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REVIEW ARTICLE

AdS black holes, holography and localization

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Abstract
I review some recent progresses in counting the number of microstates of AdS supersymmetric black holes in dimension equal or greater than four using holography. The counting is obtained by applying localization and matrix model techniques to the dual field theory. I cover in details the case of dyonic AdS_4 black holes, corresponding to a twisted compactification of the dual field theory, and I discuss the state of the art for rotating AdS_5 black holes.

Keywords Black hole entropy · Holography · AdS/CFT

Contents

1 Introduction 2
  1.1 Content of the review 3
2 AdS black holes in $d \geq 4$ 5
  2.1 AdS black holes and holography 5
    2.1.1 Kerr–Newman black holes and generalizations 6
    2.1.2 Magnetically charged black holes with a twist 7
  2.2 Computing the entropy 11
    2.2.1 The grand canonical partition function 11
    2.2.2 The supersymmetric index 13
  2.3 Entropy functionals and the attractor mechanism 14
  2.4 I-extremization 16
3 The topologically twisted index 18
  3.1 The topological twist 18
  3.2 The localization formula 21
    3.2.1 The BPS locus 22
    3.2.2 The integrand 23
    3.2.3 The contour 24

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3.2.4 Adding flavor fugacities ... 25
3.2.5 The trace interpretation ... 26
3.2.6 An example: SQED ... 26
3.3 Interpretation of the localization formula ... 27
3.3.1 Reduction to quantum mechanics ... 27
3.3.2 Reduction to two dimensions ... 29
4 The entropy of dyonic AdS_4 black holes ... 32
4.1 AdS_4 dyonic static black holes ... 32
4.1.1 Black holes in $AdS_4 \times S^7$ ... 35
4.2 The dual field theory ... 37
4.3 The ABJM Bethe vacua in the large N limit ... 40
4.4 The large N limit of the ABJM index ... 44
4.5 Matching index and entropy for ABJM ... 45
4.6 Other examples and generalizations ... 47
5 Black strings in AdS_5 ... 51
5.1 Black string solutions in $AdS_5 \times S^5$ ... 51
5.2 The topologically twisted index on $T^2 \times \Sigma_g$ ... 54
5.3 Back to Cardy ... 56
5.4 Cardy formula and black holes ... 58
6 The superconformal index ... 59
7 Electrically charged rotating black holes ... 60
7.1 The entropy functional for electrically charged rotating black holes ... 61
7.1.1 The entropy functional for Kerr–Newman black holes in $AdS_5 \times S^5$ ... 61
7.1.2 The entropy functional for Kerr–Newman black holes in diverse dimensions ... 63
7.2 Results on the quantum field theory side ... 65
8 Conclusions and comments ... 68
References ... 70

1 Introduction

We know since the seventies that black holes in general relativity are thermodynamic objects, in particular they have a temperature and an entropy (Bekenstein 1972; Bardeen et al. 1973; Hawking 1975). One of the most intriguing and celebrated relation in theoretical physics is the *Bekenstein–Hawking formula* expressing the entropy of a black hole in terms of the area A of the event horizon and the natural constants $c, \hbar, k_B, G_N$

$$S_{BH} = k_B \frac{c^3 A}{4\hbar G_N}, \tag{1.1}$$

merging in a single expression gravity, relativity, statistical mechanics and quantum theory. Using a standard statistical mechanics interpretation, we are led to write the entropy as

$$S_{BH} = k_B \log n, \tag{1.2}$$

in terms of the number n of microscopic degrees of freedom of the system, the microstates of the black hole. It is a main challenge for all theories of quantum gravity to give an explanation of the Bekenstein–Hawking formula and to identify the corresponding microstates.

The Bekenstein–Hawking formula suggests that the microstates are localized on the event horizon. This is an instance of the *holographic principle* ('t Hooft 1993; Susskind 1995), which states that a volume of space in quantum gravity can be described just in terms of boundary degrees of freedom. A concrete incarnation of this general principle is the AdS/CFT correspondence (Maldacena 1999), a cornerstone of modern theoretical physics. The correspondence explicitly identifies a theory of quantum gravity in Anti-de-Sitter space-time (AdS) with a dual conformal quantum field theory (CFT), which we may naively think of as living on the boundary of AdS. String theory provides many explicit examples of AdS backgrounds and dual CFTs. The most famous is the original example of dual pairs, type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the maximally supersymmetric gauge theory in four dimensions, $\mathcal{N} = 4$ super-Yang–Mills (SYM). Impressive checks of the correctness of this duality have been made over the last twenty years.

In this context, one of the great successes of string theory is the microscopic explanation of the entropy of certain asymptotically flat black holes. The first result was obtained in Strominger and Vafa (1996), more than twenty years ago, and has been followed by an immense literature, which would be too long to refer to. However, quite curiously, no similar results exist for asymptotically AdS black holes in dimension four or greater until very recently. Since holography suggests that the microstates of the black hole correspond to states in a dual conformal field theory, the AdS/CFT correspondence is the natural setting where to explain the black hole entropy in terms of a microscopical theory. In the past, various attempts have been made to derive the entropy of a class of rotating black holes in $\mathrm{AdS}_5 \times S^5$ in terms of states of the dual $\mathcal{N} = 4$ SYM theory in the large N limit, but none was completely successful. The more recent advent of localization techniques for supersymmetric quantum field theories, in the spirit of Nekrasov (2003b) and Pestun (2012), opens a new perspective on this problem. In this review we discuss how to use localization to derive the entropy for a class of supersymmetric black holes in AdS_4 and AdS_5 and discuss the current status for other black holes appearing in holography. We will work in dimension equal or greater than four. AdS_3 is somehow special, and well-studied in the literature, and it will not be discussed in these notes.

1.1 Content of the review

The first microscopic counting for AdS black holes in dimension equal or greater than four was performed in Benini et al. (2016b), considering asymptotically AdS_4 static supersymmetric black holes. One of the main characteristics of this class of black holes is the presence of magnetic charges that correspond to a topological twist in the dual field theory. The black holes considered in Benini et al. (2016b) can be embedded in M theory and are asymptotic to $\mathrm{AdS}_4 \times S^7$. They are dual to a topologically twisted compactification of the ABJM theory in three dimensions (Aharony et al. 2008), and their entropy scales as $N^{3/2}$ at large N, as familiar from three-dimensional holography. In this review, we will focus on this example as a prototype for many similar computations. The entropy can be extracted from the (regularized) Witten index of the quantum mechanics obtained by compactifying ABJM on a Riemann surface $\Sigma_{\mathfrak{g}}$. Holographically, the quantum mechanics describes the physics of the near horizon geometry $\mathrm{AdS}_2 \times \Sigma_{\mathfrak{g}}$ of the black holes. The index can be computed, using localization, as the three-dimensional supersymmetric partition function of ABJM on $\Sigma_{\mathfrak{g}} \times S^1$, topologically twisted along the Riemann surface $\Sigma_{\mathfrak{g}}$. From this perspective, this computation can be generalized to more general domain wall solutions interpolating between AdS_{d+n} and $\mathrm{AdS}_d \times \mathcal{M}_n$, with a topological twist along the n-dimensional compact manifold $\mathcal{M}_n$, thus providing general tests of holography.

In the second part of this review, we also discuss the analogous problem for supersymmetric rotating electrically charged black holes in AdS_d. The main difference with the previous case is the absence of a topological twist. The entropy for such black holes should be obtained by counting states with given electric charge and spin in the dual field theory and the natural observable to consider is the superconformal index, which receives contributions precisely from the BPS states of the theory. Recent results in this direction have been obtained starting with the work (Cabo-Bizet et al. 2019a; Choi et al. 2018b; Benini and Milan 2020b) and we will discuss these recent progresses obtained in various overlapping limits but all pointing towards a unified picture.

Many of the relevant field theory computations are performed using localization. This allows to reduce exact path integrals in quantum field theory to matrix models, which can be solved in the large N limit combining standard and more recent techniques. One successful approach for the physics of black holes, that works both in four (Benini et al. 2016b; Hosseini et al. 2017c) and five dimensions (Benini and Milan 2020b), involves writing the matrix model partition function as a sum of Bethe vacua (Nekrasov and Shatashvili 2009b) of an auxiliary theory. Some technical aspects of this approach are discussed in Sect. 3.

Let us stress that many derivations of the entropy for asymptotically flat black holes involve the use of the Cardy formula for the asymptotic growing of states of a two-dimensional CFT. In the localization approach for AdS black holes, we directly count the number of microstates using an index.[1] We will briefly make contact with the original Cardy approach based on a two-dimensional CFT in Sect. 5 where we discuss black strings.

This review assume some familiarity with supersymmetry and the main examples of holographic dualities in various dimensions. We assume that the reader knows that $\mathcal{N} = 4$ SYM in four dimensions is dual to $\mathrm{AdS}_5 \times S^5$, the ABJM theory to $\mathrm{AdS}_4 \times S^7$ and the so-called (2, 0) theory in six dimensions to $\mathrm{AdS}_7 \times S^4$. Some preliminary exposure to localization computation[2] would be also useful although not necessary. We discuss instead in Sect. 4.1 the elements of gauged supergravity that are needed for this review.

[1] Although some results about rotating electrically charged black holes have been obtained in a Cardy limit, which provides a generalization of the Cardy formula to higher dimensions.

[2] We refer to Marino (2011) for a nice introduction and Pestun and Zabzine (2017) for a more comprehensive review.

The review is organized as follows. In Sect. 2 we give a general overview of the various classes of supersymmetric black holes that are relevant for holography, stressing that they fall into two main classes, distinguished by the presence or absence of magnetic charges (or more precisely of a twist). We also discuss how we should compute their entropy using field theory methods and we introduce the concepts of entropy functional and attractor mechanism that prove useful in the comparison between gravity and field theory. In Sects. 3 and 4 we discuss in details the example of dyonic black holes asymptotic to $\mathrm{AdS}_4 \times S^7$ and dual to a twisted compactification of ABJM. In Sect. 3 we discuss the field theory aspects of the story, introducing the topologically twisted index and showing how to evaluate it using localization. In Sect. 4 we perform the large N limit of the resulting matrix model and we compare with gravity. In Sect. 5 we discuss black string solutions interpolating between AdS_5 and $\mathrm{AdS}_3 \times \Sigma_{\mathfrak{g}}$, as a prototype of more general domain walls interpolating between AdS spaces that can be studied with these techniques. In Sects. 6 and 7 we discuss the case of rotating electrically charged black holes in AdS_d. In Sect. 6 we discuss the field theory aspects of the story, introducing the superconformal index. Finally, in Sect. 7 we discuss the state of the art of the comparison between field theory and gravity for such black holes.

2 AdS black holes in $d \geq 4$

In this review we are interested in supersymmetric black holes that can be embedded in string theory or M-theory and are asymptotic to AdS_d vacua with a known field theory dual. There are many such black holes that can be embedded in maximally supersymmetric backgrounds. For example, we can find supersymmetric black holes in $\mathrm{AdS}_5 \times S^5$, $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$, whose dual field theories are well known. Supersymmetric black holes are extremal and have zero temperature. They also satisfy a BPS condition that relates their mass to the other conserved charges. In the limit where gravity is weakly coupled, the entropy of a black hole can be computed with the Bekenstein–Hawking formula

$$S = \frac{A}{4G_{\mathrm{N}}}, \tag{2.1}$$

where A is the area of the horizon and G_{N} is the Newton constant. We set $c = \hbar = k_{\mathrm{B}} = 1$.

We now discuss some general features of these black holes and their holographic interpretation.

2.1 AdS black holes and holography

For the purposes of holography, we can divide the known supersymmetric black holes in dimension $d \geq 4$ into two main classes, distinguished by how supersymmetry is realized and the holographic interpretation. In particular, they are distinguished by existence (or absence) of certain magnetic charges corresponding to a topological twist.

2.1.1 Kerr–Newman black holes and generalizations

The first class of black holes consists of supersymmetric electrically charged rotating black holes (Kerr–Newman). The most famous examples are the type IIB supergravity black holes asymptotic to $\mathrm{AdS}_5 \times S^5$ found in Gutowski and Reall (2004a), Gutowski and Reall (2004b), Chong et al. (2005a, b) and Kunduri et al. (2006). They depend on two angular momenta corresponding to two Cartan isometries of AdS_5

$$(j_1, j_2) \qquad U(1)^2 \subset SO(4) \subset SO(2,4), \tag{2.2}$$

and three electric charges under the Cartan isometries of S^5

$$(q_1, q_2, q_3) \qquad U(1)^3 \subset SO(6), \tag{2.3}$$

parameterizing rotations in the internal space S^5. Supersymmetry actually imposes a relation among the conserved charges, $f(j_1, j_2, q_1, q_2, q_3) = 0$, so that there are only four independent parameters.[3] These black holes preserve two real supercharges out of the original thirty-two of type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. The five-dimensional part of the metric is asymptotic to AdS_5 with $\mathbb{R} \times S^3$ as conformal boundary. As well known, type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM in four dimensions. It is then a natural expectation that the black holes correspond holographically to an ensemble of states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ that preserve the same supersymmetries and have the same electric charges and the same angular momenta. It is also natural to expect that, by counting all the 1/16 BPS states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with electric charges (q_1, q_2, q_3) and spin (j_1, j_2), we should be able to reproduce the entropy of these black holes. We will work under these assumptions. We are interested in macroscopic black holes whose entropy, when expressed in terms of field theory data, scales as $O(N^2)$, where N is the number of colors of the dual field theory.

The situation is analogous in other dimensions (Chong et al. 2005c; Cvetič et al. 2005; Chow 2008, 2010; Hristov et al. 2019a). Consider the maximally supersymmetric backgrounds $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ in M-theory. The isometry of $\mathrm{AdS}_4 \times S^7$ is $SO(2,3) \times SO(8)$ and we can find electrically charged rotating black holes depending on one angular momentum j and four electric charges (q_1, q_2, q_3, q_4) with a constraint. They preserve two real supercharges. We expect to reproduce the entropy of such black holes by counting all 1/16 BPS states of the dual field theory on $\mathbb{R} \times S^2$ with the same quantum numbers. As well known, the dual of M-theory on $\mathrm{AdS}_4 \times S^7$ is the three-dimensional ABJM theory at Chern–Simons level $k = 1$ (Aharony et al. 2008). The entropy of these black holes scales as $O(N^{3/2})$. Similarly, since the isometry of $\mathrm{AdS}_7 \times S^4$ is $SO(2,6) \times SO(5)$, there are black holes depending on three angular momenta (j_1, j_2, j_3) and two electric charges

[3] For many different families of BPS black holes, supersymmetry imposes constraints among the charges. The reason why this happens is still unclear. In the case of $\mathrm{AdS}_5 \times S^5$, BPS hairy black holes depending on all the charges have been found in Markeviciute and Santos (2019) and Markeviciute (2019).

(q_1, q_2) with a constraint, again preserving two real supercharges.[4] In this case, the entropy, which scales as $O(N^3)$, should be reproduced by counting states in the $\mathcal{N} = (2,0)$ theory in six dimensions (Witten 1995) on $\mathbb{R} \times S^5$.

Notice that all these supersymmetric black holes rotate. If we turn off the angular momenta j_i, we find singularities.

In principle, although there are not so many examples in the literature, we expect the existence of similar supersymmetric black holes in more general type II or M-theory backgrounds with an AdS_d vacuum, rotating in AdS_d and charged under the isometries of the compactification manifold. The holographic interpretation is similar. For example, for type IIB black holes in $\mathrm{AdS}_5 \times SE_5$ (Klebanov and Witten 1998), where SE_5 is a five-dimensional Sasaki–Einstein manifold, we should try to match the entropy by counting 1/4 BPS states of the dual $\mathcal{N} = 1$ superconformal field theory on $\mathbb{R} \times S^3$.

2.1.2 Magnetically charged black holes with a twist

The second class of black holes are characterized by the existence of certain magnetic charges. We should more properly refer to such black holes as solutions where supersymmetry is realized with a topological twist, as we will see. Although there are examples in higher dimensions,[5] we will focus on four dimensions, where these black holes arise naturally. Indeed we can have both magnetic and electric charges in $d = 4$ and it is natural to consider dyonic black holes. There exists a family of BPS black holes in $\mathrm{AdS}_4 \times S^7$ depending on one angular momentum j_1 in AdS_4 and on electric and magnetic charges

$$(q_1, q_2, q_3, q_4) \qquad\qquad (p^1, p^2, p^3, p^4), \tag{2.4}$$

under the abelian $U(1)^4 \subset SO(8)$ isometries of S^7 (Cacciatori and Klemm 2010; Dall'Agata and Gnecchi 2011; Hristov and Vandoren 2011; Katmadas 2014; Halmagyi 2015; Hristov et al. 2019b). Supersymmetry requires a linear constraint among the magnetic charges p^a and non-linear ones among the conserved charges, so that we have a six-dimensional family of rotating, dyonic black holes. For this class of solutions, we can also turn off rotation and have static supersymmetric black holes.[6]

In general, AdS_4 black holes with magnetic charges are qualitatively different from those with zero magnetic charge, as first noticed in Romans (1992) and

[4] Actually, only black holes with equal charges or equal momenta have been studied. However, we expect a family with at least four independent parameters to exist.

[5] There are exotic d-dimensional solutions with horizon $\mathrm{AdS}_2 \times \mathcal{M}_{d-2}$, where $\mathcal{M}_{d-2}$ is a compact manifold, with non-zero fluxes of the gauge fields on $\mathcal{M}_{d-2}$.

[6] These BPS black holes have been found in $\mathcal{N} = 2$ gauged supergravity in $d = 4$ with vector multiplets and later uplifted to M-theory. The first static example, with a hyperbolic horizon, was found in Sabra (1999) in minimal gauged supergravity. The first static spherically symmetric example was found in Cacciatori and Klemm (2010), further discussed in Dall'Agata and Gnecchi (2011), Hristov and Vandoren (2011) and generalized to the dyonic case in Katmadas (2014) and Halmagyi (2015). The rotating case has been discussed in Hristov et al. (2019b) [for other examples, see also Daniele et al. (2019)].

elaborated in Hristov et al. (2011) and Hristov (2012b). The difference is well explained using holography. Consider the black holes as solutions of an effective four-dimensional theory with a AdS_4 vacuum dual to a boundary conformal field theory. For most of this review, the CFT will be ABJM, but the following arguments apply to black holes in general compactifications and more general CFTs with at least $\mathcal{N} = 2$ supersymmetry. For our purposes, we need the terms of the effective theory describing the dynamics of the metric and of the vector fields A^a_μ corresponding to the abelian isometries of the internal manifold. Their dynamics is described by an Einstein-Maxwell theory

$$\mathcal{L} = \sqrt{g}\Big(R + g_{ab}(\phi_i)F^a_{\mu\nu}F^{\mu\nu b} + \ldots\Big) \tag{2.5}$$

where, in general, the matrix of coupling constants depends on the scalar fields ϕ_i of the theory. According to the rules of holography, gauge fields in the bulk correspond to global symmetries in the boundary CFT. For example, in the case of $AdS_4 \times S^7$ we are interested in the four fields A^a_μ corresponding to the abelian isometries $U(1)^4 \subset SO(8)$ and they couple to the field theory conserved currents $J^{\mu a}$

$$\int d^4x A^a_\mu J^{\mu a} \tag{2.6}$$

associated to the Cartan generators of the $SO(8)$ R-symmetry of ABJM at Chern–Simons level $k = 1$. Focusing for simplicity on the static case, one finds that, near the boundary, the black hole solutions behave as

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2} + r^2 ds^2_{\mathcal{M}_3} + \ldots, \\ A^a_\mu(x,r) &= \hat{A}^a_\mu(x) + \ldots, \end{aligned} \tag{2.7}$$

where r is some large radial coordinate, the ellipsis refers to terms suppressed by inverse powers of r, and x are coordinates on the boundary manifold. For spherically symmetric black holes the boundary manifold is $\mathcal{M}_3 = \mathbb{R} \times S^2$. However we can have more exotic solutions with horizon $AdS_2 \times \Sigma_g$, where Σ_g is a Riemann surface of genus g, and, in this case, $\mathcal{M}_3 = \mathbb{R} \times \Sigma_g$. Holography tells us that we should interpret (2.7) as the dual of our CFT defined on the curved manifold $\mathcal{M}_3$. What can we say about $\hat{A}^a_\mu$? Recall again the basic rules of the AdS/CFT correspondence (Klebanov and Witten 1999). Any field $\phi(x,r)$ in AdS is associated with an operator $O(x)$ in the dual CFT. If we have an expansion

$$\phi(x,r) = r^{\alpha_1}\phi_0(x) + r^{\alpha_2}\phi_1(x), \tag{2.8}$$

of the solution of the second order equations of motion for ϕ,[7] we interpret the non-normalizable piece, $\phi_0(x)$, as a deformation of the original CFT with the corresponding operator O,

[7] The values of α_i are related to the conformal dimension of O.

$$\mathcal{L}_{CFT}(x) \to \mathcal{L}_{CFT}(x) + \phi_0(x)O(x), \tag{2.9}$$

while we interpret the normalizable one, $\phi_1(x)$, as a vacuum expectation value (vev) for O, $\langle O \rangle \neq 0$. More precisely, if $\phi_0(x) \neq 0$, we are deforming the CFT with O; if $\phi_0(x) = 0$ and $\phi_1(x) \neq 0$ we have a state of the CFT with non zero vev for O. There are situations where both modes $\phi_0(x)$ and $\phi_1(x)$ are normalizable (or better have finite energy). In this case there are different possible quantizations of the same theory and we have to choose who plays the role of ϕ_0. Massless vector fields in AdS_4 allow for different types of quantizations, related to electric/magnetic duality in the bulk, and this leads to interesting applications, but this is not strictly the most important point. What is important is that, in the expansion (2.7) for A^a_μ both leading and sub-leading terms are turned on. The field A^a_μ has a leading contribution for $r \gg 1$ that approaches a constant value on the boundary $\mathcal{M}_3 = \mathbb{R} \times \Sigma_{\mathfrak{g}}$, corresponding to the magnetic charge of the black hole

$$\frac{1}{2\pi} \int_{\Sigma_{\mathfrak{g}}} F^a = p^a, \tag{2.10}$$

and sub-leading terms [the ellipsis in (2.7)] that encode information about the electric charges. This means that a dyonic black hole is holographically dual to a deformation of the dual CFT. In the natural quantization of the theory, the non-zero value of A^a_μ at the boundary corresponds to the deformation

$$\mathcal{L}_{CFT}(x) \to \mathcal{L}_{CFT}(x) + \hat{A}^a_\mu(x) J^{\mu a}(x) \,. \tag{2.11}$$

This deformation in field theory is equivalent to turning on a background gauge field for a global symmetry. For example, on S^2 we would turn on a background which is just the familiar Dirac monopole $\hat{A}^a_\mu = -\frac{1}{2} p^a \cos\theta d\phi$. On a torus T^2 we would just turn on a background constant magnetic field. Fields satisfying (2.10) can be also written explicitly for all $\Sigma_{\mathfrak{g}}$ but their expression is not particular illuminating. We will see in Sect. 3 that the deformation (2.11) is compatible with supersymmetry.

To understand better what is going on, it is useful to have a look at how supersymmetry is preserved in the presence of a generic assignment of magnetic charges. We will be very schematic here just to convey the general idea. Consider the case where our effective theory is a certain $\mathcal{N} = 2$ gauged supergravity in four dimensions, corresponding to a $\mathcal{N} = 2$ three-dimensional CFT. The effective theory contains the graviphoton field, A^R_μ, holographically dual to the $U(1)$ R-symmetry of the theory. In general, A^R_μ is a linear combination of the vector fields A^a_μ corresponding to the isometries of the internal manifold. For the solution to be supersymmetric, all fermion variations in the black hole background must be zero. The gravitino variation in $\mathcal{N} = 2$ gauged supergravity is schematically given by

$$\delta\psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon - i A^R_\mu \epsilon + \dots . \tag{2.12}$$

The magnetically charged static black holes of interest in this section satisfy the BPS condition $\delta\psi_\mu = 0$ by *cancelling the spin connection with a background field for the R-symmetry*. More precisely, we can regard the spin connection ω_μ along $\Sigma_\mathfrak{g}$ as a $U(1)$ gauge field. An explicit computation shows that ω_μ is just a monopole of charge $2 - 2\mathfrak{g}$, as the familiar relation $\frac{1}{2\pi}\int_{\Sigma_\mathfrak{g}} R = 2 - 2\mathfrak{g}$, with $R = d\omega$, clearly shows. Since A^R_μ is a linear combination of the A^a_μ, it is also a monopole, with charge given by a linear combination of the p^a. By appropriately choosing this linear combination[8] and the spinor ϵ, we can cancel the second and the third term on the right hand side in (2.12). We will come back to more precise expressions in Sect. 3. For the dyonic static black holes, restricting the index μ to lie along Σ_g, one discovers that the ellipsis cancels independently and we are left with the equation

$$\delta\psi_\mu = \partial_\mu \epsilon = 0. \tag{2.13}$$

This equation is solved by taking ϵ constant along $\Sigma_\mathfrak{g}$. One also finds that the other components of the supersymmetry variations imply that ϵ is time-independent but, in general, has a non-trivial profile in r.

This discussion can be also applied to the dual CFT. By restricting the variations to the boundary, we see that the field theory on $\mathbb{R} \times \Sigma_\mathfrak{g}$ is invariant under supersymmetry transformations with a constant spinor. The very same mechanism is at work on the boundary: we are turning on a magnetic background for the R-symmetry that compensates the spin connection. In quantum field theory, this construction is well-known (Witten 1988). It is called *topological twist*, as we will discuss in details in Sect. 3. The conclusion is that the dual CFT is deformed by the presence of magnetic fluxes for all the global and R-symmetries, and, in particular, it is topologically twisted by the magnetic flux for the R-symmetry. Notice that our argument was based on $\mathcal{N} = 2$ supersymmetry with a $U(1)$ R-symmetry. Theories can have a larger R-symmetry group, like ABJM, or many flavor symmetries. In these cases, the choice of a $U(1)$ R-symmetry is not unique. Each choice corresponds to a different twist. We can indeed think of the magnetic charges p^a as parameterizing a family of inequivalent twists. It is important to remember, however, that a linear combination of the p^a is fixed by the condition that the background for the selected $U(1)$ R-symmetry cancels the spin connection. There are only $n_V - 1$ independent magnetic charges, where n_V is the number of massless vectors. This number is $n_V = 4$ for ABJM.

The interpretation of the general rotating dyonic black hole is more complicated but similar in spirit. We can have rotation only in the spherically symmetric case, where j is the spin along S^2. Rotations in the bulk correspond to turning on an Omega-background (Nekrasov and Okounkov 2006) in the boundary theory on S^2. The theory is still topologically twisted.

All this should be contrasted with the black holes discussed in Sect. 2.1.1 where there is no cancellation between the spin connection and the R-symmetry. It can be expressed more formally in a difference between the supersymmetry algebra, as discussed in Hristov et al. (2011) and Hristov (2012b). The real discriminant among

[8] This is the linear constraint on the magnetic charges of the black hole that we mentioned before.

the two class of black holes is the topological twist, or equivalently a magnetic charge associated with the R-symmetry. There exist black holes with non-zero magnetic charges for the flavor symmetries only.[9] They correspond to a CFT in a magnetic background but with no topological twist. From the point of view of micro-state counting using holography, they are more similar in spirit to the black holes discussed in Sect. 2.1.1.

Now it is clear what we should do in order to compute the entropy of magnetically charged black holes (with a twist) using field theory: enumerate all the states with electric charges q_i and angular momentum j and the right amount of supersymmetry in the twisted CFT on $\mathbb{R} \times \Sigma_{\mathrm{g}}$. The theory is topologically twisted[10] by the magnetic background for a $U(1)$ R-symmetry and possibly deformed by magnetic fluxes for all other global symmetries.

2.2 Computing the entropy

It is reasonable to expect that we can recover the entropy of the two classes of AdS$_d$ black holes by enumerating supersymmetric states in the dual field theory on $\mathbb{R} \times \mathcal{M}_{d-2}$. For all our examples, the preserved supersymmetry $\mathcal{Q}$ satisfies an algebra of the form

$$\{\mathcal{Q}^\dagger, \mathcal{Q}\} = H - \mu_a Q_a - \nu_i J_i, \tag{2.14}$$

where H is the Hamiltonian and Q_a and J_i are the charge operators associated with the global symmetries and the angular momenta, respectively, with certain constants μ_a and ν_i that depend on the model. The explicit form of the algebra is different for different types of black holes and it will be discussed in details in the rest of this review. For the moment, we notice that the R-symmetry charge enters in the supersymmetry algebra for Kerr–Newmann black holes but not for topologically twisted ones. Supersymmetric states are annihilated by $\mathcal{Q}$ and their energy is determined by the BPS condition[11]

$$E = \mu_a q_a + \nu_i j_i\,. \tag{2.15}$$

2.2.1 The grand canonical partition function

Enumerating BPS states is equivalent to knowing the grand canonical partition function

$$\mathcal{Z}(\Delta_a, \omega_i) = \operatorname{Tr}\Big|_{\mathcal{Q}=0} e^{i(\Delta_a Q_a + \omega_i J_i)} = \sum_{q_a, j_i} c(q_a, j_i) e^{i(\Delta_a q_a + \omega_i j_i)}, \tag{2.16}$$

where the trace is taken over the Hilbert space of states on $\mathcal{M}_{d-2}$ that preserves the same amount of supersymmetry of the black hole, and Δ_a and ω_i chemical potentials conjugated to Q_a and J_i, respectively. In practical applications, $\mathcal{Z}$ is a function

[9] See Hristov et al. (2019a) for example of rotating dyonic black holes with flavor magnetic charges.

[10] And also Omega-deformed if there is rotation.

[11] It follows from the algebra that all states in the theory satisfy the BPS bound $E \geq \mu_a q_a + \nu_i j_i$.

of complex chemical potentials and converges in an appropriate domain of the complex plane for the fugacities $y_a = e^{i\Delta_a}$, $\zeta_i = e^{i\omega_i}$. In the previous formula, $c(q_a, j_i)$ is the number of supersymmetric states of electric charge q_a and angular momentum j_i. Electric and magnetic charges enter in an asymmetric way in this construction. The magnetic charges p^a enter explicitly as a set of couplings in the Lagrangian of the deformed CFT, while the electric charges q_a are introduced through chemical potentials. In particular, for magnetically charged black holes with a twist, the trace must be taken in the topologically twisted theory.

The grand canonical partition function (2.16) should also enumerate the BPS states in the dual gravity theory. Our working assumption is that, for large charges (scaling with appropriate powers of N) the supersymmetric density of states is dominated by the macroscopic black holes discussed before. Under this assumption, by the very definition of entropy, the entropy of the black hole is given by

$$S(q_a, j_i) = \log c(q_a, j_i), \tag{2.17}$$

where the dependence on the magnetic charges p^a, if present, is hidden in the form of the function c. If $\mathcal{Z}(\Delta_a, \omega_i)$ is known, the entropy can be extracted as a Fourier coefficient

$$e^{S(q_a, j_i)} = c(q_a, j_i) = \int \frac{d\Delta_a}{2\pi} \frac{d\omega_i}{2\pi} \mathcal{Z}(\Delta_a, \omega_i) e^{-i(\Delta_a q_a + \omega_i j_i)}, \tag{2.18}$$

with an appropriate integration contour. In the limit of large charges, this can be evaluated by a saddle point approximation

$$S(q_a, j_i) = \log \mathcal{Z}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i)\Big|_{\bar{\Delta}_a, \bar{\omega}_i}, \tag{2.19}$$

where $\bar{\Delta}_a$ and $\bar{\omega}_i$ are obtained by extremizing the functional

$$\mathcal{I}(\Delta_a, \omega_i) = \log \mathcal{Z}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i), \tag{2.20}$$

with respect to Δ_a and ω_i,

$$\partial_{\Delta_a} \mathcal{I}(\Delta_a, \omega_i) = \partial_{\omega_i} \mathcal{I}(\Delta_a, \omega_i) = 0 \Big|_{\bar{\Delta}_a, \bar{\omega}_i}. \tag{2.21}$$

We see that the entropy is just the Legendre transform of the logarithm of the partition function.

To understand better this point, recall that we are interested in extremal supersymmetric black holes. In particular, they have zero temperature. The standard thermodynamics relation for the grand canonical partition function of a system with temperature T,

$$\log \mathcal{Z} = -(E - TS - i\tilde{\Delta}_a q_a - i\tilde{\omega}_i j_i)/T, \tag{2.22}$$

looks singular in the zero-temperature limit. However, supersymmetric states satisfy the BPS condition (2.15), namely $E = \mu_a q_a + \nu_i j_i$. When we take the zero temperature limit, we need also to scale $\tilde{\Delta}_a(T) = -i\mu_a + \Delta_a T$ and $\omega_i(T) = -i\nu_i + \omega_i T$. In this way we obtain the Legendre transform $S = \log \mathcal{Z} - i\Delta_a q_a - i\omega_i j_i$.[12]

The entropy is also obtained via a Legendre transform in many other approaches, as the OSV conjecture (Ooguri et al. 2004) and the Sen's quantum entropy functional (Sen 2005, 2009b, a) for asymptotically flat black holes.

2.2.2 The supersymmetric index

So far everything was simple. The problem is that $\mathcal{Z}(\Delta_a, \omega_i)$ is too hard to compute, in general. For electrically charged rotating black holes in $\mathrm{AdS}_5 \times S^5$, computing $\mathcal{Z}(\Delta_a, \omega_i)$ would correspond to enumerate all the 1/16 BPS states of $\mathcal{N}=4$ SYM. For comparison, in a four-dimensional theory with $\mathcal{N}=1$ supersymmetry, this would correspond to count all the 1/4 BPS states. While almost everything is known about the counting of 1/2 BPS states in an $\mathcal{N}=1$ theory (Kinney et al. 2007; Benvenuti et al. 2007), the analogous problem for 1/4 BPS states is still open.

What we can instead compute is a supersymmetric index

$$\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) = \operatorname{Tr}\Big|_{\mathcal{Q}=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)}, \tag{2.23}$$

with the insertion of the fermionic number $(-1)^F$. Standard arguments tell us that we can also write

$$\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) = \operatorname{Tr}(-1)^F e^{-\beta\{\mathcal{Q}^\dagger, \mathcal{Q}\}} e^{i(\Delta_a Q_a + \omega_i J_i)} = Z^{\text{susy}}_{S^1 \times \mathcal{M}_{d-1}}(\Delta_a, \omega_i), \tag{2.24}$$

if Q_a and J_i commute with $\mathcal{Q}$. In the first step of the previous identification we used the fact that states with $\mathcal{Q} \neq 0$ do not contribute to the trace, since bosonic and fermionic states are paired and contribute with opposite sign.[13] In particular, the index is independent of β. In the second step, we identified the trace at temperature $1/\beta$ with the Euclidean path integral $Z^{\text{susy}}_{S^1 \times \mathcal{M}_{d-1}}$ of the theory compactified on a circle of radius β.[14] The latter partition function can be computed using localization techniques in quantum field theory as we discuss in Sect. 3.

In general, $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) \neq \mathcal{Z}(\Delta_a, \omega_i)$. First of all, the index can accommodate fugacities only for the conserved charges that commute with $\mathcal{Q}$ and, in general,

[12] For explicit examples of this zero-temperature limit from the gravitational point of view see Silva (2006), and, in particular, for AdS_5 black holes, see Cabo-Bizet et al. (2019a) and Choi et al. (2018b).

[13] This is the logic of the Witten index (Witten 1982a). For every state $|\Omega\rangle$ not annihilated by $\mathcal{Q}$ there is a state $\mathcal{Q}|\Omega\rangle$ of opposite statistics and the same value of $\{\mathcal{Q}^\dagger, \mathcal{Q}\}$. Since these states have opposite value of $(-1)^F$, their contribution cancels in the trace in (2.24). Therefore, only supersymmetric states, annihilated by $\mathcal{Q}$, contribute to the Witten index. See also the discussion in Sect. 3.3.1.

[14] It is a standard textbook result that the finite temperature partition function $\operatorname{Tr} e^{-\beta H}$ can be expressed as an Euclidean partition function with time compactified on a circle of radius β and periodic boundary conditions for bosons and anti-periodic boundary conditions for fermions. In a supersymmetric partition functions, bosons and fermions should have the same boundary conditions and this is enforced by the fermionic number $(-1)^F$. The extra insertion of $e^{\beta(\mu_a \mathcal{R}_a - \nu_i \mathcal{J}_i)}$ just introduces twisted boundary conditions along S^1 for the symmetries associated with $\mathcal{R}_a$ and $\mathcal{J}_i$.

contains less parameters than the BPS partition function. We will see that this is not a major problem for the black holes considered in this paper since they also have a constraint among charges. More importantly, the entropy should count all the BPS ground states of the theory, while the index counts bosonic ground states with a plus and fermionic ground states with a minus. However, it may happen for particular theories that the majority of ground states are of definite statistic. In this case there is no cancellation between bosonic and fermionic ground states and the index correctly reproduces the entropy in a suitable limit. This typically happens for asymptotically flat black holes in the limit of large charges, and we may hope that the same is true for asymptotically AdS black holes in the large N limit. In the case of certain asymptotically flat spherically symmetric black holes, there is an extra symmetry that implies $(-1)^F = 1$ on the relevant set of states (Sen 2009a) and one can prove that $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) = \mathcal{Z}(\Delta_a, \omega_i)$. A similar argument for asymptotically AdS black holes is more subtle and it is discussed in Sect. 2.4. More generally, we can always rely on an explicit computation, and, as we will see in the rest of this review:

- for magnetically charged black holes in AdS_4, the dual field theory is topologically twisted. $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i)$ is the so-called topologically twisted index that we define and study in Sect. 3. There is no cancellation between bosonic and fermionic ground states and the index correctly reproduces the entropy at large N, as we will see in Sect. 4;
- for electrically charged rotating black holes, we are just counting states of the CFT on $\mathbb{R} \times S^{d-1}$. $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i)$ is the superconformal index, whose properties we discuss in Sect. 6. The superconformal index is known to have large cancellations between bosons and fermions for real values of the fugacities (Kinney et al. 2007), and this would suggest that we really need to compute the original BPS partition function $\mathcal{Z}(\Delta_a, \omega_i)$. However, it has been recently realized that introducing phases for the fugacities can obstruct the cancellation between bosonic and fermionic states and that the entropy of KN black holes is indeed correctly captured by the index for complex values of the chemical potentials. We will discuss these results in Sect. 7.

Notice also that the entropy and the index of asymptotically flat black holes coincide at leading order in the charges but are in general different when corrections are included (Dabholkar et al. 2011b). We might expect the same for asymptotically AdS black holes.

2.3 Entropy functionals and the attractor mechanism

In the limit where gravity is weakly coupled, we can compute the entropy of a black hole from the area of the horizon using the Bekenstein–Hawking formula (2.1). The area can be extracted from the explicit solution of the relevant gauged supergravity, which typically contains many scalars $X^I(r)$ varying with the radial distance. In principle, the area may depend on many parameters including asymptotic moduli. However, the microscopic entropy of our black holes is just a function of the conserved charges q_a and j_i. This is explicitly realized through an *attractor mechanism*: independently of the asymptotic moduli, the scalar fields approach a value at the horizon, $X^I_*(q_a, j_i)$, that is a function of the conserved charges only. Moreover, this mechanism often allows to express the area, and therefore the entropy, in terms of the values of the scalar fields at the horizon with simple algebraic equations. This is the case of the attractor mechanism for $\mathcal{N} = 2$ supergravity discovered in Ferrara and Kallosh (1996) and Ferrara et al. (1995). It is also the idea behind Sen's quantum entropy function (Sen 2005) that allows to find the entropy of black holes with AdS_2 horizon including higher derivative corrections. In all these cases, one can define some sort of entropy functional $S(X^a, \Omega^i, q_a, j_i)$, which is a function of the conserved charges and the *horizon value* X^a, Ω^a of the scalar fields and possibly other modes, and whose extremization with respect to X^a and Ω^i reproduces the entropy. This extremization is expected to be the gravity analog of (2.19).

In order to make the comparison between gravity and field theory more manifest it is convenient to write the entropy functional as a Legendre transform

$$S(X^a, \Omega^i, q_a, j_i) = \mathcal{E}(X^a, \Omega^i) - i(X^a q_a + \Omega^i j_i), \tag{2.25}$$

where X^a and Ω^i are now interpreted as the black hole chemical potentials and $\mathcal{E}(X^a, \Omega^i)$ as the black hole gran-canonical partition function. According to standard arguments (Gibbons and Hawking 1977), $\mathcal{E}(X^a, \Omega^i)$ can be identified with the on-shell action of the Euclidean continuation of the black hole.

This description fully agrees with the field theory picture if we identify X^a and Ω^i with Δ^a and ω_i and $\mathcal{E}(X^a, \Omega^i)$ with $\log \mathcal{Z}(\Delta_a, \omega_i)$. The latter is indeed the field theory gran-canonical partition function and, according to the rule of holography, should be also identified with the on-shell action of the gravity solution corresponding to the chemical potentials Δ^a and ω_i.

The attractor mechanism has played an important role in the interpretation of the field theory result leading to the black hole entropy. For example, the attractor mechanism in N = 2 gauged supergravity (Cacciatori and Klemm 2010; Dall'Agata and Gnecchi 2011) for static dyonic black holes in $AdS_4 \times S^7$ predicts

$$S(X^a) = -\frac{2\sqrt{2}N^{\frac{3}{2}}}{3}\sum_{a=1}^{4} \mathfrak{p}_a \frac{\partial}{\partial X^a}\sqrt{X^1X^2X^3X^4} - i\sum_{a=1} X^a q_a \,, \tag{2.26}$$

with the constraint $\sum_{a=1}^{a} X^a = 2\pi$ and this result perfectly matches with the field theory computation based on the topologically twisted index (Benini et al. 2016b) as we will discuss in Sect. 4 [see formula (4.48)]. For other black holes the attractor mechanism is not known in supergravity, and the entropy functional has been written using combined field theory and gravity intuition. This approach was successfully used in Hosseini et al. (2017b) to write an entropy functional for KN black holes in $AdS_5 \times S^5$,

$$S(X^a, \Omega^i) = -i\frac{N^2}{2}\frac{X^1X^2X^3}{\Omega^1\Omega^2} - i\left(\sum_{a=1}^{3} X^a q_a + \sum_{i=1}^{2} \Omega^i j_i\right), \tag{2.27}$$

with the constraint $\sum_{a=1}^{3} X^a - \sum_{i=1}^{2} \Omega^i = 2\pi$. This result has been instrumental in the later developments and has been matched with field theory computations based on the superconformal index (Cabo-Bizet et al. 2019a; Choi et al. 2018b; Benini and Milan 2020b), as we will discuss in Sect. 7. Entropy functional for other electrically charged and rotating black holes in diverse dimensions has been later found in Hosseini et al. (2018b), Choi et al. (2020) and, in some cases, successfully compared to quantum field theory expectations. These entropy functionals can be also obtained by computing the zero-temperature limit of the on-shell action of a class of supersymmetric but non-extremal Euclidean black holes (Cabo-Bizet et al. 2019a; Cassani and Papini 2019).

A general field theory inspired formula for an entropy functional that generalizes (2.26) and (2.27) and covers all existing black hole solutions in four and five dimensions has been discussed in Hosseini et al. (2019a).

2.4 $\mathcal{I}$-extremization

We can provide a perhaps more speculative but intriguing explanation of the extremization principle (2.19) based on the renormalization group flow. All the black holes we will consider have a (possibly warped) AdS_2 factor in the near-horizon region. This suggests the existence of a superconformal quantum mechanics describing the near-horizon degrees of freedom and whose ground states correspond to the black holes microstates. The superconformal algebra associated with AdS_2 is su(1, 1|1) and contains the bosonic factor $SL(2,\mathbb{R}) \times U(1)_R$, where $U(1)_R$ is the R-symmetry. From the field theory point of view, the quantum mechanics arises from the reduction of the dual CFT on $\mathcal{M}_{d-2}$. We can see the black hole as a solution interpolating between AdS_d and AdS_2 with the dual interpretation of a renormalization group flow across dimensions where a $d-1$-dimensional CFT compatified on $\mathcal{M}_{d-2}$ flows to a infrared superconformal quantum mechanics. The index $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i)$ is invariant under renormalization group flow and can be interpreted as the Witten index of the infrared quantum mechanics.

We can write the index as

$$\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) = \text{Tr}\Big|_{\mathcal{Q}=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)} = \text{Tr}\Big|_{\mathcal{Q}=0} e^{\pi i R(\Delta,\omega)} e^{-\mathbb{I}m(\Delta_a Q_a + \omega_i J_i)}, \tag{2.28}$$

where

$$R(\Delta, \omega) = F + \frac{\mathbb{R}e\Delta_a}{\pi} Q_a + \frac{\mathbb{R}e\omega_i}{\pi} J_i\,. \tag{2.29}$$

In the examples that we will discuss the fermion number F acts precisely as a particular R-symmetry, $F \equiv R_0$. In a general supersymmetric theory with global symmetries, the R-symmetry is not unique. If R_0 is a R-symmetry and Q is a global symmetry, also $R_0 + Q$ is a R-symmetry. We see that $R(\Delta, \omega)$ is a R-symmetry of our quantum mechanics, and given the symmetries of the problem, we expect it to be the most general R-symmetry we can write. However, when a supersymmetric theory is also conformal there exists the notion of the *exact* R-symmetry, which is the one singled out by the superconformal algebra. The problem of finding the exact R-symmetry is central in supersymmetric quantum field theory and it is usually associated with an extremization problem. We have indeed c-extremization in two dimensions (Benini and Bobev 2013a), F-maximization in three dimensions (Jafferis 2012) and a-maximization in four dimensions (Intriligator and Wecht 2003). It is tempting to propose that the extremization (2.19) selects among all $R(\Delta, \omega)$ precisely the exact R-symmetry of our quantum mechanics. This principle has been called $\mathcal{I}$-*extremization* in Benini et al. (2016b, 2017). It is suggested by the fact that, in odd dimensions, the exact R-symmetry is obtained by extremizing the supersymmetric sphere partition function, which, in the case of one dimension, is just the Witten index. It would be interesting to prove or disprove this principle for generic holographic theories.

This interpretation would explain why there is no cancellation in the index between vacua of different statistic for large charges. Adapting an argument in Sen (2009a), we might expect that the microstates are invariant under the superconformal algebra su(1, 1|1) and this implies that they have zero exact R-charge. This is certainly true if the quantum mechanics consists of a set of degenerate ground states with an energy gap to the first excited state. Under this assumption and using also large N factorization of the correlation functions, the trace (2.28) becomes

$$\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) = e^{-\mathbb{I}m(\Delta_a \langle Q_a \rangle + \omega_i \langle J_i \rangle)} \operatorname{Tr} \Big|_{Q=0} 1 = e^{-\mathbb{I}m(\Delta_a q_a + \omega_i j_i)} e^{S(q_a, j_i)}, \tag{2.30}$$

where we used that, for the values of Δ_a and ω_i that select the exact R-charge, all states have $R(\Delta, \omega) = 0$. We then see that the cancelation due to $(-1)^F$ is balanced by the non-zero phases of the fugacities at the saddle point and all the ground states contribute with the same sign. The previous equation is consistent with (2.19). Indeed, by taking the logarithm of (2.30), we find

$$S(q_a, j_i) = \mathbb{R}e(\log \mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i)) \,. \tag{2.31}$$

In all our example, the extremum of (2.19) is actually real and the real part in the previous formula is superfluous.[15]

Let us also notice that c-extremization and F- and a-maximization have a well-known gravitational dual (Martelli et al. 2006; Couzens et al. 2019; Gauntlett et al. 2019a) which allows to determine the exact R-symmetry in terms of geometrical data of the supergravity background. A gravitational dual for I-extremization has been also proposed in Couzens et al. (2019) and further discussed in Gauntlett et al. (2019b), Hosseini and Zaffaroni (2019a) and Kim and Kim (2019).

[15] That the extremum of (2.19) is real is usually part of the attractor mechanism. As we will see, the fact that the extremization of (2.19) leads to a real number is equivalent to the non-linear constraint among charges imposed by supersymmetry.

3 The topologically twisted index

As we discussed in Sect. 2, magnetically charged black holes in AdS_4 are dual to topologically twisted CFT_3. In this section, we discuss the *topologically twisted index* in three dimensions, $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i)$, defined as the supersymmetric partition function $Z^{\text{susy}}_{\Sigma_g \times S^1}(\Delta_a, \omega_i)$ with a topological A-twist along Σ_g. We will discuss the case of a generic $\mathcal{N} = 2$ Yang–Mills–Chern–Simons theory in three dimensions with an R-symmetry and we will specialize to the ABJM theory in Sect. 4.

The index can be computed in many different ways. We will discuss the localization approach here, following Benini and Zaffaroni (2015, 2017). The index has been first derived by topological field theory arguments in various examples in Okuda and Yoshida (2012, 2014) and discussed in general in Nekrasov and Shatashvili (2015). In this second approach, further discussed and generalized in Gukov and Pei (2017), Okuda and Yoshida (2015), Closset and Kim (2016), Gukov et al. (2017), Closset et al. (2017a, b, 2018), the index is written as a sum of contributions coming from the Bethe vacua, the critical points of the twisted superpotential of the two-dimensional theory obtained by compactifying on S^1 (Nekrasov and Shatashvili 2009b, a). We will discuss the connections between the two approaches in Sect. 3.3.2. The reader that is not interested in quantum field theory properties of the index can just have a quick look at Sect. 3.1, the first part of Sects. 3.2 and 3.3.2.

3.1 The topological twist

Consider an $\mathcal{N} = 2$ quantum field theory in three dimensions. The $\mathcal{N} = 2$ supersymmetry multiplets are:

- the vector multiplet, $(A_\mu, \lambda, \sigma, D)$, where λ is a Dirac spinor and σ and D are real scalars. D is an auxiliary field;
- the chiral multiplet, (ϕ, ψ, F), where ψ is a Dirac spinor and ϕ and F are complex scalars. F is an auxiliary field.

These multiplets can be obtained by dimensional reduction from the corresponding $\mathcal{N} = 1$ multiplets in four dimensions. We assume that the theory has an R-symmetry[16]

$$\lambda \to e^{-i\alpha}\lambda, \qquad (\phi, \psi, F) \to (e^{ir_\phi \alpha}\phi, e^{i(r_\phi - 1)\alpha}\psi, e^{i(r_\phi - 2)\alpha}F), \tag{3.1}$$

with charges r_ϕ for the chiral fields. The charges should be integrally quantized, as we will discuss in the following. We want to define a supersymmetric theory on $\Sigma_g \times S^1$ using a non-trivial background for the R-symmetry. Let us start, for simplicity, with the case of $S^2 \times S^1$ with metric

[16] The sign of the charges is somehow unconventional (for example, λ and ϵ have charge -1) but we keep it for consistency with Benini and Zaffaroni (2015, 2017).

$$ds^2 = R^2(d\theta^2 + \sin\theta^2 d\phi^2) + \beta^2 dt^2, \tag{3.2}$$

and a non-trivial R-symmetry background field A_μ^R.

To see if supersymmetry is preserved we can use the approach of Festuccia and Seiberg (2011). We promote the metric $g_{\mu\nu}$ and the R-symmetry background field A_μ^R to dynamical fields by coupling the theory to supergravity, using an appropriate off-shell formulation. The theory coupled to gravity is invariant under supersymmetry transformations with a local spinorial parameter $\epsilon(x)$. We can recover the rigid theory by freezing the supergravity multiplet to a background value. This can be done, for example, by sending the Planck mass to infinity. In this process we set all supergravity fermionic fields to zero, while keeping a non-trivial background for the metric and A_μ^R, and possibly some auxiliary fields. The rigid theory so obtained is no more invariant under local supersymmetries. However, it is still invariant under those transformations that preserve the background fields. The supersymmetry variation of the bosonic supergravity fields is automatically zero since it is proportional to the supergravity fermions that vanish in the background. On the other hand, the vanishing of the fermionic variations gives a differential equation for $\epsilon(x)$. The solutions to this equation determine the rigid supersymmetries that are preserved by the curved background.[17]

In any supergravity with a R-symmetry gauge field, when all the other supergravity fields are set to zero, the fermionic variations have the universal form

$$\delta\psi_\mu = D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon + iA_\mu^R\epsilon = 0\,. \tag{3.3}$$

In three dimensions, we can choose $\gamma_a = \sigma_a$, where σ_a are the Pauli matrices. The non-trivial components of the spin connection are easily computed[18] to be $\omega^{12} = -\cos\theta d\phi$. If we take $\gamma_3\epsilon = \epsilon$, so that $\gamma_{12}\epsilon = i\epsilon$, we see that the background field

$$A^R = \frac{1}{2}\cos\theta d\phi \tag{3.4}$$

precisely cancels the spin connection. The equation reduces to

$$\delta\psi_\mu = \partial_\mu\epsilon = 0, \tag{3.5}$$

which is solved by a constant spinor ϵ. We thus see that the background (3.4) allows to define a supersymmetric theory on $S^2 \times S^1$. The generalization of the above discussion to $\Sigma_{\mathfrak{g}} \times S^1$ is straightforward: we just turn on a background for the R-symmetry with $A^R = -\omega/2$ and everything else works in the same way.

This way of preserving supersymmetry corresponds to a *topological twist* along $\Sigma_{\mathfrak{g}}$ in the sense of Witten (1988, 1991). To make contact with the language used in

[17] These conditions impose constraints on the manifold and the choice of background fields. For examples related to our context see Festuccia and Seiberg (2011), Klare et al. (2012), Dumitrescu et al. (2012), Closset et al. (2013) and Hristov et al. (2013).

[18] We use the frame $e^1 = Rd\theta, e^2 = R\sin\theta d\phi$ and $e^3 = \beta dt$.

Witten (1988, 1991), we can interpret the background field for A^R as effectively changing the spin of the fields in the theory, thus transforming ϵ into a scalar.

A supersymmetric Lagrangian for a Yang–Mills–Chern–Simons theory with gauge group G and chiral matter transforming in a representation $\mathcal{R}$ on $\Sigma_g \times S^1$ can be written as $\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm mat} + \mathcal{L}_{\rm W}$ with[19]

$$\begin{aligned}
\mathcal{L}_{\rm YM} &= \operatorname{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D^2 - \frac{i}{2}\lambda^\dagger\gamma^\mu D_\mu\lambda - \frac{i}{2}\lambda^\dagger[\sigma,\lambda]\right] \\
\mathcal{L}_{\rm CS} &= -\frac{ik}{4\pi}\operatorname{Tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \lambda^\dagger\lambda + 2D\sigma\right] \\
\mathcal{L}_{\rm mat} &= D_\mu\phi_i^\dagger D^\mu\phi_i + \phi_i^\dagger\left(\sigma^2 + iD - r_\phi F_{12}^R\right)\phi_i + F_i^\dagger F_i \\
&\quad + i\psi_i^\dagger(\gamma^\mu D_\mu - \sigma)\psi_i - i\psi_i^\dagger\lambda\phi_i + i\phi_i^\dagger\lambda^\dagger\psi_i \\
\mathcal{L}_W &= i\left(\frac{\partial W}{\partial\Phi_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\Phi_i\partial\Phi_j}\psi_j^{c\dagger}\psi_i + \frac{\partial\overline{W}}{\partial\Phi_i^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\Phi_i^\dagger\partial\Phi_j^\dagger}\psi_j^\dagger\psi_i^c\right),
\end{aligned} \tag{3.6}$$

where the superpotential $W(\phi_i)$ is a holomorphic function of R-charge two and the fields A_μ, σ, D act on the matter fields in the appropriate representation. Here the derivative D_μ is covariantized with respect to the spin and gauge connection and also to the R-symmetry background A^R. As usual in Euclidean signature, fields and their conjugate, ϕ and $\phi^\dagger$ for example, should be considered as independent variables. Notice that a vev for the scalar field σ gives mass to the matter fields ϕ_i and ψ_i. This kind of coupling is typical of three dimensions and called *real mass* to distinguish it from the mass terms that can be introduced through the superpotential W. One can check that the Lagrangian is invariant under the following supersymmetry transformations

$$\begin{aligned}
QA_\mu &= \tfrac{i}{2}\lambda^\dagger\gamma_\mu\epsilon & Q\lambda &= +\tfrac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma \\
\widetilde{Q}A_\mu &= \tfrac{i}{2}\tilde{\epsilon}^\dagger\gamma_\mu\lambda & \widetilde{Q}\lambda^\dagger &= -\tfrac{1}{2}\tilde{\epsilon}^\dagger\gamma^{\mu\nu}F_{\mu\nu} + \tilde{\epsilon}^\dagger D + i\tilde{\epsilon}^\dagger\gamma^\mu D_\mu\sigma \\
QD &= -\tfrac{i}{2}D_\mu\lambda^\dagger\gamma^\mu\epsilon + \tfrac{i}{2}[\lambda^\dagger\epsilon,\sigma] & Q\lambda^\dagger &= 0 \qquad Q\sigma = -\tfrac{1}{2}\lambda^\dagger\epsilon \\
\widetilde{Q}D &= \tfrac{i}{2}\tilde{\epsilon}^\dagger\gamma^\mu D_\mu\lambda + \tfrac{i}{2}[\sigma,\tilde{\epsilon}^\dagger\lambda] & \widetilde{Q}\lambda &= 0 \qquad \widetilde{Q}\sigma = -\tfrac{1}{2}\tilde{\epsilon}^\dagger\lambda\,.
\end{aligned}$$

for the vector multiplet fields and

[19] This can be obtained for example by taking the rigid limit of supergravity in the background (3.2), as suggested in Festuccia and Seiberg (2011).

$$\begin{array}{llllll} Q\phi & = 0 & Q\psi & = \left(i\gamma^\mu D_\mu\phi + i\sigma\phi\right)\epsilon & \widetilde{Q}\psi & = \tilde{\epsilon}^c F \\ \widetilde{Q}\phi & = -\tilde{\epsilon}^\dagger\psi & \widetilde{Q}\psi^\dagger & = \tilde{\epsilon}^\dagger\left(-i\gamma^\mu D_\mu\phi^\dagger + i\phi^\dagger\sigma\right) & Q\psi^\dagger & = -\epsilon^{c\dagger}F^\dagger \\ Q\phi^\dagger & = \psi^\dagger\epsilon & QF & = \epsilon^{c\dagger}\left(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi\right) & \widetilde{Q}F & = 0 \\ \widetilde{Q}\phi^\dagger & = 0 & \widetilde{Q}F^\dagger & = \left(-iD_\mu\psi^\dagger\gamma^\mu - i\psi^\dagger\sigma + i\phi^\dagger\lambda^\dagger\right)\tilde{\epsilon}^c & QF^\dagger & = 0\,. \end{array}$$

for the chiral multiplets. To future purposes, we also define $\mathcal{Q} = Q + \widetilde{Q}$.

Notice that the Lagrangian (3.6) and the transformations of supersymmetry are almost identical to the flat space ones with the further covariantization with respect to the metric and the background R-symmetry. This is not always the case in curved space, where extra terms should be included to maintain supersymmetry. In general, a Lagrangian is not invariant under flat space supersymmetry transformations when defined on a curved space because covariant derivatives do not commute anymore. With a topological twist, the spinor ϵ is covariantly constant and the problem is milder.

3.2 The localization formula

The *topologically twisted index* is just the $\Sigma_{\mathfrak{g}} \times S^1$ path integral of the theory discussed in the previous section. We can evaluate it using localization. The basic idea of localization is simple. Let us review it briefly, referring to Pestun and Zabzine (2017) and Marino (2011) for more details.[20] In a theory with a fermionic symmetry squaring to zero (or to a bosonic symmetry of the theory)[21] we can deform the action with a $\mathcal{Q}$-exact term, $\mathcal{Q}V$, where V is a fermionic functional invariant under all the symmetries. The new action is still $\mathcal{Q}$ invariant and the path integral independent of the deformation

$$Z^{\text{susy}}(t) = \int e^{-S+t\mathcal{Q}V}\,; \qquad\qquad \frac{d}{dt}Z^{\text{susy}} = \int e^{-S+t\mathcal{Q}V}\mathcal{Q}V = 0\,, \tag{3.7}$$

since $\mathcal{Q}$ acts as a total derivative. The path integral can be then computed for $t \to \infty$ and, if we choose V cleverly and we are lucky, it reduces to a sum over saddle points of a classical contribution and a one-loop determinant,

$$Z^{\text{susy}}(t=\infty) = \sum_{\text{saddle points}} e^{-S_{\text{classical}}}\,\frac{\text{det fermions}}{\text{det bosons}}\,. \tag{3.8}$$

In many supersymmetric gauge theories on Euclidean manifolds, this approach successfully reduces the path integral to the evaluation of a matrix model.

In our theory, $\mathcal{L}_{\text{YM}}, \mathcal{L}_{\text{mat}}$ and $\mathcal{L}_{\text{W}}$ are not only $\mathcal{Q}$-closed but also $\mathcal{Q}$-exact. For example, up to total derivatives,

[20] These references cover the developments after Pestun (2012). The idea of localization in physics is much older and it has been applied to many systems since Witten (1982b).

[21] In our case $\mathcal{Q}^2$ is a linear combination of a gauge transformation and a rotation along S^1.

$$\tilde{\epsilon}^\dagger \epsilon \mathcal{L}_{\rm YM} = \mathcal{Q}\widetilde{\mathcal{Q}}\,{\rm Tr}\left(\tfrac{1}{2}\lambda^\dagger\lambda + 2D\sigma\right). \tag{3.9}$$

Therefore we can put arbitrary coefficients in front of the various $\mathcal{Q}$-exact terms in the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm CS} + \frac{1}{\lambda^2}\mathcal{L}_{\rm mat} + \frac{1}{\eta^2}\mathcal{L}_{\rm W}\,, \tag{3.10}$$

and the path integral is independent of g, λ, η. We can then take the limit $g, \lambda, \eta \to 0$ and evaluate the path integral in the saddle point approximation. Notice in particular that $\mathcal{L}_{\rm W}$ is $\mathcal{Q}$-exact. This means that the partition function is independent of the precise form of the interactions in the Lagrangian. The superpotential is important nevertheless for determining the global symmetries of the theory, which enter in the partition function through chemical potentials.

We now give the localization formula for the topologically twisted index. Since the computation is complicated and subtle, we just provide the final formula, referring to Pestun and Zabzine (2017) and Marino (2011) for a general introduction to localization and to Benini and Zaffaroni (2015, 2017) for the details of this particular computation.

The path integral of the topologically twisted theory on $\Sigma_{\mathfrak{g}} \times S^1$ for a $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group G can be written as a contour integral

$$Z_{\Sigma_{\mathfrak{g}}\times S^1} = \frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma}\oint_C Z_{\rm int}(u, \mathfrak{m}), \tag{3.11}$$

of a meromorphic form of Cartan-valued variables u, summed over a lattice Γ of magnetic fluxes. W is just the order of the Weyl group of G. We will explain all the ingredients in the following, referring to Benini and Zaffaroni (2015, 2017) for proofs. Notice that from now on we drop the superscript susy from the partition functions.

3.2.1 The BPS locus

We have a family of saddle points labeled by the vev of the scalar field σ, the value of the Wilson line A_t along S^1 and a quantized magnetic flux $\mathfrak{m}$ along $\Sigma_{\mathfrak{g}}$. As standard in localization computation, these saddle points can be found as the locus where the fermionic variations vanish. The gaugino BPS equations read

$$Q\lambda = \left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D\right)\epsilon + i\gamma^\mu\epsilon D_\mu\sigma = 0, \tag{3.12}$$

and are solved by setting the two terms on the right-hand side to zero. The second term in (3.12), $D_\mu\sigma$, vanishes for constant commuting adjoint fields σ and A_t. With a gauge transformation, we can transform them in the Cartan subalgebra. We can combine these fields in a complex Cartan-valued quantity

$$u = A_t + i\beta\sigma\,. \tag{3.13}$$

The Wilson line A_t is periodic, invariant under a shift of any element χ of the co-root lattice Γ, $A_t \sim A_t + 2\pi\chi$,[22] the physical object being the holonomy e^{iA_t}. So u naturally lives on a cylinder and it is convenient to define the quantity $x = e^{iu}$. For a $U(1)$ theory, u lives on the cylinder $S^1 \times \mathbb{R}$ and x on the punctured plane $\mathbb{C}^*$. The first term in (3.12), using $\gamma_{12}\epsilon = i\epsilon$, implies that the auxiliary field D is proportional to the gauge field strength along $\Sigma_{\mathfrak{g}}$, $D = iF_{12}$, and both live in the Cartan subalgebra. The curvature along $\Sigma_{\mathfrak{g}}$ is quantized

$$\frac{1}{2\pi}\int_{\Sigma_{\mathfrak{g}}} F = \mathfrak{m} \in \Gamma \tag{3.14}$$

where Γ is again the co-root lattice. For a $U(1)$ theory $\mathfrak{m}$ is just an integer.

The path integral involves a sum over saddle points and is therefore given as an integral over u and a sum over the magnetic fluxes $\mathfrak{m}$. Both variables live in the Cartan subalgebra and are only defined up to an action of the Weyl group, the surviving gauge symmetry. This explains the factor $1/|W|$ in (3.11). For a $U(N)$ theory the co-root lattice is just $\Gamma = \mathbb{Z}^N$ and the Weyl group is the permutation group of N elements with $|W| = N!$

3.2.2 The integrand

The contribution to the saddle point of the classical action comes only from the Chern–Simons term[23]

$$Z^{\rm CS}_{\rm class}(u) = x^{k\mathfrak{m}} \equiv \prod_{i=1}^{r} x_i^{k\mathfrak{m}_i} \tag{3.15}$$

where $x = e^{iu}$.

The one-loop determinant receives contributions from the vector multiplets and the chiral multiplets. The vector multiplet contribution is

$$Z^{\rm gauge}_{1-{\rm loop}}(u) = \prod_{\alpha\in G} (1 - x^{\alpha})^{1-\mathfrak{g}}\,(i\,du)^r \tag{3.16}$$

where α are the roots of G and, for convenience, we included the integration measure $(du)^r$ in this expression. The chiral multiplet contribution is

$$Z^{\rm chiral}_{1-{\rm loop}}(u,\mathfrak{m}) = \prod_{\rho\in\mathfrak{R}} \left(\frac{x^{\rho/2}}{1 - x^{\rho}}\right)^{\rho(\mathfrak{m}) + (\mathfrak{g}-1)(r_\phi - 1)} \tag{3.17}$$

[22] The co-root lattice is defined by the requirement that $e^{2\pi i\chi}$ acts as the identity on any representation of the group G, and defines the weight lattice of the Langland (or S-dual) group $\check{G}$.

[23] This expression follows from a holomorphic recombination of the terms $A_t \wedge F_{12}$ and σD in the Chern–Simons action, using $D = iF_{12}$ and $u = A_t + i\beta\sigma$.

where $\mathfrak{R}$ is the representation under the gauge group G, ρ are the corresponding weights and r_ϕ is the R-charge of the field. These expressions arise by taking ratios of determinants for fermionic and bosonic fields, computed by expanding in modes on $\Sigma_\mathfrak{g} \times S^1$. Due to supersymmetry, most of the modes cancel between bosons and fermions and we are left with the contribution of a set of zero-modes (indeed a convenient way to perform this computation is via an index theorem). For a chiral multiplet these zero modes contribute

$$\prod_{\rho\in\mathfrak{R}} \prod_{n=-\infty}^{\infty} \left(\frac{2\pi i}{\beta} n + i\rho(\frac{A_t}{\beta} + i\sigma)\right)^{-(\rho(\mathfrak{m})+(\mathfrak{g}-1)(r_\phi-1))} . \tag{3.18}$$

The term in bracket represents the mass of a chiral multiplet mode due to the coupling to σ, which acts as a real mass, to the Wilson line A_t and to the KK momentum n along the circle S^1. The exponent is the multiplicity of the zero-mode that can be easily obtained using the Riemann–Roch theorem. Notice that this multiplicity must be an integer and therefore the R-charges r_ϕ must be quantized.[24] This infinite product needs to be regularized. In (3.17) we chose a parity invariant regularization. There are other possible ones.[25]

The full integrand is

$$Z_{\text{int}}(u,\mathfrak{m}) = Z_{\text{pert}}(u,\mathfrak{m}) \left(\det_{ab} \frac{\partial^2 \log Z_{\text{pert}}(u,\mathfrak{m})}{\partial i u_a \partial \mathfrak{m}_b}\right)^g , \tag{3.19}$$

where

$$Z_{\text{pert}}(u,\mathfrak{m}) = Z^{\text{CS}}_{\text{class}}(u,\mathfrak{m}) Z^{\text{gauge}}_{1-\text{loop}}(u) Z^{\text{chiral}}_{1-\text{loop}}(u,\mathfrak{m}) . \tag{3.20}$$

The determinant term exists only on a Riemann surface of genus $\mathfrak{g} > 0$ and arise from the integration of the extra $\mathfrak{g}$ fermionic zero-modes existing on these surfaces.

3.2.3 The contour

The integrand (3.19) is a meromorphic form in the Cartan variables u with poles at $x^\rho = 1$, the points in the BPS locus where chiral multiplets become massless, and at the boundaries $x = 0$ and $x = \infty$ of the moduli space. The partition function is obtained by using the residue theorem. Supersymmetry will choose the correct integration contour and tell us which poles to include. One might hope that we need to integrate over some simple contour, like the unit circle in the plane x, but one actually discovers that the contour is highly non-trivial and depends on the charges

[24] On $S^2 \times S^1$ the R-charges r_ϕ must be integer. In the case of a higher genus Riemann surface it is enough to require that the quantity $(1-\mathfrak{g})(r_\phi - 1)$ is an integer.

[25] Parity acts as $u \to -u$ and (3.17) is obviously invariant. A gauge invariant regularization breaking parity is used in Closset and Kim (2016) and Closset et al. (2017a, b, 2018). The latter has the advantage of clarifying subtle sign issues and simplifying the mapping of parameters between dual theories. However, even for theories with zero CS, in the gauge invariant regularization one has to introduce extra effective CS contact terms, which makes the physical interpretation less transparent.

of the matter fields. For example, for a $U(1)$ theory with chiral fields of charge Q_i and Chern–Simons level k, defining the effective Chern–Simons level[26]

$$k_{\text{eff}}(\sigma) = k + \frac{1}{2} \sum_i Q_i^2 \, \text{sign} \, (Q_i \sigma) \,, \tag{3.21}$$

the rule is to take the residues of the poles created by fields with positive charge $Q_i > 0$, the residue at the origin $x = 0$ if $k_{\text{eff}}(\infty) < 0$ and the residue at infinity $x = \infty$ if $k_{\text{eff}}(-\infty) > 0$. The rule for a generic gauge group can be written in terms of the so-called Jeffrey–Kirwan (JK) residue (Jeffrey and Kirwan 1995), a prescription for dealing with poles arising from multiple intersecting hyperplane singularities. To explain it properly will lead us too far and we refer to Benini and Zaffaroni (2015, 2017) for details. The JK residue also appears in localization computations for elliptic genera in two dimensions, quantum mechanics, and various other partition functions (Benini et al. 2014; Hori et al. 2015; Closset et al. 2015).

The reader may object that we are supposed to integrate over the BPS locus, which is the whole complex plane, and not to perform a contour integral in u. Luckily again supersymmetry comes to a rescue. On $\Sigma_{\mathrm{g}} \times S^1$ there are gaugino zero-modes that contribute an extra term in the integrand in addition to the one-loop determinant. It turns out that the full integrand is a total derivative in $\bar{u}$ and we can reduce the integral over the u plane to a contour integral around the singularities, as discussed in details in Benini and Zaffaroni (2015).

3.2.4 Adding flavor fugacities

If the theory has a flavor symmetry group F acting on the chiral fields, we can introduce extra parameters in a supersymmetric way. We can just gauge the flavor symmetry and then freeze all the bosonic fields to background values that are preserved by supersymmetry. The background bosonic fields give rise to supersymmetric couplings in the Lagrangian. The analysis of fermionic variations is identical to the one performed in Sect. 3.2.1 for gauge symmetries. We need to solve (3.12) for a background multiplet, $(A_\mu^F, \lambda^F, \sigma^F, D^F)$. The result is that we can turn on in a supersymmetric way a constant value for σ^F and A_t^F which we combine into a complex quantity $u_F = A_t^F + i\beta\sigma^F$, and a background magnetic flux $\mathfrak{m}_F$ with $D^F = i\mathfrak{m}_F$. σ^F appears in the Lagrangian as a real mass for the chiral fields. In three dimensions, any gauge theory with a $U(1)$ factor with field strength F has also a *topological symmetry* associated with the current $J = *F$, which is automatically conserved. We can similarly introduce parameters u_T and $\mathfrak{m}_T$ for the topological symmetry.

The path integral is then a function of x_F, x_T and $\mathfrak{m}_F, \mathfrak{m}_T$. In the localization formula we just need to replace the one-loop determinant of a chiral field with

[26] This is actually the Chern–Simons level that one sees at one-loop after integrating out the matter fields (they have mass σ at a generic point of the BPS locus).

$$Z^{\text{chiral}}_{1-\text{loop}}(u,\mathfrak{m};u_F,\mathfrak{m}_F)=\prod_{\rho\in\mathfrak{R}}\left(\frac{x^{\rho/2}x_F^{\nu/2}}{1-x^\rho x_F^\nu}\right)^{\rho(\mathfrak{m})+\nu(\mathfrak{m}^F)+(\mathfrak{g}-1)(r_\phi-1)} \tag{3.22}$$

where $x_F=e^{iu_F}$ and ν is the weight of the chiral field under the flavor symmetry F. There is no modification to the vector multiplet determinant. A $U(1)$ topological symmetry just contributes a classical term

$$x^{\mathfrak{m}_T}x_T^{\mathfrak{m}} \tag{3.23}$$

to the classical action.

3.2.5 The trace interpretation

As any path integral that involves an S^1 factor, the topologically twisted index can be written as a trace[27]

$$Z_{\Sigma_\mathfrak{g}\times S^1}(x_G,\mathfrak{m}_G)=\ \mathrm{Tr}\,(-1)^F e^{iA_t^G J^G}e^{-\beta H_\mathfrak{g}} \tag{3.24}$$

where $H_\mathfrak{g}$ is the Hamiltonian of the topologically twisted theory on $\Sigma_\mathfrak{g}$, in the presence of magnetic fluxes $\mathfrak{m}_G=(\mathfrak{m}_F,\mathfrak{m}_T)$ and a supersymmetric background $x_G=(x_F,x_T)$ for the global symmetries, whose conserved charges have been denoted as J^G. The Hamiltonian $H_\mathfrak{g}$ explicitly depends on the magnetic fluxes $\mathfrak{m}_G$ and the real masses σ^G. If sufficiently real masses are turned on, the spectrum of $H_\mathfrak{g}$ is discrete and the trace is well-defined.

3.2.6 An example: SQED

To explain all the ingredients, we can give a simple example of the final formula, using supersymmetric QED. This is a $U(1)$ theory with two chiral multiplets Q and $\tilde{Q}$ of charges ± 1 (electron and positron), and no Chern–Simons couplings. Since there is no superpotential, we have many possible choices of integer R-charges for the fields. We choose to assign R-charge $+1$ to both Q and $\tilde{Q}$. There is an axial flavor symmetry $U(1)_A$ acting on Q and $\tilde{Q}$ with equal charges and a topological symmetry $U(1)_T$. The charges of the chiral fields are

$$\begin{array}{ccccc} & U(1)_g & U(1)_T & U(1)_A & U(1)_R \\ Q & 1 & 0 & 1 & 1 \\ \tilde{Q} & -1 & 0 & 1 & 1 \end{array} \tag{3.25}$$

The topological symmetry acts only on non-perturbative states constructed with monopole operators. We introduce a gauge variable x, with associated magnetic flux $\mathfrak{m}$, and flavor and topological variables $y=x_F$ and $\xi=x_T$, with associated background fluxes $\mathfrak{n}=\mathfrak{m}_F$ and $\mathfrak{t}=\mathfrak{m}_T$. According to our rules, the partition function on $S^2\times S^1$ is

[27] See footnote 14. The factor $e^{iA_t^G J^G}$ represents the insertion of a Wilson line A_t^G.

$$Z(y,\xi,\mathfrak{n},\mathfrak{t}) = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{dx}{2\pi i x} x^{\mathfrak{t}} (-\xi)^{\mathfrak{m}} \left(\frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{1-xy}\right)^{\mathfrak{m}+\mathfrak{n}} \left(\frac{x^{-\frac{1}{2}} y^{\frac{1}{2}}}{1-x^{-1}y}\right)^{-\mathfrak{m}+\mathfrak{n}} \tag{3.26}$$

where we included an extra $(-1)^{\mathfrak{m}}$, which can be reabsorbed in the definition of ξ, for later convenience.[28] Notice that gauge and flavor variables enter in a similar way in this formula. The main difference is that x and $\mathfrak{m}$ are integrated and summed over, while y,ξ and $\mathfrak{n},\mathfrak{t}$ are background parameters.

Our prescription instructs us to take the residues from the field Q with positive gauge charge, whose pole is at $x=\frac{1}{y}$. By computing residues and resumming the result, one finds

$$Z(y,\xi,\mathfrak{n},\mathfrak{t}) = \left(\frac{y}{1-y^2}\right)^{2\mathfrak{n}-1} \left(\frac{\xi^{\frac{1}{2}} y^{-\frac{1}{2}}}{1-\xi y^{-1}}\right)^{\mathfrak{t}-\mathfrak{n}+1} \left(\frac{\xi^{-\frac{1}{2}} y^{-\frac{1}{2}}}{1-\xi^{-1} y^{-1}}\right)^{-\mathfrak{t}-\mathfrak{n}+1} . \tag{3.27}$$

One recognizes here the product of three factors of the form (3.22) that we can associate with chiral multiplets. Indeed it is well known that the mirror theory to SQED is a Wess–Zumino model with fields $M,T,\tilde{T}$ and a cubic superpotential $W=MT\tilde{T}$ (Aharony et al. 1997).

3.3 Interpretation of the localization formula

We can give an interpretation of the localization formula for the theory on $\Sigma_{\mathfrak{g}}\times S^1$ in two different ways that correspond to two different dimensional reductions of the three-dimensional theory. Compactification on $\Sigma_{\mathfrak{g}}$ gives rise to a quantum mechanics and compactification on S^1 to a two-dimensional (2, 2) supersymmetric theory.

3.3.1 Reduction to quantum mechanics

Compactifying on $\Sigma_{\mathfrak{g}}$, we obtain a supersymmetric quantum mechanics describing an infinite number of KK modes on $\Sigma_{\mathfrak{g}}$. These are particles living on the Riemann surface in the presence of a magnetic field for the R-symmetry and magnetic fluxes $\mathfrak{m}_G$ for the global symmetries. These magnetic fields create Landau levels. The trace (3.24) can be interpreted as the Witten index (Witten 1982a) of this particular quantum mechanics. Let us understand this concept better.

The quantum mechanics in question has $\mathcal{N}=2$ supersymmetry. With no background for the global symmetries, the algebra of supersymmetry is simply $\{\bar{Q},Q\}=H_{\mathfrak{g}}$, where Q is a complex supercharge. The index is just

$$\mathrm{Tr}\,(-1)^F e^{-\beta H_{\mathfrak{g}}}, \tag{3.28}$$

and, according to standard arguments, is independent of β. Indeed, any state ψ with $H_{\mathfrak{g}}\neq 0$ has a non-zero partner $Q\psi$ with the same energy and opposite statistic and

[28] For a more careful discussion of sign ambiguities see Closset et al. (2017b). They will not play any important role in this review.

their contributions cancel in the trace. Therefore the only contribution comes from ground states.[29] The index is then clearly independent of β and it is an integer counting the number of ground states with signs (plus for bosonic ones, minus for fermionic ones). When we turn on backgrounds for the global symmetries, the supersymmetry algebra is modified to $\{\bar{Q}, Q\} = H_{\mathfrak{g}} - \sigma^G J^G$, where J^G is the conserved charge associated to the global symmetry.[30] Using (3.24), we can now write the index as follows

$$\mathrm{Tr}\,(-1)^F e^{iA_t^G J^G} e^{-\beta H_{\mathfrak{g}}} = \mathrm{Tr}\,(-1)^F e^{i(A_t^G + i\beta\sigma^G)J^G} e^{-\beta\{\bar{Q},Q\}} = \sum_n g(n) x_G^n, \tag{3.29}$$

where $g(n)$ is the number of supersymmetric states, $Q\psi = 0$, with charge n under the global symmetry and, as usual, $x_G = e^{i(A_t^G + i\beta\sigma^G)}$. In deriving this expression, we used again the fact that states with $Q\psi \neq 0$ are paired by supersymmetry and have the same energy and charge. This time, the states that contribute to the trace are chiral states with $H_{\mathfrak{g}} = \sigma_G J^G$. In this way, we have obtained an equivariant index, where the supersymmetric states are graded according to their charge by powers of the fugacity x_G. Notice also that this argument shows that the topologically twisted index is an holomorphic function of the fugacities, as we already found using localization.

Let us also notice that the integrand of the localization formula has a simple Hamiltonian interpretation. There are two type of multiplets in the $\mathcal{N} = 2$ quantum mechanics we are discussing: the chiral multiplet containing a complex scalar ϕ and a spinor as dynamical fields, and the Fermi multiplet containing only a spinor (Hori et al. 2015). The Landau levels on $\Sigma_{\mathfrak{g}}$ give rise to zero-modes with multiplicities dictated by the Riemann–Roch theorem. One can see that the zero-modes organize themselves into $\rho(\mathfrak{m}) + (\mathfrak{g}-1)(r_\phi - 1)$ chiral multiplets if $\rho(\mathfrak{m}) + (\mathfrak{g}-1)(r_\phi - 1) > 0$, and $|\rho(\mathfrak{m}) + (\mathfrak{g}-1)(r_\phi - 1)|$ Fermi multiplets if $\rho(\mathfrak{m}) + (\mathfrak{g}-1)(r_\phi - 1) < 0$, where for simplicity we set the flavor fugacities to zero. We can now compute the index for Fermi and chiral multiplets. For the Fermi multiplet the Hilbert space is a fermionic Fock space, and assigning charge $-\frac{\rho}{2}$ and fermion number 0 to the vacuum, the index is

$$\frac{1 - x^\rho}{x^{\frac{\rho}{2}}}\,. \tag{3.30}$$

For the chiral multiplet the Hilbert space is the product of a bosonic Fock space generated by $\phi, \phi^\dagger$ and a fermionic Fock space; assigning fermion number 1 to the vacuum, the index is

[29] By the algebra of supersymmetry $H_{\mathfrak{g}}\psi = 0$ is equivalent to $Q\psi = 0$ and, therefore, ground states not necessarily have a partner.

[30] See, for example, Hori et al. (2015) or Appendix C of Benini et al. (2016b).

$$(-x^{-\frac{\rho}{2}} + x^{\frac{\rho}{2}}) \sum_{n\geq 0} x^{n\rho} \sum_{m\geq 0} x^{-m\rho} = \frac{x^{\frac{\rho}{2}}}{1 - x^{\rho}} \,. \tag{3.31}$$

Raising these quantities to a power corresponding to the multiplicity, and taking into account the different signs for the two types of multiplets, we recover exactly the contribution (3.17) of a three-dimensional chiral multiplet to the partition function.

In particular, the localization formula for the topologically twisted index is just the sum over many topological sectors labelled by $\mathfrak{m}$ of the localization formula for the Witten index of $\mathcal{N} = 2$ quantum mechanics found in Hori et al. (2015).

3.3.2 Reduction to two dimensions

We can alternatively reduce our three-dimensional theory on S^1 and obtain a (2, 2) supersymmetric theory containing all the KK modes on S^1. At a generic point of the Coulomb branch where $\sigma \neq 0$, all the non-Cartan gauge bosons and the chiral multiplets are massive.[31] We can integrate them out and write a Lagrangian for the Cartan modes of the vector multiplets. In two dimensions, a vector multiplet can be described using a twisted chiral multiplet Σ and its interactions is described by a twisted superpotential $\int d\theta_+ d\bar{\theta}_- \mathcal{W}$.[32]

It is interesting to observe that such twisted superpotential $\mathcal{W}$ enters explicitly in the integrand of the localization formula (Closset and Kim 2016; Closset et al. 2017b). Indeed, the dependence on the gauge flux $\mathfrak{m}$ can be explicitly written as

$$\sum_{\mathfrak{m}\in\Gamma} \int \frac{dx_i}{2\pi i x_i}\, Q(x)\, e^{i\mathfrak{m}_i \frac{\partial \mathcal{W}}{\partial u_i}}, \tag{3.32}$$

where $Q(x)$ is a meromorphic function independent of $\mathfrak{m}$. The function $\mathcal{W}$, up to an overall normalization and sign ambiguities that we fix for convenience, is given by[33]

$$\mathcal{W}(u) = \frac{k}{2}\sum_i u_i^2 + \sum_{\mathfrak{R}} \left(\frac{1}{2} g_2(\rho(u)) - \mathrm{Li}_2(e^{i\rho(u)}) \right), \tag{3.33}$$

with $g_2(u) = \frac{u^2}{2} - \pi u + \frac{\pi^2}{3}$. As argued in Nekrasov and Shatashvili (2009b, 2015), this can be interpreted as the effective twisted superpotential of the two-dimensional theory obtained by compactifying on S^1. The first term in (3.33) is the classical contribution coming from the CS term, and the second is the sum of all the

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[31] Due to the KK mass or their coupling to σ.

[32] Σ has a scalar as its lowest component and it satisfies $\bar{D}_+\Sigma = D_-\Sigma = 0$. See Witten (1993).

[33] Polylogarithms $\mathrm{Li}_s(z)$ are defined by $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ for $|z| < 1$ and by analytic continuation outside the disk. Notice, in particular, that $\mathrm{Li}_1(z) = -\log(1-z)$. For $s \geq 1$, there exists a the branch cut that we take along $[1, +\infty)$. They satisfy $\partial_u \mathrm{Li}_s(e^{iu}) = i\,\mathrm{Li}_{s-1}(e^{iu})$ and, for $0 < \mathbb{R}eu < 2\pi$, $\mathrm{Li}_s(e^{iu}) + (-1)^s \mathrm{Li}_s(e^{-iu}) = -\frac{(2i\pi)^s}{s!} B_s\left(\frac{u}{2\pi}\right) \equiv i^{s-2} g_s(u)$, where $B_s(u)$ are the Bernoulli polynomials. In this paper we need, in particular, $g_2(u) = \frac{u^2}{2} - \pi u + \frac{\pi^2}{3}$ and $g_3(u) = \frac{u^3}{6} - \frac{\pi}{2}u^2 + \frac{\pi^2}{3}u$. One then sees that, for $0 < \mathbb{R}eu < 2\pi$ and $\mathbb{I}mu \ll 0$, $\mathrm{Li}_s(e^{iu}) \sim i^{s-2} g_s(u)$. Notice also that $\mathcal{W}$ is a multi-valued function but, since the action is defined up to integer multiples of $2\pi i$, the path integral and all physical observables are single valued.

perturbative contributions of massive fields, including the infinite tower of KK modes. Indeed, a one-loop diagram for a mode of mass m contributes a term proportional to $i(\Sigma + m)(\log(\Sigma + m)/2\pi - 1)$ to $\mathcal{W}$ and there are no higher order corrections (Witten 1993). The contribution of the KK modes of a chiral multiplet, whose mass depends on σ, the Wilson line A_t and the KK momentum n, can be resummed to

$$i \sum_{n \in \mathbb{Z}} (u + 2\pi n) \left(\log \frac{u + 2\pi n}{2\pi} - 1 \right) = -\text{Li}_2(e^{i\rho(u)}) \,. \tag{3.34}$$

The other term in the round bracket in (3.33) is local and it is due to our choice of a parity invariant regularization.[34] Using the asymptotic expansion of the polylogarithms, we find that the content of the bracket in (3.33) behaves, for large σ, as

$$\frac{\rho(u)^2}{4} \, \text{sign} \, (\rho(\sigma)) \,. \tag{3.35}$$

This can be interpreted as a one-loop effective Chern–Simons term obtained by integrating out a field of mass $\rho(\sigma)$.[35]

The Jeffrey–Kirwan prescription typically selects poles in the integrand of the localization formula that are contained in a half-lattice $\mathfrak{m}_i \geq M$ (or $\mathfrak{m}_i \leq M$) for some cut-off M. We can then use the geometric series to resum the integrand in (3.32)

$$\int \frac{dx_i}{2\pi i x_i} \frac{Q(x) e^{iM \frac{\partial \mathcal{W}}{\partial u_i}}}{\prod_i \left(1 - e^{i \frac{\partial \mathcal{W}}{\partial u_i}}\right)} , \tag{3.36}$$

and evaluate the index by taking the residues at the poles

$$\exp\left(i \frac{\partial \mathcal{W}}{\partial u_i} \right) = 1. \tag{3.37}$$

The cut-off M disappears in the process. The solutions to (3.37) are the so-called *Bethe vacua* of the two-dimensional theory. They play an important role in the Bethe/gauge correspondence (Nekrasov and Shatashvili 2009a, b).

We then find the following general characterization of the topologically twisted index as a sum over Bethe vacua

$$Z_{\Sigma_g \times S^1} = \sum_{x^*} \frac{Q(x^*)}{\det_{ij}(-\partial^2_{u_i u_j} \mathcal{W}(x^*))} , \tag{3.38}$$

where x^* are the solutions of (3.37). The expression for the topologically twisted index as a sum over Bethe vacua was first derived by topological field theory arguments in Okuda and Yoshida (2012, 2014), Nekrasov and Shatashvili (2015)

[34] For a choice of a gauge invariant regularization and an extensive discussion of other issues related to definition of $\mathcal{W}$ see Closset et al. (2017b).

[35] For a field of mass m and charge Q_i the one-loop effective Chern–Simons term is $k_{\text{eff}} = k + \frac{1}{2} Q_i^2 \, \text{sign} \, (m)$.

and Okuda and Yoshida (2015). In the context of localization, this expression for the index has been derived and generalized in Closset and Kim (2016) and Closset et al. (2017a, b, 2018). The expression in (3.38) can be also written as

$$Z_{\Sigma_g\times S^1}=\sum_{x^*}\mathcal{H}(x^*)^{\mathfrak{g}-1}, \tag{3.39}$$

in terms of a *handle-gluing operator* $\mathcal{H}(x)=e^{\Omega(x)}\det_{ij}\partial^2_{u_iu_j}\mathcal{W}(x)$ and an *effective dilaton* $\Omega(x)$ whose complete characterization in terms of field theory data can be found in the above mentioned papers. Here we just notice that, for genus $\mathfrak{g}>0$, the Hessian of $\mathcal{W}$ enters at the power $\mathfrak{g}-1$. Indeed, the determinant in (3.19) contributes $\mathfrak{g}$ extra powers of the Hessian that combine with the denominator in (3.38).

A very interesting result of Closset et al. (2017a, b, 2018) is the generalization of formula (3.39) to three-dimensional manifolds that are not a direct product. For example, the supersymmetric partition function on a three-manifold $\mathcal{M}_3$ that is an S^1 fibration of Chern class p over a Riemann surface $\Sigma_\mathfrak{g}$ can be written as a sum over the very same set of Bethe vacua,

$$Z_{\mathcal{M}_3}=\sum_{x^*}\mathcal{F}(x^*)^p\mathcal{H}(x^*)^{\mathfrak{g}-1}, \tag{3.40}$$

with a *fibering operator* $\mathcal{F}(x)$ that can be expressed in terms of field theory data. There exists a similar result for the partition function on more general three-dimensional manifolds and also for some selected four-dimensional ones (Closset et al. 2017a, b, 2018). The particular case of the formula for the four-dimensional superconformal index plays a role in the physics of AdS_5 black holes (Benini and Milan 2020a, b), as discussed in Sect. 7.2.

Let us give a couple of examples of Bethe vacua. For a pure $\mathcal{N}=2$ Chern–Simons theory with gauge group $SU(2)$, the expression for the partition function (3.19) is

$$Z=\frac{(-1)^{\mathfrak{g}-1}}{2}\sum_{\mathfrak{m}\in\mathbb{Z}}\int_{\mathrm{JK}}\frac{dx}{2\pi ix}\,(2k)^{\mathfrak{g}}x^{2k\mathfrak{m}}\left[\frac{(1-x^2)^2}{x^2}\right]^{1-\mathfrak{g}}, \tag{3.41}$$

where we used $x_i=(x,1/x)$ and $\mathfrak{m}_i=(\mathfrak{m},-\mathfrak{m})$. The twisted superpotential receives contribution only from the classical action: $\mathcal{W}=\sum_i ku_i^2/2=ku^2$. The Bethe vacua (3.37) are then $x^{2k}=1$ with solutions the $2k$-roots of unity. Formula (3.38) gives, up to an ambiguous sign,

$$Z=\left(\frac{\bar{k}+2}{2}\right)^{\mathfrak{g}-1}\sum_{j=1}^{\bar{k}+1}\left(\sin\frac{\pi i}{\bar{k}+2}j\right)^{2-2\mathfrak{g}}, \tag{3.42}$$

where $\bar{k}=k-2$. This is the well-known Verlinde formula for the CS partition function on $\Sigma_\mathfrak{g}\times S^1$.[36] Notice that the root $x=1$ is not included in the sum: as a

[36] Since σ and λ are massive and free, they can be integrated out leading to a shift in the CS coupling. An $\mathcal{N}=2$ Chern–Simons theory is thus equivalent to a bosonic CS theory with level $\bar{k}=k-2$.

general rule, the Bethe vacua that are also zeros of the Vandermonde determinant are not physical.

In the presence of matter, the Bethe equations (3.37) are more complicated. In the SQED example discussed in Sect. 3.2.6, the Bethe equation is

$$\frac{\xi(y-x)}{1-xy} = 1, \tag{3.43}$$

with solution $x = (1-\xi y)/(y-\xi)$. It is easy to see that (3.38) correctly reproduces (3.27). For a general theory with gauge group G, the Bethe equations (3.37) cannot be analytically solved.

4 The entropy of dyonic AdS_4 black holes

In this section we will derive microscopically the entropy of a family of BPS static dyonic black holes in $AdS_4 \times S^7$ (Benini et al. 2016b, 2017). These solutions have been found in $\mathcal{N}=2$ gauged supergravity in four-dimensions (Cacciatori and Klemm 2010; Dall'Agata and Gnecchi 2011; Hristov and Vandoren 2011; Katmadas 2014; Halmagyi 2015) and later uplifted to M-theory. We then start by briefly discussing the main features of $\mathcal{N}=2$ gauged supergravity in four dimensions. This will be also useful to write an entropy functional. We then consider the large N limit of the topologically twisted index for the ABJM theory in three dimensions and show that it reproduces the entropy of the dual black holes.

We will only consider the case of static black holes in this section. A field theory derivation for the entropy of dyonic rotating black holes in $AdS_4 \times S^7$ (Hristov et al. 2019b) is still missing.

4.1 AdS_4 dyonic static black holes

BPS black holes in AdS_4 can be found by studying an effective four-dimensional $\mathcal{N}=2$ gauged supergravity. We start discussing the main features of the effective theory.

The $\mathcal{N}=2$ supergravity multiplets are:

- the graviton multiplet, whose bosonic components are the metric $g_{\mu\nu}$ and a vector field A^0_μ, called graviphoton;
- the vector multiplet, whose bosonic components are a vector A^i_μ and a complex scalar z;
- the hypermultiplet, whose bosonic components are four real scalars q^α.

For simplicity, we will restrict to $\mathcal{N}=2$ gauged supergravities with n_V vector multiplets and no hypermultiplets.[37] This will be enough to describe the black holes in $AdS_4 \times S^7$. The theory contains n_V+1 vector multiplets A^Λ_μ and n_V complex

[37] Theory with hypermultiplets have been considered for matching the entropy of black holes in massive type IIA and other models (Hosseini et al. 2017a; Benini et al. 2018; Bobev et al. 2018).

scalar fields z_i, where $\Lambda = 0, 1, \ldots, n_V$ and $i = 1, \ldots, n_V$. The Lagrangian can be written in terms of a holomorphic prepotential $\mathcal{F}(X^\Lambda)$, which is a homogeneous function of degree two, and a vector of magnetic and electric Fayet–Iliopoulos (FI) parameters (g^Λ, g_Λ). $X^\Lambda(z_i)$ are a set of $n_V + 1$ homogeneous coordinates on the scalar manifold. The theory is invariant under rescaling of the X^Λ and one can identify the physical scalar fields with $z_i = X^i/X^0$.[38] It is also convenient to define $\mathcal{F}_\Lambda \equiv \partial_\Lambda \mathcal{F}$. The theory is fully covariant under a $Sp(2n_V + 2)$ group of electric/magnetic dualities acting on $(X^\Lambda, \mathcal{F}_\Lambda)$ and (g^Λ, g_Λ) as symplectic vectors.

The action of the bosonic part of the theory reads (Andrianopoli et al. 1997)

$$S^{(4)} = \frac{1}{8\pi G_{\mathrm{N}}^{(4)}} \int_{\mathbb{R}^{3,1}} \left[\frac{1}{2} R^{(4)} \star_4 1 + \frac{1}{2} \mathbb{I}m \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge \star_4 F^\Sigma + \frac{1}{2} \mathbb{R}e \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma \right.$$
$$\left. - g_{i\bar{j}} Dz^i \wedge \star_4 D\bar{z}^{\bar{j}} - V(z,\bar{z}) \star_4 1 \right].$$

The metric on the scalar manifold is given by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}(z,\bar{z}) \,. \tag{4.1}$$

Here, $\mathcal{K}(z,\bar{z})$ is the Kähler potential and it reads

$$e^{-\mathcal{K}(z,\bar{z})} = i\left(\bar{X}^\Lambda \mathcal{F}_\Lambda - X^\Lambda \bar{\mathcal{F}}_\Lambda\right) . \tag{4.2}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}$ of the gauge kinetic term is a function of the vector multiplet scalars and is given by

$$\mathcal{N}_{\Lambda\Sigma} = \bar{\mathcal{F}}_{\Lambda\Sigma} + 2i \frac{\mathbb{I}m \mathcal{F}_{\Lambda\Delta} \mathbb{I}m \mathcal{F}_{\Sigma\Theta} X^\Delta X^\Theta}{\mathbb{I}m \mathcal{F}_{\Delta\Theta} X^\Delta X^\Theta} \,. \tag{4.3}$$

Finally, the scalar potential reads

$$V(z,\bar{z}) = g^{i\bar{j}} D_i \mathcal{L} \bar{D}_{\bar{j}} \bar{\mathcal{L}} - 3|\mathcal{L}|^2, \tag{4.4}$$

where $\mathcal{L} = e^{\mathcal{K}/2}\left(X^\Lambda g_\Lambda - \mathcal{F}_\Lambda g^\Lambda\right)$ and $D_i \mathcal{L} = \partial_i \mathcal{L} + \partial_i \mathcal{K} \mathcal{L}/2$.

The ansatz for a static dyonic black hole with horizon $\Sigma_{\mathfrak{g}}$ is of the form[39]

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)}\left(dr^2 + V(r)^2 ds^2_{\Sigma_{\mathfrak{g}}}\right)$$
$$A^\Lambda = a_0(r) dt + a_1(r) A_{\Sigma_{\mathfrak{g}}},$$

where $A_{\Sigma_{\mathfrak{g}}}$ is the gauge potential for a magnetic flux on $\Sigma_{\mathfrak{g}}$. For example, for $\Sigma_{\mathfrak{g}} = S^2$ we can take $A_{S^2} = -\cos\theta d\phi$. We assume that the scalar fields z^i have only

[38] Other choices of gauge fixing for the rescaling symmetry are possible, corresponding to field redefinitions.

[39] We normalize the metric on $\Sigma_{\mathfrak{g}}$ such that the scalar curvature is 2κ, where $\kappa = 1$ for S^2, $\kappa = 0$ for T^2, and $\kappa = -1$ for $\Sigma_{\mathfrak{g}}$ with $\mathfrak{g} > 1$. The volume is then $\mathrm{Vol}(\Sigma_{\mathfrak{g}}) = 2\pi\eta$ where $\eta = 2|\mathfrak{g} - 1|$ for $\mathfrak{g} \neq 1$ and $\eta = 1$ for $\mathfrak{g} = 1$.

radial dependence. We are interested in solutions that are asymptotic to AdS_4 for large values of the radial coordinate,

$$e^{U(r)} \sim r, \qquad V(r) \sim r^2, \qquad r \gg 1, \tag{4.5}$$

and approach a regular horizon $AdS_2 \times \Sigma_{\mathfrak{g}}$ at some fixed value $r = r_0$,

$$e^{U(r)} \sim r - r_0, \qquad V(r) \sim e^{U(r)}, \qquad r \sim r_0 \,. \tag{4.6}$$

Notice that we can also interpret these black holes as domain walls interpolating between AdS_4 and $AdS_2 \times \Sigma_{\mathfrak{g}}$. The AdS_2 factor suggests the existence of a superconformal quantum mechanics describing the horizon microstates. We expect that this is the IR limit of the quantum mechanics discussed in Sect. 3.3.1.

There are two conserved quantities

$$\int_{\Sigma_{\mathfrak{g}}} F^\Lambda = \mathrm{Vol}(\Sigma_{\mathfrak{g}})\, p^\Lambda \;, \qquad \int_{\Sigma_{\mathfrak{g}}} G_\Lambda = \mathrm{Vol}(\Sigma_{\mathfrak{g}})\, q_\Lambda \;, \tag{4.7}$$

where $G_\Lambda = 8\pi G_{\mathrm{N}}\, \delta(\mathscr{L} d\mathrm{vol}_4)/\delta F^\Lambda$, corresponding to the magnetic and electric charges of the black hole. Under $Sp(2n_V + 2)$ they transform as a symplectic vector (p^Λ, q_Λ). In a frame with purely electric gauging g_Λ, the magnetic and electric charges are quantized as follows

$$\mathrm{Vol}(\Sigma_{\mathfrak{g}})\, p^\Lambda\, g_\Lambda \in 2\pi\mathbb{Z} \;, \qquad \frac{\mathrm{Vol}(\Sigma_{\mathfrak{g}})\, q_\Lambda}{4G_{\mathrm{N}}^{(4)}\, g_\Lambda} \in 2\pi\mathbb{Z}, \tag{4.8}$$

not summed over Λ.

As discussed in Sect. 2.1.2, supersymmetry is realized with a topological twist. In particular, the Killing spinors ϵ_A, $A = 1, 2$, only depend on the radial coordinate. The BPS equations give a set of ordinary differential equations for the functions $U, V, a_0, a_1, z_i, \epsilon_A$ that are explicitly given in Cacciatori and Klemm (2010), Dall'Agata and Gnecchi (2011), Hristov and Vandoren (2011), Katmadas (2014) and Halmagyi (2015). For our purposes, the only important point is that the gravitino variation contains, among other pieces,

$$\delta\psi_{\mu A} = \partial_\mu \epsilon_A + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon_A + \frac{i}{2} g_\Lambda A_\mu^\Lambda (\sigma^3)_A{}^B \epsilon_B + \dots \,. \tag{4.9}$$

The vanishing of this variation, when the index μ is restricted to $\Sigma_{\mathfrak{g}}$, requires that A_μ^Λ cancels the spin connection and one obtains

$$\sum_\Lambda g_\Lambda p^\Lambda = -\kappa, \tag{4.10}$$

where, with standard notations, $\kappa = 1$ for horizon S^2, $\kappa = 0$ for T^2 and $\kappa = -1$ for $\mathfrak{g} > 1$. We see that a linear combination of the magnetic charges is fixed by the twist. In a general theory with also magnetic FI the previous condition is replaced by

$$\sum_{\Lambda}(g_{\Lambda}p^{\Lambda} - g^{\Lambda}q_{\Lambda}) = -\kappa \tag{4.11}$$

which is manifestly symplectic invariant. The previous discussion closely parallels the field theory analysis of the supersymmetries preserved by a topological twist in Sect. 3.1.[40]

It has been noticed in Dall'Agata and Gnecchi (2011) that the BPS equations of gauged supergravity for the near-horizon geometry can be put in the form of *attractor equations*.[41] The BPS equations are indeed equivalent to the extremization of the quantity

$$\mathcal{I}_{\text{sugra}}(X^{\Lambda}) = -i\frac{\text{Vol}(\Sigma_{\mathfrak{g}})}{4G_{\text{N}}^{(4)}}\frac{q_{\Lambda}X^{\Lambda} - p^{\Lambda}\mathcal{F}_{\Lambda}}{g_{\Lambda}X^{\Lambda} - g^{\Lambda}\mathcal{F}_{\Lambda}}, \tag{4.12}$$

with respect to the horizon-value of the symplectic sections X^{Λ}, combined with the requirement that the value of $\mathcal{I}_{\text{sugra}}$ at the critical point $\bar{X}^{\Lambda}$ is *real*. In general, in gauged supergravity, $\mathcal{F}(X^{\Lambda})$ is a homogeneous function of degree two, so we can equivalently define $\hat{Y}^{\Lambda} \equiv X^{\Lambda}/(g_{\Sigma}X^{\Sigma} - g^{\Sigma}\mathcal{F}_{\Sigma})$ and extremize

$$\mathcal{I}_{\text{sugra}}(\hat{Y}^{\Lambda}) = i\frac{\text{Vol}(\Sigma_{\mathfrak{g}})}{4G_{\text{N}}^{(4)}}\left(p^{\Lambda}\mathcal{F}_{\Lambda}(\hat{Y}) - q_{\Lambda}\hat{Y}^{\Lambda}\right). \tag{4.13}$$

The extremization of (4.12) gives a set of algebraic equations for the value of the physical scalars z^i at the horizon, and the entropy of the black hole is given by evaluating the functional (4.12) at its extremum

$$S_{\text{BH}}(p^{\Lambda}, q_{\Lambda}) = \mathcal{I}_{\text{sugra}}(\bar{X}^{\Lambda}). \tag{4.14}$$

4.1.1 Black holes in $\text{AdS}_4 \times S^7$

M-theory on $\text{AdS}_4 \times S^7$ can be consistently truncated to an $\mathcal{N} = 2$ gauged supergravity containing the four vectors parameterizing the Cartan subgroup of the $SO(8)$ isometry of S^7. In the language of gauged supergravity, one is the graviphoton and the other three give rise to a model with three vector multiplets, $n_V = 3$. By explicitly reducing M-theory on $\text{AdS}_4 \times S^7$, one can determine the prepotential[42]

$$\mathcal{F} = -2i\sqrt{X^0X^1X^2X^3}, \tag{4.15}$$

and the FI, $g_{\Lambda} \equiv g$, $g^{\Lambda} = 0$, that are purely electric. With these notations the AdS_4 vacuum has radius $L^2 = 1/2g^2$.

[40] This is not a coincidence (Klare et al. 2012): when holography applies, solving the Killing spinor equations in bulk near the AdS boundary gives a set of constraints on the boundary theory that are equivalent to those obtained with the approach proposed in Festuccia and Seiberg (2011).

[41] The attractor mechanism for AdS_4 static black holes in $\mathcal{N} = 2$ gauged supergravity is discussed in Cacciatori and Klemm (2010), Dall'Agata and Gnecchi (2011), Chimento et al. (2015). For some recent progress for dyonic rotating black holes see Hristov et al. (2019b).

[42] See for example Hristov (2012a).

We can introduce four magnetic and electric charges (p^Λ, q_Λ). However, two of these charges are determined by supersymmetry. Indeed, one is fixed by the twisting condition (4.10) that gives a linear constraint among the magnetic charges. For purely magnetic black holes (Cacciatori and Klemm 2010), this is the only constraint and we find a three-dimensional family of solutions. They are particularly simple since all the scalars z_i are real. We can write, for example, the solution for a black hole with S^2 horizon

$$ds^2 = -\frac{1}{2}e^{\mathcal{K}(X)}\left(r - \frac{c}{r}\right)^2 dt^2 + 2\frac{e^{-\mathcal{K}(X)}\,dr^2}{\left(r-\frac{c}{r}\right)^2} + 2e^{-\mathcal{K}(X)}\,r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),$$
$$F^\Lambda_{\theta\phi} = p^\Lambda\,\sin\theta \tag{4.16}$$

where the real sections are given by $X^\Lambda = \frac{1}{4} - \frac{\beta_\Lambda}{r}$ and the parameters β^Λ and c are determined in terms of the magnetic charges by

$$c = 4\left(\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2\right) - \frac{1}{2}, \quad -\sqrt{2}p^\Lambda - \frac{1}{2} = 16\beta_\Lambda^2 - 4\sum\nolimits_\Sigma \beta_\Sigma^2, \quad \sum_\Lambda \beta_\Lambda = 0$$

where we also set $L = 1$ or, equivalently, $g = 1/\sqrt{2}$. The generic dyonic black holes found in Katmadas (2014) and Halmagyi (2015) are more complicated and we will not report here the form of the solution. For dyonic black holes there is an extra constraint on the charges that follows from the requirement that the entropy computed through the attractor mechanism (4.14) is a real number. This constraint is highly non linear in the charges and leaves a six-dimensional family of black holes.

The entropy can be written by using (4.14), even without knowing the explicit form of the metric. The final expression can be written in a symplectic invariant form

$$S_{\rm BH}(p^\Lambda, q_\Lambda) = \frac{\mathrm{Vol}(\Sigma_{\mathfrak{g}})}{8\sqrt{2}G_{\rm N}^{(4)}}\frac{\sqrt{I_4(\Gamma,\Gamma,G,G) \pm \sqrt{I_4(\Gamma,\Gamma,G,G)^2 - 64 I_4(\Gamma)I_4(G)}}}{I_4(G)}$$

where $\Gamma = (p^\Lambda, q_\Lambda)$ and $G = (g^\Lambda, g_\Lambda)$ are symplectic vectors containing the charges and the FI parameters, and I_4 is a quartic polynomial, known as the quartic invariant, whose explicit expression can be found in Katmadas (2014) and Halmagyi (2015). In the simplest case of a purely magnetic black hole with $p^1 = p^2 = p^3 \equiv -p/(2g)$, $p^0 = (3p-2)/(2g)$ and horizon S^2, we find an expression of the form

$$S_{\rm BH} \sim \sqrt{-1 + 6p - 6p^2 + \sqrt{(6p-1)(-1+2p)^3}}\,. \tag{4.17}$$

Notice that this expression is quite complicated, especially if compared with simple forms of the entropy as a function of charges that one can find for some asymptotically flat black holes.

The solutions with spherical horizon can be generalized by adding rotation (Hristov et al. 2019b). For completeness we report the form of the entropy

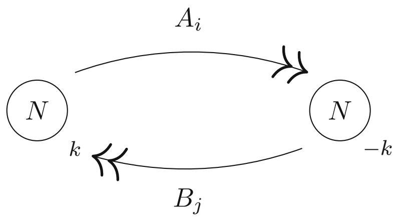
$$S_{\mathrm{BH}}(p^{\Lambda}, q_{\Lambda}, j) = \pi \frac{\sqrt{I_4(\Gamma,\Gamma,G,G) \pm \sqrt{I_4(\Gamma,\Gamma,G,G)^2 - 64 I_4(G)(I_4(\Gamma)+j^2)}}}{\sqrt{8} G_{\mathrm{N}}^{(4)} I_4(G)}$$

where j is the angular momentum. This time supersymmetry imposes three constraints on the charges and leaves again a six-dimensional family of rotating black holes.

4.2 The dual field theory

The CFT dual to $\mathrm{AdS}_4 \times S^7$ is the so-called ABJM theory (Aharony et al. 2008). We briefly discuss its properties and then we write the corresponding topological twisted index using the rules discussed in Sect. 3.

The ABJM theory describes the low-energy dynamics of N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ (Aharony et al. 2008). It is a three-dimensional supersymmetric Chern–Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ (the subscripts are the CS levels) and matter in bifundamental representation. The matter content, in $\mathcal{N} = 2$ notations, is described by the quiver diagram



where $i, j = 1, 2$ and nodes represent gauge groups and arrows represent bifundamental chiral multiplets. The theory has a quartic superpotential

$$W = \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right) . \tag{4.18}$$

The ABJM theory has a number of interesting properties:

- the theory has $\mathcal{N} = 6$ superconformal symmetry, non-perturbatively enhanced to $\mathcal{N} = 8$ for $k = 1, 2$;
- it has an $SU(4)$ R-symmetry, enhanced to $SO(8)$ for $k = 1, 2$;
- for $N \gg k^5$ the theory is well-described by a weakly coupled M-theory background, $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$;
- the free energy on S^3 can be computed using localization and scales as $O(N^{3/2})$ in the M-theory limit (Drukker et al. 2011)

$$F_{S^3} = \log Z_{S^3} = \frac{\pi\sqrt{2}}{3}\sqrt{k} N^{3/2} . \tag{4.19}$$

For a review of these properties see Klebanov and Torri (2010) and Marino (2011).

We will consider the case $k=1$ where the theory has maximal supersymmetry, $SO(8)$ R-symmetry and is dual to $\mathrm{AdS}_4 \times S^7$. The four abelian symmetries of the theory, $U(1)^4 \subset SO(8)$ correspond, in $\mathcal{N}=2$ notation, to an R-symmetry and three global symmetries. There are many choices of $U(1)$ R-symmetry corresponding to different decompositions $SO(8) \rightarrow U(1)_R \times U(1)^3$. In order to write the index we need to select one with integer charges. Introducing a natural basis of $U(1)$ R-symmetries,

$$\begin{array}{ccccc} & R_1 & R_2 & R_3 & R_4 \\ A_1 & 2 & 0 & 0 & 0 \\ A_2 & 0 & 2 & 0 & 0 \\ B_1 & 0 & 0 & 2 & 0 \\ B_2 & 0 & 0 & 0 & 2 \end{array} \tag{4.20}$$

we can for example choose the R-symmetry $\sum_a R_a/2$ that has integer charges. The remaining three $U(1)$s combine to give three flavor symmetries, say $(R_a - R_4)/2$ for $a = 1,2,3$.[43]

Our general rules for the index allow to introduce a number of independent fluxes and fugacities equal to the number of global symmetries. We then introduce three magnetic fluxes $\mathfrak{p}$ and three fugacities y for the three flavor symmetries of ABJM. It will be convenient to choose a redundant but democratic parameterization of these quantities. We assign a flux and a fugacity, $\mathfrak{p}_a$ and y_a with $a = 1,2,3,4$, to each of the fields A_1, A_2, B_1, B_2 in the order indicated. The index is given by

$$\begin{aligned} Z = & \frac{1}{(N!)^2} \sum_{\mathfrak{m},\tilde{\mathfrak{m}} \in \mathbb{Z}^N} \int_{\mathcal{C}} \prod_{i=1}^{N} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} x_i^{k\mathfrak{m}_i} \tilde{x}_i^{-k\tilde{\mathfrak{m}}_i} \prod_{i \neq j}^{N} \left[\left(1 - \frac{x_i}{x_j}\right) \left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right) \right]^{1-\mathfrak{g}} \\ & \prod_{i,j=1}^{N} \prod_{a=1,2} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_a}}{1 - \frac{x_i}{\tilde{x}_j} y_a} \right)^{\mathfrak{m}_i - \tilde{\mathfrak{m}}_j - \mathfrak{p}_a + 1 - \mathfrak{g}} \prod_{a=3,4} \left(\frac{\sqrt{\frac{\tilde{x}_j}{x_i} y_a}}{1 - \frac{\tilde{x}_j}{x_i} y_a} \right)^{\tilde{\mathfrak{m}}_j - \mathfrak{m}_i - \mathfrak{p}_a + 1 - \mathfrak{g}} \\ & \times \left(\det_{AB} \frac{\partial^2 \mathcal{W}}{\partial u_A \partial u_B} \right)^{\mathfrak{g}}, \end{aligned} \tag{4.21}$$

where we used the rules of Sect. 3.2.2. Notice that the Hessian of $\mathcal{W}$ should be computed using the $2N$ variables $u_A = (u_i, \tilde{u}_i)$. We have also included the CS term k in order to make clear where the terms come from but soon we will set $k = 1$. As

[43] We are cheating a little bit here. The ABJM theory has also two topological symmetries, T_1 and T_2, associated with the two $U(1)$ gauge groups. This apparently makes a total of five $U(1)$ global symmetries. However $T_1 + T_2$ is decoupled, and the baryonic symmetry that rotates A_i and B_i with opposite charges is actually gauged. More precisely, due the CS term, a linear combination of $T_1 - T_2$ and the baryonic symmetry differ by a gauge transformation and are therefore equivalent. In the index we could introduce extra fluxes and fugacities for T_1 and T_2 but these can be re-absorbed by a shift of the fluxes and a rescaling of the integration variables. See Benini et al. (2016b) for more details.

already said, the fugacities are not independent. Since the superpotential (4.18) must have charge zero under a global symmetry we must set

$$\prod_{a=1}^{4} y_a = 1\,. \tag{4.22}$$

This translates into a constraint for the corresponding (complexified) chemical potentials Δ_a, $y_a = e^{i\Delta_a}$,

$$\sum_a \Delta_a \in 2\pi\mathbb{Z}, \tag{4.23}$$

since the Δ_a are only defined modulo 2π.[44] Similarly, the four fluxes $\mathfrak{p}_a$ are not independent. To understand our parameterization, let us compare the chiral fields contributions in (4.21) with (3.22). Identifying exponents we have

$$-\mathfrak{p}_a + 1 - \mathfrak{g} = \mathfrak{m}_a^F + (\mathfrak{g} - 1)(r_a - 1), \tag{4.24}$$

where $\mathfrak{m}_a^F$ is an assignment of background fluxes for the global symmetry and r_a the R-charge of the a-th field. Since W has charge zero under global symmetries and charge two under R-symmetries, we have $\sum_{a=1}^4 \mathfrak{m}_a^F = 0$ and $\sum_{a=1}^4 r_a = 2$, so that

$$\sum_{a=1}^{4} \mathfrak{p}_a = 2(1 - \mathfrak{g})\,. \tag{4.25}$$

The dependence of our index on three magnetic fluxes and three fugacities fits well with the family of black holes discussed in Sect. 4.1.1 that have three magnetic and three electric charges. (4.25) is clearly the analog of (4.10) and already suggests the following identification between parameters $p^\Lambda \to -\kappa\mathfrak{p}_a/(2g(1-\mathfrak{g}))$.

The index can be written as a sum over Bethe vacua (3.38). The twisted superpotential (3.33) reads[45]

$$\mathcal{W} = \sum_{i=1}^{N} \frac{k}{2}\left(\tilde{u}_i^2 - u_i^2\right) + \sum_{i,j=1}^{N}\left[\sum_{a=3,4} \mathrm{Li}_2\left(e^{i(\tilde{u}_j - u_i + \Delta_a)}\right) - \sum_{a=1,2} \mathrm{Li}_2\left(e^{i(\tilde{u}_j - u_i - \Delta_a)}\right)\right] \tag{4.26}$$

and the Bethe vacua equations are

[44] In the notation of Sect. 3.2.4, $\Delta_a = A^F_{t\,a} + i\beta\sigma^F_a$, where $A^F_{t\,a}$ and σ^F_a are the backgrounds for the a-th symmetry. The periodicity of Δ_a is due to the periodicity of the Wilson line $A^F_{t\,a}$.

[45] We used the of polylogarithm identities given in footnote 33 in order to recombine the terms in (3.33), and discarded terms that do not contribute to the Bethe equations (3.37). We also introduced an extra minus sign in the definition of $\mathcal{W}$ in order to match the original conventions in Benini et al. (2016b) and Hosseini and Zaffaroni (2016). It is easy to check directly that the equations (4.27) give the position of the poles of the integrand after we sum the geometric series in m_i and $\tilde{m}_i$ and that, with the given definition of $\mathcal{W}$, (4.27) are equivalent to (3.37).

$$x_i^k \prod_{j=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)} = \tilde{x}_j^k \prod_{i=1}^{N} \frac{\left(1 - y_3 \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_4 \frac{\tilde{x}_j}{x_i}\right)}{\left(1 - y_1^{-1} \frac{\tilde{x}_j}{x_i}\right)\left(1 - y_2^{-1} \frac{\tilde{x}_j}{x_i}\right)} = 1. \tag{4.27}$$

In the large N limit we expect that just one Bethe vacuum dominates the partition function.

4.3 The ABJM Bethe vacua in the large N limit

We want to study the solutions of (4.27) in the large N limit (Benini et al. 2016b). By running numerics, one discovers that the imaginary parts of the solutions u_i and $\tilde{u}_i$ grow with N,

$$u_i = iN^\alpha t_i + v_i \qquad \tilde{u}_i = iN^\alpha t_i + \tilde{v}_i \tag{4.28}$$

and are equal for the two sets, while the real parts remain bounded. As usual, in the large N limit, the distributions of u_i and $\tilde{u}_i$ become almost continuous and we introduce a parameter $t(i/N) = t_i$, defined in an interval $[t_-, t_+]$. We also introduce two functions of t, $v(t)$ and $\tilde{v}(t)$, defined implicitly by $v(i/N) = v_i$, $\tilde{v}(i/N) = \tilde{v}_i$, and a normalized density

$$\rho(t) = \frac{1}{N}\frac{di}{dt}, \qquad \int_{t_-}^{t_+} \rho(t) dt = 1\,. \tag{4.29}$$

The interesting feature of this model is that $\mathcal{W}$ becomes a *local* functional,[46]

$$\mathcal{W}[\rho(t), \delta v(t)] = iN^{1+\alpha} \int dt\, t\rho(t)\delta v(t) + iN^{2-\alpha} \int dt \rho(t)^2 \sum_{a=1}^{4} g_3(-\epsilon_a \delta v(t) + \Delta_a), \tag{4.30}$$

where $\delta v(t) = \tilde{v}(t) - v(t)$, $g_3(u) = \frac{1}{6}u^3 - \frac{1}{2}\pi u^2 + \frac{\pi^2}{3}u$ and $\epsilon_a = 1$ for $a = 1, 2$ and $\epsilon_a = -1$ for $a = 3, 4$. We also assumed that Δ_a are real and

$$0 < -\epsilon_a \delta v(t) + \Delta_a < 2\pi \tag{4.31}$$

for all a. The first term in $\mathcal{W}$ comes from the Chern–Simons interaction and the second is the contribution of matter fields. The derivation of (4.30) is given in Benini et al. (2016b). Here we just mention few facts.

- $\mathcal{W}$ is local because of the exponential terms $e^{i(\tilde{u}_j - u_i \pm \Delta_a)}$ in the arguments of polylogs in (4.26). Due to (4.28), for $j > i$ the polylogs are exponentially suppressed in the large N limit. For $i > j$ the exponential is large but we can use the identity $\mathrm{Li}_2(e^{iu}) + \mathrm{Li}_2(e^{-iu}) = \frac{1}{2}u^2 - \pi u + \frac{\pi^2}{3}$, valid for $\mathbb{R}e u \in [0, 2\pi]$, (see footnote 33) to transform it into a polynomial plus exponentially suppressed terms. As a consequence, up to polynomial terms, the main contribution comes for values of the indices $i \sim j$ and makes the functional local.

46 This is similar to other matrix models solved using localization in three and five dimensions (Herzog et al. 2011; Jafferis et al. 2011; Jafferis and Pufu 2014; Minahan et al. 2013).

- Terms with higher powers of N cancel. For more general $\mathcal{N} = 2$ theories this is not automatic and imposes conditions on the matter content of the theories for which this method works (Hosseini and Zaffaroni 2016).
- Polynomial terms coming from this manipulation or Chern–Simons terms that are not in (4.30) happily combine into a contribution $\sum_{i=1}^{N} 2\pi n_i u_i + 2\pi \tilde{n}_i \tilde{u}_i$ to $\mathcal{W}$, where n_i and $\tilde{n}_i$ are integers. These angular ambiguities disappear in the Bethe equations (3.37).

In general, the two contributions in (4.30) have different powers of N. They compete and give a sensible functional with a minimum only for $\alpha = 1/2$. We then see that $\mathcal{W}$ scales as $N^{3/2}$ as predicted by holography for AdS_4 black holes. We will then set $\alpha = 1/2$ from now on.

In order to extremize (4.30) we add to $\mathcal{W}$ a Lagrange multiplier term

$$-iN^{3/2}\mu\left(\int \rho(t)dt - 1\right) \tag{4.32}$$

that enforces the normalization condition (4.29). Differentiating $\mathcal{W}$ with respect to $\delta v(t)$ and $\rho(t)$, we obtain a pair of algebraic equations

$$t - \rho(t)\sum_{a=1}^{4} \epsilon_a g_3'(-\epsilon_a \delta v(t) + \Delta_a) = 0, \tag{4.33}$$

$$t\delta v(t) + 2\rho(t)\sum_{a=1}^{4} g_3(-\epsilon_a \delta v(t) + \Delta_a) = \mu, \tag{4.34}$$

which can be easily solved in terms of rational functions of t. The solution is depicted in Fig. 1 together with the numerical solution for large N.

From the figure we see that $\rho(t)$ and $\delta v(t)$ are piece-wise continuous functions of t. The solution in (4.33) and (4.34) only covers the central part of these functions. Numerics suggest that there are two external intervals, which we dub *tails*, where $\delta v(t)$ is actually constant in the large N limit. It turns out that such constant value corresponds to the saturation of the inequality (4.31) for some value of a, $\delta v(t) = \epsilon_a \Delta_a$. The inequalities (4.31) are necessary to restrict to a particular determination of the multi-valued polylog functions and the saturation corresponds to the position of the cuts. The numerics suggest that, once v_i and $\tilde{v}_i$ hit the cut, their value is frozen. The value for $\rho(t)$ in the tails can be obtained by *solving* its equation of motion, (4.34), setting $\delta v(t)$ to the constant value $\epsilon_a \Delta_a$ and *ignoring* the equation of motion for $\delta v(t)$, (4.33), which would be inconsistent. The end-points of the interval, t_- and t_+, are finally determined by $\rho(t_\pm) = 0$.

Obviously, the equation of motion for $\delta v(t)$, (4.33), must be satisfied at finite N. The main correction to $\delta v(t)$ and to its equation comes from the terms with $i = j$ in (4.26). Such terms contribute

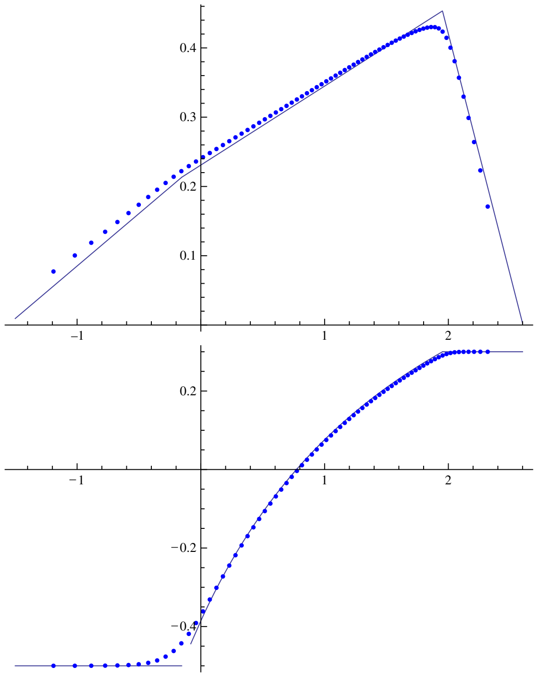


Fig. 1 Plots of the density of eigenvalues $\rho(t)$ and the function $\delta v(t)$ for $N = 75$, $\Delta_1 = 0.3$, $\Delta_2 = 0.4$, $\Delta_3 = 0.5$ with $\sum_a \Delta_a = 2\pi$ and $k = 1$. The blue dots represent the numerical simulation, while the solid grey line is the analytical result

$$\delta \mathcal{W} = N \int dt \rho(t) \left[\sum_{a=3,4} \mathrm{Li}_2\left(e^{i(\delta v(t)+\Delta_a)}\right) - \sum_{a=1,2} \mathrm{Li}_2\left(e^{i(\delta v(t)-\Delta_a)}\right) \right] . \tag{4.35}$$

Notice that these terms are suppressed compared to $\mathcal{W}$.[47] They contribute a term

$$-\frac{1}{\sqrt{N}} \left[\sum_{a=1}^{4} \epsilon_a \log\left(1 - e^{i(\delta v(t) - \epsilon_a \Delta_a)}\right) \right], \tag{4.36}$$

to the right-hand side of the Eq. (4.33) for $\delta v(t)$. Such a correction is generically of order $1/\sqrt{N}$. However, on the tails, since $\delta v(t) = \epsilon_b \Delta_b$ for some b, one of the logarithms blows up and the correction can be effectively of order one. Indeed, the equation of motion for $\delta v(t)$ can be satisfied if

$$\delta v(t) = \epsilon_b \left(\Delta_b - e^{-\sqrt{N} Y_b(t)} \right) , \tag{4.37}$$

where $Y_b(t)$ is a quantity of order one. In this case, on the tail, the equation becomes

[47] This is a standard argument in the context of matrix models: $\sum_{i \neq j}^{N} = O(N^2)$ while $\sum_{i=1}^{N} = O(N)$. Notice that the contribution to $\mathcal{W}$ comes from terms with i almost equal to j and contributions to $\delta \mathcal{W}$ from terms with $i = j$.

$$t - \rho(t) \sum_{a=1}^{4} \epsilon_a g_3'(-\epsilon_a \delta v(t) + \Delta_a) = \epsilon_b Y_b(t), \tag{4.38}$$

not summed over b, which determines the value of $Y_b(t)$. Notice that, quite remarkably, the correction to $\delta v(t)$ is not power-like but exponentially small. The equations for $\rho(t)$ and the value of $\mathcal{W}$ on the solution are not affected by these corrections in the large N limit since $\mathrm{Li}_2(z)$ is finite for $z \to 1$. However, these corrections are important for evaluating the index.

The explicit solution is as follows (Benini et al. 2016b). Let us first take Δ_a real. Using the periodicity of Δ_a, we can always restrict to the case where $0 \le \Delta_a \le 2\pi$. We will also assume that $\Delta_1 \le \Delta_2$, $\Delta_3 \le \Delta_4$. The constraint (4.23) can be satisfied only for $\sum_a \Delta_a = 0, 2\pi, 4\pi, 6\pi, 8\pi$ and we need to consider all possible cases. We find a solution for $\sum_a \Delta_a = 2\pi$. We have a central region where

$$\begin{aligned} \rho &= \frac{2\pi\mu + t(\Delta_3\Delta_4 - \Delta_1\Delta_2)}{(\Delta_1+\Delta_3)(\Delta_2+\Delta_3)(\Delta_1+\Delta_4)(\Delta_2+\Delta_4)} \\ \delta v &= \frac{\mu(\Delta_1\Delta_2 - \Delta_3\Delta_4) + t\sum_{a<b<c}\Delta_a\Delta_b\Delta_c}{2\pi\mu + t(\Delta_3\Delta_4 - \Delta_1\Delta_2)} \end{aligned} \qquad -\frac{\mu}{\Delta_4} < t < \frac{\mu}{\Delta_2}\,. \tag{4.39}$$

When δv hits $-\Delta_3$ on the left the solution becomes

$$\rho = \frac{\mu + t\Delta_3}{(\Delta_1+\Delta_3)(\Delta_2+\Delta_3)(\Delta_4-\Delta_3)}, \quad \delta v = -\Delta_3\,, \quad -\frac{\mu}{\Delta_3} < t < -\frac{\mu}{\Delta_4}, \tag{4.40}$$

with the exponentially small correction $Y_3 = (-t\Delta_4 - \mu)/(\Delta_4 - \Delta_3)$, while when δv hits Δ_1 on the right the solution becomes

$$\rho = \frac{\mu - t\Delta_1}{(\Delta_1+\Delta_3)(\Delta_1+\Delta_4)(\Delta_2-\Delta_1)}, \qquad \delta v = \Delta_1\,, \quad \frac{\mu}{\Delta_2} < t < \frac{\mu}{\Delta_1}, \tag{4.41}$$

with $Y_1 = (t\Delta_2 - \mu)/(\Delta_2 - \Delta_1)$. It turns out that, for $\sum_{a=1}^4 \Delta_a = 0, 4\pi, 8\pi$, equations (4.33) and (4.34) have no regular solutions. There is also a solution for $\sum_a \Delta_a = 6\pi$ which, however, is obtained by the previous one by a discrete symmetry of the index: $\Delta_a \to 2\pi - \Delta_a$ $\left(y_a \to y_a^{-1}\right)$.

We can also evaluate the twisted superpotential on the solution and find

$$\widetilde{\mathcal{W}}(\Delta) = \frac{2iN^{3/2}}{3}\sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}, \qquad \Delta_a \in [0, 2\pi]\,, \qquad \sum_{a=1}^{4}\Delta_a = 2\pi\,. \tag{4.42}$$

The result for a different determination of the Δ_a is obtained by periodicity: just replace Δ_a with $[\Delta_a] = (\Delta_a \bmod 2\pi)$. We can also extend by holomorphicity the result to complex Δ_a.

It is interesting to observe that $\widetilde{\mathcal{W}}(\Delta)$ has the same functional dependence of two important physical quantities appearing in the study of ABJM and its dual $\mathrm{AdS}_4 \times S^7$. One is of purely field theory origin. It is known that, for any $\mathcal{N} = 2$ theory, there exists a family of supersymmetric Lagrangians on S^3 parameterized by arbitrary R-charges of the chiral fields (Hama et al. 2011; Jafferis 2012). When the theory is superconformal, the resulting partition function has an extremum precisely at the exact R-symmetry of the theory (Jafferis 2012). For ABJM at $k = 1$, the S^3 free energy reads (Jafferis et al. 2011)

$$F_{S^3}(r) = \frac{4\pi N^{3/2}}{3}\sqrt{2r_1 r_2 r_3 r_4}\,, \tag{4.43}$$

where r_a are a general assignment of R-charges for the fields A_1, A_2, B_1, B_2 satisfying $\sum_{a=1}^4 r_a = 2$. We see that (Hosseini and Zaffaroni 2016)

$$\widetilde{\mathcal{W}}(\Delta_a) = \frac{\pi i}{2} F_{S^3}\left(\frac{\Delta_a}{\pi}\right). \tag{4.44}$$

The second quantity, of supergravity origin, is the prepotential of the $\mathcal{N} = 2$ gauged supergravity describing the low energy theory of the holographic dual. Restricted to the $U(1)^4$ gauge sector, the prepotential is given by (4.15) and we see that

$$\widetilde{\mathcal{W}}(\Delta_a) = -\frac{\sqrt{2}N^{3/2}}{3}\mathcal{F}(\Delta_a)\,. \tag{4.45}$$

This will be important for comparison with the attractor mechanism.

4.4 The large N limit of the ABJM index

Using (3.38), up to an irrelevant overall factor, the index is given by

$$\begin{aligned}&\left(\det_{AB}\frac{\partial^2 \mathcal{W}}{\partial u_A \partial u_B}\right)^{\mathfrak{g}-1}\prod_{i\neq j}^{N}\left(1 - x_i^*/x_j^*\right)^{1-\mathfrak{g}}\left(1-\tilde{x}_i^*/\tilde{x}_j^*\right)^{1-\mathfrak{g}}\\ &\times\prod_{i,j=1}^{N}\prod_{a=1,2}\left(\frac{\sqrt{x_i^*\, y_a/\tilde{x}_j^*}}{1-x_i^*\, y_a/\tilde{x}_j^*}\right)^{-\mathfrak{p}_a+1-\mathfrak{g}}\prod_{b=3,4}\left(\frac{\sqrt{\tilde{x}_j^*\, y_b/x_i^*}}{1-\tilde{x}_j^*\, y_b/x_i^*}\right)^{-\mathfrak{p}_b+1-\mathfrak{g}},\end{aligned} \tag{4.46}$$

where x_i^* and $\tilde{x}_i^*$ is the large N Bethe vacuum found in the previous section. By taking the large N limit of (4.46), after some manipulations, one finds

$$\begin{aligned}\log Z = &- N^{\frac{3}{2}}\int dt\, \rho(t)^2\left[(1-\mathfrak{g})\frac{2\pi^2}{3} + \sum_{a=1}^4 \left(\mathfrak{p}_a - 1 + \mathfrak{g}\right) g_3'\left(-\epsilon_a \delta v(t) + \Delta_a\right)\right]\\ &- N^{\frac{3}{2}}\sum_{a=1}^4 \mathfrak{p}_a \int_{\delta v \approx \varepsilon_a \Delta_a} dt\, \rho(t)\, Y_a(t),\end{aligned} \tag{4.47}$$

up to corrections of order $N \log N$. The first contribution in the first line of (4.47) comes from the Vandermonde determinant and the second from the matter contribution. The second line in (4.47) comes from the tails. Since the logarithm of the one-loop determinant of the chiral fields is singular on such regions, we need to take into account the exponentially small corrections Y_a to the tails. The exponent $-\mathfrak{p}_a + 1 - \mathfrak{g}$ of the one-loop determinant is corrected to $-\mathfrak{p}_a$ by an analogous and subtle contribution from the determinant of the Hessian of $\mathcal{W}$.[48]

By plugging in the explicit solution for $\rho(t), \delta v(t)$ and $Y_a(t)$ we find

$$\log Z = -\frac{2\sqrt{2}}{3} N^{\frac{3}{2}} \sum_{a=1}^{4} \mathfrak{p}_a \frac{\partial}{\partial \Delta_a} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \, . \tag{4.48}$$

Notice that

$$\log Z = i \sum_{a=1}^{4} \mathfrak{p}_a \frac{\partial}{\partial \Delta_a} \widetilde{\mathcal{W}}(\Delta) \, . \tag{4.49}$$

One can show that this remarkable identity can be extended to other $\mathcal{N} = 2$ quiver theories. Indeed it can be proved using only the equations of motion for ρ and δv, and taking into account the extra terms on the tails (Hosseini and Zaffaroni 2016).

4.5 Matching index and entropy for ABJM

We can now extract the degeneracy of states from the index using (2.19). We introduce four electric charges $\mathfrak{q}_a$, adapted to the democratic basis of charges we are using, and we set $j_i = 0$ since we are interested in static black holes. The degeneracy of states with given electric charge is then obtain by extremizing the functional

$$\mathcal{I}(\Delta) = \log Z(\Delta) - i \sum_{a=1}^{4} \mathfrak{q}_a \Delta_a, \tag{4.50}$$

which implicitly depends on the magnetic charges $\mathfrak{p}_a$ through Z. This is the $\mathcal{I}$-extremization principle introduced in Benini et al. (2016b, 2017). For purely magnetic black holes we just extremize $\log Z$. For a generic dyonic static black hole, we obtain

$$S_{\text{micro}}(\mathfrak{p}_a, \mathfrak{q}_a) = \log Z - i \sum_{a=1}^{4} \mathfrak{q}_a \Delta_a = i \sum_{a=1}^{4} \left(\mathfrak{p}_a \frac{\partial}{\partial \Delta_a} \widetilde{\mathcal{W}}(\Delta) - \mathfrak{q}_a \Delta_a \right), \tag{4.51}$$

evaluated at its critical point in Δ_a with the constraints $\sum_{a=1}^{4} \Delta_a = 2\pi$ and $\mathbb{R}e\Delta_a \in [0, 2\pi]$.

By an explicit computation one can see that this expression correctly reproduces the entropy of the black holes in Cacciatori and Klemm (2010), Dall'Agata and Gnecchi (2011), Hristov and Vandoren (2011), Katmadas (2014) and Halmagyi (2015). This can be checked more easily by comparing with the attractor equations (4.14). Using (4.12) we find

[48] See Benini et al. (2016b) for details.

$$S_{\mathrm{BH}}(\mathfrak{p}^{\Lambda},\mathfrak{q}_{\Lambda}) = -\frac{i\mathrm{Vol}(\Sigma_{\mathfrak{g}})}{4G_{\mathrm{N}}^{(4)}}\frac{q_{\Lambda}X^{\Lambda}-p^{\Lambda}F_{\Lambda}}{g_{\Lambda}X^{\Lambda}-g^{\Lambda}F_{\Lambda}} = i\sum_{\Lambda=0}^{3}\left(\hat{\mathfrak{p}}^{\Lambda}\frac{\partial}{\partial\hat{X}_{\Lambda}}\widetilde{\mathcal{W}}(\hat{X}) - \hat{\mathfrak{q}}_{\Lambda}\hat{X}^{\Lambda}\right). \tag{4.52}$$

Here we used the data of the relevant gauged supergravity [see (4.15)]

$$g_{\Lambda}\equiv g\,, \qquad g^{\Lambda}=0\,, \qquad \mathcal{F}=-2i\sqrt{X^0X^1X^2X^3}, \tag{4.53}$$

we defined adapted scalar fields

$$\hat{X}^{\Lambda} = \frac{2\pi X^{\Lambda}}{\sum_{\Lambda=0}^{3}X^{\Lambda}}, \tag{4.54}$$

satisfying $\sum_{\Lambda=0}^{3}\hat{X}^{\Lambda}=2\pi$, and enforced the Dirac quantization condition of fluxes (4.8) by defining the integers $\hat{\mathfrak{p}}^{\Lambda}$ and $\hat{\mathfrak{q}}_{\Lambda}$,

$$\mathrm{Vol}(\Sigma_{\mathfrak{g}})\,p^{\Lambda}\,g_{\Lambda} = -2\pi\hat{\mathfrak{p}}^{\Lambda}\,, \qquad \frac{\mathrm{Vol}(\Sigma_{\mathfrak{g}})\,q_{\Lambda}}{4G_{\mathrm{N}}g_{\Lambda}} = 2\pi\hat{\mathfrak{q}}_{\Lambda}. \tag{4.55}$$

Finally, we used the known holographic dictionary (see for example Marino 2011)

$$\frac{1}{2g^2G_N^{(4)}} = \frac{2\sqrt{2}}{3}N^{3/2}. \tag{4.56}$$

Using the identification[49]

$$\begin{aligned} a=1,2,3,4 &\rightarrow \Lambda=0,1,2,3,\\ \Delta_a &\rightarrow \hat{X}^{\Lambda},\\ (\mathfrak{p}_a,\mathfrak{q}_a) &\rightarrow (\hat{\mathfrak{p}}^{\Lambda},\hat{\mathfrak{q}}_{\Lambda}), \end{aligned} \tag{4.57}$$

we see that $S_{\mathrm{micro}}(\mathfrak{p}_a,\mathfrak{q}_a) = S_{\mathrm{BH}}(\mathfrak{p}^{\Lambda},\mathfrak{q}_{\Lambda})$. The extremization of (4.51) is thus completely equivalent to the attractor mechanism in gauged supergravity. The correspondence, up to constants, is

$$\begin{aligned} \widetilde{\mathcal{W}}(\Delta) &\rightarrow \mathcal{F}(X^{\Lambda}),\\ \mathcal{I}(\Delta) &\rightarrow \mathcal{I}_{\mathrm{sugra}}(X^{\Lambda})\,. \end{aligned} \tag{4.58}$$

Notice that the functional $\mathcal{I}(\Delta)$ is only defined up to integer multiples of $2\pi i$, due to the presence of $\log Z$. So the microscopical entropy should be properly defined as $S_{\mathrm{micro}}(\mathfrak{p}_a,\mathfrak{q}_a) = \mathcal{I}\ (\mathrm{mod}\ 2\pi i)$.

In field theory, $\mathcal{I}$ only depends on three independent chemical potentials, which we can choose to be Δ_a with $a=1,2,3$. The extremization of $\mathcal{I}$ determines them in terms of three electric charges, $q_a - q_4$,

[49] Notice that the identification is consistent with the twisting condition (4.10) and the constraint $\sum_{a=1}^{4}\Delta_a = 2\pi$.

$$\frac{\partial \log Z}{\partial \Delta_a} = i(\mathfrak{q}_a - \mathfrak{q}_4) , \qquad a = 1, 2, 3 . \tag{4.59}$$

The entropy can be then expressed in terms of $\mathfrak{q}_a - \mathfrak{q}_4$ as

$$\log Z - i \sum_{a=1}^{3} (\mathfrak{q}_a - \mathfrak{q}_4) \Delta_a \ (\text{mod}\, 2\pi i) . \tag{4.60}$$

In these derivations we used that $\mathfrak{q}_a$ are integers and $\sum_{a=1}^{4} \Delta_a = 2\pi$.

Correspondingly, in gravity, the family of black holes depends only on three independent electric charges, as discussed in Sect. 2.3. Indeed, the requirement that $S_{\text{BH}}(\mathfrak{p}^\Lambda, \mathfrak{q}_\Lambda)$ is real gives a constraints on the charges. For given magnetic charges $\mathfrak{p}_a$ and flavor electric charges $\mathfrak{q}_a - \mathfrak{q}_4$, there is at most one value of $\mathfrak{q}_4$ that leads to black hole with regular horizon.

We can see the left hand side of (4.59) as determining the average electric charge $\mathfrak{q}_a - \mathfrak{q}_4$ in our statistical ensemble at large N. The index only depends on the global symmetries of the theory and there are three of them. Correspondingly, with our method, we cannot determine the average electric charge associated with the R-symmetry. However, this value is fixed by the BPS equations in gravity. It would be interesting to find a purely field theoretical method for testing this prediction for the fourth charge. Notice that, with the permutationally invariant definition of $\mathcal{I}$ given in (4.50), the value of q_4 predicted by gravity is precisely the one that makes the critical value of $\mathcal{I}$ real.

Let us conclude this section with a comment. We computed the entropy as a Legendre transform of the grand canonical partition function, according to a natural microscopic point of view. However, in holography, the partition function of the boundary field theory can be also identified with the (holographically renormalized) on-shell action of the bulk theory. By consistency, we should be able to extract the same information from the gravitational on-shell action. The agreement of these points of view has been discussed in Azzurli et al. (2018), Halmagyi and Lal (2018) and Cabo-Bizet et al. (2018). In particular, the authors of Bobev et al. (2020) found a family of smooth Euclidean solutions with boundary $\Sigma_g \times S^1$ whose on-shell action coincides with the topologically twisted index of ABJM for general values of the fugacities and that contains one element with an AdS_2 geometry that can be wick-rotated to the original Lorentzian black hole.

4.6 Other examples and generalizations

The large N limit of the topologically twisted index can be evaluated for other $\mathcal{N} = 2$ quiver Chern–Simons theories with an AdS_4 dual. There is a large class of Yang–Mills–Chern–Simons theories with fundamental and bi-fundamental chiral fields that have been proposed as duals of M-theory and massive type IIA compactifications. Most of these are obtained by dimensionally reducing a *parent* four-dimensional quiver gauge theory with an $\text{AdS}_5 \times SE_5$ dual, where SE_5 is a five-dimensional Sasaki–Einstein manifold, adding Chern–Simons terms and flavoring with fundamentals. Holography predicts that the twisted index scales as $N^{3/2}$ for theories dual to M-theory on seven-dimensional Sasaki–Einstein manifolds, and as $N^{5/3}$ for a class of theories dual to massive type IIA on warped six-manifolds.[50] For a large class of these theories the large N method discussed in Sect. 4.3 applies and the twisted index has been evaluated in Hosseini and Zaffaroni (2016), Hosseini and Mekareeya (2016), Jain and Ray (2019) and Jain (2019). Quite remarkably, in all these theories the on-shell twisted superpotential coincides with the corresponding S^3 free energy (Hosseini and Zaffaroni 2016)

$$\widetilde{\mathcal{W}}(\Delta_a) = \frac{i\pi}{d} F_{S^3}\left(\frac{\Delta_a}{\pi}\right), \tag{4.61}$$

where $d = 2$ for M-theory examples and $d = 3$ in massive type IIA ones.[51] The partition function is given by the following *index theorem* (Hosseini and Zaffaroni 2016)[52]

$$\log Z = (1-\mathfrak{g})\left(\frac{di}{\pi}\widetilde{\mathcal{W}}(\Delta_I) + i\sum_I \left[\left(\frac{\mathfrak{p}_I}{1-\mathfrak{g}} - \frac{\Delta_I}{\pi}\right)\frac{\partial \widetilde{\mathcal{W}}(\Delta_I)}{\partial \Delta_I}\right]\right). \tag{4.62}$$

It is often possible, as for ABJM, to choose a convenient redundant parameterization for the chemical potentials such that $\widetilde{\mathcal{W}}$ becomes a homogeneous functions of degree d. For this choice the twisted index is given again by

$$\log Z = i\sum_a \mathfrak{p}_a \frac{\partial}{\partial \Delta_a}\widetilde{\mathcal{W}}(\Delta) \ . \tag{4.63}$$

The entropy of dyonic static black holes in such theories is obtained by taking the Legendre transform of (4.63). Comparison with the attractor mechanism in the form (4.13) then requires that

$$\widetilde{\mathcal{W}}(\Delta) \sim \mathcal{F}(X^\Lambda), \tag{4.64}$$

under some map $\Delta_a \to X^\Lambda$. (4.64) is however oversimplified. The effective low energy theory for M or type IIA compactifications on general manifolds is an $\mathcal{N} = 2$ gauged supergravity containing massless vector multiplets associated with the global symmetries of the CFT. In general, such gauged supergravity contains also hypermultiplets and a number of vector multiplet that is larger than the number of global symmetries. Therefore, in (4.64), the number of Δ_a does not match the

[50] See Hanany et al. (2009), Hanany and Zaffaroni (2008), Martelli and Sparks (2008), Gaiotto and Jafferis (2012), Benini et al. (2010), Jafferis and Tomasiello (2008), Herzog et al. (2011), Gulotta et al. (2012), Crichigno et al. (2013) and Martelli and Sparks (2009) for examples of theories with M-theory dual and Guarino et al. (2015) and Fluder and Sparks (2016) for theories with massive type IIA dual.

[51] The twisted index depends on chemical potentials for the global symmetries. As for ABJM, we can use a redundant basis where we assign a chemical potential Δ_a to each field ϕ_a. For each term W_I in the superpotential we must require $\sum_{a\in W_I}\Delta_a = 2\pi p$ with $p \in \mathbb{Z}$ for all the fields ϕ_a appearing in W_I. It turns out that, as for ABJM, a large N solution exists only for $p = 1$ (up to equivalent solutions). On the other hand the S^3 free energy is a function of the R-charges of the fields, and these are constrained to satisfy $\sum_{a\in W_I} r_a = 2$. The identity (4.61) makes sense because $p = 1$.

[52] See Hosseini (2018–02) for more details and applications.

number of sections X^Λ. In general, the hypermultiplets have a VEV at the horizon and give a mass to some of the vector fields. If n_V is the number of vector multiplets and n_H is the number of hypermultiplets, $n_V - n_H$ is the number of massless vectors, corresponding to the number of global symmetries and therefore to the number of independent Δ_a. The BPS equations for the hyperinos are typically algebraic and give linear constraints among the sections X^Λ. By solving these constraints we can obtain an effective prepotential $\mathcal{F}_{\text{eff}}(X^\Lambda)$ for the massless vector multiplets only. If we believe that this further truncated theory correctly describes the horizon of the black hole, (4.64) should hold with $\mathcal{F}(X^\Lambda)$ replaced by $\mathcal{F}_{\text{eff}}(X^\Lambda)$.[53]

Two notable applications where this works perfectly are the following. A well-known example of massive type IIA background is the warped $\text{AdS}_4 \times Y_6$ flux vacua of massive type IIA dual to the $\mathcal{N} = 2$ $U(N)$ gauge theory with three adjoint multiplets and a Chern–Simons coupling k (Guarino et al. 2015). It corresponds to an internal manifold Y_6 with the topology of S^6. The theory has two global symmetries and there exists a family of black holes depending on two magnetic charges (Guarino 2017). The entropy of such black holes has been matched with the prediction of the twisted index in Hosseini et al. (2017a) and Benini et al. (2018). For this example

$$\widetilde{\mathcal{W}}(\Delta_a) \sim F_{S^3}(\Delta_a) \sim \mathcal{F}_{\text{eff}}(\Delta_a) \sim (\Delta_1\Delta_2\Delta_3)^{2/3}, \qquad \sum_{a=1}^{3} \Delta_a = 2\pi\,. \tag{4.65}$$

The second example involves the so-called *universal black hole* (Azzurli et al. 2018). This is dual to the *universal twist* (Benini et al. 2016a; Bobev and Crichigno 2017) defined by a set of magnetic fluxes $\mathfrak{p}_a$ proportional to the exact R-symmetry of the CFT. This black hole is a solution of minimal gauged supergravity and, as such, can be embedded in all M-theory and massive type IIA compactifications, thus explaining the name universal. Hence it provides an infinite family of examples. Since $\widetilde{\mathcal{W}}$ is proportional to the S^3 free energy [see (4.61)] and the latter is extremized at the exact R-symmetry, it follows easily from (4.62) that

$$S_{\text{BH}}(\mathfrak{p}_a) = (\mathfrak{g} - 1)F_{S^3}\,. \tag{4.66}$$

This relation agrees with the gravitational prediction based on minimal gauged supergravity (Azzurli et al. 2018).

There have been progresses in various directions and also open problems:

- For theories with M-theory dual the large N method discussed above works only when the bi-fundamental fields transform in a real representation of the gauge group and the total number of fundamentals is equal to the total number of anti-fundamentals. The same restriction has been found in Jafferis et al. (2011) for the large N evaluation of the S^3 free energy. It is not known yet how to take the large N limit for the S^3 free energy or the twisted index in the case of chiral quivers. Also in the case of vector-like quivers, where the method works, there is

[53] This happens for example for the mass deformed ABJM model (Bobev et al. 2018), where the value of one Δ_a is fixed by the mass deformation and, in gravity, one X^Λ is fixed by the hyperino conditions.

an extra complication since baryonic symmetries are invisible in the large N limit (accidental flat directions of the matrix model). It is still not clear how to introduce and study baryonic fluxes in the large N limit of the twisted index.

- A gravitational dual of $\mathcal{I}$-extremization was proposed in Couzens et al. (2019), generalizing similar results for a- and c-extremization (Martelli et al. 2006; Gauntlett et al. 2019a). In this approach, the entropy of an M theory black hole with AdS_2 horizon is obtained by extremizing a geometric quantity associated with a supersymmetric background where some of the equations of motion have been relaxed. As a result, the entropy can be expressed in terms of simple geometrical data of the internal manifold and can be computed also in the cases where the explicit solution is not known. In particular, this provides an explicit formula for the entropy and the exact R-symmetry of the associated superconformal mechanics for an infinite class of models based on toric Sasaki-Einstein manifolds. The agreement of the gravitational picture with the available field theory results at large N has been discussed in Gauntlett et al. (2019b), Hosseini and Zaffaroni (2019a) and Kim and Kim (2019).
- The microstate counting for dyonic rotating black holes (Hristov et al. 2019b) is still an open problem. The chemical potential dual to rotation in the bulk is associated with an Omega-background for the boundary theory on $S^2 \times S^1$. The topologically twisted index in an Omega-background is known (Benini and Zaffaroni 2015) but it gives rise to a complicated matrix model. For particular values of the Ω-deformation, it has been written as a sum over Bethe vacua (Closset et al. 2018). This could be useful to solve the matrix model in the large N limit. The result is expected to match the entropy functional proposed in Hosseini et al. (2019a) by gluing gravitational blocks.
- For asymptotically flat black holes, there is a huge literature including higher derivative corrections and highly detailed precision tests. For asymptotically AdS black holes, the story is just begun. At the moment, $1/N$ corrections to the twisted index for ABJM and the above massive type IIA background have been computed numerically focusing on the universal logarithmic correction (Liu et al. 2018a, b, c; Jeon and Lal 2017; Pando Zayas and Xin 2019). The ABJM matrix model provides a quantum corrected entropy functional that would be interesting to study further. In particular, it would be interesting to find the analog of standard results and conjectures about the quantum entropy of asymptotically flat back holes, like the OSV conjecture (Ooguri et al. 2004), the Sen's quantum entropy functional (Sen 2009b, a) and localization in supergravity (Dabholkar et al. 2011a, 2013),[54] to cite only few of them. In particular, some interesting results about localization in $AdS_2 \times S^2$ have been already found in Hristov et al. (2018, 2019c).
- The topologically twisted index has been extended to five-dimensions (Hosseini et al. 2018c; Crichigno et al. 2018) and successfully compared with the entropy of AdS_6 black holes with horizon $AdS_2 \times \Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ (Suh 2019a; Hosseini et al. 2018a; Suh 2019b, 2018; Fluder et al. 2019). The expression for the entropy is given by a generalization of the index theorem (4.63) and (4.62).

[54] See Iqbal et al. (2008) for an earlier localization computation in gravity.

Other interesting related results and solutions can be found in Cabo-Bizet et al. (2017), Toldo and Willett (2018), Gang and Kim (2019), Gang et al. (2020), Hosseini et al. (2019c) and Bae et al. (2020).

5 Black strings in AdS_5

The methods introduced in Sect. 4 can be generalized to other dimensions and can be used to provide general tests of holography. In particular, they can be applied to domain wall solutions interpolating between AdS_{d+n} and $AdS_d \times \mathcal{M}_n$, with a topological twist along the n-dimensional compact manifold $\mathcal{M}_n$. In this section, making a brief digression from the main subject of these notes, we discuss the example of black strings in AdS_5.

More precisely, we consider a family of black strings in AdS_5 with near horizon geometry $AdS_3 \times \Sigma_{\mathfrak{g}}$. They correspond holographically to a twisted compactification of a four-dimensional theory flowing to a two-dimensional CFT in the IR. The properties of this CFT, and, in particular, its elliptic genus, can be computed using a topologically twisted index for four-dimensional theories on $\Sigma_{\mathfrak{g}} \times T^2$. We can make contact with the physics of black holes by compactifying the black string on a circle, as in the standard example (Strominger and Vafa 1996). In this way we can also make contact with a Cardy formula approach to the microstate counting. Other examples of similar tests of holography related to twisted compactifications can be found in Toldo and Willett (2018), Crichigno et al. (2018) and Hosseini et al. (2018c).

5.1 Black string solutions in $AdS_5 \times S^5$

We consider solutions of a five-dimensional $\mathcal{N} = 2$ effective gauged supergravity with abelian vector multiplets of the form

$$\begin{aligned} ds^2 &= e^{2f(r)}(-dt^2 + dz^2) + e^{-2f(r)}dr^2 + e^{2g(r)}ds^2_{\Sigma_{\mathfrak{g}}} \\ A^\Lambda &= p^\Lambda A_{\Sigma_{\mathfrak{g}}}, \end{aligned}$$

where $A_{\Sigma_{\mathfrak{g}}}$ is the gauge potential for a magnetic flux on $\Sigma_{\mathfrak{g}}$ and supersymmetry is preserved with a twist along $\Sigma_{\mathfrak{g}}$. We are interested in solutions that are asymptotic to AdS_5 for large values of the radial coordinate,

$$e^{f(r)} \sim r\,, \qquad e^{g(r)} \sim r, \qquad r \gg 1 \tag{5.1}$$

and approach a regular horizon $AdS_3 \times \Sigma_{\mathfrak{g}}$ at some fixed value $r = r_0$,

$$e^{f(r)} \sim r - r_0\,, \qquad e^{g(r)} \sim \text{constant}\,, \qquad r \sim r_0\,. \tag{5.2}$$

These solutions can be interpreted as black strings extended in the direction z or, equivalently, as domain walls interpolating between AdS_5 and $AdS_3 \times \Sigma_{\mathfrak{g}}$. They are holographically dual to a twisted compactification of a $SCFT_4$ on $\Sigma_{\mathfrak{g}}$ that flows in the IR to a two-dimensional (0, 2) $SCFT_2$ associated with the horizon factor AdS_3.

Solutions that can be embedded in $AdS_5 \times S^5$ have been found in Benini and Bobev (2013b), using a five-dimensional gauged supergravity with three abelian gauge fields associated with the isometries $U(1)^3 \subset SO(6)$ of S^5. The solution depends on three fluxes $\mathfrak{p}_a$ constrained by the twisting condition

$$\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 = 2 - 2\mathfrak{g}, \tag{5.3}$$

and corresponds to a twisted compactification of $\mathcal{N} = 4$ SYM on $\Sigma_\mathfrak{g}$. Holography suggests that this theory flows to an IR two-dimensional SCFT. The value of the central charge of the IR $SCFT_2$ in the large N limit can be extracted from the solution using standard arguments (Brown and Henneaux 1986; Henningson and Skenderis 1998) and reads (Benini and Bobev 2013b)

$$c = \frac{3R_{AdS_3}}{2G_N^{(3)}} = 12N^2 \frac{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}{\mathfrak{p}_1^2 + \mathfrak{p}_2^2 + \mathfrak{p}_3^2 - 2\mathfrak{p}_1\mathfrak{p}_2 - 2\mathfrak{p}_2\mathfrak{p}_3 - 2\mathfrak{p}_3\mathfrak{p}_1} \,. \tag{5.4}$$

This result can be successfully compared with field theory (Benini and Bobev 2013b), where central charges can be computed using extremization techniques. Let us briefly explain how this works. Consider first the case of $\mathcal{N} = 1$ four-dimensional SCFTs and $\mathcal{N} = 4$ SYM as an example. In $\mathcal{N} = 1$ language, the theory contains three chiral multiplets ϕ_i with the superpotential

$$W = \operatorname{Tr}\left(\phi_3[\phi_1, \phi_2]\right). \tag{5.5}$$

We introduce a generic R-charge assignment Δ_a for the three chiral fields ϕ_i. Since the superpotential has R-charge 2, we must have $\Delta_1 + \Delta_2 + \Delta_3 = 2$. The exact R-symmetry of an $\mathcal{N} = 1$ superconformal theory can be obtained by extremizing a trial a-charge

$$a(\Delta_a) = \frac{9}{32} \operatorname{Tr} R(\Delta_a)^3 - \frac{3}{32} \operatorname{Tr} R(\Delta_a), \tag{5.6}$$

where $R(\Delta_a)$ is the matrix of R-charges of the fermionic fields. This construction is known as a-maximization (Intriligator and Wecht 2003). For $\mathcal{N} = 4$ SYM at large N we find[55]

$$a(\Delta_a) = \frac{9}{32} N^2 \left(1 + \sum_{a=1}^{3} (\Delta_a - 1)^3\right) = \frac{27}{32} N^2 \Delta_1\Delta_2\Delta_3, \tag{5.7}$$

where the first contribution in the bracket comes from the gauginos and the second from the fermions in the chiral multiplets. This expression is trivially extremized for $\Delta_1 = \Delta_2 = \Delta_3 = 2/3$, the exact R-charges of the fields ϕ_i. The critical value of $a(\Delta)$ is the central charge a of the $SCFT_4$. Similarly, the exact R-symmetry and the right-moving central charge of a two-dimensional (0, 2) SCFT are obtained by extremizing the trial quantity (Benini and Bobev 2013a)

[55] For $\mathcal{N} = 4$ SYM $\operatorname{Tr} R$ is identically zero. For theories with an AdS dual, $\operatorname{Tr} R = 0$ in the large N limit.

$$c_r(\Delta_a) = 3\,\mathrm{Tr}\,\gamma_3 R(\Delta_a)^2, \tag{5.8}$$

where γ_3 is the two-dimensional chirality operator and $R(\Delta_a)$ the matrix of R-charges of the massless fermionic fields. This construction is known as c-extremization (Benini and Bobev 2013a). For the twisted compactification of $\mathcal{N}=4$ SYM, the quantity c_r can be computed using topological arguments. We just need to know the difference between the number of fermionic zero modes with positive and negative chirality. This is easily computed from the Riemann–Roch theorem as in Sect. 3.3.1. We then find at large N (Benini and Bobev 2013b)

$$\begin{aligned} c_r(\Delta_a) &= -3\,N^2\left(1-\mathfrak{g}+\sum_{a=1}^{3}(\mathfrak{p}_a-1+\mathfrak{g})(\Delta_a-1)^2\right) \\ &= -3N^2(\Delta_1\Delta_2\mathfrak{p}_3+\Delta_2\Delta_3\mathfrak{p}_1+\Delta_3\Delta_1\mathfrak{p}_2), \end{aligned} \tag{5.9}$$

where again the first contribution in the first line bracket comes from gauginos and the second from matter fields. One can easily check that the extremization of (5.9) with respect to Δ_a reproduces (5.4). The agreement is valid in the large N limit where $c = c_l = c_r$.[56]

Notice that, in the large N limit, the c_r trial central charge can be written as (Hosseini et al. 2017c)

$$c_r(\Delta_a) = -\frac{32}{9}\sum_{a=1}^{3}\mathfrak{p}_a\frac{\partial a(\Delta_a)}{\partial \Delta_a}\,. \tag{5.10}$$

It is interesting to observe that formula (5.10), with a suitable parameterization for the fluxes and the R-charges, holds for all the twisted compactifications of $\mathcal{N}=1$ SCFT_4 dual to $\mathrm{AdS}_5\times SE_5$, where SE_5 is a toric Sasaki–Einstein manifold (Hosseini et al. 2017c).[57]

In the case of black strings the information about states is encoded in the *elliptic genus* of the two-dimensional (0, 2) CFT

$$Z(y,q) = \mathrm{Tr}(-1)^F q^{L_0}\prod_I y_I^{J_I}, \tag{5.11}$$

where $q = e^{2\pi i\tau}$, with τ the modular parameter of T^2, y_I are fugacities for the global symmetries and L_0 the left-moving Virasoro generator. Since the index is independent of the scale, we can evaluate it in the UV, where it becomes the topologically twisted index, defined as the partition function on $\Sigma_\mathfrak{g}\times T^2$, with a topological twist along $\Sigma_\mathfrak{g}$.

56 $c_r - c_l$ is equal to the gravitational anomaly $k = \mathrm{Tr}\gamma_3$ which is of order one in the large N limit (Benini and Bobev 2013a).

57 We refer to Hosseini et al. (2017c) for a proof, to Amariti et al. (2018) for more detailed expressions, and to the appendices of Hosseini et al. (2018c) and Hosseini and Zaffaroni (2019b) for an alternative derivation based on the anomaly polynomial.

5.2 The topologically twisted index on $T^2 \times \Sigma_\mathfrak{g}$

For an $\mathcal{N} = 1$ four-dimensional gauge theory with a non-anomalous U(1) R-symmetry, the topologically twisted index is a function of $q = e^{2\pi i \tau}$, where τ is the modular parameter of T^2, fugacities y_I for the global symmetries and flavor magnetic fluxes $\mathfrak{m}_I^F$ on $\Sigma_\mathfrak{g}$ parameterizing the twist. It can be computed using localization and it is given by an elliptic generalization of the formulae in Sect. 3.2 (Closset and Shamir 2014; Benini and Zaffaroni 2015). Explicitly, for a theory with gauge group G and a set of chiral multiplets transforming in representations $\mathfrak{R}_I$ of G with R-charge r_I, the topologically twisted index is given by a contour integral of a meromorphic form

$$Z(\mathfrak{p}, y) = \frac{1}{|W|} \sum_{\mathfrak{m} \in \Gamma_\mathfrak{h}} \oint_{\mathcal{C}} \prod_{\text{Cartan}} \left(\frac{dx}{2\pi i x} \eta(q)^{2(1-\mathfrak{g})} \right) \prod_{\alpha \in G} \left[\frac{\theta_1(x^\alpha; q)}{i\eta(q)} \right]^{1-\mathfrak{g}} \\ \times \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \left[\frac{i\eta(q)}{\theta_1(x^{\rho_I} y_I; q)} \right]^{\rho_I(\mathfrak{m}) + (\mathfrak{g}-1)(r_I - 1) + \mathfrak{m}_I^F} \left(\det_{ij} \frac{\partial^2 \log Z_{\text{pert}}(u, \mathfrak{m})}{\partial i u_i \partial \mathfrak{m}_j} \right)^{\mathfrak{g}}, \tag{5.12}$$

where α are the roots of G, ρ_I the weights of the representation $\mathfrak{R}_I$ and $|W|$ denotes the order of the Weyl group. In this formula, $\theta_1(x; q)$ is a Jacobi theta function and $\eta(q)$ is the Dedekind eta function. The zero-mode gauge variables $x = e^{iu}$ parameterize the Wilson lines on the two directions of the torus

$$u = 2\pi \oint_{A-\text{cycle}} A - 2\pi\tau \oint_{B-\text{cycle}} A, \tag{5.13}$$

and are defined modulo

$$u_i \sim u_i + 2\pi n + 2\pi m \tau, \qquad n, m \in \mathbb{Z}. \tag{5.14}$$

As in three dimensions, the result is summed over a lattice of gauge magnetic fluxes $\mathfrak{m}$ living in the co-root lattice $\Gamma_\mathfrak{h}$ of the gauge group G and the contour of integration selects the Jeffrey–Kirwan prescription for taking the residues. In four dimensions there is a one-loop contribution from the Cartan components of the vector multiplets. One can show that the integrand in (5.12) is a well-defined meromorphic function of x on the torus provided that the gauge and the gauge-flavor anomalies vanish. The index has a trace interpretation as a sum over a Hilbert space of states on $\Sigma_\mathfrak{g} \times S^1$

$$Z(\mathfrak{n}, y, q) = \text{Tr}_{\Sigma_\mathfrak{g} \times S^1} (-1)^F q^{H_L} \prod_I y_I^{J_I}. \tag{5.15}$$

This trace reduces to the elliptic genus of the two-dimensional theory obtained by the twisted compactification on $\Sigma_\mathfrak{g}$.

We now consider the index for $\mathcal{N}=4$ SYM. The superpotential (5.5) imposes the following constraints on the chemical potentials Δ_a and the flavor magnetic fluxes $-\mathfrak{p}_a+1-\mathfrak{g}=(\mathfrak{g}-1)(r_a-1)+\mathfrak{m}_a^F$ associated with the fields ϕ_a,

$$\sum_{a=1}^{3}\Delta_a\in 2\pi\mathbb{Z}, \qquad \sum_{a=1}^{3}\mathfrak{p}_a=2-2\mathfrak{g}\,. \tag{5.16}$$

The topologically twisted index for the SYM theory with gauge group $SU(N)$ is then given by

$$Z=\frac{1}{N!}\sum_{\substack{\mathfrak{m}\,\in\,\mathbb{Z}^N,\\ \sum_i\mathfrak{m}_i=0}}\int_{\mathcal{C}}\prod_{i=1}^{N-1}\frac{dx_i}{2\pi i x_i}\eta(q)^{2(N-1)(1-\mathfrak{g})}\prod_{i,j=1}^{N}\left(\frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)}\right)^{1-\mathfrak{g}} \prod_{i,j=1}^{N}\prod_{a=1}^{3}\left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{\mathfrak{m}_i-\mathfrak{m}_j-\mathfrak{p}_a+1-\mathfrak{g}}\left(\det_{ab}\frac{\partial^2\log Z_{\text{pert}}(u,\mathfrak{m})}{\partial i u_a\partial\mathfrak{m}_b}\right)^{g}, \tag{5.17}$$

with $y_a=e^{i\Delta_a}$. The Bethe vacua are determined by

$$e^{i\frac{\partial\mathcal{W}}{\partial u_i}}=\prod_{j=1}^{N}\prod_{a=1}^{3}\frac{\theta_1\left(\frac{x_j}{x_i}y_a;q\right)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}=1\,. \tag{5.18}$$

The Bethe equations (5.18) have a remarkably simple solution (Hosseini et al. 2017c; Hong and Liu 2018)[58]

$$u_k=-\frac{2\pi\tau}{N}\left(k-\frac{N+1}{2}\right). \tag{5.19}$$

These Bethe vacua will play a role also in the physics of AdS_5 black holes (Benini and Milan 2020a, b), as discussed in Sect. 7.2.

This time the index is too hard to solve in the large N limit. We can study instead the *high temperature limit* corresponding to a shrinking of the torus given by $\tau=i\beta/(2\pi)$ with $\beta\to 0^+$ (Hosseini et al. 2017c).[59] This limit is also particularly interesting from the field theory point of view because it controls the asymptotic growing of the number of states with the dimension (Cardy 1986). Using (5.19) it is easy to compute, at leading order in β, (Hosseini et al. 2017c)

[58] This solution was found in the high temperature limit in Hosseini et al. (2017c) and proved to be exact for all τ in Hong and Liu (2018). Using $SL(2,\mathbb{Z})$, the authors of Hong and Liu (2018) have found many other solutions of the Bethe equations for $\mathcal{N}=4$ SYM at finite τ. Non-standard solutions corresponding to continua of Bethe vacua have been found in Arabi Ardehali et al. (2019). The solution (5.19) extends to more general quivers (Hosseini et al. 2017c; González Lezcano and Pando Zayas 2020; Lanir et al. 2020).

[59] Notice that β is not really a temperature, but rather a parameterization of the modular parameter of the torus.

$$\widetilde{\mathcal{W}}(\Delta) = \frac{i(N^2-1)}{2\beta}\Delta_1\Delta_2\Delta_3,$$
$$\log Z(\mathfrak{p}, \Delta) = i\sum_{a=1}^{3} \mathfrak{p}_a \frac{\partial\widetilde{\mathcal{W}}}{\partial\Delta_a}, \tag{5.20}$$

where $\widetilde{\mathcal{W}}(\Delta)$ is the on-shell value of the twisted superpotential. As in the three-dimensional case, the result is valid for $\sum_a \Delta = 2\pi$. We see a striking similarity with the black hole case, in particular equation (4.63). Moreover, the twisted superpotential is proportional to the trial a-central charge (5.7) of the four-dimensional SCFT. This statement is the four-dimensional analog of (4.61).

Comparing with (5.10) we find the *Cardy formula*

$$\log Z(\mathfrak{p}, \Delta) = \frac{\pi^2}{6\beta}c_r(\Delta) = \frac{\pi i}{12\tau}c_r(\Delta), \tag{5.21}$$

valid at leading order in β and at large N.

This result can be extended to all $\mathcal{N}=1$ $SCFT_4$ dual to $AdS_5 \times SE_5$ vacua (Hosseini et al. 2017c). It can be also generalized to the case of the *refined* topologically twisted index on $S^2 \times T^2$

$$Z(\mathfrak{n}, y, q) = \mathrm{Tr}_{S^2\times S^1}(-1)^F q^{H_L}\zeta^{2J}\prod_I y_I^{J_I}, \tag{5.22}$$

where $\zeta = e^{i\omega/2}$ is a fugacity for the angular momentum. The result is (Hosseini et al. 2019b)

$$\log Z(\mathfrak{p}, \Delta) = \frac{\pi^2}{6\beta}\left(c_r(\Delta) - \frac{8\omega^2}{9\pi^2}a(\mathfrak{p})\right), \tag{5.23}$$

valid at leading order in β and at large N.

The example discussed in this section can be actually generalized to many other flows interpolating between AdS_{3+n} and $AdS_3 \times \mathcal{M}_n$ where supersymmetry is preserved along the compact manifold $\mathcal{M}_n$ with a topological twist. Examples of compactifications of the (2, 0) theory in six dimensions on the product of two Riemann surfaces $\Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ are discussed for example in Hosseini et al. (2018c).

5.3 Back to Cardy

Similarly to what is done for generic CFT_2 (Cardy 1986), we can extract information on the growing of the number of supersymmetric states with the energy from the asymptotic behavior (5.21) of the elliptic genus.

From the definition of the index as a trace (5.22), we see that the number of supersymmetric states with momentum n, electric charge $\mathfrak{q}_a$ under the Cartan subgroup of $SO(6)$ and angular momentum j can be extracted as a Fourier coefficient

$$d(\mathfrak{p},n,j,\mathfrak{q}) = -i\int_{i\mathbb{R}}\frac{d\beta}{2\pi}\int_0^{2\pi}\frac{d\Delta_a}{2\pi}\,Z(\mathfrak{p},\Delta)\,e^{\beta n-i\sum_{a=1}^{3}\Delta_a\mathfrak{q}_a-i\omega j}\delta\left(\sum_{a=1}^{3}\Delta=2\pi\right), \tag{5.24}$$

where $\beta=-2\pi i\tau$ and the corresponding integration is over the imaginary axis.

In the limit of large charges, we can use the saddle point approximation. Consider first $\mathfrak{q}=0$ and $j=0$. The number of supersymmetric states with charges $(\mathfrak{p},n)$ can be obtained by extremizing

$$\mathcal{I}(\beta,\Delta)\equiv\log Z(\mathfrak{p},\Delta)+\beta n \tag{5.25}$$

with respect to Δ and β. Given (5.21), we see that the extremization with respect to Δ is the c-extremization principle (Benini and Bobev 2013a, b) and sets the trial right-moving central charge $c_r(\Delta)$ to its *exact* value $c_{\mathrm{CFT}}(\mathfrak{p})$ given in (5.4). Extremizing $\mathcal{I}(\beta,\Delta)$ with respect to β yields

$$\beta(\mathfrak{p},n)=\pi\sqrt{\frac{c_{\mathrm{CFT}}(\mathfrak{p})}{6n}}. \tag{5.26}$$

Plugging back (5.26) into $\mathcal{I}(\beta,\Delta)$, we find for the entropy of states

$$S(\mathfrak{p},n)\equiv\log d(\mathfrak{p},n,0,0)=\mathcal{I}\big|_{\mathrm{crit}}(\beta,\Delta)=2\pi\sqrt{\frac{n\,c_{\mathrm{CFT}}(\mathfrak{p})}{6}}. \tag{5.27}$$

This is obviously nothing else than Cardy formula (Cardy 1986).[60]

One can generalize the previous computation to the case $\mathfrak{q}_a\neq0$ and $j\neq0$ by extremizing

$$\mathcal{I}(\beta,\Delta)\equiv\log Z(\mathfrak{p},\Delta)+\beta n-i\sum_a\mathfrak{q}_a\Delta_a-i\omega j\,. \tag{5.28}$$

After some manipulations, the result can be expressed in the form Hosseini et al. (2019b, 2020)

$$S(\mathfrak{p},\mathfrak{q},n)=2\pi\sqrt{\frac{c_{\mathrm{CFT}}}{6}\left(n+\frac{1}{2}\sum_{I,K}q_IA^{-1}_{IK}(\mathfrak{p})q_K-\frac{27}{16a(\mathfrak{p})}j^2\right)}, \tag{5.29}$$

where the index I runs over a set of independent global symmetries J_K and $A_{IK}=\mathrm{Tr}\,\gamma_3J_IJ_K$ is the 't Hooft anomaly matrix of the two-dimensional theory. This result holds for a generic $\mathcal{N}=1$ quiver with a Sasaki-Einstein dual. The particular combination of n, electric charges and angular momentum appearing in (5.29) is related to the properties of the elliptic genus and is familiar from the physics of asymptotically flat black holes (Dijkgraaf et al. 2000; Kraus and Larsen 2007; Manschot and Moore 2010).[61]

60 For a further discussion about the asymptotic behavior see Hosseini (2018–02).

61 Our elliptic genus is actually a meromorphic function of Δ_a and this leads to a more complicated structure of the corresponding modular forms, related to wall-crossing phenomena. For asymptotically flat

5.4 Cardy formula and black holes

There is standard argument to obtain a black holes from a black string: compactify along the circle inside AdS_3 and add a momentum n along it. More precisely, one replaces the near-horizon geometry of the five-dimensional black string with $\mathrm{BTZ} \times \Sigma_{\mathfrak{g}_1}$, where the metric for the extremal BTZ reads (Bañados et al. 1992)

$$ds_3^2 = \frac{1}{4}\left(\frac{-dt^2 + dr^2}{r^2}\right) + \rho\left[dz + \left(-\frac{1}{4} + \frac{1}{2\rho r}\right)dt\right]^2 . \tag{5.30}$$

Here, the parameter ρ is related to the electric charge n. This solution is *locally* equivalent to AdS_3, since there exists locally only one constant curvature metric in three dimensions, and solves the same BPS equations; however, BTZ and AdS_3 are inequivalent globally. Compactifying the full five-dimensional black string of Benini and Bobev (2013b) on the circle with the extra momentum we obtain a static BPS black hole in four dimensions, with magnetic charges $\mathfrak{p}_a$ and electric charge n. This can be thought as a domain wall that interpolates between an $\mathrm{AdS}_2 \times \Sigma_{\mathfrak{g}_1}$ near-horizon region and a complicated asymptotic *non-AdS* $_4$ vacuum (Hristov 2014). By another standard argument, the entropy of such a black hole is given by the number of states of the CFT with momentum n, and is therefore given by the Cardy formula (5.27)

$$S_{\mathrm{BH}}(\mathfrak{p}, n) = \mathcal{I}\big|_{\mathrm{crit}}(\beta, \Delta) = 2\pi\sqrt{\frac{n\, c_{\mathrm{CFT}}(\mathfrak{p})}{6}} . \tag{5.31}$$

One can see that this prediction matches the black holes entropy computed from supergravity (Hristov 2014). Moreover, a family of rotating dyonic black strings in $\mathrm{AdS}_5 \times S^5$ was found in Hosseini et al. (2019b) and the entropy of the compactified black hole successfully compared with (5.29). An analogous matching for black strings in $\mathrm{AdS}_7 \times S^4$ was discussed in Hosseini et al. (2020).

It is also interesting to observe that, for static black holes, the field theory extremization of $\mathcal{I}$ becomes again equivalent to the attractor mechanism in the reduced four-dimensional gauged supergravity. The dimensional reduced supergravity has one more vector coming from the reduction on the circle, prepotential $\mathcal{F} \sim \frac{X^1X^2X^3}{X^0}$, and purely electric FI parameters $g_0 = 0$ and $g_i = g$ (Hristov 2014). Under the identifications $\hat{X}^0 \rightarrow \beta, \hat{X}^i \rightarrow i\Delta_a, (\mathfrak{q}_\Lambda, \mathfrak{p}^\Lambda) \rightarrow (-n, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, 0, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ and $\mathcal{F}(\hat{X}) \rightarrow \widetilde{\mathcal{W}}(\Delta)$, the attractor functional (4.13) can be identified with the $\mathcal{I}$-functional (5.25).

Footnote 61 continued
black holes this leads to interesting behaviours (see for example Dabholkar et al. 2012). It would be interesting to see if the same happens for the black holes discussed in Sect. 5.4 obtained by compactifying the black string.

6 The superconformal index

As we discussed in Sect. 2, we should be able to obtain the entropy of supersymmetric Kerr–Newman black holes in AdS_{d+1} by counting states in the dual CFT on $S^d \times \mathbb{R}$. The relevant supersymmetric quantity to consider is the *superconformal index*, which enumerates BPS states on $S^d \times \mathbb{R}$ (Romelsberger 2006; Kinney et al. 2007).

We start by considering $\mathcal{N} = 1$ supersymmetric field theories on $S^3 \times \mathbb{R}$. We refer to Festuccia and Seiberg (2011) for an explicit description of the supersymmetric Lagrangian on $S^3 \times \mathbb{R}$ and the corresponding supersymmetry transformations. The superconformal index is defined as the trace

$$I(p, q, u) = \mathrm{Tr}_{S^3}(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} p^{J_1+\frac{R}{2}} q^{J_2+\frac{R}{2}} u^J \tag{6.1}$$

on the Hilbert space of states on S^3. Here J_1 and J_2 generate rotations on S^3, R is the R-symmetry generator, and J denotes collectively the global symmetries. $\mathcal{Q}$ is a supercharge with $J_1 = J_2 = -1/2$ and $R = 1$ and satisfies the algebra

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = \Delta - J_1 - J_2 - \frac{3}{2}R, \tag{6.2}$$

where Δ generates translation along $\mathbb{R}$. We introduced fugacities, p, q, and u, for all the generators that commute with $\mathcal{Q}$, which are $J_1 + R/2$, $J_2 + R/2$ and the global symmetries J. The quantity (6.1) is a Witten index in the sense discussed in Sect. 2.2.2 and it is therefore independent of β and invariant under continuous small deformations of the Lagrangian. Despite the name, the supersymmetric index (6.1) makes sense also for non-conformal theories. Since we are interested in holography, we will just consider the case of conformal theories where Δ can be identified with the dilatation operator.

The index (6.1) can be also written as the Euclidean supersymmetric partition function on $S^3 \times S^1$, as discussed in Sect. 2.2.2, and computed using localization. The precise relation among the two quantities involves a prefactor

$$\mathcal{Z}^{\text{susy}}_{S^3\times S^1}(p, q, u) = e^{-\beta E_{c.e.}} I(p, q, u)\,, \tag{6.3}$$

where the quantity $E_{c.e}$, called supersymmetric Casimir energy (Assel et al. 2014, 2015; Lorenzen and Martelli 2015; Benetti Genolini et al. 2017; Martelli and Sparks 2016; Closset et al. 2019), is due to the regularization of the one-loop determinants and it can be interpreted as the vacuum expectation value of the Hamiltonian.

The superconformal index can be computed explicitly through localization or, being invariant under continuous deformations, just by enumerating the gauge invariant states annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ in the weakly coupled UV theory. The result for an $\mathcal{N} = 1$ theory with gauge group G and chiral matter in the representation $\mathcal{R}_a$ and R-charge r_a is (Romelsberger 2006; Kinney et al. 2007; Dolan and Osborn 2009)

$$I(p,q,u) = \frac{(p,p)_\infty^r (q,q)_\infty^r}{|W|} \prod_{i=1}^r \oint \frac{dz_i}{2\pi i z_i} \frac{\prod_a \prod_{\rho\in\mathcal{R}_a} \Gamma((pq)^{r_a/2} z^{\rho_a} u^{\nu_a}; p,q)}{\prod_\alpha \Gamma(z^{\rho_\alpha}; p,q)}, \quad (6.4)$$

where r is the rank of the gauge group, α the roots, ρ_a are the weights of the representation $\mathcal{R}_a$, ν_a the flavor weights, and $|W|$ the order of the gauge group. The integration is over the Cartan subgroup of G and is taken over the unit circle for all variables z_i. The special functions appearing in the previous formula are the elliptic Gamma function (Felder and Varchenko 2000)

$$\Gamma(z;p,q) = \prod_{n,m=0}^\infty \frac{1-p^{n+1}q^{m+1}/z}{1-p^n q^n z}\,, \qquad |p|<1\,, |q|<1\,, \quad (6.5)$$

and the q-Pochhammer symbol

$$(z;q)_\infty = \prod_n^\infty (1-q^n z)\,, \qquad |q|<1\,. \quad (6.6)$$

In the localization approach, numerator and denominator of (6.4) arise as one-loop determinants for chiral matter multiplets and vector multiplets, respectively. The reader can compare the formula for the superconformal index (6.4) with the analogous one for the topologically twisted one (3.11) and look for analogies and differences.[62]

The superconformal index can be defined and computed through localization also in other dimensions (Bhattacharya et al. 2008; Pestun and Zabzine 2017). For three-dimensional theories the elliptic Gamma functions are replaced by hyperbolic ones and there is an extra sum over magnetic fluxes, as in (3.11), due to the existence of local BPS monopole operators in three dimensions (Kim 2009; Imamura and Yokoyama 2011; Kapustin and Willett 2011; Dimofte et al. 2013).

7 Electrically charged rotating black holes

In this section we discuss the case of Kerr–Newman black holes in various dimensions. These are electrically charged rotating black holes without a twist. As argued in Sect. 2, they are qualitatively different from the magnetically charged black holes discussed in Sect. 4.

In general, we should be able to recover the entropy of the electrically charged rotating black holes in AdS_d from the BPS partition function (2.16) that counts supersymmetric states of the dual CFT on $S^{d-2} \times \mathbb{R}$. Since the black holes preserve just two real supercharges, we need to count 1/16 BPS states and this is a hard problem. Old attempts to evaluate the BPS partition function of $\mathcal{N}=4$ SYM (Grant et al. 2008; Chang and Yin 2013; Yokoyama 2014) reached the somehow disappointing conclusion that there is a subset of states whose number grows with N but slower than the entropy of the black holes. Alternatively, one may try to

[62] One main difference is the sum over magnetic fluxes in the (3.11). From a more technical point of view, another big difference is the integration contour.

replace the BPS partition function with the corresponding index (2.23). The appropriate index is the superconformal one, defined in the previous section, that can be expressed and computed as a supersymmetric Euclidean path integral on $S^{d-2} \times S^1$. It is known that, for generic real fugacities, the superconformal index is a quantity of order one in the large N limit (Kinney et al. 2007) and, as such, it does not reproduce the entropy which grows with powers of N. As a difference with the twisted index, already in the large N limit, there is a large cancellation between bosonic and fermionic supersymmetric states and $\mathcal{Z}_{\text{index}}(\Delta_a, \omega_i) \neq \mathcal{Z}(\Delta_a, \omega_i)$. All these (partial) results have stood for long time as puzzles about supersymmetric electrically charged rotating black holes.

However, the entropy functionals for many Kerr–Newman black holes in different dimensions can be expressed in terms of quantities with a clear field theory interpretation (Hosseini et al. 2017b, 2018b; Choi et al. 2020), thus suggesting that the entropy can be always reproduced by a field theory computation. These entropy functionals also suggest complex value for the chemical potentials Δ_a and ω_i. As stressed in Choi et al. (2018b) and Benini and Milan (2020b), the computation in Kinney et al. (2007) is valid only for *real fugacities* and the introduction of phases in the fugacities can obstruct the cancellation at large N and lead to an enhancement of the entropy. This is also in the spirit of the $\mathcal{I}$-extremization principle discussed in Sect. 2.4. Recent results for AdS_5 and other dimensions, started with the work of Cabo-Bizet et al. (2019a), Choi et al. (2018b) and Benini and Milan (2020b), confirm this point of view and lead to various derivations of the entropy using the index, as we discuss in this section.

7.1 The entropy functional for electrically charged rotating black holes

For many Kerr–Newman black holes, the entropy, as a function of electric charges q_a and angular momenta j_i, can be written as a Legendre transform

$$S(q_a, j_i) = \log \mathcal{Z}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i) \qquad (7.1)$$

of a quantity $\log \mathcal{Z}(\Delta_a, \omega_i)$ related to anomalies or free energies of the dual CFT. This was derived in Hosseini et al. (2017b) for five-dimensional black holes and generalized to other dimensions in Hosseini et al. (2018b) and Choi et al. (2020).[63] We consider first the example of $AdS_5 \times S^5$ and we come back later to the general case.

7.1.1 The entropy functional for Kerr–Newman black holes in $AdS_5 \times S^5$

As discussed in Sect. 2.1.1, the Kerr–Newman black holes in $AdS_5 \times S^5$ depend on three charges q_1, q_2, q_3 associated with $U(1)^3 \subset SO(6)$, the Cartan subgroup of the isometry of S^5, and two angular momenta j_1 and j_2 in AdS_5. Supersymmetry requires a non-linear constraint among these conserved charges, whose form we discuss below, and leaves a four-dimensional family of BPS black holes (Gutowski

[63] A proposal for non BPS black holes can be found in Larsen et al. (2020).

and Reall 2004a, b; Chong et al. 2005a, b; Kunduri et al. 2006). The entropy can be compactly written as (Kim and Lee 2006)

$$S_{\rm BH}(q_a,j_i) = 2\pi\sqrt{q_1q_2+q_1q_3+q_2q_3-\frac{\pi}{4G_{\rm N}^{(5)}g^3}(j_1+j_2)}, \tag{7.2}$$

where $G_{\rm N}^{(5)}$ is the five-dimensional Newton constant and g the gauge coupling of the five-dimensional effective supergravity. Holography relates these quantities to the number of colors of the dual field theory, $\mathcal{N}=4$ SYM in four dimensions, through $N^2=\pi/(2G_{\rm N}^{(5)}g^3)$. We see that black holes with charges and angular momenta of order $O(N^2)$ have an entropy of order $O(N^2)$.

Quite remarkably, the entropy (7.2) can be written as the Legendre transform of a very simple quantity (Hosseini et al. 2017b)

$$S_{\rm BH}(q_a,j_i) = -i\frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} - i\left(\sum_{a=1}^{3}\Delta_a q_a + \sum_{i=1}^{2}\omega_i j_i\right)\Big|_{\text{extremum }\bar{\Delta}_a,\bar{\omega}_i}, \tag{7.3}$$

where the chemical potentials are constrained by

$$\Delta_1+\Delta_2+\Delta_3-\omega_1-\omega_2 = \pm 2\pi\,. \tag{7.4}$$

This constraint resembles the analogous one, (4.42), for black holes in ABJM. The quantity in (7.3) is extremized for *complex values* of the chemical potentials Δ_a and ω_i. However, quite remarkably, the on-shell value (7.3) becomes real once we impose the non-linear constraint among charges imposed by supersymmetry. In fact, a simple way of characterize the constraint on charges is to identify it with the imaginary part of entropy functional (7.3) at its extremum. The fact that the critical value for Δ_a and ω_i are complex will also play an important for the field theory interpretation of the result. The two choice of signs in (7.4) lead to the same final result. Formally, this is due to the fact that (7.3) is a holomorphic homogeneous function of the chemical potentials of degree one.[64]

The quantity

$$\log\mathcal{Z}(\Delta_a,\omega_i) = -i\frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} \tag{7.5}$$

must be interpreted as a grand canonical partition function, and, as discussed in 2.3, should be related to the on-shell action of the Euclidean black hole. This has been proved in Cabo-Bizet et al. (2019a) by taking the zero-temperature limit of the on-shell action of a family of supersymmetric complexified non-extremal Euclidean solutions. In this approach, one can also derive the complex value for the chemical

[64] Consider the two functionals $S_\pm = S_{\rm BH} - i\Lambda(\sum_{a=1}^{3}\Delta_a - \sum_{i=1}^{2}\omega_i \mp 2\pi)$, where the constraint is enforced by the Lagrange multiplier Λ. Given the homogeneity of $S_{\rm BH}$, the extremal value is $S_\pm = \pm 2\pi i\Lambda$. Since q_a and j_i are real, it is immediate to see that, if $(\Delta_a,\omega_i,\Lambda)$ is an extremum of S_+, then $(-\bar{\Delta}_a,-\bar{\omega}_i,\bar{\Lambda})$ is an extremum of S_- and the extremal values are related by $S_+=\bar{S}_-$. Therefore the extremization with different constraints (7.4) give the same results for the real part of the entropy.

potentials at the saddle point.[65] The constraints $\sum_a \Delta_a - \sum_i \omega_i = \pm 2\pi$ arise due to regularity conditions to be imposed on the Killing spinors.

7.1.2 The entropy functional for Kerr–Newman black holes in diverse dimensions

The expression for the entropy functionals for Kerr–Newman black holes in diverse dimensions is schematically given in Table 1.

The content of the table refers to electrically charged rotating black holes in each dimension that can be embedded in a maximally supersymmetric string theory or M-theory background. In addition to the $\mathrm{AdS}_5 \times S^5$ black holes in type IIB, there are analogous $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ black holes in M-theory (Chow 2008; Cvetič et al. 2005; Chong et al. 2005c). The dual field theories are well known: the ABJM theory in three dimensions and the (2, 0) theory in six dimensions. In addition, there are black holes in the warped $\mathrm{AdS}_6 \times_W S^4$ background of massive type IIA (Brandhuber and Oz 1999; Chow 2010). The dual CFT is the $\mathcal{N} = 1$ five-dimensional fixed point associated with D4-D8-O8 branes in type IIA found in Seiberg (1996). Notice that, in six dimensions, the maximal superconformal algebra has only sixteen supercharges instead of thirty-two and the superconformal theory with such an algebra is not unique. Among the theories with an AdS dual, the D4-D8-O8 system is somehow the simplest and most studied.

The chemical potentials in the table refer to the isometries of the internal manifold.[66] Notice that the chemical potentials are always subject to a constraint. Indeed, as already discussed in Sect. 2, in all dimensions, we expect the existence of a family of supersymmetric black holes depending on the possible electric charges and spins with a constraint among them.[67] This explains the constraint among chemical potentials. The family of black holes with generic charges and spins allowed by the constraint is known only for $\mathrm{AdS}_5 \times S^5$ (and $\mathrm{AdS}_4 \times S^7$) and (7.1) has been fully checked only in these cases (Hosseini et al. 2017b). In all other dimensions the relation (7.1) has been checked for the solutions available in the literature (Hosseini et al. 2018b; Choi et al. 2020).

In the second column of the table, there is a quantity, $F(\Delta_a)$, with a clear field theory interpretation. The reader can recognize the S^3 free energy of ABJM in the first row and the trial a-charge of $\mathcal{N} = 4$ SYM in the second row, see (4.43) and (5.7), which are both functions of trial R-charges satisfying

[65] The chemical potentials are extracted following the logic discussed at the end of Sect. 2.2.1.

[66] In the case of the D4-D8-O8 system, the field theory has $SU(2)_R \times SU(2) \times E_{n_f+1}$ symmetry, where $SU(2)_R \times SU(2)$ is realized by the isometry of the warped S^4 and E_{n_f+1} by the theory on the N_f physical D8 branes. We introduced a symmetric notation for the chemical potentials associated with $SU(2)_R \times SU(2)$ following the notations of Hosseini et al. (2018c). The entropy functional discussed in Choi et al. (2020) refers to the case $\Delta_1 = \Delta_2$.

[67] Supersymmetric hairy AdS_5 black holes depending on all the charges has been recently found in Markeviciute and Santos (2019) and Markeviciute (2019). Their entropy seems to be subleading compared to the Kerr–Newman black hole.

Table 1 Entropy functionals for electrically charged rotating black holes

$AdS_4 \times S^7$	$F(\Delta_a) = \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$	$\log \mathcal{Z}(\Delta_a, \omega_i) = -\frac{4\sqrt{2}N^{3/2}}{3}\frac{\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}}{\omega_1}$
	$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 2$	$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - \omega_1 = 2\pi$
$AdS_5 \times S^5$	$F(\Delta_a) = \Delta_1\Delta_2\Delta_3$	$\log \mathcal{Z}(\Delta_a, \omega_i) = -i\frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2}$
	$\Delta_1 + \Delta_2 + \Delta_3 = 2$	$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi$
$AdS_6 \times_W S^4$	$F(\Delta_a) = (\Delta_1\Delta_2)^{3/2}$	$\log \mathcal{Z}(\Delta_a, \omega_i) \sim N^{5/2}\frac{(\Delta_1\Delta_2)^{3/2}}{\omega_1\omega_2}$
	$\Delta_1 + \Delta_2 = 2$	$\Delta_1 + \Delta_2 - \omega_1 - \omega_2 = 2\pi$
$AdS_7 \times S^4$	$F(\Delta_a) = (\Delta_1\Delta_2)^3$	$\log \mathcal{Z}(\Delta_a, \omega_i) = i\frac{N^3}{24}\frac{(\Delta_1\Delta_2)^3}{\omega_1\omega_2\omega_3}$
	$\Delta_1 + \Delta_2 = 2$	$\Delta_1 + \Delta_2 - \omega_1 - \omega_2 - \omega_3 = 2\pi$

For simplicity of notations, we opted for a uniform notation for all dimensions, involving some sign redefinitions in comparison to Hosseini et al. (2017b, 2018b) and Choi et al. (2020), to which we refer for more precise statements

$$\sum_a \Delta_a = 2\,. \tag{7.6}$$

In general, $F(\Delta_a)$ is the sphere free energy for odd-dimensional CFTs, and a particular combination of t'Hooft anomaly coefficients in even dimensions.[68] In all cases, the quantity $\log \mathcal{Z}(\Delta_a, \omega_i)$ can be obtained by taking the quotient of $F(\Delta)$ by the product of all angular momentum chemical potentials and by replacing the R-charge constraint (7.6) with

$$\sum_a \Delta_a - \sum_i \omega_i = 2\pi\,. \tag{7.7}$$

This constraint is strongly reminiscent of the analogous constraint (4.42) for magnetically charged black holes. Notice that $\log \mathcal{Z}(\Delta_a, \omega_i)$ is a sort of equivariant generalization of $F(\Delta)$ with respect to rotations. Indeed, the expression for $\log \mathcal{Z}(\Delta_a, \omega_i)$ for AdS_5 and AdS_7 can be also directly obtained by an equivariant integration of the six-dimensional and eight-dimensional anomaly polynomial for $\mathcal{N} = 4$ SYM and the (2, 0) theory in six dimensions, respectively (Bobev et al. 2015).[69]

Various observations made for $AdS_5 \times S^5$ generalize to the other dimensions. First, the quantity in (7.1) is extremized for complex values of the chemical

[68] It is also curious to observe that $F(\Delta)$ is the on-shell value of the twisted superpotential of the CFT in three and four-dimensions, as discussed in Sects. 4 and 5, and the on-shell Seiberg–Witten prepotential in five- and six-dimensional computations (Hosseini et al. 2018c).

[69] The expression for $\log \mathcal{Z}(\Delta_a, \omega_i)$ for AdS_4 and AdS_6 can be instead related to a small ϵ limit of the partition functions on $\mathbb{R}^2_{\epsilon_1} \times S^1$ and $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2} \times S^1$, respectively, where $\epsilon_i \propto \omega_i$ are equivariant parameters for rotations in the $\mathbb{R}^2$ planes. The sphere and twisted partitions functions in three and five dimensions can be obtained by gluing together these basic building blocks in the spirit of Pasquetti (2012), Beem et al. (2014), Nieri et al. (2015), Gukov et al. (2017), Pasquetti (2017), Nekrasov (2003a), Bershtein et al. (2017), Kim et al. (2013), Hosseini et al. (2018c), Qiu and Zabzine (2017) and Festuccia et al. (2020). This point of view has been applied to the physics of black holes in Hosseini et al. (2018c), (2019a), Choi et al. (2019) and Choi and Hwang (2020).

potentials Δ_a and ω_i but the on-shell value (7.1) is real, as an entropy must be. Secondly, there is a sign ambiguity in the constraint (7.7) that can be replaced by $\sum_a \Delta_a + \sum_i \omega_i = -2\pi$ with no differences. Thirdly, the expression of $\log \mathcal{Z}(\Delta_a, \omega_i)$ can be explicitly derived in gravity by taking the zero-temperature limit of the on-shell action of a family of supersymmetric Euclidean solutions (Cassani and Papini 2019).

Finally, we expect that (7.1) corresponds to the attractor mechanism in the relevant gauged supergravity. Unfortunately, the attractor mechanism in generic dimensions and, specifically, for electrically charged rotating black holes is not known. AdS_5 black holes with equal angular momenta can be dimensionally reduced to static black holes in four dimensions and, in this case, one can show that (7.1) corresponds to the attractor mechanism in four-dimensional gauged supergravity (Hosseini et al. 2017b). This was actually the observation that led to write the entropy functional (7.1).

7.2 Results on the quantum field theory side

Various field theory derivations of the extremization principles (7.3) have been recently proposed for AdS_5. All these results are valid in overlapping limits and the connection between different approaches still to be understood, but all seems to indicate that the entropy is correctly accounted by the large N limit of the superconformal index. Partial results in other dimensions also confirm the content of Table 1.

Consider first the case of $AdS_5 \times S^5$. The dual field theory is $\mathcal{N} = 4$ SYM. In the language of $\mathcal{N} = 1$ supersymmetry, it contains a vector multiplet W_α and three chiral multiplets ϕ_a subject to the superpotential (5.5). We introduce three R-symmetries $\mathcal{R}_a$ associated with $U(1)^3 \subset SO(6)$. $\mathcal{R}_a$ assign charge 2 to ϕ_a and zero to the ϕ_b with $b \neq a$. The exact R-symmetry is $R = (\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3)/3$ and the global symmetries are $q_a = (\mathcal{R}_a - R)/2$, with associated fugacities u_a. Only two global symmetries are independent since $\sum_{a=1}^3 q_a = 0$ and, as a consequence, $\prod_{a=1}^3 u_a = 1$. Defining $y_a = (pq)^{1/3} u_a$ we can write the superconformal index (6.1) as

$$I(\Delta_a, \omega_i) = \mathrm{Tr}\Big|_{\mathcal{Q}=0} (-1)^F p^{J_1} q^{J_2} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3} = \mathrm{Tr}\Big|_{\mathcal{Q}=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)}, \qquad (7.8)$$

with $y_a = e^{i\Delta_a}$, $p = e^{i\omega_1}$, $q = e^{i\omega_2}$ and $Q_a = \mathcal{R}_a/2$. The fugacities are constrained by $\prod_{a=1}^3 y_a = pq$, and the index depends only on four independent parameters, as the family of BPS black holes.

Due to cancellations between bosonic and fermionic supersymmetric states, the result obtained from the index can only be a lower bound on the number of BPS states. However, we may expect that, as for magnetically charged black holes, for large N, the result saturates the entropy. Most of the computations for the superconformal index in the old literature has been performed for *real fugacities* and give results of order $O(1)$ for large N. However, the extremization principle discussed above strongly suggests that we should look at the behavior of the index as a function of *complex* chemical potentials.

Agreement with the gravity result (7.5)

$$\log I(\Delta_a, \omega_i) = -i\frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2}, \tag{7.9}$$

with $\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = \pm 2\pi$ has been obtained analytically, up to now, in two partially overlapping limits:

- Large N and equal angular momenta (Benini and Milan 2020a, b). In the large N limit, the index has a Stokes behavior as a functions of the chemical potentials, and it can give a contribution to the entropy of order $O(N^2)$ along the right direction in the complex plane. A crucial technical ingredient in this approach involves writing the superconformal index as a sum over Bethe vacua. As shown in Closset et al. (2017a, b, 2018), the supersymmetric partition function of many three- and four-dimensional manifolds can be expressed as a sum over two-dimensional Bethe vacua using the formalism that we briefly discussed in Sect. 3.3.2. A formula for the superconformal index was obtained in Closset et al. (2017a) and generalized to unequal fugacities for the angular momenta in Benini and Milan (2020a). Schematically, it allows to write the index as in (3.38)

$$\mathcal{I} = \sum_{x^*} \frac{Q(x^*)}{\det_{ij}(-\partial^2_{u_i u_j} W(x^*))}, \tag{7.10}$$

  where x^* are the Bethe vacua of the two-dimensional theory obtained by reduction on T^2, and $Q(x)$ is a suitable function whose expression can be found in Closset et al. (2017a) and Benini and Milan (2020a).[70] The Bethe vacua of $\mathcal{N} = 4$ SYM on T^2 have been already discussed in Sect. 5.2 for a different purpose and are explicitly given by solutions of (5.18).[71] It is argued in Benini and Milan (2020b) that, in the large N limit, the particular Bethe vacuum (5.19) dominates for sufficiently large charges and reproduces the entropy of the AdS_5 black holes. The same result has been reproduced by directly analysing the saddle point of the integrand (6.4) in Cabo-Bizet and Murthy (2019). The extension to the case of unequal angular momenta is discussed in Benini et al. (2020a). Although it is difficult to evaluate and compare the contribution of all the Bethe vacua, one can show that there is a natural choice that leads precisely to the gravity result (7.5).
- The Cardy limit, corresponding to $\omega_i \ll 1$ at fixed complex valued of Δ_a (Choi et al. 2018a, b). This limit corresponds to large black holes with electric charges and angular momenta scaling as

[70] The four-manifold $S^1 \times S^3$ can be considered as a torus fibration over a two-dimensional manifold using S^1 and a circle inside S^3 for the T^2 fiber. For details see Closset et al. (2017a) and Benini and Milan (2020a).

[71] As argued in Arabi Ardehali et al. (2019), there can exist non-standard solutions corresponding to continua of Bethe vacua and the formula (7.10) must be accordingly generalized.

$$q_a \sim \frac{1}{\omega^2}, \qquad j_i \sim \frac{1}{\omega^3}, \qquad \omega_1 \sim \omega_2 \sim \omega \to 0\,. \tag{7.11}$$

It is crucial that the chemical potentials are complex. As argued in Choi et al. (2018a, b), the imaginary parts of the fugacities at the saddle point introduce phases that optimally obstruct the cancellation between bosonic and fermionic states. The number of states accounted by the index in the Cardy limit correctly reproduces the entropy of large AdS_5 black holes and the $\omega_i \ll 1$ limit of the extremization formula (7.3). This approach has been further refined and generalized in Honda (2019), Arabi Ardehali (2019) and Arabi Ardehali et al. (2019).

Numerical analysis confirming the $O(N^2)$ behavior of the index for complex chemical potential has been performed in Murthy (2020) and Agarwal et al. (2020).

In all these approaches, there seems to exist instabilities when decreasing the charges, which might suggest the contribution of other types of black holes. Given also the recently found supersymmetric hairy black holes in AdS_5 (Markeviciute and Santos 2019; Markeviciute 2019), we may expect a rich structure of the index/partition function still to be uncovered.

It was observed in Hosseini et al. (2017b, 2018b) that the quantity (7.9) and its analogous for AdS_7 given in Table 1 matches the expression for the supersymmetric Casimir energy $E_{c.e}$ quoted in the literature with the *precise* coefficient for both $\mathcal{N}=4$ SYM and the (2, 0) theory (see for example Bobev et al. 2015). This observation was strengthened in Cabo-Bizet et al. (2019a) by considering a modified supersymmetric partition function on $S^3 \times S^1$ implementing the constraint (7.4) and showing that the corresponding supersymmetric Casimir energy $E_{c.e}$ has still the expression (7.9).[72] It is not completely clear why the supersymmetric Casimir energy, which corresponds to the energy of the vacuum of the CFT (Assel et al. 2015), should be related to the entropy of the black hole. It would be intriguing if this a consequence of some modular properties of the partition function, still to be understood.

The large N holographic expectation for theories with a Sasaki–Einstein dual is (Hosseini et al. 2018b)

$$\log I(\Delta_a, \omega_i) = -i\frac{16}{27}\frac{a(\Delta)}{\omega_1\omega_2} = -i\frac{N^2}{12}\sum_{a,b,c=1}^{d} f_{abc}\frac{\Delta_a\Delta_b\Delta_c}{\omega_1\omega_2}, \tag{7.12}$$

with the constraint $\sum_{a=1}^{d} \Delta - \omega_1 - \omega_2 = \pm 2\pi$, where $a(\Delta)$ is the trial central charge defined in Sect. 5.1 and f_{abc} are the cubic t'Hooft anomaly coefficients for a basis of d independent R-charges. The coefficients f_{abc} have a natural dual gravitational interpretation. They arise as intersection numbers of cycles in the internal manifold (Benvenuti et al. 2006), and, from an effective field theory perspective, as Chern–Simons terms in the corresponding gauged supergravity in five dimensions. In particular, they determine completely the structure of $\mathcal{N}=1$ gauged supergravity

[72] The motivation for using this partition function comes from holography, since the constraint (7.4) explicitly arises in the Euclidean description of the AdS_5 black holes (Cabo-Bizet et al. 2019a).

including the prepotential. The field theory computation of the index has been extended to other $\mathcal{N}=1$ superconformal theories and gives results consistent with (7.12). In particular, the Cardy limit for a generic $\mathcal{N}=1$ superconformal theory has been studied in Kim et al. (2019) and Cabo-Bizet et al. (2019b) with the result[73]

$$\log I(\omega_i) \underset{\omega_i \to 0}{=} 4\pi^2 i \frac{3\omega_1 + 3\omega_2 \pm 2\pi}{27\omega_1\omega_2}(3c-5a) + 4\pi^2 i \frac{\omega_1 + \omega_2 \pm 2\pi}{\omega_1\omega_2}(a-c) + O(1)$$

when all flavor symmetries are turned off. This formula generalize a previous result by Di Pietro and Komargodski (2014) for the standard index with real fugacities. Flavor fugacities have been introduced in Amariti et al. (2019) and consistency with (7.12) in the large N limit, where $c=a$, checked for many toric models. The large N limit of the index for equal angular momenta has been studied in Lanir et al. (2020) and Cabo-Bizet et al. (2020) with results again consistent with (7.12).[74] The case of unequal angular momenta is discussed in Benini et al. (2020a).

These results have been generalized and extended to other dimensions. In particular, the entropy of Kerr–Newman black holes in AdS_4 has been reproduced in the Cardy limit in Choi et al. (2019), Nian and Pando Zayas (2020) and Choi and Hwang (2020), one of the method involving factorization of the partition function. The case of AdS_6 and AdS_7 have been analysed in Choi and Kim (2019), Kantor et al. (2020) and Nahmgoong (2019). Other interesting developments can be found in Benini et al. (2020b), Bobev and Crichigno (2019) and Goldstein et al. (2020).

8 Conclusions and comments

At the end of our journey, it is time to recapitulate. We have seen that, using holography, the entropy of supersymmetric AdS black holes and black objects with large charges can be correctly accounted by the evaluation of the relevant index in the dual conformal field theory, which just enumerates the corresponding microstates. This solves a long standing puzzle about supersymmetric black holes in AdS and their holographic interpretation. Many results have been obtained for most of the existing black objects in maximally supersymmetric string theory backgrounds in diverse dimensions. One can reasonably expect that the agreement will persist for the more technically involved case of black objects with arbitrary rotations and magnetic charges and of black objects in string backgrounds with reduced supersymmetry.

It is interesting to observe that, in all dimensions, a single function $F(\Delta)$ controls the entropy of most of the black holes and black objects asymptotic to a maximally supersymmetric AdS vacuum, with or without magnetic charges or rotation. For comparison, the partition function for Kerr–Newman black holes and magnetically

[73] The authors use a slightly modified index where $(-1)^F$ is replaced by $(-1)^R$. This replacement has the same effect as introducing complex fugacities for the flavor symmetries. The prescription is the same introduced in Cabo-Bizet et al. (2019a) for the modified supersymmetric partition function on $S^3 \times S^1$.

[74] See also González Lezcano and Pando Zayas (2020).

Table 2 Entropy functionals for magnetically charged static spherically symmetric black objects with a twist

$AdS_4 \times S^7$	$F(\Delta_a) = \sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$	$\log \mathcal{Z} = -\frac{2\sqrt{2}N^{3/2}}{3}\sum_{a=1}^{4}\mathfrak{p}_a \frac{\partial \mathcal{F}(\Delta)}{\partial \Delta_a}$
	$\Delta_1+\Delta_2+\Delta_3+\Delta_4=2$	$\Delta_1+\Delta_2+\Delta_3+\Delta_4=2\pi$
$AdS_5 \times S^5$	$F(\Delta_a) = \Delta_1\Delta_2\Delta_3$	$\log \mathcal{Z} = -\frac{N^2}{2\beta}\sum_{a=1}^{3}\mathfrak{p}_a \frac{\partial \mathcal{F}(\Delta)}{\partial \Delta_a}$
	$\Delta_1+\Delta_2+\Delta_3=2$	$\Delta_1+\Delta_2+\Delta_3=2\pi$
$AdS_6 \times_W S^4$	$F(\Delta_a) = (\Delta_1\Delta_2)^{3/2}$	$\log \mathcal{Z} \sim N^{5/2}\sum_{a,b=1}^{2}\mathfrak{p}_a\tilde{\mathfrak{p}}_b \frac{\partial^2 \mathcal{F}(\Delta)}{\partial \Delta_a \partial \Delta_b}$
	$\Delta_1+\Delta_2=2$	$\Delta_1+\Delta_2=2\pi$
$AdS_7 \times S^4$	$F(\Delta_a) = (\Delta_1\Delta_2)^2$	$\log \mathcal{Z} \sim \frac{N^3}{\beta}\sum_{a,b=1}^{2}\mathfrak{p}_a\tilde{\mathfrak{p}}_b \frac{\partial^2 \mathcal{F}(\Delta)}{\partial \Delta_a \partial \Delta_b}$
	$\Delta_1+\Delta_2=2$	$\Delta_1+\Delta_2=2\pi$

The cases of $AdS_5 \times S^5$ and $AdS_7 \times S^4$ correspond to black strings. The Legendre transform of $\log \mathcal{Z}$ reproduces the entropy of the static black hole obtained by reduction on a circle, as discussed in Sect. 5. Details and normalizations for AdS_6 and AdS_7 can be found in Hosseini et al. (2018c), Crichigno et al. (2018), Hosseini et al. (2018a) and Suh (2019b)

charged black objects with a twist in various dimensions is reported in Table 1 and Table 2, respectively.

We see that the function $F(\Delta)$ determines the entropy of all such black holes. A general entropy functional built out of $F(\Delta)$ that covers all existing black holes and generalises the content of Tables 1 and 2 to the case of arbitrary rotations and magnetic charges has been discussed in Hosseini et al. (2019a) in analogy with the factorization properties of supersymmetric partition functions.

The function $F(\Delta)$ has various interpretations. In gravity, it determines the effective gauged supergravity action for the massless vectors, being the prepotential in four dimensions. In field theory, it is related to the anomalies of the dual CFT in even dimensions, and the round sphere partition function in odd dimensions. On a more technical side, there is a third interpretation in terms of the twisted superpotential of the two-dimensional theory obtained by reducing the CFT on circle or tori, as discussed at length in Sects. 3 and 5.[75]

In this review we discussed black holes in the supergravity approximation, where the charges are large in units of the number of colors of the dual CFT. For asymptotically flat black holes impressive computations and precision tests have been made beyond the supergravity limit. The story for AdS black hole has just begun.

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[75] For five and six-dimensional CFTs, this should be replaced with the Seiberg–Witten prepotential (Hosseini et al. 2018c; Crichigno et al. 2018).

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