq_i^*v \neq 0$ for some $i \in \{1,\ldots,n\}$, then $s(p_i) = s(q_i) = v$ and $r(p_i) = r(q_i)$.

Thus p_i and q_i are both paths from v to $r(p_i)$. Since v is a line point there can only be one such path and so $p_i = q_i$. Furthermore, since p_i contains no bifurcations we have $vp_iq_i^*v = v$ (see the proof of Lemma 3.2.3). Thus $x = (\sum k_i)v$ and so $\mathrm{End}(L_K(E)v) \cong (vL_K(E)v)^{Op} = Kv$. Since Kv is a field with identity element v, every nonzero element of $\mathrm{End}(L_K(E)v)$ is invertible and thus is an automorphism.

Now let a be an arbitrary nonzero element in $L_K(E)v$. Since $L_K(E)$ has local units, $L_K(E)a \neq 0$. Furthermore, since $L_K(E)$ is semiprime we have $(L_K(E)a)^2 \neq 0$, and so there exist $b, c \in L_K(E)$ such that $(ca)(ba) \neq 0$. Define $\phi : L_K(E)v \to L_K(E)v$ by $\phi(x) = x(ba)$. Then $\phi(a) = aba \neq 0$ and so ϕ is a nonzero endomorphism, and therefore an automorphism. Thus, since $v \in L_K(E)v$, we must have $v = d(ba)$ for some $d \in L_K(E)$. Therefore $v \in L_K(E)a$ and so, by Lemma 1.1.6, $L_K(E)v$ is minimal.

Conversely, suppose that $L_K(E)v$ is minimal. Suppose by way of contradiction that $T(v)$ contains vertices with bifurcations, and choose a bifurcation vertex $u \in T(v)$ such that the path p connecting u and v is of the shortest length possible. Since p contains no bifurcations, by Lemma 3.2.3 we have $L_K(E)u \cong L_K(E)v$, and so $L_K(E)u$ is minimal. By Proposition 3.2.7, there cannot be a cycle based at u.

Let e be an edge in E^1 with $s(e) = u$. We claim that $L_K(E)u = L_K(E)ee^* \oplus C$, where $C = \{x - xee^* : x \in L_K(E)u\}$. To show this, first take $y \in L_K(E)u$. Then $y = yee^* + y - yee^* \in L_K(E)ee^* + C$. Now take $z \in L_K(E)ee^* + C$. Then, for some $r, s \in L_K(E)$, $z = ree^* + su - suee^* = ree^*u + su - suee^*u \in L_K(E)u$, and so $L_K(E)u = L_K(E)ee^* + C$. To show the sum is direct, suppose that $z \in L_K(E)ee^* \cap C$, so that $z = t_1ee^* = t_2u - t_2uee^*$ for some $t_1, t_2 \in L_K(E)$. Then $t_2u = t_1ee^* + t_2ee^*$, and so multiplying on the right by e gives $t_2e = t_1e + t_2e$. Thus $t_1e = 0$ and therefore $z = 0$, showing the sum is direct.

Suppose that $C = 0$. Then, taking $x = u$ in the definition of C, we must have $u - ee^* = 0$. Now, since u is a bifurcation, there must exist an edge $f \in E^1$ such that $s(f) = u$ but $e \neq f$. Thus $f = uf = ee^*f = 0$, which is absurd. Therefore C is nonzero, and thus $L_K(E)u$ is not minimal, a contradiction. Thus v must be a line point. $\square$

Proposition 3.2.8 leads to the following lemma from [AMMS1, Proposition 4.1]

Lemma 3.2.9. *Let E be an arbitrary graph. Then*

$$\sum_{u \in P_l(E)} L_K(E)u \subseteq \operatorname{soc}(L_K(E)).$$

The reverse containment does not hold in general.

Proof. By Proposition 3.2.8, we know that $L_K(E)u$ is a minimal left ideal for any vertex $u \in P_l(E)$ and is therefore contained in the socle. To show that the converse containment is not true, we give the following counterexample. Let E be the graph

$$\bullet^v \xleftarrow{\ e\ } \bullet^z \xrightarrow{\ f\ } \bullet^w$$

By Lemma 2.2.9, $L_K(E) \cong \mathbb{M}_2(K) \oplus \mathbb{M}_2(K)$. By Lemma 1.1.10, $\mathbb{M}_2(K)$ is simple and so the only minimal left ideals of $\mathbb{M}_2(K) \oplus \mathbb{M}_2(K)$ are $\mathbb{M}_2(K) \oplus \{0\}$ and $\{0\} \oplus \mathbb{M}_2(K)$. Thus $\operatorname{soc}(L_K(E)) \cong \mathbb{M}_2(K) \oplus \mathbb{M}_2(K)$ and so $L_K(E)$ coincides with its socle. However, $\operatorname{soc}(L_K(E)) = L_K(E) \neq \sum_{u \in P_l(E)} L_K(E)u = L_K(E)v + L_K(E)w$, since for instance $z \notin L_K(E)v + L_K(E)w$. To see this, suppose that $z = xv + yw$ for some $x, y \in L_K(E)$. Then $z = z^2 = xvz + ywz = 0$, a contradiction. $\square$

So far we have shown that any principal left ideal of $L_K(E)$ generated by a line point u is contained in the socle of $L_K(E)$, but we have not quite given a precise formulation of the socle. The following theorem, from [AMMS1, Theorem 3.4], brings us one step closer to doing so. Though the original proof is given for the row-finite case, it is easily generalised to the arbitrary case by applying the relevant generalised results.

Theorem 3.2.10. *Let E be an arbitrary graph and let x be an element of $L_K(E)$ such that $L_K(E)x$ is a minimal left ideal. Then there exists a vertex $v \in P_l(E)$ such that $L_K(E)x \cong L_K(E)v$ as left $L_K(E)$-modules.*

Proof. Consider $x \in L_K(E)$. By Proposition 2.2.11 we have two cases; we show that the second case is not possible.

Suppose that there exist elements $y, z \in L_K(E)$ such that yxz is a nonzero element in

$$wL_K(E)w = \left\{ \sum_{i=-m}^{n} k_i c^i \text{ for } m, n \in \mathbb{N} \text{ and } k_i \in K \right\},$$

where c is a cycle without exits in E based at a vertex $w \in E^0$. For ease of notation, let $\lambda = yxz \in wL_K(E)w$. Since $L_K(E)yx$ is a nonzero left ideal contained in $L_K(E)x$ and $L_K(E)x$ is minimal, we must have $L_K(E)yx = L_K(E)x$. Furthermore, we can define a map $\phi_z : L_K(E)x \to L_K(E)xz$ by $\phi_z(a) = az$ for all $a \in L_K(E)x$. Clearly ϕ_z is a nonzero epimorphism. Also, since $L_K(E)x$ is minimal and $\ker(\phi_z) \neq L_K(E)x$ (since $0 \neq yxz \in \mathrm{Im}(\phi_z)$), we have $\ker(\phi_z) = \{0\}$ and so ϕ_z is a monomorphism. Therefore $L_K(E)x \cong L_K(E)xz = L_K(E)yxz = L_K(E)\lambda$, and so $L_K(E)\lambda$ is a minimal left ideal of $L_K(E)$.

We now show that $(wL_K(E)w)\lambda$ is a minimal left ideal in the subring $wL_K(E)w$. By Lemma 1.1.6 it suffices to show that, for any nonzero $a \in (wL_K(E)w)\lambda$, we have $\lambda \in (wL_K(E)w)a$. Since $a \in L_K(E)\lambda$ and $L_K(E)\lambda$ is minimal in $L_K(E)$, we have $L_K(E)a = L_K(E)\lambda$, and so $\lambda \in L_K(E)a$. Therefore $\lambda = w\lambda \in wL_K(E)a = (wL_K(E)w)a$, as required.

It is straightforward to see that the function $\phi : wL_K(E)w \to K[t, t^{-1}]$ given by $\phi(w) = 1$, $\phi(c) = t$ and $\phi(c^*) = t^{-1}$ (and expanded linearly) is an isomorphism. This implies that $\phi((wL_K(E)w)\lambda)$ is minimal in $K[t, t^{-1}]$. However, $K[t, t^{-1}]$ has no minimal left ideals. To see this, suppose that $f(t) = \sum_{i=k}^{l} a_i t^i$ and $g(t) = \sum_{j=m}^{n} b_j t^j$ are two nonzero elements of $R = K[t, t^{-1}]$. Without loss of generality, we can suppose that $a_k \neq 0$ and $b_m \neq 0$, so that $f(t)g(t) = a_k b_m t^{k+m} + \textit{ higher powers} \neq 0$. Thus R is an integral domain. Now suppose that R contains a minimal left ideal I and let x be a nonzero element of I. Since $x^2 \in I$ and I is minimal, $I = Rx^2$, and so $x = yx^2$ for some $y \in R$. Since R is an integral domain, this gives $1 = yx \in I$ and so $I = R$. Thus R is a field. However, this is a contradiction, since it is easy to see that not all elements in R have an inverse (for example, $1 + t$). Thus $K[t, t^{-1}]$ has no minimal left ideals, and so the second case of Proposition 2.2.11 is not possible, as claimed.

Therefore we must be in the first case of Proposition 2.2.11, and so there exist elements $y, z \in L_K(E)$ such that $yxz = kv \neq 0$, for some $v \in E^0$ and $k \in K$. Now $L_K(E)v = L_K(E)k^{-1}kv \subseteq L_K(E)kv$ and so $L_K(E)v = L_K(E)kv$. Using the same argument as in the second paragraph of the proof, we have $L_K(E)x \cong L_K(E)yxz = L_K(E)kv$ and so $L_K(E)x \cong L_K(E)v$ as left $L_K(E)$-modules, as required. Finally, since $L_K(E)v$ is therefore minimal, by Proposition 3.2.8 we have $v \in P_l(E)$. $\square$

Now we come to the main result of this section, where we describe precisely the structure of the socle of a Leavitt path algebra.

Theorem 3.2.11. *Let E be an arbitrary graph. Then* $\mathrm{soc}(L_K(E)) = I(P_l(E)) = I(H)$*, where H is the hereditary saturated closure of $P_l(E)$.*

Proof. First, we show that $\mathrm{soc}(L_K(E)) \subseteq I(P_l(E))$. Let I be a minimal left ideal of $L_K(E)$. Since $L_K(E)$ is semiprime, by Corollary 3.1.6 there exists an idempotent $\alpha \in L_K(E)$ such that $I = L_K(E)\alpha$. Furthermore, by Theorem 3.2.10 we have $L_K(E)\alpha \cong L_K(E)u$ for some $u \in P_l(E)$. Thus there exists a left $L_K(E)$-module isomorphism $\phi : L_K(E)\alpha \to L_K(E)u$ and we can find elements $x, y \in L_K(E)$ such that $\phi(\alpha) = xu$ and $\phi^{-1}(u) = y\alpha$, giving

$$\alpha = \phi^{-1}\phi(\alpha) = \phi^{-1}(xu) = xu\,\phi^{-1}(u) = xuy\alpha.$$

Thus $\alpha = x(u)y\alpha \in I(P_l(E))$, and so $I = L_K(E)\alpha \subseteq I(P_l(E))$ and therefore $\mathrm{soc}(L_K(E)) \subseteq I(P_l(E))$.

For the converse containment, take a vertex $v \in P_l(E)$. By Lemma 3.2.9, we have $L_K(E)v \subseteq \mathrm{soc}(L_K(E))$ and so, since $\mathrm{soc}(L_K(E))$ is a two-sided ideal, $L_K(E)vL_K(E) \subseteq \mathrm{soc}(L_K(E))$. Since this is true for all $v \in P_l(E)$, we have $L_K(E)P_l(E)L_K(E) = I(P_l(E)) \subseteq \mathrm{soc}(L_K(E))$, and so $\mathrm{soc}(L_K(E)) = I(P_l(E))$. Finally, Lemma 2.2.2 gives $I(P_l(E)) = I(H)$, where H is the hereditary saturated closure of $P_l(E)$. $\square$

Theorem 3.2.11 leads immediately to the following useful corollary.

Corollary 3.2.12. *For an arbitrary graph E, the Leavitt path algebra $L_K(E)$ has nonzero socle if and only if $P_l(E) \neq \emptyset$.*

Example 3.2.13. We now use Theorem 3.2.11 to compute the socle of some familiar Leavitt path algebras.

(i) *The finite line graph* M_n. Every vertex in M_n is a line point, and so by Theorem 3.2.11 we have $\mathrm{soc}(L_K(M_n)) = I(P_l(M_n)) = I((M_n)^0) = L_K(M_n)$. Thus, since $L_K(M_n) \cong \mathbb{M}_n(K)$, we also have that $\mathrm{soc}(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$ for all $n \in \mathbb{N}$.

(ii) *The rose with* n *leaves* R_n. The graph R_n contains a single vertex v that is the base of n cycles; in particular, v is not a line point. Thus $P_l(R_n) = \emptyset$ and so $\mathrm{soc}(R_n) = 0$. Thus, since $L_K(R_n) \cong L(1,n)$, we also have that $\mathrm{soc}(L(1,n)) = 0$ for all $n \in \mathbb{N}$.

(iii) *The infinite clock graph* C_∞. In this case, the line points of C_∞ are the radial vertices v_i, so that $P_l(C_\infty) = \{v_i\}_{i=1}^\infty$. Thus we have $\mathrm{soc}(L_K(C_\infty)) = I(\{v_i\}_{i=1}^\infty)$. Recall from Example 2.1.7 the isomorphism $\phi : L_K(C_\infty) \to \bigoplus_{i=1}^\infty \mathbb{M}_2(K) \oplus KI_{22}$ that maps each vertex v_i to $(E_{11})_i$, the element of $\bigoplus_{i=1}^\infty \mathbb{M}_2(K)$ with E_{11} in the i^{th} component and zeros elsewhere. Thus $\mathrm{soc}(\bigoplus_{i=1}^\infty \mathbb{M}_2(K) \oplus KI_{22})$ is the two-sided ideal generated by the set $\{(E_{11})_i\}_{i=1}^\infty$. Note that this ideal contains any matrix unit $(E_{mn})_j$, since $(E_{mn})_j = (E_{m1})_j(E_{11})_j(E_{1n})_j$, and since such matrix units generate $\bigoplus_{i=1}^\infty \mathbb{M}_2(K)$ we have

$$\mathrm{soc}\left(\bigoplus_{i=1}^\infty \mathbb{M}_2(K) \oplus KI_{22}\right) = \bigoplus_{i=1}^\infty \mathbb{M}_2(K).$$

3.3 Quotient Graphs and Graded Ideals

In Section 3.4 we will examine the socle series of a Leavitt path algebra, a concept that naturally extends the socle. In doing so we will need to consider quotient rings of the form $L_K(E)/I$, where I is a graded ideal of $L_K(E)$. Thus, in this section we will examine some properties of graded ideals I of $L_K(E)$ and quotient rings of the form $L_K(E)/I$. In particular, we show in Theorem 3.3.8 that, for any graded ideal I of $L_K(E)$, $L_K(E)/I$ is isomorphic to the Leavitt path algebra of a 'quotient graph' of E, a concept we define below.

Many of the results in this section are thanks to Tomforde, whose paper [To] gives many valuable results regarding the ideal structure of a Leavitt path algebra. We begin with the following definitions.

Definition 3.3.1. Let E be a graph and let H be a hereditary saturated subset of E^0. The set of **breaking vertices** of H, denoted B_H, is defined to be the set

$$B_H = \{v \in E^0 \backslash H : v \text{ is an infinite emitter and } 0 < |s^{-1}(v) \cap \ r^{-1}(E^0 \backslash H)| < \infty\}.$$

In other words, a breaking vertex is an infinite emitter that emits an infinite number of edges into H, while emitting only a finite number of edges into the rest of the graph. Note that if E is row-finite then B_H is always empty.

Furthermore, we say that (H, S) is an **admissible pair of** E if H is a hereditary saturated subset of E^0 and $S \subseteq B_H$.

We use these definitions to define the quotient graph $E \backslash (H, S)$.

Definition 3.3.2. Let E be an arbitrary graph and let (H, S) be an admissible pair of E. The **quotient graph** $E \backslash (H, S)$ is defined as follows. Let B'_H be a set of duplicates of B_H, and write $B'_H = \{v' : v \in B_H\}$. Let $S' = \{v' \in B'_H : v \in S\}$. We define

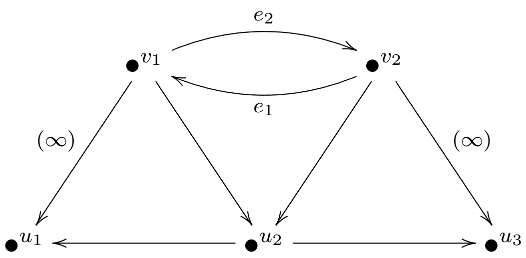
$$(E \backslash (H, S))^0 = (E^0 \backslash H) \cup (B'_H \backslash S') \quad \text{and}$$

$$(E \backslash (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H \backslash S\}.$$

Furthermore, the source and range functions $s_{E \backslash (H,S)}$ and $r_{E \backslash (H,S)}$ coincide with s_E and r_E when applied to $\{e \in E^1 : r(e) \notin H\}$, while we define $s_{E \backslash (H,S)}(e') = s_E(e)$ and $r_{E \backslash (H,S)}(e') = (r_E(e))'$. If $S = \emptyset$, we often write $E \backslash (H, S)$ as simply $E|H$.

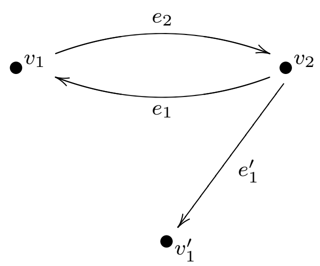
Thus, to form the quotient graph $E \backslash (H, S)$ we first remove all vertices $u \in H$ and all edges $e \in E^1$ with $r(e) \in H$. Then, for each breaking vertex $v \in B_H \backslash S$, we add a new vertex v' to the graph. Furthermore, for each edge e with $r(e) = v$, we add a new edge e' to the graph, running from $s(e)$ to v'. Note that this construction implies that every $v' \in B'_H \backslash S'$ is a sink.

Example 3.3.3. Consider the following graph E:

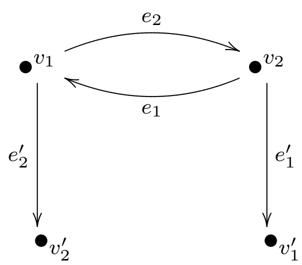


where (∞) denotes an infinite number of edges. Let $H = \{u_1, u_2, u_3\}$. Then H is clearly a hereditary saturated subset of E^0. Furthermore, both v_1 and v_2 emit an infinite number of edges into H and a single edge into $E^0 \backslash H$. Thus $B_H = \{v_1, v_2\}$.

If we choose $S = \{v_2\}$, then the quotient graph $E\backslash(H, S)$ looks like:



Furthermore, if we choose $S = \emptyset$ then $E\backslash(H, S) = E|H$ looks like:



Definition 3.3.4. Let E be a graph and let H be a hereditary saturated subset of E^0. For any $v \in B_H$, we define

$$v^H := v - \sum_{\substack{s(e)=v,\\ r(e)\notin H}} ee^*.$$

Note that, by the definition of a breaking vertex, this sum must be finite and is therefore well-defined. Using the fact that $e_i e_i^* e_j e_j^* = \delta_{ij}\, e_i e_i^*$ (by the (CK1) relation), it is easy to see that v^H is an idempotent.

Definition 3.3.5. Let E be an arbitrary graph. For any admissible pair (H,S) of E, we denote by $I_{(H,S)}$ the two-sided ideal in $L_K(E)$ generated by the sets $\{u : u \in H\}$ and $\{v^H : v \in S\}$. Note that if S is empty then $I_{(H,S)} = I(H)$.

The following proposition is from [To, Lemma 5.6] and describes the structure of an ideal of the form $I_{(H,S)}$. Here we have greatly expanded the proof for clarity.

Proposition 3.3.6. *Let E be an arbitrary graph. For any admissible pair (H,S) of E we have*

$$I_{(H,S)} = \operatorname{span}\big(\{\alpha\beta^* : r(\alpha) = r(\beta) \in H\} \cup \{\alpha w^H \beta^* : r(\alpha) = r(\beta) = w \in S\}\big)$$

where each $\alpha, \beta \in E^*$. Furthermore, $I_{(H,S)}$ is a graded ideal of $L_K(E)$.

Proof. Let J denote the right-hand side of the above equation. It is clear that every element in J is in the ideal generated by $\{v : v \in H\} \cup \{w^H : w \in S\}$, that is, $J \subseteq I_{(H,S)}$. To show the converse containment, let $x \in I_{(H,S)}$, so that

$$x = \sum_i a_i v_i b_i + \sum_j c_j w_j^H d_j,$$

where each $a_i, b_i, c_j, d_j \in L_K(E)$, each $v_i \in H$, each $w_j \in S$ and the sums are finite. By Lemma 2.1.8 we know that every element in $L_K(E)$ is of the form $\sum_i k_i p_i q_i^*$, where each $p_i, q_i \in E^*$ and each $k_i \in K$. Thus, omitting the scalars k_i for ease of notation, we can write the above expression as

$$x = \sum_i (p_{1_i} q_{1_i}^*) v_i (p_{2_i} q_{2_i}^*) + \sum_j (p_{1_j} q_{1_j}^*) w_j^H (p_{2_j} q_{2_j}^*),$$

where each $p, q \in E^*$.

Take a nonzero term $y = (p_1 q_1^*) v (p_2 q_2^*)$ from the first sum. Since y is nonzero, we must have $s(q_1) = s(p_2) = v$, and so $y = p_1 q_1^* p_2 q_2^*$. Since $q_1^* p_2 \neq 0$, Lemma 2.1.10 tells us that either $p_2 = q_1\gamma$ or $q_1 = p_2\tau$ for some paths γ, τ in E. For the former case, we have $y = p_1 q_1^* (q_1\gamma) q_2^* = p_1 \gamma q_2^*$. Since $s(p_2) = v$ and $r(p_2) = r(\gamma)$, we have $r(\gamma) \in T(v)$. Thus $r(\gamma) \in H$, by the hereditary nature of H. So, taking $\alpha = p_1\gamma$ and $\beta = q_2$, we have $y = \alpha r(\gamma) \beta^* \in J$. For the latter case, we have $y = p_1 \tau^* q_2^*$, and a similar argument shows that again $y \in J$.

Now take a nonzero term $z = (p_1 q_1^*) w^H (p_2 q_2^*)$ from the second sum above. Letting $M = \{e \in E^1 : s(e) = w,\ r(e) \notin H\}$, we can write $z = (p_1 q_1^*)(w - \sum_{e \in M} ee^*)(p_2 q_2^*)$ (from the definition of w^H). Again, since z is nonzero we must have $s(q_1) = s(p_2) = w$ and so

$$z = p_1 q_1^* p_2 q_2^* - \sum_{e \in M} p_1 q_1^* ee^* p_2 q_2^*.$$

We now consider three different cases.

Case 1: $l(q_1) = l(p_2) = 0$. In this case, since z is nonzero we must have $q_1 = p_2 = w$, and so $z = p_1 w q_2^* - \sum_{e \in M} p_1 ee^* q_2^*$. Thus, letting $\alpha = p_1$ and $\beta = q_2$ we have $z = \alpha w^H \beta^* \in J$.

Case 2: $l(q_1) = 0$, $l(p_2) > 0$. Let f be the initial edge of p_2, so that $p_2 = f p_2'$. Thus $z = p_1 f p_2' q_2^* - \sum_{e \in M} p_1 ee^* f p_2' q_2^*$. If $r(f) \notin H$ then $f \in M$ (since $s(f) = w$), and so using the fact that $e^* f = 0$ for all $e \in M$ such that $e \neq f$, we have

$$z = p_1 f p_2' q_2^* - p_1 f f^* f p_2' q_2^* = p_1 f p_2' q_2^* - p_1 f p_2' q_2^* = 0,$$

contradicting the fact that z is nonzero. Thus $r(f) \in H$ and so $f \notin M$. Therefore $e^* f = 0$ for all $e \in M$ and so $z = p_1 f p_2' q_2^*$. Since $r(f) = s(p_2') \in H$, we have $r(p_2') \in H$, by the hereditary nature of H. Thus, letting $\alpha = p_1 f p_2'$ and $\beta = q_2$, we have $z = \alpha r(p_2') \beta^* \in J$. Using a similar argument, we can see that $z \in J$ for the case that $l(q_1) > 0$ and $l(p_2) = 0$.

Case 3: $l(q_1) > 0$, and $l(p_2) > 0$. Let $p_1 = f p_2'$ and $q_1 = g q_1'$, where $f, g \in E^1$. If $f \neq g$, then $z = p_1 (q_1')^* g^* f p_2' q_2^* - \sum_{e \in M} p_1 (q_1')^* g^* ee^* f p_2' q_2^* = 0$ (since $g^* e = 0$ and/or $e^* f = 0$ for all $e \in M$), a contradiction. Thus $f = g$ and so $z = p_1 (q_1')^* p_2' q_2^* - \sum_{e \in M} p_1 (q_1')^* f^* ee^* f p_2' q_2^*$. As in Case 2, if $r(f) \notin H$ then $z = p_1 (q_1')^* p_2' q_2^* - p_1 (q_1')^* p_2' q_2^* = 0$, a contradiction. So $r(f) \in H$, $f \in M$ and we have $z = (p_1 (q_1')^*) r(f) (p_2' q_2^*)$. Thus using the same argument as in Case 2, we have $z \in J$.

Thus $x \in J$, and so $I_{(H,S)} \subseteq J$, as required.

To see that $I_{(H,S)}$ is graded, note that each term $\alpha\beta^*$, where $r(\alpha) = r(\beta) \in H$, is homogeneous of degree $|\alpha| - |\beta|$. Furthermore, for any $v \in S$, v^H is by definition an element of degree 0, so again we have that each term $\alpha v^H \beta$ is homogeneous of degree $|\alpha| - |\beta|$. Thus each element in $I_{(H,S)}$ can be expressed as the sum of homogeneous elements of the form $\alpha\beta^*$ or $\alpha v^H \beta^*$. Since each of these homogeneous elements is also in $I_{(H,S)}$, by definition, $I_{(H,S)}$ is therefore a graded ideal. □

If H is a hereditary and saturated subset of E^0, then taking $S = \emptyset$ and applying Proposition 3.3.6 we get

$$I(H) = \text{span}(\{\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}).$$

Thus Proposition 3.3.6 allows us to describe precisely the elements of an ideal $I_{(H,S)}$ (or $I(H)$) in a relatively simple way. This will prove valuable in future results.

If (H, S) is an admissible pair in E then, by definition, $I_{(H,S)}$ is generated by the set of vertices $u \in H$ and the set of elements v^H for which $v \in S$. It is natural to ask if $I_{(H,S)}$ also contains vertices that are not in H, and elements v^H for which $v \notin S$. The following proposition, which has been adapted from the beginning of the proof of [To, Theorem 5.7], shows that this is in fact not possible.

Proposition 3.3.7. *Let E be an arbitrary graph and let (H, S) be an admissible pair of E. Then $I_{(H,S)} \cap E^0 = H$ and $\{v \in B_H : v^H \in I_{(H,S)}\} = S$.*

Proof. We begin this proof by setting up a homomorphism between $L_K(E)$ and $L_K(E\backslash(H, S))$ that we will refer to again in later proofs, particularly the proof of Theorem 3.3.12. Define $\phi : L_K(E) \to L_K(E\backslash(H, S))$ on the generators of $L_K(E)$ as follows:

$$\phi(v) = \begin{cases} v & \text{if } v \in (E^0\backslash H)\backslash(B_H\backslash S) \\ v + v' & \text{if } v \in B_H\backslash S \\ 0 & \text{if } v \in H, \end{cases}$$

$$\phi(e) = \begin{cases} e & \text{if } r(e) \in (E^0\backslash H)\backslash(B_H\backslash S) \\ e + e' & \text{if } r(e) \in B_H\backslash S \\ 0 & \text{if } r(e) \in H \end{cases}$$

and

$$\phi(e^*) = \begin{cases} e^* & \text{if } r(e) \in (E^0\backslash H)\backslash(B_H\backslash S) \\ e^* + (e')^* & \text{if } r(e) \in B_H\backslash S \\ 0 & \text{if } r(e) \in H. \end{cases}$$

Extend ϕ linearly and multiplicatively. To begin, we must check that ϕ preserves the Leavitt path algebra relations on $L_K(E)$, a rather technical and tedious procedure. However, for the sake of completeness we will show this process in full for this particular proof. For ease of notation, we set $(E^0\backslash H)\backslash(B_H\backslash S) = T$.

First, we check that the (A1) relation holds, i.e. that $\phi(v_i)\phi(v_j) = \delta_{ij}\,\phi(v_i)$ for all $v_i, v_j \in E^0$. We must examine several different cases:

Case 1: $v_i, v_j \in T$. Then $\phi(v_i)\phi(v_j) = v_iv_j = \delta_{ij}\,v_i = \delta_{ij}\,\phi(v_i)$.

Case 2: $v_i \in T, v_j \in B_H\backslash S$. Then $\phi(v_i)\phi(v_j) = v_i(v_j + v_j') = \delta_{ij}\,v_i = \delta_{ij}\,\phi(v_i)$ (we know that $v_i \neq v_j'$ since $v_j' \notin L_K(E)$). A similar argument shows the relation holds for $v_i \in B_H\backslash S, v_j \in T$.

Case 3: $v_i, v_j \in B_H\backslash S$. Then $\phi(v_i)\phi(v_j) = (v_i + v_i')(v_j + v_j') = v_iv_j + v_i'v_j' = \delta_{ij}\,(v_i + v_i') = \delta_{ij}\,\phi(v_i)$.

Case 4: Either v_i or $v_j \in H$. Then $\phi(v_i)\phi(v_j) = 0 = \delta_{ij}\,\phi(v_i)$.

Next, we check that the (A2) relations hold. First, we check that $\phi(s(e))\phi(e) = \phi(e)$ for all $e \in E^1$.

Case 1: $s(e), r(e) \in T$. Then $\phi(s(e))\phi(e) = s(e)e = e = \phi(e)$.

Case 2: $s(e) \in T, r(e) \in B_H\backslash S$. Then $\phi(s(e))\phi(e) = s(e)(e + e') = e + e' = \phi(e)$, since $s(e') = s(e)$.

Case 3: $s(e) \in B_H\backslash S, r(e) \in T$. Then $\phi(s(e))\phi(e) = (s(e) + s(e)')e = s(e)e = e = \phi(e)$ (we know that $s(e)'e = 0$ since every $v' \in B_H'$ is a sink).

Case 4: $s(e), r(e) \in B_H\backslash S$. Then $\phi(s(e))\phi(e) = (s(e) + s(e)')(e + e') = s(e)e + s(e)e' = e + e' = \phi(e)$.

Case 5: $s(e) \in H$. Then, since H is hereditary, $r(e) \in H$ and so $\phi(s(e))\phi(e) = 0 = \phi(e)$.

Case 6: $s(e) \in E^0\backslash H, r(e) \in H$. Then $\phi(s(e))\phi(e) = 0 = \phi(e)$.

Next, we check that $\phi(e)\phi(r(e)) = \phi(e)$ for all $e \in E^1$.

Case 1: $r(e) \in T$. Then $\phi(e)\phi(r(e)) = er(e) = e = \phi(e)$.

Case 2: $r(e) \in B_H \backslash S$. Then $\phi(e)\phi(r(e)) = (e+e')(r(e)+r(e)') = er(e)+e'r(e') = e + e' = \phi(e)$.

Case 3: $r(e) \in H$. Then $\phi(e)\phi(r(e)) = 0 = \phi(e)$.

By very similar arguments, we can show that $\phi(r(e))\phi(e^*) = \phi(e^*)$ and that $\phi(e^*)\phi(s(e)) = \phi(e^*)$ for all $e \in E^1$.

Next we check that the (CK1) relation holds, i.e. that $\phi(e_i^*)\phi(e_j) = \delta_{ij}\phi(r(e_i))$ for all $e_i, e_j \in E^1$.

Case 1: $r(e_i), r(e_j) \in T$. Then $\phi(e_i^*)\phi(e_j) = e_i^* e_j = \delta_{ij} r(e_i) = \delta_{ij}\phi(r(e_i))$.

Case 2: $r(e_i) \in T, r(e_j) \in B_H \backslash S$. Then $\phi(e_i^*)\phi(e_j) = e_i^*(e_j + e_j') = \delta_{ij} r(e_i) = \delta_{ij}\phi(r(e_i))$ (we know that $e_i \neq e_j'$ since $e_j' \notin L_K(E)$). A similar argument shows that the relation holds for $r(e_i) \in B_H \backslash S, r(e_j) \in T$.

Case 3: $r(e_i), r(e_j) \in B_H \backslash S$. Then $\phi(e_i^*)\phi(e_j) = (e_i^* + (e_i')^*)(e_j + e_j') = e_i^* e_j + (e_i')^* e_j' = \delta_{ij}(r(e_i) + r(e_i)') = \delta_{ij}\phi(r(e_i))$.

Case 4: Either $r(e_i)$ or $r(e_j) \in H$. Then $\phi(e_i^*)\phi(e_j) = 0 = \delta_{ij}\phi(r(e_i))$.

Finally, we check that the (CK2) relation holds, i.e. that $\phi(v - \sum_{s_E(e)=v} ee^*) = 0$ for all regular vertices $v \in E^0$. Specifically, v is not a breaking vertex.

If $v \in H$, $s(e) = v$ implies that $r(e) \in H$ since H is hereditary. So

$$\phi\left(v - \sum_{s_E(e)=v} ee^*\right) = \phi(v) - \sum_{s_E(e)=v} \phi(e)\phi(e^*) = 0.$$

Otherwise, we can assume that $v \in T$. Thus we have

$$\begin{aligned}
\phi\left(v - \sum_{s_E(e)=v} ee^*\right) &= \phi(v) - \sum_{\substack{s_E(e)=v \\ r(e)\in T}} \phi(ee^*) - \sum_{\substack{s_E(e)=v \\ r(e)\in B_H\setminus S}} \phi(ee^*) - \sum_{\substack{s_E(e)=v \\ r(e)\in H}} \phi(ee^*) \\
&= v - \sum_{\substack{s_E(e)=v \\ r(e)\in T}} ee^* - \sum_{\substack{s_E(e)=v \\ r(e)\in B_H\setminus S}} (e+e')(e^*+(e')^*) \\
&= v - \sum_{\substack{s_E(e)=v \\ r(e)\in T}} ee^* - \sum_{\substack{s_E(e)=v \\ r(e)\in B_H\setminus S}} (ee^*+e'(e')^*) \\
&= v - \sum_{s_{E\setminus(H,S)}(e)=v} ee^* \\
&= 0, \text{ for the following reason.}
\end{aligned}$$

We know that v must emit at least one edge e with $r(e) \notin H$, because otherwise the saturated property of H would imply that $v \in H$. Thus v is not a sink in $E\setminus(H,S)$. Furthermore, since v is not an infinite emitter in E, and since v must emit only a finite number of new edges e' in $E\setminus(H,S)$, v is not an infinite emitter in $E|H$. Thus v is a regular vertex in $E\setminus(H,S)$ and so we are able to apply the (CK2) relation in the final step above. Thus ϕ preserves the Leavitt path algebra relations on E and is therefore a K-algebra homomorphism.

We now show that $I_{(H,S)} \subseteq \ker(\phi)$. By definition, $I_{(H,S)}$ is generated by the sets $\{v : v \in H\}$ and $\{v^H : v \in S\}$, so it suffices to show that all such generating elements are mapped to 0 under ϕ. We know that $\phi(v) = 0$ for all $v \in H$. Now consider an element v^H, where $v \in S$. Then, using the same argument as we did when checking the (CK2) relation, we have

$$\phi(v^H) = \phi\left(v - \sum_{\substack{s_E(e)=v \\ r(e)\notin H}} ee^*\right) = v - \sum_{\substack{s_E(e)=v \\ r(e)\in T}} ee^* - \sum_{\substack{s_E(e)=v \\ r(e)\in B_H\setminus S}} (ee^*+e'(e')^*),$$

noting that $\phi(v) = v$ since $v \in T = (E^0\setminus H)\setminus(B_H\setminus S)$. Note that since $v \in S \subseteq B_H$, v must be a regular vertex in $E\setminus(H,S)$, by the definition of a breaking vertex. Furthermore, we have

$$\sum_{\substack{s_E(e)=v \\ r(e)\in T}} ee^* - \sum_{\substack{s_E(e)=v \\ r(e)\in B_H\setminus S}} (ee^*+e'(e')^*) = \sum_{s_{E\setminus(H,S)}(e)=v} ee^*,$$

and so by the (CK2) relation $\phi(v^H) = 0$, as required. Thus $I_{(H,S)} \subseteq \ker(\phi)$.

Now suppose there exists $w \in I_{(H,S)} \cap E^0$ such that $w \notin H$. Then either $\phi(w) = w$ or $\phi(w) = w + w'$ (by the definition of ϕ), a contradiction since $I_{(H,S)} \subseteq \ker(\phi)$. Thus $I_{(H,S)} \cap E^0 = H$. Similarly, suppose there exists $v \in B_H \backslash S$ such that $v^H \in I_{(H,S)}$. Then

$$\phi(v^H) = \phi\left(v - \sum_{s_E(e)=v, r(e)\notin H} ee^* \right) = (v + v') - \sum_{s_{E\backslash(H,S)}(e)=v} ee^* = v'$$

(following the same argument as above). Once again, this contradicts that $I_{(H,S)} \subseteq \ker(\phi)$ and so $\{v \in B_H : v^H \in I_{(H,S)}\} = S$, completing the proof. □

Note that if we take S to be the empty set, the statement of Proposition 3.3.7 simplifies to $I(H) \cap E^0 = H$ for all hereditary saturated subsets H of E^0.

Now we come to perhaps the most important result of this section, which shows that, for any admissible pair (H, S) of a graph E, the quotient ring $L_K(E)/I_{(H,S)}$ is in fact isomorphic to the Leavitt path algebra of the quotient graph $E\backslash(H, S)$. This powerful result is from [To, Theorem 5.7(2)]. Here we have greatly expanded the proof for clarity.

Theorem 3.3.8. *Let E be an arbitrary graph and let (H, S) be an admissible pair of E. Then*

$$L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H, S)).$$

Proof. Define $\varphi : L_K(E\backslash(H, S)) \to L_K(E)$ on the generators of $L_K(E\backslash(H, S))$ as follows:

$$\varphi(v) = \begin{cases} v & \text{if } v \in (E^0\backslash H)\backslash(B_H\backslash S) \\ \sum_{s(e)=v, r(e)\notin H} ee^* & \text{if } v \in B_H\backslash S \\ v^H & \text{if } v = v' \in B'_H\backslash S', \end{cases}$$

$$\varphi(e) = \begin{cases} e & \text{if } r(e) \in (E^0\backslash H)\backslash(B_H\backslash S) \\ e\,\varphi(r(e)) & \text{if } r(e) \in B_H\backslash S \\ e\,\varphi(r(e)') & \text{if } e = e' \end{cases}$$

and

$$\varphi(e^*) = \begin{cases} e^* & \text{if } r(e) \in (E^0\backslash H)\backslash(B_H\backslash S) \\ \varphi(r(e))\, e^* & \text{if } r(e) \in B_H\backslash S \\ \varphi(r(e)')\, e^* & \text{if } e = e' \end{cases}$$

Extend φ linearly and multiplicatively. Furthermore, define $\varphi^* : L_K(E\backslash(H,S)) \to L_K(E)/I_{(H,S)}$ by $\varphi^*(x) = \varphi(x) + I_{(H,S)}$. It can be verified that φ, and therefore φ^*, preserves the Leavitt path algebra relations on $L_K(E\backslash(H,S))$. This a straightforward but tedious process, with several subcases for each relation, so here we will just provide a sample calculation: we show that the (CK1) relation $e_i^* e_j = \delta_{ij}\, r(e_i)$ is preserved for the case $r(e_i), r(e_j) \in B_H\backslash S$. In this case we have

$$\begin{aligned}
\varphi(e_i^*)\varphi(e_j) &= \varphi(r(e_i))\, e_i^* e_j\, \varphi(r(e_j)) \\
&= \delta_{ij}\, \varphi(r(e_i))\, r(e_i)\, \varphi(r(e_i)) \\
&= \delta_{ij} \left(\textstyle\sum_{s(f_i)=r(e_i), r(f_i)\notin H} f_i f_i^* \right) r(e_i) \left(\textstyle\sum_{s(f_i)=r(e_i), r(f_i)\notin H} f_i f_i^* \right) \\
&= \delta_{ij} \left(\textstyle\sum_{s(f_i)=r(e_i), r(f_i)\notin H} f_i f_i^* \right) \\
&= \delta_{ij}\, \varphi(r(e_i)),
\end{aligned}$$

as required. Checking that these relations are preserved ensures that φ^* is indeed a K-homomorphism.

To show that φ^* is a monomorphism, we will apply the Graded Uniqueness Theorem (Theorem 2.2.13). We know that $I_{(H,S)}$ is $\mathbb{Z}$-graded (by Proposition 3.3.6), and so $L_K(E)/I_{(H,S)}$ is $\mathbb{Z}$-graded. Furthermore, φ (and therefore φ^*) is a graded homomorphism, since it takes generating elements to elements of equal degree. To show that $\varphi^*(v) \neq 0$ for all $v \in (E\backslash(H,S))^0$, it suffices to show that $\varphi(v) \notin I_{(H,S)}$ for all $v \in (E\backslash(H,S))^0$. Suppose that $v \in (E^0\backslash H)\backslash(B_H\backslash S)$. Then $\varphi(v) = v \notin I_{(H,S)}$, since $I_{(H,S)} \cap E^0 = H$ and $v \notin H$ (by Proposition 3.3.7). Now suppose that $v \in B_H\backslash S$. Then $\varphi(v) = \sum_{s(e)=v, r(e)\notin H} ee^*$. Suppose that $\varphi(v) \in I_{(H,S)}$ and choose a fixed edge f for which $s(f) = v$ and $r(f) \notin H$. Then $f^*\varphi(v)f = \sum_{s(e)=v, r(e)\notin H} f^*ee^*f = r(f) \in I_{(H,S)}$, since $I_{(H,S)}$ is a two-sided ideal. However, this implies $r(f) \in H$, a contradiction, and so $\varphi(v) \notin I_{(H,S)}$. Furthermore, if $v' \in B_H'\backslash S'$ then $\varphi(v') = v^H \notin I_{(H,S)}$, since $v^H \in I_{(H,S)}$ implies $v \in S$ (again by Proposition 3.3.7), a contradiction since

$v \in B_H \backslash S$. Thus $\varphi(v) \notin I_{(H,S)}$ for all $v \in (E\backslash(H,S))^0$, so we can apply the Graded Uniqueness Theorem to obtain that φ^* is indeed a monomorphism.

To show that φ^* is an epimorphism, note that $L_K(E)/I_{(H,S)}$ is generated by elements of the form $\alpha\beta^* + I_{(H,S)}$, where $\alpha, \beta \in E^*$, $r(\alpha) = r(\beta)$ and $\alpha\beta^* \notin I_{(H,S)}$. Suppose $|\alpha| = |\beta| = 0$, so that $\alpha\beta^* = v$ for some $v \in E^0$. Now if $v \in H$ then $v \in I_{(H,S)}$, a contradiction. So $v \notin H$. Now suppose $|\alpha| > 0$. If α contains an edge e such that $r(e) \in H$, then $r(\alpha) \in H$, since H is hereditary, and so $\alpha\beta^* \in I_{(H,S)}$, a contradiction. Thus α, and similarly β, contains no edges such that $r(e) \in H$. Thus $L_K(E)/I_{(H,S)}$ is generated by the set

$$\{v + I_{(H,S)} : v \notin H\} \cup \{e + I_{(H,S)} : r(e) \notin H\} \cup \{e^* + I_{(H,S)} : r(e) \notin H\}.$$

Since $\varphi^*(x) = \varphi(x) + I_{(H,S)}$, it suffices to show that the set $\{v : v \notin H\} \cup \{e : r(e) \notin H\} \cup \{e^* : r(e) \notin H\}$ is in the image of φ. If $v \in (E^0\backslash H)\backslash(B_H\backslash S)$, then $v = \varphi(v)$. If $v \in B_H\backslash S$, then

$$v = \sum_{\substack{s(e)=v,\\ r(e)\notin H}} ee^* + v^H = \varphi(v) + \varphi(v').$$

Similarly, if $r(e) \in (E^0\backslash H)\backslash(B_H\backslash S)$, then $e = \varphi(e)$ (and $e^* = \varphi(e^*)$). If $r(e) \in B_H\backslash S$, then by the above equation we have

$$e = er(e) = e(\varphi(r(e)) + \varphi(r(e)')) = e\varphi(r(e)) + e\varphi(r(e)') = \varphi(e) + \varphi(e')$$

(and $e^* = \varphi(e^*) + \varphi((e')^*)$). Thus φ^* is an epimorphism. Therefore φ^* is an isomorphism, so we have $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$ as required. □

Note that if we take S to be the empty set then Theorem 3.3.8 simplifies to $L_K(E)/I(H) \cong L_K(E|H)$.

So far we have been exclusively considering graded ideals of the form $I_{(H,S)}$. However, as the following theorem shows, any graded ideal of $L_K(E)$ is in fact of the form $I_{(H,S)}$ for some admissible pair (H,S) of E. This result has been adapted from [To, Theorem 5.7(1)].

Theorem 3.3.9. *Let E be an arbitrary graph and let I be a graded ideal of $L_K(E)$. If we let $H = I \cap E^0$ and $S = \{w \in B_H : w^H \in I\}$, then $I = I_{(H,S)}$.*

Proof. Let I be a graded ideal of $L_K(E)$ and let H and S be the two sets described above. Clearly we have $I_{(H,S)} \subseteq I$, from the definition of $I_{(H,S)}$. By Theorem 3.3.8, there exists an isomorphism $\varphi^* : L_K(E\backslash(H,S)) \to L_K(E)/I_{(H,S)}$. Let $\pi : L_K(E)/I_{(H,S)} \to L_K(E)/I$ be the quotient map, so that $\pi(x + I_{(H,S)}) = x + I$. Note that this map is well-defined, since $I_{(H,S)} \subseteq I$. Consider $\pi\varphi^* : L_K(E\backslash(H,S)) \to L_K(E)/I$. Now, since I is graded so too is $L_K(E)/I$. Furthermore, both π and φ^* are graded (by definition), and so $\pi\varphi^*$ is also graded.

We wish to show that $\pi\varphi^*(v) \neq 0$ for any $v \in (E\backslash(H,S))^0$. Note that $\pi\varphi^*(v) = \pi(\varphi(v) + I_{(H,S)}) = \varphi(v) + I$, so it suffices to show that $\varphi(v) \notin I$ for all $v \in (E\backslash(H,S))^0$. We proceed in a similar fashion to the proof of Theorem 3.3.8. Suppose $v \in (E^0\backslash H)\backslash(B_H\backslash S)$. Then $\varphi(v) = v \notin I$, since $H = I \cap E^0$ (by definition) and $v \notin H$. Now suppose $v \in B_H\backslash S$. Then $\varphi(v) = \sum_{s(e)=v, r(e)\notin H} ee^*$. Suppose that $\varphi(v) \in I$ and choose a fixed edge f for which $s(f) = v$ and $r(f) \notin H$. Then $f^*\varphi(v)f = \sum_{s(e)=v,r(e)\notin H} f^*ee^*f = r(f) \in I$, since I is a two-sided ideal. However, this implies $r(f) \in H$, a contradiction, and so $\varphi(v) \notin I$. Furthermore, if $v' \in B'_H\backslash S'$ then $\varphi(v') = v^H \notin I$, since $v^H \in I$ implies $v \in S$ (by the definition of S), a contradiction as $v \in B_H\backslash S$. Therefore we have $\pi\varphi^*(v) \neq 0$ for any $v \in (E\backslash(H,S))^0$, as required.

Thus, since $\pi\varphi^*$ is a graded homomorphism between two graded rings, we can apply Theorem 2.2.13 to give that $\pi\varphi^*$ is injective. Since φ^* is an isomorphism, this implies that π is injective. Thus π must be the identity map and so $L_K(E)/I_{(H,S)} = L_K(E)/I$ and therefore $I_{(H,S)} = I$, as required. □

Note that if E is a row-finite graph, then E^0 cannot contain any breaking vertices and so the set S in the statement of Theorem 3.3.9 will always be empty. Thus in the row-finite case we have that $I = I(H)$ for any graded ideal I of $L_K(E)$, where $H = I \cap E^0$.

Since Theorem 3.3.9 tells us that all graded ideals of $L_K(E)$ are of the form $I_{(H,S)}$ for some admissible pair (H,S) of E, and Proposition 3.3.6 describes the structure of such an ideal, we can now describe the structure of any graded ideal of $L_K(E)$. We state this explicitly in the following corollary.

Corollary 3.3.10. *Let I be a graded ideal of $L_K(E)$. Then*

$$I = \text{span}\big(\{\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\} \cup \{\alpha w^H \beta^* : r(\alpha) = r(\beta) = w \in S\}\big),$$

where $H = I \cap E^0$ and $S = \{w \in B^H : w^H \in I\}$.

Theorem 3.3.9 also leads to the following useful corollary.

Corollary 3.3.11. *For an arbitrary graph E, the Jacobson radical $J(L_K(E)) = 0$.*

Proof. We know that $L_K(E)$ is $\mathbb{Z}$-graded and that E^0 is a set of local units for $L_K(E)$, with each element of E^0 homogeneous. Thus, by Lemma 1.1.8 we have that $J = J(L_K(E))$ is a graded ideal. Furthermore, Theorem 3.3.9 tells us that $J = J_{(H,S)}$, where $H = J \cap E^0$ and $S = \{w \in B^H : w^H \in J\}$. However, by Lemma 1.1.7 we know that $J(R)$ cannot contain any nonzero idempotents, and so $H = \emptyset$. By the definition of B_H, we must also have that $S = \emptyset$, and thus $J(L_K(E)) = 0$. □

We finish this section with a result that will prove useful when examining the socle series of a Leavitt path algebra in Section 3.4. This proof is based on the homomorphism $\phi : L_K(E) \to L_K(E\backslash(H,S))$ that we defined in the proof of Proposition 3.3.7, as well as the isomorphism given in Theorem 3.3.8. This result is stated in a simpler form in [ARM1, Theorem 1.7(ii)] and the reader is referred to Tomforde's [To, Theorem 5.7]. However, Tomforde does not prove this result explicitly, and so we provide details of the proof here.

Theorem 3.3.12. *Let E be an arbitrary graph and let H be a hereditary saturated subset of E^0. Then there is an algebra epimorphism $\phi : L_K(E) \to L_K(E\backslash(H,S))$ for which* $\ker(\phi) = I_{(H,S)}$.

Proof. Recall the homomorphism $\phi : L_K(E) \to L_K(E\backslash(H,S))$ from the proof of Proposition 3.3.7. To show that ϕ is an epimorphism, it suffices to show that ϕ maps onto the set of generators of $L_K(E\backslash(H,S))$; that is, each vertex, edge and ghost edge of $L_K(E\backslash(H,S))$ is in the image of ϕ. We begin by checking the vertices.

Case 1: $v \notin B_H\backslash S$. Then $\phi(v) = v$.

Case 2: $v' \in B'_H \backslash S'$. Then we have $\phi(v^H) = v'$ (see the final paragraph of the proof of Proposition 3.3.7).

Case 3: $v \in B_H \backslash S$. Then $\phi(v - v^H) = (v + v') - v' = v$.

Next, we check the edges.

Case 1: $r(e) \notin B_H \backslash S$. Then $\phi(e) = e$.

Case 2: $r(e') = v' \in B'_H \backslash S$. Then $\phi(ev^H) = (e + e')v' = e'$.

Case 3: $r(e) = v \in B_H \backslash S$. Then $\phi(e - ev^H) = (e + e') - e' = e$.

Similar arguments show that the ghost edges of $E\backslash(H, S)$ are also in the image of ϕ. Thus ϕ is an epimorphism, as required.

Since ϕ is an epimorphism, we have that $L_K(E)/\ker(\phi) \cong L_K(E\backslash(H, S))$. We denote this isomorphism by $\bar{\phi} : L_K(E)/\ker(\phi) \to L_K(E\backslash(H, S))$, where $\bar{\phi}(x + \ker(\phi)) = \phi(x)$. To complete the proof, we must show that $\ker(\phi) = I_{(H,S)}$. From the proof of Proposition 3.3.7, we know that $I_{(H,S)} \subseteq \ker(\phi)$. To show that we have equality, we first show that the isomorphism

$$\varphi^* \bar{\phi} : L_K(E)/\ker(\phi) \to L_K(E\backslash(H, S)) \to L_K(E)/I_{(H,S)}$$

sends $x + \ker(\phi)$ to $x + I_{(H,S)}$ for all $x \in L_K(E)$, where φ^* is the isomorphism defined in the proof of Theorem 3.3.8. To show this is true, it suffices to show it for the generators of $L_K(E)$, that is, the set $E^0 \cup E^1 \cup (E^1)^*$. For $v \in H$, we have $v \in I_{(H,S)} \subseteq \ker(\phi)$, so trivially $\varphi^* \bar{\phi}(v + \ker(\phi)) = v + I_{(H,S)}$, since $v + \ker(\phi)$ and $v + I_{(H,S)}$ are the zero elements of $L_K(E)/\ker(\phi)$ and $L_K(E)/I_{(H,S)}$, respectively. The same is true for all $e \in E^1$ and $e^* \in (E^1)^*$ with $r(e) \in H$.

For a generating element y that is not contained in $I_{(H,S)}$, it suffices to show that $\varphi\phi(y) = y$, since in that case we have $\varphi^* \bar{\phi}(y + \ker(\phi)) = \varphi^*(\phi(y)) = \varphi\phi(y) + I_{(H,S)} = y + I_{(H,S)}$. Consider $v \in (E^0\backslash H)\backslash(B_H\backslash S)$. Then $\varphi\phi(v) = \varphi(v) = v$. Similarly, we have $\varphi\phi(e) = \varphi(e) = e$ for all $e \in E^1$ (and $\varphi\phi(e^*) = \varphi(e^*) = e^*$ for all $e^* \in (E^1)^*$) with $r(e) \in (E^0\backslash H)\backslash(B_H\backslash S)$.

Now consider $v \in B_H \backslash S$. Then $\varphi\phi(v) = \varphi(v + v') = \sum_{s(e)=v, r(e)\notin H} ee^* + v^H = v$, by the definition of v^H. Thus, for any $e \in E^1$ with $r(e) \in B_H \backslash S$ we have $\varphi\phi(e) = \varphi(e + e') = e(\varphi(r(e)) + \varphi(r(e)')) = er(e) = e$. Similarly, for any $e^* \in (E^1)^*$ with $r(e) \in B_H \backslash S$ we have $\varphi\phi(e^*) = e^*$.

Thus $\varphi^*(\bar{\phi}(x + \ker(\phi))) = x + I_{(H,S)}$ for all $x \in L_K(E)$, as required. Suppose that $\ker(\phi) \neq I_{(H,S)}$, so that there exists some $a \in \ker(\phi)$ for which $a \notin I_{(H,S)}$. In that case $\varphi^*(\bar{\phi}(a + \ker(\phi))) = a + I_{(H,S)}$, which is impossible since $a + \ker(\phi)$ is zero in $L_K(E)/\ker(\phi)$ while $a + I_{(H,S)}$ is a nonzero element of $L_K(E)/I_{(H,S)}$. Thus $\ker(\phi) = I_{(H,S)}$, as required. □

Note that this proof relies on the fact that we already know that $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$ from Theorem 3.3.8. If we could show that $\ker(\phi) = I_{(H,S)}$ directly, then this would also prove $L_K(E)/I_{(H,S)} \cong L_K(E\backslash(H,S))$, making Theorem 3.3.8 redundant. However, while we can easily show that $I_{(H,S)} \subseteq \ker(\phi)$, it is not clear how to show that $\ker(\phi) \subseteq I_{(H,S)}$ without appealing to Theorem 3.3.8.

3.4 The Socle Series of a Leavitt Path Algebra

Definition 3.4.1. Let R be any ring and let $\tau = 2^{|R|}$. The **Loewy left ascending socle series**, or simply **left socle series**, of R is the well-ordered ascending chain of two-sided ideals

$$0 = S_0 \leq S_1 \leq \cdots \leq S_\alpha \leq S_{\alpha+1} \leq \cdots \qquad (\alpha < \tau)$$

where, for each $\alpha < \tau$,

$$\begin{array}{rcll} S_{\alpha+1}/S_\alpha &=& \mathrm{soc}_l(R/S_\alpha) & \text{if } \gamma = \alpha+1 \text{ is not a limit ordinal, and} \\ S_\gamma &=& \bigcup_{\alpha<\gamma} S_\alpha & \text{if } \gamma \text{ is a limit ordinal.} \end{array}$$

For each $\alpha < \tau$, S_α is called the **α-th left socle** of R (and in particular, $S_1 = \mathrm{soc}_l(R)$). The least ordinal λ for which $S_\lambda = S_{\lambda+1}$ is called the **left Loewy length** of R, denoted $l(R)$. If $R = S_\alpha$ for some α, then R is said to be a **left Loewy ring** (of length α).

Starting with the right socle of R, we can define the right socle series of R (and related terms) similarly.

Although the left and right socle series may differ in general, we will show in Corollary 3.4.8 that they coincide for Leavitt path algebras. (Note that we already know $\mathrm{soc}_l(L_K(E)) = \mathrm{soc}_r(L_K(E))$ by Corollary 3.2.2.) Thus, since we will henceforth only be concerned with the socle series of Leavitt path algebras, there is no need to specify 'left' or 'right' when using terms related to the socle series.

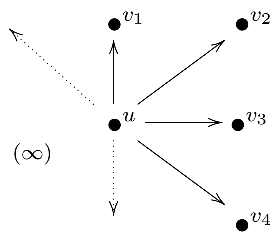
In this section we give several results regarding the socle series of an arbitrary Leavitt path algebra $L_K(E)$. In Theorem 3.4.7 we describe the α-th socle of $L_K(E)$ for all ordinals α, and describe precisely when $L_K(E)$ is a Loewy ring of length λ. Furthermore, in Theorem 3.4.12 we show that for any ordinal λ there exists a graph E for which $L_K(E)$ is a Loewy ring of length λ.

Example 3.4.2. We begin by examining the socle series of some familiar Leavitt path algebras.

(i) *The finite line graph M_n.* We saw in Example 3.2.13 that $\mathrm{soc}(L_K(M_n)) = S_1 = L_K(M_n)$. Thus $L_K(M_n)$, and therefore $\mathbb{M}_n(K)$, is a Loewy ring of length 1 (for all $n \in \mathbb{N}$).

(ii) *The rose with n leaves R_n.* In Example 3.2.13 we showed that $\mathrm{soc}(L_K(R_n)) = S_1 = 0$. By definition $S_2/S_1 = \mathrm{soc}(L_K(R_n)/S_1)$, and so $S_2 = \mathrm{soc}(L_K(R_n)) = 0$. Thus $S_\alpha = 0$ for all ordinals α, and in particular $L_K(R_n)$, and therefore $L(1, n)$, is certainly not a Loewy ring for any $n \in \mathbb{N}$.

(iii) *The infinite clock graph C_∞.* Recall that C_∞ looks like:



We saw in Example 3.2.13 that $\mathrm{soc}(L_K(C_\infty)) = I(H)$, where $H = \{v_i\}_{i=1}^{\infty}$. Since H is a hereditary saturated subset of E^0 (recalling that the saturated condition does not apply at infinite emitters), we can apply Theorem 3.3.8 to get $L_K(C_\infty)/I(H) \cong L_K(C_\infty|H) = L_K(\{u\}) \cong K$. Now, since the only ideals of K are $\{0\}$ and K, we have $\mathrm{soc}(K) = K$, and so $S_2/I(H) = \mathrm{soc}(L_K(C_\infty)/I(H)) = L_K(C_\infty)/I(H)$. Thus $S_2 = L_K(C_\infty)$ and so $L_K(C_\infty)$ is a Loewy ring with Loewy length 2.

We now look at a new example that will be integral to the proof of Theorem 3.4.12. This example is a combination of Examples 2.1, 2.5, 2.6 and 2.7 from [ARM1].

Example 3.4.3. We define a sequence of graphs P_n as follows. First, let P_0 be the graph consisting of a single vertex and no edges:

$$P_0 : \quad \bullet^{v}$$

Next, let P_1 be the 'infinite line' graph

$$P_1 : \qquad \bullet^{v_{1,1}} \longrightarrow \bullet^{v_{1,2}} \longrightarrow \bullet^{v_{1,3}} \longrightarrow \bullet^{v_{1,4}} \cdots\cdots>$$

Now we construct the graph P_2 from P_1 by adding a second row of vertices $\{v_{2,j} : j \in \mathbb{N}\}$ and edges from $v_{2,j}$ to $v_{2,j+1}$ for each $j \in \mathbb{N}$, effectively adding a second 'infinite line' graph. We then connect the two rows of vertices by adding an edge from $v_{2,j}$ to $v_{1,1}$, for each $j \in \mathbb{N}$, giving the graph

$$P_2 : \quad \begin{tikzcd} \bullet^{v_{1,1}} \arrow[r] & \bullet^{v_{1,2}} \arrow[r] & \bullet^{v_{1,3}} \arrow[r] & \bullet^{v_{1,4}} \arrow[r, dotted] & {} \\ \bullet^{v_{2,1}} \arrow[r] \arrow[u] & \bullet^{v_{2,2}} \arrow[r] \arrow[ul] & \bullet^{v_{2,3}} \arrow[r] \arrow[ull] & \bullet^{v_{2,4}} \arrow[r, dotted] \arrow[ulll] & {} \end{tikzcd}$$

In general, we construct the graph P_{i+1} from the graph P_i by adding vertices $\{v_{1+1,j} : j \in \mathbb{N}\}$ and, for each $j \in \mathbb{N}$, an edge from $v_{i+1,j}$ to $v_{i+1,j+1}$ and an edge from $v_{i+1,j}$ to $v_{i,1}$.

Now, $L_K(P_0) \cong K$, and so $\mathrm{soc}(L_K(P_0)) = L_K(P_0)$. Thus $L_K(P_0)$ is a Loewy ring with $l(L_K(P_0)) = 1$. In the graph P_1, every vertex is a line point, and so by Theorem 3.2.11 we have $\mathrm{soc}(L_K(P_1)) = I((P_1)^0) = L_K(P_1)$. Thus $L_K(P_1)$ is also a Loewy ring with $l(L_K(P_1)) = 1$.

For the graph P_2, the set of line points is the top row of vertices $H = \{v_{1,j} : j \in \mathbb{N}\}$. Note that H is both hereditary and saturated. Thus $\mathrm{soc}(L_K(P_2)) = I(H)$. Furthermore, note that the quotient graph $P_2|H$ consists of the 'bottom row' of vertices and edges and is clearly isomorphic as a graph to P_1. Thus, by Theorem 3.3.8 we have

$$L_K(P_2)/I(H) \cong L_K(P_2|H) \cong L_K(P_1),$$

and since $L_K(P_1)$ is a Loewy ring with $l(L_K(P_1)) = 1$, $L_K(P_2)$ is therefore a Loewy ring with $l(L_K(P_2)) = 2$.

Using induction, it is easy to see that $L_K(P_n)$ is a Loewy ring with $l(L_K(P_n)) = n$ for all $n \in \mathbb{N}$: to begin, we know this statement is true for $n = 1$ and $n = 2$. Now assume it is true for $n = i$ and consider the graph P_{i+1}. The line points of P_{i+1} are again the set $H = \{v_{1,j} : j \in \mathbb{N}\}$, and $P_{i+1}|H$ is isomorphic to P_i. Thus, as above, $L_K(P_{i+1}|H) \cong L_K(P_i)$, and since $L_K(P_i)$ is a Loewy ring with $l(L_K(P_i)) = i$ (by our assumption), $L_K(P_{i+1})$ is therefore a Loewy ring with $l(L_K(P_{i+1})) = i + 1$, as required.

If we view P_i as being contained in P_{i+1} for each $i \in \mathbb{N}$, then $\{P_i\}_{i\in\mathbb{N}}$ is an ascending chain of graphs. Thus, with ω denoting the first infinite ordinal, we can form the graph

$$P_\omega = \bigcup_{i<\omega} P_i.$$

Then, using the same argument as above, $P_\omega|P_l(P_\omega) \cong P_\omega$, with $S_1 \subset S_2 \subset S_3 \subset \cdots \subset \bigcup_{i<\omega} S_i = L_K(P_\omega)$, and $L_K(P_\omega)$ is a Loewy ring with Loewy length ω.

We now define a sequence of graphs Q_n that are very similar to the graphs P_n, except for one subtle but important difference. This example is from [ARM1, Example 2.8].

Example 3.4.4. Let Q_1 be the infinite line graph:

$$Q_1 : \qquad \begin{tikzcd} \bullet^{w_{1,1}} \arrow[r] & \bullet^{w_{1,2}} \arrow[r] & \bullet^{w_{1,3}} \arrow[r] & \bullet^{w_{1,4}} \arrow[r, dotted] & {} \end{tikzcd}$$

As in the previous example, we now add a second 'infinite line' graph, but this time we connect the two rows of vertices from the lower to the upper by adding an edge from $w_{1,j}$ to $w_{2,1}$ for each $j \in \mathbb{N}$, giving the graph

$$Q_2 : \qquad \begin{tikzcd} \bullet^{w_{2,1}} \arrow[r] & \bullet^{w_{2,2}} \arrow[r] & \bullet^{w_{2,3}} \arrow[r] & \bullet^{w_{2,4}} \arrow[r, dotted] & {} \\ \bullet^{w_{1,1}} \arrow[r] \arrow[u] & \bullet^{w_{1,2}} \arrow[r] \arrow[ul] & \bullet^{w_{1,3}} \arrow[r] \arrow[ull] & \bullet^{w_{1,4}} \arrow[r, dotted] \arrow[ulll] & {} \end{tikzcd}$$

In general, we construct the graph Q_{i+1} from the graph Q_i by adding vertices $\{w_{1+1,j} : j \in \mathbb{N}\}$ and, for each $j \in \mathbb{N}$, an edge from $w_{i+1,j}$ to $w_{i+1,j+1}$ and an edge from $w_{i,j}$ to $w_{i+1,1}$. Contrast this with the construction of P_{i+1}, in which we add an edge from $v_{i+1,j}$ to $v_{i,1}$ for each $j \in \mathbb{N}$. Despite this difference, it is clear that the graph Q_i is isomorphic to the graph P_i for each $i \in \mathbb{N}$. Thus the Leavitt path algebra $L_K(Q_n)$ is a Loewy ring with $l(L_K(Q_n)) = n$ for all $n \in \mathbb{N}$.

Once again, viewing Q_i as being contained in Q_{i+1} for each $i \in \mathbb{N}$, we can form the graph $Q_\omega = \bigcup_{i<\omega} Q_i$. This is where the two examples diverge. For each $i \in \mathbb{N}$, $P_l(P_i) = \{v_{1,j} : j \in \mathbb{N}\}$ (which is independent of i) and so $P_l(P_\omega) = \{v_{1,j} : j \in \mathbb{N}\}$. However, $P_l(Q_i) = \{w_{i,j} : j \in \mathbb{N}\}$ for each $i \in \mathbb{N}$, and so Q_ω has no line points. Therefore $\mathrm{soc}(L_K(Q_\omega)) = \{0\}$, and so $S_\alpha = \{0\}$ for each α. Thus, while $L_K(P_\omega)$ is a Loewy ring, its counterpart $L_K(Q_\omega)$ is not.

We now give a definition that is an integral part of Theorem 3.4.7. This definition is from [ARM1, Definition 3.1], although this version differs from the published version for reasons that will be explained after the proof of Theorem 3.4.7.

Definition 3.4.5. Let E be an arbitrary graph and let $L_K(E)$ be its associated Leavitt path algebra. Recall the definitions of the quotient graph $E\backslash(H, S)$ and v^H from Section 3.3. For each ordinal γ, we define transfinitely a subset V_γ of E^0 as follows.

(i) V_1 is the hereditary saturated closure of the set $P_l(E)$.

Suppose $\gamma > 1$ is any ordinal and that the sets V_α have been defined for all $\alpha < \gamma$. Let S_α denote the α-th socle of $L_K(E)$ and define $B_\alpha := \{w \in B_{V_\alpha} : w^{V_\alpha} \in S_\alpha\}$.

(ii) If $\gamma = \alpha + 1$ is a non-limit ordinal, then $V_\gamma = E^0 \cap I(V'_{\alpha+1})$, defining

$$V'_{\alpha+1} = V_\alpha \cup W_\alpha \cup Z_\alpha,$$

where

$$W_\alpha = \{w \in E^0\backslash V_\alpha : w \text{ is a line point in the quotient graph } E\backslash(V_\alpha, B_\alpha)\}$$

and

$$Z_\alpha = \left\{ v - \sum_{\substack{s(e)=v,\\ r(e)\notin V_\alpha}} ee^* : v \in B_{V_\alpha} \backslash B_\alpha \right\}$$

(iii) If γ is a limit ordinal, then $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$.

Lemma 3.4.6. *Each subset V_γ (as defined in Definition 3.4.5) is a hereditary saturated subset of E^0.*

Proof. We know that the set of line points of E must be a hereditary subset of E^0 since, given a vertex $v \in P_l(E)$, every vertex $w \in T(v)$ must also be a line point, by definition. Thus V_1, the hereditary saturated closure of $P_l(E)$, must be a hereditary saturated subset of E^0.

If γ is a non-limit ordinal, then $V_\gamma = E^0 \cap I(V'_{\alpha+1})$, where $I(V'_{\alpha+1})$ is as defined above. Since $I(V'_{\alpha+1})$ is an ideal, V_γ must be a hereditary saturated subset, by Lemma 2.2.1.

For the case where γ is a limit ordinal, take a vertex $v \in V_\gamma$ and a vertex $w \in T(v)$. Since $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$, we must have $v \in V_\alpha$ for some $\alpha < \gamma$, and since V_α is hereditary, we have $w \in V_\alpha$ and so $w \in V_\gamma$. Now suppose that u is a regular vertex in E^0 such that, for each $e_i \in s^{-1}(u)$, we have $r(e_i) \in V_\gamma$. Since $V_\alpha \subseteq V_{\alpha+1}$ for each $\alpha < \gamma$, there must exist some $\alpha < \gamma$ for which $r(e_i) \in V_\alpha$ for all $e_i \in s^{-1}(u)$. Then, since V_α is saturated, we must have that $u \in V_\alpha$ and thus $u \in V_\gamma$, as required. □

We now come to the main result of this section. This theorem is from [ARM1, Theorem 3.2], although it differs from the published version, which the author found to be incorrect for a number of reasons. After correspondence with one of the authors of the paper, the theorem was adjusted to the current version below. The differences between versions and why the changes were made will be discussed after the proof. The proof has also been expanded to clarify some of the arguments used.

Theorem 3.4.7. *Let E be an arbitrary graph and let $L_K(E)$ be its associated Leavitt path algebra. For each ordinal α, let S_α denote the α-th socle of $L_K(E)$, and let V_α and B_α be the subsets of E^0 and B_{V_α}, respectively, defined in Definition 3.4.5. Then* *(i)* S_α *is a graded ideal for each* α*;*

(ii) $V_\alpha = E^0 \cap S_\alpha$ *for each* α*;*

(iii) $S_\alpha = I_{(V_\alpha, B_\alpha)}$ *for each* α*;*

(iv) $L_K(E)/S_\alpha \cong L_K(E\backslash(V_\alpha, B_\alpha))$ *as graded* K*-algebras for each* α*; and*

(v) $L_K(E)$ *is a Loewy ring of length* λ *if and only if* λ *is the smallest ordinal such that* $E^0 = V_\lambda$*.*

Proof. We prove (i)-(iv) simultaneously by transfinite induction. For $\gamma = 1$, V_1 has been defined as the saturated closure of the set of line points of E. Thus, by Theorem 3.2.11, we have $S_1 = \mathrm{soc}(L_K(E)) = I(V_1)$. By Proposition 3.3.6, S_1 is a graded ideal (proving (i)), and by Proposition 3.3.7 we have $S_1 \cap E^0 = V_1$ (proving (ii)). Since $S_1 = I(V_1) = I_{(V_1,\emptyset)}$, Proposition 3.3.7 gives $B_1 = \{w \in B_{V_1} : w^{V_1} \in S_1\} = \emptyset$ and thus $S_1 = I_{(V_1,B_1)}$, proving (iii). Now (iv) follows directly from (iii) and Theorem 3.3.8, and so we have shown that (i)-(iv) hold for the case $\gamma = 1$.

Now suppose that $\gamma > 1$ and that properties (i)-(iv) hold for all $\alpha < \gamma$. Suppose that γ is a non-limit ordinal, so that $\gamma = \alpha + 1$ for some α, and suppose that $V_\alpha \neq E^0$ (so that $S_\alpha \neq L_K(E)$.) Recall (from Definition 3.3.2) the sets B_{V_α}, B'_{V_α} and the quotient graph $L_K(E\backslash(V_\alpha, B_\alpha))$. Let $\phi : L_K(E) \to L_K(E\backslash(V_\alpha, B_\alpha))$ be the epimorphism defined in the proof of Proposition 3.3.7 and $V'_{\alpha+1}$ be the set defined in Definition 3.4.5. Consider $\phi(V'_{\alpha+1}) = \phi(V_\alpha) \cup \phi(W_\alpha) \cup \phi(Z_\alpha)$, a subset of $L_K(E\backslash(V_\alpha, B_\alpha))$. By the definition of ϕ, we have $\phi(V_\alpha) = \{0\}$.

Let $W_\alpha = \{v_1, v_2, \ldots, w_1, w_2, \ldots\}$, where each $v_i \in (E^0\backslash V_\alpha)\backslash(B_{V_\alpha}\backslash B_\alpha)$ and each $w_i \in B_{V_\alpha}\backslash B_\alpha$. Thus

$$\phi(W_\alpha) = \{v_1, v_2, \ldots, w_1 + w'_1, w_2 + w'_2, \ldots\}.$$

Recalling from the proof of Proposition 3.3.7 that $\phi(u_i - \sum_{s(e)=u_i, r(e)\notin V_\alpha} ee^*) = u'_i$ for each $u_i \in B_{V_\alpha}\backslash B_\alpha$, we also have $\phi(Z_\alpha) = B'_{V_\alpha}\backslash B'_\alpha$. Thus

$$\phi(V'_{\alpha+1}) = \{v_1, v_2, \ldots, w_1 + w'_1, w_2 + w'_2, \ldots\} \cup (B'_{V_\alpha}\backslash B'_\alpha).$$

Now, since each $w_i' \in B'_{V_\alpha} \backslash B'_\alpha$, each $w_i = (w_i + w_i') - w_i' \in I(\phi(V'_{\alpha+1}))$, and so

$$I(\phi(V'_{\alpha+1})) = I(\{v_1, v_2, \ldots, w_1, w_2, \ldots\} \cup (B'_{V_\alpha} \backslash B'_\alpha)) = I(W_\alpha \cup (B'_{V_\alpha} \backslash B'_\alpha)).$$

By definition, W_α is the set of all line points in $E\backslash(V_\alpha, B_\alpha)$ that are also vertices in the original graph E. Furthermore, the only new vertices introduced into $E\backslash(V_\alpha, B_\alpha)$ are the set $B'_{V_\alpha} \backslash B'_\alpha$, which are sinks (and therefore line points) by definition. Thus $P_l(E\backslash(V_\alpha, B_\alpha)) = W_\alpha \cup (B'_{V_\alpha} \backslash B'_\alpha)$ and so, by Theorem 3.2.11,

$$\operatorname{soc}(L_K(E\backslash(V_\alpha, B_\alpha)) = I(P_l(E\backslash(V_\alpha, B_\alpha))) = I(W_\alpha \cup (B'_{V_\alpha} \backslash B'_\alpha)) = I(\phi(V'_{\alpha+1})).$$

Now, by our induction hypothesis we have $I_{(V_\alpha, B_\alpha)} = S_\alpha$, and so by Theorem 3.3.8 we have $L_K(E)/S_\alpha \cong L_K(E\backslash(V_\alpha, B_\alpha))$. Specifically, the function $\bar{\phi} : L_K(E)/S_\alpha \to L_K(E\backslash(V_\alpha, B_\alpha))$ with $\bar{\phi}(x + S_\alpha) = \phi(x)$ is an isomorphism. Thus, from the socle series definition we have

$$S_{\alpha+1}/S_\alpha = \operatorname{soc}(L_K(E)/S_\alpha) \cong \operatorname{soc}(L_K(E\backslash(V_\alpha, B_\alpha))) = I(\phi(V'_{\alpha+1})) = \phi(I(V'_{\alpha+1})).$$

Thus $\bar{\phi}(S_{\alpha+1}/S_\alpha) = \phi(I(V'_{\alpha+1}))$, and so

$$S_{\alpha+1}/S_\alpha = \bar{\phi}^{-1}(\phi(I(V'_{\alpha+1}))) = I(V'_{\alpha+1})/S_\alpha,$$

giving $S_{\alpha+1} = I(V'_{\alpha+1})$. Thus $V_{\alpha+1} = I(V'_{\alpha+1}) \cap E^0 = S_{\alpha+1} \cap E^0$, proving (ii).

By our inductive hypothesis, S_α and $\operatorname{soc}(L_K(E\backslash(V_\alpha, B_\alpha))$ are graded. Furthermore, $S_{\alpha+1}/S_\alpha \cong \operatorname{soc}(L_K(E\backslash(V_\alpha, B_\alpha)))$ and so $S_{\alpha+1}$ is graded, proving (i). Since $S_{\alpha+1}$ is graded, by Theorem 3.3.9 $S_{\alpha+1} = I_{(H,S)}$, where $H = S_{\alpha+1} \cap E^0$ and $S = \{w \in B_H : w^H \in S_{\alpha+1}\}$. From above, we have $H = V_{\alpha+1}$ and so $S = B_{\alpha+1}$ (by the definition of $B_{\alpha+1}$). Thus $S_{\alpha+1} = I_{(V_{\alpha+1}, B_{\alpha+1})}$, proving (iii). Again, (iv) follows directly from (iii) and Theorem 3.3.8.

Thus we have shown properties (i)-(iv) for when γ is not a limit ordinal. If γ is a limit ordinal, then by definition $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$ and $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$. Since each S_α is graded, S_γ is also graded, proving (i). Furthermore, if $V_\alpha = S_\alpha \cap E^0$ for each $\alpha < \gamma$ then it follows that $V_\gamma = S_\gamma \cap E^0$, proving (ii). As above, the fact that $S_\gamma = I_{(V_\gamma, B_\gamma)}$ follows from (i) and (ii) and the definition of B_γ, and (iv) follows directly from (iii) and Theorem 3.3.8. Thus we have established (i)-(iv) for all γ.

Finally, note that $L_K(E)$ is a Loewy ring of length λ if and only if λ is the smallest ordinal for which $S_\lambda = L_K(E)$, by definition. By Lemma 2.2.3, $S_\lambda = L_K(E)$ if and only if $S_\lambda \cap E^0 = E^0$, that is, $V_\lambda = E^0$ (by (ii)). Thus $L_K(E)$ is a Loewy ring of length λ if and only if λ is the smallest ordinal for which $V_\lambda = E^0$, proving (v). $\square$

The primary error in the original proof of [ARM1, Theorem 3.2] was the assumption that $S_\alpha = I(V_\alpha)$ rather than $S_\alpha = I_{(V_\alpha,B_\alpha)}$. While the property $S_\alpha = I(V_\alpha)$ was not stated explicitly in the theorem itself, the assumption is implied when [ARM1, Theorem 1.7(ii)] is invoked to give $L_K(E)/S_\alpha \cong L_K(E|V_\alpha)$ during the induction process. As shown in the proof above, the fact that $V_\alpha = E^0 \cap S_\alpha$ (together with Theorem 3.3.9) implies directly that $S_\alpha = I_{(V_\alpha,B_\alpha)}$, and $I_{(V_\alpha,B_\alpha)} \neq I(V_\alpha)$ unless $B_\alpha = \emptyset$, which is not true in general. Thus we have changed the proof of Theorem 3.4.7 accordingly and have added the statement $S_\alpha = I_{(V_\alpha,B_\alpha)}$ as property (iii) for clarity. Furthermore, [ARM1, Theorem 3.2(3)] states that '$L_K(E)/S_\alpha \cong L_K(E|V_\alpha)$ as graded K-algebras for each K'; here we have changed that to '$L_K(E)/S_\alpha \cong L(E\backslash(V_\alpha, B_\alpha))$ as graded K-algebras for each α' in property (iv).

Furthermore, recall that we define W_α in Definition 3.4.5 as

$$W_\alpha = \{w \in E^0\backslash V_\alpha : w \text{ is a line point in the quotient graph } E\backslash(V_\alpha, B_\alpha)\},$$

and that this definition allows us to conclude in the proof of Theorem 3.4.7 that $P_l(E\backslash(V_\alpha, B_\alpha)) = W_\alpha \cup (B'_{V_\alpha}\backslash B'_\alpha)$, an equality that is central to the proof. In [ARM1, Definition 3.1], the corresponding set is defined as

$$\{w \in E^0\backslash V_\alpha : \text{every bifurcation vertex } u \in T_E(w)\backslash V_\alpha \text{ has at most one edge } e \text{ with } s(e) = u \text{ and } r(e) \notin V_\alpha\}.$$

However, such vertices will not necessarily be line points in the quotient graph $E\backslash(V_\alpha, B_\alpha)$, since there is the possibility that a new edge e' with $s(e') = u \in T_E(w)\backslash V_\alpha$ will be added in the construction of $E\backslash(V_\alpha, B_\alpha)$, making u a bifurcation in the quotient graph. Hence we have modified the definition in our version.

As promised at the beginning of this section, we now show that the left and right socle series of a Leavitt path algebra coincide.

Corollary 3.4.8. *Let E be an arbitrary graph. For any ordinal $\alpha < 2^{|L_K(E)|}$, the α-th left socle of $L_K(E)$ is equal to the α-th right socle of $L_K(E)$.*

Proof. We proceed using transfinite induction. For ease of notation we will denote α-th left socle of $L_K(E)$ by S_α and the α-th right socle of $L_K(E)$ by T_α. For the case $\alpha = 1$, we have $S_1 = \mathrm{soc}_l(L_K(E)) = \mathrm{soc}_r(L_K(E)) = T_1$, by Corollary 3.2.2.

Now let $1 < \alpha < 2^{|L_K(E)|}$ and suppose that $S_\beta = T_\beta$ for all $\beta < \alpha$. Moreover, suppose that $\alpha = \beta+1$, where β is not a limit ordinal. Then, applying Corollary 3.2.2 and Theorem 3.4.7 (iv), we have

$$\begin{aligned} S_\alpha/S_\beta = \mathrm{soc}_l(L_K(E)/S_\beta) &\cong \mathrm{soc}_l(L_K(E\backslash(V_\beta, B_\beta))) \\ &= \mathrm{soc}_r(L_K(E\backslash(V_\beta, B_\beta))) \cong \mathrm{soc}_r(L_K(E)/T_\beta) = T_\alpha/T_\beta, \end{aligned}$$

and since $S_\beta = T_\beta$ we therefore have $S_\alpha = T_\alpha$.

Now suppose that α is a limit ordinal. Then we have $S_\alpha = \bigcup_{\beta<\alpha} S_\beta = \bigcup_{\beta<\alpha} T_\beta = T_\alpha$, completing the proof. □

We now proceed with several ring-theoretic results related to the socle series of a Leavitt path algebra. Because some of these results rely on Theorem 3.4.7, the proofs have had to be subtly adjusted. However, these adjustments have not led to any changes in the results themselves. The first result is from [ARM1, Proposition 3.3].

Proposition 3.4.9. *Let E be an arbitrary graph and let S_α be the α-th socle of $L_K(E)$. Each S_α is a von Neumann regular ring.*

Proof. It is known (see for example [J2, pages 65, 90]) that for a semiprime ring R, $\mathrm{soc}(R)$ is a direct sum of simple rings T_i and that each T_i is the directed union of full matrix rings over division rings. By the remark on p.67 of [L1], a matrix ring over a division ring is von Neumann regular, and thus a directed union of matrix rings over division rings must be von Neumann regular. Since $\mathrm{soc}(R)$ is a direct sum of such rings, it must also be von Neumann regular. Now we know that $L_K(E)$ is semiprime by Proposition 3.2.1, and so $S_1 = \mathrm{soc}(L_K(E))$ is von Neumann regular.

We proceed by transfinite induction. Suppose that $\gamma > 1$ and assume that S_α is von Neumann regular for each $\alpha < \gamma$. Suppose γ is not a limit ordinal, so that $\gamma = \alpha+1$ for some α. By Theorem 3.4.7 (iv), we have $L_K(E)/S_\alpha \cong L_K(E\backslash(V_\alpha, B_\alpha))$, and so $S_{\alpha+1}/S_\alpha = \mathrm{soc}(L_K(E)/S_\alpha) \cong \mathrm{soc}(L_K(E\backslash(V_\alpha, B_\alpha)))$. Since the socle of a Leavitt path algebra is von Neumann regular (by the paragraph above), we have that $S_{\alpha+1}/S_\alpha$ is von Neumann regular. Furthermore, S_α is von Neumann regular by our inductive hypothesis and so $S_{\alpha+1}$ is von Neumann regular (by Lemma 1.1.9).

If γ is a limit ordinal, then $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$. Take $a \in S_\gamma$. Then $a \in S_\alpha$ for some $\alpha < \gamma$, and since S_α is von Neumann regular by our inductive hypothesis there exists $x \in S_\alpha \subseteq S_\gamma$ for which $a = axa$. Thus S_γ is von Neumann regular, completing the proof. $\square$

Proposition 3.4.9, together with the yet-to-come Theorem 4.2.3, leads to the following corollary.

Corollary 3.4.10. *Let E be an arbitrary graph. If $L_K(E)$ is a Loewy ring, then E is acyclic and $L_K(E)$ is locally K-matricial and von Neumann regular.*

Proof. If $L_K(E)$ is a Loewy ring then $L_K(E) = S_\alpha$ for some α. Thus $L_K(E)$ is von Neumann regular (by Proposition 3.4.9) and so, by Theorem 4.2.3, E is acyclic and $L_K(E)$ is locally K-matricial. $\square$

Note that the converse is not true: recall the graph Q_ω from Example 3.4.3, which was acyclic but not a Loewy ring since $S_\alpha = \{0\}$ for all α. However, the following corollary (from [ARM1, Corollary 3.5]) shows that we have equivalence when E^0 is finite.

Corollary 3.4.11. *Let E be a graph for which E^0 is finite. The following statements are equivalent.*

(i) *$L_K(E)$ is a Loewy ring*

(ii) *E is acyclic*

(iii) *$L_K(E)$ is von Neumann regular.*

Furthermore, if E^1 is also finite, then the previous conditions are equivalent to

(iv) $L_K(E)$ is semisimple (in this case, we have $l(L_K(E)) = 1$).

Proof. Theorem 4.2.3 gives (ii) $\iff$ (iii), while Corollary 3.4.10 gives (i)$\Rightarrow$(ii). To show (ii)$\Rightarrow$(i), suppose that E is acyclic. Since E^0 is finite, this implies that E must contain at least one sink, and so $P_l(E) \neq \emptyset$. Recall from Definition 3.4.5 that $V_1 = \overline{P_l(E)} \neq \emptyset$. If $V_1 = E^0$ then $L_K(E)$ is a Loewy ring (of length 1) by Theorem 3.4.7 (v). If not, then we can form the quotient graph $E\backslash(V_1, B_1)$. This graph must also be acyclic, since the only edges added in the construction of the quotient graph end in sinks, by definition. Now, if the added vertices $v' \in B'_{V_1}\backslash B'_1$ are the only sinks in $E\backslash(V_1, B_1)$, then the graph $(E\backslash(V_1, B_1)) \setminus ((B'_{V_1}\backslash B'_1) \cup \{e' \in (E\backslash(V_1, B_1))^1\})$ contains no sinks, a contradiction since this is also a finite and acyclic graph. Thus $E\backslash(V_1, B_1)$ must contain a vertex from the original graph E that is a sink in $E\backslash(V_1, B_1)$.

Now, by definition V_2 contains the sets V_1 and

$$W_1 = \{w \in E^0\backslash V_1 : w \text{ is a line point in the quotient graph } E\backslash(V_1, B_1)\}.$$

By our observation above, $W_1 \neq \emptyset$ and so $V_1 \subset V_2$, giving $|V_2| > |V_1|$. Again, either $V_2 = E^0$, in which case we are done, or we can repeat the above argument to show that $|V_3| > |V_2|$, and so on. Since E^0 is finite, this ascending chain of subsets of E^0 must stop, eventually giving $V_n = E^0$ for some $n \in \mathbb{N}$. Thus $L_K(E)$ is a Loewy ring by Theorem 3.4.7 (v).

Now suppose that E^1 is finite and E is acyclic. Then, by Lemma 2.2.9, $L_K(E)$ is isomorphic to a direct sum of matrix rings over K. Since each matrix ring is simple (by Lemma 1.1.10), $L_K(E)$ is therefore the direct sum of simple left ideals and so is semisimple, showing (ii)$\Rightarrow$(iv). If $L_K(E)$ is semisimple, then $\text{soc}(L_K(E)) = L_K(E)$ and so $l(L_K(E)) = 1$, as required. Thus $L_K(E)$ is a Loewy ring, showing (iv)$\Rightarrow$(i) and completing the proof. $\square$

We now come to the second main result of this section, which is from [ARM1, Theorem 4.1].

Theorem 3.4.12. *For every ordinal λ and any field K, there exists an acyclic graph P_λ for which $L_K(P_\lambda)$ is a Loewy ring of length λ.*

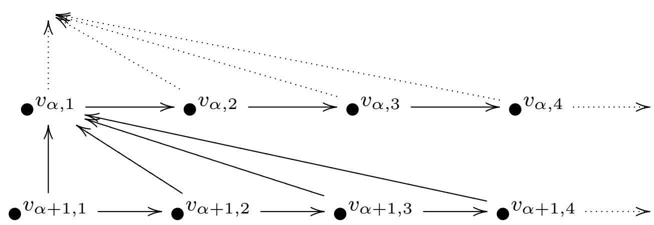
Proof. We construct a series of graphs P_n that transfinitely extends the series introduced in Example 3.4.3. For $\lambda = 1$, we choose $E = P_1$, the 'infinite line' graph

$$P_1 : \qquad \bullet^{v_{1,1}} \longrightarrow \bullet^{v_{1,2}} \longrightarrow \bullet^{v_{1,3}} \longrightarrow \bullet^{v_{1,4}} \dashrightarrow .$$

P_1 is clearly acyclic.

Now suppose that $\lambda \geq 2$ is any ordinal, and suppose that the graphs P_α have been defined for all $\alpha < \lambda$ and that each P_α has Loewy length α. There are three possibilities for λ. First, suppose that $\lambda = \alpha + 1$, where α is not a limit ordinal. Then, in a similar manner to Example 3.4.3, we construct the graph $P_{\alpha+1}$ from P_α by adding vertices $\{v_{\alpha+1,j} : j \in \mathbb{N}\}$ and, for each $j \in \mathbb{N}$, an edge from $v_{\alpha+1,j}$ to $v_{\alpha+1,j+1}$ and an edge from $v_{\alpha+1,j}$ to $v_{\alpha,1}$, giving

$P_{\alpha+1}$:

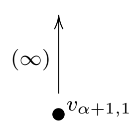


Secondly, if λ is a limit ordinal then we define

$$P_\lambda = \bigcup_{\alpha<\lambda} P_\alpha.$$

Finally, suppose that $\lambda = \alpha + 1$, where α is a limit ordinal. Then we construct the graph $P_{\alpha+1}$ from P_α by adding a single vertex $v_{\alpha+1,1}$ and, for each $\beta < \alpha$, an edge from $v_{\alpha+1,1}$ to $v_{\beta,1}$, giving

$P_{\alpha+1}$: $\quad P_\alpha \quad \cup$



Note that in each case P_λ is acyclic (as required) and P_α is a subgraph of P_λ for all $\alpha < \lambda$, giving a chain of subgraphs $P_1 \subset P_2 \subset \cdots \subset P_{\lambda-1} \subset P_\lambda$.

We now show by transfinite induction that $l(L_K(P_\alpha)) = \alpha$ for each ordinal α. We do this by showing that α is the smallest ordinal for which $V_\alpha = P_\alpha^0$ and then applying Theorem 3.4.7 (v). For $\alpha = 1$, we have $V_1 = P_l(P_1) = P_1^0$, since every vertex in P_1 is a line point.

Now let λ be any ordinal greater than 1 and suppose that $L_K(P_\alpha)$ is a Loewy ring of length α for all $\alpha < \lambda$. Since each P_α can be viewed as a subgraph of P_λ, this is equivalent to assuming that $V_\alpha = P_\alpha^0$ for each $\alpha < \lambda$, where each V_α is a subset of P_λ^0.

Suppose that $\lambda = \beta + 1$, where β is not a limit ordinal. Now $V_\beta = P_\beta^0$, and since there are no infinite emitters going into V_β we have $B_{V_\beta} = \emptyset$. Thus it is easy to see that the definition of V_λ simplifies to $V_\lambda = E^0 \cap I(V_\beta \cup W_\beta)$, where W_β is the set

$$W_\beta = \{w \in P_\lambda^0 \backslash V_\beta : w \text{ is a line point in } P_\lambda | V_\beta\}.$$

Now, since $B_{V_\beta} = \emptyset$ we have $P_\lambda|V_\beta \cong P_1$, and so every vertex in the set $(P_\lambda|V_\beta)^0 = \{v_{\beta+1,j} : j \in \mathbb{N}\}$ is a line point. Thus $V_\lambda = E^0 \cap I(P_\beta^0 \cup \{v_{\beta+1,j} : j \in \mathbb{N}\}) = E^0 \cap I(P_\lambda^0) = P_\lambda^0$. Furthermore, since $V_\beta \neq P_\lambda^0$, λ is indeed the smallest ordinal for which $V_\lambda = P_\lambda^0$, as required.

Now suppose that $\lambda = \beta + 1$, where β is a limit ordinal. Since $v_{\beta+1,1}$ emits an infinite number of edges into $V_\beta = P_\beta^0$ and no edges into the rest of the graph, we again have $B_{V_\beta} = \emptyset$. Furthermore, every vertex in the graph $P_\lambda|V_\beta \cong P_0$ is a line point,[1] and so, by the same argument as above, we have that λ is the smallest ordinal for which $V_\lambda = P_\lambda^0$, as required.

Finally, suppose that λ is a limit ordinal. Then $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha = \bigcup_{\alpha<\lambda} P_\alpha^0 = P_\lambda^0$, completing the proof. $\square$

We finish this section with a result from [ARM1, Theorem 4.2] that shows there exists an upper bound on the Loewy length of the Leavitt path algebra of a row-finite graph. The proof of this theorem refers to the set W_α from Definition 3.4.5.

[1] Recall from Example 3.4.3 that P_0 is the graph consisting of a single vertex and no edges.

As noted earlier, this definition is different from the one seen in [ARM1, Definition 3.1] and so we have had to modify the following proof accordingly.

Theorem 3.4.13. *Suppose that the Leavitt path algebra $L_K(E)$ is a Loewy ring. If E is a row-finite graph then $L_K(E)$ must have Loewy length $\leq \omega_1$, the first uncountable ordinal.*

Proof. Suppose that $L_K(E)$ has Loewy length greater than ω_1. Let S_{ω_1} be the ω_1-st socle of $L_K(E)$ and let V_{ω_1} be the set of vertices defined in Definition 3.4.5. Recall from the definition that V_{ω_1+1} contains the set

$$W_{\omega_1} = \{w \in E^0 \backslash V_{\omega_1} : w \text{ is a line point in } E|V_{\omega_1}\},$$

noting that $B_{\omega_1} = \emptyset$ since E is row-finite. Consider a fixed $w \in W_{\omega_1}$. Now w is not a line point in E, for otherwise we would have $w \in V_1 \subset V_{\omega_1}$. Furthermore, since $L_K(E)$ is a Loewy ring then E must be acyclic (by Corollary 3.4.10), and in particular there cannot be a cycle based at any vertex in $T_E(w)$. Thus $T_E(w)$ must contain at least one bifurcation.

Let $U = \{u_1, u_2, \ldots\}$ be the set of bifurcation vertices in $T_E(w)$ that are also contained in $T_{E|V_{\omega_1}}(w)$ (though indeed they are not bifurcations in $E|V_{\omega_1}$ since w is a line point in $E|V_{\omega_1}$). Since E is row-finite, for a fixed positive integer n the number of paths of length n with source w is finite, and so the number of vertices in $T_E(w)$ is at most countable and thus $|U|$ is at most countable. Furthermore, U is not empty. To see this, suppose that U is empty and let p be a path of minimum length in E from w to a bifurcation $u \in T_E(w)$. Since the only vertices removed in the construction of $E|V_{\omega_1}$ are in the set V_{ω_1}, if $u \notin T_{E|V_{\omega_1}}(w)$ then there must be a vertex $v \in p^0$ for which $v \in V_{\omega_1}$ (noting that we may have $v = u$). However, since there are no bifurcations between w and v, the saturated nature of V_{ω_1} implies that $w \in V_{\omega_1}$, a contradiction. Thus U is not empty. Note that, by definition, each $u_i \notin V_{\omega_1}$.

As noted above, each $u_i \in U$ cannot be a bifurcation in $E|V_{\omega_1}$. Thus each u_i must emit at most one edge into $E^0 \backslash V_{\omega_1}$ and so, since it is a bifurcation in E, u_i

emits at least one edge into V_{ω_1}. For each $u_i \in U$, let $s^{-1}(u_i) = \{e_{i_1}, \ldots, e_{i_{k(i)}}\}$ (a finite set since E is row-finite), and define

$$J_i = \{e_{i_j} \in s^{-1}(u_i) : r(e_{i_j}) \in V_{\omega_1}\}.$$

Note that each of these sets is nonempty since each u_i emits at least one edge into V_{ω_1}, as explained above. From the definition of J_i we have $r(J_i) \subseteq V_{\omega_1}$. Thus, since $V_{\omega_1} = \bigcup_{\alpha<\omega_1} V_\alpha$, for each $u_i \in U$ we have $r(J_i) \subseteq V_{\alpha_i}$ for some $\alpha_i < \omega_1$.

Let $\gamma = \sup\{\alpha_i : i = 1, 2, \ldots\}$, so that $\bigcup_{u_i \in U} r(J_i) \subseteq V_\gamma$ (noting that $\gamma < \omega_1$, since U is countable). Thus the quotient graph $E|V_\gamma$ contains none of the edges in $\bigcup_{u_i \in U} J_i$. Since each u_i emits at most one edge into $E^0 \backslash V_{\omega_1}$, and therefore at most one edge into $E^0 \backslash V_\gamma$, each u_i must be a line point in $E|V_\gamma$. Thus, by Theorem 3.2.11, each $u_i \in \operatorname{soc}(L_K(E|V_\gamma)$. Recall the definition of $\phi : L_K(E) \to L_K(E|V_\gamma)$ from Proposition 3.3.7. Since each $u_i \notin V_{\omega_1}$ we have $u_i \notin V_\gamma$ and so $\phi(u_i) = u_i$. Letting $\bar{\phi} : L_K(E)/S_\gamma \to L_K(E|V_\gamma)$ be the isomorphism defined by $\bar{\phi}(x + S_\gamma) = \phi(x)$, we therefore have $\bar{\phi}^{-1}(u_i) = u_i + S_\gamma$. Thus we have

$$S_{\gamma+1}/S_\gamma = \operatorname{soc}(L_K(E)/S_\gamma) \cong \operatorname{soc}(L_K(E|V_\gamma),$$

and specifically $\bar{\phi}^{-1}\big[\operatorname{soc}(L_K(E|V_\gamma))\big] = S_{\gamma+1}/S_\gamma$.

Since each $u_i \in \operatorname{soc}(L_K(E|V_\gamma))$, we have $u_i + S_\gamma \in S_{\gamma+1}/S_\gamma$. Thus each $u_i \in S_{\gamma+1}$, and so $u_i \in S_{\gamma+1} \cap E^0 = V_{\gamma+1} \subseteq V_{\omega_1}$, contradicting the fact that each $u_i \notin V_{\omega_1}$. Thus $L_K(E)$ has Loewy length $\leq \omega_1$, as required. □

One may be tempted to think that ω, the first countable ordinal, would be an upper bound for the Loewy length of $L_K(E)$ in the case that E is row-finite. However, [ARM1, Example 4.3] constructs a series of row-finite graphs P_α for which the Loewy length of $L_K(P_\alpha) = \alpha$ for each $\alpha < \omega_1$, thus showing that ω_1 is indeed the best possible upper bound.

Chapter 4

Regular and Self-Injective Leavitt Path Algebras

In this chapter we define various notions of 'regularity' for a ring and examine Leavitt path algebras with these properties in Sections 4.2 and 4.3. Furthermore, in Section 4.4 we examine Leavitt path algebras that are self-injective; that is, injective as left (or right) modules over themselves. To begin, we define the construction of a particular K-subalgebra of a Leavitt path algebra that will be integral to proving our main result in Section 4.2.

4.1 The Subalgebra Construction

In this section, we define a subalgebra $B(X)$ of $L_K(E)$ for a given graph E and finite subset of nonzero elements $X \subseteq L_K(E)$. Furthermore, we show in Proposition 4.1.7 that $L_K(E)$ is in fact the directed union of such algebras. This subalgebra construction is given by Abrams and Rangaswamy in [AR], and we follow their work closely for the majority of this section. To begin, we introduce the concept of a graph E_F; this definition is given in [AR, Definition 2], which in turn is based on an idea presented by Raeburn and Szymański in [RS].

Definition 4.1.1. Let E be a graph, and let F be a finite subset of E^1. We define $s(F)$ to be the set of all vertices in E^0 that are the source of at least one edge in F, and similarly $r(F)$ to be the set of all vertices that are the range of at least one edge in F. We construct a new graph, E_F, in two parts. First, we define the vertices:

$$E_F^0 = F \cup W_F \cup Z_F,$$

where

$$W_F = r(F) \cap s(F) \cap s(E^1 \backslash F) \quad \text{and} \quad Z_F = r(F) \backslash s(F).$$

In other words, each edge in F becomes a vertex in our new graph. In addition, we include all vertices which are both the source and range of at least one edge in F as well as the source of at least one edge that is *not* in F (the set W_F), as well as all vertices that are the range of at least one edge in F but not the source of any edge in F (the set Z_F). Now we define the edges of E_F:

$$E_F^1 = \{(f, x) \in F \times E_F^0 : r(f) = s(x)\},$$

following the convention that $s(v) = v$ when $x = v$ is a vertex from our original graph E (i.e. when $x \in W_F \cup Z_F \subseteq E_F^0$). In other words, E_F^1 is the set of ordered pairs (f, x) of edges $f \in F$ and vertices $x \in E_F^0$ for which either fx forms a path in our original graph E (if $x \in F$), or x is the range vertex for f in E (if $x \in W_F \cup Z_F$).

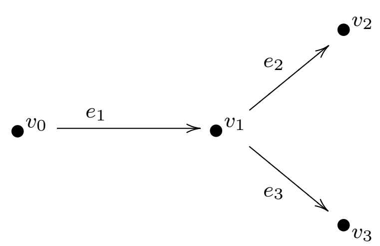
Finally, we define the source and range functions of E_F:

$$s((f, x)) = f \quad \text{and} \quad r((f, x)) = x \quad \text{for all } (f, x) \in E_F^1$$

Note that, since F is finite, the graph E_F must also be finite. Also, any vertices in the sets W_F or Z_F become sinks in our new graph. We illustrate the construction of E_F with the following example.

Example 4.1.2. Let E be the graph

$E =$



and let F be the set of edges $\{e_1, e_2\}$. Then $W_F = \{v_1\}$ and $Z_F = \{v_2\}$, and so $E_F^0 = \{e_1, e_2, v_1, v_2\}$. Thus $E_F^1 = \{(e_1, e_2), (e_1, v_1), (e_2, v_2)\}$, and so we have

$$E_F = \begin{tikzcd} \bullet^{e_1} \arrow[r, "{(e_1,e_2)}"] \arrow[d, "{(e_1,v_1)}"'] & \bullet^{e_2} \arrow[d, "{(e_2,v_2)}"] \\ \bullet^{v_1} & \bullet^{v_2} \end{tikzcd}$$

The following lemma is from [AR, Lemma 1].

Lemma 4.1.3. *Let E be an acyclic graph. Then, for any finite subset F of E^1, the graph E_F is acyclic.*

Proof. By definition, any vertex $v \in E_F^0$ is a sink unless it is of the form $v = e \in F$. Since $r((x, y)) = y$ (where $(x, y) \in E_F^1$), any cycle in E_F must be of the form $(e_1, e_2)(e_2, e_3), \ldots, (e_n, e_1)$, where $e_1, e_2, e_3, \ldots, e_n \in F$. However, (e, f) is an edge in E_F only if $r(e) = s(f)$ in E, by definition. Thus $e_1 e_2 e_3 \ldots e_n$ must be a cycle in E. Therefore, E_F can only be acyclic if E is acyclic. $\square$

The homomorphism $\phi : L_K(E_F) \to L_K(E)$ defined in the following proposition (from [AR, Proposition 1]) is integral to our definition of $B(X)$.

Proposition 4.1.4. *Let E be an arbitrary graph and let F be a finite subset of E^1. Then there is an algebra homomorphism $\phi : L_K(E_F) \to L_K(E)$ with the following properties:*

(i) $F \cup F^* \subseteq \mathrm{Im}(\phi)$ *(where $F^* = \{e^* : e \in F\}$);*

(ii) $r(F) \subseteq \mathrm{Im}(\phi)$*; and*

(iii) if w is not a sink in E and $s_E^{-1}(w) \subseteq F$, then $w \in \mathrm{Im}(\phi)$.

Proof. We begin by defining $\phi : L_K(E_F) \to L_K(E)$ on the generators of E_F as follows:

$$\phi(w) = \begin{cases} ee^* & \text{if } w = e \in F \\ w - \sum_{f \in F, s(f)=w} ff^* & \text{if } w \in W_F \\ w & \text{if } w \in Z_F, \end{cases}$$

$$\phi(h) = \begin{cases} eff^* & \text{if } h = (e,f), f \in F \\ e - \sum_{f \in F, s(f)=r(e)} eff^* & \text{if } h = (e, r(e)), r(e) \in W_F \\ e & \text{if } h = (e, r(e)), r(e) \in Z_F, \end{cases}$$

and

$$\phi(h^*) = (\phi(h))^* \quad \text{for all } h^* \in (E^1)^*$$

Extend ϕ linearly and multiplicatively. It is a straightforward yet tedious process to check that ϕ preserves the Leavitt path algebra relations on $L_K(E)$. As a sample of the calculations required, we will check that the (CK1) relation holds for $h_i, h_j \in E_F^1$, for the case $h_i = (e_i, r(e_i))$, $h_j = (e_j, r(e_j))$, with $r(e_i), r(e_j) \in W_F$. We want to check that $\phi(h_i^*)\phi(h_j) = \delta_{ij}\, \phi(r(h_i)) = \delta_{ij}\, \phi(r(e_i))$, since $r(h_i) = r(e_i)$ by definition. Thus

$$\begin{aligned} \phi(h_i^*)\phi(h_j) &= \left(e_i^* - \sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* e_i^* \right) \left(e_j - \sum_{\substack{f_j \in F, \\ s(f_j)=r(e_j)}} e_j f_j f_j^* \right) \\ &= e_i^* e_j - \sum_{\substack{f_j \in F, \\ s(f_j)=r(e_j)}} e_i^* e_j f_j f_j^* - \sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* e_i^* e_j \\ &\quad + \left(\sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* \right) e_i^* e_j \left(\sum_{\substack{f_j \in F, \\ s(f_j)=r(e_j)}} f_j f_j^* \right) \end{aligned}$$

Note that $e_i^* e_j$ appears in every term in the above expression, which simplifies to $\delta_{ij}(r(e_i))$ by the (CK1) relation in $L_K(E)$. Note also that the (CK1) relation simplifies the last term in the above expression to

$$\delta_{ij} \left(\sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* \right) \left(\sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* \right) = \delta_{ij} \left(\sum_{\substack{f_i \in F, \\ s(f_i)=r(e_i)}} f_i f_i^* \right).$$

Thus the above expression simplifies to

$$\begin{aligned}\phi(h_i^*)\phi(h_j) &= \delta_{ij}(r(e_i)) - 2\,\delta_{ij}\left(\sum_{\substack{f_i\in F,\\ s(f_i)=r(e_i)}} f_if_i^*\right) + \delta_{ij}\left(\sum_{\substack{f_i\in F,\\ s(f_i)=r(e_i)}} f_if_i^*\right)\\ &= \delta_{ij}(r(e_i)) - \delta_{ij}\left(\sum_{\substack{f_i\in F,\\ s(f_i)=r(e_i)}} f_if_i^*\right)\\ &= \delta_{ij}\,\phi(r(e_i))\end{aligned}$$

as required. Similar calculations can be made for the other Leavitt path algebra relations, and for each subcase contained within. We are now ready to show that properties (i), (ii) and (iii) hold for our definition of ϕ.

To show that $F\cup(F)^*\subseteq \mathrm{Im}(\phi)$, it suffices to show that every $f\in F$ is contained in $\mathrm{Im}(\phi)$, since if $f\in\mathrm{Im}(\phi)$ then $f^*\in\mathrm{Im}(\phi)$, by definition. If $r(f)$ is a sink, then $r(f)\in r(F)\backslash s(F)=Z_F$, and so $f=\phi((f,r(f)))$. Now suppose that $\emptyset\neq s_E^{-1}(r(f))\subseteq F$, so that $r(f)$ emits edges only into F. Let $s_E^{-1}(r(f))=\{g_1,\dots,g_n\}$, which is a finite set since each $g_i\in F$ and F is finite. For each $g_i\in s_E^{-1}(r(f))$ we have $fg_ig_i^*=\phi((f,g_i))$, and so , applying the (CK2) relation, we have $f=fr(f)=f\big(\sum_i g_ig_i^*\big)=\sum_i fg_ig_i^*=\sum_i\phi((f,g_i))\in\mathrm{Im}(\phi)$.

Now suppose that $r(f)$ is not a sink and emits edges only into $E^1\backslash F$. This again implies that $r(f)\in r(F)\backslash s(F)=Z_F$ and so $f=\phi((f,r(f))$. Thus the only remaining case is that $r(F)$ emits edges into both F and $E^1\backslash F$. In this case, $r(f)\in r(F)\cap s(F)\cap s(E^1\backslash F)=W_F$. Let $\{g_1,\dots,g_m\}$ be the subset of edges in F for which $s(g_i)=r(f)$. As above, we have $fg_ig_i^*=\phi((f,g_i))$ for each g_i. Thus

$$f=\big(\textstyle\sum_i fg_ig_i^*\big)+\big(f-\sum_i fg_ig_i^*\big)=\sum_i\phi((f,g_i))+\phi((f,r(f))\in\mathrm{Im}(\phi)$$

since $r(f)\in W_F$. Thus we have established property (i).

Now suppose $v=r(f)$ for some $f\in F$. Then $v=f^*f\in\mathrm{Im}(\phi)$ by (i), establishing property (ii). Finally, suppose that w is a vertex that is not a sink in E and $s_E^{-1}(w)=\{f_1,\dots,f_n\}\subseteq F$. Then $f_if_i^*\in\mathrm{Im}(\phi)$ for each f_i, by (i), and so the (CK2) relation gives $w=\sum_i f_if_i^*\in\mathrm{Im}(\phi)$, establishing property (iii). □

We can apply Theorem 2.2.15 to prove the following lemma.

Lemma 4.1.5. *Let E be a graph, let F be a finite subset of E^1 and let $\phi : L_K(E_F) \to L_K(E)$ be the homomorphism defined in Proposition 4.1.4. If E is acyclic, then ϕ is a monomorphism.*

Proof. Recall that, for each $w \in E_F^0$, we have $\phi(w) = ee^*$ if $w = e \in F$, $\phi(w) = w$ if $w \in Z_F$ and $\phi(w) = w - \sum_{f \in F, s(f)=w} ff^*$ if $w \in W_F$. For the former two cases, it is clear that $\phi(w) \neq 0$; for the latter case, recall that $W_F = r(F) \cap s(F) \cap s(E^1 \backslash F)$, so that w emits at least one edge that is in $E^1 \backslash F$ and thus $\phi(w) \neq 0$ by the (CK2) relation. Therefore $\phi(v) \neq 0$ for every vertex $v \in E_F^0$. If E is acyclic, then Lemma 4.1.3 gives that E_F is acyclic, so it is trivially true that ϕ maps each cycle without exits to a non-nilpotent homogeneous element of nonzero degree. Thus, by Theorem 2.2.15, ϕ is a monomorphism. □

Having defined our homomorphism ϕ, we are almost ready to construct the K-subalgebra $B(X)$ of $L_K(E)$ defined in Definition 4.1.6, which will play an important role in the proof of Theorem 4.2.3. We first have a few preliminary definitions.

Let E be an arbitrary graph and let $X = \{a_1, \ldots, a_n\}$ be a finite subset of nonzero elements of $L_K(E)$. By Lemma 2.1.8, we can write each a_r in the form

$$a_r = \sum_{i=1}^{s(r)} k_{r_i} v_{r_i} + \sum_{j=1}^{t(r)} l_{r_j} p_{r_j} q_{r_j}^*$$

where each k_{r_i}, l_{r_j} is a nonzero element of K, each $v_{r_i} \in E^0$ and each $p_{r_j}, q_{r_j} \in E^*$. Additionally, for each $j \in \{1, \ldots, t(r)\}$, at least one of p_{r_j} or q_{r_j} has length 1 or greater (since the case in which both paths have zero length is covered in the first sum).

Let F denote the set of edges that appear in the representation of some p_{r_j} or q_{r_j} for $1 \leq j \leq t(r)$, $1 \leq r \leq n$. Furthermore, let S be the set of vertices

$$S = \big\{v_{r_1}, \ldots, v_{r_{s(r)}} : 1 \leq r \leq n\big\}.$$

Thus F, F^* and S are the sets of all edges and vertices, respectively, that appear in the representation of our elements in X. Note that both F and S must be finite.

We now partition S into four disjoint subsets as follows:

$$S_1 = S \cap r(F),$$

and, defining $T = S \backslash S_1$,

$$S_2 = \{v \in T : s_E^{-1}(v) \subseteq F \text{ and } s_E^{-1}(v) \neq \emptyset\},$$

$$S_3 = \{v \in T : s_E^{-1}(v) \cap F = \emptyset\}, \text{ and}$$

$$S_4 = \{v \in T : s_E^{-1}(v) \cap F \neq \emptyset \text{ and } s_E^{-1}(v) \cap (E^1 \backslash F) \neq \emptyset\}.$$

In other words, S_1 is the set of all vertices in S that are the range of some edge in F. For those vertices in S that are not the range of some edge in F, we then have three cases: vertices that emit edges only into F, vertices that emit no edges into F, and vertices that emit edges into both F and $E^1 \backslash F$; these three cases make up the subsets S_2, S_3 and S_4, respectively.

Finally, let E_F be the graph corresponding to our set of edges F, as defined in Definition 4.1.1, and let $\phi : L_K(E_F) \to L_K(E)$ be the homomorphism defined in the proof of Proposition 4.1.4. We are now ready to construct our subalgebra $B(X)$ of $L_K(E)$, as defined in [AR, Definition 3].

Definition 4.1.6. Let E be an arbitrary graph and let $X = \{a_1, \dots, a_n\}$ be any finite subset of nonzero elements of $L_K(E)$. Let S_3, S_4 and ϕ be as defined above. Define $B(X)$ to be the K-subalgebra of $L_K(E)$ generated by the set $\mathrm{Im}(\phi) \cup S_3 \cup S_4$, so that

$$B(X) = \langle \mathrm{Im}(\phi), S_3, S_4 \rangle.$$

We finish this section by describing several important properties of our subalgebra $B(X)$, in a result from [AR, Proposition 2]. In particular, we show that $L_K(E)$ is the directed union of such subalgebras.

Proposition 4.1.7. *Let E be an arbitrary graph and let $X = \{a_1, \dots, a_n\}$ be any finite subset of nonzero elements of $L_K(E)$. Let F, S_3, S_4 and ϕ be as defined above. For $w \in S_4$, let u_w denote the (nonzero) element $w - \sum_{f \in F, s(f) = w} ff^*$. Then* *(i)* $X \subseteq B(X)$*;*

(ii) $B(X) = \mathrm{Im}(\phi) \oplus \big(\bigoplus_{v_i \in S_3} Kv_i\big) \oplus \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big)$*;*

(iii) The collection $\{B(X) : X \subseteq L_K(E),\ X \textit{ finite}\}$ *is an upward-directed set of subalgebras of* $L_K(E)$*; and*

(iv) $L_K(E) = \varinjlim_{\{X \subseteq L_K(E),\ X \textit{ finite}\}} B(X)$*.*

Proof. To prove (i), recall that the set X is generated by the subsets F, F^* and $S = S_1 \cup S_2 \cup S_3 \cup S_4$, as defined above. By Proposition 4.1.4, we have $F \cup F^* \subseteq \mathrm{Im}(\phi) \subseteq B(X)$ (property (i)), $S_1 \subseteq r(F) \subseteq \mathrm{Im}(\phi) \subseteq B(X)$ (property (ii)) and $S_2 \subseteq \mathrm{Im}(\phi) \subseteq B(X)$ (property (iii)). Finally, $S_3 \cup S_4 \subseteq B(X)$, by definition, and so $X \subseteq B(X)$, as required.

To prove (ii), first note that since $S_3 \subseteq E^0$, it is a set of pairwise orthogonal idempotents, and so $\sum_{v_i \in S_3} Kv_i = \bigoplus_{v_i \in S_3} Kv_i$. Furthermore, the set $\{u_{w_j} : w_j \in S_4\}$ is also a set of pairwise orthogonal idempotents, as follows: let $w_i, w_j \in S_4$. Then

$$\begin{aligned}
u_{w_i} u_{w_j} &= \left(w_i - \sum_{f_i \in F, s(f_i) = w_i} f_i f_i^*\right)\left(w_j - \sum_{f_j \in F, s(f_j) = w_j} f_j f_j^*\right) \\
&= \delta_{ij}\, w_i - 2\,\delta_{ij}\left(\sum_{f_i \in F, s(f_i) = w_i} f_i f_i^*\right) + \delta_{ij}\left(\sum_{f_i \in F, s(f_i) = w_i} f_i f_i^* f_i f_i^*\right) \\
&= \delta_{ij}\left(w_i - \sum_{f_i \in F, s(f_i) = w_i} f_i f_i^*\right) \\
&= \delta_{ij}\, u_{w_i}
\end{aligned}$$

as required. Thus $\sum_{w_j \in S_4} Ku_{w_j} = \bigoplus_{w_j \in S_4} Ku_{w_j}$.

We now show that the sum $\mathrm{Im}(\phi) + \big(\bigoplus_{v_i \in S_3} Kv_i\big) + \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big)$ is direct. We begin by showing that $\big(\bigoplus_{v_i \in S_3} Kv_i\big) \cap \mathrm{Im}(\phi) = \{0\}$. Let $v \in S_3$. By the definition of S_3 we have $v \notin r(F) \cup s(F)$, and so $v \notin W_F \cup Z_F$. We now show that v is orthogonal to each element $\phi(x)$, where $x \in (E_F^0) \cup (E_F^1) \cup (E_F^1)^*$. If $x = e \in F$, then $v \cdot \phi(x) = vee^* = 0$, since $v \notin s(F)$. If $x = w \in W_F$, then $v \neq w$ (since

$v \notin W_F$) and so $v \cdot \phi(x) = v(w - \sum_{f \in F, s(f)=w} ff^*) = 0$. If $x = w \in Z_F$, then again $v \neq w$ (since $v \notin Z_F$) and so $v \cdot \phi(x) = vw = 0$. Similarly, it is easy to see that $\phi(x) \cdot v = 0$ for each of the above three cases. Thus v is orthogonal to each element in $\phi(E_F^0)$. Now suppose $h = (x, y) \in E_F^1$. Then $\phi(h) = \phi(xhy) = \phi(x)\phi(h)\phi(y)$ and $\phi(h^*) = \phi(yh^*x) = \phi(y)\phi(h^*)\phi(x)$. Since $x, y \in E_F^0$, v is therefore orthogonal to $\phi(h)$ and $\phi(h^*)$. Therefore v is orthogonal to each generator of $\mathrm{Im}(\phi)$, and so Kv_i is orthogonal to $\mathrm{Im}(\phi)$ for each $v_i \in S_3$. Since each v_i is an idempotent, we therefore have $\big(\bigoplus_{v_i \in S_3} Kv_i\big) \cap \mathrm{Im}(\phi) = \{0\}$, as required.

Now we show that $\big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big) \cap \mathrm{Im}(\phi) = \{0\}$. Let $w \in S_4$. By the definition of S_4 we have $w \notin r(F)$, and so again $w \notin W_F \cup Z_F$. Again, we must show that u_w is orthogonal to each element $\phi(x)$, where $x \in (E_F^0) \cup (E_F^1) \cup (E_F^1)^*$. If $x = e \in F$, then $u_w \cdot \phi(x) = (w - \sum_{f \in F, s(f)=w} ff^*)ee^* = \delta_{w,s(e)}ee^* - \delta_{w,s(e)}ee^*ee^* = 0$, using the (CK1) relation. If $x = w' \in W_F$, then $w' \neq w$ (since $w \notin W_F$), and so $u_w \cdot \phi(w') = (w - \sum_{f \in F, s(f)=w} ff^*)(w' - \sum_{f' \in F, s(f')=w'} f'(f')^*) = 0$. If $x = w' \in Z_F$, then again $w \neq w'$ (since $w \notin Z_F$) and so $u_w \cdot \phi(x) = (w - \sum_{f \in F, s(f)=w} ff^*)w' = 0$. Similarly, it is easy to see that $\phi(x) \cdot u_w = 0$ for each of the above three cases. Thus, using the same logic as above, we have that u_w is orthogonal to each generator of $\mathrm{Im}(\phi)$, and thus Ku_{w_j} is orthogonal to $\mathrm{Im}(\phi)$ for each $w_j \in S_4$. As shown above, $\{u_{w_j} : w_j \in S_4\}$ is a set of pairwise orthogonal idempotents, and so $\big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big) \cap \mathrm{Im}(\phi) = \{0\}$.

Now take $v \in S_3$ and $w \in S_4$. Since $S_3 \cap S_4 = \emptyset$, we have $v \neq w$ and so $v \cdot u_w = v(w - \sum_{f \in F, s(f)=w} ff^*) = 0 = (w - \sum_{f \in F, s(f)=w} ff^*)v = u_w \cdot v$. Thus $\big(\bigoplus_{v_i \in S_3} Kv_i\big) \cap \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big) = \{0\}$. Therefore the three sets are mutually orthogonal, and so $\mathrm{Im}(\phi) + \big(\bigoplus_{v_i \in S_3} Kv_i\big) + \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big) = \mathrm{Im}(\phi) \oplus \big(\bigoplus_{v_i \in S_3} Kv_i\big) \oplus \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big)$, as required.

Now we need to show that this direct sum is indeed equal to $B(X)$. For ease of notation, let $\mathrm{Im}(\phi) \oplus \big(\bigoplus_{v_i \in S_3} Kv_i\big) \oplus \big(\bigoplus_{w_j \in S_4} Ku_{w_j}\big) = A$. It is clear that $\mathrm{Im}(\phi) \subseteq B(X)$ and $\bigoplus_{v_i \in S_3} Kv_i \subseteq B(X)$, by definition. Let $w \in S_4$. Then for each $f \in F$ with $s(f) = w$ we have $ff^* = \phi(f) \in \mathrm{Im}(\phi)$, and so $u_w = w - \sum_{f \in F, s(f)=w} ff^* \in B(X)$. Thus $A \subseteq B(X)$. To show that $B(X) \subseteq A$, it suffices to show that each of its generating elements is contained in A. It is clear that $\mathrm{Im}(\phi) \subseteq A$ and $S_3 \subseteq A$. Furthermore, if $w \in S_4$, then $w = u_w + \sum_{f \in F, s(f)=w} ff^* \in A$, since

$\sum_{f\in F, s(f)=w} ff^* \in \mathrm{Im}(\phi)$, as shown above. Thus $B(X) = A$, as required.

To show that the collection $Z = \{B(X) : X \subseteq L_K(E),\ X \text{ finite}\}$ is an upward-directed set of subalgebras of $L_K(E)$, we need to show that every pair of elements in Z has an upper bound; that is, for every pair of finite subsets $X_1, X_2 \subseteq L_K(E)$, we can find a finite subset $X_3 \in L_K(E)$ such that $B(X_1) \subseteq B(X_3)$ and $B(X_2) \subseteq B(X_3)$ (see Appendix A). Now, if X is finite then the sets F, S_3 and S_4 are finite, by construction. Then, as noted earlier, E_F is finite, and so $L_K(E_F)$ is a finitely-generated K-algebra. Thus $\mathrm{Im}(\phi)$, and therefore $B(X)$, is a finitely-generated K-algebra for each finite subset X of $L_K(E)$. Let T_1, T_2 be finite generating sets for X_1 and X_2 respectively, and let $T = T_1 \cup T_2$. Then for each generating element $t \in T_1$, we have $t \in T \subseteq B(T)$ (by (i)) and so $B(X_1) \subseteq B(T)$. Similarly, $B(X_2) \subseteq B(T)$, as required.

Finally, let $M = \varinjlim_{\{X \subseteq L_K(E),\ X \text{ finite}\}} B(X)$ (for ease of notation) and suppose that $L_K(E) \neq M$. Then there must exist a finite subset $X \subseteq L_K(E)$ such that $X \nsubseteq M$. However, since $X \in B(X)$ (by (i)) and M is the limit of the upward-directed set of all such subalgebras $B(X)$ (by (iii)), this is impossible. Thus we have $L_K(E) = M$, as required, completing the proof. $\square$

4.2 Regularity Conditions for Leavitt Path Algebras

Recall that a ring R is said to be von Neumann regular if, for every $x \in R$, there exists $y \in R$ such that $x = xyx$. We now introduce the concept of 'π-regularity' and related variations on this definition.

Definition 4.2.1. Let R be a ring.

(i) R is said to be **π-regular** if, for every $x \in R$, there exist $y \in R$ and $n \in \mathbb{N}$ such that $x^n = x^n y x^n$.

(ii) R is said to be **left π-regular** (resp. right π-regular) if, for every $x \in R$, there exist $y \in R$ and $n \in \mathbb{N}$ such that $x^n = yx^{n+1}$ (resp. $x^n = x^{n+1}y$).

(iii) R is said to be **strongly π-regular** if it is both left and right π-regular.

It is clear that any ring R that is von Neumann regular is also π-regular since, taking $n = 1$, for every $x \in R$ there exists a $y \in R$ such that $x^n = x^n y x^n$. However, the converse is not true. Consider, for example, the ring $R = \mathbb{Z}/4\mathbb{Z}$. Now R is π-regular, since $\bar{2}^2 = \bar{0} = \bar{2}^2\,\bar{1}\,\bar{2}^2$ and $\bar{3}^2 = \bar{1} = \bar{3}^2\,\bar{1}\,\bar{3}^2$. However, it is clear that $\bar{2}$ has no von Neumann regular inverse, and so R is not von Neumann regular.

Furthermore, if R is a unital strongly π-regular ring then [CY, Lemma 6] tells us that for every element $x \in R$ there exist $y \in R$ and $n \in \mathbb{N}$ such that $xy = yx$ and $x^{n+1}y = x^n = yx^{n+1}$. It is then straightforward to show that if R is strongly π-regular then R is also π-regular (see the proof of Theorem 4.2.3 (iv)$\Rightarrow$(v)). On the other hand, consider the ring $R = \mathrm{End}_K(V)$, where V is a vector space over a field K with infinite basis $\{x_i\}_{i=1}^{\infty}$. It is well-known that R is von Neumann regular (see for example [Ri, Example (c), p.131]) and is therefore π-regular. However, if we let $f : V \to V$ be the shift transformation defined by $f(x_1) = 0$ and $f(x_{i+1}) = f(x_i)$ for $i > 1$, then we have $\ker(f) = Kx_1$, $\ker(f^2) = Kx_1 \oplus Kx_2$ and in general $\ker(f^n) = \bigoplus_{i=1}^{n} Kx_i$. If there were to exist a $g \in R$ for which $f^n = gf^{n+1}$, we would have $\ker(gf^{n+1}) \supseteq \ker(f^{n+1}) = \bigoplus_{i=1}^{n+1} Kx_i \supset \ker(f^n)$, which is impossible. Thus R is not strongly π-regular, and so in general the property π-regular does not necessarily imply strongly π-regular.

The following lemma (from [AR, Lemma 2]) is useful in the context of Leavitt path algebras.

Lemma 4.2.2. *Let R be a ring with local units. Then R is strongly π-regular if and only if the subring eRe is strongly π-regular, for every nonzero idempotent $e \in R$.*

Proof. Suppose that R is strongly π-regular and let $x \in eRe$ for some idempotent $e \in R$. Since x is an element of R, there exist $y, z \in R$ such that $x^n = yx^{n+1}$ and $x^m = x^{m+1}z$, for some $m, n \in \mathbb{N}$. Furthermore, since $x \in eRe$ we have $x = xe = ex$, and so $x^n = ex^n = e(yx^{n+1}) = eyex^{n+1}$. Thus there exists an element $y' = eye \in eRe$ for which $x^n = y'x^{n+1}$. Similarly, we can find an element $z' = eze \in eRe$ such that $x^m = x^{m+1}z'$, and so eRe is strongly π-regular.

Conversely, suppose that eRe is strongly π-regular for every idempotent $e \in R$ and let $x \in R$. Since R has local units, there exists an idempotent $f \in R$ such that $x \in fRf$. Since fRf is strongly π-regular, there exist $y, z \in fRf$ for which $x^n = yx^{n+1}$ and $x^m = x^{m+1}z$, for some $m, n \in \mathbb{N}$. However, since y, z are elements of R, this implies that R is also strongly π-regular, completing the proof. $\square$

We now proceed to our main result for this section (from [AR, Theorem 1]), which shows, perhaps surprisingly, that the properties von Neumann regular, π-regular and strongly π-regular are equivalent for Leavitt path algebras. We also finally show that $L_K(E)$ is locally matricial if and only if E is acyclic, a result first mentioned in Section 2.2 (see page 56). Here we utilise the subalgebra $B(X)$ introduced in Section 4.1.

Theorem 4.2.3. *Let E be an arbitrary graph. The following statements are equivalent:*

(i) $L_K(E)$ is von Neumann regular

(ii) $L_K(E)$ is π-regular

(iii) E is acyclic

(iv) $L_K(E)$ is locally matricial

(v) $L_K(E)$ is strongly π-regular.

Proof. (i)⇒(ii): This is immediate, since any von Neumann regular ring is π-regular.

(ii)⇒(iii): Suppose that $L_K(E)$ is π-regular and that there exists a cycle c based at a vertex v in E. Let $x = v + c \in L_K(E)$. Since $L_K(E)$ is π-regular, there exists a $y \in L_K(E)$ and $n \in \mathbb{N}$ such that $x^n y x^n = x^n$. Note that $xv = x = vx$ and so, letting $a = vyv$, we have $x^n a x^n = x^n(vyv)x^n = x^n y x^n = x^n$. Now break a into its graded components, so that

$$a = \sum_{i=s}^{t} a_i,$$

where $s, t \in \mathbb{Z}, a_s \neq 0, a_t \neq 0$ and $\deg a_i = i$ for all $s \leq i \leq t$. Now $vav = v(vyv)v = vyv = a$, and so $\sum_{i=s}^{t} va_iv = \sum_{i=s}^{t} a_i$. Since $\deg(v) = 0$, equating graded components gives $va_iv = a_i$ for each $s \leq i \leq t$.

Now, applying the binomial expansion and using the fact that v is an idempotent and $cv = c = vc$, we have

$$x^n = (v+c)^n = \sum_{k=0}^{n} \binom{n}{k} v^{n-k} c^k = v + \sum_{k=1}^{n} \binom{n}{k} c^k,$$

and so $x^n a x^n = x^n$ expands to

$$\left(v + \sum_{k=1}^{n} \binom{n}{k} c^k \right) \left(\sum_{i=s}^{t} a_i \right) \left(v + \sum_{k=1}^{n} \binom{n}{k} c^k \right) = v + \sum_{k=1}^{n} \binom{n}{k} c^k. \quad (*)$$

Since $\deg(c) > 0$, we have $\deg(c^k) > 0$ for all $1 \leq k \leq n$, and so the lowest-degree term on the left-hand side is va_sv. Since the term of lowest degree on the right-hand side is v, we have $va_sv = v$ and thus $a_s = v$. This implies $s = 0$, and so we can write $a = \sum_{i=0}^{t} a_i$, with $a_0 = v$. Now suppose that c is a cycle of length m, so that $\deg(c^k) = km$. With the exception of the first term, every term on the right-hand side contains a power of c, and so every term on the right-hand side is of degree km, where $0 \leq k \leq n$. Note that on the left-hand side, the leftmost terms of each bracket multiply to give $v\big(\sum_{i=0}^{t} a_i\big)v = \sum_{i=0}^{t} a_i$, and so each a_i appears in the expansion of the left-hand side. Thus, equating terms of equal degree on both sides, we have that $a_i \neq 0$ only if $i = km$ for some $0 \leq k \leq n$.

We now use induction to establish that $a_{km} = f_k(c)$ for each $0 \leq k \leq n$, where $f_k(c)$ is a polynomial in c with integer coefficients. For $k = 0$, we know that $a_0 = v = c^0$, as required. For $k = 1$, we equate components of degree m on both sides of $(*)$, giving

$$va_mv + \binom{n}{1} ca_0 + a_0 \binom{n}{1} c = \binom{n}{1} c$$

and so, since $a_0 = v$, we have $a_m + nc + nc = nc$. Thus $a_m = -nc$, which is certainly a polynomial in c with integer coefficients. Now suppose $l > 1$ and suppose that $a_{km} = f_k(c)$, where $f_k(c)$ is a polynomial in c with integer coefficients, for all $0 \le k < l$. We now equate terms of degree lm on both sides of (*), giving

$$\begin{aligned} a_{lm} &+ \binom{n}{1} c \left(a_{(l-1)m} + a_{(l-2)m} \binom{n}{1} c + a_{(l-3)m} \binom{n}{2} c^2 + \cdots + a_0 \binom{n}{l-1} c^{l-1} \right) \\ &+ \binom{n}{2} c^2 \left(a_{(l-2)m} + a_{(l-3)m} \binom{n}{2} c + \cdots + a_0 \binom{n}{l-2} c^{l-2} \right) \\ &+ \binom{n}{3} c^3 \left(a_{(l-3)m} + a_{(l-4)m} \binom{n}{1} c + \cdots + a_0 \binom{n}{l-3} c^{l-3} \right) + \cdots + \binom{n}{l} c^l a_0 \\ &= \binom{n}{l} c^l. \end{aligned}$$

By our induction hypothesis, $a_m, \ldots, a_{(l-1)m}$ are all polynomials in c with integer coefficients and so, rearranging the above equation for a_{lm}, it is clear that a_{lm} is a polynomial in c with integer coefficients.

So we can conclude that for every nonzero homogeneous component a_i of a, we have $a_i c = c a_i$, and so $ac = ca$. Thus

$$(v+c)^n = (v+c)^n a (v+c)^n = a (v+c)^n (v+c)^n = a(v+c)^{2n}.$$

Let i be maximal with respect to the property $a_i(v+c)^{2n} \neq 0$. (We know such an i exists, since $a_0(v+c)^{2n} = (v+c)^{2n} \neq 0$.) Thus the term of maximum degree of $a(v+c)^{2n}$ is $a_i c^{2n}$, with degree $i + 2nm$, while the term of maximum degree of $(v+c)^n$ is c^n, with degree nm. This contradiction shows that c cannot exist, and so E must be acyclic.

(iii)$\Rightarrow$(iv): Recall from Definition 2.2.10 that $L_K(E)$ is locally matricial if it is the direct limit of an upward-directed set of subalgebras, each of which is isomorphic to a finite direct sum of finite-dimensional matrix rings over K. Let $\{B(X) : X \subseteq L_K(E),\ X \text{ finite}\}$ be the upward-directed set of subalgebras of $L_K(E)$ defined in Proposition 4.1.7 (iii). By Proposition 4.1.7 (iv), we know the direct limit of this set is $L_K(E)$. Thus, by Proposition 4.1.7 (ii) it suffices to show that $B(X) = \mathrm{Im}(\phi) \oplus \left(\bigoplus_{v_i \in S_3} K v_i \right) \oplus \left(\bigoplus_{w_j \in S_4} K u_{w_j} \right)$ is isomorphic to a finite direct sum of finite-dimensional matrix rings over K for each finite subset $X \subseteq L_K(E)$.

First, note that if E is acyclic then E_F must be acyclic, by Lemma 4.1.3. Furthermore, note that the only vertices in E_F that are not sinks are those of the form

$e \in F$, and that these vertices only emit edges to their range vertices $r(e)$ or to other vertices of the form $f \in F$ (in the case that ef forms a path in E). Since F is finite, E_F must therefore be row-finite (and finite, as noted earlier). Thus, by Lemma 2.2.9 we have $L_K(E_F) \cong \bigoplus_{i=1}^{l} M_{m_i}(K)$ for some $m_i, \ldots, m_l \in \mathbb{N}$. Now, by Lemma 4.1.5, the restricted homomorphism $\bar{\phi} : L_K(E_F) \to \mathrm{Im}(\phi)$ is an isomorphism, and thus $\mathrm{Im}(\phi) \cong \bigoplus_{i=1}^{l} M_{m_i}(K)$.

Furthermore, each term in the direct sum $\bigoplus_{v_i \in S_3} Kv_i$ is isomorphic to $K \cong M_1(K)$, and similarly each term in the direct sum $\bigoplus_{w_j \in S_4} Ku_{w_j}$ is isomorphic to $M_1(K)$. Thus $B(X)$ is isomorphic to a finite direct sum of finite-dimensional matrix rings over K for each finite subset $X \subseteq L_K(E)$, as required.

(iv)$\Rightarrow$(i): If $L_K(E)$ is locally matricial, then every element of $L_K(E)$ is contained in a subring $S \cong \bigoplus_{i=1}^{l} M_{m_i}(K)$. It is well known that any ring of this form is von Neumann regular (see for example [L1], Proposition 4.27), and so every $x \in L_K(E)$ has a von Neumann regular inverse.

(iv)$\Rightarrow$(v): As above, if $L_K(E)$ is locally matricial then every element $x \in L_K(E)$ is contained in a subring $S \cong \bigoplus_{i=1}^{l} M_{m_i}(K)$. Now, any ring of this form is a unital left (and right) artinian ring, and so considering the descending chain of left ideals $Sx \supseteq Sx^2 \supseteq Sx^3 \supseteq \ldots$, we must have $Sx^n = Sx^{n+1}$ for some $n \in \mathbb{N}$. Thus, since S is unital, we have $x^n \in Sx^n = Sx^{n+1}$, and so $x^n = yx^{n+1}$ for some $y \in S \subseteq L_K(E)$. Since S is also right artinian, we can similarly show that there exists $z \in L_K(E)$ such that $x^m = x^{m+1}z$ for some $m \in \mathbb{N}$. Thus $L_K(E)$ is strongly π-regular, as required.

(v)$\Rightarrow$(ii): Let $x \in L_K(E)$. Since $L_K(E)$ has local units, $x \in eL_K(E)e$ for some idempotent $e \in L_K(E)$. If $L_K(E)$ is strongly π-regular, then by Lemma 4.2.2 we have that $eL_K(E)e$ is strongly π-regular. Since $eL_K(E)e$ is unital, we can apply [CY, Lemma 6], and so there exists an element $y \in eL_K(E)e$ and $n \in \mathbb{N}$ such that $xy = yx$ and $x^{n+1}y = x^n = yx^{n+1}$. Thus $x^n = x^{n+1}y = (x^n)xy = (x^{n+1}y)xy = x^{n+2}y^2$, since x and y commute. Repeating this process, we get

$$x^n = x^{n+2}y^2 = x^{n+3}y^3 = \ldots = x^{2n}y^n$$

and so, using $xy = yx$ again, we have $x^n = (x^n x^n)y^n = x^n y^n x^n$. Since $y^n \in L_K(E)$, we have that $L_K(E)$ is π-regular, as required. $\square$

Example 4.2.4. We now apply Theorem 4.2.3 to our familiar examples of Leavitt path algebras.

(i) *The finite line graph M_n.* Since M_n is acyclic, $L_K(M_n) \cong \mathbb{M}_n(K)$ is von Neumann regular, π-regular and strongly π-regular for all $n \in \mathbb{N}$. As we are already aware, Theorem 4.2.3 also confirms that $L_K(M_n)$ is locally matricial.

(ii) *The rose with n leaves R_n.* Since R_n contains n cycles, $L_K(R_n) \cong L(1, n)$ is not von Neumann regular, π-regular, strongly π-regular or locally matricial for any $n \in \mathbb{N}$.

(iii) *The infinite clock graph C_∞.* Since C_∞ is acyclic, $L_K(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$ is von Neumann regular, π-regular, strongly π-regular and, of course, locally matricial.

4.3 Weakly Regular Leavitt Path Algebras

A ring R is said to be **right weakly regular** (resp. **left weakly regular**) if $I^2 = I$ for every right (resp. left) ideal I of R. This concept was first introduced by Ramamurthi in [Ram]. We begin with some general properties of weakly regular rings before moving on to look at weakly regular Leavitt path algebras. This first proposition is from [Ram, Proposition 5].

Proposition 4.3.1. *Let R be a ring with local units. If R is right weakly regular, then every two-sided ideal I of R is right weakly regular and the quotient R/I is right weakly regular. On the other hand, if R contains a two-sided ideal I such that both I and R/I are right weakly regular, then R is also right weakly regular.*

Proof. Suppose that R is right weakly regular. Let I be a two-sided ideal of R and let J be a right ideal of I. Clearly $J^2 \subseteq J$, so it suffices to show that $a \in J^2$ for any $a \in J$. Now, aR is a right ideal of R, and so $aR = (aR)^2 = aRaR \subseteq aI$, since I is a two-sided ideal. Furthermore, since R has local units, $a = ae$ for some idempotent $e \in R$. Thus $a = ae \in aR = (aR)^2 = (aR)^4 \subseteq (aI)^2 \subseteq J^2$, as required. Now, any right ideal of R/I is of the form M/I, where M is a right ideal of R containing I. Thus $(M/I)^2 = M^2/I = M/I$, and so R/I is right weakly regular.

Now suppose that R contains a two-sided ideal I such that both I and R/I are right weakly regular. Let J be a right ideal of R and let $a \in J$. Again, let e be a local unit for a, so that $a = ae \in aR$. Since R/I is right weakly regular we have $(aR)^2/I = (aR/I)^2 = aR/I$, and so there must exist $b \in (aR)^2$ such that $b + I = a + I$, i.e. $(a - b) \in I$. Since $(a - b)R \subseteq I$ and I is right weakly regular, we have $(a - b) \in (a - b)R = ((a - b)R)^2$. Furthermore, since $b \in (aR)^2 = aRaR$ we have $b = ag$ for some $g \in RaR$, and thus $(a - b)R = (ae - ag)R = a(e - g)R \subseteq aR$. Thus $a = (a - b) + b \in ((a - b)R)^2 + (aR)^2 \subseteq (aR)^2 + (aR)^2 \subseteq (aR)^2 \subseteq J^2$. Thus $J \subseteq J^2$ and so R is right weakly regular. $\square$

The following proposition gives two useful equivalences to the property that R is right weakly regular. The equivalence (i) $\iff$ (ii) is from [Ram, Proposition 1], while the equivalence (ii) $\iff$ (iii) is from [ARM2, Theorem 3.1].

Proposition 4.3.2. *Let R be a ring with local units. The following statements are equivalent:*

(i) R is right weakly regular.

(ii) For all $a \in R$ there exists $x \in RaR$ such that $a = ax$.

(iii) For every two-sided ideal I of R, the left R-module R/I is flat.

Proof. (i)$\Rightarrow$(ii): Suppose that R is right weakly regular. Then, for any $a \in R$, we have $aR = aRaR$. Since R has local units, $a \in aR = aRaR$, and so there exists $x \in RaR$ such that $a = ax$.

(ii)$\Rightarrow$(iii): Let I be a two-sided ideal of R. Since R has local units, R is flat as a left R-module (by Corollary 1.2.16). Thus, viewing I as a submodule of R, by Proposition 1.2.17 it suffices to show that if Y is a right ideal of R then $I \cap YR = YI$. Now $YI \subseteq I$ and $YI \subseteq YR$, so $YI \subseteq I \cap YR$. Next suppose that $y \in I \cap YR$. By (ii), there exists $x \in RyR$ such that $y = yx$. Since $y \in YR$, we have $y = yx \in YRRyR$.

Since Y is a right ideal, $YRR \subseteq Y$. Furthermore, $yR \subseteq I$, since $y \in I$. Thus $YRRyR \subseteq YI$ and so $y \in YI$. Therefore $I \cap YR = YI$, as required.

(iii)⇒(ii): Let $a \in R$. Then RaR is a two-sided ideal of R, and so by (iii) R/RaR is flat as a left R-module. Since R is flat as a left R-module (by Proposition 1.2.17) we have that $RaR \cap YR = YRaR$ for every right ideal Y of R. Specifically, taking $Y = aR$, we have $RaR \cap aRR = aRRaR$. Since R has local units, there exists an idempotent $e \in R$ for which $eae = a = ae^2$. Thus $a \in RaR \cap aRR$, and so $a \in aRRaR \subseteq aRaR$. Therefore there exist $r_i, s_i \in R$ such that $a = \sum_{i=1}^n ar_ias_i = a\big(\sum_{i=1}^n r_ias_i\big)$. Letting $x = \sum_{i=1}^n r_ias_i$, we have $a = ax$ for $x \in RaR$, as required.

(ii)⇒(i): Assume that for all $a \in R$ there exists $x \in RaR$ such that $a = ax$, and let I be a right ideal of R. Then, for any $b \in I$, there exist $r_i, s_i \in R$ such that $b = b\big(\sum_{i=1}^n r_ibs_i\big) = \sum_{i=1}^n br_ibs_i$ and so $b \in I^2$. Since $I^2 \subseteq I$, we have $I^2 = I$, as required. □

The following proposition from [ARM2, Proposition 3.11] shows that the property of being right weakly regular is preserved by subrings eRe and matrix rings.

Proposition 4.3.3. *Let R be a ring with local units. The following statements are equivalent:*

(i) R is right weakly regular.

(ii) The subring eRe is right weakly regular for all idempotents $e \in R$.

(iii) The matrix ring $\mathbb{M}_n(R)$ is right weakly regular for all $n \in \mathbb{N}$.

Proof. (i)⇒(ii): Suppose that R is right weakly regular and that $e \in R$ is an idempotent. Let $eae \in eRe$, where $a \in R$. By Proposition 4.3.2 there exists $x \in ReaeR$ for which $eae = eaex$. Let $x = \sum_{i=1}^n b_i(eae)c_i$, where each $b_i, c_i \in R$. Since e is an idempotent, we have $eae = (eae)e = (eae\sum_{i=1}^n b_i(eae)c_i)e = eae\sum_{i=1}^n(eb_ie)(eae)(ec_ie)$. Let $y = \sum_{i=1}^n(eb_ie)(eae)(ec_ie)$. Thus we have found an element $y \in (eRe)eae(eRe)$ for which $eae = eaey$, and so eRe is right weakly regular (by Proposition 4.3.2).

(ii)⇒(i): Let $a \in R$. Since R has local units, $a \in eRe$ for some idempotent $e \in R$. By our assumption, eRe is right weakly regular, and so there exists $x \in (eRe)a(eRe)$

for which $a = ax$. However, $(eRe)a(eRe) \subseteq RaR$, so that $x \in RaR$ and thus R is right weakly regular (again by Proposition 4.3.2).

(i)⇒(iii): This follows from the analogous result for unital rings in [Tu, Proposition 20.4(ii)]. We can generalise it to rings with local units by applying Proposition 4.3.2.

(iii)⇒(i): For the case $n = 1$ we have $\mathbb{M}_n(R) \cong R$, and so R must be right weakly regular by our assumption. □

Proposition 4.3.3 leads to the following theorem from [ARM2, Theorem 3.12], which shows that the property 'right weakly regular' is Morita invariant.

Theorem 4.3.4. *Let R and S be rings with local units that are Morita equivalent. Then R is right weakly regular if and only if S is right weakly regular.*

Proof. Suppose that R is right weakly regular. It suffices to show that eSe is right weakly regular for every idempotent $e \in S$, since S is then right weakly regular by Proposition 4.3.3. By Theorem 1.3.7, there exists a surjective Morita context (R, S, N, M). Since $e \in S = MN$, we have $e = \sum_{i=1}^{n} x_i y_i$, where each $x_i \in M$ and each $y_i \in N$. Define $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ so that, in matrix notation with t denoting transpose, $e = \mathbf{x}\mathbf{y}^t$. Note that the element $u = \mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x}$ is an idempotent in $\mathbb{M}_n(R)$, since

$$u^2 = (\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x})(\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x}) = \mathbf{y}^t(\mathbf{x}\mathbf{y}^t)(\mathbf{x}\mathbf{y}^t)(\mathbf{x}\mathbf{y}^t)\mathbf{x} = \mathbf{y}^t e^3\mathbf{x} = \mathbf{y}^t e\mathbf{x} = \mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x} = u.$$

Define the map $\phi : u\,\mathbb{M}_n(R)u \to eSe$ by $\phi(uAu) = e(\mathbf{x}A\mathbf{y}^t)e$. (Note that $\mathbf{x}A\mathbf{y}^t \in MRN \subseteq MN = S$, since M is a right R-module.) First we must check that ϕ is well-defined. Suppose that $A, B \in \mathbb{M}_n(R)$ with $uAu = uBu$. Then $\phi(uAu) = e(\mathbf{x}A\mathbf{y}^t)e = e^2(\mathbf{x}A\mathbf{y}^t)e^2 = \mathbf{x}\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x}A\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x}\mathbf{y}^t = \mathbf{x}uAu\mathbf{y}^t = \mathbf{x}uBu\mathbf{y}^t = \cdots = e(\mathbf{x}B\mathbf{y}^t)e = \phi(uBu)$, as required. Now we show that ϕ is a ring homomorphism. Clearly ϕ is additive. To check the multiplicative property, consider

$uAu, uBu \in u\,\mathbb{M}_n(R)u$. Then

$$\begin{aligned}\phi(uAu)\phi(uBu) &= (e\mathbf{x}A\mathbf{y}^te)(e\mathbf{x}B\mathbf{y}^te)\\ &= e\mathbf{x}A\mathbf{y}^te\mathbf{x}B\mathbf{y}^te\\ &= e\mathbf{x}A(\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x})B\mathbf{y}^te\\ &= e\mathbf{x}AuB\mathbf{y}^te\\ &= \phi(u(AuB)u)\\ &= \phi((uAu)(uBu))\end{aligned}$$

as required.

Now we show that ϕ is injective. Suppose $\phi(uAu) = e\mathbf{x}A\mathbf{y}^te = 0$ for some $uAu \in u\mathbb{M}_n(R)u$. Then $uAu = (\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x})A(\mathbf{y}^t\mathbf{x}\mathbf{y}^t\mathbf{x}) = \mathbf{y}^t(e\mathbf{x}A\mathbf{y}^te)\mathbf{x} = 0$, and so $\ker(\phi) = \{0\}$, as required.

Finally, we show that ϕ is surjective. Consider $ese = \mathbf{x}\mathbf{y}^ts\mathbf{x}\mathbf{y}^t \in eSe$, where $s \in S$. Note that $\mathbf{y}^ts\mathbf{x}$ is an $n \times n$ matrix, and each $y_isx_j \in NSM \subseteq NM = R$, since N is a right S-module. Thus $\mathbf{y}^ts\mathbf{x} \in \mathbb{M}_n(R)$. Letting $\mathbf{y}^ts\mathbf{x} = C$, we have

$$ese = e(ese)e = e\mathbf{x}\mathbf{y}^ts\mathbf{x}\mathbf{y}^te = e(\mathbf{x}C\mathbf{y}^t)e = \phi(uCu)$$

and so ϕ is surjective. Thus ϕ is an isomorphism, and since $\mathbb{M}_n(R)$ is right weakly regular (by Proposition 4.3.3), eSe is right weakly regular, as required. □

We now start to examine weakly regular rings in the context of Leavitt path algebras. We begin by showing that, for any Leavitt path algebra, the properties 'right weakly regular' and 'left weakly regular' are in fact equivalent. The proof here expands on the proof given in [ARM2, Theorem 3.15], (i) $\iff$ (iii).

Lemma 4.3.5. *Let E be an arbitrary graph. Then $L_K(E)$ is right weakly regular if and only if it is left weakly regular.*

Proof. For any element $\alpha = k_1p_1q_1^* + \cdots + k_np_nq_n^* \in L_K(E)$, where each $k_i \in K$ and each $p_i, q_i \in E^*$, denote by α^* the element

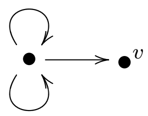
$$\alpha^* := k_1q_1p_1^* + \cdots + k_nq_np_n^*.$$

It is easy to see that for any $\alpha, \beta \in L_K(E)$ we have $(\alpha\beta)^* = \beta^*\alpha^*$. Let I be a right ideal of $L_K(E)$ and define $I^* := \{\alpha^* : \alpha \in I\}$. If $a, b \in I$ then $a^* - b^* = (a-b)^* \in I^*$, since $a - b \in I$. Furthermore, if $a \in I$ and $x \in L_K(E)$ then $xa^* = (ax^*)^* \in I^*$, since $ax^* \in I$. Thus I^* is a left ideal of $L_K(E)$. Similarly, if I is a left ideal of $L_K(E)$ then I^* is a right ideal of $L_K(E)$.

Suppose that $L_K(E)$ is right weakly regular, and consider a left ideal J of $L_K(E)$. Then J^* is a right ideal of $L_K(E)$, and so $(J^*)^2 = J^*$. Take an arbitrary element $a \in J$. Then $a^* = \sum_{i=1}^n x_i^* y_i^*$, where each $x_i, y_i \in J$. Thus $a = (a^*)^* = \sum_{i=1}^n (x_i^* y_i^*)^* = \sum_{i=1}^n y_i x_i \in J^2$, and so $J \subseteq J^2$. Therefore $J = J^2$ and so $L_K(E)$ is left weakly regular. A similar argument shows the reverse implication. □

We now give an example of a Leavitt path algebra that is right weakly regular. This example is from [ARM2, Example 3.2(ii)].

Example 4.3.6. Consider the following graph E:



Since E satisfies Condition (K), [G2, Theorem 4.2] tells us that every ideal of $L_K(E)$ is graded. Since E is row-finite, for any graded ideal I of $L_K(E)$ we have $I = I(H)$, where $H = I \cap E^0$ (by Theorem 3.3.9). Furthermore, H is a hereditary saturated subset of E^0 (by Lemma 2.2.1), and so the only ideals in $L_K(E)$ are those generated by hereditary saturated subsets of E^0. Specifically, we have precisely three ideals: 0, $L_K(E)$ and $I = I(\{v\})$.

Clearly $L_K(E)/L_K(E)$ is flat as a left $L_K(E)$-module. Furthermore, $L_K(E)/0 = L_K(E)$ is flat by Corollary 1.2.16. Finally, note that $P_l(E) = \{v\}$, and so by Theorem 3.2.11 we have $\operatorname{soc}(L_K(E)) = I$. Now, [ARM2, Corollary 2.24] states that if R is a semiprime ring with local units then $R/\operatorname{soc}(R)$ is flat as a left R-module. Since $L_K(E)$ is semiprime (by Proposition 3.2.1), $L_K(E)/I$ is flat. Thus we can apply Proposition 4.3.2 (iii)⇒(i) to obtain that $L_K(E)$ is right weakly regular.

Not every Leavitt path algebra is right weakly regular, as the following examples (from [ARM2, Example 3.3]) illustrate.

Example 4.3.7. Consider the graph

$$E: \quad \begin{tikzcd} \bullet^u \arrow[loop left] \end{tikzcd}$$

We know that $L_K(E) \cong K[x, x^{-1}]$ from Example 2.1.6. Let $J = \langle 1 + x \rangle$ be the two-sided ideal generated by $1+x$ in $K[x, x^{-1}]$. Now, if $J = J^2$ then we would have $1 + x = f(x)(1 + x)^2$ for some $f(x) \in K[x, x^{-1}]$, which is impossible. Thus $L_K(E)$ is not right (or left) weakly regular.

Now consider the graph

$$F: \quad \begin{tikzcd} \bullet^u \arrow[loop left] \arrow[r] & \bullet^v \end{tikzcd}$$

Letting $H = \{v\}$ we have $F|H \cong E$, and so $L_K(F)/I(H) \cong L_K(E)$ by Theorem 3.3.8. From above, we know that $L_K(E)$ is not right weakly regular, and so $L_K(F)/I(H)$ is not right weakly regular. Thus, by Proposition 4.3.1, $L_K(F)$ is not right weakly regular.

We now begin to work our way towards Proposition 4.3.10, which shows that any graded ideal of a Leavitt path algebra is itself isomorphic to a Leavitt path algebra. This result, while being interesting in its own right, will also be useful when determining which Leavitt path algebras are right weakly regular. To begin, we need the following definition.

Definition 4.3.8. Let E be an arbitrary graph, let H be a nonempty, hereditary saturated subset of E^0 and let $S \subseteq B_H$. We denote by $\tilde{F}_E(H, S)$ the collection of all finite paths $\alpha = e_1 \dots e_n$ (where each $e_i \in E^1$) such that $s(\alpha) \notin H$, $r(\alpha) \in H \cup S$, and $r(e_i) \notin H \cup S$ for $i = 1, \dots, n-1$. Informally, $\tilde{F}_E(H, S)$ is the set of all finite paths in E that begin outside H and end in $H \cup S$ (with only the final edge entering $H \cup S$). Now we define

$$F_E(H, S) = \tilde{F}_E(H, S) \backslash \{e \in E^1 : s(e) \in S, r(e) \in H\}.$$

In other words, $F_E(H, S)$ is the set $\tilde{F}_E(H, S)$ with all paths of length one going directly from S to H removed.

We can use the set $F_E(H,S)$ to construct a new graph ${}_HE_S$. First, we create a copy of $F_E(H,S)$ and denote this by $\bar{F}_E(H,S) = \{\bar{\alpha} : \alpha \in F_E(H,S)\}$. Then we define the graph ${}_HE_S = ({}_HE_S^0, {}_HE_S^1, s', r')$ as follows:

${}_HE_S^0 := H \cup S \cup F_E(H,S)$.

${}_HE_S^1 := \{e \in E^1 : s(e) \in H\} \cup \{e \in E^1 : s(e) \in S \text{ and } r(e) \in H\} \cup \bar{F}_E(H,S)$.

For every $\bar{\alpha} \in \bar{F}_E(H,S)$, $s'(\bar{\alpha}) = \alpha$ and $r'(\bar{\alpha}) = r(\alpha)$.

For the other edges in ${}_HE_S^1$, $s'(e) = s(e)$ and $r'(e) = r(e)$.

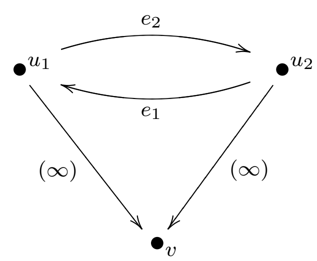
Note that for any $\bar{\alpha} \in \bar{F}_E(H,S)$ we have $s'(\bar{\alpha}) = \alpha \in F_E(H,S) \subseteq {}_HE_S^0$ and $r'(\bar{\alpha}) = r(\alpha) \in H \cup S \subseteq {}_HE_S^0$. Similarly, for any other edge $e \in {}_HE_S^1$ we have $s'(e) \in H \cup S \subseteq {}_HE_S^0$ and $r'(e) \in H \subseteq {}_HE_S^0$, and so the source and range functions are well-defined.

We now note some properties of the graph ${}_HE_S$. First, note that ${}_HE_S$ contains the restriction graph

$$E_H := (H, \{e \in E^1 : s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Note also that every vertex in $S \subseteq {}_HE_S^0$ is an infinite emitter, emitting an infinite number of edges into H and no other edges. On the other hand, each vertex $\alpha \in F_E(H,S) \subseteq {}_HE_S^0$ is by definition a source that emits exactly one edge $\bar{\alpha}$ with range in $H \cup S$. Moreover, since H is hereditary, if a cycle c in ${}_HE_S$ contains a vertex in H then all vertices of c must be in H. Thus any cycle in the graph ${}_HE_S$ must come from the restriction graph E_H. These properties will prove useful in the proof of Proposition 4.3.10. However, we first give an example to illustrate the construction of ${}_HE_S$.

Example 4.3.9. Consider the following graph E:



where the (∞) symbol indicates that there are infinitely many edges from u_1 to v and from u_2 to v. Let $H = \{v\}$ (which is clearly a hereditary and saturated subset of E^0), giving $B_H = \{u_1, u_2\}$. Furthermore, let $S = B_H$. Then $F_E(H, S) = \{e_1, e_2\}$, and so ${}_HE_S$ is the graph

$$\begin{tikzcd}
\bullet^{e_1} \arrow[d, "\bar{e}_1"'] & & \bullet^{e_2} \arrow[d, "\bar{e}_2"] \\
\bullet^{u_1} \arrow[dr, "(\infty)"'] & & \bullet^{u_2} \arrow[dl, "(\infty)"] \\
& \bullet_v &
\end{tikzcd}$$

Recall from Theorem 3.3.9 that any graded ideal I of $L_K(E)$ is generated by the hereditary saturated subset $H = I \cap E^0$ and the set $\{v^H : v \in S\}$, where $S = \{w \in B_H : w^H \in I\}$. We denote this by $I = I_{(H,S)}$.

The following proposition is from [ARM2, Proposition 3.7], which is the algebraic analogue of [DHS, Lemma 1.6] and a generalisation of [AP, Lemma 1.2] to arbitrary graphs. However, when examining this proposition the author discovered an error that leaves the proof incomplete. Furthermore, the proof of [DHS, Lemma 1.6] was discovered to contain a similar error. At the time of writing, these errors are yet to be resolved. We will mention these problems when they arise in the proof and show that they can be avoided in the row-finite case (so that [AP, Lemma 1.2] is still valid).

Proposition 4.3.10. *Let E be an arbitrary graph. For any graded ideal $I = I_{(H,S)}$ of the Leavitt path algebra $L_K(E)$, there exists an isomorphism $\phi : L_K({}_HE_S) \to I_{(H,S)}$.*

Proof. Define $\phi : L_K({}_HE_S) \to I_{(H,S)}$ on the generators of $L_K({}_HE_S)$ as follows:

$$\phi(v) = \begin{cases} v & \text{if } v \in H \\ v^H & \text{if } v \in S \\ \alpha\alpha^* & \text{if } v = \alpha \in F_E(H,S),\ r(\alpha) \in H \\ \alpha r(\alpha)^H \alpha^* & \text{if } v = \alpha \in F_E(H,S),\ r(\alpha) \in S, \end{cases}$$

$$\phi(e) = \begin{cases} e & \text{if } s(e) \in H \\ e & \text{if } s(e) \in S, r(e) \in H \\ \alpha & \text{if } e = \bar{\alpha} \in \bar{F}_E(H,S),\ r(\alpha) \in H \\ \alpha r(\alpha)^H & \text{if } e = \bar{\alpha} \in \bar{F}_E(H,S),\ r(\alpha) \in S, \end{cases}$$

and

$$\phi(e^*) = \begin{cases} e^* & \text{if } s(e) \in H \\ e^* & \text{if } s(e) \in S, r(e) \in H \\ \alpha^* & \text{if } e = \bar{\alpha} \in \bar{F}_E(H,S),\ r(\alpha) \in H \\ r(\alpha)^H \alpha^* & \text{if } e = \bar{\alpha} \in \bar{F}_E(H,S),\ r(\alpha) \in S. \end{cases}$$

Note that, by Proposition 3.3.6, $\mathrm{Im}(\phi)$ is indeed contained in $I_{(H,S)}$. Extend ϕ linearly and multiplicatively. As usual, it can be shown that ϕ preserves the Leavitt path algebra relations on $L_K({}_HE_S)$. We will check the (CK2) relation, i.e. that $\phi(v - \sum_{s'(e)=v} ee^*) = 0$ for all regular vertices $v \in {}_HE_S^0$, as an example. Note that if $v \in S$ then v is an infinite emitter and so the (CK2) relation does not apply.

Case 1: $v \in H$. Note that every edge emitted by v in E^1 is contained in the restriction graph E_H and is therefore in ${}_HE_S^1$. Thus $\sum_{s'(e)=v} ee^* = \sum_{s(e)=v} ee^*$, and so $\phi(v - \sum_{s'(e)=v} ee^*) = v - \sum_{s(e)=v} ee^* = 0$, by the (CK2) relation in $L_K(E)$.

Case 2: $v = \alpha \in F_E(H,S)$ with $r(\alpha) \in H$. Then α only emits the edge $\bar{\alpha}$, and so $\phi(\alpha - \bar{\alpha}\bar{\alpha}^*) = \alpha\alpha^* - \alpha\alpha^* = 0$.

Case 3: $v = \alpha \in F_E(H,S)$ with $r(\alpha) \in S$. Again, α only emits the edge $\bar{\alpha}$, and so $\phi(\alpha - \bar{\alpha}\bar{\alpha}^*) = \alpha r(\alpha)^H \alpha^* - (\alpha r(\alpha)^H)(r(\alpha)^H)\alpha^*) = 0$, since $r(\alpha)^H$ is an idempotent.

Thus the (CK2) relation is preserved by ϕ.

To show that ϕ is a monomorphism we apply Theorem 2.2.15. From the definition of ϕ it is clear that $\phi(v) \neq 0$ for each $v \in {}_HE_S^0$. Furthermore, the only cycles in ${}_HE_S$ come from the restriction graph E_H, as noted above. From the definition of ϕ we see that generating elements from E_H are mapped to themselves, so that any cycle without exits c in $L_K({}_HE_S)$ is mapped to itself (but seen as an element in $I_{(H,S)}$). Since c is a non-nilpotent homogeneous element of nonzero degree in $I_{(H,S)}$, ϕ is therefore a monomorphism by Theorem 2.2.15.

Now we show that ϕ is an epimorphism. Recall from Proposition 3.3.6 that

$$I_{(H,S)} = \operatorname{span}\big(\{\alpha\beta^* : r(\alpha) = r(\beta) \in H\} \cup \{\alpha w^H \beta^* : r(\alpha) = r(\beta) = w \in S\}\big),$$

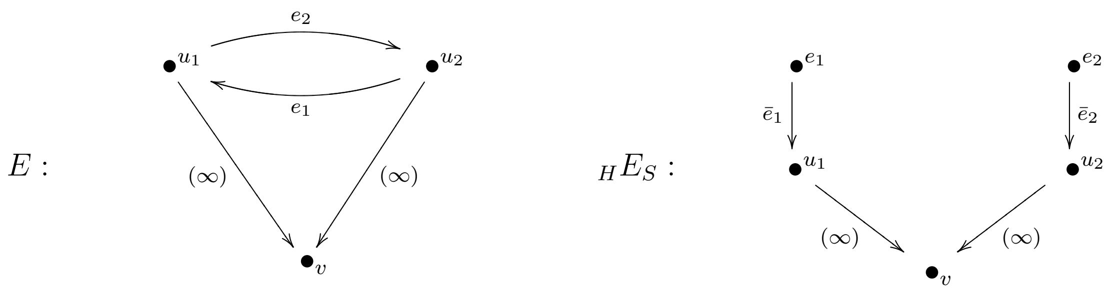
where each $\alpha, \beta \in E^*$. Thus, to show the surjectivity of ϕ it is enough to find inverse images for these generators. Note that for any $x \in {}_H E_S^1$ we have $\phi(x^*) = [\phi(x)]^*$ and so, for any $\alpha \in E^*$, if we can find $y \in L_K({}_H E_S)$ for which $\phi(y) = \alpha$ then $\phi(y^*) = \alpha^*$. Thus it suffices to find inverse images for elements of the form α and $\beta r(\beta)^H$ with $r(\alpha) \in H$ and $r(\beta) \in S$ (noting that, if $r(\beta_1) = r(\beta_2) = w \in S$, then $\beta_1 w^H (\beta_2 w^H)^* = \beta_1 w^H w^H \beta_2^* = \beta_1 w^H \beta_2^*$, since w^H is an idempotent). Note also that these α and β are paths in our original graph E, rather than our constructed graph ${}_H E_S$.

We begin with an arbitrary path $\alpha \in E^*$ with $r(\alpha) \in H$. Let $\alpha = f_1 \dots f_m$, where each $f_i \in E^1$. Suppose that $s(\alpha) \in H$, so that $s(f_i) \in H$ for each H (by the hereditary nature of H). Thus each $f_i \in {}_H E_S^1$ with $\phi(f_i) = f_i$ (by definition) and so $\alpha = \phi(f_i) \dots \phi(f_m) = \phi(\alpha)$.

Now suppose that $s(\alpha) \notin H$ and suppose $r(f_1) \in H$. Then, as above we have $\phi(f_i) = f_i$ for $i = 2, \dots, m$. If $s(f_1) \in S$, then $f_1 \in {}_H E_S^1$ and $\phi(f_1) = f_1$, again giving $\alpha = \phi(\alpha)$. If $s(f_1) \notin S$, then f_1 is a path of length 1 contained in $F_E(H, S)$ and so $\bar{f}_1 \in {}_H E_S^1$ with $\phi(\bar{f}_1) = f_1$. Thus $\alpha = \phi(\bar{f}_1 f_2 \dots f_m)$. Note that $\bar{f}_1 f_2 \dots f_m$ is a nonzero element of $L_K({}_H E_S)$ since $r(\bar{f}_1) = r(f_1) = s(f_2)$.

Now we suppose that $r(f_1) \notin H$. Let n be the smallest integer such that $1 < n \leq m$ and $r(f_n) \in H$. (We know that such an n exists since $r(f_m) \in H$.) As above, we have $\phi(f_i) = f_i$ for $i = n+1, \dots, m$. However, it is finding the inverse image for $f_1 \dots f_n$ that poses a problem. If $s(f_n) \in S$ then $\phi(f_n) = f_n$ as above, but beyond this it is not clear how to proceed. In the proof of [ARM2, Proposition 3.7] it is stated that any edge from a vertex in S must end in a vertex in H, which is true for the graph ${}_H E_S$ but not necessarily true for the original graph E. (See for example the graph E in Example 4.3.9, where the edges e_1, e_2 have both source and range in $S = \{u_1, u_2\}$.) The reliance on this fact renders the remainder of the proof invalid.

On the other hand, the proof of [DHS, Lemma 1.6] appears to get around this problem by writing $f_1 \dots f_n$ as a concatenation of subpaths $\alpha_1 \dots \alpha_k$, where the final edge (and only the final edge) of each $\alpha_1, \dots, \alpha_{k-1}$ has range in S. Since $s(f_i) \notin H$ for any $i = 1, \dots, n$ (by the minimality of n), each $\alpha_i \in F_E(H, S)$ for $i = 1, \dots, k-1$. Furthermore, either $\alpha_k \in F_E(H, S)$ or α_k is a single edge from S to H, in which case $\phi(\alpha_k) = \alpha_k$. The proof asserts that we therefore have either $\alpha = \phi(\bar{\alpha_1}) \dots \phi(\bar{\alpha_k})\phi(f_{n+1}) \dots \phi(f_m)$ (in the former case) or $\alpha = \phi(\bar{\alpha_1}) \dots \phi(\bar{\alpha_{k-1}})\phi(\alpha_k)$ $\phi(f_{n+1}) \dots \phi(f_m)$ (in the latter case). Aside from the fact that $\phi(\bar{\alpha_i}) = \alpha_i r(\alpha_i)^H$ rather than simply α_i, the most significant problem is that $\bar{\alpha_1} \dots \bar{\alpha_k}$ is not a nonzero element in $L_K({}_HE_S)$, since it is impossible for two edges $\bar{\beta}_1, \bar{\beta}_2 \in \bar{F}_E(H, S) \subseteq {}_HE_S^1$ to be adjacent. (Recall that for any edge $\beta \in \bar{F}_E(H, S)$, we define $s(\bar{\beta}) = \beta$, which is a source in our graph ${}_HE_S$ by definition.) We refer again to Example 4.3.9, in which e_1, e_2 are adjacent edges in our graph E, while $\bar{e_1}, \bar{e_2}$ are not:



Indeed, if we let $\alpha = e_2e_1f$ in the above example, where f is one of the (infinite number of) edges from u_1 to v, it is not clear what the inverse image of α is. A similar problem arises when we attempt to find the inverse image of an element of the form $\beta r(\beta)^H$, where $r(\beta) \in S$.

However, in the case that E is row-finite the proof simplifies greatly and it is possible to show that ϕ is an epimorphism, as we now show. Note that if E is row-finite there are no breaking vertices and so $S = \emptyset$. Thus the set $F_E(H, S)$ is simply the set of all positive paths $\alpha = e_1 \dots e_n$ for which each $e_i \in E^1$, $r(\alpha) \in H$ and $s(e_i) \notin H$ for each $i = 1, \dots n$. Furthermore, $I_{(H,S)} = I(H)$, which is generated by elements of the form $\alpha\beta^*$, with $r(\alpha) = r(\beta) \in H$. As above, to show that ϕ is an epimorphism it suffices to find an inverse image for $\alpha = f_1 \dots f_m$ with $r(\alpha) \in H$. If $s(\alpha) \in H$, then $\alpha = \phi(\alpha)$, as was shown in the more general case. Suppose $s(\alpha) \notin H$ and let n be the smallest integer such that $1 < n \leq m$ and $r(f_n) \in H$. If $n < m$, then $\alpha_1 = f_1 \dots f_n \in F_E(H, S)$, while $s(f_i) \in H$ for each $i = n+1, \dots, m$.

Thus $\alpha = \alpha_1 f_{n+1} \dots f_m = \phi(\bar{\alpha_1})\phi(f_{n+1}) \dots \phi(f_m)$. If $n = m$, then $\alpha \in F_E(H,S)$ and so $\alpha = \phi(\bar{\alpha})$. Thus ϕ is an epimorphism, and therefore an isomorphism, as required. $\square$

While we have only proved that Proposition 4.3.10 holds in the case that E is row-finite, we will proceed as in [ARM2] and assume that the following results, some of which rely on Proposition 4.3.10, hold for an arbitrary graph E (unless stated otherwise). As a side note, [ARM2, Proposition 3.7] states that ϕ is a graded isomorphism, which is not necessarily true. To see this, recall that $\phi(\bar{\alpha}) = \alpha$ for all $\bar{\alpha} \in \bar{F}_E(H,S)$ with $r(\alpha) \in H$. Now, $\bar{\alpha}$ is an element of degree 1 in $L_K({}_HE_S)$, since $\bar{\alpha} \in {}_HE_S^1$, whereas α is an element of degree $l(\alpha)$ in $L_K(E)$, and $l(\alpha)$ is not necessarily 1. However, this observation does not affect any subsequent results.

We now proceed to work our way toward the main theorem of this section, Theorem 4.3.15. To begin, we give the following useful theorem, which is a combination of results from Tomforde [To] and Goodearl [G2]. Recall that a ring R is said to be an **exchange ring** if, given any element $x \in R$, there exists an idempotent $e \in xR$ such that $e = x + s - xs$ for some $x \in R$. Note that if R is unital then we have $1 - e = 1 - (x + s - xs) = (1 - x)(1 - s) \in (1 - x)R$, and so this definition is consistent with the more familiar unital definition.

Theorem 4.3.11. *Let E be an arbitrary graph. The following statements are equivalent:*

(i) Every ideal of $L_K(E)$ is graded;

(ii) $L_K(E)$ is an exchange algebra; and

(iii) E satisfies Condition (K).

Proof. (i) $\iff$ (iii) is from [To, Theorem 6.16], while (ii) $\iff$ (iii) is from [G2, Theorem 4.2]. $\square$

The following proposition is from [ARM2, Proposition 3.8].

Proposition 4.3.12. *Let E be an arbitrary graph. If E satisfies Condition (K), then the Leavitt path algebra $L_K(E)$ is right weakly regular.*

Proof. Let I be a two-sided ideal of $L_K(E)$. Since E satisfies Condition (K), we know that I is a graded ideal by Theorem 4.3.11. By Proposition 4.3.10, I is isomorphic to a Leavitt path algebra, and in particular I has local units. We proceed by showing that I satisfies condition (ii) of Theorem 1.2.19; that is, for all $x \in I$ there exists $f \in \mathrm{Hom}_{L_K(E)}(L_K(E), I)$ such that $f(x) = x$. (Note that we can apply Theorem 1.2.19 since every Leavitt path algebra is an E^0-free left $L_K(E)$-module with basis E^0 – see page 52.)

Fix $a \in I$. Since I has local units, there exists an idempotent $e \in I$ for which $a = ae$. Define $\rho_e : L_K(E) \to I$ by $\rho_e(x) = xe$. Clearly this is a homomorphism of left $L_K(E)$-modules, and furthermore $\rho_e(a) = ae = a$, as required. Thus we can apply Theorem 1.2.19 to give that $L_K(E)/I$ is a flat $L_K(E)$-module. Finally, by Proposition 4.3.2 we have that $L_K(E)$ is right weakly regular. □

The following proposition from [ARM2, Proposition 3.9] shows that the converse of Proposition 4.3.12 is true in the row-finite case.

Proposition 4.3.13. *Let E be a row-finite graph. If the Leavitt path algebra $L_K(E)$ is right weakly regular, then the graph E satisfies Condition (K).*

Proof. We begin by showing that if $L_K(E)$ is right weakly regular then every cycle in E has an exit. Suppose, by way of contradiction, that there exists a cycle c without exits in E, and let H be the hereditary saturated closure of the vertices of c. By [AAPS, Proposition 3.6(iii)] we have $I(H) \cong \mathbb{M}_n(K[x, x^{-1}])$ for some $n \in \mathbb{N} \cup \{\infty\}$. Now, since $L_K(E)$ is right weakly regular, so too is $I(H)$ (by Proposition 4.3.1), and thus $\mathbb{M}_n(K[x, x^{-1}])$ is right weakly regular. Consider $E_{11} \in \mathbb{M}_n(K[x, x^{-1}])$, the matrix unit with 1 in the $(1, 1)$ position and zeros elsewhere. Since E_{11} is an idempotent, we have that $E_{11}\mathbb{M}_n(K[x, x^{-1}])E_{11}$ is right weakly regular by Proposition 4.3.3. Note that $E_{11}\mathbb{M}_n(K[x, x^{-1}])E_{11}$ consists of those matrices for which the only nonzero entry is in the $(1, 1)$ position, and so is isomorphic to $K[x, x^{-1}]$. However, we know that $K[x, x^{-1}]$ is not right weakly regular (see Example 4.3.7), a contradiction, and so E contains no cycles without exits.

Now we show that if $L_K(E)$ is right weakly regular then E must satisfy Condition (K). We proceed in a similar manner to the proof of Lemma 2.3.4: suppose, by way of contradiction, that there exists a $v \in E^0$ such that $CSP(v) = \{p\}$. If p is not a cycle, it is easy to see that there exists a cycle based at v whose edges are a subset of the edges of p, contradicting the fact that $CSP(v) = \{p\}$. Thus p is a cycle and so, by the above paragraph, there must exist exits $e_1, \dots, e_m$ for p.

Let A be the set of all vertices in p. Now $r(e_i) \notin A$ for any $i = 1, \dots, m$, for otherwise we would have another closed simple path based at v distinct from p. Let $X = \{r(e_i) : i = 1, \dots, m\}$ and let H be the hereditary saturated closure of X. Recall the definition of $G_n(X)$ from Lemma 1.4.9. Suppose that $A \cap H \neq \emptyset$, and let n be the minimum natural number for which $A \cap G_n(X) \neq \emptyset$.

Let $w \in A \cap G_n(X)$ and suppose that $n > 0$. By the minimality of n, we have $w \notin G_{n-1}(X)$. Thus, by the definition of $G_n(X)$, w must be a regular vertex and $r(s^{-1}(w)) \subseteq G_{n-1}(X)$, so that w only emits edges into $G_{n-1}(X)$. Since w is a vertex in p, there must exist an edge f such that $s(f) = w$ and $r(f) \in A$. Thus $r(f) \in A \cap G_{n-1}(X)$, contradicting the minimality of n. Therefore we must have $n = 0$, and so $w \in G_0(X) = T(X)$ (by definition). Thus, for some $i = 1, \dots, m$, there is a path q from $r(e_i)$ to w. Since w is in the cycle p, and e_i is an exit for p, there must also be a path p' from w to $r(e_i)$, and so $p'q$ is a closed path based at w. However, this implies that $|CSP(v)| \geq 2$, a contradiction.

Thus $H \cap A = \emptyset$, and in particular $H \neq E^0$. Since E is row-finite, $B_H = \emptyset$ and so we have

$$(E|H)^0 = E^0 \backslash H, \text{ and } (E|H)^1 = \{e \in E^1 : r(e) \notin H\}.$$

Since $H \cap A = \emptyset$, we have $A \subseteq (E|H)^0$. Let $p = f_1 \dots f_k$. Since $r(f_j) \in A$ for each f_j (by definition), we have $r(f_j) \notin H$ and so $\{f_1, \dots, f_k\} \subseteq (E|H)^1$. Thus p can be viewed as a cycle in $E|H$. Furthermore, for each exit e_i of p we have $r(e_i) \in X \subseteq H$ by definition, and so $e_i \notin (E|H)^1$. Thus p is a cycle without exits in $E|H$. By Theorem 3.3.8 we have $L_K(E|H) \cong L_K(E)/I(H)$, and since $L_K(E)$ is right weakly regular then so too is $L_K(E|H)$, by Proposition 4.3.1. However, this implies that every cycle in $E|H$ has an exit (from the first part of this proof), a contradiction. Thus $L_K(E)$ satisfies Condition (K), as required. $\square$

Using the fact that right weakly regular is a Morita invariant property, we can use the desingularisation process to extend Proposition 4.3.13 to countable graphs, as shown in [ARM2, Proposition 3.14].

Proposition 4.3.14. *Let E be a countable but not necessarily row-finite graph. Then E satisfies Condition (K) if and only if the Leavitt path algebra $L_K(E)$ is right weakly regular.*

Proof. Suppose that E satisfies Condition (K). Then by Proposition 4.3.12, $L_K(E)$ is right weakly regular.

Conversely, suppose that $L_K(E)$ is right weakly regular. Since E is countable, we can apply the desingularisation process (see Definition 2.4.1) to obtain a row-finite desingularisation F of E. By Theorem 2.4.5, $L_K(E)$ and $L_K(F)$ are Morita equivalent, and so, by Theorem 4.3.4, we have that $L_K(F)$ is right weakly regular. Since F is row-finite, this implies that F satisfies Condition (K) (by Proposition 4.3.13). Thus $L_K(F)$ is an exchange ring by Theorem 4.3.11, and since the exchange property is a Morita invariant for rings with local units (see [AGS, Theorem 2.1]), $L_K(E)$ is also an exchange ring. Finally, this implies that E satisfies Condition (K), by Theorem 4.3.11. □

Proposition 4.3.14 is futhermore generalised to arbitrary graphs in [ARM2, Theorem 3.15], following the proof of [G2, Theorem 4.2]. However, this proof requires a large amount of background theory regarding direct limits of Leavitt path algebras and so we will omit it here.

We now come to the main theorem of this section (from [ARM2, Theorem 3.15]), which summarises the results we have seen thus far.

Theorem 4.3.15. *Let E be an arbitrary graph. The following statements are equivalent:*

(i) The Leavitt path algebra $L_K(E)$ is a right weakly regular ring.

(ii) The graph E satisfies Condition (K).

(iii) The Leavitt path algebra $L_K(E)$ is a left weakly regular ring.

(iv) The Leavitt path algebra $L_K(E)$ is an exchange ring.

(v) Every ideal of $L_K(E)$ is graded.

(vi) Every ideal of $L_K(E)$ is isomorphic to a Leavitt path algebra.

(vii) Every ideal of $L_K(E)$ has local units.

Proof. The equivalences (i) $\iff$ (iv) $\iff$ (v) are from Theorem 4.3.11, while (i) $\iff$ (iii) comes from Lemma 4.3.5.

(i)$\Rightarrow$(ii): This generalisation of Proposition 4.3.14 comes from [ARM2, Theorem 3.15], as mentioned above.

(ii)$\Rightarrow$(vi): If E satisfies Condition (K) then by [G2, Theorem 3.8] every ideal of $L_K(E)$ is graded. Thus every ideal of $L_K(E)$ is isomorphic to a Leavitt path algebra, by Proposition 4.3.10.

(vi)$\Rightarrow$(vii): This is immediate, since every Leavitt path algebra has local units.

(vii)$\Rightarrow$(i): Suppose that every ideal of $L_K(E)$ has local units and consider an arbitrary element $a \in L_K(E)$. Since $L_K(E)$ has local units, $a = eae$ for some idempotent $e \in L_K(E)$, and so $a \in L_K(E)aL_K(E)$. Since $L_K(E)aL_K(E)$ is a two-sided ideal, it has local units, and so there exists $u \in L_K(E)aL_K(E)$ for which $a = au$. Thus, by Proposition 4.3.2, $L_K(E)$ is right weakly regular, as required. $\square$

Example 4.3.16. We now apply Theorem 4.3.15 to our familiar examples of Leavitt path algebras to determine if they are weakly regular.

(i) *The finite line graph M_n.* Since M_n is acyclic, it satisfies Condition (K), and so $L_K(M_n) \cong \mathbb{M}_n(K)$ is both left and right weakly regular for all $n \in \mathbb{N}$.

(ii) *The rose with n leaves R_n.* For $n = 1$, the vertex v in R_1 is the base of exactly one closed simple path, and so R_1 does not satisfy Condition (K). Thus $L_K(R_1) \cong K[x, x^{-1}]$ is not left or right weakly regular, confirming what we saw in Example 4.3.7. However, for $n > 1$ the graph R_n does satisfy Condition (K), and so $L_K(R_n) \cong L(1, n)$ is both left and right weakly regular.

(iii) *The infinite clock graph C_∞.* Since C_∞ is acyclic, $L_K(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$ is both left and right weakly regular.

4.4 Self-Injective Leavitt Path Algebras

Recall from Lemma 2.2.4 that every Leavitt path algebra is projective as a left (and right) module over itself. However, the same is not true of injectivity. Thus it is natural to ask when a Leavitt path algebra is injective as a left (or right) module over itself; that is, when it is left (or right) self-injective. In this section we build towards Theorem 4.4.7, which shows that for any Leavitt path algebra the properties 'left self-injective' and 'right self-injective' are equivalent, and furthermore gives graph theoretic conditions on E that are equivalent to $L_K(E)$ being left (and right) self-injective.

Our first result is from [ARM2, Proposition 4.1].

Proposition 4.4.1. *Let E be an arbitrary graph. If $L_K(E)$ is left (or right) self-injective then $L_K(E)$ is von Neumann regular and the graph E is acyclic.*

Proof. Let $e \in L_K(E)$ be an idempotent, and recall that $L_K(E)e$ is a direct summand of $L_K(E)$ (by Lemma 1.2.3 (i)). Since $L_K(E)$ is injective as a left $L_K(E)$-module, so too is $L_K(E)e$ (by Lemma 1.2.12) and thus, by [L2, Theorem 13.1], we have that $\mathrm{End}_{L_K(E)}(L_K(E)e)$ is left self-injective. Since $\mathrm{End}_{L_K(E)}(L_K(E)e) \cong (eL_K(E)e)^{Op}$ (by Lemma 1.2.2), $(eL_K(E)e)^{Op}$ is therefore left self-injective. Thus, by [L2, Corollary 13.2(2)] we have that $(eL_K(E)e)^{Op}/J((eL_K(E)e)^{Op})$ is von Neumann regular. Note that if a ring R^{Op} is von Neumann regular then, for any $a \in R$, there exists an $x \in R$ such that $a = a \cdot x \cdot a = axa$, and so R is also von Neumann regular. In particular, we have that $eL_K(E)e/J(eL_K(E)e)$ is von Neumann regular.

Now, by [J2, Proposition 3.7.1], we have $J(eL_K(E)e) = eJ(L_K(E))e$. However, $J(L_K(E)) = \{0\}$ (by Corollary 3.3.11) and so $J(eL_K(E)e) = \{0\}$. Thus we have $eL_K(E)e/J(eL_K(E)e) = eL_K(E)e$ and so $eL_K(E)e$ is von Neumann regular for any idempotent $e \in R$.

Let $x \in L_K(E)$. Since $L_K(E)$ has local units, there exists an idempotent $f \in R$ such that $x \in fL_K(E)f$. Since $fL_K(E)f$ is von Neumann regular, there exists $y \in fL_K(E)f$ such that $x = yxy$, and so $L_K(E)$ is von Neumann regular. Finally, by Theorem 4.2.3, E must be acyclic. $\square$

In Proposition 4.4.4 we give the somewhat surprising result that if a Leavitt path algebra $L_K(E)$ is left (or right) self-injective then the corresponding graph E must be row-finite. This is the first time in this thesis we have seen a property of $L_K(E)$ imply row-finiteness on E. To set up this proposition, we first give two preliminary results. The first of these results requires the following definition.

Suppose that V is a left vector space over a division ring D. The **dual vector space** of V, denoted V^*, is the set of homomorphisms from V to D; that is, $\mathrm{Hom}_D(V, D)$. Furthermore, V^* is a right vector space over D. The following theorem, known as the 'Erdös-Kaplansky Theorem', gives a formulation for the dimension of V^*. This theorem is given as Theorem 2 on p. 237 of Jacobson's [J1] and Exercise 7.3(d) of Bourbaki's [Bo].

Theorem 4.4.2 (The Erdös-Kaplansky Theorem)**.** *Let V be a left vector space with infinite basis $\{b_i : i \in I\}$ over a division ring D. Then the dimension of V^* as a right vector space over D is given by*

$$\dim(V^*) = \mathrm{card}(V^*) = \mathrm{card}(D)^{\mathrm{card}(I)}.$$

The Erdös-Kaplansky Theorem has the following useful application, which we will use in the proof of Proposition 4.4.4. Suppose that K is a field and I is an infinite index set. Using the fact that $\mathrm{Hom}_K(K, K) \cong K$ and applying Proposition 1.2.4, we have $K^I \cong \mathrm{Hom}_K(K, K)^I \cong \mathrm{Hom}_K(K^{(I)}, K)$. Since we can view $K^{(I)}$ as a left vector space over K (with an infinite basis indexed by I), we have $\mathrm{Hom}_K(K^{(I)}, K) = (K^{(I)})^*$. Thus, applying the Erdös-Kaplansky Theorem we have

$$\dim(K^I) = \dim((K^{(I)})^*) = \mathrm{card}(K)^{\mathrm{card}(I)}.$$

Now let E be an arbitrary graph and let X be a collection of paths in E. We say that X is an **set of independent paths** if no path in X is an initial subpath of any other path in X. The following related lemma has been adapted from the proof of [ARM2, Proposition 4.4].

Lemma 4.4.3. *Let E be an arbitrary graph and let X be a set of independent paths in E. Then the set of left ideals $\{L_K(E)pp^* : p \in X\}$ is $L_K(E)$-independent – that* *is,* $L_K(E)pp^* \cap \sum_{q \in X, q \neq p} L_K(E)qq^* = \{0\}$ *for all* $p \in X$ *or, equivalently, that the sum of these left ideals is a direct sum.*

Proof. Suppose that $rpp^* = \sum_{q \in X, q \neq p} r_q qq^*$ for some $p \in X$ and $r, r_q \in L_K(E)$ (with only a finite number of r_q nonzero). Since no path in X is an initial subpath of any other path in X, we have $q^*p = 0$ for all $q \in X, q \neq p$ (by Lemma 2.1.10). Thus $rp = rpp^*p = \sum_{q \in X, q \neq p} r_q qq^*p = 0$, and so $rpp^* = (rp)p^* = 0$, as required. □

With these two preliminary results established we can now prove the following result from [ARM2, Proposition 4.4].

Proposition 4.4.4. *If a Leavitt path algebra* $L_K(E)$ *is left (or right) self-injective, then the graph* E *must be row-finite.*

Proof. Suppose by way of contradiction that $v \in E^0$ is an infinite emitter. For each $n \in \mathbb{N}$, define $Y_n = \{p \in E^* : s(p) = v, l(p) = n\}$, and let α_n be the cardinality of Y_n. Note that $Y_1 = s^{-1}(v)$, which has infinite cardinality since v is an infinite emitter. Let $Y = \bigcup_{n \in \mathbb{N}} Y_n$, so that Y is the set of all paths in E with source v. Then Y has infinite cardinality $\sigma = \sup\{\alpha_n : n \in \mathbb{N}\}$.

Now, elements of $vL_K(E)v$ are of the form $\sum_{i=1}^n k_i p_i q_i^*$, where each $p_i, q_i \in E^*$ with $s(p_i) = v = s(q_i)$ and each $k_i \in K$. Thus each $p_i, q_i \in Y$, and so, since the cardinality of the set of all finite subsets of Y is again σ, the K-dimension of $vL_K(E)v$ must be $\leq \sigma$. This observation will prove useful later in the proof.

For the first part of this proof, we wish to find a subset X of Y with cardinality σ such that the set of left ideals $\{L_K(E)pp^* : p \in X\}$ is $L_K(E)$-independent. First note that, for each $n \in \mathbb{N}$, the set $\{L_K(E)pp^* : p \in Y_n\}$ is $L_K(E)$-independent. To see this, note that all paths in Y_n are of length n, so that no path in Y_n is an initial subpath of any other path in Y_n. Thus the result follows from Lemma 4.4.3. Therefore, if $\alpha_n = |Y_n| = \sigma$ for some $n \in \mathbb{N}$, we can choose $X = Y_n$.

If not, then we must have $\alpha_n < \sigma$ for all $n \in \mathbb{N}$. Note that it is not always the case that $\alpha_{n+1} > \alpha_n$, since not every path in Y_n is necessarily a subpath of a path in Y_{n+1}. Thus we define a strictly increasing subsequence $\{\alpha_{i_n} : n < \omega\}$ as follows: let $\alpha_{i_1} = \alpha_1$, and define i_2 to be the smallest integer for which $\alpha_{i_2} > \alpha_{i_1}$.

In general, if α_{i_n} is chosen for some n, we define i_{n+1} to be the smallest integer for which $\alpha_{i_{n+1}} > \alpha_{i_n}$. Note that, since $\alpha_{i_1} = \alpha_1$ is infinite, this is a sequence of strictly increasing infinite cardinalities.

We now construct a sequence of sets T_n of independent paths. First, we define $T_1 = s^{-1}(v) = Y_1$, which is a set of independent paths since it is a set of distinct edges. By the minimality of i_2, the number of paths of length $i_2 - 1$, i.e. α_{i_2-1}, must be less than α_{i_2}. Thus, remembering that α_{i_2} is an infinite cardinal, there must exist a path $p_2 \in Y_{i_2-1}$ such that $r(p_2)$ emits α_{i_2} edges. Let $r(p_2) = v_2$ and let $s^{-1}(v_2) = \{e_\beta^{(2)} : \beta < \alpha_{i_2}\}$. Now we define

$$T_2 = \{p_2 e_\beta^{(2)} : \beta < \alpha_{i_2}\} \cup (T_1 \backslash \{q : q \text{ is an initial subpath of } p_2\}).$$

Note that the removal of the set $\{q : q$ is an initial subpath of $p_2\}$ ensures that T_2 is also a set of independent paths.

Now let $k \in \mathbb{N}$ and suppose that T_i has been defined (and is a set of independent paths of length at most i_j) for all $j \leq k$. As above, there must exist a path $p_{k+1} \in Y_{i_{k+1}-1}$ such that $r(p_{k+1})$ emits $\alpha_{i_{k+1}}$ edges. Again, let $r(p_{k+1}) = v_{k+1}$ and let $s^{-1}(v_{k+1}) = \{e_\beta^{(k+1)} : \beta < \alpha_{i_{k+1}}\}$. Now we define

$$T_{k+1} = \{p_{k+1} e_\beta^{(k+1)} : \beta < \alpha_{i_{k+1}}\} \cup (T_k \backslash \{q : q \text{ is an initial subpath of } p_{k+1}\}).$$

Again, the removal of the set $\{q : q$ is an initial subpath of $p_{k+1}\}$ ensures that T_{k+1} is a set of independent paths. Thus T_n is defined (and is a set of independent paths) for all $n \in \mathbb{N}$. Furthermore, for any $n \in \mathbb{N}$, $\{L_K(E)pp^* : p \in T_n\}$ is an $L_K(E)$-independent set of left ideals, by Lemma 4.4.3. Note also that each T_n is a set of paths of length i_n or less.

However, it may not necessarily be the case that $T = \bigcup_{n<\omega} T_n$ is a set of independent set of paths, since for example a path in T_2 may still be an initial subpath of p_4. Thus, for each $n \in \mathbb{N}$, we define

$$W_n = T_n \backslash \{q : q \text{ is an initial subpath of } p_m \text{ for some } m = 2, 3, \ldots\},$$

which ensures that $W = \bigcup_{n<\omega} W_n$ is a set of independent paths. To see this, let q_i, q_j be two paths in W and let m, n be the smallest integers for which $q_i \in T_m$

and $q_j \in T_n$, respectively. Suppose, without loss of generality, that $m \leq n$. If $m = n$, then $q_i, q_j \in T_m$, which we know is a set of independent paths. So suppose that $m < n$. Now q_j must be of the form $q_j = p_n e_\beta^{(n)}$ (for some $\beta < \alpha_{i_n}$), since otherwise $q_j \in T_{n-1}$ (by the construction of T_n), contradicting the minimality of n. In particular, q_j has length i_n. Similarly, the minimality of m gives that $p_i = p_m e_\gamma^{(m)}$ (for some $\gamma < \alpha_{i_m}$) and so p_i has length i_m. Thus p_i cannot be a subpath of q_j, since $m < n$ implies $i_m < i_n$. Conversely, q_j cannot be an initial subpath of q_i, since this would imply that q_j is an initial subpath of p_m, which is impossible by our construction of W_n. Thus W is a set of independent paths.

Note that by construction every path in W has source v, so by Lemma 4.4.3 we have that $\{L_K(E)qq^* : q \in W\}$ is an $L_K(E)$-independent family of left ideals contained in $L_K(E)v$. Note also that each T_n, and thus each W_n, has cardinality α_{i_n}, and so the cardinality of $W = \sup\{\alpha_{i_n} : n \in \mathbb{N}\} = \sigma$. Thus letting $W = X$, we have found a subset X of Y with cardinality σ such that the set of left ideals $\{L_K(E)pp^* : p \in X\}$ is $L_K(E)$-independent, as required.

Now, define

$$S = \sum_{p \in X} L_K(E)pp^* = \bigoplus_{p \in X} L_K(E)pp^* \subseteq L_K(E)v.$$

We know that $L_K(E)v$ is a direct summand of $L_K(E)$ (by Lemma 2.1.9), and so since $L_K(E)$ is injective as a left $L_K(E)$-module, so too is $L_K(E)v$ (by Lemma 1.2.12). Consider the inclusion map $\phi : S \to L_K(E)v$ and let $f \in \mathrm{Hom}_{L_K(E)}(S, L_K(E)v)$. Since $L_K(E)v$ is injective, there exists $h \in \mathrm{Hom}_{L_K(E)}(L_K(E)v, L_K(E)v)$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\phi"] \arrow[d, "f"'] & L_K(E)v \arrow[dl, dotted, "h"] \\ & L_K(E)v & \end{tikzcd}$$

That is, $h\phi = f$. Thus, if we define

$$\phi^* : \mathrm{Hom}_{L_K(E)}(L_K(E)v, L_K(E)v) \to \mathrm{Hom}_{L_K(E)}(S, L_K(E)v)$$

by $\phi^*(g) = g\phi$ for all $g \in \mathrm{Hom}_{L_K(E)}(L_K(E)v, L_K(E)v)$, then ϕ^* is an epimorphism. Then we have

$$\begin{aligned}\mathrm{Hom}_{L_K(E)}(S, L_K(E)v) \supseteq\ & \mathrm{Hom}_{L_K(E)}(S, S) \\ &= \mathrm{Hom}_{L_K(E)}\left(\bigoplus_{p\in X} L_K(E)pp^*, \bigoplus_{p\in X} L_K(E)pp^*\right) \\ &\cong \prod_{p\in X} \mathrm{Hom}_{L_K(E)}\left(L_K(E)pp^*, \bigoplus_{p\in X} L_K(E)pp^*\right),\end{aligned}$$

the final isomorphism coming from Proposition 1.2.4. Now, for each $k \in K$ and a fixed $i \in I$, we can define $\lambda_k^{(i)} \in \mathrm{Hom}_{L_K(E)}\left(L_K(E)pp^*, \bigoplus_{p\in X} L_K(E)pp^*\right)$ by $\lambda_k^{(i)}(x) = (w_j)_{j\in I}$, where $w_i = kx$ and $w_j = 0$ for $j \neq i$. Thus, setting $F^{(i)} = \{\lambda_k^{(i)} : k \in K\}$, we have $F^{(i)} \cong K$. Therefore

$$\prod_{p\in X} \mathrm{Hom}_{L_K(E)}\left(L_K(E)pp^*, \bigoplus_{p\in X} L_K(E)pp^*\right) \supseteq \prod_{p\in X} F_p^{(i)} \cong \prod_{p\in X} K_p,$$

where each $F_p^{(i)} = F^{(i)}$ and $K_p = K$. Now, by the Erdös-Kaplansky Theorem, $\prod_{p\in X} K_p$ has K-dimension $\mathrm{card}(K)^{\mathrm{card}(X)} = \mathrm{card}(K)^\sigma$ and so, by the above inequalities, $\mathrm{Hom}_{L_K(E)}(S, L_K(E)v)$ has K-dimension $\geq \mathrm{card}(K)^\sigma$. However, by Lemma 1.2.2, $\mathrm{Hom}_{L_K(E)}(L_K(E)v, L_K(E)v) \cong vL_K(E)v$, which has K-dimension $\leq \sigma < \mathrm{card}(K)^\sigma$, as observed earlier. This contradicts the fact that ϕ^* is an epimorphism, and so E must be row-finite. $\square$

Let R be a ring and let $n \in \mathbb{N}$. An R-module M is said to have **uniform dimension** n if M contains a direct sum of n nonzero submodules and no such collection larger than this. This notion features in the proof of the following proposition, which is from [ARM2, Proposition 4.5].

Proposition 4.4.5. *Let $L_K(E)$ be a left (resp. right) self-injective Leavitt path algebra, and let a be an arbitrary element of $L_K(E)$. Then the left ideal $L_K(E)a$ (resp. right ideal $aL_K(E)$) cannot contain an infinite set of $L_K(E)$-independent left (resp. right) ideals of $L_K(E)$.*

Proof. If $L_K(E)$ is left self-injective, then by Proposition 4.4.4 the graph E must be row-finite. Let $a \in L_K(E)$. Write $a = \sum_{j=1}^n k_j p_j q_j^*$, where $p_j, q_j \in E^*$ and $k_j \in K$, and let $V = \{s(p_j), s(q_j) : j = 1, \ldots, n\}$. By Lemma 2.1.12, $e = \sum_{v \in V} v$ is a local unit for a, and in particular we have $L_K(E)a \subseteq L_K(E)e$. We show that $L_K(E)e$ has finite uniform dimension.

By way of contradiction, suppose that $L_K(E)e$ contains an infinite family of independent submodules $\{A_i : i \in I\}$, where I is an infinite index set, and let $S = \bigoplus_{i \in I} A_i$. Note that every element of $eL_K(E)e$ is of the form $\sum_{j=1}^{m} l_j a_j b_j^*$, where $s(a_j), s(b_j) \in V$ for each $j = 1, \ldots, m$. Thus $eL_K(E)e = \bigoplus_{v \in V} vL_K(E)v$. For any $v \in V$, the cardinality of the set of paths of a fixed length n beginning with v must be finite (since E is row-finite), so the cardinality of the set of all paths of finite length beginning with v is at most countably infinite. Since $vL_K(E)v$ is generated by finite paths beginning with v, the K-dimension of $vL_K(E)v$ is at most countable, and thus the K-dimension of $eL_K(E)e$ is at most countable.

We now proceed as in the proof of Proposition 4.4.4. Using a similar argument, we can show $\mathrm{Hom}_{L_K(E)}(S, L_K(E)e) \supseteq \prod_{i \in I} F_i$, where each $F_i \cong K$. Furthermore, since $L_K(E)$ is left self-injective, the direct summand $L_K(E)e$ is an injective left $L_K(E)$-module, and so again we have an epimorphism

$$\phi^* : \mathrm{Hom}_{L_K(E)}(L_K(E)e, L_K(E)e) \to \mathrm{Hom}_{L_K(E)}(S, L_K(E)e).$$

However, as noted above, $\mathrm{Hom}_{L_K(E)}(L_K(E)e, L_K(E)e) \cong eL_K(E)e$ has countable K-dimension, while $\prod_{i \in I} F_i$ has K-dimension $\mathrm{card}(K)^{\mathrm{card}(I)}$ (by the Erdös-Kaplansky Theorem), which is uncountably infinite since I is infinite. Thus we have a contradiction, and so $L_K(E)e$, and therefore $L_K(E)a$, has finite uniform dimension. □

The following proposition is from [ARM2, Proposition 4.6].

Proposition 4.4.6. *For any graph E, if the Leavitt path algebra $L_K(E)$ is left (or right) self-injective, then every infinite path in E contains a line point.*

Proof. Suppose that γ is an infinite path in E that contains no line points. Now, since $L_K(E)$ is left self-injective, E must be acyclic, by Proposition 4.4.1. Thus γ must contain an infinite number of bifurcation vertices $\{v_i : i = 1, 2, 3, \ldots\}$, and so we can write γ as a concatenation of a series of countably many paths $\gamma_1\gamma_2\gamma_3\ldots$, where $r(\gamma_i) = v_i$ for each $i = 1, 2, 3, \ldots$. Furthermore, let $s(\gamma) = v$.

For each $n \in \mathbb{N}$, let $p_n = \gamma_1\gamma_2 \dots \gamma_n\gamma_n^* \dots \gamma_2^*\gamma_1^*$. Note that p_n is an idempotent in $L_K(E)v$. Suppose that $p_n = xp_{n+1}$ for some $x \in L_K(E)$ and some $n \in \mathbb{N}$. Since v_n is a bifurcation, there must exist an edge f_n with $s(f_n) = v_n$ such that f_n is not equal to the initial edge of γ_{n+1}. Thus $\gamma_{n+1}^* f_n = 0$, and so

$$0 \neq \gamma_1\gamma_2 \dots \gamma_n f_n = p_n\gamma_1\gamma_2 \dots \gamma_n f_n = xp_{n+1}\gamma_1\gamma_2 \dots \gamma_n f_n = x\gamma_1\gamma_2 \dots \gamma_{n+1}\gamma_{n+1}^* f = 0,$$

a contradiction. Thus, in particular we have $p_n \neq vp_{n+1} = p_{n+1}$, so that $p_n - p_{n+1} \neq 0$ for all $n \geq 1$.

We now show that $\{p_n - p_{n+1} : n = 1, 2, 3, \dots\}$ is a set of mutually orthogonal idempotents in $L_K(E)v$. First, consider $p_j p_i$ with $j > i$. Then

$$\begin{aligned} p_j p_i &= (\gamma_1\gamma_2 \dots \gamma_j\gamma_j^* \dots \gamma_2^*\gamma_1^*)(\gamma_1\gamma_2 \dots \gamma_i\gamma_i^* \dots \gamma_2^*\gamma_1^*) \\ &= \gamma_1\gamma_2 \dots \gamma_j\gamma_j^* \dots \gamma_{i+1}^*(\gamma_i^* \dots \gamma_2^*\gamma_1^*\gamma_1\gamma_2 \dots \gamma_i)\gamma_i^* \dots \gamma_2^*\gamma_1^* \\ &= \gamma_1\gamma_2 \dots \gamma_j\gamma_j^* \dots \gamma_{i+1}^*(v_i)\gamma_i^* \dots \gamma_2^*\gamma_1^* \\ &= p_j. \end{aligned}$$

Similarly, if $j < i$ then we have $p_j p_i = p_i$. In particular, $p_{i+1}p_i = p_{i+1}$. Note also that $p_i p_i = p_i$ for any $i \geq 1$. Thus

$$(p_i - p_{i+1})^2 = p_i p_i - p_i p_{i+1} - p_{i+1} p_i + p_{i+1} p_{i+1} = p_i - 2p_{i+1} + p_{i+1} = p_i - p_{i+1},$$

while for $j > i$,

$$(p_j - p_{j+1})(p_i - p_{i+1}) = p_j p_i - p_j p_{i+1} - p_{j+1} p_i + p_{j+1} p_{i+1} = p_j - p_j - p_{j+1} + p_{j+1} = 0.$$

Note that if $j = i + 1$, we still have $p_j p_{i+1} = p_j p_j = p_j$ in the expression above, as required. Similarly, $(p_j - p_{j+1})(p_i - p_{i+1}) = 0$ for $j < i$.

Thus $\{p_n - p_{n+1} : n = 1, 2, 3, \dots\}$ is a set of nonzero, mutually orthogonal idempotents in $L_K(E)v$, and so $\{L_K(E)(p_n - p_{n+1}) : n = 1, 2, 3, \dots\}$ is a countably infinite independent family of left ideals in $L_K(E)v$. However, this contradicts Proposition 4.4.5, and so every infinite path in E must contain a line point. □

We now come to the main result of this section, which is from [ARM2, Theorem 4.7].

Theorem 4.4.7. *Let E be an arbitrary graph and let K be any field. The following statements are equivalent:*

(i) $L_K(E)$ is left self-injective.

(ii) $L_K(E)$ is right self-injective.

(iii) The graph E is row-finite, acyclic and every infinite path contains a line point.

(iv) $L_K(E)$ is semisimple.

Proof. (i)$\Rightarrow$(iii): This follows directly from Propositions 4.4.1, 4.4.4 and 4.4.6.

(iii)$\Rightarrow$(iv): We begin by showing that $\overline{P_l(E)} = E^0$. Suppose, by way of contradiction, that there exists $v \in E^0$ such that $v \notin \overline{P_l(E)}$. Since v is not a line point, v cannot be a sink, and so $s^{-1}(v) \neq \emptyset$. Now if $r(s^{-1}(v)) \subseteq \overline{P_l(E)}$, then the saturated property of $\overline{P_l(E)}$ would imply that $v \in \overline{P_l(E)}$, a contradiction. Thus there must exist some edge $e_1 \in s^{-1}(v)$ for which $w = r(e_1) \notin \overline{P_l(E)}$. Repeating this argument, we can find an edge $e_2 \in s^{-1}(w)$ for which $x = r(e_2) \notin \overline{P_l(E)}$, and so on. Since E is acyclic, we can create an infinite path $\gamma = e_1e_2e_3\dots$ for which $r(e_i) \notin \overline{P_l(E)}$ for each e_i. However, this contradicts the fact that every infinite path in E must contain a line point. Thus $\overline{P_l(E)} = E^0$, and so we have $I(\overline{P_l(E)}) = L_K(E)$. By Theorem 3.2.11, this implies that $\mathrm{soc}_l(L_K(E)) = L_K(E)$, and so $L_K(E)$ is the direct sum of minimal left ideals. Thus $L_K(E)$ is semisimple.

(iv)$\Rightarrow$(i): If $L_K(E)$ is semisimple then it is a direct sum of minimal left ideals, and so $\mathrm{soc}_l(L_K(E)) = L_K(E)$. Thus, by [L1, Theorem 2.8], every left $L_K(E)$-module is injective. In particular, $L_K(E)$ is left self-injective.

Similarly, we can show that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii), since Propositions 4.4.1, 4.4.4 and 4.4.6 also hold for when $L_K(E)$ is right self-injective. Furthermore, if $L_K(E)$ is semisimple, then $\mathrm{soc}_l(L_K(E)) = L_K(E) = \mathrm{soc}_r(L_K(E))$ (by Corollary 3.2.2), and so we can apply [L1, Theorem 2.8] again to yield that $L_K(E)$ is right self-injective. $\square$

Example 4.4.8. We now apply Theorem 4.4.7 to our familiar examples of Leavitt path algebras to determine if they are self-injective.

(i) *The finite line graph* M_n. Since M_n is row-finite, acyclic and contains no infinite paths, $L_K(M_n) \cong \mathbb{M}_n(K)$ is both left and right self-injective (and also semisimple) for all $n \in \mathbb{N}$.

(ii) *The rose with* n *leaves* R_n. For each $n \in \mathbb{N}$, R_n contains n cycles and so $L_K(R_n) \cong L(1, n)$ is neither left nor right self-injective.

(iii) *The infinite clock graph* C_∞. Since C_∞ is not row-finite, we have that $L_K(C_\infty) \cong \bigoplus_{i=1}^{\infty} \mathbb{M}_2(K) \oplus KI_{22}$ is neither left nor right self-injective.

Appendix A

Direct Limits

A.1 Direct Limits

A set A is said to be an **upward-directed set** if there is a partial ordering $\leq$ on A such that, for any pair $a, b \in A$, there exists $c \in A$ such that $a \leq c$ and $b \leq c$.

Let I be an upward-directed index set and let $\{R_i : i \in I\}$ be a family of (not necessarily unital) rings. Furthermore, for each pair $i, j \in I$ with $i \leq j$, let $\varphi_{ij} : R_i \to R_j$ be a ring homomorphism. We say that $(R_i, \varphi_{ij})_I$ is a **direct system of rings, indexed by** I if, for all $i, j, k \in I$ with $i \leq j \leq k$, we have $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$; that is, the following diagram commutes:

$$\begin{tikzcd} R_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & & R_j \arrow[dl, "\varphi_{jk}"] \\ & R_k & \end{tikzcd}$$

Definition A.1.1. Let $(R_i, \varphi_{ij})_I$ be a direct system of rings and let R be a ring for which there exists a ring homomorphism $\varphi_i : R_i \to R$ for each $i \in I$. We say that (R, φ_i), or simply R, is a **direct limit** of the system if the following two conditions are satisfied:

(i) For each pair $i, j \in I$ with $i \leq j$, we have $\varphi_i = \varphi_j\varphi_{ij}$; that is, the following diagram commutes:

$$\begin{tikzcd} R_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"'] & & R_j \arrow[dl, "\varphi_j"] \\ & R & \end{tikzcd}$$

(ii) If S is a ring for which there exist ring homomorphisms $\mu_i : R_i \to S$ such that $\mu_i = \mu_j \varphi_{ij}$ for all $i, j \in I$ with $i \leq j$, then there exists a unique ring homomorphism $\mu : R \to S$ such that $\mu_i = \mu\varphi_i$ for each $i \in I$; that is, the following diagram commutes:

$$\begin{tikzcd} R_i \arrow[rr, "\varphi_i"] \arrow[dr, "\mu_i"'] & & R \arrow[dl, "\mu"] \\ & S & \end{tikzcd}$$

Now suppose that $(\bar{R}, \bar{\varphi}_i)$ is another ring and set of ring homomorphisms that satisfy conditions (i) and (ii). Then there exists a unique homomorphism $\mu : R \to \bar{R}$ such that $\bar{\varphi}_i = \mu\varphi_i$ for all $i \in I$. Similarly, there exists a unique homomorphism $\mu' : \bar{R} \to R$ such that $\varphi_i = \mu'\bar{\varphi}_i$ for all $i \in I$. Thus we have $\varphi_i = \mu'\mu\varphi_i$, giving (by the uniqueness) $\mu'\mu = 1_R$, and $\bar{\varphi}_i = \mu\mu'\bar{\varphi}_i$, giving $\mu\mu' = 1_{\bar{R}}$. Thus μ is an isomorphism and so $R \cong \bar{R}$. A direct limit is therefore unique up to isomorphism, and so we can uniquely denote this limit by $\varinjlim(R_i, \varphi_{ij})$.

Note that if I is an upward-directed index set and $\{R_i : i \in I\}$ is an ascending chain of rings – that is, $R_i \subseteq R_{i+1}$ for each $i \in I$ – then defining φ_{ij} to be the inclusion map from R_i to R_j (for each pair $i, j \in I$ with $i \leq j$), we have that $(R_i, \varphi_{ij})_I$ is a direct system. In this case we usually drop the φ_{ij} from the notation and write the direct limit of the family as simply $\varinjlim_{i\in I} R_i$. It is straightforward to show that $\varinjlim_{i\in I} R_i = \bigcup_{i\in I} R_i$, the directed union of the family.

We illustrate the concept of a direct limit with the following useful example. Let R be a ring with local units, so that there exists a set of idempotents $I \subseteq R$ for which, given any finite subset $\{x_1 \dots, x_n\} \subseteq R$, there exists $e \in I$ such that $x_i \in eRe$ for each $i = 1, \dots, n$. We define a partial ordering $\leq$ on I by writing $e \leq f$ if $e \in fRf$. (Note that $e \leq f$ is equivalent to $eRe \subseteq fRf$.) Furthermore, I

is an upward-directed set: given any pair $e, f \in I$, there must exist $g \in I$ such that $e, f \in gRg$ (by the definition of local units), so that $e \leq g$ and $f \leq g$.

Lemma A.1.2. *Let R be a ring with local units. Let I be the set of local units and let $\leq$ be the partial ordering defined above. For each pair $e, f \in I$ with $e \leq f$, define $\varphi_{ef} : eRe \to fRf$ and $\varphi_e : eRe \to R$ to be the inclusion ring homomorphisms. Then $R = \varinjlim(eRe, \varphi_{ef})$.*

Proof. For each $e, f, g \in I$ with $e \leq f \leq g$ we clearly have $\varphi_{eg} = \varphi_{fg}\varphi_{ef}$, and so $(eRe, \varphi_{ef})_I$ is a direct system of rings. Furthermore, for each pair $e, f \in I$ with $e \leq f$ we clearly have $\varphi_e = \varphi_f\varphi_{ef}$, so that the following diagram commutes:

$$\begin{tikzcd} eRe \arrow[rr, "\varphi_{ef}"] \arrow[dr, "\varphi_e"'] & & fRf \arrow[dl, "\varphi_f"] \\ & R & \end{tikzcd}$$

Thus we have satisfied condition (i) of the direct limit definition.

Now suppose there exists a ring S and ring homomorphisms $\mu_e : eRe \to S$ such that $\mu_e = \mu_f\varphi_{ef}$ for all $e, f \in I$ with $e \leq f$. For any $x \in R$, choose $e \in I$ such that $x \in eRe$ (such an element exists since I is a set of local units), and let $\mu(x) = \mu_e(x)$, thus defining a map $\mu : R \to S$. Note that our choice of e is not unique, so we must check that this map is well-defined. Suppose there exists $f \in I$ with $f \neq e$ such that $x \in fRf$. Since I is an upward-directed set, there exists $g \in I$ such that $e \leq g$ and $f \leq g$, and so

$$\mu_e(x) = \mu_g(\varphi_{eg}(x)) = \mu_g(x) = \mu_g(\varphi_{fg}(x)) = \mu_f(x)$$

and thus μ is well-defined. Furthermore, given $x, y \in R$ there exists $e \in I$ for which $x+y \in eRe$ and $xy \in eRe$, and so $\mu(x+y) = \mu_e(x+y) = \mu_e(x)+\mu_e(y) = \mu(x)+\mu(y)$ (since μ_e is a ring homomorphism) and similarly $\mu(xy) = \mu(x)\mu(y)$. Thus μ is a ring homomorphism.

Now, for each $e \in I$ we have $\mu(\varphi_e(x)) = \mu(x) = \mu_e(x)$ for all $x \in eRe$, so that the following diagram commutes:

$$\begin{tikzcd} eRe \arrow[rr, "\varphi_e"] \arrow[dr, "\mu_e"'] & & R \arrow[dl, "\mu"] \\ & S & \end{tikzcd}$$

Finally, to show that μ is unique, suppose that $\nu : R \to S$ is also a ring homomorphism with $\nu\varphi_e = \mu_e$ for all $e \in I$. Let $x \in R$ and choose $f \in I$ such that $x \in fRf$. Then

$$\nu(x) = \nu(\varphi_f(x)) = \mu_f(x) = \mu(x)$$

and so $\nu = \mu$. Thus we have satisfied condition (ii) of the direct limit definition and so $R = \varinjlim(R_i, \varphi_{ef})$, up to isomorphism. $\square$

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Index