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Lebesgue Points of Multi-Dimensional Functions

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Abstract: Lebesgue and Walsh-Lebesgue points are introduced for higher dimensional functions and it is proved that a.e. point is a (Walsh)-Lebesgue point of a function f from the space $L(\log L)^{d-1}$. Every function $f \in L(\log L)^{d-1}$ is Fejér summable at each (Walsh)-Lebesgue point.

Keywords: Lebesgue point, Walsh-Lebesgue point, Walsh functions, Fejér-summability.

1 Introduction

IT WAS PROVED by Fejér [1] that the $(C,1)$ or Fejér means of the one-dimensional trigonometric Fourier series of a continuous function converge uniformly to the function. The same problem for integrable functions was investigated by Lebesgue [2]. He proved that for every integrable function f,

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k f(x) \to f(x) \quad \text{as} \quad n \to \infty$$

at each Lebesgue point of f, where $s_k f$ denotes the kth partial sum of the Fourier series of f. Almost every point is a Lebesgue point of f (see Zygmund [3] or Butzer and Nessel [4]).

The concept of Lebesgue points was extended to the one-dimensional Walsh system by the author in [5], the points are called Walsh-Lebesgue points in this case. The definition of Walsh-Lebesgue points is not a simple adaptation of the one of Lebesgue points, it needs new ideas, because the Walsh-Fejér kernels differ entirely from the trigonometric Fejér kernel. It was proved there that a.e. point

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is a Walsh-Lebesgue point of a one-dimensional integrable function. Moreover, the Fejér means of the Walsh-Fourier series of $f \in L_1[0,1)$ converge to f at each Walsh-Lebesgue point. The a.e. convergence of the Fejér means was proved earlier by Fine [6] (see also Schipp [7]).

In this paper we generalize the definition of Lebesgue and Walsh-Lebesgue points for higher dimensions. We prove that a.e. point is a (Walsh)-Lebesgue point of $f \in L(\log L)^{d-1}$. The Fejér means of the Walsh-Fourier series of $f \in L(\log L)^{d-1}$ converge to f at each (Walsh)-Lebesgue point.

2 Lebesgue Points

For a set $\mathbb{X} \neq \emptyset$ let $\mathbb{X}^d$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself d-times. We briefly write $L_p(\mathbb{X}^d)$ instead of $L_p(\mathbb{X}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{X}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leq \infty)$, where λ is the Lebesgue measure and $\mathbb{X}$ denotes the torus $\mathbb{T} = [-1/2, 1/2]$ or the unit interval $[0,1)$.

In the one-dimensional case Lebesgue differentiation theorem says that

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x)$$

for a.e. $x \in \mathbb{T}$, where $f \in L_1(\mathbb{T})$. This motivates the next definition. A point $x \in \mathbb{T}$ is called a *Lebesgue point* of a function f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt = 0.$$

Using Lebesgue differentiation theorem we can prove in the usual way that a.e. point $x \in \mathbb{T}$ is a Lebesgue point of $f \in L_1(\mathbb{T})$ (see e.g. Butzer and Nessel [4] or Stein and Weiss [8]).

Feichtinger and Weisz [9] extended the definition of Lebesgue points to higher dimensions as follows. The *strong Hardy-Littlewood maximal function* is defined by

$$M_s f(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f| \, d\lambda,$$

where $f \in L_1(\mathbb{T}^d)$, $x \in \mathbb{T}^d$ and the supremum is taken over all rectangles $I \subset \mathbb{T}^d$ with sides parallel to the axes. It is known that in the one-dimensional case the maximal function is of weak type $(1,1)$, i.e.,

$$\sup_{\rho > 0} \rho \lambda(M_s f > \rho) \leq C_1 \|f\|_1, \qquad (f \in L_1(\mathbb{T})).$$

However, for higher dimensions there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ a.e. Thus M_s cannot be of weak type $(1,1)$ if $d > 1$, but we have

$$\sup_{\rho>0} \rho \lambda(M_s f > \rho) \leq C_d + C_d \|f\|_{L_1(\log L)^{d-1}}, \tag{1}$$

where C_d is depending only on d. Moreover,

$$\|M_s f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d), 1 < p \leq \infty, d \geq 1). \tag{2}$$

For these results see Zygmund [3], Stein [10] or Weisz [11, p. 71]. Set $\log^+ u = 1_{\{u>1\}} \log u$. Recall that a function f is in the set $L_1(\log L)^k(\mathbb{T}^d)$ if

$$\|f\|_{L_1(\log L)^k} := \int_{\mathbb{T}^d} |f| (\log^+ |f|)^k \, d\lambda < \infty.$$

If $k = 0$ then $L_1(\log L)^k(\mathbb{T}^d) = L_1(\mathbb{T}^d)$. We can say that the role of $L_1(\mathbb{T})$ in one dimension is played in higher dimensions by $L_1(\log L)^{d-1}(\mathbb{T}^d)$.

Inequalities (1) and (2) imply

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1+h_1} \dots \int_{x_d}^{x_d+h_d} f(t)\, dt = f(x)$$

for a.e. $x \in \mathbb{T}^d$, where $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ or $f \in L_p(\mathbb{T}^d)$ $(1 < p \leq \infty)$. Note that $L_1(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d)$ $(1 < p \leq \infty)$. Here $h \to 0$ is understood in the Pringsheim's sense, i.e., $h_j \to 0$ for all $j = 1, \dots, d$.

A point $x \in \mathbb{T}^d$ is called a *Lebesgue point* of f if $M_s f(x)$ is *finite* and

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1+h_1} \dots \int_{x_d}^{x_d+h_d} |f(t) - f(x)|\, dt = 0.$$

The next theorem is proved in Feichtinger and Weisz [9].

Theorem 1 *Almost every point $x \in \mathbb{T}^d$ is a Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$.*

3 Fejér Means of Fourier Series

For a one-dimensional integrable function f the nth Fourier coefficient is defined by

$$\hat{f}(n) = \int_{\mathbb{T}} f(t) e^{-2\pi \imath n t} \, dt \qquad (n \in \mathbb{Z}).$$

The nth partial sum of the trigonometric Fourier series of f is given by

$$s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi \imath k x} \qquad (n \in \mathbb{N}).$$

One of the deepest results in harmonic analysis is Carleson's theorem [12, 13]:

$$s_n f \to f \qquad \text{a.e. as } n \to \infty,$$

whenever $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$. This theorem does not hold, if $p = 1$. However, some summability results can be obtained in this case, too.

The *Fejér-means* of f are defined by

$$\sigma_n f(x) := \frac{1}{n+1} \sum_{k=0}^{n} s_k f(x) = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) \hat{f}(k) e^{2\pi \imath k x} = \int_{\mathbb{T}} f(t) K_n(x+t)\, dt,$$

where $x \in \mathbb{T}, n \in \mathbb{N}$ and K_n denote the *Fejér kernels*. As mentioned in the introduction, Lebesgue [2] proved for all $f \in L_1(\mathbb{T})$ that

$$\sigma_n f \to f \qquad \text{at each Lebesgue point of } f \text{ as } n \to \infty.$$

In the multi-dimensional case let the nth Fourier coefficient of a function $f \in L_1(\mathbb{T}^d)$ be defined by

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(t) e^{-2\pi \imath n \cdot t}\, dt \qquad (n \in \mathbb{Z}^d),$$

where $u \cdot x := \sum_{k=1}^{d} u_k x_k$, $(x = (x_1, \ldots, x_d) \in \mathbb{R}^d, u = (u_1, \ldots, u_d) \in \mathbb{R}^d)$. Denote by $s_n f$ the nth *partial sum* of the trigonometric Fourier series of f:

$$s_n f(x) := \sum_{j=1}^{d} \sum_{k_j=-n_j}^{n_j} \hat{f}(k) e^{2\pi \imath k \cdot x} \qquad (n \in \mathbb{N}^d).$$

Under $\sum_{j=1}^{d} \sum_{k_j=-n_j}^{n_j}$ we mean the sum $\sum_{k_1=-n_1}^{n_1} \cdots \sum_{k_d=-n_d}^{n_d}$.

Carleson's result does not hold for higher dimensions (see Fefferman [14]). The only known result is that

$$s_{n,\ldots,n} f \to f \qquad \text{a.e. as } n \to \infty, \tag{3}$$

whenever $f \in L_p(\mathbb{T}^d)$ $(1 < p < \infty)$ (Fefferman [15]).

Now we introduce the *Fejér-means* of f by

$$\sigma_n f(x) := \frac{1}{\prod_{i=1}^d (n_i+1)} \sum_{j=1}^d \sum_{k_j=0}^{n_j} s_k f(x) = \sum_{j=1}^d \sum_{k_j=-n_j}^{n_j} \Big(\prod_{i=1}^d \Big(1 - \frac{|k_i|}{n_i+1}\Big)\Big) \hat{f}(k) e^{2\pi \imath k\cdot x},$$

$(x \in \mathbb{T}^d, n \in \mathbb{N}^d)$. In the following theorem we generalize Lebesgue's theorem just mentioned (see Feichtinger and Weisz [9]).

Theorem 2 *For all Lebesgue points of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ we have*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x).$$

4 Walsh-Lebesgue Points

The definition of the Walsh-Lebesgue points should fulfill the next two requirements: a.e. point is a Walsh-Lebesgue point of an integrable function and the Walsh-Fejér means of an integrable function converge at all Walsh-Lebesgue points. The proof of the one-dimensional version of Theorem 2 is based on the fact, that the Fejér kernels K_n can be estimated by an integrable, on $[0,1/2]$ non-increasing function K_n' such that $\|K_n'\|_1 \le C$ for all $n \in \mathbb{N}$. Recall that

$$K_n'(x) = C(n+1)\mathbf{1}_{[0,1/(n+1)]} + \frac{C}{(n+1)x^2}\mathbf{1}_{[1/(n+1),1/2]}.$$

This does not hold for the Walsh-Fejér kernels K_{2^n} (for the definition see the next section), because

$$K_{2^n}(x) = \frac{1}{2}\Big(2^{-n} D_{2^n}(x) + \sum_{k=0}^n 2^{k-n} D_{2^n}(x \dot{+} e_k)\Big),$$

where

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if} \quad x \in [0,2^{-n}), \\ 0, & \text{if} \quad x \in [2^{-n},1) \end{cases}$$

are the Walsh-Dirichlet kernels, $\dot{+}$ denotes the dyadic addition and $e_k := 2^{-k-1}$. It is easy to see that if K_n' denotes the smallest non-increasing function for which $K_n \le K_n'$ then $\|K_{2^n}'\|_1 = Cn$. Because of this difference of the Fejér and Walsh-Fejér kernels, a new definition of Lebesgue points is needed in the dyadic case.

By a *dyadic interval* we mean one of the form $[k2^{-n},(k+1)2^{-n})$ for some $k,n \in \mathbb{N}$, $0 \le k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ be the dyadic interval of length 2^{-n} which contains x. A Cartesian product of d dyadic intervals is called a *dyadic rectangle*. For $n \in \mathbb{N}^d$ and $x \in [0,1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$, where $n = (n_1, \ldots, n_d)$ and $x = (x_1, \ldots, x_d)$. The σ-algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0,1)^d\}$ will be denoted by $\mathscr{F}_n$ $(n \in \mathbb{N}^d)$. Let E_n denote the conditional expectation operator with respect to $\mathscr{F}_n$. Obviously, if $f \in L_1[0,1)^d$ then $(E_n f, n \in \mathbb{N}^d)$ is a martingale.

Butzer and Wagner [16] introduced the dyadic derivative of f with the limit of

$$\mathbf{d}_n f(x) := \sum_{k=0}^{n-1} 2^{k-1}(f(x) - f(x \dotplus e_k)) \qquad (x \in [0,1))$$

as $n \to \infty$. For $f \in L_1[0,1)$ let $F(x) := \int_{I_n(x)} f$ and investigate the function

$$\mathbf{d}_n F(x) = \sum_{k=0}^{n-1} 2^{k-1} \Big(\int_{I_n(x)} f - \int_{I_n(x \dotplus e_k)} f \Big).$$

Since the first terms on the right hand side can be well handled, in the definition of Walsh-Lebesgue points we will consider the second terms, only. We can prove (see Schipp, Wade, Simon and Pál [17] or Weisz [11]) that $\lim_{n\to\infty} \mathbf{d}_n F(x) = 0$ a.e. Since $2^n \int_{I_n(x)} f = E_n f(x)$, by the corresponding martingale theorem $\lim_{n\to\infty} E_n f(x) = f(x)$ a.e. Thus

$$\frac{1}{2} \sum_{k=0}^{n} 2^k \int_{I_n(x \dotplus e_k)} f = (2^n - \frac{1}{2}) \int_{I_n(x)} f - \mathbf{d}_n F(x)$$

tends to $f(x)$ for a.e. $x \in [0,1)$ as $n \to \infty$.

Motivated by this fact, the author introduced the one-dimensional *Walsh-Lebesgue points* in [5] as follows: $x \in [0,1)$ is a Walsh-Lebesgue point of $f \in L_1[0,1)$, if

$$\lim_{n\to\infty} \sum_{k=0}^{n} 2^k \int_{I_n(x \dotplus e_k)} |f(t) - f(x)|\, dt = 0.$$

We proved in [5] that a.e. point $x \in [0,1)$ is a Walsh-Lebesgue point of an integrable function f.

In the multi-dimensional case a point $x \in [0,1)^d$ is a *Walsh-Lebesgue point* of $f \in L_1[0,1)^d$, if

$$\lim_{n\to\infty} \sum_{j=1}^{d} \sum_{k_j=0}^{n_j} 2^k \int_{I_n(x \dotplus e_k)} |f(t) - f(x)|\, dt = 0, \tag{4}$$

where $2^k := 2^{k_1} \cdots 2^{k_d}$ and $e_k := (e_{k_1}, \ldots, e_{k_d})$. If we define

$$V_n f(x) := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j} 2^{k-n} E_n f(x \dot{+} e_k),$$

then it is easy to see that x is a Walsh-Lebesgue point of f if and only if

$$\lim_{n\to\infty} V_n(|f - f(x)|)(x) = 0,$$

because $E_n f(x) = 2^n \int_{I_n(x)} f$. We ([18]) have shown the next theorem for the operator

$$V f := \sup_{n\in\mathbb{N}^d} |V_n f|.$$

Theorem 3 *For all* $1 < p \leq \infty$

$$\|V f\|_p \leq C_p \|f\|_p \qquad (f \in L_p[0,1)^d)$$

and

$$\sup_{\rho>0} \rho \lambda (V f > \rho) \leq C \|f\|_{L_1(\log L)^{d-1}} \qquad (f \in L_1(\log L)^{d-1}[0,1)^d). \tag{5}$$

It is easy to show that (4) holds for every Walsh polynomials and $x \in [0,1)^d$. Since the Walsh polynomials are dense in $L_1(\log L)^{d-1}[0,1)^d$, (5) and the usual density argument (see Marcinkievicz and Zygmund [19]) imply

Corollary 1 *If* $f \in L_1(\log L)^{d-1}[0,1)^d$ *then*

$$\lim_{n\to\infty} \sum_{j=1}^{d} \sum_{k_j=0}^{n_j} 2^k \int_{I_n(x \dot{+} e_k)} |f(t) - f(x)|\, dt = 0 \qquad \text{a.e. } x \in [0,1)^d,$$

thus a.e. point is a Walsh-Lebesgue point of f.

5 Fejér Means of Walsh-Fourier Series

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k},$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $(0 \le n_k < 2)$.

The Kronecker product $(w_n, n \in \mathbb{N}^d)$ of d Walsh systems is said to be the *d-dimensional Walsh system*. Thus

$$w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d)$$

where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in [0,1)^d$.

The nth Fourier coefficient and the partial sum of $f \in L_1[0,1)^d$ are introduced by

$$\hat{f}(n) := \int_{[0,1)^d} f w_n \, d\lambda \qquad (n \in \mathbb{N}^d)$$

and

$$s_n f := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j-1} \hat{f}(k) w_k \qquad (n \in \mathbb{N}^d).$$

It is known that $s_{2^{n_1},\ldots,2^{n_d}} f = E_n f$ $(n \in \mathbb{N}^d)$ and

$$s_{2^{n_1},\ldots,2^{n_d}} f \to f \qquad \text{in } L_p\text{-norm as } n \to \infty,$$

if $f \in L_p[0,1)^d$ $(1 \le p < \infty)$. If $p > 1$ then the convergence holds also a.e. (see e.g. Schipp, Wade, Simon and Pál [17] or Weisz [20]).

The one-dimensional Carleson's theorem was extended to Walsh-Fourier series by Billard [21] and Sjölin [22]: if $f \in L_p[0,1)$ $(1 < p < \infty)$ then

$$s_n f \to f \qquad \text{a.e. as } n \to \infty.$$

The a.e. convergence of $s_n f$ is not true in the multi-dimensional case (Fefferman [14, 15]), however, the analogue of (3) holds: for $f \in L_2[0,1)^d$

$$s_{n,\ldots,n} f \to f \qquad \text{a.e. as } n \to \infty,$$

(Móricz [23] or Schipp, Wade, Simon and Pál [17]). In contrary to the trigonometric case, it is unknown whether this result holds for functions in $L_p[0,1)^d$, $1 < p < 2$.

To obtain convergence results for $L_1[0,1)$ or $L(\log L)^{d-1}[0,1)^d$ functions we introduce the *Fejér means* of f by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^d \sum_{k_j=1}^{n_j} s_k f = \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \left(\prod_{i=1}^d \left(1 - \frac{k_i}{n_i}\right)\right) \hat{f}(k) w_k.$$

If

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) w_k \qquad (n \in \mathbb{N})$$

denotes the one-dimensional *Fejér kernels*, then

$$\sigma_n f(x) = \int_{[0,1)^d} f(t) (K_{n_1}(x_1 \dot{+} t_1) \cdots K_{n_d}(x_d \dot{+} t_d))\, dt.$$

The Fejér means of f converge to f a.e. if $f \in L(\log L)^{d-1}[0,1)^d$ (see Fine [6] and Schipp [7] for the one-dimensional case, i.e., for integrable functions and Weisz [11] for the multi-dimensional case). For Vilenkin-Fourier series these results are due to Simon [24]. The next result concerning Walsh-Lebesgue points characterizes the set of convergence and was proved by the author in [5] for one dimension and in [18] for higher dimensions.

Theorem 4 *If $f \in L_1(\log L)^{d-1}[0,1)^d$ then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all Walsh-Lebesgue points of f.

Note that the convergence $\lim_{n\to\infty} \sigma_n f = f$ a.e. cannot be extended to all $f \in L_1[0,1)^d$ (see Gát [25, 26]) and so Theorem 4 is not true for all $f \in L_1[0,1)^d$.

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References

[1] L. Fejér, “Untersuchungen über fouriersche reihen,” *Math. Annalen*, vol. 58, pp. 51–69, 1904.

[2] H. Lebesgue, "Recherches sur la convergence des séries de fourier," *Math. Annalen*, vol. 61, pp. 251–280, 1905.

[3] A. Zygmund, *Trigonometric Series*, 3rd ed. London: Cambridge Press, 2002.

[4] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*. Basel: Birkhäuser Verlag, 1971.

[5] F. Weisz, "Convergence of singular integrals," *Ann. Univ. Sci. Budapest. Sect. Math.*, vol. 32, pp. 243–256, 1989.

[6] N. J. Fine, "Cesáro summability of Walsh-Fourier series," *Proc. Nat. Acad. Sci. USA*, vol. 41, pp. 558–591, 1955.

[7] F. Schipp, "Über gewissen Maximaloperatoren," *Ann. Univ. Sci. Budapest Sect. Math.*, vol. 18, pp. 189–195, 1975.

[8] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton, N.J.: Princeton Univ. Press, 1971.

[9] H. G. Feichtinger and F. Weisz, "Wiener amalgams and pointwise summability of fourier transforms and fourier series," *Math. Proc. Camb. Phil. Soc.*, vol. 140, pp. 509–536, 2006.

[10] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*. Princeton, N.J.: Princeton Univ. Press, 1993.

[11] F. Weisz, *Summability of Multi-dimensional Fourier Series and Hardy Spaces*. Dordrecht, Boston, London: Kluwer Academic Publishers, 2002, vol. Mathematics and Its Applications.

[12] L. Carleson, "On convergence and growth of partial sums of Fourier series," *Acta Math.*, vol. 116, pp. 135–157, 1966.

[13] R. A. Hunt, "On the convergence of Fourier series," in *Proc. konf Orthogonal Expansions and their Continuous Analogues*, Edwardsville, Ill., 1967, pp.235-255, Illinois Univ. Press Carbondale, 1968.

[14] C. Fefferman, "On the divergence of multiple Fourier series," *Bull. Amer. Math. Soc.*, vol. 77, pp. 191–195, 1971.

[15] ——, "On the convergence of multiple Fourier series," *Bull. Amer. Math. Soc.*, vol. 77, pp. 744–745, 1971.

[16] P. L. Butzer and H. J. Wagner, "Walsh series and the concept of a derivative," *Applic. Anal.*, vol. 3, pp. 29–46, 1973.

[17] F. Schipp, W. R. Wade, P. Simon, and J. Pál, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*. Bristol, New York: Adam Hilger, 1990.

[18] F. Weisz, "Walsh-lebesgue points of multi-dimensional functions," *Anal. Math.*, vol. 34, pp. 307–324, 2008.

[19] J. Marcinkiewicz and A. Zygmund, "On the summability of double Fourier series," *Fund. Math.*, vol. 32, pp. 122–132, 1939.

[20] F. Weisz, *Martingale Hardy Spaces and their Applications in Fourier Analysis*. Berlin: Springer, 1994, vol. 1568 of Lecture Notes in Math.

[21] P. Billard, "Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $l^2[0,1]$," *Studia Math.*, vol. 28, pp. 363–388, 1967.

[22] P. Sjölin, "An inequality of Paley and convergence a.e. of walsh-fourier series," *Arhiv für Math.*, vol. 8, pp. 551–570, 1969.

[23] F. Móricz, "On the convergence of double orthogonal series," *Anal. Math.*, vol. 2, pp. 287–304, 1976.

[24] P. Simon, "Investigations with respect to the Vilenkin system," *Ann. Univ. Sci. Budapest Sect. Math.*, vol. 27, pp. 87–101, 1985.

[25] G. Gát, "On the divergence of the $(C,1)$ means of double Walsh-Fourier series," *Proc. Amer. Math. Soc.*, vol. 128, pp. 1711–1720, 2000.

[26] ——, "Divergence of the $(C,1)$ means of d-dimensional Walsh-Fourier series," *Anal. Math.*, vol. 27, pp. 157–171, 2001.