LECTURES ON EQUIVARIANT STABLE HOMOTOPY THEORY

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Contents

We review some foundations for equivariant stable homotopy theory in the context of *orthogonal G-spectra*. The main reference for this theory is the AMS memoir [16] by Mandell and May; the appendices of the paper [10] by Hill, Hopkins and Ravenel contain further material, in particular on the norm construction. At many places, however, our exposition is substantially different from these two sources, compare Remark 2.7. We do not develop model category aspects of the theory; the relevant references here are again Mandell-May [16], Hill-Hopkins-Ravenel [10] and the thesis of Stolz [26]. For a general, framework independent, introduction to equivariant stable homotopy theory, one may consult the survey articles by Adams [1] and Greenlees-May [8].

We restrict our attention to finite groups (as opposed to compact Lie groups) throughout, which allows to simplify the treatment at various points. For a detailed exposition of orthogonal G-spectra in the generality of compact Lie groups, the reader may want to consult Chapter 3 of the author's book [21]. Also, we implicitly only deal with the 'complete universe' (which can be seen from the fact that we stabilize with respect to multiples of the regular representation).

These notes were originally assembled on the occasion of a series of lectures at the Universitat Autònoma de Barcelona in October, 2010, and then subsequently expanded. They are still incomplete and certainly contain typos, but hopefully not too many mathematical errors. At some places, proper credit is also still missing, and will be added later. This survey paper makes no claim to originality. If there is anything new it may be the particular model for the real bordism spectrum MR as a commutative equivariant orthogonal ring spectrum in Example 2.14.

Before we start, let us fix some notation and conventions. By a 'space' we mean a *compactly generated space* in the sense of McCord [17], i.e., a k-space (also called a *Kelley space*) that also satisfies the weak

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Hausdorff condition. Section 7.9 of tom Dieck's textbook [28] and Appendix A of the author's book [21] are two detailed references about compactly generated spaces.

For a finite dimensional $\mathbb{R}$-vector space V we denote by S^V the one-point compactification; we consider S^V as a based space with basepoint at infinity. If V is endowed with a scalar product, we denote by $D(V)$ the unit ball and by $S(V)$ the unit sphere of V. If no other scalar product is specified, then the vector space $\mathbb{R}^n$ is always endowed with the standard scalar product

$$\langle x, y\rangle \ = \ \sum_{i=1}^{n} x_i y_i \ .$$

We write S^n for $S^{\mathbb{R}^n}$, the one-point compactification of $\mathbb{R}^n$.

For a finite group G we denote by ρ_G the regular representation of G, i.e., the free vector space $\mathbb{R}[G]$ with orthonormal basis G. By $\mathcal{O}(G)$ we denote the orbit category of G , i.e., the category with objects the cosets G/H for all subgroups H of G and with morphisms the homomorphisms of left G-sets.

I would like to thanks John Greenlees for being a reliable consultant on equivariant matters and for patiently answering many of my questions.

1. Orthogonal spectra

Starting from the next section, our category of G-spectra will be the category of orthogonal spectra with G-action. So before adding group actions, we first review non-equivariant orthogonal spectra.

Definition 1.1. An *orthogonal spectrum* consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$,
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$,
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m}$$

defined as the composition

$$(1.2) \qquad X_n \wedge S^m \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge S^{m-2}} \ \cdots \ \xrightarrow{\sigma_{n+m-2} \wedge \ S^1} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $O(n) \times O(m)$-equivariant. Here the orthogonal group $O(m)$ acts on S^m since this is the one-point compactification of $\mathbb{R}^m$, and $O(n) \times O(m)$ acts on the target by restriction, along orthogonal sum, of the $O(n+m)$-action. We refer to the space X_n as the *n-th level* of the orthogonal spectrum X.

A *morphism* $f : X \longrightarrow Y$ of orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$ for all $n \geq 0$. We denote the category of orthogonal spectra by $\mathcal{S}p$.

An *orthogonal ring spectrum* R consists of the following data:

- a sequence of pointed spaces R_n for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on R_n for each $n \geq 0$
- $O(n) \times O(m)$-equivariant *multiplication maps* $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ for $n, m \geq 0$, and
- $O(n)$-equivariant *unit maps* $\iota_n : S^n \longrightarrow R_n$ for all $n \geq 0$.

This data is subject to the following conditions:

(Associativity) The square

$$\begin{tikzcd} R_n \wedge R_m \wedge R_p \arrow[r, "\mathrm{Id} \wedge \mu_{m,p}"] \arrow[d, "\mu_{n,m} \wedge \mathrm{Id}"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\ R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"'] & R_{n+m+p} \end{tikzcd}$$

commutes for all $n, m, p \geq 0$.

(Unit) The two composites

$$R_n \cong R_n \wedge S^0 \xrightarrow{R_n \wedge \iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \cong S^0 \wedge R_n \xrightarrow{\iota_0 \wedge R_n} R_0 \wedge R_n \xrightarrow{\mu_{0,n}} R_n$$

are the identity for all $n \geq 0$.

(Multiplicativity) The composite

$$S^{n+m} \ \cong \ S^n \wedge S^m \ \xrightarrow{\iota_n \wedge \iota_m} \ R_n \wedge R_m \ \xrightarrow{\mu_{n,m}} \ R_{n+m}$$

equals the unit map $\iota_{n+m} : S^{n+m} \longrightarrow R_{n+m}$. (where the first map is the canonical homeomorphism sending $(x, y) \in S^{n+m}$ to $x \wedge y$ in $S^n \wedge S^m$).

(Centrality) The diagrams

$$\begin{tikzcd} R_m \wedge S^n \arrow[r, "R_m \wedge \iota_n"] \arrow[d, "\mathrm{twist}"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"] & R_{m+n} \arrow[d, "\chi_{m,n}"] \\ S^n \wedge R_m \arrow[r, "\iota_n \wedge R_m"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

commutes for all $m, n \geq 0$. Here $\chi_{m,n} \in O(m+n)$ denotes the permutation matrix of the shuffle permutation which moves the first m elements past the last n elements, keeping each of the two blocks in order; in formulas,

$$\chi_{m,n}(i) \ = \ \begin{cases} i + n & \text{for } 1 \leq i \leq m, \\ i - m & \text{for } m + 1 \leq i \leq m + n. \end{cases}$$

An orthogonal ring spectrum R is *commutative* if the square

$$\begin{tikzcd} R_m \wedge R_n \arrow[r, "\mu_{m,n}"] \arrow[d, "\mathrm{twist}"'] & R_{m+n} \arrow[d, "\chi_{m,n}"] \\ R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

commutes for all $m, n \geq 0$. Note that this commutativity diagram implies the centrality condition above.

Remark 1.3. (i) The higher-dimensional unit maps $\iota_n : S^n \longrightarrow R_n$ for $n \geq 2$ are determined by the unit map $\iota_1 : S^1 \longrightarrow R_1$ and the multiplication as the composite

$$S^n = S^1 \wedge \ldots \wedge S^1 \xrightarrow{\iota_1 \wedge \ldots \wedge \iota_1} R_1 \wedge \ldots \wedge R_1 \xrightarrow{\mu_{1,\ldots,1}} R_n \ .$$

The centrality condition implies that this map is Σ_n-equivariant, but we require that ι_n is even $O(n)$-equivariant.

(ii) As the terminology suggests, the orthogonal ring spectrum R has an underlying orthogonal spectrum. We keep the spaces R_n and orthogonal group actions and define the missing structure maps $\sigma_n : R_n \wedge$

$S^1 \longrightarrow R_{n+1}$ as the composite $\mu_{n,1} \circ (R_n \wedge \iota_1)$. Associativity implies that the iterated structure map $\sigma^m : R_n \wedge S^m \longrightarrow R_{n+m}$ equals the composite

$$R_n \wedge S^m \xrightarrow{R_n \wedge \iota_m} R_n \wedge R_m \xrightarrow{\mu_{n,m}} R_{n+m} .$$

So the iterated structure map is $O(n) \times O(m)$-equivariant, and we have in fact obtained an orthogonal spectrum.

(iii) Using the internal smash product of orthogonal spectra one can identify the 'explicit' definition of an orthogonal ring spectrum which we just gave with a more 'implicit' definition of an orthogonal spectrum R together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ (where $\mathbb{S}$ is the sphere spectrum) which are suitably associative and unital. The 'explicit' and 'implicit' definitions of orthogonal ring spectra coincide in the sense that they define isomorphic categories.

A *morphism* $f : R \longrightarrow S$ of orthogonal ring spectra consists of $O(n)$-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps in the sense that $f_{n+m}\mu_{n,m} = \mu_{n,m}(f_n \wedge f_m)$ and $f_n\iota_n = \iota_n$.

Example 1.4 (Sphere spectrum)**.** The orthogonal *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, where the orthogonal group acts as the one-point compactification of its natural action on $\mathbb{R}^n$. The map $\sigma_n : S^n \wedge S^1 \longrightarrow S^{n+1}$ is the canonical homeomorphism. This is a commutative orthogonal ring spectrum with the identity map of S^n as the n-th unit map and the canonical homeomorphism $S^n \wedge S^m \longrightarrow S^{n+m}$ as multiplication map. The sphere spectrum is the *initial* orthogonal ring spectrum: if R is any orthogonal ring spectrum, then a unique morphism of orthogonal ring spectra $\mathbb{S} \longrightarrow R$ is given by the collection of unit maps $\iota_n : S^n \longrightarrow R_n$.

The category of right $\mathbb{S}$-modules is isomorphic to the category of orthogonal spectra, via the forgetful functor mod-$\mathbb{S} \longrightarrow \mathcal{S}p$. Indeed, if X is a orthogonal spectrum then the associativity condition shows that there is at most one way to define action maps

$$X_n \wedge S^m \longrightarrow X_{n+m} ,$$

namely as the iterated structure map σ^m, and these do define the structure of a right $\mathbb{S}$-module on X.

Primary invariants of an orthogonal spectrum are its homotopy groups: the *k-th homotopy group* of a orthogonal spectrum X is defined as the colimit

$$\pi_k(X) = \operatorname{colim}_n \pi_{k+n}(X_n)$$

taken over the *stabilization maps* $\iota : \pi_{k+n}(X_n) \longrightarrow \pi_{k+n+1}(X_{n+1})$ defined as the composite

$$\pi_{k+n}(X_n) \xrightarrow{-\wedge S^1} \pi_{k+n+1}\left(X_n \wedge S^1\right) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1}(X_{n+1}) .$$

For large enough n, the set $\pi_{k+n}X_n$ has a natural abelian group structure and the stabilization maps are homomorphisms, so the colimit $\pi_k X$ inherits a natural abelian group structure. The stable homotopy category can be obtained from the category of orthogonal spectra by formally inverting the class of π_*-isomorphisms.

Now we get to the smash product of orthogonal spectra. We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spectra (X, Y) to an orthogonal spectrum Z as a collection of based $O(p) \times O(q)$-equivariant maps

$$b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$$

for $p, q \geq 0$, such that the *bilinearity diagram*

$$\begin{tikzcd}
 & X_p \wedge Y_q \wedge S^1 \arrow[r, "X_p \wedge \mathrm{twist}"] \arrow[dl, "X_p \wedge \sigma_q"'] \arrow[d, "b_{p,q} \wedge S^1"] & X_p \wedge \ S^1 \wedge Y_q \arrow[d, "\sigma_p \wedge Y_q"] \\
X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge S^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\
 & Z_{p+q+1} & Z_{p+1+q} \arrow[l, "1 \times \chi_{1,q}"]
\end{tikzcd} \tag{1.5}$$

commutes for all $p, q \geq 0$.

We can then define a smash product of X and Y as a universal example of an orthogonal spectrum with a bimorphism from X and Y. More precisely, a smash product for X and Y is a pair $(X \wedge Y, i)$ consisting of an orthogonal spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every orthogonal spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \longrightarrow \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q} \tag{1.6}$$

is bijective.

We have to show that a universal bimorphism out of any pair of orthogonal spectra exists; in other words: we have to construct a smash product $X \wedge Y$ from two given orthogonal spectra X and Y. We want $X \wedge Y$ to be the universal recipient of a bimorphism from (X, Y), and this pretty much tells us what we have to do. For $n \geq 0$ we define the n-th level $(X \wedge Y)_n$ as the coequalizer, in the category of pointed $O(n)$-spaces, of two maps

$$\alpha_X,\, \alpha_Y \ : \ \bigvee_{p+1+q=n} O(n)_+ \wedge_{O(p)\times 1 \times O(q)} X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \ .$$

The wedges run over all non-negative values of p and q which satisfy the indicated relations. The map α_X takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p+1, q)$ using the map

$$\sigma_p^X \wedge \mathrm{Id} \ : \ X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ X_{p+1} \wedge Y_q$$

and inducing up. The other map α_Y takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p, 1+q)$ using the composite

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{Id} \wedge \mathrm{twist}} X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{Id} \wedge \sigma_q^Y} X_p \wedge Y_{q+1} \xrightarrow{\mathrm{Id} \wedge \chi_{q,1}} X_p \wedge Y_{1+q}$$

and inducing up.

The structure map $(X \wedge Y)_n \wedge S^1 \longrightarrow (X \wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \wedge S^1 \ \longrightarrow \ O(n+1)_+ \wedge_{O(p)\times O(q+1)} X_p \wedge Y_{q+1}$$

induced from $\mathrm{Id} \wedge \sigma_q^Y : X_p \wedge Y_q \wedge S^1 \longrightarrow X_p \wedge Y_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by moving S^1 past Y_q, using the structure map of X (instead of that of Y) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \xrightarrow{x \wedge y \mapsto 1 \wedge x \wedge y} \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \xrightarrow{\text{projection}} (X \wedge Y)_{p+q}$$

forms a bimorphism – and in fact a universal one.

Very often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (1.6) as the *universal property* of the smash product of orthogonal spectra.

The smash product $X \wedge Y$ is a functor in both variables. It is also symmetric monoidal, i.e., there are natural associativity respectively symmetry isomorphisms

$$(X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z) \qquad \text{respectively} \qquad X \wedge Y \longrightarrow Y \wedge X$$

and unit isomorphisms $\mathbb{S} \wedge X \cong X \cong X \wedge \mathbb{S}$.

We can obtain all the isomorphisms of the symmetric monoidal structure just from the universal property. Let us choose, for each pair of orthogonal spectra (X, Y), a smash product $X \wedge Y$ and a universal bimorphism $i = \{i_{p,q}\} : (X, Y) \longrightarrow X \wedge Y$. For the construction of the associativity isomorphism we notice that the family

$$\left\{ X_p \wedge\ Y_q \wedge Z_r \xrightarrow{i_{p,q} \wedge Z_r} (X \wedge Y)_{p+q} \wedge Z_r \xrightarrow{i_{p+q,r}} ((X \wedge Y) \wedge Z)_{p+q+r} \right\}_{p,q,r \geq 0}$$

and the family

$$\left\{ X_p \wedge\ Y_q \wedge Z_r \xrightarrow{X_p \wedge i_{q,r}} X_p \wedge (Y \wedge Z)_{q+r} \xrightarrow{i_{p,q+r}} (X \wedge (Y \wedge Z))_{p+q+r} \right\}_{p,q,r \geq 0}$$

both have the universal property of a *tri*morphism (whose definition is hopefully clear) out of X, Y and Z. The uniqueness of representing objects gives a unique isomorphism of orthogonal spectra

$$\alpha_{X,Y,Z} \ : \ (X \wedge Y) \wedge Z \ \cong X \wedge (Y \wedge Z)$$

such that $(\alpha_{X,Y,Z})_{p+q+r} \circ i_{p+q,r} \circ (i_{p,q} \wedge Z_r) = i_{p,q+r} \circ (X_p \wedge i_{q,r})$.

The symmetry isomorphism $\tau_{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ corresponds to the bimorphism

$$\left\{ X_p \wedge\ Y_q \xrightarrow{\text{twist}} Y_q \wedge\ X_p \xrightarrow{i_{q,p}} (Y \wedge X)_{q+p} \xrightarrow{\chi_{q,p}} (Y \wedge X)_{p+q} \right\}_{p,q \geq 0} .$$

The block permutation $\chi_{q,p}$ is crucial here: without it the bilinearity diagram (1.5) would not commute and we would not have a bimorphism. If we restrict the composite $\tau_{Y,X} \circ \tau_{X,Y}$ in level $p+q$ along the map $i_{p,q} : X_p \wedge Y_q \longrightarrow (X \wedge Y)_{p+q}$ we get $i_{p,q}$ again. Thus $\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{Id}_{X \wedge Y}$ and $\tau_{Y,X}$ is inverse to $\tau_{X,Y}$.

The upshot is that the associativity and symmetry isomorphisms make the smash product of orthogonal spectra into a symmetric monoidal product with the sphere spectrum $\mathbb{S}$ as unit object. This product is *closed* symmetric monoidal in the sense that the smash product is adjoint to an internal Hom spectrum (that we discuss in Example 5.11 below), i.e., there is an adjunction isomorphism

$$\mathrm{Hom}(X \wedge Y, Z) \ \cong \ \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \ .$$

We remark again that orthogonal ring spectra are the same as monoid objects in the symmetric monoidal category of orthogonal spectra with respect to the smash product.

2. Equivariant orthogonal spectra

In the rest of these notes we let G denote a finite group. Much of what we explain can be generalized to compact Lie groups, or to even more general classes of groups, but we'll concentrate on the finite group case throughout.

Definition 2.1.

- An *orthogonal G-spectrum* is an orthogonal spectrum equipped with a G-action through automorphisms of orthogonal spectra.
- An *orthogonal G-ring spectrum* is an orthogonal ring spectrum equipped with a G-action through automorphisms of orthogonal ring spectra.

- A morphism of orthogonal G-spectra (respectively orthogonal G-ring spectra) is a morphism of underlying orthogonal spectra (respectively orthogonal ring spectra) that commutes with the group action.

If we unravel the definitions, we obtain that an orthogonal G-spectrum consists of pointed spaces X_n for $n \geq 0$, a based left $O(n) \times G$-action on X_n and based structure maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ that are G-equivariant with respect to the given G-actions on X_n and X_{n+1} and the trivial G-action on the sphere S^1. Of course, this data is again subject to the condition that the iterated structure maps $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$ are $O(n) \times O(m)$-equivariant. The iterated structure map σ^m is then automatically G-equivariant with respect to the given G-actions on X_n and X_{n+m} and the trivial G-action on S^m.

Readers familiar with other accounts of equivariant stable homotopy theory may wonder immediately why no orthogonal representations of the group G show up in the definition of equivariant spectra. The reason is that they are secretly already present: the actions of the orthogonal groups encode enough information so that we can evaluate an orthogonal G-spectrum on a G-representation. We will now spend some time explaining this in detail.

In the following, an *inner product space* is a finite dimensional real vector space equipped with a scalar product. For every orthogonal spectrum X and inner product space V of dimension n we define $X(V)$, the *value of X on V*, as

$$X(V) \;=\; \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n \tag{2.2}$$

where $\mathbb{R}^n$ has the standard scalar product and $\mathbf{L}(\mathbb{R}^n, V)$ is the space of linear isometries from $\mathbb{R}^n$ to V. The orthogonal group $O(n)$ acts simply transitively on $\mathbf{L}(\mathbb{R}^n, V)$ by precomposition, and $X(V)$ is the coequalizer of the two $O(n)$-actions on $\mathbf{L}(\mathbb{R}^n, V)_+ \wedge X_n$. If $V = \mathbb{R}^n$ then there is a canonical homeomorphism

$$X_n \;\longrightarrow\; X(\mathbb{R}^n) \;, \quad x \;\longmapsto\; [\mathrm{Id}, x] \;. \tag{2.3}$$

In general, any choice of isometry $\varphi : \mathbb{R}^n \longrightarrow V$ (which amounts to a choice of orthonormal basis of V) gives rise to a homeomorphism

$$[\varphi, -] \;:\; X_n \;\longrightarrow\; X(V) \;, \quad x \;\longmapsto\; [\varphi, x] \;.$$

Now let us consider a finite group G and an orthogonal G-spectrum X and suppose that V is a G-representation (i.e., G acts on V by linear isometries). Then $X(V)$ becomes a G-space by the rule

$$g \cdot [\varphi, x] \;=\; [g\varphi, gx] \;.$$

We want to stress that the underlying space of $X(V)$ depends, up to homeomorphism, only on the dimension of the representation V. However, the G-action on V influences the G-action on $X(V)$.

The iterated structure maps $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$ of an orthogonal G-spectrum X now become special cases of *generalized structure maps*

$$\sigma_{V,W} \;:\; X(V) \wedge S^W \;\longrightarrow\; X(V \oplus W) \;. \tag{2.4}$$

To define $\sigma_{V,W}$ we set $m = \dim(W)$ and choose an isometry $\gamma : \mathbb{R}^m \longrightarrow W$. Then

$$\sigma_{V,W}([\varphi, x] \wedge w) \;=\; [\varphi \oplus \gamma,\, \sigma^m(x \wedge \gamma^{-1}(w))] \quad \text{in} \quad \mathbf{L}(\mathbb{R}^{n+m}, V \oplus W)_+ \wedge_{O(n+m)} X_{n+m} \;=\; X(V \oplus W) \;.$$

We omit the verification that the map $\sigma_{V,W}$ is well defined and independent of the choice of γ. It is straightforward from the definitions that the generalized structure maps are G-equivariant where – in contrast to the 'ordinary' structure maps $X_n \wedge S^m \longrightarrow X_{n+m}$ – here the group G also acts on the representation sphere S^W. The generalized structure map $\sigma_{V,W}$ is also $O(V) \times O(W)$-equivariant, so altogether it is equivariant for the semi-direct product group $G \ltimes (O(V) \times O(W))$ formed from the conjugation action of G on

$O(V)$ and $O(W)$. Finally, the generalized structure maps are also associative: If we are given a third inner product space U, then the square

$$\begin{tikzcd} X(V)\wedge S^W\wedge S^U \arrow[r, "\sigma_{V,W}\wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\cong"'] & X(V\oplus W)\wedge S^U \arrow[d, "\sigma_{V\oplus W,U}"] \\ X(V)\wedge S^{W\oplus U} \arrow[r, "\sigma_{V,W\oplus U}"'] & X(V\oplus W\oplus U) \end{tikzcd} \tag{2.5}$$

commutes.

We end this section by introducing a piece of notation that will be convenient later. For an orthogonal G-spectrum, G-representations V, W and a based map $f : S^V \longrightarrow X(V)$ (not necessarily equivariant), we denote by $f \diamond W : S^{V\oplus W} \longrightarrow X(V \oplus W)$ the composite

$$S^{V\oplus W} \ \cong\ S^V\wedge S^W \xrightarrow{\ f\wedge S^W\ } X(V)\wedge S^W \xrightarrow{\ \sigma_{V,W}\ } X(V\oplus W)\ . \tag{2.6}$$

We refer to $f \diamond W$ as the *stabilization of f by W*. The associativity property of the generalized structure maps implies the associativity property

$$(f\diamond W)\diamond U \ =\ f\diamond (W\oplus U)\ :\ S^{V\oplus W\oplus U} \ \longrightarrow\ X(V\oplus W\oplus U)\ .$$

Remark 2.7. Let us clarify the relationship between our current definition of an orthogonal G-spectrum and the one used by Mandell and May in [16] and Hill, Hopkins and Ravenel in [10]. As we shall explain, the two concepts are *not* the same, but the two categories are equivalent. This equivalence of categories is first discussed in [16, Thm. V.1.5], and also appears in [10, Prop. A.19]. This is not the first time such a non-obvious equivalence of categories appears in equivariant homotopy theory. Segal's notion of a Γ-space has two equivariant generalizations in the presence of a finite group G. Segal developed the equivariant version in the preprint [22], but this paper was never published. In [24], Shimakawa published a detailed account of the theory of Γ_G-spaces, the Γ-space analogue of the $\mathscr{I}_G$-spectra of [16]; in [25], Shimakawa observed that the category of Γ_G-spaces is equivalent to the category of Γ-G-spaces (i.e., Γ-spaces with G-action, the analog of orthogonal spectra with G-action). This equivalence is a close analogue, but with G-sets as opposed to G-representations, of the equivalence we are about to discuss now.

For us, an orthogonal G-spectrum is simply an orthogonal spectrum with action by the group G; in particular, our equivariant spectra do not initially assign values to general G-representations. Let us denote, for the course of this remark, the category of orthogonal spectra with G-action by G-$\mathcal{S}p^O$.

The definition of an orthogonal G-spectrum used by Mandell and May refers to a universe U, i.e., a certain infinite dimensional real inner product space with G-action by linear isometries. However, one upshot of this discussion is that, up to equivalence of categories, the equivariant orthogonal spectra of [16] are nevertheless independent of the universe. Mandell and May denote by $\mathscr{V}(U)$ the class of all finite dimensional G-representations that admit a G-equivariant, isometric embedding into the universe U. An $\mathscr{I}_G$-spectrum Y, or orthogonal G-spectrum, in the sense of [16, II Def. 2.6], consists of the following data:

(i) a based G-space $Y(V)$ for every G-representation V in the class $\mathscr{V}(U)$,

(ii) a continuous based G-map

$$\mathbf{L}(V,W)_+\wedge Y(V) \ \longrightarrow\ Y(W) \tag{2.8}$$

for every pair of G-representations V and W in $\mathscr{V}(U)$ of the same dimension (where Mandell and May write $\mathscr{I}_G^{\mathscr{V}}(V,W)$ for $\mathbf{L}(V,W)$),

(iii) continuous based G-maps

$$\sigma_{V,W}\ :\ Y(V)\wedge S^W \ \longrightarrow\ Y(V\oplus W)$$

for all pairs of G-representation V and W in $\mathscr{V}(U)$.

This data is subject to the following conditions:

(a) the action maps (2.8) of the isometries on the values of Y have to be unital and associative;
(b) the action maps (2.8) of the isometries on the values of Y and on representations spheres have to be compatible with the structure maps $\sigma_{V,W}$, i.e., the squares

$$\begin{tikzcd} \mathbf{L}(V,V')_+ \wedge \mathbf{L}(W,W')_+ \wedge Y(V) \wedge S^W \arrow[r, "\oplus\wedge\sigma_{V,W}"] \arrow[d] & \mathbf{L}(V\oplus W, V'\oplus W')_+ \wedge Y(V\oplus W) \arrow[d] \\ Y(V') \wedge S^{W'} \arrow[r, "\sigma_{V',W'}"'] & Y(V'\oplus W') \end{tikzcd} \tag{2.9}$$

commute.
(c) the morphism $\sigma_{V,0} : Y(V) \wedge S^0 \longrightarrow Y(V\oplus 0)$ is the composite of the natural isomorphisms $Y(V)\wedge S^0 \cong Y(V)$ and $Y(V) \cong Y(V\oplus 0)$, and the associativity diagram (2.5) commutes.

A morphism $f : Y \longrightarrow Z$ of $\mathscr{I}_G$-spectra consists of a based continuous G-map $f(V) : Y(V) \longrightarrow Z(V)$ for every V in $\mathscr{V}(U)$, strictly compatible with the action by the isometries and the structure maps $\sigma_{V,W}$. We denote the category of $\mathscr{I}_G$-spectra by $\mathscr{I}_G$-$\mathcal{S}p$.

The definition of $\mathscr{I}_G$-spectra above can be cast into an isomorphic, but more compact form, as enriched functors on a topological G-category $\mathscr{J}_G$, compare Theorem II.4.3 of [16] (we also discuss this reformulation in Example 5.5 below). In the formulation as enriched functors on $\mathscr{J}_G$, the structure on the collection of G-spaces $Y(V)$ consists of continuous based G-map

$$\mathscr{J}_G^{\mathscr{V}}(V,W) \wedge Y(V) \;\longrightarrow\; Y(W)$$

for every pair of G-representations V and W in $\mathscr{V}(U)$ (of possibly different dimensions), where $\mathscr{J}_G^{\mathscr{V}}(V,W)$ is the Thom G-space of the orthogonal complement bundle over the G-space $\mathbf{L}(V,W)$. This formulation combines the actions (2.8) of the linear isometries and the structure maps $\sigma_{V,W}$ into a single piece of structure, and also simplifies the compatibility conditions. The definition of orthogonal spectra as enriched functors on the topological G-category $\mathscr{J}_G$ is also the one used by Hill, Hopkins and Ravenel in [10, Def. A.13].

We explain the inverse equivalences of categories

$$G\text{-}\mathcal{S}p^O \quad \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \quad \mathscr{I}_G\text{-}\mathcal{S}p\ .$$

A $\mathscr{I}_G$-spectrum Y has an ‘underlying’ orthogonal spectrum with G-action $\mathbb{U}Y$. Indeed, all trivial G-representations belong to the class $\mathscr{V}(U)$ for any universe U, so an $\mathscr{I}_G$-spectrum Y has a value at the trivial representation $\mathbb{R}^n$, and we set $(\mathbb{U}Y)_n = Y(\mathbb{R}^n)$. For $V = W = \mathbb{R}^n$, the action (2.8) of the isometries specializes to an $O(n)$-action on $(\mathbb{U}Y)_n$. The map $\sigma_{\mathbb{R}^n,\mathbb{R}^m} : Y(\mathbb{R}^n) \wedge S^m \longrightarrow Y(\mathbb{R}^{n+m})$ is the iterated structure map of the orthogonal spectrum $\mathbb{U}Y$, and it is $O(n)\times O(m)$-equivariant by the special case $V = V' = \mathbb{R}^n$ and $W = W' = \mathbb{R}^m$ of (2.9).

Conversely, given an orthogonal spectrum with G-action X, we can evaluate it on any G-representation as in (2.2) and equip it with generalized structure maps $\sigma_{V,W}$ as in (2.4). The action (2.8) of a linear isometry $\psi : V \longrightarrow W$ is given by $\psi \wedge [\varphi, x] \mapsto [\psi\varphi, x]$. Altogether, this defines an $\mathscr{I}_G$-spectrum $\mathbb{P}X$ from the orthogonal spectrum with G-action X. The underlying orthogonal G-spectrum $\mathbb{U}\mathbb{P}X$ gives back what we started with; more precisely, the canonical homeomorphism (2.3) is a natural isomorphism between X and $\mathbb{U}\mathbb{P}X$.

In the other direction, a natural isomorphism from an $\mathscr{I}_G$-spectrum Y to $\mathbb{P}\mathbb{U}Y$ is obtained as follows. For G-representations V and W of the same dimension the isometry action (2.8) factors over a G-map

$$\mathbf{L}(V,W)_+ \wedge_{O(V)} Y(V) \;\longrightarrow\; Y(W)$$

that is an equivariant homeomorphism. In the special case $V = \mathbb{R}^n$ we obtain a G-homeomorphism

$$(\mathbb{U}Y)(W) \;=\; \mathbf{L}(\mathbb{R}^n, W)_+ \wedge_{O(n)} Y(\mathbb{R}^n) \;\longrightarrow\; Y(W)$$

which is the W-component of a natural isomorphism $\mathbb{P}\mathbb{U}Y \cong Y$. So the forgetful functor $\mathbb{U}$ and the functor $\mathbb{P}$ of 'extensions to non-trivial G-representations' are inverse equivalences of categories.

Since our Definition 2.1 and Definition II.2.6 of [16] define equivalent categories, it is mainly a matter of taste and convenience in which one to work. The author prefers the present definition because the objects are freed of all unnecessary baggage. As we explained, the value of an equivariant spectrum on a general n-dimensional G-representation V can be recovered canonically from the value at the trivial representation $\mathbb{R}^n$ by the formula $X(V) = \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$, so there is no need to drag the redundant values along. A related point is that in the language of $\mathscr{I}_G$-spectra the 'equivariant' smash product (see Theorem II.3.1 of [16]) may seem more mysterious than it actually is. In fact, in our present setup, the 'equivariant' smash product is simply the smash product of the underlying non-equivariant orthogonal spectra with diagonal group action.

2.1. **Basic examples.**

Example 2.10 (Sphere spectrum)**.** The *equivariant sphere spectrum* $\mathbb{S}$ is given by

$$\mathbb{S}_n \;=\; S^n$$

with action by $O(n)$ from the natural action on $\mathbb{R}^n$ and with trivial action of the group G. This does *not* mean, however, that G acts trivially on the value $\mathbb{S}(V)$ of $\mathbb{S}$ on a general G-representation V. Indeed, the map

$$\mathbb{S}(V) \;=\; \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} S^n \;\longrightarrow\; S^V \;, \quad [\varphi, x] \;\longmapsto\; \varphi(x)$$

is a G-equivariant homeomorphism to the representation sphere of V (which has non-trivial G-action if and only if V has).

Example 2.11 (Suspension spectra)**.** Every pointed G-space A gives rise to a *suspension spectrum* $\Sigma^\infty A$ via

$$(\Sigma^\infty A)_n \;=\; A \wedge S^n \;.$$

The orthogonal group acts through the action on S^n, the group G acts through the action on A, and the structure maps are the canonical homeomorphism $(A \wedge S^n) \wedge S^1 \xrightarrow{\cong} A \wedge S^{n+1}$. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$ (where G necessarily acts trivially on S^0). If we evaluate the suspension spectrum on a G-representation V we obtain

$$(\Sigma^\infty A)(V) \;\cong\; A \wedge S^V \;.$$

This homeomorphism is G-equivariant with respect to the diagonal G-action on the right hand side.

Example 2.12 (Non-equivariant spectra)**.** Every (non-equivariant) orthogonal spectrum X gives rise to a G-spectrum by letting G act trivially. As in the example of the sphere spectrum above, this does *not* mean, however, that G acts trivially on $X(V)$ for a general G-representation V. For example, if the underlying inner product space of V is $\mathbb{R}^n$, then $X(V)$ is X_n with G-action through the representation homomorphism $G \longrightarrow O(n)$.

At first sight, one might think that trivial G-actions should not lead to any interesting new phenomena, but in fact, the opposite is true: orthogonal spectra spectra with trivial G-action underlie *global equivariant* stable homotopy types, where the adjective 'global' refers to simultaneous and compatible actions of all compact Lie groups. We will not discuss the global theory in these notes, but we refer the reader to Chapter 4 of [21] for a detailed exposition of this rich variant of equivariant stable homotopy theory.

Example 2.13 (Eilenberg-Mac Lane spectra)**.** Let M be a $\mathbb{Z}G$-module, i.e., an abelian group M with an additive G-action. The *Eilenberg-Mac Lane spectrum* HM is defined by

$$(HM)_n \;=\; M[S^n] \;,$$

the reduced M-linearization of the n-sphere. The orthogonal group acts through the action on S^n, and the group G acts through its action on M. The structure map $\sigma_n : (HM)_n \wedge S^1 \longrightarrow (HM)_{n+1}$ is given by

$$M[S^n] \wedge S^1 \longrightarrow M[S^{n+1}] , \quad \left(\sum\nolimits_i a_i \cdot x_i\right) \wedge y \longmapsto \sum\nolimits_i a_i \cdot (x_i \wedge y) .$$

If V is a G-representation, then we have a G-equivariant homeomorphism

$$HM(V) \cong M[S^V]$$

where G acts diagonally on the right, through the action on M and on S^V.

The underlying non-equivariant space of $M[S^n]$ is an Eilenberg-Mac Lane space of type (M,n). But more is true: namely $M[S^n]$ is an equivariant Eilenberg-Mac Lane space for the coefficient system (i.e., contravariant functor $\mathcal{O}(G) \longrightarrow \mathcal{A}b$) associated to M that assigns the H-fixed points M^H to the coset G/H. Indeed, since G acts trivially on S^n we have $(M[S^n])^H = (M^H)[S^n]$ for every subgroup H of G. Hence the homotopy groups of $(M[S^n])^H$ vanish in dimensions different from n and the map

$$M^H \longrightarrow \pi_n(M[S^n]^H)$$

that sends $m \in M^H$ to the homotopy class of the H-map

$$m \cdot - \ : \ S^n \longrightarrow M[S^n]$$

is an isomorphism of abelian groups. In particular we see that under this isomorphism the inclusion maps $M[S^n]^H \longrightarrow M[S^n]^K$ correspond to the inclusion $M^H \longrightarrow M^K$, so this is an isomorphism of contravariant functors on the orbit category $\mathcal{O}(G)$. But even more than that is true. As we shall discuss in Example 4.40 below, the Eilenberg-Mac Lane spectrum HM is even an Ω-G-spectrum, and its collection of 0th homotopy groups realizes the Mackey functor associated to the $\mathbb{Z}G$-module M.

The Eilenberg-Mac Lane functor H can be made into a lax symmetric monoidal functor with respect to the tensor product of $\mathbb{Z}G$-modules (with diagonal G-action) and the smash product of orthogonal G-spectra (with diagonal G-action). Indeed, if M and N are $\mathbb{Z}G$-modules, a natural morphism of orthogonal G-spectra

$$HM \wedge HN \longrightarrow H(M \otimes N)$$

is obtained, by the universal property (1.6), from the bilinear morphism

$$\begin{aligned}(HM)_m \wedge (HN)_n \ &= \ M[S^m] \wedge N[S^n] \\ &\longrightarrow \ (M \otimes N)[S^{m+n}] \ = \ (H(M \otimes N))_{m+n}\end{aligned}$$

given by

$$\left(\sum_i m_i \cdot x_i\right) \wedge \left(\sum_j n_j \cdot y_j\right) \longmapsto \sum_{i,j} (m_i \otimes n_j) \cdot (x_i \wedge y_j) .$$

A unit map $\mathbb{S} \longrightarrow H\mathbb{Z}$ is given by the inclusion of generators, and it is equivariant with respect to the trivial G-action on $\mathbb{Z}$.

As a formal consequence, the Eilenberg-Mac Lane functor H turns a G-ring A into an orthogonal G-ring spectrum with multiplication map

$$HA \wedge HA \longrightarrow H(A \otimes A) \xrightarrow{H\mu} HA ,$$

where $\mu : A \otimes A \longrightarrow A$ is the multiplication in A, i.e., $\mu(a \otimes b) = ab$.

Example 2.14 (Real cobordism)**.** The Thom spectrum representing stably almost complex cobordism, has a natural structure of C_2-orthogonal ring spectrum, where the action of the cyclic group C_2 of order two comes from complex conjugation on the coefficients of unitary matrices.

We first consider the collection of pointed C_2-spaces $MU = \{MU_n\}_{n \geq 0}$ defined by

$$MU_n \ = \ EU(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n} ,$$

the Thom space of the tautological complex vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n$ over $BU(n) = EU(n)/U(n)$. Here $U(n)$ is the n-th unitary group consisting of automorphisms of $\mathbb{C}^n$ preserving the standard hermitian scalar product.

There are multiplication maps

$$\mu_{n,m} \ : \ MU_n \wedge MU_m \ \longrightarrow \ MU_{n+m}$$

which are induced from the identification $\mathbb{C}^n \oplus \mathbb{C}^m \cong \mathbb{C}^{n+m}$ which is equivariant with respect to the group $U(n) \times U(m)$, viewed as a subgroup of $U(n+m)$ by direct sum of linear maps. For $n \geq 0$ there are unit maps $\iota_n : S^{\mathbb{C}^n} \longrightarrow MU_n$ using the 'vertex map' $U(n) \longrightarrow EU(n)$. The collection of spaces MU_n does *not* form an orthogonal spectrum since we only get structure maps $MU_n \wedge S^{\mathbb{C}} \longrightarrow MU_{n+1}$ involving a 2-sphere $S^{\mathbb{C}}$. The natural structure that the collection of spaces MU has is that of a 'real spectrum', as we explain in Example 7.11 below. We have to modify the construction somewhat to end up with an orthogonal spectrum.

We set

$$MR_n \ = \ \mathrm{map}(S^{i\mathbb{R}^n}, MU_n)$$

where i stands for the imaginary unit. The orthogonal group act by conjugation (via the complexification map $O(n) \longrightarrow U(n)$ on MU_n). The group C_2 acts on $i\mathbb{R}$ by sign, on MU_n by complex conjugation and on the space MR_n by conjugation.

Then the product of MU combined with smashing maps gives $C_2 \times O(n) \times O(m)$-equivariant maps

$$\begin{array}{ccc} MR_n \wedge MR_m = \mathrm{map}(S^{i\mathbb{R}^n}, MU_n) \wedge \mathrm{map}(S^{i\mathbb{R}^m}, MU_m) & \longrightarrow & \mathrm{map}(S^{i\mathbb{R}^{n+m}}, MU_{n+m}) \cong MR_{n+m} \\ f \quad \wedge \quad g & \longmapsto & f \cdot g = \mu_{n,m} \circ (f \wedge g) \ . \end{array}$$

We make MR into an orthogonal C_2-ring spectrum via the unit maps $S^n \longrightarrow (MR)_n = \mathrm{map}(S^{i\mathbb{R}^n}, MU_n)$ which is adjoint to

$$S^n \wedge S^{i\mathbb{R}^n} \ \cong \ S^{\mathbb{C}^n} \ \xrightarrow{\iota_n} \ MU_n \ .$$

Here we use the C_2-equivariant decomposition $\mathbb{C}^n = 1 \cdot \mathbb{R}^n \oplus i \cdot \mathbb{R}^n$ to identify $S^{\mathbb{C}^n}$ with the smash product of a 'real' and 'imaginary' n-sphere. Since the multiplications of MU and MR are commutative, the centrality condition is automatically satisfied. The resulting orthogonal C_2-ring spectrum is called the *real bordism spectrum*.

The value of the orthogonal spectrum underlying MR on a real inner product space V is given by

$$MR(V) \ = \ \mathrm{map}(S^{iV}, EU(V_{\mathbb{C}})_+ \wedge_{U(V_{\mathbb{C}})} S^{V_{\mathbb{C}}}) \ ,$$

where $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ is the complexification of V, with induced hermitian scalar product.

The (non-equivariant) homotopy groups of MR are given by

$$\pi_k(MR) = \mathrm{colim}_n \ \pi_{k+n}\, \mathrm{map}(S^{i\mathbb{R}^n}, MU_n) \ \cong \ \mathrm{colim}_n \, \pi_{k+2n}(EU(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n}) \ ;$$

so by Thom's theorem they are isomorphic to the ring of cobordism classes of stably almost complex k-manifolds. The underlying non-equivariant spectrum of MR is the complex cobordism spectrum. So even though the individual spaces MR_n are not Thom spaces, the orthogonal spectrum which they form altogether has the 'correct' stable homotopy type.

In Example 7.11 we will reinterpret the $RO(C_2)$-graded equivariant homotopy groups of MR as

$$\pi_k^{C_2}(MR) \ \cong \ \mathrm{colim}_n \ [S^{k+n\mathbb{C}}, MU_n]^{C_2} \ .$$

As we shall also discuss in Example 7.11, the geometric fixed points $\Phi^{C_2} MR$ of MR are stably equivalent to the unoriented cobordism spectrum MO.

Essentially the same construction gives a commutative orthogonal C_2-ring spectrum MSR whose underlying non-equivariant spectrum is a model for special unitary cobordism and whose geometric fixed points are a model for oriented cobordism MSO.

3. Equivariant homotopy groups

The 0-*th equivariant homotopy group* $\pi_0^G(X)$ of an orthogonal G-spectrum X is defined as the colimit

$$\pi_0^G(X) \;=\; \operatorname{colim}_n\, [S^{n\rho_G}, X(n\rho_G)]^G \;, \tag{3.1}$$

where ρ_G is the regular representation of G, $n\rho_G = \rho_G \oplus \cdots \oplus \rho_G$ (n copies) and $[-,-]^G$ means G-equivariant homotopy classes of based G-maps. The colimit is taken along stabilization by the regular representation

$$-\diamond \rho_G \;:\; [S^{n\rho_G}, X(n\rho_G)]^G \;\longrightarrow\; [S^{(n+1)\rho_G}, X((n+1)\rho_G)]^G$$

where the stabilization was defined in (2.6) as $f \diamond \rho_G = \sigma_{n\rho_G,\rho_G}(f \wedge S^{\rho_G})$.
If k is positive, we define

$$\pi_k^G(X) \;=\; \pi_0^G(\Omega^k X) \;;$$

if k is negative, we define

$$\pi_k^G(X) \;=\; \pi_0^G(\mathrm{sh}^{-k} X) \;.$$

Obviously, the definition of equivariant homotopy groups makes essential use of the fact that we can evaluate an orthogonal G-spectrum on a representation (in this case, on multiples of the regular representation), and that we have generalized structure maps relating these values.

First we observe that the colimit $\pi_0^G(X)$ is indeed naturally an abelian group. The regular representation decomposes as $\rho_G = (\rho_G)^G \oplus \bar\rho_G \cong \mathbb{R} \oplus \bar\rho_G$, where $\bar\rho_G$ is the reduced regular representation, the kernel of the augmentation map

$$\rho_G = \mathbb{R}[G] \;\longrightarrow\; \mathbb{R} \;, \quad \sum_{g\in G} \lambda_g \cdot g \;\longmapsto\; \sum_{g\in G} \lambda_g \;.$$

So the representation sphere $S^{n\rho_G}$ decomposes G-equivariantly as a smash product $S^n \wedge S^{n\bar\rho_G}$. For $n \geq 1$ we can use the trivial suspension coordinate to define a group structure on the set $[S^{n\rho_G}, A]^G$. For $n \geq 2$ there are two independent trivial suspension coordinates, so the group structure is abelian. Hence the colimit $\pi_0^G(X)$ inherits an abelian group structure.

It will be important for the development of the theory to know that a based G-map $f : S^V \longrightarrow X(V)$, for any G-representation V, gives rise to an unambiguously defined element $\langle f\rangle$ in $\pi_0^G(X)$ as follows. First we consider a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$. We let $W - \varphi(V)$ denote the orthogonal complement inside W of the image $\varphi(V)$. Given a G-map $f : S^V \longrightarrow X(V)$ we define another G-map $\varphi_* f : S^W \longrightarrow X(W)$ as the composite

$$S^W \;\cong\; S^{V\oplus(W-\varphi(V))} \;\xrightarrow{\;f\diamond(W-\varphi(V))\;}\; X(V\oplus(W-\varphi(V))) \;\cong\; X(W) \tag{3.2}$$

where we have used φ twice to identify $V \oplus (W - \varphi(V))$ with W. If $\psi : W \longrightarrow U$ is another G-isometric embedding, then we have

$$\psi_*(\psi_* f) \;=\; (\psi\varphi)_* f \;.$$

We observe that if φ is bijective (i.e., an equivariant isometry), then $\varphi_* f$ becomes the 'φ-conjugate' of f, i.e., the composite

$$S^W \;\xrightarrow{\;\varphi^{-1}\;}\; S^V \;\xrightarrow{\;f\;}\; X(V) \;\xrightarrow{\;X(\varphi)\;}\; X(W) \;.$$

This construction also generalizes the stabilization by a representation. Indeed, when $i : V \longrightarrow V \oplus W$ is the inclusion of the first summand, then $i_* f = f \diamond W$, the stabilization of f by W in the sense of (2.6).

Given a G-map $f : S^V \longrightarrow X(V)$, we choose a linear isometric embedding $j : V \longrightarrow m\rho_G$ for suitably large m and obtain an element

$$\langle f\rangle \;=\; [j_* f] \;\in\; \pi_0^G(X) \;.$$

Clearly, for G-homotopic maps f and f', the maps $j_* f$ and $j_* f'$ are again G-homotopic. It is more subtle to see that $\langle f\rangle$ does not depend on the choice of embedding j, but we will show this now.

Proposition 3.3. *Let X be a G-spectrum, V a G-representation and $f : S^V \longrightarrow X(V)$ a based G-map.*

(i) *The class* $\langle f\rangle = [j_* f]$ *in* $\pi_0^G(X)$ *is independent of the choice of linear isometric embedding* j.
(ii) *For every* G*-equivariant linear isometric embedding* $\varphi : V \longrightarrow W$ *we have*

$$\langle \varphi_* f\rangle \;=\; \langle f\rangle \qquad \textit{in} \quad \pi_0^G(X).$$

(iii) *For every* G*-representation* W *we have* $\langle f \diamond W\rangle = \langle f\rangle$.

Proof. We start by proving a special case of (ii); loosely speaking we show that conjugation of the G-map $g : S^W \longrightarrow X(W)$ by an automorphism of the representation W is homotopically trivial after stabilization with W. In more detail: given an automorphism $\varphi : W \longrightarrow W$ (i.e., a G-equivariant linear isometry), the map g and its conjugate $\varphi_* g$ are not generally homotopic, but:

Claim: For every based continuous G-map $g : S^W \longrightarrow X(W)$ the two maps

$$g \diamond W \ , \ (\varphi_* g) \diamond W \ : \ S^{W\oplus W} \ \longrightarrow \ X(W \oplus W)$$

are G-homotopic.

To prove the claim we let $i : W \longrightarrow W \oplus W$ be the inclusion of the first summand. We define

$$\psi \ : \ [0,1] \ \longrightarrow \ [0,1] \qquad \text{by} \qquad \psi(x) = \sqrt{1-x^2}$$

and consider the continuous map

$$H \ : \ W \times [0,1] \ \longrightarrow \ W \oplus W \ , \quad (w,t) \ \longmapsto \begin{cases} (\psi(2t)\cdot\varphi(w),\ 2t\cdot w) & \text{for } 0 \le t \le 1/2, \\ ((2t-1)\cdot w,\ \psi(2t-1)\cdot w) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then H is a homotopy, through G-equivariant isometric embeddings, from $i \circ \varphi$, via the second summand inclusion, to i. In particular, $H_t : W \longrightarrow W \oplus W$ defined by $H_t(w) = H(w,t)$ is a G-equivariant linear isometric embedding for all $t \in [0,1]$. So

$$t \ \longmapsto (H_t)_* g \ : \ S^{W\oplus W} \longrightarrow \ X(W \oplus W)$$

is the desired continuous 1-parameter family of G-equivariant based maps, from

$$(H_0)_* g \ = \ (i \circ \varphi)_* g \ = \ i_*(\varphi_* g) \ = \ (\varphi_* g) \diamond W$$

to $(H_1)_* g = i_* g = g \diamond W$.

(i) Let $j : V \longrightarrow m\rho$ and $j' : V \longrightarrow m'\rho$ be two equivariant linear isometric embeddings. We first discuss the case where $m = m'$. We choose an equivariant isometry $\varphi : m\rho \longrightarrow \ m\rho$ such that $\varphi j = j'$. Then we have

$$[j'_* f] \ = \ [\varphi_*(j_* f)] \ = \ [(\varphi_*(j_* f)) \diamond m\rho] \ = \ [(j_* f) \diamond m\rho] \ = \ [j_* f]$$

by the claim above for $W = m\rho$ and $g = j_* f : S^{m\rho} \longrightarrow X(m\rho)$. In general we can suppose without loss of generality that $m' = m+n \ge m$. We let $i : m\rho \longrightarrow m\rho \oplus n\rho = (m+n)\rho$ the inclusion of the first summand. Then we have

$$(ij)_* f \ = \ i_*(j_* f) \ = \ (j_* f) \diamond n\rho$$

and hence

$$[j'_* f] \ = \ [(ij)_* f] \ = \ [j_* f] \ ,$$

where the first equation is the special case of the previous paragraph.

(ii) If $j : W \longrightarrow m\rho$ is an equivariant linear isometric embedding, then so is $j\varphi : V \longrightarrow m\rho$. Since we can use any equivariant isometric embedding to define the class $\langle f\rangle$, we get

$$\langle \varphi_* f\rangle \ = \ [j_*(\varphi_* f)] \ = \ [(j\varphi)_* f] \ = \ \langle f\rangle \ .$$

Part (iii) is a special case of (ii) because $f \diamond W = i_* f$ for the inclusion $i : V \longrightarrow V \oplus W$ of the first summand. $\square$

Definition 3.4. A morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is a $\underline{\pi}_*$-*isomorphism* if the induced map $\pi_k^H(f) : \pi_k^H(X) \longrightarrow \pi_k^H(Y)$ is an isomorphism for all integers k and all subgroups H of G. We define the G-*equivariant stable homotopy category* $\mathrm{Ho}(\mathcal{S}p_G)$ as the category obtained from the category $\mathcal{S}p_G$ of orthogonal G-spectra by formally inverting the $\underline{\pi}_*$-isomorphisms.

The class of $\underline{\pi}_*$-isomorphisms takes part in several model structures on the category of orthogonal G-spectra: Mandell and May establish a 'projective' stable model structure in [16, III Thm. 4.2]. Stolz constructs a stable '$\mathbb{S}$-model structure' with the same equivalences, but more cofibrations, in [26, Thm. 2.3.27]. Hill, Hopkins and Ravenel, finally, provide the stable 'positive complete model structure' in [10, Prop. B.63]. Hence the tools of homotopical algebra are available for studying and manipulating the G-equivariant stable homotopy category $\mathrm{Ho}(\mathcal{S}p_G)$.

Functoriality. We can now discuss the functoriality of the G-equivariant homotopy groups with respect to change of the group G. We let $\alpha : K \longrightarrow G$ be any group homomorphism. We denote by α^* the restriction functor from G-spaces to K-spaces (or from G-representations to K-representations) along α, i.e., $\alpha^* X$ (respectively $\alpha^* V$) is the same topological space as X (respectively the same inner product space $\alpha^* V$) endowed with K-action via

$$k \cdot x \;=\; \alpha(k) \cdot x \;.$$

Given an orthogonal G-spectrum X we denote by $\alpha^* X$ the orthogonal K-spectrum with the same underlying orthogonal spectrum as X, but with K-action obtained by restricting the G-action along α. We note that for every G-representation V, the K-spaces $\alpha^*(X(V))$ and $(\alpha^* X)(\alpha^* V)$ are equal (not just isomorphic).

We can define a *restriction map*

$$\alpha^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^K(\alpha^* X) \tag{3.5}$$

by restricting everything in sight from the group G to K along α. More precisely, given a G-map $f : S^{n\rho_G} \longrightarrow X(n\rho_G)$ we can consider the K-map

$$\alpha^* f \;:\; S^{\alpha^*(n\rho_G)} = \alpha^*\left(S^{n\rho_G}\right) \;\longrightarrow\; \alpha^*(X(n\rho_G)) = (\alpha^* X)(\alpha^*(n\rho_G)) \;.$$

As explained in Proposition 3.3, such a map defines an element in the 0-th K-equivariant homotopy group of $\alpha^* X$, and we set

$$\alpha^* \langle f \rangle \;=\; \langle \alpha^* f \rangle \;\in\; \pi_0^K(\alpha^* X) \;.$$

Proposition 3.3 (iii) and the relation

$$\alpha^*(f \diamond \rho_G) \;=\; (\alpha^* f) \diamond (\alpha^* \rho_G)$$

guarantee that the outcome only depends on the class of f in $\pi_0^G(X)$.

Clearly the restriction map is additive and for a second group homomorphism $\beta : L \longrightarrow K$ we have $\beta^*(\alpha^* X) = (\alpha\beta)^* X$ and

$$\beta^* \circ \alpha^* \;=\; (\alpha\beta)^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^L\left((\alpha\beta)^* X\right) \;.$$

For later reference we give another interpretation of the restriction map along an inner automorphism. For $g \in G$ we denote by

$$c_g \;:\; G \;\longrightarrow\; G \;, \quad \gamma \;\longmapsto\; c_g(\gamma) \;=\; g^{-1}\gamma g$$

the conjugation automorphism by g. We observe that for every orthogonal G-spectrum X the map

$$l_g^X \;:\; c_g^* X \;\longrightarrow\; X \;, \quad x \;\longmapsto\; gx$$

given by left multiplication by g is an isomorphism of orthogonal G-spectra from the restriction of X along c_g to X.

Proposition 3.6. *For every G-spectrum X and every $g \in G$, the maps*

$$c_g^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^G(c_g^* X) \qquad \textit{and} \qquad \pi_0^G(l_g^X) \;:\; \pi_0^G(c_g^* X) \;\longrightarrow\; \pi_0^G(X)$$

are inverse to each other.

Proof. We consider a G-map $f : S^V \longrightarrow X(V)$ that represents a class in $\pi_0^G(X)$ (for example, for V can be a multiple of the regular representation). The diagram of G-maps

$$\begin{tikzcd}
c_g^*(S^V) \arrow[rr, "c_g^* f"] \arrow[d, equal] & & c_g^*(X(V)) \arrow[r, equal] \arrow[d, "l_g^{X(V)}"'] & (c_g^*X)(c_g^*V) \arrow[d, "l_g^X(c_g^*V)"] \\
S^{c_g^*V} \arrow[r, "l_g^V"] \arrow[rrr, bend right, "(l^V_{g^{-1}})_* f"'] & S^V \arrow[r, "f"] & X(V) \arrow[r, "X(l^V_{g^{-1}})"] & X(c_g^*V)
\end{tikzcd}$$

commutes, where $l_g^V, l_g^{X(V)}$ and l_g^X are the left multiplication maps on the representation V, the space $X(V)$ respectively the spectrum X. The left square commutes because f is a G-map, and the right square commutes because the G-action on $X(V) = \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$ was defined diagonally, using the G-action on V and on X_n. So we get

$$\pi_0^G(l_g^X)\langle c_g^* f\rangle \ = \ \langle l_g^X(c_g^*V) \circ (c_g^* f)\rangle \ = \ \langle (l^V_{g^{-1}})_* f\rangle \ = \ \langle f\rangle \ .$$

The last equation holds by Proposition 3.3 (ii) because $l^V_{g^{-1}} : V \longrightarrow c_g^*V$ is an isomorphism of G-representations. □

If X is a G-spectrum and H subgroup of G, we denote by $\pi_k^H(X)$ the H-equivariant homotopy group of the underlying H-spectrum of X. The collections of groups $\pi_k^H(X)$, for $H \subseteq G$, have a lot of extra structure, known as a *Mackey functor*, as H varies over the subgroups of G, It suffices to explain this structure for $k = 0$, and two thirds of the structure maps are a special case of the functoriality of the equivariant homotopy groups in the group.

Restriction. We let H be a subgroup of G. As the name suggests, we obtain a *restriction map*

$$\mathrm{res}^G_H \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^H(X)$$

by restricting everything in sight from the group G to H. More formally, we let $i : H \longrightarrow G$ denote the inclusion and we define

$$\mathrm{res}^G_H \ = \ i^* \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^H(i^*X) = \pi_0^H(X) \ .$$

We have $\mathrm{res}^G_G = \mathrm{Id}$ and restriction is transitive, i.e., for subgroups $K \subseteq H \subseteq G$ we have $\mathrm{res}^H_K \circ \mathrm{res}^G_H = \mathrm{res}^G_K$.

Conjugation. For every subgroup H of G and every element $g \in G$ the conjugation map

$$c_g \ : \ H \ \longrightarrow \ H^g = g^{-1}Hg \ , \quad h \ \longmapsto \ c_g(h) \ = \ g^{-1}hg$$

is a group homomorphism; moreover, left multiplication by g is an isomorphism

$$l_g^X \ : \ c_g^* X \ \longrightarrow \ X$$

of orthogonal H-spectra from the restriction of the underlying H-spectrum of X along c_g to the underlying H-spectrum. We denote the composite

$$\pi_0^{H^g}(X) \ \xrightarrow{\ c_g^*\ } \ \pi_0^H(c_g^*X) \ \xrightarrow{\ \pi_0^H(l_g^X)\ } \ \pi_0^H(X) \tag{3.7}$$

by g_* and refer to it as the *conjugation map*.

Conjugation is transitive. Indeed, for $g, \bar{g} \in G$ we have $c_{g\bar{g}} = c_{\bar{g}} \circ c_g : H \longrightarrow H^{g\bar{g}}$ and thus $c^*_{g\bar{g}} = c_g^* \circ c^*_{\bar{g}}$ as maps from $\pi_0^{H^{g\bar{g}}} X$ to $\pi_0^H(c_g^*(c_{\bar{g}}^* X)) = \pi_0^H(c^*_{g\bar{g}} X)$. So we deduce

$$\begin{aligned}
g_* \circ \bar{g}_* \ &= \ \pi_0^H(l_g^X) \circ c_g^* \circ \pi_0^{H^g}(l_{\bar{g}}^X) \circ c_{\bar{g}}^* \\
&= \ \pi_0^H(l_g^X) \circ \pi_0^H(c_g^*(l_{\bar{g}}^X)) \circ c_g^* \circ c_{\bar{g}}^* \ = \ \pi_0^H(l^X_{g\bar{g}}) \circ c^*_{g\bar{g}} \ = \ (g\bar{g})_*
\end{aligned}$$

as maps $\pi_0^{H^{g\bar g}}(X) \longrightarrow \pi_0^H(X)$. Here the second equality is the naturality of the restriction homomorphism c_g^*, and the third equality uses that $l_g^X \circ c_g^*(l_{\bar g}^X) = l_{g\bar g}^X$ as morphisms $c_{g\bar g}^* X \longrightarrow X$.

Conjugation yields an action of the Weyl group. Indeed, if g normalizes H, then $H^g = H$ and g_* is an automorphism of the group $\pi_0^H(X)$. If moreover g belongs to H, then g_* is the identity automorphism of $\pi_0^H(X)$ by Proposition 3.6. So the action of the normalizer $N_G H$ of H on $\pi_0^H(X)$ factors over the Weyl group $WH = N_G H/H$.

Now we discuss various properties of the homotopy groups of G-spectra, for example that looping and suspending a spectrum shifts homotopy groups, a long exact sequences of homotopy groups associated to a mapping cone, or that homotopy groups commute with sums and products.

Example 3.8 (Loop and suspension by representations)**.** Let X be a G-spectrum and V a representation. The *loop spectrum* $\Omega^V X$ is defined by

$$(\Omega^V X)_n \ = \ \Omega^V(X_n) \ = \ \mathrm{map}(S^V, X_n) \ ,$$

the based mapping space from the sphere S^V to the n-th level of X. The group $O(n)$ acts through its action on X_n and G acts by conjugation, i.e., via $({}^g\varphi)(v) = g\cdot\varphi(g^{-1}v)$ for $g : S^V \longrightarrow X_n$, $v \in S^V$ and $g \in G$. The structure map is given by the composite

$$\mathrm{map}(S^V, X_n)\wedge S^1 \ \longrightarrow \ \mathrm{map}(S^V, X_n\wedge S^1) \ \xrightarrow{\mathrm{map}(S^V,\sigma_n)} \ \mathrm{map}(S^V, X_{n+1})$$

where the first is an assembly map that sends $\varphi\wedge t \in \mathrm{map}(S^V, X_n)\wedge S^1$ to the map sending $v \in S^V$ to $\varphi(v)\wedge t$.

The *suspension* $S^V \wedge X$ is defined by

$$(S^V\wedge X)_n \ = \ S^V \wedge X_n \ ,$$

the smash product of the sphere S^V with the n-th level of X. The group $O(n)$ acts through its action on X_n and G acts diagonally, through the actions on S^V and X_n. The structure map is given by the composite

$$(S^V\wedge X)_n\wedge S^1 \ = \ S^V\wedge X_n\wedge S^1 \ \xrightarrow{S^V\wedge\sigma_n} \ S^V\wedge X_{n+1} \ = \ (S^V\wedge X)_{n+1}$$

For the values on a G-representation V we have

$$(\Omega^V X)(W) \ \cong \ \mathrm{map}(S^V, X(W)) \qquad \text{respectively} \qquad (S^V\wedge X)(W) \ \cong \ S^V\wedge X(W) \ .$$

Both constructions are special cases of mapping spectra from and smash products with a based G-spaces, compare Example 5.2. We obtain an adjunction between $S^V\wedge -$ and Ω^V as the special case $A = S^V$ of (5.3) below.

Example 3.9 (Shift by representations)**.** Let V be a G-representation. The V*-shift* $\mathrm{sh}^V X$ of a G-spectrum X is given in level n by the G-space

$$(\mathrm{sh}^V X)_n \ = \ X(V\oplus\mathbb{R}^n) \ .$$

The orthogonal group $O(n)$ acts through the monomorphism $\mathrm{Id}_V\oplus - : O(\mathbb{R}^n)\longrightarrow O(V\oplus\mathbb{R}^n)$. The structure maps of $\mathrm{sh}^V X$ are the generalized structure maps for X. We observe that $(\mathrm{sh}^m X)_n = X(\mathbb{R}^m\oplus\mathbb{R}^n)$ is canonically isomorphic to X_{m+n}, which explains the name 'shift'. On the n-level of $\mathrm{sh}^V X$ the group $O(V)$ acts via the inclusion $-\oplus\mathrm{Id}_{\mathbb{R}^n} : O(V)\longrightarrow O(V\oplus\mathbb{R}^n)$. These levelwise actions commute with all other structure, so they constitute a continuous left action of the group $O(V)$ on the spectrum $\mathrm{sh}^V X$.

The suspension and the shift of an equivariant spectrum are related by a natural morphism $\lambda_X : S^V\wedge X\longrightarrow \mathrm{sh}^V X$ in level n as the composite

$$S^V\wedge X_n \xrightarrow{\text{twist}} X_n\wedge S^V \xrightarrow{\sigma_{n,V}} X(\mathbb{R}^n\oplus V) \xrightarrow{X(\tau_{\mathbb{R}^n,V})} X(V\oplus\mathbb{R}^n) = (\mathrm{sh}^V X)_n \ . \tag{3.10}$$

It follows that for every G-representation W the map $\lambda_X(W) : S^V \wedge X(W) \longrightarrow (\mathrm{sh}^V X)(W)$ is the composite

$$S^V \wedge X(W) \xrightarrow{\text{twist}} X(W) \wedge S^V \xrightarrow{\sigma_{W,V}} X(W \oplus V) \xrightarrow{X(\tau_{V,W})} X(V \oplus W) = (\mathrm{sh}^V X)(W) \ .$$

As an example, the shift of a suspension spectrum is another suspension spectrum:

$$\mathrm{sh}^V(\Sigma^\infty A) \ \cong \ \Sigma^\infty(A \wedge S^V) \ .$$

For another G-representation W we have

$$(\mathrm{sh}^V X)(W) \ \cong \ X(V \oplus W)$$

by a natural G-equivariant homeomorphism, and hence $\mathrm{sh}^U(\mathrm{sh}^V X)$ is isomorphic to $\mathrm{sh}^{V\oplus U} X$.

Now we show that looping, suspending and shifting a G-spectrum shifts the $RO(G)$-graded homotopy groups. In particular, looping, suspending and shifting by a trivial representation shifts the $\mathbb{Z}$-graded equivariant homotopy groups. The loop homomorphism starts from the bijection

$$\alpha \ : \ [S^{k+n\rho_G}, \Omega^V X(n\rho_G)]^G \ \cong \ [S^{V+k+n\rho_G}, X(n\rho_G)]^G$$

defined by sending a representing G-map $f : S^{k+n\rho_G} \longrightarrow \Omega^V X(n\rho_G)$ to the class of the adjoint $\hat{f} : S^{V+k+n\rho_G} \longrightarrow X(n\rho_G)$ given by $\hat{f}(s \wedge t) = f(t)(s)$, where $s \in S^V$, $t \in S^{k+n\rho_G}$. As n varies, these particular isomorphisms are compatible with stabilization maps, so they induce an isomorphism

$$\alpha \ : \ \pi_k^G(\Omega^V X) \xrightarrow{\cong} \pi_{V+k}^G(X) \qquad (3.11)$$

on colimits. In the special case $V = \mathbb{R}$ this becomes a natural isomorphism $\alpha : \pi_k^G(\Omega X) \cong \pi_{1+k}^G(X)$.

The maps

$$S^V \wedge - \ : \ [S^{k+n\rho_G}, X(n\rho_G)]^G \ \longrightarrow \ [S^{V+k+n\rho_G}, S^V \wedge X(n\rho_G)]^G$$

given by smashing from the left with the identity of S^V are compatible with the stabilization process for the equivariant homotopy groups for X respectively $S^V \wedge X$, so upon passage to colimits they induce a natural map of homotopy groups

$$S^V \wedge - \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_{V+k}^G(S^V \wedge X) \ ,$$

which we call the *suspension homomorphism*.

The *shift homomorphism*

$$\mathrm{sh}^V \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_V^G(\mathrm{sh}^V X)$$

is defined by sending the class represented by a G-map $f : S^{n\rho_G} \longrightarrow X(n\rho_G)$ to the class $\langle i_* f \rangle$, where $i : n\rho_G \longrightarrow V \oplus n\rho_G$ is the inclusion of the second summand and

$$i_* f \ : \ S^{V+n\rho_G} \ \longrightarrow \ X(V \oplus n\rho_G) = (\mathrm{sh}^V X)(n\rho_G)$$

is as in (3.2). If we stabilize f to $f \diamond \rho_G$ then $i_* f$ changes to $(i_* f) \diamond \rho_G$. So the assignment $\mathrm{sh}^V[f] = \langle i_* f \rangle$ is well-defined.

Proposition 3.12. *For every orthogonal G-spectrum X, every integer k and every G-representation V the loop homomorphism*

$$\alpha \ : \ \pi_k^G(\Omega^V X) \ \longrightarrow \ \pi_{V+k}^G(X) \ ,$$

the suspension homomorphism

$$S^V \wedge - \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_{V+k}^G(S^V \wedge X)$$

and the shift homomorphism

$$\mathrm{sh}^V \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_{V+k}^G(\mathrm{sh}^V X)$$

are isomorphisms. Moreover, the unit $\eta_X : X \longrightarrow \Omega^V(S^V \wedge X)$ and counit $\epsilon_X : S^V \wedge \Omega^V X \longrightarrow X$ of the adjunction, and the morphism $\lambda_X : S^V \wedge X \longrightarrow \mathrm{sh}^V X$ and its adjoint $\tilde{\lambda}_X : X \longrightarrow \Omega^V(\mathrm{sh}^V X)$ are $\underline{\pi}_*$-isomorphisms.

Proof. For the course of the proof we abbreviate the regular representation of G as $\rho = \rho_G$. We already justified why the loop morphism α is an isomorphism. We deal with the shift homomorphism next. There is a tautological map in the opposite direction: we send the class in $\pi_V^G(\mathrm{sh}^V X)$ represented by a G-map $f : S^{V+n\rho} \longrightarrow (\mathrm{sh}^V X)(n\rho)$ to the class

$$\langle f : S^{V+n\rho} \longrightarrow X(V \oplus n\rho)\rangle \ \in \ \pi_0^G(X) \ .$$

In other words: we don't change the representing map at all and only rewrite the target $(\mathrm{sh}^V X)(n\rho)$ as $X(V \oplus n\rho)$. This is clearly compatible with stabilization by the regular representation, so it descends to a well-defined map $\pi_V^G(\mathrm{sh}^V X) \longrightarrow \pi_0^G(X)$. The two maps are inverse to each other by Proposition 3.3 (ii).

The composite

$$\pi_k^G(X) \xrightarrow{S^V\wedge -} \pi_{V+k}^G(S^V\wedge X) \xrightarrow{\pi_{V+k}^G(\lambda_X)} \pi_{V+k}^G(\mathrm{sh}^V X)$$

is the shift homomorphism, and hence bijective. So the suspension homomorphism is injective. To show that the suspension homomorphism is surjective, we let $f : S^{V+k+n\rho} \longrightarrow S^V\wedge X(n\rho)$ represent an element in $\pi_{V+k}^G(S^V\wedge X)$. Then the map

$$\lambda_X(n\rho)\circ f \ : \ S^{V+k+n\rho} \ \longrightarrow \ X(V \oplus n\rho)$$

represents an element in $\pi_k^G(X)$. The map

$$S^V \wedge (\lambda_X(n\rho)\circ f) \ : \ S^{V+V+k+n\rho} \ \longrightarrow \ S^V \wedge X(V \oplus n\rho)$$

differs from $V \diamond f$ (the stabilization of f along the embedding $V \longrightarrow V \oplus V$ as the second summand) by the twist homeomorphism $\tau_{V,V} : S^{V+V} \longrightarrow S^{V+V}$ that interchanges the two summands. The twist homeomorphism $\tau_{V,V}$ is equivariantly homotopic to the homeomorphism $S^{V+(-\mathrm{Id})} : S^{V+V} \longrightarrow S^{V+V}$. We conclude that

$$\begin{aligned} S^V \wedge \langle \lambda_X(n\rho)\circ f \circ S^{(-\mathrm{Id})+k+n\rho}\rangle \ &= \ \langle (S^V \wedge (\lambda_X(n\rho)\circ f))\circ S^{V+(-\mathrm{Id})+k+n\rho}\rangle \\ &= \ \langle (S^V \wedge (\lambda_X(n\rho)\circ f))\circ (\tau_{V,V} \wedge S^{k+n\rho})\rangle \ = \ \langle V \diamond f\rangle \ = \ [f] \end{aligned}$$

So the suspension homomorphism is surjective, and hence bijective. This concludes the proof that the loop homomorphism, the suspension homomorphism and the shift homomorphism are isomorphisms.

The composite

$$\pi_k^G(X) \xrightarrow{\pi_k^G(\eta_X)} \pi_k^G(\Omega^V(S^V\wedge X)) \xrightarrow{\alpha} \pi_{V+k}^G(S^V\wedge X)$$

coincides with the suspension isomorphism. Since the loop and suspension homomorphisms are isomorphisms, the adjunction unit $\eta_X : X \longrightarrow \Omega^V(S^V \wedge X)$ induces isomorphisms on π_k^G for all integers k. The composite

$$\pi_k^G(X) \xrightarrow{\pi_k^G(\tilde\lambda_X)} \pi_k^G(\Omega^V(\mathrm{sh}^V X)) \xrightarrow{\alpha} \pi_{V+k}^G(\mathrm{sh}^V X)$$

coincides with the shift isomorphism. Because the loop homomorphism and the shift homomorphism are isomorphisms, we conclude that the morphism $\tilde\lambda_X : X \longrightarrow \Omega^V(\mathrm{sh}^V X)$ induces isomorphisms on π_k^G for all integers k.

The composite

$$\pi_k^G(\Omega^V X) \xrightarrow{S^V\wedge -} \pi_{V+k}^G(S^V\wedge \Omega^V X) \xrightarrow{\pi_{V+k}^G(\epsilon_X)} \pi_{V+k}^G(X)$$

coincides with the loop isomorphism. So ϵ_X induces isomorphisms on π_{V+k}^G for all integers k. Replacing X by $\mathrm{sh}^V X$ shows that $\pi_{V+k}^G(\epsilon_{\mathrm{sh}^V X})$ is an isomorphism for all $k \in \mathbb{Z}$. Now

$$S^V\wedge \Omega^V(\mathrm{sh}^V X) \ = \ \mathrm{sh}^V(S^V\wedge \Omega^V X)$$

and $\epsilon_{\mathrm{sh}^V X} = \mathrm{sh}^V(\epsilon_X)$. So the following diagram commutes by naturality of the shift isomorphism:

$$\begin{tikzcd}
\pi_k^G(S^V \wedge \Omega^V X) \arrow[r, "\mathrm{sh}^V", "\cong"'] \arrow[dd, "\pi_k^G(\epsilon_X)"'] & \pi_{V+k}^G(\mathrm{sh}^V(S^V \wedge \Omega^V X)) \arrow[d, equal] \arrow[dd, bend left, "\pi_{V+k}^G(\mathrm{sh}^V \epsilon_X)"] \\
& \pi_{V+k}^G(S^V \wedge \Omega^V(\mathrm{sh}^V X)) \arrow[d, "\pi_{V+k}^G(\epsilon_{\mathrm{sh}^V X})"'] \\
\pi_k^G(X) \arrow[r, "\cong", "\mathrm{sh}^V"'] & \pi_{V+k}^G(\mathrm{sh}^V X)
\end{tikzcd}$$

We conclude that $\pi_k^G(\epsilon_X)$ is an isomorphism, because the other maps are.

The composite

$$\pi_k^G(X) \xrightarrow{S^V \wedge -} \pi_{V+k}^G(S^V \wedge X) \xrightarrow{\pi_{V+k}^G(\lambda_X)} \pi_{V+k}^G(\mathrm{sh}^V X)$$

coincides with the shift isomorphism. Because the suspension and the shift homomorphism are isomorphisms, we conclude that the morphism λ_X induces isomorphisms on π_{V+k}^G for all integers k. Replacing X by $\mathrm{sh}^V X$ shows that $\pi_{V+k}^G(\lambda_{\mathrm{sh}^V X})$ is an isomorphism for all $k \in \mathbb{Z}$. Now $S^V \wedge \mathrm{sh}^V X = \mathrm{sh}^V(S^V \wedge X)$, but the two morphisms $\lambda_{\mathrm{sh}^V X}$ and $\mathrm{sh}^V(\lambda_X)$ do *not* coincide. Instead, they differ by the involution τ of $\mathrm{sh}^V(\mathrm{sh}^V X)$ that interchanges the two shift coordinates. So the following diagram commutes by naturality of the shift isomorphism:

$$\begin{tikzcd}
\pi_k^G(S^V \wedge X) \arrow[r, "\mathrm{sh}^V", "\cong"'] \arrow[dd, "\pi_k^G(\lambda_X)"'] & \pi_{V+k}^G(\mathrm{sh}^V(S^V \wedge X)) \arrow[d, "\pi_{V+k}^G(\lambda_{\mathrm{sh}^V X})"'] \arrow[dd, bend left, "\pi_{V+k}^G(\mathrm{sh}^V \lambda_X)"] \\
& \pi_{V+k}^G(\mathrm{sh}^V(\mathrm{sh}^V X)) \arrow[d, "\pi_{V+k}^G(\tau)"'] \\
\pi_k^G(\mathrm{sh}^V X) \arrow[r, "\cong", "\mathrm{sh}^V"'] & \pi_{V+k}^G(\mathrm{sh}^V(\mathrm{sh}^V X))
\end{tikzcd}$$

We conclude that $\pi_k^G(\lambda_X)$ is an isomorphism, because the other maps are.

The restriction of $\Omega^V X$, $S^V \wedge X$ and $\mathrm{sh}^V X$ to a subgroup H of G is again $\Omega^V X$, $S^V \wedge X$ or $\mathrm{sh}^V X$, respectively, where now V denotes the underlying H-representation of V. So by applying the previous arguments to the underlying H-spectrum of X and the underlying H-representation of V proves that $\pi_k^H(\eta_X)$, $\pi_k^H(\epsilon_X)$, $\pi_k^H(\lambda_X)$ and $\pi_k^H(\tilde{\lambda}_X)$ are isomorphisms for every subgroup H of G; hence η_X, ϵ_X, λ_X and $\tilde{\lambda}_X$ are $\underline{\pi}_*$-isomorphisms. □

As a word of warning we remark that the analog of the map λ_X in the world of *symmetric* G-spectra (with a G-set in place of the G-representation V) is not generally a $\underline{\pi}_*$-isomorphism. This phenomenon can be traced back to Proposition 3.3 which has no counterpart in the world of symmetric G-spectra.

Now we introduce an important concept, the notion of 'G-Ω-spectra', which encode equivariant infinite loop spaces.

Definition 3.13. An orthogonal G-spectrum X is a *G-Ω-spectrum* if for every subgroup H of G and every pair of H-representations V, W the map $\tilde{\sigma}_{V,W} : X(V) \longrightarrow \Omega^W X(V \oplus W)$ which is adjoint to the generalized structure map $\sigma_{V,W} : X(V) \wedge S^W \to X(V \oplus W)$ is a weak H-equivalence.

G-Ω-spectra do not come up so frequently in nature. Some examples are given by Eilenberg-Mac Lane spectra of $\mathbb{Z}G$-modules (see Examples 2.13 and 4.40) and spectra that arise from very special Γ-G-spaces by evaluation on spheres.

Shifting preserves G-Ω-spectra: if X is a G-Ω-spectrum and U, V and W are representations of a subgroup H of G, then the map

$$\tilde{\sigma}_{U,W} \ : \ (\mathrm{sh}^V X)(U) \ \longrightarrow \ \Omega^W(\mathrm{sh}^V X)(U \oplus W)$$

for the spectrum $\mathrm{sh}^V X$ is H-homeomorphic to the map

$$\tilde{\sigma}_{V\oplus U,W} \ : \ X(V \oplus U) \ \longrightarrow \ \Omega^W X(V \oplus U \oplus W)$$

for X, and hence a weak H-equivalence.

Proposition 3.14. *For every G-Ω-spectrum X, every $k \geq 0$ and every subgroup H of G the map*

$$\pi_k(X_0^H) \ \longrightarrow \ \pi_k^H(X)$$

is an isomorphism.

Proof. In the special case where $V = n\rho_H$ and $W = \rho_H$ are multiples of the regular representation, the defining property of a G-Ω-spectrum specializes to the fact that the maps

$$\tilde{\sigma}_{n\rho_H,\rho_H} \ : \ X(n\rho_H) \ \longrightarrow \ \Omega^{\rho_H} X((n+1)\rho_H)$$

are H-weak equivalences. If we loop by $S^{n\rho_H}$ and take H-equivariant homotopy classes, we see that the stabilization map

$$- \diamond \rho_H \ : \ [S^{n\rho_H}, X(n\rho_H)]^H \ \longrightarrow \ [S^{(n+1)\rho_H}, X((n+1)\rho_H)]^H$$

is bijective. The group $\pi_0^H(X)$ is the colimit of this sequence, so the upshot is that for every G-Ω-spectrum X the map

$$\pi_0(X_0^H) \ \longrightarrow \ \pi_0^H(X)$$

is an isomorphism. Under this isomorphism the restriction map $\pi_0^H(X) \longrightarrow \pi_0^K(X)$ for $K \subset H$ corresponds to the map induced by the inclusion $X_0^H \longrightarrow X_0^K$ on path components. □

Construction 3.15. We can now indicate how a G-spectrum can be naturally approximated, up to $\underline{\pi}_*$-isomorphism, by a G-Ω-spectrum. For this purpose we introduce a functor called Q as the mapping telescope of the sequence

(3.16)

$$X \ \xrightarrow{\tilde{\lambda}_X} \ \Omega^\rho \, \mathrm{sh}^\rho X \ \xrightarrow{\Omega^\rho(\tilde{\lambda}_{\mathrm{sh}^\rho X})} \ \cdots \ \longrightarrow \ \Omega^{m\rho} \, \mathrm{sh}^{m\rho} X \ \xrightarrow{\Omega^{m\rho}(\tilde{\lambda}_{\mathrm{sh}^{m\rho} X})} \ \Omega^{(m+1)\rho} \, \mathrm{sh}^{(m+1)\rho} X \ \longrightarrow \ \cdots \ .$$

Here $\rho = \rho_G$ is the regular representation of G and $\tilde{\lambda}_X : X \longrightarrow \Omega^V \, \mathrm{sh}^V X$ is the adjoint of the morphism $\lambda_X : S^V \wedge X \longrightarrow \mathrm{sh}^V X$ defined in (3.10). This construction comes with a canonical natural morphism $\lambda_X^\infty : X \longrightarrow QX$, the embedding of the initial term into the mapping telescope.

Every morphism in the sequence defining QX is a $\underline{\pi}_*$-isomorphism by Proposition 3.12. So the morphism $\lambda_X^\infty : X \longrightarrow QX$ is also a $\underline{\pi}_*$-isomorphism. One has to work a little more to show that the spectrum QX is a G-Ω-spectrum.

Mapping cone and homotopy fiber. The *(reduced) mapping cone* Cf of a morphism of based G-spaces $f : A \longrightarrow B$ is defined by

$$Cf \ = \ ([0,1] \wedge A) \cup_f B \ .$$

Here the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of A. The group G acts trivially on the interval. The mapping cone comes with an inclusion $i : B \longrightarrow Cf$ and a projection

$$p \ : \ Cf \ \longrightarrow \ S^1 \wedge A$$

the projection sends B to the basepoint and is given on $[0,1]\wedge A$ by $p(x,a)=\mathbf{t}(x)\wedge a$ where $\mathbf{t}:[0,1]\longrightarrow S^1$ is given by $\mathbf{t}(x)=\frac{2x-1}{x(1-x)}$. What is relevant about the map $\mathbf{t}$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and the circle S^1, and that it satisfies $\mathbf{t}(1-x)=-\mathbf{t}(x)$.

The *homotopy fiber* is the construction 'dual' to the mapping cone. The homotopy fiber of a morphism $f:A\longrightarrow B$ of based spaces is the fiber product

$$F(f)\ =\ *\times_B B^{[0,1]}\times_B A\ =\ \{(\lambda,a)\in B^{[0,1]}\times A\mid \lambda(0)=*,\ \lambda(1)=f(a)\}\ ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. Again the group G acts trivially on the interval. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B\xrightarrow{\ i\ }F(f)\xrightarrow{\ p\ }A\ ;$$

the map p is the projection to the second factor and the value of the map i on a based loop $\omega:S^1\longrightarrow B$ is

$$i(\omega)=(\omega\circ\mathbf{t},*)\ .$$

Proposition 3.17. *Let $f:A\longrightarrow B$ be a map of based G-spaces. Then the composites*

$$A\xrightarrow{\ f\ }B\xrightarrow{\ i\ }Cf\qquad\text{and}\qquad F(f)\xrightarrow{\ p\ }A\xrightarrow{\ f\ }B$$

are naturally based G-null-homotopic. Moreover, the diagram

$$\begin{tikzcd} & CA\cup_f CB \arrow[dl, "p_A\cup *"'] \arrow[dr, "*\cup p_B"] & \\ S^1\wedge A \arrow[rr, "\tau\wedge f"'] & & S^1\wedge B \end{tikzcd}$$

commutes up to natural, based G-homotopy, where τ is the sign involution of S^1 given by $x\mapsto -x$.

The proof of Proposition 3.17 is by elementary and explicit homotopies, and we omit it.

Lemma 3.18. *Let $f:A\longrightarrow B$ and $\beta:Z\longrightarrow B$ be morphisms of based G-spaces such that the composite $i\beta:Z\longrightarrow Cf$ is equivariantly null-homotopic. Then there exists a based G-map $h:S^1\wedge Z\longrightarrow S^1\wedge A$ such that $(S^1\wedge f)\circ h:S^1\wedge Z\longrightarrow S^1\wedge B$ is equivariantly homotopic to $S^1\wedge\beta$.*

Proof. Let $H:[0,1]\times Z\longrightarrow Cf$ be a based, equivariant null-homotopy of the composite $i\beta:Z\longrightarrow Cf$, i.e., H takes $0\times Z$ and $[0,1]\times z_0$ to the basepoint and $H(1,x)=i(\beta(x))$ for all $x\in Z$. The composite $p_AH:[0,1]\times Z\longrightarrow S^1\wedge A$ then factors as $p_AH=hp_Z$ for a unique G-map $h:S^1\wedge Z\longrightarrow S^1\wedge A$. We claim that h has the required property.

To prove the claim we need the G-homotopy equivalence $p_Z\cup *:CZ\cup_{1\times Z}CZ\longrightarrow S^1\wedge Z$ which collapses the second cone. We obtain a sequence of equalities and G-homotopies

$$\begin{aligned}(S^1\wedge f)\circ h\circ(p_Z\cup *)\ &=\ (S^1\wedge f)\circ(p_A\cup *)\circ(H\cup C(\beta))\\ &=\ (\tau\wedge B)\circ(\tau\wedge f)\circ(p_A\cup *)\circ(H\cup C(\beta))\\ &\simeq\ (\tau\wedge B)\circ(*\cup p_B)\circ(H\cup C(\beta))\\ &=\ (\tau\wedge B)\circ(S^1\wedge\beta)\circ(*\cup p_Z)\\ &=\ (S^1\wedge\beta)\circ(\tau\wedge Z)\circ(*\cup p_Z)\ \simeq\ (S^1\wedge\beta)\circ(p_Z\cup *)\end{aligned}$$

Here $H\cup C(\beta):CZ\cup_{1\times Z}CZ\longrightarrow Cf\cup_B CB\cong CA\cup_f CB$ and τ is the sign involution of S^1. The two homotopies result from Proposition 3.17 applied to f respectively the identity of Z, and we used the naturality of various constructions. Since the map $p_Z\cup *$ is a G-homotopy equivalence, this proves that the map $(S^1\wedge f)\circ h$ is homotopic to $S^1\wedge\beta$. $\square$

Now we can introduce mapping cones and homotopy fibers for orthogonal G-spectra. The *mapping cone* Cf of a morphism of orthogonal G-spectra $f : X \longrightarrow Y$ is defined by

$$(Cf)_n \;=\; C(f_n) \;=\; ([0,1] \wedge X_n) \cup_f Y_n \ , \tag{3.19}$$

the reduced mapping cone of $f_n : X_n \longrightarrow Y_n$. The orthogonal group $O(n)$ acts on $(Cf)_n$ through the given action on X_n and Y_n and trivially on the interval. The inclusions $i_n : Y_n \longrightarrow C(f_n)$ and projections $p_n : C(f_n) \longrightarrow S^1 \wedge X_n$ assemble into morphisms of orthogonal G-spectra $i : Y \longrightarrow Cf$ and $p : Cf \longrightarrow S^1 \wedge X$. For every G-representation V, the G-space $(Cf)(V)$ is naturally G-homeomorphic to the mapping cone of the G-map $f(V) : X(V) \longrightarrow Y(V)$.

We define a *connecting homomorphism* $\delta : \pi^G_{1+k}(Cf) \longrightarrow \pi^G_k(X)$ as the composite

$$\pi^G_{1+k}(Cf) \xrightarrow{\pi^G_{1+k}(p)} \pi^G_{1+k}(S^1 \wedge X) \;\cong\; \pi^G_k(X) \ , \tag{3.20}$$

where the first map is the effect of the projection $p : Cf \longrightarrow S^1 \wedge X$ on homotopy groups, and the second map is the inverse of the suspension isomorphism $S^1 \wedge - : \pi^G_k(X) \longrightarrow \pi^G_{1+k}(S^1 \wedge X)$.

The homotopy fiber $F(f)$ of the morphism $f : X \longrightarrow Y$ is the orthogonal spectrum defined by

$$F(f)_n \;=\; F(f_n) \ ,$$

the homotopy fiber of $f_n : X_n \longrightarrow Y_n$. The group $G \times O(n)$ acts on $F(f)_n$ through the given action on X_n and Y_n and trivially on the interval. Put another way, the homotopy fiber is the pullback in the cartesian square of orthogonal G-spectra:

$$\begin{tikzcd}
F(f) \arrow[r, "p"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\
Y^{[0,1]} \arrow[r, "{\omega \mapsto (\omega(0),\omega(1))}"'] & Y \times Y
\end{tikzcd}$$

The inclusions $i_n : \Omega Y_n \longrightarrow F(f)_n$ and projections $p_n : F(f)_n \longrightarrow X_n$ assemble into morphisms of orthogonal G-spectra $i : \Omega Y \longrightarrow F(f)$ and $p : F(f) \longrightarrow X$. For every G-representation V, the G-space $F(f)(V)$ is naturally G-homeomorphic to the homotopy fiber of the G-map $f(V) : X(V) \longrightarrow Y(V)$. We define a *connecting homomorphism* $\delta : \pi^G_{1+k}(Y) \longrightarrow \pi^G_k(F(f))$ as the composite

$$\pi^G_{1+k}(Y) \xrightarrow{\alpha^{-1}} \pi^G_k(\Omega Y) \xrightarrow{\pi^G_k(i)} \pi^G_k(F(f)) \ ,$$

where $\alpha : \pi^G_k(\Omega Y) \longrightarrow \pi^G_{1+k}(Y)$ is the loop isomorphism (3.11).

Proposition 3.21. *For every morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the long sequences of abelian groups*

$$\cdots \longrightarrow \pi^G_k(X) \xrightarrow{f_*} \pi^G_k(Y) \xrightarrow{i_*} \pi^G_k(Cf) \xrightarrow{\delta} \pi^G_{k-1}(X) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi^G_k(X) \xrightarrow{f_*} \pi^G_k(Y) \xrightarrow{\delta} \pi^G_{k-1}(F(f)) \xrightarrow{p_*} \pi^G_{k-1}(X) \longrightarrow \cdots$$

are exact.

Proof. We start with exactness of the first sequence at $\pi^G_k(Y)$. The composite of $f : X \longrightarrow Y$ and the inclusion $Y \longrightarrow Cf$ is equivariantly null-homotopic, so it induces the trivial map on π^G_k. It remains to show that every element in the kernel of $i_* : \pi^G_k(Y) \longrightarrow \pi^G_k(Cf)$ is in the image of f_*. Let $\beta : S^{k+n\rho} \longrightarrow Y(n\rho)$ represent an element in the kernel. By increasing n, if necessary, we can assume that the composite of β with the inclusion $i : Y(n\rho) \longrightarrow (Cf)(n\rho) = C(f(n\rho))$ is equivariantly null-homotopic. By Lemma 3.18

there is a G-map $h : S^1 \wedge S^{k+n\rho} \longrightarrow S^1 \wedge X(n\rho)$ such that $(S^1 \wedge f(n\rho)) \circ h$ is G-homotopic to $S^1 \wedge \beta$. The composite

$$\tilde{h} \ : \ S^{k+n\rho+1} \ \cong \ S^{k+n\rho} \wedge S^1 \ \xrightarrow{\tau_{S^{k+n\rho},S^1}} \ S^1 \wedge S^{k+n\rho} \ \xrightarrow{\ h\ } \ S^1 \wedge X(n\rho) \ \xrightarrow{\tau_{S^1,X(n\rho)}} \ X(n\rho) \wedge S^1$$

then has the property that $(f(n\rho) \wedge S^1) \circ \tilde{h}$ is G-homotopic to $\beta \wedge S^1$. The composite

$$S^{k+n\rho+1} \ \xrightarrow{\tilde{h}} \ X(n\rho) \wedge S^1 \ \xrightarrow{\sigma_{n\rho,\mathbb{R}}} \ X(n\rho \oplus \mathbb{R})$$

represents an equivariant homotopy class $\langle \sigma_{n\rho,\mathbb{R}} \circ \tilde{h} \rangle$ in $\pi_k^G(X)$ and we have

$$\begin{aligned} \pi_k^G(f)\langle \sigma_{n\rho,\mathbb{R}} \circ \tilde{h} \rangle \ &= \ \langle f(n\rho \oplus \mathbb{R}) \circ \sigma_{n\rho,\mathbb{R}} \circ \tilde{h} \rangle \ = \ \langle \sigma_{n\rho,\mathbb{R}} \circ (f(n\rho) \wedge S^1) \circ \tilde{h} \rangle \\ &= \langle \sigma_{n\rho,\mathbb{R}} \circ (\beta \wedge S^1) \rangle \ = \ \langle \beta \diamond \mathbb{R} \rangle \ = \ \langle \beta \rangle \ . \end{aligned}$$

So the class represented by β is in the image of $f_* : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$.

We now deduce the exactness at $\pi_k^G(Cf)$ and $\pi_{k-1}^G(X)$ by comparing the mapping cone sequence for $f : X \longrightarrow Y$ to the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$ (shifted to the left). We observe that the collapse map

$$* \cup p \ : \ Ci \cong CY \cup_f CX \ \longrightarrow \ S^1 \wedge X$$

is an equivariant homotopy equivalence, and thus induces an isomorphism of equivariant homotopy groups. Indeed, a homotopy inverse

$$r \ : \ S^1 \wedge X \ \longrightarrow \ CY \cup_f CX$$

is defined by the formula

$$r(s \wedge x) \ = \ \begin{cases} \quad (2s, x) \quad \ \in CX & \text{for } 0 \leq s \leq 1/2 \text{, and} \\ (2 - 2s, f(x)) \in CY & \text{for } 1/2 \leq s \leq 1, \end{cases}$$

which is to be interpreted levelwise. We omit the explicit G-homotopies $r(* \cup p) \simeq \mathrm{Id}$ and $(* \cup p)r \simeq \mathrm{Id}$.

Now we consider the diagram

$$\begin{tikzcd} Cf \arrow[r, "i_i"] \arrow[dr, "p"'] & Ci \arrow[r, "p_i"] \arrow[d, "*\cup p"'] & S^1 \wedge Y \\ & S^1 \wedge X \arrow[ur, "\tau \wedge f"'] & \end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$. The left triangle commutes on the nose and the right triangle commutes up to G-homotopy, by Proposition 3.17. We get a diagram

$$\begin{tikzcd} \pi_k^G(Y) \arrow[r, "i_*"] \arrow[d, equal] & \pi_k^G(Cf) \arrow[r, "(i_i)_*"] \arrow[d, equal] & \pi_k^G(Ci) \arrow[r, "\delta"] \arrow[d, "(S^{-1}\wedge -)\circ(*\cup p)_*"', "\cong"] & \pi_{k-1}^G(Y) \arrow[d, equal] \\ \pi_k^G(Y) \arrow[r, "i_*"'] & \pi_k^G(Cf) \arrow[r, "\delta"'] & \pi_{k-1}^G(X) \arrow[r, "f_*"'] & \pi_{k-1}^G(Y) \end{tikzcd}$$

whose left and middle squares commutes. The right square commutes up to sign (the degree of the sign involution $\tau : S^1 \longrightarrow S^1$), using the naturality of the suspension isomorphism. By the previous paragraph, applied to $i : Y \longrightarrow Cf$ instead of f, the upper row is exact at $\pi_k^G(Cf)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_k^G(Cf)$. But the morphism f was arbitrary, so when applied to $i : Y \longrightarrow Cf$ instead of f, we obtain that the upper row is exact at $\pi_k^G(Ci)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_{k-1}^G(X)$. This finishes the proof of exactness of the first sequence.

Now we come to why the second sequence is exact. For every $n \geq 0$ the sequence $F(f)(n\rho) = F(f(n\rho)) \longrightarrow X(n\rho) \longrightarrow Y(n\rho)$ is an equivariant homotopy fiber sequence. So for every based G-CW-complex A, the long sequence of based sets

$$\cdots \longrightarrow [A, \Omega Y(n\rho)]^G \xrightarrow{\delta} [A, F(f(n\rho))]^G \xrightarrow{[A,p(n\rho)]^G} [A, X(n\rho)]^G \xrightarrow{[A,f(n\rho)]^G} [A, Y(n\rho)]^G$$

is exact. We take $A = S^{k+n\rho}$ and form the colimit over n. Since sequential colimits are exact the resulting sequence of colimits is again exact, and that proves the second claim. □

Corollary 3.22. (i) *For every family of orthogonal G-spectra $\{X^i\}_{i \in I}$ and every integer k the canonical map*

$$\bigoplus_{i \in I} \underline{\pi}_k(X^i) \longrightarrow \underline{\pi}_k\left(\bigvee_{i \in I} X^i\right)$$

is an isomorphism of Mackey functors.

(ii) *For every finite indexing set I, every family $\{X^i\}_{i \in I}$ of orthogonal G-spectra and every integer k the canonical map*

$$\underline{\pi}_k\left(\prod_{i \in I} X^i\right) \longrightarrow \prod_{i \in I} \underline{\pi}_k(X^i)$$

is an isomorphism of Mackey functors.

(iii) *For every finite family of orthogonal G-spectra the canonical morphism from the wedge to the product is a $\underline{\pi}_*$-isomorphism.*

Proof. (i) We first show the special case of two summands. If X and Y are two orthogonal G-spectra, then the wedge inclusion $i_X : X \longrightarrow X \vee Y$ has a retraction. So for every subgroup H of G the associated long exact homotopy group sequence of Proposition 3.21 (i) splits into short exact sequences

$$0 \longrightarrow \pi_k^H(X) \xrightarrow{\pi_k^H(i_X)} \pi_k^H(X \vee Y) \xrightarrow{\text{incl}} \pi_k^H(C(i_X)) \longrightarrow 0 .$$

The mapping cone $C(i_X)$ is isomorphic to $(CX) \vee Y$ and thus G-homotopy equivalent to Y. So we can replace $\pi_k^H(C(i_X))$ by $\pi_k^H(Y)$ and conclude that $\pi_k^H(X \vee Y)$ splits as the sum of $\pi_k^H(X)$ and $\pi_k^H(Y)$, via the canonical map. The case of a finite indexing set I now follows by induction, and the general case follows since homotopy groups of orthogonal G-spectra commute with filtered colimits.

(ii) The functor $X \mapsto [S^{k+n\rho_G}, X(n\rho_G)]$ commutes with products. For finite indexing sets product are also sums, which commute with filtered colimits.

(iii) This is a direct consequence of (i) and (ii). More precisely, for finite indexing set I and every integer k the composite map

$$\bigoplus_{i \in I} \pi_k^H(X^i) \longrightarrow \pi_k^H(\bigvee_{i \in I} X^i) \longrightarrow \pi_k^H(\prod_{i \in I} X^i) \longrightarrow \prod_{i \in I} \pi_k^H(X^i)$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map. □

As a word of warning we remark that the functors π_k^G and $\underline{\pi}_k$ do not preserve arbitrary products; the problem is that the sequential colimit involved in the definition of π_k^G does not commute with arbitrary products.

4. Wirthmüller isomorphism and transfers

In this section we establish the *Wirthmüller isomorphism* and discuss the closely related *transfer maps* on equivariant homotopy groups. The restriction functor from G-spectra to H-spectra has both a left adjoint $G \ltimes_H -$ and a right adjoint $\mathrm{map}^H(G,-)$. In classical representation theory of finite groups, the algebraic analogues of the left and the right adjoint are naturally isomorphic. In equivariant stable homotopy theory, the best we can hope for is a natural $\underline{\pi}_*$-isomorphism, and that is the content of the Wirthmüller isomorphism, compare Theorem 4.9 below.

We start with an auxiliary lemma.

Lemma 4.1. *Let H be a finite group, W an H-representation and $w \in W$ an H-fixed point. We define the 'radius 1 scanning map' around w by*

$$s[w] \ : \ S^W \ \longrightarrow \ S^W \ , \quad x \ \longmapsto \ \begin{cases} \frac{x-w}{1-|x-w|} & \text{for } |x-w|<1, \text{ and} \\ \infty & \text{for } |x-w| \geq 1. \end{cases}$$

Then the scanning map $s[w]$ is H-equivariantly based homotopic to the identity.

Proof. The homotopy

$$[0,1] \times S^W \ \longrightarrow \ S^W \ , \quad (t,x) \ \longmapsto \ s[t\cdot w](x)$$

interpolates between $s[0]$ and $s[w]$. Another homotopy then interpolates between the identity and the scaling map $s[0]$. □

Construction 4.2 (Transfer)**.** We let H be a subgroup of a finite group G. We choose a G-representation W and a G-equivariant injection

$$j \ : \ G/H \ \longrightarrow \ W \ .$$

Such an injection is determined by the point $w = j(H)$, the image of the preferred coset, and any point of W whose stabilizer group is H does the job. By scaling the function j, if necessary, we can assume without loss of generality that the embedding is *wide*, i.e., the open unit balls around the image points $i(gH) = g \cdot w$ are pairwise disjoint.

This data determines a G-equivariant *transfer map* as follows. The G-map

$$j \ : \ G \times_H D(W) \ \longrightarrow \ W \ , \quad [g,x] \ \longmapsto \ g\cdot(w+x)$$

is an embedding on the open unit balls. So we get a G-equivariant Thom-Pontryagin collapse map

$$t_H^G \ : \ S^W \ \longrightarrow \ G \ltimes_H S^W \tag{4.3}$$

that sends the complement of $j(G \times_H \mathring{D}(W))$ to the basepoint at infinity and is otherwise given by the formula

$$t_H^G(g\cdot(w+x)) \ = \ \frac{g\cdot x}{1-|x|} \ ,$$

where $|x|<1$. The map depends on the choice of G-representation W and the wide equivariant embedding j, but we do not record this dependence in the notation.

Now we need some more notation in order to state and prove the key unstable ingredient for the Wirthmüller isomorphism, namely Proposition 4.5 below. We let H be a subgroup of a finite group G. Then the restriction functor i^* from based G-spaces to based H-space has a left adjoint $G \ltimes_H -$ and a right adjoint $\mathrm{map}^H(G,-)$. A natural based G-map

$$\Psi_B \ : \ G \ltimes_H B \ \longrightarrow \ \mathrm{map}^H(G,B) \tag{4.4}$$

is defined by

$$\Psi_B(g \ltimes b)(\gamma) \ = \ \begin{cases} \gamma g b & \text{if } \gamma g \in H, \text{ and} \\ * & \text{if } \gamma g \notin H. \end{cases}$$

For a based H-space B and a based G-space A, the *shearing isomorphism* is the G-equivariant homeomorphism

$$(G \ltimes_H B) \wedge A \;\cong\; G \ltimes_H (B \wedge i^*A) \ , \quad (g \ltimes b) \wedge a \;\longmapsto\; g \ltimes (b \wedge (g^{-1}a)) \ .$$

Similarly, the *assembly map* is the G-map

$$\alpha \ : \ \mathrm{map}^H(G, B) \wedge A \;\longrightarrow\; \mathrm{map}^H(G, B \wedge i^*A) \ , \quad \alpha(f \wedge a)(g) \;=\; f(g) \wedge ga \ .$$

It is straightforward to check that all these maps make the following square commute:

$$\begin{tikzcd}
(G \ltimes_H B) \wedge A \arrow[r, "\Psi_B \wedge A"] \arrow[d, "\mathrm{shear}"', "\cong"] & \mathrm{map}^H(G, B) \wedge A \arrow[d, "\alpha"] \\
G \ltimes_H (B \wedge i^*A) \arrow[r, "\Psi_{B \wedge i^*A}"'] & \mathrm{map}^H(G, B \wedge i^*A)
\end{tikzcd}$$

In the situation where $A = S^W$ is the sphere of a G-representation W into which G/H embeds, the transfer (4.3) gives rise to another G-map $\tau_B : \mathrm{map}^H(G, B) \wedge S^W \longrightarrow G \ltimes_H (B \wedge S^{i^*W})$ defined as the composite

$$\begin{aligned}
\mathrm{map}^H(G, B) \wedge S^W \ &\xrightarrow{\mathrm{Id} \wedge t_H^G} \ \mathrm{map}^H(G, B) \wedge (G \ltimes_H S^{i^*W}) \\
&\xrightarrow{\mathrm{shear}} \ G \ltimes_H \left(i^*(\mathrm{map}^H(G, B)) \wedge S^{i^*W} \right) \\
&\xrightarrow{G \ltimes_H (\epsilon \wedge S^{i^*W})} \ G \ltimes_H (B \wedge S^{i^*W}) \ .
\end{aligned}$$

Here ϵ is the adjunction counit.

Proposition 4.5. *Let H be a subgroup of a finite group G, and B a based H-space. Then the following diagram commutes up to G-equivariant based homotopy:*

$$\begin{tikzcd}
(G \ltimes_H B) \wedge S^W \arrow[r, "\Psi_B \wedge S^W"] \arrow[d, "\mathrm{shear}"', "\cong"] & \mathrm{map}^H(G, B) \wedge S^W \arrow[d, "\alpha"] \arrow[dl, "\tau_B"'] \\
G \ltimes_H (B \wedge S^{i^*W}) \arrow[r, "\Psi_{B \wedge S^{i^*W}}"'] & \mathrm{map}^H(G, B \wedge S^{i^*W})
\end{tikzcd}$$

Proof. We start by showing that the upper left triangle in the proposition commutes up to G-homotopy. Since $G \ltimes_H -$ is left adjoint to the restriction functor, it suffices to show that the composite

$$\begin{aligned}
B \wedge S^{i^*W} \ &\xrightarrow{\psi_B \wedge t_H^G} \ \mathrm{map}^H(G, B) \wedge (G \ltimes_H S^{i^*W}) \\
&\xrightarrow{\mathrm{shear}} \ G \ltimes_H \left(i^*(\mathrm{map}^H(G, B)) \wedge S^{i^*W} \right) \\
&\xrightarrow{G \ltimes_H (\epsilon \wedge S^{i^*W})} \ G \ltimes_H (B \wedge S^{i^*W})
\end{aligned}$$

is H-equivariantly homotopic to the adjunction unit, where we have expanded the definition of τ_B. Expanding the definition of the transfer map t_H^G identifies this composite with the map

$$B \wedge S^{i^*W} \ \xrightarrow{\mathrm{Id} \wedge s[w]} \ B \wedge S^{i^*W} \ \xrightarrow{1 \ltimes -} \ G \ltimes_H (B \wedge S^{i^*W}) \ ,$$

the radius 1 scanning map $s[w]$ around the distinguished H-fixed point $w = j(H)$, followed by the adjunction unit. By Lemma 4.1, the map $s[w]$ is H-equivariantly homotopic to the identity, so the claim follows.

Now we show the commutativity of the lower right triangle. Since $\mathrm{map}^H(G,-)$ is right adjoint to the restriction functor, it suffices to show that the composite

$$\begin{aligned}\mathrm{map}^H(G,B)\wedge S^W \ &\xrightarrow{\mathrm{Id}\wedge t_H^G}\ \mathrm{map}^H(G,B)\wedge (G\ltimes_H S^{i^*W})\\ &\xrightarrow{\text{shear}}\ G\ltimes_H \left(i^*(\mathrm{map}^H(G,B))\wedge S^{i^*W}\right)\\ &\xrightarrow{G\ltimes_H(\epsilon\wedge S^{i^*W})}\ G\ltimes_H (B\wedge S^{i^*W})\ \xrightarrow{\mathrm{proj}_H}\ B\wedge S^{i^*W}\end{aligned}$$

is H-equivariantly homotopic to $\epsilon\wedge\mathrm{Id} : \mathrm{map}^H(G,B)\wedge S^W \longrightarrow B\wedge S^W$, where again we have expanded the definition of τ_B. This composite equals the map

$$\epsilon\wedge s(w)\ :\ \mathrm{map}^H(G,B)\wedge S^W\ \longrightarrow\ B\wedge S^W\ ;$$

so again, the claim follows because the scanning map $s[w]$ is H-equivariantly homotopic to the identity. □

Now we can establish the Wirthmüller isomorphism. This isomorphism first appeared in [29, Thm. 2.1] in the more general context of compact Lie groups. Wirthmüller attributes parts of the ideas to tom Dieck and his statement that G-spectra define a 'complete G-homology theory', amounts to Theorem 4.9 when Y is a suspension spectrum. The generalization of Wirthmüller's isomorphism to arbitrary H-spectra is due to Lewis and May [15, II Thm. 6.2]. Our proof is essentially Wirthmüller's original argument, but specialized to finite groups and adapted to orthogonal spectra, which simplifies the exposition somewhat.

Let H be a subgroup of G. Then the restriction functor from orthogonal G-spectra to orthogonal H-spectra has a left and a right adjoint, and both are essentially given by applying the space level adjoints $G\ltimes_H -$ and $\mathrm{map}^H(G,-)$ levelwise.

Construction 4.6. We let H be a subgroup of G and Y an orthogonal H-spectrum. The *coinduced G-spectrum* is defined levelwise, i.e., by $(\mathrm{map}^H(G,Y))_n = \mathrm{map}^H(G,Y_n)$ with induced action by the orthogonal group and induced structure maps. If V is a G-representation, then the G-space $\mathrm{map}^H(G,Y)(V)$ is canonically isomorphic to $\mathrm{map}^H(G,Y(i^*V))$. Indeed, a G-equivariant homeomorphism

$$\begin{aligned}\mathrm{map}^H(G,Y)(V)\ &=\ \mathbf{O}(\mathbb{R}^n,V)\wedge_{O(n)}\mathrm{map}^H(G,Y_n)\\ &\longrightarrow\ \mathrm{map}^H(G,\mathbf{O}(\mathbb{R}^n,V)\wedge_{O(n)}Y_n) = \mathrm{map}^H(G,Y(i^*V))\end{aligned}\tag{4.7}$$

is given by

$$[\varphi,f]\ \longmapsto\ \{\,g\mapsto [l_g\circ\varphi,\ f(g)]\,\}$$

where $\dim(V)=n$, $\varphi:\mathbb{R}^n\longrightarrow V$ is a linear isometry and $f:G\longrightarrow Y_n$ an H-map and $l_g : V\longrightarrow V$ is left translation by $g\in G$. Under the identification (4.7), the generalized structure map $\sigma_{V,W}$ of the spectrum $\mathrm{map}^H(G,Y)$ becomes the composite

$$\mathrm{map}^H(G,Y(i^*V))\wedge S^W\ \xrightarrow{\ \alpha\ }\ \mathrm{map}^H(G,Y(i^*V)\wedge S^{i^*W})\ \xrightarrow{\mathrm{map}^H(G,\sigma_{i^*V,i^*W})}\ \mathrm{map}^H(G,Y(i^*(V\oplus W)))\ .$$

The left adjoint to the restriction functor from G-spectra to H-spectrum is constructed in a similar way. For an orthogonal H-spectrum Y we denote by $G\ltimes_H Y$ the *induced G-spectrum* with n-th level given by $(G\ltimes_H Y)_n = G\ltimes_H Y_n$, with induced action by the orthogonal group and induced structure maps. If V is a G-representation, then the map

$$G\ltimes_H Y(i^*V)\ \cong\ (G\ltimes_H Y)(V)\ ,\quad g\ltimes[\varphi,y]\ \longmapsto\ [l_g\circ\varphi, g\ltimes y]\tag{4.8}$$

is a preferred G-equivariant homeomorphism. Under the identification (4.8), the generalized structure map $\sigma_{V,W}$ of the spectrum $\mathrm{map}^H(G,Y)$ becomes the composite

$$(G\ltimes_H Y(i^*V))\wedge S^W\ \xrightarrow{\text{shear}}\ G\ltimes_H (Y(i^*V)\wedge S^{i^*W})\ \xrightarrow{G\ltimes_H\sigma_{i^*V,i^*W}}\ G\ltimes_H Y(i^*(V\oplus W))\ .$$

The G-maps (4.4) for the various levels Y_n form a morphism of orthogonal G-spectra $\Psi_Y : G \ltimes_H Y \longrightarrow \mathrm{map}^H(G, Y)$.

Theorem 4.9 (Wirthmüller isomorphism)**.** *Let H be a subgroup of a finite group G, and Y an orthogonal H-spectrum. Then the morphism*

$$\Psi_Y \ : \ G \ltimes_H Y \ \longrightarrow \ \mathrm{map}^H(G, Y)$$

is a $\underline{\pi}_*$-isomorphism.

Proof. This is a relatively straightforward consequence of Proposition 4.5. We let K be any subgroup of G, and we start by showing the injectivity of $\pi_k^K(\Psi_Y)$; we give the argument for $k \geq 0$, the other cases being similar. We let $x \in \pi_k^K(G \ltimes_H Y)$ be a class in the kernel of $\pi_k^K(\Psi_Y)$ and we represent it by a based K-map

$$f \ : \ S^{\mathbb{R}^k \oplus V} \ \longrightarrow \ (G \ltimes_H Y)(V)$$

for a suitable K-representation V. By increasing V and stabilizing f, if necessary, we can assume that V is underlying a G-representation. Then we use the homeomorphism (4.8) to rewrite the target of f as

$$(G \ltimes_H Y)(V) \ \cong \ G \ltimes_H Y(i^*V) \ .$$

By increasing V even further, if necessary, we can assume that in addition the composite

$$S^{\mathbb{R}^k \oplus V} \ \xrightarrow{\ f\ } \ G \ltimes_H Y(i^*V) \ \xrightarrow{\Psi_{Y(i^*V)}} \mathrm{map}^H(G, Y(i^*V))$$

is K-equivariantly null-homotopic. Hence the composite

$$S^{\mathbb{R}^k \oplus V \oplus W} \ \xrightarrow{f \wedge S^W} \ (G \ltimes_H Y(i^*V)) \wedge S^W \ \xrightarrow{\Psi_{Y(i^*V)} \wedge S^W} \mathrm{map}^H(G, Y(i^*V)) \wedge S^W$$

is K-equivariantly null-homotopic as well.

By Proposition 4.5 the composite

$$(G \ltimes_H Y(i^*V)) \wedge S^W \ \xrightarrow{\Psi_{Y(i^*V)} \wedge S^W} \mathrm{map}^H(G, Y(i^*V)) \wedge S^W \ \xrightarrow{\tau_{Y(i^*V)}} \ G \ltimes_H (Y(i^*V) \wedge S^{i^*W})$$

is G-equivariantly – and hence also K-equivariantly – homotopic to the shearing homeomorphism; so already the map $f \wedge S^W : S^{\mathbb{R}^k \oplus V \oplus W} \longrightarrow (G \ltimes_H Y(i^*V)) \wedge S^W$ is K-equivariantly null-homotopic. In particular, f represents the trivial element in $\pi_k^K(G \ltimes_H Y)$, and so the map $\pi_k^K(\Psi_Y)$ in injective.

The argument for surjectivity is similar. We let

$$g \ : \ S^{\mathbb{R}^k \oplus V} \ \longrightarrow \ \mathrm{map}^H(G, Y)(V)$$

be a based K-map that represents any given element of $\pi_k^K(\mathrm{map}^H(G, Y))$, for a suitable K-representation V. By increasing V and stabilizing g, if necessary, we can assume that V is underlying a G-representation. Then we use the homeomorphism (4.7) to rewrite the target of g as

$$\mathrm{map}^H(G, Y)(V) \ \cong \ \mathrm{map}^H(G, Y(i^*V)) \ .$$

The composite

$$\begin{aligned} S^{\mathbb{R}^k \oplus V \oplus W} \ \xrightarrow{g \wedge S^W} \ \mathrm{map}^H(G, Y(i^*V)) \wedge S^W \ &\xrightarrow{\tau_{Y(i^*V)}} G \ltimes_H (Y(i^*V) \wedge S^{i^*W}) \\ &\xrightarrow{G \ltimes_H (\sigma_{i^*V, i^*W})} G \ltimes_H Y(i^*(V \oplus W)) \end{aligned}$$

represents an element $x \in \pi_k^K(G \ltimes_H Y)$. By naturality of the maps Ψ and Proposition 4.5 for $B = Y(i^*V)$, the composite

$$\begin{aligned} &\mathrm{map}^H(G, Y(i^*V)) \wedge S^W \ \xrightarrow{\tau_{Y(i^*V)}} G \ltimes_H (Y(i^*V) \wedge S^{i^*W}) \\ &\qquad \xrightarrow{G \ltimes_H (\sigma_{i^*V, i^*W})} G \ltimes_H Y(i^*(V \oplus W)) \ \xrightarrow{\Psi_{Y(i^*(V \oplus W))}} \ \mathrm{map}^H(G, Y(i^*(V \oplus W))) \end{aligned}$$

is G-equivariantly – and hence also K-equivariantly – homotopic to the composite

$$\mathrm{map}^H(G, Y(i^*V)) \wedge S^W \xrightarrow{\alpha} \mathrm{map}^H(G, Y(i^*V) \wedge S^{i^*W}) \xrightarrow{\mathrm{map}^H(G,\sigma_{i^*V,i^*W})} \mathrm{map}^H(G, Y(i^*(V \oplus W))) .$$

Up to the identification (4.7), this last composite is the generalized structure map of the G-spectrum $\mathrm{map}^H(G, Y)$, so this shows that the original class represented by g is the image of the class x. So the map $\pi_k^K(\Psi_Y)$ is surjective, hence bijective. □

The G-equivariant homotopy groups of the coinduced G-spectrum $\mathrm{map}^H(G, Y)$ are isomorphic to the H-equivariant homotopy groups of Y, by a simple adjointness argument. We claim hat for every integer k the composite

$$\pi_k^G(\mathrm{map}^H(G, Y)) \xrightarrow{\mathrm{res}^G_H} \pi_k^H(\mathrm{map}^H(G, Y)) \xrightarrow{\pi_k^H(\mathrm{ev})} \pi_k^H(Y) \tag{4.10}$$

is an isomorphism, where $\mathrm{ev} : \mathrm{map}^H(G, Y) \longrightarrow Y$ is evaluation at $1 \in G$ (also known as the adjunction counit). Indeed, for every $n \geq 0$, the G-equivariant homeomorphism (4.7)

$$\mathrm{map}^H(G, Y)(n\rho_G) \cong \mathrm{map}^H(G, Y(i^*(n\rho_G)))$$

and the adjunction provide a natural bijection

$$[S^{k+n\rho_G}, \mathrm{map}^H(G, Y)(n\rho_G)]^G \cong [S^{k+i^*(n\rho_G)}, Y(i^*(n\rho_G))]^H .$$

These bijections are compatible with stabilization as n increases, and assemble into an isomorphism of abelian groups

$$\pi_k^G(\mathrm{map}^H(G, Y)) = \mathrm{colim}_n\, [S^{k+n\rho_G}, \mathrm{map}^H(G, Y)(n\rho_G)]^G \cong \mathrm{colim}_n\, [S^{k+i^*(n\rho_G)}, Y(i^*(n\rho_G))]^H .$$

The restricted representation $i^*(\rho_G)$ is H-isomorphic to $[G : H] \cdot \rho_H$, so the sequence $\{i^*(n\rho_G)\}_{n\geq 0}$ of restricted regular representations is isomorphic to a cofinal subsequence of the sequence $\{m\rho_H\}_{m\geq 0}$. Hence the colimit on the right hand side is isomorphic to the group $\pi_k^H(Y)$.

For every subgroup H of G and every orthogonal H-spectrum Y we define a morphism

$$\mathrm{pr} : G \ltimes_H Y \longrightarrow Y \tag{4.11}$$

as the projection onto the preferred wedge summand $H \ltimes_H Y$ in $G \ltimes_H Y$. In other words, the n-th level $\mathrm{pr}_n : G \ltimes_H Y_n \longrightarrow Y_n$ is defined by

$$\mathrm{pr}_n(g \ltimes y) = \begin{cases} gy & \text{if } g \in H, \text{ and} \\ * & \text{if } g \notin H. \end{cases}$$

The projection is a morphism of orthogonal H-spectra (but it is *not* G-equivariant). The next result is now essentially a corollary of the Wirthmüller isomorphism.

Proposition 4.12. *For every subgroup H of G and every orthogonal H-spectrum Y the composite*

$$\pi_*^G(G \ltimes_H Y) \xrightarrow{\mathrm{res}^G_H} \pi_*^H(G \ltimes_H Y) \xrightarrow{\pi_*^H(\mathrm{pr})} \pi_*^H(Y)$$

is an isomorphism.

Proof. The projection pr factors as the composite

$$G \ltimes_H Y \xrightarrow{\Psi_Y} \mathrm{map}^H(G, Y) \xrightarrow{\mathrm{ev}} Y ,$$

where the second morphism is evaluation at 1 (hence the counit of the adjunction); the map in question is thus equal to

$$\pi_*^H(\mathrm{pr}) \circ \mathrm{res}^G_H = \pi_*^H(\mathrm{ev}) \circ \pi_*^H(\Psi_Y) \circ \mathrm{res}^G_H = \pi_*^H(\mathrm{ev}) \circ \mathrm{res}^G_H \circ \pi_*^G(\Psi_Y) .$$

Since $\pi_*^H(\mathrm{ev}) \circ \mathrm{res}^G_H$ is an isomorphism by (4.10), and $\pi_*^G(\Psi_Y)$ is the Wirthmüller isomorphism (Theorem 4.9), this proves the claim. □

Now we discuss the transfer maps of equivariant homotopy groups. For a subgroup H of G we construct two kinds of transfer maps, the *external transfer* Tr_H^G that is defined and natural for orthogonal H-spectra, and the *internal transfer* tr_H^G that is defined and natural for orthogonal G-spectra. In order to distinguish the two kinds of transfer maps we use a capital 'T' for the external transfer and a lower case 't' for the internal transfer.

Our definition of the external transfer Tr_H^G is essentially as the 'inverse of the Wirthmüller isomorphism', modulo the identification (4.10) of $\pi_k^G(\mathrm{map}^H(G,Y))$ with $\pi_k^H(Y)$.

Definition 4.13. Let H be a subgroup of a finite groups G.

(i) For an orthogonal H-spectrum Y the *external transfer*

$$\mathrm{Tr}_H^G \;:\; \pi_*^H(Y) \longrightarrow \pi_*^G(G \ltimes_H Y) \tag{4.14}$$

is defined as the inverse of the isomorphism $\pi_*^H(\mathrm{pr}) \circ \mathrm{res}_H^G$.

(ii) For an orthogonal G-spectrum X the *internal transfer*

$$\mathrm{tr}_H^G \;:\; \pi_*^H(X) \longrightarrow \pi_*^G(X) \tag{4.15}$$

is defined as the composite

$$\pi_0^H(X) \xrightarrow{\mathrm{Tr}_H^G} \pi_0^G(G \ltimes_H X) \xrightarrow{\pi_0^G(\mathrm{act})} \pi_0^G(X)$$

of the external transfer for the underlying H-spectrum of X and the effect of the action morphism $G \ltimes_H X \longrightarrow X$ on G-equivariant homotopy groups.

The definition of the external transfer as the inverse of some easily understood map allows for rather formal proofs of various properties of the transfer maps. Along these lines we will show below the transitivity of the transfer maps, the compatibility with restriction along epimorphisms, and the double coset formula. On the other hand, Definition 4.13 does not reveal the geometric interpretation of the transfer as a Thom-Pontryagin construction – which is usually taken as the definition of the transfer. We will reconcile these two approaches now.

Construction 4.16. We relate that rather abstract definition of the transfer to the more concrete traditional definition via an equivariant Thom-Pontryagin construction. In fact, this interpretation is already implicit in the proof of the Wirthmüller isomorphism, which identifies the inverse as coming from the transfer map (4.3)

$$t_H^G \;:\; S^W \longrightarrow G \ltimes_H S^{i^*W} \;.$$

We let H be a subgroup of G and Y an orthogonal H-spectrum. We let V be an H-representation and $f : S^V \longrightarrow Y(V)$ an H-equivariant based map that represents a class in $\pi_0^H(Y)$. By enlarging V, if necessary, we can assume that $V = i^*W$ is the underlying H-representation of a G-representation W. By enlarging W, if necessary, we can assume moreover that there exists a G-equivariant injection

$$j \;:\; G/H \longrightarrow W \;,$$

which amounts to a choice of vector $j(eH)$ in W whose stabilizer group is H. As we explained in Construction 4.2, an associated Thom-Pontryagin collapse map gives rise to the G-equivariant transfer map t_H^G. The composite

$$S^W \xrightarrow{t_H^G} G \ltimes_H S^{i^*W} \xrightarrow{G\ltimes_H f} G \ltimes_H Y(i^*W) \cong_{(4.8)} (G \ltimes_H Y)(W)$$

is then a G-equivariant based map; we claim that it represents the external transfer, i.e.,

$$\langle (G \ltimes_H f) \circ t_H^G \rangle \;=\; \mathrm{Tr}_H^G \langle f \rangle \qquad \text{in} \qquad \pi_0^G(G \ltimes_H Y) \;.$$

To see this we contemplate the commutative diagram of based H-maps:

$$\begin{tikzcd}
S^W \arrow[r, "t_H^G"] \arrow[dr, bend right, "s[w]"'] & G \ltimes_H S^{i^*W} \arrow[r, "G\ltimes_H f"] \arrow[d, "\mathrm{pr}"] & G \ltimes_H Y(i^*W) \arrow[d, "\mathrm{pr}"] \\
& S^{i^*W} \arrow[r, "f"'] & Y(i^*W)
\end{tikzcd}$$

The composite $\mathrm{pr}\circ t_H^G : S^W \longrightarrow S^W$ is the radius 1 scanning map $s[w]$ around the preferred vector $w = j(H)$, so $\mathrm{pr} \circ t_H^G$ is H-equivariantly homotopic to the identity of S^W by Lemma 4.1. We conclude that

$$\pi_0^H(\mathrm{pr})\left(\mathrm{res}_H^G\langle (G \ltimes_H f) \circ t_H^G\rangle\right) \;=\; \langle \mathrm{pr} \circ (G \ltimes_H f) \circ t_H^G\rangle \;=\; \langle f \circ s[w]\rangle \;=\; \langle f\rangle \qquad \text{in} \qquad \pi_0^H(Y) \ .$$

The external transfer is defined as the inverse of $\pi_0^H(\mathrm{pr}) \circ \mathrm{res}_H^G$, so this proves the claim.

Now we prove various properties of the external and internal transfer maps. We start with transitivity with respect to a nested triple of groups $K \leq H \leq G$. In this situation, restricting a G-action to a K-action can be done in two steps, through an intermediate H-action. So the left adjoint $G \ltimes_K -$ is canonically isomorphic to the composite of the two partial left adjoints:

$$\kappa \;:\; G \ltimes_H (H \ltimes_K Y) \;\cong\; G \ltimes_K Y \ , \quad g \ltimes (h \ltimes y) \;\longmapsto\; (gh) \ltimes y \ ,$$

and similarly for the various right adjoints.

Proposition 4.17. *The external transfer maps are transitive, i.e., for nested subgroups $K \leq H \leq G$ and every orthogonal K-spectrum Y the composite*

$$\pi_*^K(Y) \xrightarrow{\mathrm{Tr}_K^H} \pi_*^K(H \ltimes_K Y) \xrightarrow{\mathrm{Tr}_H^G} \pi_*^G(G \ltimes_H (H \ltimes_K Y)) \xrightarrow{\pi_*^G(\kappa)} \pi_*^G(G \ltimes_K Y)$$

agrees with the external transfer Tr_K^G. The internal transfer maps are transitive, i.e.,

$$\mathrm{tr}_H^G \circ \mathrm{tr}_K^H \;=\; \mathrm{tr}_K^G \;:\; \pi_*^K(X) \;\longrightarrow\; \pi_*^G(X)$$

for every orthogonal G-spectrum X.

Proof. The square

$$\begin{tikzcd}
G \ltimes_H (H \ltimes_K Y) \arrow[r, "\kappa"] \arrow[d, "\mathrm{pr}_H^G"'] & G \ltimes_K Y \arrow[d, "\mathrm{pr}_K^G"] \\
H \ltimes_K Y \arrow[r, "\mathrm{pr}_K^H"'] & Y
\end{tikzcd}$$

commutes, where we decorate the wedge summand projections by the groups involved. So

$$\begin{aligned}
\pi_*^K(\mathrm{pr}_K^G) \circ \mathrm{res}_K^G \circ \pi_*^G(\kappa) \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \;&=\; \pi_*^K(\mathrm{pr}_K^G) \circ \pi_*^K(\kappa) \circ \mathrm{res}_K^G \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \\
&=\; \pi_*^K(\mathrm{pr}_K^H) \circ \pi_*^K(\mathrm{pr}_H^G) \circ \mathrm{res}_K^H \circ \mathrm{res}_H^G \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \\
&=\; \pi_*^K(\mathrm{pr}_K^H) \circ \mathrm{res}_K^H \circ \pi_*^H(\mathrm{pr}_H^G) \circ \mathrm{res}_H^G \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \\
&=\; \pi_*^K(\mathrm{pr}_K^H) \circ \mathrm{res}_K^H \circ \mathrm{Tr}_K^H \;=\; \mathrm{Id} \ .
\end{aligned}$$

Since the composite $\pi_*^G(\kappa) \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H$ is inverse to $\pi_*^K(\mathrm{pr}_K^G) \circ \mathrm{res}_K^G$, it equals Tr_K^G. The transitivity of the internal transfer maps follows by naturality because for every orthogonal G-spectrum X the square

$$\begin{tikzcd}
G \ltimes_H (H \ltimes_K X) \arrow[r, "\kappa"] \arrow[d, "G\ltimes_H(\mathrm{act}_K^H)"'] & G \ltimes_K X \arrow[d, "\mathrm{act}_K^G"] \\
G \ltimes_H X \arrow[r, "\mathrm{act}_H^G"'] & X
\end{tikzcd}$$

commutes. Indeed:

$$\begin{aligned}
\mathrm{tr}_H^G \circ \mathrm{tr}_K^H &= \pi_*^G(\mathrm{act}_H^G) \circ \mathrm{Tr}_H^G \circ \pi_*^H(\mathrm{act}_K^H) \circ \mathrm{Tr}_K^H \\
&= \pi_*^G(\mathrm{act}_H^G) \circ \pi_*^G(G \ltimes_H \mathrm{act}_K^H) \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \\
&= \pi_*^G(\mathrm{act}_K^G) \circ \pi_*^G(\kappa) \circ \mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H \;=\; \mathrm{act}_K^G \circ \mathrm{Tr}_K^G \;=\; \mathrm{tr}_K^G \ .
\end{aligned}$$
□

Now we study how transfer maps interact with the restriction homomorphism (3.5) of equivariant homotopy groups. The following proposition explains what happens when a transfer is restricted along an epimorphism. The double coset formula (4.22) below explains what happens when a transfer is restricted to a subgroup. Every group homomorphism is the composite of an epimorphism and a subgroup inclusion, so together this can be used to rewrite the composite of a transfer map with the restriction homomorphism along an arbitrary group homomorphism.

We let $\alpha : K \longrightarrow G$ be a surjective homomorphism of finite groups, H a subgroup of G and $L = \alpha^{-1}(H)$. For every based H-space A the map

$$K \ltimes_L ((\alpha|_L)^* A) \longrightarrow \alpha^*(G \ltimes_H A) \ , \quad k \ltimes a \longmapsto \alpha(k) \ltimes a$$

is a K-equivariant homeomorphism. For an orthogonal H-spectrum Y, these isomorphisms for the various levels together define an isomorphism of orthogonal K-spectra

$$\xi \ : \ K \ltimes_L ((\alpha|_L)^* Y) \longrightarrow \alpha^*(G \ltimes_H Y) \ .$$

The next proposition shows that transfer maps are compatible in a straightforward way with restriction maps along epimorphisms.

Proposition 4.18. *Let $\alpha : K \longrightarrow G$ be a surjective homomorphism of finite groups, H a subgroup of G and $L = \alpha^{-1}(H)$.*

(i) *For every orthogonal H-spectrum Y the following square commutes:*

$$\begin{tikzcd}
\pi_*^H(Y) \arrow[rr, "\mathrm{Tr}_H^G"] \arrow[d, "(\alpha|_L)^*"'] & & \pi_*^G(G \ltimes_H Y) \arrow[d, "\alpha^*"] \\
\pi_*^L((\alpha|_L)^*(Y)) \arrow[r, "\mathrm{Tr}_L^K"'] & \pi_*^K(K \ltimes_L ((\alpha|_L)^* Y)) \arrow[r, "\pi_*^K(\xi)"'] & \pi_*^K(\alpha^*(G \ltimes_H Y))
\end{tikzcd}$$

(ii) *For every orthogonal G-spectrum X the following square commutes:*

$$\begin{tikzcd}
\pi_*^H(X) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "(\alpha|_L)^*"'] & \pi_*^G(X) \arrow[d, "\alpha^*"] \\
\pi_*^L(\alpha^* X) \arrow[r, "\mathrm{tr}_L^K"'] & \pi_*^K(\alpha^* X)
\end{tikzcd}$$

Proof. (i) The composite

$$K \ltimes_L ((\alpha|_L)^* Y) \xrightarrow{\ \xi\ } \alpha^*(G \ltimes_H Y) \xrightarrow{(\alpha|_L)^*(\mathrm{pr}_H^G)} (\alpha|_L)^*(Y)$$

equals the wedge summand projection pr_L^K, so

$$\begin{aligned}
\pi_*^K(\xi) \circ \mathrm{Tr}_L^K \circ \pi_*^L((\alpha|_L)^*(\mathrm{pr}_H^G)) \circ \mathrm{res}_L^K &= \pi_*^K(\xi) \circ \mathrm{Tr}_L^K \circ \pi_*^L(\mathrm{pr}_L^K) \circ \pi_*^L(\xi^{-1}) \circ \mathrm{res}_L^K \\
&= \pi_*^K(\xi) \circ \mathrm{Tr}_L^K \circ \pi_*^L(\mathrm{pr}_L^K) \circ \mathrm{res}_L^K \circ \pi_*^K(\xi^{-1}) \\
&= \pi_*^K(\xi) \circ \pi_*^K(\xi^{-1}) \;=\; \mathrm{Id}
\end{aligned}$$

is the identity of $\pi_*^K(\alpha^*(G\ltimes_H Y))$. Thus

$$\begin{aligned}\pi_*^K(\xi)\circ \mathrm{Tr}_L^K\circ(\alpha|_L)^*\circ\pi_*^H(\mathrm{pr}_H^G)\circ\mathrm{res}^G_H \ &= \ \pi_*^K(\xi)\circ\mathrm{Tr}_L^K\circ\pi_*^L((\alpha|_L)^*(\mathrm{pr}_H^G))\circ(\alpha|_L)^*\circ\mathrm{res}^G_H\\ &= \ \pi_*^K(\xi)\circ\mathrm{Tr}_L^K\circ\pi_*^L((\alpha|_L)^*(\mathrm{pr}_H^G))\circ\mathrm{res}^K_L\circ\alpha^* \ = \ \alpha^* \ .\end{aligned}$$

Precomposing with the external transfer Tr_H^G and using Proposition 4.12 shows the first claim. Part (ii) follows from part (i) and the fact that

$$K\ltimes_L((\alpha|_L)^*(Y)) \ \xrightarrow{\ \xi\ } \ \alpha^*(G\ltimes_H Y) \ \xrightarrow{\alpha^*(\mathrm{act})} \ \alpha^* Y$$

is the action map of K on $(\alpha|_L)^*(Y)$. □

A special case of an epimorphism is the conjugation map

$$c_g \ : \ H \ \longrightarrow \ H^g=g^{-1}Hg \ , \quad h \ \longmapsto \ c_g(h) \ = \ g^{-1}hg$$

induced by an element $g\in G$ of the ambient group. We recall from (3.7) that the conjugation map $g_*:\pi_0^{H^g}(X)\longrightarrow\pi_0^H(X)$ is defined as the composite

$$\pi_0^{H^g}(X) \ \xrightarrow{\ c_g^*\ } \ \pi_0^H(c_g^*X) \ \xrightarrow{\pi_0^H(l_g^X)} \ \pi_0^H(X)$$

where $l_g^X : c_g^*X \longrightarrow X$ is left translation by g. Proposition 4.18 (ii) applied to $K=G$ and the inner automorphism $\alpha=c_g:G\longrightarrow G$ implies the relation

$$\mathrm{tr}_H^G\circ g_* \ = \ \mathrm{tr}_H^G\circ\pi_*^H(l_g^X)\circ c_g^* \ = \ \pi_*^G(l_g^X)\circ\mathrm{tr}_H^G\circ c_g^* \ = \ \pi_*^G(l_g^X)\circ c_g^*\circ\mathrm{tr}_{H^g}^G \ = \ \mathrm{tr}_{H^g}^G$$

as maps from $\pi_*^{H^g}(X)$ to $\pi_*^G(X)$, for every orthogonal G-spectrum X. The last step is the fact that inner automorphisms induce the identity, compare Proposition 3.6.

Now we prove the *double coset formula* for the restriction of a transfer to a subgroup. We let K and H be subgroups of G and A a based H-space. For every $g\in G$, the map

$$\kappa_g \ : \ K\ltimes_{K\cap{}^gH}(\mathrm{res}^{{}^gH}_{K\cap{}^gH}(c_g^*A)) \ \longrightarrow \ G\ltimes_H A \ , \quad k\ltimes a \ \longmapsto \ (kg)\ltimes a$$

is K-equivariant. As usual, $c_g:{}^gH\longrightarrow H$ is the conjugation homomorphism given by $c_g(\gamma)=g^{-1}\gamma g$. If we let g vary in a set of K-H-double coset representatives, the combined map

$$\bigvee_{[g]\in K\backslash G/H} K\ltimes_{K\cap{}^gH}(\mathrm{res}^{{}^gH}_{K\cap{}^gH}(c_g^*A)) \ \xrightarrow{\ \bigvee\kappa_g\ } \ \mathrm{res}^G_K(G\ltimes_H A)$$

is a K-equivariant homeomorphism. All this is natural, so we can apply the constructions and maps levelwise to an orthogonal H-spectrum Y and obtain an analogous morphism of orthogonal K-spectra

$$\kappa_g \ : \ K\ltimes_{K\cap{}^gH}(\mathrm{res}^{{}^gH}_{K\cap{}^gH}(c_g^*Y)) \ \longrightarrow \ G\ltimes_H Y \ ,$$

which gives a wedge decomposition of the underlying K-spectrum of $G\ltimes_H Y$ when g runs over a set of representatives of all K-H-double cosets.

Proposition 4.19 (External double coset formula)**.** *For all subgroups K and H of G and every orthogonal H-spectrum Y the relation*

$$\mathrm{res}^G_K\circ\mathrm{Tr}_H^G \ = \ \sum_{[g]\in K\backslash G/H}\pi_*^K(\kappa_g)\circ\mathrm{Tr}^K_{K\cap{}^gH}\circ\mathrm{res}^{{}^gH}_{K\cap{}^gH}\circ c_g^*$$

holds as maps $\pi_*^H(Y)\longrightarrow\pi_*^K(G\ltimes_H Y)$.

Proof. For $g \in G$ we denote by

$$\mathrm{pr}_g \ : \ G \ltimes_H Y \ \longrightarrow \ K \ltimes_{K \cap {}^g H} c_g^*(Y)$$

the morphism of orthogonal K-spectra that is left inverse to κ_g and sends all K-H-double cosets other than KgH to the basepoint. The morphism of orthogonal $(K \cap {}^g H)$-spectra $c_g^*(\mathrm{pr}_H^G) : c_g^*(G \ltimes_H Y) \longrightarrow c_g^*(Y)$ equals the composite

$$c_g^*(G \ltimes_H Y) \xrightarrow{\ l_g\ } G \ltimes_H Y \xrightarrow{\ \mathrm{pr}_g\ } K \ltimes_{K \cap {}^g H} c_g^*(Y) \xrightarrow{\ \mathrm{pr}^K_{K \cap {}^g H}\ } c_g^*(Y)$$

where l_g is left multiplication by g. So

$$\begin{aligned}
\mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ c_g^* \circ \pi_*^H(\mathrm{pr}_H^G) \circ \mathrm{res}^G_H \ &= \ \pi_*^{K \cap {}^g H}(c_g^*(\mathrm{pr}_H^G)) \circ \mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ c_g^* \circ \mathrm{res}^G_H \\
&= \ \pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \pi_*^{K \cap {}^g H}(\mathrm{pr}_g) \circ \pi_*^{K \cap {}^g H}(l_g) \circ \mathrm{res}^G_{K \cap {}^g H} \circ c_g^* \\
&= \ \pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \pi_*^{K \cap {}^g H}(\mathrm{pr}_g) \circ \mathrm{res}^G_{K \cap {}^g H} \circ \pi_*^G(l_g) \circ c_g^* \\
&= \ \pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \pi_*^{K \cap {}^g H}(\mathrm{pr}_g) \circ \mathrm{res}^G_{K \cap {}^g H}
\end{aligned}$$

We have used various naturality properties and, in the last equation, that inner automorphisms induce the identity (Proposition 3.6). The external transfer Tr_H^G is inverse to $\pi_*^H(\mathrm{pr}_H^G) \circ \mathrm{res}^G_H$, so precomposition with this external transfer gives

$$\begin{aligned}
\mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ c_g^* \ &= \ \pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \pi_*^{K \cap {}^g H}(\mathrm{pr}_g) \circ \mathrm{res}^G_{K \cap {}^g H} \circ \mathrm{Tr}_H^G \\
&= \ \pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \mathrm{res}^K_{K \cap {}^g H} \circ \pi_*^K(\mathrm{pr}_g) \circ \mathrm{res}^G_K \circ \ \mathrm{Tr}_H^G \ .
\end{aligned}$$

Similarly, the transfer map $\mathrm{Tr}^K_{K \cap {}^g H}$ is inverse to $\pi_*^{K \cap {}^g H}(\mathrm{pr}^K_{K \cap {}^g H}) \circ \mathrm{res}^K_{K \cap {}^g H}$, so postcomposition with this external transfer gives

$$\mathrm{Tr}^K_{K \cap {}^g H} \circ \mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ c_g^* \ = \ \pi_*^K(\mathrm{pr}_g) \circ \mathrm{res}^G_K \circ \ \mathrm{Tr}_H^G \tag{4.20}$$

as maps $\pi_*^H(Y) \longrightarrow \pi_*^K(K \ltimes_{K \cap {}^g H} c_g^*(Y))$.

The underlying orthogonal K-spectrum of $G \ltimes_H Y$ is the wedge, indexed over K-H-double coset representatives, of the images of the idempotent endomorphisms $\kappa_g \circ \mathrm{pr}_g$ of $G \ltimes_H Y$. Since equivariant homotopy groups takes wedges to sums, the identity of $\pi_*^K(G \ltimes_H Y)$ is the sum of the effects of these idempotents. So we can postcompose the relation (4.20) with $\pi_*^K(\kappa_g)$ and sum over double coset representatives to get the desired formula:

$$\sum_{[g] \in K \backslash G/H} \pi_*^K(\kappa_g) \circ \mathrm{Tr}^K_{K \cap {}^g H} \circ \mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ c_g^* \ = \ \sum_{[g] \in K \backslash G/H} \pi_*^K(\kappa_g \circ \mathrm{pr}_g) \circ \mathrm{res}^G_K \circ \mathrm{Tr}_H^G \ = \ \mathrm{res}^G_K \circ \mathrm{Tr}_H^G \ . \qquad \square$$

The double coset formula for the internal transfer maps follows from the external one by naturality arguments.

Proposition 4.21 (Internal double coset formula)**.** *For all subgroups K and H of G and every orthogonal G-spectrum X the relation*

$$\mathrm{res}^G_K \circ \mathrm{tr}_H^G \ = \ \sum_{[g] \in K \backslash G/H} \mathrm{tr}^K_{K \cap {}^g H} \circ \mathrm{res}^{{}^g H}_{K \cap {}^g H} \circ g_*$$

holds as maps $\pi_*^H(X) \longrightarrow \pi_*^K(X)$.

Proof. We denote by $\eta^G : G \ltimes_H X \longrightarrow X$ the G-action morphism. If we apply the map $\pi_*^K(\eta^G)$ to the external double coset formula, then the left hand side becomes the composite of internal transfer and restriction (using that restriction is natural for the G-morphism η^G). Now we simplify the summands on the right hand side of the external double coset formula. For every $g \in G$ the square of K-morphisms

$$\begin{tikzcd} K \ltimes_{K\cap {}^g H} (c_g^* X) \arrow[r, "\eta^K"] \arrow[d, "\kappa_g"'] & c_g^* X \arrow[d, "l_g^X"] \\ G \ltimes_H X \arrow[r, "\eta^G"'] & X \end{tikzcd}$$

commutes, where as usual l_g is left multiplication by g. So we get

$$\begin{aligned} \pi_*^K(\eta^G)\circ\pi_*^K(\kappa_g)\circ \mathrm{Tr}^K_{K\cap {}^g H}\circ \mathrm{res}^{^g H}_{K\cap {}^g H}\circ c_g^* &= \pi_*^K(l_g^X)\circ\pi_*^K(\eta^K)_*\circ \mathrm{Tr}^K_{K\cap {}^g H}\circ \mathrm{res}^{^g H}_{K\cap {}^g H}\circ c_g^* \\ &= \pi_*^K(l_g^X)\circ \mathrm{tr}^K_{K\cap {}^g H}\circ \mathrm{res}^{^g H}_{K\cap {}^g H}\circ c_g^* \\ &= \mathrm{tr}^K_{K\cap {}^g H}\circ \mathrm{res}^{^g H}_{K\cap {}^g H}\circ \pi_*^{^g H}(l_g^X)\circ c_g^* \\ &= \mathrm{tr}^K_{K\cap {}^g H}\circ \mathrm{res}^{^g H}_{K\cap {}^g H}\circ g_* \end{aligned}$$

The second equality is the definition of the internal transfer for the spectrum $c_g^* X$. The third equality is the fact that transfer and restriction are natural for the G-morphism $l_g^X : c_g^* X \longrightarrow X$. The final relation is the definition (3.7) of the conjugation map $g_* : \pi_*^H(X) \longrightarrow \pi_*^{^g H}(X)$. □

The restriction, conjugation and transfer maps make the homotopy groups $\pi_k^H(X)$ for varying H into a *Mackey functor*. We recall that a Mackey functor for a group G consists of the following data:

- an abelian group $M(H)$ for every subgroup H of G,
- conjugation maps $g_\star : M(H) \longrightarrow M({}^g H)$ for $H \subset G$ and $g \in G$, where ${}^g H = gHg^{-1}$,
- restriction maps $\mathrm{res}^H_K : M(H) \longrightarrow M(K)$ for $K \subset H \subset G$,
- transfer maps $\mathrm{tr}^H_K : M(K) \longrightarrow M(H)$ for $K \subset H \subset G$.

This data has to satisfy the following conditions. The unit conditions

$$\mathrm{res}^H_H = \mathrm{Id}_{M(H)} \qquad \text{and} \qquad h_\star = \mathrm{Id}_{M(H)} \quad \text{for } h \in H$$

and transitivity conditions

$$g_\star \circ g'_\star = (gg')_\star\ , \quad \mathrm{res}^K_L \circ \mathrm{res}^H_K = \mathrm{res}^H_L \qquad \text{and} \qquad \mathrm{res}^H_K \circ g_\star = g_\star \circ \mathrm{res}^{H^g}_{K^g}$$

express that facts that the restriction and conjugation maps assemble into a contravariant functor on the orbit category $\mathcal{O}(G)$ of G. The unit conditions $\mathrm{tr}^H_H = \mathrm{Id}_{M(H)}$ and transitivity conditions

$$\mathrm{tr}^H_K \circ \mathrm{tr}^K_L = \mathrm{tr}^H_L \quad \text{and} \quad \mathrm{tr}^H_K \circ g_\star = g_\star \circ \mathrm{tr}^{H^g}_{K^g}$$

express the fact that the transfer and conjugation maps form a covariant functor on the orbit category of G. Finally, restriction and transfer are related by the double coset formula. It says that for every pair of subgroups K, K' of H the relation

$$\mathrm{res}^H_{K'} \circ \mathrm{tr}^H_K = \sum_{[h]\in K'\backslash H/K} \mathrm{tr}^{K'}_{K'\cap {}^h K} \circ \mathrm{res}^{^h K}_{K'\cap {}^h K} \circ h_\star \tag{4.22}$$

holds as maps $M(K) \longrightarrow M(K')$. Here $[h]$ runs over a set of representatives for the double cosets for $K'\backslash H/K$.

Example 4.23. To get used to the double coset formula, the reader is advised to calculate some instances in specific examples. We do one sample calculation here. We let $G = \Sigma_{n+1}$ be the symmetric group on $n+1$ letters, for some $n \geq 1$. We set $H = K = \Sigma_n$, which we think of as the subgroup of those permutations in Σ_{n+1} that fix the last element element $n+1$. There are then two Σ_n-Σ_n double cosets in Σ_{n+1}, and the identity permutation 1 and the transposition $t = (n, n+1)$ are convenient coset representatives. The double coset formula for $\mathrm{res}^{\Sigma_{n+1}}_{\Sigma_n} \circ \mathrm{tr}^{\Sigma_{n+1}}_{\Sigma_n}$ thus has two terms that we identify now. The identity permutation contributes the term $\mathrm{tr}^{\Sigma_n}_{\Sigma_n} \circ 1_\star \circ \mathrm{res}^{\Sigma_n}_{\Sigma_n}$, which is the identity. The transposition $(n, n+1)$ satisfies

$$\Sigma_n \cap {}^t(\Sigma_n) \ = \ (\Sigma_n)^t \cap \Sigma_n \ = \ \Sigma_{n-1} \ ,$$

the subgroup of those permutations in Σ_{n+1} that fix the two last elements n and $n+1$. So the double coset specializes to the expression

$$\mathrm{res}^{\Sigma_{n+1}}_{\Sigma_n} \circ \mathrm{tr}^{\Sigma_{n+1}}_{\Sigma_n} \ = \ \mathrm{Id} \ + \ \mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}} \circ t_\star \circ \mathrm{res}^{\Sigma_n}_{\Sigma_{n-1}} \ .$$

Remark 4.24. The definition (3.1) of equivariant homotopy groups of an orthogonal G-spectrum has room for an extra parameter. Indeed, we can use a G-representation U instead of the regular representation, and modify (3.1) to

$$\pi_0^{G,U}(X) \ = \ \mathrm{colim}_n \, [S^{nU}, X(nU)]^G \ ,$$

where the colimit is taken along $- \diamond U$, stabilization by U. In order to end up with abelian groups we should assume that $U^G \neq 0$. We can then define a (G, U)-equivariant stable homotopy category by formally inverting the morphisms of orthogonal G-spectra that induce isomorphisms on the groups $\pi_k^{H,U}$ for all integers k and all subgroups H of G.

If we stabilize with a representation U that does not contain all irreducible G-representations, then some aspects of the theory change. For example, Proposition 3.3 does not hold in full generality anymore, but only for G-representation V that embed into nU for some $n \geq 1$. Also, the Wirthmüller isomorphism may fail for the U-based homotopy groups $\pi_*^{G,U}$, i.e., the morphism $\Psi_Y : G \ltimes_H Y \longrightarrow \mathrm{map}^H(G, Y)$ need not in general induce isomorphisms in $\pi_*^{G,U}$. However, an inspection of the proof of Theorem 4.9 shows that

$$\pi_*^{G,U}(\Psi_Y) \ : \ \pi_*^{G,U}(G \ltimes_H Y) \ \longrightarrow \ \pi_*^{G,U}(\mathrm{map}^H(G, Y))$$

is an isomorphism if G/H admits a G-equivariant injection into nU for some $n \geq 1$.

Also we can in general *not* construct the transfer maps (4.14) and (4.15) because we may not be able to embed a coset G/H equivariantly into a sum of copies of U. So the U-based homotopy groups $\pi_0^{G,U}(X)$ will typically admit some, but not all transfers, and they do not form full Mackey functors.

If $\bar{U}$ is another G-representation such that U embeds into a sum of copies of $\bar{U}$, then the analog (with $\bar{U}$ instead of the regular representation) of Proposition 3.3 lets us define a preferred homomorphism

$$\pi_0^{G,U}(X) \ \longrightarrow \ \pi_0^{G,\bar{U}}(X) \ , \quad [f : S^{nU} \longrightarrow X(nU)] \ \longmapsto \ \langle f \rangle \ .$$

This homomorphism is natural in X and compatible with restriction to subgroups and with those transfers that exists for $\pi_0^{G,U}$. If $\bar{U}$ also embeds into a sum of copies of U, we get an inverse homomorphism in the other direction by exchanging the roles of U and $\bar{U}$. So up to canonical natural isomorphism, the group $\pi_0^{G,U} X$ only depends on the 'universe' generated by U, i.e., on the infinite dimensional G-representation ∞U, the direct sum of countably many copies of U. The universes ∞U and $\infty \bar{U}$ are G-isometrically isomorphic if and only the same irreducible representations embed in U and $\bar{U}$. So $\pi_0^{G,U} X$, and hence the (G, U)-equivariant stable homotopy category, only depends on the class of irreducible representations contained in U. Somewhat less obviously, the group $\pi_0^{G,U} X$, and in fact the entire G-equivariant stable homotopy theory based on the universe ∞U, depends on even less, namely only on the set of those subgroups of G that occur as stabilizers of vectors in ∞U (or what is the same, the set of subgroups $H \leq G$ such that G/H embeds G-equivariantly into ∞U). [14, Thm. 1.2].

In these notes, we focus on the most interesting case where $U = \rho_G$ is the regular representation. Then $\infty \rho_G$ is a complete universe (i.e., every G-representation embeds into it) and we arrive at what is often referred to as 'genuine' equivariant stable homotopy theory, with full Mackey functor structure on the equivariant homotopy groups.

The other extreme is where $U = \mathbb{R}$ is a trivial 1-dimensional representation. Then $\infty U = \mathbb{R}^\infty$ has trivial G-action and is thus called a trivial universe. The homotopy groups $\pi_0^{G,\mathbb{R}} X$ do not support any non-trivial transfers and the $(G, \mathbb{R})$-equivariant stable homotopy category is often referred to as the 'naive' equivariant stable homotopy category.

Another case that comes up naturally is the natural representation of the symmetric group Σ_m on $\mathbb{R}^m$, by permutation of coordinates; for $m \geq 3$ the corresponding universe is neither complete nor trivial. The corresponding equivariant homotopy groups arise naturally as target of power operation, compare Remark 9.3 below.

Construction 4.25 (External multiplication)**.** If X and Y are orthogonal G-spectra, we let G act diagonally on the smash product $X \wedge Y$. We now define an external product

$$\wedge \; : \; \pi_k^G(X) \; \times \; \pi_l^G(Y) \; \longrightarrow \; \pi_{k+l}^G(X \wedge Y) \tag{4.26}$$

on the G-equivariant homotopy groups. The definition is essentially straightforward, but there is one subtlety in showing that the product is well-defined.

We suppose that $f : S^{k+V} \longrightarrow X(V)$ and $g : S^{l+W} \longrightarrow Y(W)$ represent classes in $\pi_k^G(X)$ respectively $\pi_l^G(Y)$, where V and W are G-representations. Then we denote by $f \cdot g$ the composite

$$S^{k+l+V+W} \xrightarrow{\mathrm{Id} \wedge \tau_{l,V} \wedge \mathrm{Id}} S^{k+V} \wedge S^{l+W} \xrightarrow{f \wedge g} X(V) \wedge Y(W) \xrightarrow{\iota_{V,W}} (X \wedge Y)(V+W) \; .$$

Here $i_{V,W} : X(V) \wedge Y(W) \longrightarrow (X \wedge Y)(V \oplus W)$ is the (V, W)-component of the universal bimorphism from (X, Y) to $X \wedge Y$. When we stabilize the representing maps by another G-representation U, we have the relations

$$f \cdot (g \diamond U) \; = \; (f \cdot g) \diamond U \; = \; \alpha_* \left((f \diamond U) \cdot g \right)$$

where $\alpha : V \oplus U \oplus W \longrightarrow V \oplus W \oplus U$ is the automorphism that interchanges U and W; here Proposition 3.3 is used one more time. The upshot is that the definition

$$[f] \cdot [g] \; = \; [f \cdot g]$$

is well-defined. One also checks that the product is biadditive and associative in the sense that for all orthogonal G-spectrum X, Y and Z the diagram commutes (where we suppress the associativity isomorphism for the smash product):

$$\begin{tikzcd} \pi_k^G(X) \times \pi_l^G(Y) \times \pi_j^G(Z) \arrow[r, "\mathrm{Id} \times \wedge"] \arrow[d, "\wedge \times \mathrm{Id}"'] & \pi_k^G(X) \times \pi_{l+j}^G(Y \wedge Z) \arrow[d, "\wedge"] \\ \pi_{k+l}^G(X \wedge Y) \times \pi_j^G(Z) \arrow[r, "\wedge"'] & \pi_{k+l+j}^G(X \wedge Y \wedge Z) \end{tikzcd}$$

The external pairing is also unital, relative to the class $1 \in \pi_0^G(\mathbb{S})$ represented by the identity of $S^0 = \mathbb{S}(0)$. More precisely, for every orthogonal G-spectrum X, the following two composites are the identity of $\pi_k^G(X)$:

$$\pi_k^G(X) \xrightarrow{1 \wedge -} \pi_k^G(\mathbb{S} \wedge X) \xrightarrow{\cong} \pi_k^G(X)$$
$$\pi_k^G(X) \xrightarrow{- \wedge 1} \pi_k^G(X \wedge \mathbb{S}) \xrightarrow{\cong} \pi_k^G(X)$$

Finally, the internal multiplication is commutative in the graded sense, i.e., the following diagram commutes for all orthogonal G-spectra X and Y:

$$\begin{tikzcd} \pi_k^G(X) \times \pi_l^G(Y) \arrow[r, "\wedge"] \arrow[d, "\mathrm{twist}"'] & \pi_{k+l}^G(X \wedge Y) \arrow[d, "(-1)^{kl} \circ \pi_{k+l}^G(\tau_{X,Y})"] \\ \pi_l^G(Y) \times \pi_k^G(X) \arrow[r, "\wedge"'] & \pi_{l+k}^G(Y \wedge X) \end{tikzcd}$$

Here $\tau_{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetric isomorphism of the smash product.

The next proposition records how the external multiplication interacts with restriction and transfer maps.

Proposition 4.27. *Let H be a subgroup of a finite group G.*

(i) *For all orthogonal G-spectra X and Y and every homomorphism $\alpha : K \longrightarrow G$, the following square commutes:*

$$\begin{tikzcd}
\pi_k^G(X) \times \pi_l^G(Y) \arrow[rr, "\wedge"] \arrow[d, "\alpha^* \times \alpha^*"'] & & \pi_{k+l}^G(X \wedge Y) \arrow[d, "\alpha^*"] \\
\pi_k^K(\alpha^*(X)) \times \pi_l^K(\alpha^*(Y)) \arrow[r, "\wedge"'] & \pi_{k+l}^K(\alpha^*(X) \wedge \alpha^*(Y)) \arrow[r, equal] & \pi_{k+l}^K(\alpha^*(X \wedge Y))
\end{tikzcd}$$

(ii) *Let X be an orthogonal G-spectrum and Y an orthogonal H-spectrum. Define the G-equivariant shearing isomorphism*

$$\chi \ : \ X \wedge (G \ltimes_H Y)) \ \xrightarrow{\ \cong\ } \ G \ltimes_H (X \wedge Y) \qquad by \qquad x \wedge [g, y] \ \longmapsto \ [g, g^{-1}x \wedge y] \ .$$

Then the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X) \times \pi_l^H(Y) \arrow[r, "\mathrm{Id} \times \mathrm{Tr}_H^G"] \arrow[d, "\mathrm{res}_H^G \times \mathrm{Id}"'] & \pi_k^G(X) \times \pi_l^G(G \ltimes_H Y) \arrow[r, "\wedge"] & \pi_{k+l}^G(X \wedge (G \ltimes_H Y)) \arrow[d, "\chi_*"] \\
\pi_k^H(X) \times \pi_l^H(Y) \arrow[r, "\wedge"'] & \pi_{k+l}^H(X \wedge Y) \arrow[r, "\mathrm{Tr}_H^G"'] & \pi_{k+l}^G(G \ltimes_H (X \wedge Y))
\end{tikzcd}$$

(iii) *For all orthogonal G-spectra X and Y, and all classes $x \in \pi_k^G(X)$ and $y \in \pi_l^G(Y)$, the relation*

$$x \wedge \mathrm{tr}_G^H(y) \ = \ \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \wedge y)$$

holds in $\pi_{k+l}^G(X \wedge Y)$.

Proof. Part (i) is straightforward from the definitions.

(ii) We observe that the shearing isomorphism makes the following diagram commute:

$$\begin{tikzcd}
X \wedge (G \ltimes_H X) \arrow[rr, "\chi"] \arrow[dr, "X \wedge \mathrm{pr}_Y"'] & & G \ltimes (X \wedge Y) \arrow[dl, "\mathrm{pr}_{X \wedge Y}"] \\
& X \wedge Y &
\end{tikzcd}$$

Here pr_Y and $\mathrm{pr}_{X \wedge Y}$ are the projections to the preferred coset, defined in (4.11). Then the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X) \times \pi_l^H(Y) \arrow[r, "\mathrm{Id} \times \mathrm{Tr}_H^G"] \arrow[d, "\mathrm{res}_H^G \times \mathrm{Id}"'] & \pi_k^G(X) \times \pi_l^G(G \ltimes_H Y) \arrow[r, "\wedge"] \arrow[d, "\mathrm{res}_H^G \times \mathrm{Id}"] & \pi_{k+l}^G(X \wedge (G \ltimes_H Y)) \arrow[r, "\chi_*"] \arrow[dd, "\mathrm{res}_H^G"'] & \pi_{k+l}^G(G \ltimes_H (X \wedge Y)) \arrow[dd, "\mathrm{res}_H^G"] \\
\pi_k^H(X) \times \pi_l^H(Y) \arrow[r, "\mathrm{Id} \times \mathrm{Tr}_H^G"] \arrow[ddr, equal, bend right] & \pi_k^H(X) \times \pi_l^G(G \ltimes_H Y) \arrow[d, "\mathrm{Id} \times \mathrm{res}_H^G"] & & \\
& \pi_k^H(X) \times \pi_l^H(G \ltimes_H Y) \arrow[r, "\wedge"'] \arrow[d, "\mathrm{Id} \times (\mathrm{pr}_Y)_*"] & \pi_{k+l}^H(X \wedge (G \ltimes_H Y)) \arrow[r, "\chi_*"'] \arrow[d, "(X \wedge \mathrm{pr}_Y)_*"'] & \pi_{k+l}^G(G \ltimes_H (X \wedge Y)) \arrow[dl, bend left, "(\mathrm{pr}_{X \wedge Y})_*)"] \\
& \pi_k^H(X) \times \pi_l^H(Y) \arrow[r, "\wedge"'] & \pi_{k+l}^H(X \wedge Y) &
\end{tikzcd}$$

The lower left triangle is the fact that Tr_H^G was defined as the inverse to the map $(\mathrm{pr}_Y)_* \circ \mathrm{res}^G_H$. The upper middle square commutes by part (i), applied to the inclusion $H \longrightarrow G$. The right vertical composite $(\mathrm{pr}_{X\wedge Y})_* \circ \mathrm{res}^G_H$ is inverse to the external transfer $\mathrm{Tr}_H^G : \pi^H_{k+l}(X \wedge Y) \longrightarrow \pi^G_{k+l}(G \ltimes_H (X \wedge Y))$, by definition. So postcomposing with the external transfer yields the desired commutativity.

(iii) The shearing isomorphism makes the following diagram commute:

$$\begin{tikzcd}
X \wedge (G \ltimes_H X) \arrow[rr, "\chi"] \arrow[dr, "X \wedge \eta_Y"'] & & G \ltimes (X \wedge Y) \arrow[dl, "\eta_{X\wedge Y}"] \\
& X \wedge Y &
\end{tikzcd}$$

Here η_Y and $\eta_{X\wedge Y}$ are the action maps, i.e. adjunction counits. So part (ii) yields

$$\begin{aligned}
\mathrm{tr}_H^G(\mathrm{res}^G_H(x) \wedge y) \ &= \ (\eta_{X\wedge Y})_*(\mathrm{Tr}_H^G(\mathrm{res}^G_H(x) \wedge y)) \\
&= \ (\eta_{X\wedge Y})_*(\chi_*(x \wedge \mathrm{Tr}_H^G(y))) \\
&= \ (X \wedge \eta_Y)_*(x \wedge \mathrm{Tr}_H^G(y)) \\
&= \ x \wedge (\eta_Y)_*(\mathrm{Tr}_H^G(y)) \ = \ x \wedge \mathrm{tr}_H^G(y) \ . \qquad \square
\end{aligned}$$

The formula for the interaction of multiplication and transfers are sometimes referred to as *reciprocity*. The graded commutativity of the external product and the naturality of transfers imply that the analogous reciprocity relation also holds on the other side: for $x \in \pi^G_k(X)$ and $y \in \pi^H_l(Y)$, we have

$$\begin{aligned}
\mathrm{tr}_H^G(y) \wedge x \ &= \ (-1)^{kl} \cdot \pi^G_{k+l}(\tau_{X,Y})(x \wedge \mathrm{tr}_H^G(y)) \\
&= \ (-1)^{kl} \cdot \pi^G_{k+l}(\tau_{X,Y})(\mathrm{tr}_H^G(\mathrm{res}^G_H(x) \wedge y)) \\
&= \ (-1)^{kl} \cdot \mathrm{tr}_H^G(\pi^G_{k+l}(\tau_{X,Y})(\mathrm{res}^G_H(x) \wedge y)) \ = \ \mathrm{tr}_H^G(y \wedge \mathrm{res}^G_H(x)) \ .
\end{aligned}$$

Construction 4.28 (Multiplication by the equivariant stems)**.** The equivariant stable stems $\pi^G_* = \pi^G_*(\mathbb{S})$ form a graded ring with a certain commutativity property that acts on the homotopy groups of every other G-spectrum X. We define the action as the composite

$$\pi^G_k(X) \times \pi^G_l \ \xrightarrow{\ \wedge\ } \ \pi^G_{k+l}(X \wedge \mathbb{S}) \ \xrightarrow{\ \cong\ } \ \pi^G_{k+l}(X) \ ,$$

and denote it simply by a 'dot'

$$\cdot \ : \ \pi^G_k(X) \times \pi^G_l \ \longrightarrow \ \pi^G_{k+l}(X) \ . \tag{4.29}$$

Expanding the definitions gives the following explicit description of this action. Suppose $f : S^{k+W} \longrightarrow X(W)$ and $g : S^{l+V} \longrightarrow S^V$ represent classes in $\pi^G_k(X)$ respectively $\pi^G_l(\mathbb{S})$. Then $[f] \cdot [g] = [f \cdot g]$, where $f \cdot g$ is the composite

$$S^{k+l+V+W} \ \xrightarrow{\mathrm{Id} \wedge \tau_{l,W} \wedge \mathrm{Id}} \ S^{k+V} \wedge S^{l+W} \ \xrightarrow{f \wedge g} \ X(V) \wedge S^W \ \xrightarrow{\sigma_{V,W}} \ X(V+W) \ .$$

The various properties of the external multiplication (4.26) imply corresponding properties of this action: the dot product is biadditive, unital, and associative, Finally, in the case $X = \mathbb{S}$ the internal multiplication in the equivariant homotopy groups of spheres is commutative in the graded sense, i.e., we have $xy = (-1)^{kl} yx$ for $x \in \pi^G_k$ and $y \in \pi^G_l$. We will prove this as a special case of a more subtle commutativity property of the external product of '$RO(G)$-graded homotopy groups', see (4.34).

The action of the equivariant stable stems on the homotopy groups of a G-spectrum is compatible with restriction to subgroups, i.e., for all $H \subseteq G$ we have

$$\mathrm{res}^G_H(x \cdot y) \ = \ \mathrm{res}^G_H(x) \cdot \mathrm{res}^G_H(y)$$

for $x \in \pi_k^G(X)$ and $y \in \pi_l^G$, by Proposition 4.27 (i). Proposition 4.27 (iii) specializes to the reciprocity formulas

$$\mathrm{tr}_H^G(x) \cdot y \;=\; \mathrm{tr}_H^G(x \cdot \mathrm{res}_H^G(y)) \qquad \text{and} \qquad x \cdot \mathrm{tr}_H^G(y) \;=\; \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \cdot y)\ . \tag{4.30}$$

Remark 4.31. The Mackey functors that arise in algebra, for example in group cohomology, often have the special property that restriction followed by transfer to the same subgroup is multiplication by the index. In out context, however, $\mathrm{tr}_K^G \circ \mathrm{res}_K^G$ is *not* in general multiplication by the index $[G : K]$. Indeed, the special case of Frobenius reciprocity with $y = 1$ says that

$$\mathrm{tr}_K^G(\mathrm{res}_K^G(x)) \;=\; x \cdot \mathrm{tr}_K^G(1)$$

for all $a \in \pi_k^G(X)$. The element $\mathrm{tr}_K^G(1) \in \pi_0^G$ is different from $[G : K] \cdot 1$, but the map

$$\pi_0^G \;=\; \pi_0^G(\mathbb{S}) \;\longrightarrow\; \pi_0^G(H\mathbb{Z}) \;\cong\; \mathbb{Z}$$

induced by the unit morphism $\mathbb{S} \longrightarrow H\mathbb{Z}$ takes $\mathrm{tr}_K^G(1)$ to the index $[G : K]$. So the relation $\mathrm{tr}_K^G(\mathrm{res}_K^G(x)) = [G : K] \cdot x$ does hold in the homotopy Mackey functor of every $H\mathbb{Z}$-module spectrum.

$RO(G)$**-graded homotopy groups.** There is a way to index homotopy groups by representations that is commonly referred to as $RO(G)$*-graded homotopy groups*. In this note we will *not* actually grade by the group $RO(G)$, i.e., by *isomorphism classes* of representations, but rather by actual representations. Lewis and Mandell [12, App. A] show that a strict $RO(G)$-grading is possible, but it involves coherence issues that are resolvable because a certain 'coherence cycle' is a coboundary, see [12, Prop. A.5]. For a comprehensive account of the intricacies of $RO(G)$-gradings we recommend Dugger's paper [6].

For an orthogonal G-spectrum X and a G-representation V we define

$$\pi_V^G(X) \;=\; \pi_0^G(\Omega^V X) \;\cong\; \mathrm{colim}_n\, [S^{V+n\rho_G}, X(n\rho_G)]^G\ .$$

With this definition we have $\pi_k^G(X) \cong \pi_{\mathbb{R}^k}^G(X)$. The '$RO(G)$-graded' homotopy groups admit an external product

$$\cdot\ :\ \pi_V^G(X) \times \pi_W^G(Y) \;\longrightarrow\; \pi_{V+W}^G(X \wedge Y) \tag{4.32}$$

that is a straightforward generalization of the action (4.29) of the equivariant stable stems on the equivariant homotopy groups of a G-spectrum. Suppose $f : S^{V+n\rho} \longrightarrow X(n\rho)$ and $g : S^{W+m\rho} \longrightarrow Y(m\rho)$ represent classes in $\pi_V^G(X)$ respectively $\pi_W^G(Y)$. Then we denote by $f \cdot g$ the composite

$$\begin{aligned} S^{(V+W)+(n+m)\rho} &\xrightarrow{\mathrm{Id}\wedge\tau_{W,n\rho}\wedge\mathrm{Id}} S^{V+n\rho} \wedge S^{W+m\rho} \xrightarrow{f\wedge g} X(n\rho) \wedge Y(m\rho) \\ &\xrightarrow{i_{n\rho,m\rho}} (X \wedge Y)(n\rho + m\rho) \;=\; (X \wedge Y)((n+m)\rho)\ . \end{aligned}$$

The justification that the assignment $[f] \cdot [g] = [f \cdot g]$ is a well-defined, biadditive, unital and associative is the same as for the action map (4.29) above.

The Frobenius property of the $RO(G)$-graded multiplication has the form:

$$\mathrm{tr}(x) \cdot y \;=\; \mathrm{tr}_H^G(x \cdot \mathrm{res}_H^G(y))$$

where V and W are G-representations, $x \in \pi_{i^*V}^H(X)$ and $y \in \pi_W^G\mathbb{S}$. The transfer maps on the left hand side is the one associated to the representation V, the one on the right hand side is the one for $V \oplus W$.

The external product (4.32) also has a certain commutativity property:

Proposition 4.33. *For all orthogonal G-spectra X and Y and all G-representations V and W the square*

$$\begin{tikzcd}
\pi^G_V(X)\times\pi^G_W(Y) \arrow[r, "\cdot"] \arrow[dd, "\text{twist}"'] & \pi^G_{V+W}(X\wedge Y) \arrow[d, "\pi^G_{V+W}(\tau_{X,Y})"] \\
& \pi^G_{V+W}(Y\wedge X) \arrow[d, "\tau^*_{W,V}"] \\
\pi^G_W(Y)\times\pi^G_V(X) \arrow[r, "\cdot"'] & \pi^G_{W+V}(Y\wedge X)
\end{tikzcd}$$

commutes. Here $\tau_{X,Y} : X\wedge Y \longrightarrow Y\wedge X$ is the symmetry isomorphism of the smash product and $\tau_{V,W} : V\oplus W \longrightarrow W\oplus V$ is the isometry $\tau_{V,W}(v,w) = (w,v)$. In particular, the external product in $\mathbb{Z}$-graded equivariant homotopy groups satisfies

$$y\cdot x \ = \ (-1)^{kl}\cdot(\tau_{X,Y})_*(x\cdot y) \tag{4.34}$$

for $x\in\pi^G_k(X)$ and $y\in\pi^G_l(Y)$.

Proof. For representing maps $f : S^{V+n\rho}\longrightarrow X(n\rho)$ and $g : S^{W+m\rho}\longrightarrow Y(m\rho)$ the diagram

$$\begin{tikzcd}
S^{(V+W)+(n+m)\rho} \arrow[r, "\mathrm{Id}\wedge\tau_{W,n\rho}\wedge\mathrm{Id}"] \arrow[d, "\tau_{V,W}\wedge\mathrm{Id}"'] \arrow[rrr, bend left, "f\cdot g"] & S^{(V+n\rho)+(W+m\rho)} \arrow[r, "f\wedge g"] \arrow[dd, "\tau_{V+n\rho,W+m\rho}"'] & X(n\rho)\wedge Y(m\rho) \arrow[r, "i"] \arrow[dd, "\text{twist}"] & (X\wedge Y)((n+m)\rho) \arrow[d, "(X\wedge Y)(\tau_{n\rho,m\rho})"] \\
S^{(W+V)+(n+m)\rho} \arrow[d, "\mathrm{Id}\wedge\tau_{n\rho,m\rho}"'] & & & (X\wedge Y)((m+n)\rho) \arrow[d, "\tau_{X,Y}((m+n)\rho)"] \\
S^{(W+V)+(m+n)\rho} \arrow[r, "\mathrm{Id}\wedge\tau_{V,m\rho}\wedge\mathrm{Id}"'] \arrow[rrr, bend right, "g\cdot f"'] & S^{(W+m\rho)+(V+n\rho)} \arrow[r, "g\wedge f"'] & Y(m\rho)\wedge X(n\rho) \arrow[r, "i"'] & (Y\wedge X)((m+n)\rho)
\end{tikzcd}$$

commutes. Passage to homotopy classes gives

$$\begin{aligned}
[g\cdot f] \ &= \ \Big[\tau_{X,Y}((m+n)\rho)\circ(\tau_{n\rho,m\rho})_*\Big((f\cdot g)\circ(\tau^{-1}_{V,W}\wedge\mathrm{Id})\Big)\Big] \\
&= \ (\tau_{X,Y})_*\Big[(\tau_{n\rho,m\rho})_*\Big((f\cdot g)\circ(\tau^{-1}_{V,W}\wedge\mathrm{Id})\Big)\Big] \\
&= \ (\tau_{X,Y})_*\left[(f\cdot g)\circ(\tau_{W,V}\wedge\mathrm{Id})\right] \ = \ (\tau_{X,Y})_*(\tau^*_{W,V}[f\cdot g]) \ ,
\end{aligned}$$

as claimed.; the third equation uses Proposition 3.3. If $V=\mathbb{R}^k$ and $W=\mathbb{R}^l$ are trivial G-representations, then precomposition by $\tau_{W,V} : \mathbb{R}^{l+k}\longrightarrow\mathbb{R}^{k+l}$ induces multiplication by $(-1)^{kl}$ on π^G_{k+l}. So the commutativity relation becomes $y\cdot x = (-1)^{kl}\cdot(\tau_{X,Y})_*(x\cdot y)$. □

When we change the group along a homomorphism $\alpha : K\longrightarrow G$, the '$RO(G)$-grading' changes accordingly. Indeed, by applying the restriction α^* to representing G-maps we obtain a well-defined restriction homomorphism

$$\alpha^* \ : \ \pi^G_V(X) \ \longrightarrow \ \pi^K_{\alpha^*V}(\alpha^*X)$$

generalizing the restriction map (3.5).

The conjugation map gets an extra twist in the $RO(G)$-graded context coming from the fact that also the indexing representation changes. For a subgroup H of G, a G-spectrum X and $g \in G$, there are really two different kinds of conjugation maps

$$g_\star \ : \ \pi_V^H(X) \xrightarrow{c_g^*} \pi^{^gH}_{c_g^* V}(X) \qquad \text{and} \qquad g_* \ : \ \pi_V^H(X) \xrightarrow{c_g^*} \pi_V^{^gH}(X) \ .$$

The first map $g_\star$ is defined for any H-representation V as the composite

$$\pi_V^H(X) \xrightarrow{c_g^*} \pi^{^gH}_{c_g^* V}(c_g^* X) \xrightarrow{(l_g^X)_*} \pi^{^gH}_{c_g^* V}(X) \ .$$

The second map g_* is only defined if V is the restriction to H of a G-representation; g_* is then the composite

$$\pi_V^H(X) \xrightarrow{g_\star} \pi^{^gH}_{c_g^* V}(X) \xrightarrow{(l_g^V)_*} \pi_V^{^gH}(X) \ ,$$

where $l_g^V : c_g^* V \longrightarrow V$ is left multiplication by $g \in G$, which is an isomorphism of ^gH-representations. If $V = \mathbb{R}^k$ with trivial G-action, then $c_g^* V$ has trivial action, l_g^V is the identity and so $g_\star$ and g_* coincide and both specialize to the conjugation map in the integer graded context (3.7).

There are also $RO(G)$-graded external and internal transfer maps for a subgroup $H \subset G$. These transfer maps take the form

$$\mathrm{Tr}_H^G \ : \ \pi^H_{i^* V}(Y) \longrightarrow \pi_V^G(G \ltimes_H Y) \qquad \text{respectively} \qquad \mathrm{tr}_H^G \ : \ \pi^H_{i^* V}(X) \longrightarrow \pi_V^G(X) \tag{4.35}$$

for an H-spectrum Y respectively a G-spectrum X. Here V is a G-representation and $i^* V$ is the underlying H-representation. We emphasize that there are in general many G-representations with the same underlying H-representation, so there can be many different $RO(G)$-graded transfer maps with the same source but different targets.

These $RO(G)$-graded transfers (4.35) can either be defined by adding the representation V to the construction in the special case (4.14) above. Alternatively, we can define this more general transfer from the previous transfer (4.14) as the composite

$$\pi^H_{i^* V}(Y) \ = \ \pi_0^H(\Omega^{i^* V} Y) \xrightarrow{\mathrm{Tr}_H^G} \pi_0^G(G \ltimes_H (\Omega^{i^* V} Y)) \longrightarrow \pi_0^G(\Omega^V(G \ltimes_H Y)) \ = \ \pi_V^G(G \ltimes_H Y) \ .$$

Here the second map is induced by the morphism of G-spectra $G \ltimes_H (\Omega^{i^* V} Y) \longrightarrow \Omega^V(G \ltimes_H Y)$ that is adjoint to the H-morphism

$$\Omega^{i^* V} Y \xrightarrow{\Omega^{i^* V}(\eta)} \Omega^{i^* V}(i^*(G \ltimes_H Y)) \ = \ i^*(\Omega^V(G \ltimes_H Y)) \ .$$

In the $RO(G)$-graded setting, there are also external and internal double coset formulas; they look almost the same as in the integer graded context, but a little more care has to be taken with respect to the indexing representations. The proof is then almost the same as in Proposition 4.19.

Proposition 4.36 (External $RO(G)$-graded double coset formula)**.** *For all subgroups K and H of G, every orthogonal H-spectrum Y and every G-representation V we have*

$$\mathrm{res}_K^G \circ \mathrm{Tr}_H^G \ = \ \sum_{[g] \in K \backslash G / H} (\kappa_g)_* \circ \mathrm{Tr}^K_{K \cap {^gH}} \circ \mathrm{res}^{^gH}_{K \cap {^gH}} \circ (l_g^V)_* \circ c_g^*$$

as maps $\pi_V^H(Y) \longrightarrow \pi_V^K(G \ltimes_H Y)$.

The internal double coset formula follows from the external one by naturality arguments, in much the same way as in the integer graded situation in Proposition 4.21, but paying attention to change of indexing representations. The final formula has the exact same form.

Proposition 4.37 (Internal $RO(G)$-graded double coset formula)**.** *For all subgroups K and H of G, every orthogonal G-spectrum X and every G-representation V we have*

$$\mathrm{res}^G_K \circ \mathrm{tr}^G_H \ = \ \sum_{[g]\in K\backslash G/H} \mathrm{tr}^K_{K\cap {}^gH} \circ \mathrm{res}^{{}^gH}_{K\cap {}^gH} \circ g_*$$

as maps $\pi^H_V(X) \longrightarrow \pi^K_V(X)$.

Now we discuss the homotopy groups of some of the sample G-spectra with special attention to the Mackey functor structure. We will discuss the 0-th equivariant stable stems $\pi^G_0(\mathbb{S})$ in some detail in Section 6, after proving the tom Dieck splitting. The upshot is Theorem 6.16, due to Segal, that identifies $\pi^G_0(\mathbb{S})$ with the Burnside ring $A(G)$.

Example 4.38 (Cyclic group of order 2)**.** We review some of the features discussed so far in the first non-trivial case, i.e., for the cyclic group C_2 of order 2. The orbit category of $C_2 = \{1,\tau\}$ is displayed below on the left (where only non-identity morphisms are drawn). To the right of it are the values and structure maps of a Mackey functor M for the group C_2

$$\mathcal{O}(C_2) \ : \quad \begin{tikzcd} C_2/C_2 \\ C_2/e \arrow[u] \arrow["\tau"', loop below] \end{tikzcd} \qquad\qquad \begin{tikzcd} M(C_2) \arrow[d, "\mathrm{res}"', shift right] \\ M(e) \arrow[u, dashed, "\mathrm{tr}"', shift right] \arrow["c_\tau", loop below] \end{tikzcd}$$

The transfer map is drawn with a dashed arrow since it does not correspond to any morphism in the orbit category and is a genuinely stable phenomenon. In this case there is only one interesting instance of the double coset formula, namely for $H = C_2$ and $K = K' = e$, and that specializes to the relation

$$\mathrm{res}\circ\mathrm{tr} \ = \ 1 + c_\tau \ .$$

The group C_2 has two irreducible representations, both 1-dimensional, namely the trivial representation 1 and the sign representation σ. The regular representation is isomorphic to $\mathbb{C}$ with action by complex conjugation, and it decomposes as $\rho_{C_2} \cong 1+\sigma$. So the representation ring $RO(C_2)$ is free abelian of rank 2, and the $RO(C_2)$-grading can be turned into a bigrading. We use the 'motivic' grading convention and write

$$\pi^{C_2}_{p,q}(X) \ = \ \pi^{C_2}_{p-q}(\Omega^{q\sigma}X) \ = \ \mathrm{colim}_n [S^{(p-q)+q\sigma+n\rho},\ X(n\rho)]^{C_2} \ .$$

The convention reflects the fact that the underlying non-equivariant sphere of $S^{(p-q)+q\sigma}$ has dimension p. By Proposition 4.33 the bigraded product in $\pi^{C_2}_{*,*}$ has a certain commutativity property, namely

$$y\cdot x \ = \ (-1)^{(p-q)(k-l)}\varepsilon^{ql}\cdot x\cdot y \tag{4.39}$$

for $x\in\pi^{C_2}_{p,q}$ and $y\in\pi^{C_2}_{k,l}$, where $\varepsilon = \langle\tau_{\sigma,\sigma}\rangle \in \pi^{C_2}_{0,0}$ is the class of the twist automorphism of $\tau_{\sigma,\sigma}$ of $S^{\sigma+\sigma}$.

The inclusion of equivariant maps into all maps gives a restriction homomorphism

$$i^* \ : \ \pi^{C_2}_{p,q} \ \longrightarrow \ \pi^{\mathrm{s}}_p$$

to the equivariant to the non-equivariant stable stems. We can also define a 'geometric fixed point map'

$$\Phi \ : \ \pi^{C_2}_{p,q} \ \longrightarrow \ \pi^{\mathrm{s}}_{p-q}$$

by sending the class of a C_2-map $f : S^{(p-q)+q\sigma+n\rho} \longrightarrow S^{n\rho}$ to the class of the fixed point map

$$f^{C_2} \ : \ S^{p-q+n} \ \cong \ (S^{(p-q)+q\sigma+n\rho})^{C_2} \ \longrightarrow \ (S^{n\rho})^{C_2} \ \cong S^n \ ,$$

using that the sign representation has no nonzero fixed points and identifying $\rho^{C_2} \cong \mathbb{R}$. The map Φ is a special case of a more general geometric fixed point map $\Phi^G : \pi^G_{V+k}(X) \longrightarrow \pi_{\dim(V)+k}(\Phi^G X)$ that we discuss in (7.4) below.

As we shall show in Theorem 6.16, the group $\pi^{C_2}_{0,0}$ is isomorphic to the Burnside ring $A(C_2)$ and it is free abelian of rank 2 with basis given by the class 1 and $t = \mathrm{tr}(1)$, the image of the generator 1 under the transfer map

$$\mathrm{tr} \ : \ \pi^{\mathrm{s}}_0 = \pi^e_0(\mathbb{S}) \ \longrightarrow \ \pi^{C_2}_0(\mathbb{S}) \ = \ \pi^{C_2}_{0,0} \ .$$

For $p = q = 0$ the combined map

$$(i^*, \Phi) \ : \ \pi^{C_2}_{0,0} \ \longrightarrow \ \pi^{\mathrm{s}}_0 \times \pi^{\mathrm{s}}_0$$

is thus a monomorphism and can be used to deduce relations in the ring $\pi^{C_2}_{0,0}$.

The class 1 is represented by the identity of S^0 and we have $i^*(1) = 1$ and $\Phi(1) = 1$. We can find an unstable representative of the element t by going back to the definition of the transfer. We can embed C_2 equivariantly into the sign representation σ by sending $1 \in C_2$ to $1 \in \mathbb{R}$ and sending $\gamma \in C_2$ to $-1 \in \mathbb{R}$. The open balls of radius 1 around 1 and -1 are disjoint, so the transfer is represented unstably by the composite

$$t \ : \ S^\sigma \ \xrightarrow{\text{collapse}} \ \frac{C_2 \times [-1,+1]}{C_2 \times \{+1,-1\}} \ \xrightarrow{\cong} \ (C_2)_+ \wedge \frac{[-1,+1]}{\{+1,-1\}} \ \xrightarrow{\cong} \ (C_2)_+ \wedge S^\sigma \ \xrightarrow{\text{act}} \ S^\sigma \ .$$

The underlying non-equivariant endomorphism of $S^\sigma = S^1$ has degree 2, so we have $i^*(t) = 2$. The map t takes the two fixed points 0 and ∞ of S^σ to the basepoint, so $\Phi(t) = 0$.

The class ε represented by the twist automorphism of $\tau : S^\sigma \wedge S^\sigma$ satisfies $i^*(\varepsilon) = -1$ (since S^σ is non-equivariantly a 1-sphere) and $\Phi(\varepsilon) = 1$ (since the fixed point map τ^{C_2} is the twist map of the 0-sphere S^0, which is the identity). So we must have

$$\varepsilon \ = \ 1 - t \quad \text{in} \quad \pi^{C_2}_{0,0} \ .$$

Evidently we have $\varepsilon^2 = 1$, so we also obtain the multiplicative relation $t^2 = 2t$ in $\pi^{C_2}_{0,0}$.

Now that we understand the ring $\pi^{C_2}_{0,0}$ we turn to some non-zero bigrading. The Hopf map

$$\eta \ : \ S(\mathbb{C}^2) \ \longrightarrow \ \mathbb{C}\mathrm{P}^1 \ , \quad (x,y) \ \longmapsto \ [x:y]$$

is C_2-equivariant with respect to complex conjugation on the coordinates of the unit sphere $S(\mathbb{C}^2)$ and the projective line $\mathbb{C}\mathrm{P}^1$. The unit sphere $S(\mathbb{C}^2)$ is equivariantly homeomorphic to the representation sphere $S^{\sigma+\rho}$, where σ is the sign representation on $\mathbb{R}$. Moreover, $\mathbb{C}\mathrm{P}^1$ is equivariantly homeomorphic to S^ρ, so we can interpret the projection map as a C_2-map $S^{\sigma+\rho} \longrightarrow S^\rho$ that represents an element

$$\eta_{C_2} \ \in \ \pi^{C_2}_\sigma(\mathbb{S}) = \pi^{C_2}_{1,1} \ .$$

The commutativity relation (4.39) specializes to $\eta^2_{C_2} = \varepsilon \cdot \eta^2_{C_2}$.

Under the restriction map $i^* : \pi^{C_2}_{1,1} \longrightarrow \pi^{\mathrm{s}}_1$ to the non-equivariant stable 1-stem, the class η_{C_2} maps to the Hopf map η. However, in contrast to its non-equivariant image, the C_2-class η_{C_2} is neither torsion nor nilpotent. Indeed, the image of η_{C_2} under the geometric fixed point map (7.4)

$$\Phi \ : \ \pi^{C_2}_{1,1} \ \longrightarrow \ \pi^s_0$$

is represented by the fixed points of the map $S^{\mathbb{C}} \wedge \eta$ which turns out to be

$$S(\mathbb{R}^2) \ \xrightarrow{\eta^{C_2}} \ \mathbb{R}\mathrm{P}^1 \ .$$

This is the real Hopf map, so we have $\Phi(\eta_{C_2}) = 2$ in $\pi^{\mathrm{s}}_0 \cong \mathbb{Z}$. Since Φ is a ring homomorphism this shows that all powers $\eta^m_{C_2}$ are elements of infinite order in $\pi^{C_2}_{m,m}$.

We consider the commutative square

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] & S(\mathbb{C}^2) \arrow[r, "\eta"] \arrow[d] & \mathbb{C}\mathrm{P}^1 \arrow[d] & {[x:y]} \arrow[d, mapsto] \\ (y,x) & S(\mathbb{C}^2) \arrow[r, "\eta"'] & \mathbb{C}\mathrm{P}^1 & {[y:x]} \end{tikzcd}$$

of C_2-spaces and equivariant maps. The left vertical map has degree 1 as a non-equivariant map and degree -1 on fixed points $S(\mathbb{C}^2)^{C_2} = S(\mathbb{R}^2)$; so the left vertical map represents $-\varepsilon$ in $\pi_{0,0}^{C_2}$. The right vertical map has degree -1 as a non-equivariant map, and it has degree -1 on fixed points, so it represents the class -1 in $\pi_{0,0}^{C_2}$. Hence the commutative square implies the relation $-\varepsilon \cdot \eta_{C_2} = -\eta_{C_2}$; equivalently, we have

$$t \cdot \eta_{C_2} \ = \ (1-\varepsilon) \cdot \eta_{C_2} \ = \ 0$$

in $\pi_{1,1}^{C_2}$. Under the restriction map i^* this relation becomes the familiar relation $2 \cdot \eta = 0$ in the non-equivariant 1-stem.

Example 4.40 (Eilenberg-Mac Lane spectra)**.** In Example 2.13 we introduced the Eilenberg-Mac Lane spectrum HM of a $\mathbb{Z}G$-module M. Now we discuss the homotopy groups of HM as a Mackey functor. From M we can obtain a Mackey functor $\underline{M}$ with values

$$\underline{M}(H) \ = \ M^H \ ;$$

the contravariant functoriality is by inclusion of fixed points and conjugation. The covariant functoriality is given by algebraic transfer, i.e., for $K \subset H$ the map $\mathrm{tr}_K^H : M^K \longrightarrow M^H$ is given by

$$\mathrm{tr}_K^H(m) \ = \ \sum_{hK \in H/K} hm \ .$$

As we discussed above, the G-space $HM_n = M[S^n]$ is an equivariant Eilenberg-Mac Lane space for the underlying contravariant functor of $\underline{M}$. Moreover, the equivariant Eilenberg-Mac Lane spectrum HM of a $\mathbb{Z}G$-module is even an Ω-G-spectrum. Indeed, when we assign to a finite G-set S the (discrete) G-space $M[S]$ with diagonal G-action, then we obtain a very special G-Γ-space. So Segal's equivariant Γ-space machine applies and shows that HM is a G-Ω-spectrum for the Mackey functor $\underline{M}$ (see Proposition 4.3 of [22], or [24, Thm. B] for a published version). Dos Santos reproves this result in [19] with different methods. Either of these approaches shows that for every G-representation V the G-space $HM(V) = M[S^V]$ is an equivariant Eilenberg-Mac Lane space of type $(\underline{M}, V)$, i.e., the G-space $\mathrm{map}(S^V, M[S^V])$ has homotopically discrete fixed points for all subgroups of G and the natural map

$$M^H \ \longrightarrow \ [S^V, M[S^V]]^H = \pi_0 \, \mathrm{map}^H(S^V, M[S^V])$$

sending $m \in M^H$ to the homotopy class of $m \cdot - : S^V \longrightarrow M[S^V]$ is an isomorphism. More generally, for every G-representation V and every based G-CW-complex L the map

$$M[L] \ \longrightarrow \ \mathrm{map}(S^V, M[L \wedge S^V])$$

adjoint to the assembly map $M[L] \wedge S^V \longrightarrow M[L \wedge S^V]$ is a G-weak equivalence.

As for every G-Ω-spectrum, the map

$$\pi_k(M^K) = \pi_k((HM_0)^K) \ \longrightarrow \ \pi_k^K(HM)$$

is an isomorphism for all $k \geq 0$. Thus the homotopy Mackey functor $\underline{\pi}_k(HM)$ is trivial for $k > 0$. For negative k, the group $\pi_k^K(HM)$ is isomorphic to $\pi_0((M[S^{-k}])^K)$, and hence trivial. Moreover, we have $\pi_0^K(HM) \cong \pi_0((HM_0)^K) \cong M^K$. This is natural in the subgroup K, and for $K \subset H$ the restriction maps $\pi_0^H(HM) \longrightarrow \pi_0^K(HM)$ correspond to the inclusion $M^H \longrightarrow M^K$. Moreover, we have 'transfer=transfer', i.e., the topologically defined transfer $\mathrm{tr}_K^H : \pi_0^K(HM) \longrightarrow \pi_0^H(HM)$ corresponds to the algebraic transfer $M^K \longrightarrow M^H$. So in summary, we have obtained isomorphism of Mackey functors

$$\underline{\pi}_0(HM) \;\cong\; \underline{M}\ .$$

The Mackey functors $\underline{M}$ arising from $\mathbb{Z}G$-modules M are special, for example because all restriction maps are injective. A general Mackey functor N also has an Eilenberg-Mac Lane G-spectrum HN that satisfies

$$\underline{\pi}_*(HN) \;\cong\; \begin{cases} N & \text{for } n = 0 \text{, and} \\ 0 & \text{else.} \end{cases}$$

Moreover, any two orthogonal G-spectra with this property are related by a chain of $\underline{\pi}_*$-isomorphisms. In other words, the Eilenberg-Mac Lane spectrum of a Mackey functor is unique up to preferred isomorphism in the equivariant stable homotopy category $\mathrm{Ho}(\mathcal{S}p_G)$. However, in contrast to the special Mackey functor arising from $\mathbb{Z}G$-modules, I am not aware of an explicit construction of HN for a general Mackey functor N. The first construction (in the context of Lewis-May-Steinberger spectra) is due to Lewis, May and McClure [13] and proceeds by defining an 'ordinary' homology theory, defined on equivariant spectra, with coefficients in the Mackey functor and then using a general representability theorem. A different construction was later given by dos Santos and Nie [20, Thm. 4.22].

Similarly as in the non-equivariant context, Eilenberg-Mac Lane spectra represent 'ordinary' (as opposed to 'generalized') homology and cohomology. More specifically this means that for every Mackey functor N and every G-CW-complex A the homotopy group $\pi_k(HN^A)$ of the mapping spectrum is naturally isomorphic to the Bredon cohomology group $H_G^{-k}(A, N)$ of A with coefficients in the underlying contravariant $\mathcal{O}(G)$-functor of N, and the group $\pi_k(A \wedge HN)$ is naturally isomorphic to the Bredon homology group $H_k^G(A, N)$ of A with coefficients in the underlying covariant $\mathcal{O}(G)$-functor of N.

Example 4.41. Here is a specific example of a $K(\underline{M}, V)$. We let the group C_2 act on $\mathbb{C}\mathrm{P}^\infty$ by complex conjugation. We claim that $\mathbb{C}\mathrm{P}^\infty$ is an Eilenberg-Mac Lane space of type $(\rho, \underline{\mathbb{Z}})$, and hence C_2-homotopy equivalent to the space $H\mathbb{Z}(S^\rho) = \mathbb{Z}[S^\rho]$ (here $\rho = \rho_{C_2}$ is the regular representation). Since $\mathbb{C}\mathrm{P}^\infty$ is a non-equivariant $K(\mathbb{Z}, 2)$, the underlying space of $\mathrm{map}(S^\rho, \mathbb{C}\mathrm{P}^\infty)$ is homotopically discrete with components given by $\mathbb{Z}$. To get at the homotopy type of the C_2-fixed points we map out of the C_2-cofibration

$$S^1 = (S^\rho)^{C_2} \;\longrightarrow\; S^\rho$$

and investigate the resulting Serre fibration

$$\mathrm{map}^{C_2}(S^\rho/S^1, \mathbb{C}\mathrm{P}^\infty) \;\longrightarrow\; \mathrm{map}^{C_2}(S^\rho, \mathbb{C}\mathrm{P}^\infty) \;\longrightarrow\; \mathrm{map}^{C_2}(S^1, \mathbb{C}\mathrm{P}^\infty) \cong \mathrm{map}(S^1, (\mathbb{C}\mathrm{P}^\infty)^{C_2})\ .$$

The space S^ρ/S^1 is C_2-homeomorphic to $(C_2)_+ \wedge S^2$, hence the fiber $\mathrm{map}^{C_2}(S^\rho/S^1, \mathbb{C}\mathrm{P}^\infty)$ is homeomorphic to

$$\mathrm{map}^{C_2}((C_2)_+ \wedge S^2, \mathbb{C}\mathrm{P}^\infty) \;\cong\; \Omega^2\mathbb{C}\mathrm{P}^\infty\ ,$$

hence homotopically discrete with π_0 isomorphic to $\mathbb{Z}$. Since $(\mathbb{C}\mathrm{P}^\infty)^{C_2} \cong \mathbb{R}\mathrm{P}^\infty$, the base is homeomorphic to $\Omega\mathbb{R}\mathrm{P}^\infty$, hence homotopically discrete with π_0 isomorphic to $\mathbb{F}_2$. Finally, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \cong [S^\rho/S^1, \mathbb{C}\mathrm{P}^\infty]^{C_2} \;\longrightarrow\; [S^\rho, \mathbb{C}\mathrm{P}^\infty]^{C_2} \;\longrightarrow\; [S^1, \mathbb{R}\mathrm{P}^\infty] \cong \mathbb{F}_2 \;\longrightarrow\; 0$$

does not split since the C_2-map

$$S^\rho \cong \mathbb{C}\mathrm{P}^1 \xrightarrow{\text{incl}} \mathbb{C}\mathrm{P}^\infty$$

is an element of infinite order in the middle group whose restriction to fixed points is the inclusion $S^1 \cong \mathbb{R}\mathrm{P}^1 \longrightarrow \mathbb{R}\mathrm{P}^\infty$, hence the generator of $[S^1, \mathbb{R}\mathrm{P}^\infty]$ and such that twice this class is the image of the generator from the left group.

Example 4.42. We close this section with an example of a morphism of G-spectra that is a π_*-isomorphism of underlying non-equivariant spectra, but not an equivariant $\underline{\pi}_*$-isomorphism We consider the rationalized sphere spectrum $\mathbb{SQ}$, defined as the homotopy colimit in $\mathrm{Ho}(\mathcal{S}p_G)$ of the sequence

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \xrightarrow{\cdot 3} \mathbb{S} \xrightarrow{\cdot 4} \cdots .$$

The unit map $\mathbb{S} \longrightarrow H\mathbb{Q}$ to the Eilenberg-Mac Lane spectrum of the trivial $\mathbb{Z}G$-module $\mathbb{Q}$ extends to a morphism $\mathbb{SQ} \longrightarrow H\mathbb{Q}$ that is a π_*-isomorphism of underlying non-equivariant spectra. However, we have

$$\pi_0^G(\mathbb{SQ}) \ = \ \mathbb{Q} \otimes \pi_0^G(\mathbb{S}) \ \cong \ \mathbb{Q} \otimes A(G) \ ,$$

the rationalized Burnside ring. For every non-trivial group G, $\pi_0^G(\mathbb{SQ})$ is thus non isomorphic to $\pi_0^G(H\mathbb{Q}) = \mathbb{Q}$.

5. Constructions with equivariant spectra

We discuss various constructions which produce new equivariant orthogonal spectra from old ones.

Example 5.1 (Limits and colimits)**.** The category of orthogonal G-spectra has all limits and colimits, and they are defined levelwise. Let us be a bit more precise and consider a functor $F : J \longrightarrow \mathcal{S}p_G$ from a small category J to the category of orthogonal G-spectra. Then we define an orthogonal G-spectrum $\mathrm{colim}_J\, F$ in level n by

$$(\mathrm{colim}_J\, F)_n \ = \ \mathrm{colim}_{j\in J}\, F(j)_n \ ,$$

the colimit being taken in the category of pointed $G \times O(n)$-spaces. The structure map is the composite

$$(\mathrm{colim}_{j\in J}\, F(j)_n) \wedge S^1 \cong \mathrm{colim}_{j\in J}(F(j)_n \wedge S^1) \xrightarrow{\mathrm{colim}_J\, \sigma_n} \mathrm{colim}_{j\in J}\, F(j)_{n+1} \ ;$$

here we exploit that smashing with S^1 is a left adjoint, and thus the natural map $\mathrm{colim}_{j\in J}(F(j)_n \wedge S^1) \longrightarrow (\mathrm{colim}_{j\in J}\, F(j)_n) \wedge S^1$ is an isomorphism, whose inverse is the first map above.

The argument for inverse limits is similar, but we have to use that structure maps can also be defined in the adjoint form. We can take

$$(\lim_J F)_n \ = \ \lim_{j\in J} F(j)_n \ ,$$

and the structure map is adjoint to the composite

$$\lim_{j\in J} F(j)_n \xrightarrow{\lim_J \hat{\sigma}_n} \lim_{j\in J} \Omega(F(j)_{n+1}) \cong \Omega\left(\lim_{j\in J} F(j)_{n+1}\right) \ .$$

Limits and colimits commute with evaluation at a G-representation V, i.e., the G-space $(\mathrm{colim}_J\, F)(V)$ (respectively $(\lim_J F)(V)$) is a colimit (respectively limit) of the composite of F with the functor of evaluation at V.

The inverse limit, calculated levelwise, of a diagram of orthogonal G-ring spectra and homomorphisms is again an orthogonal G-ring spectrum. In other words, equivariant ring spectra have limits and the forgetful functor to G-spectra preserves them. Equivariant ring spectra also have *co*-limits, but they are not preserved by the forgetful functor.

Example 5.2 (Smash products with and functions from G-space)**.** If A is pointed G-space and X a G-spectrum, we can define two new G-spectra $A \wedge X$ and X^A by smashing with A or taking maps from A levelwise; the structure maps and actions of the orthogonal groups do not interact with A.

In more detail we set

$$(A \wedge X)(V) \ = \ A \wedge X(V) \qquad \text{respectively} \qquad (X^A)(V) \ = \ X(V)^A \ = \ \mathrm{map}(A, X(V)) \ .$$

The group $O(V)$ acts through its action on $X(V)$. The structure map is given by the composite

$$(A \wedge X)(V) \wedge S^W \ = \ A \wedge X(V) \wedge S^W \xrightarrow{A \wedge \sigma_{V,W}} A \wedge X(V \oplus W) \ = \ (A \wedge X)(V \oplus W)$$

respectively by the composite

$$X(V)^A \wedge S^W \longrightarrow (X(V) \wedge S^W)^A \xrightarrow{(\sigma_{V,W})^A} X(V \oplus W)^A$$

where the first is an assembly map that sends $\varphi \wedge w \in X(V)^A \wedge S^W$ to the map sending $a \in A$ to $\varphi(a) \wedge w$. The second is application of $\mathrm{map}(A, -)$ to the structure map of X. The group G acts on $(A \wedge X)(V) = A \wedge X(V)$ diagonally, through the actions on A and $X(V)$. In the other case the group G acts on $(X^A)(V) = \mathrm{map}(A, X(V))$ by conjugation, i.e., via $({}^g\varphi)(a) = g \cdot \varphi(g^{-1}a)$ for $g : A \longrightarrow X(V)$, $a \in A$ and $g \in G$. For example, the spectrum $A \wedge \mathbb{S}$ is equal to the suspension spectrum $\Sigma^\infty A$.

Just as the functors $A \wedge -$ and $\mathrm{map}(A, -)$ are adjoint on the level of based G-spaces, the two functors just introduced are an adjoint pair on the level of G-spectra. The adjunction

$$\hat{} \; : \; \mathcal{S}p_G(X, Y^A) \xrightarrow{\cong} \mathcal{S}p_G(A \wedge X, Y) \tag{5.3}$$

takes a morphism $f : X \longrightarrow Y^A$ to the morphism $\hat{f} : A \wedge X \longrightarrow Y$ whose V-th level $\hat{f}(V) : A \wedge X(V) \longrightarrow Y(V)$ is given by $\hat{f}(V)(a \wedge x) = f(V)(x)(a)$.

We note that if X is a G-Ω-spectrum and A a based G-CW-complex, then X^A is again a G-Ω-spectrum. Indeed, the mapping space functor $\mathrm{map}(A, -)$ takes the G-weak equivalence $\tilde{\sigma}_{V,W} : X(V) \longrightarrow \Omega^W X(V \oplus W)$ to a G-weak equivalence

$$X^A(V) = \mathrm{map}(A, X(V)) \xrightarrow{\mathrm{map}(A, \tilde{\sigma}_{V,W})} \mathrm{map}(A, \Omega^W X(V \oplus W)) \cong \Omega^W (X^A(V \oplus W)) .$$

Loop and suspension with a representation sphere are the special case $A = S^V$ of the previous construction. As we discussed above, the adjunction unit $X \longrightarrow \Omega^V(S^V \wedge X)$ and counit $S^V \wedge \Omega^V Y \longrightarrow Y$ are then $\underline{\pi}_*$-isomorphisms, see Proposition 3.12. As we discussed in (3.11), the group $\pi_k^G(\Omega^m X)$ is naturally isomorphic to $\pi_{m+k}^G(X)$, and the group $\pi_{m+k}^G(S^m \wedge X)$ is naturally isomorphic to $\pi_k^G(X)$; so looping and suspending (by trivial representation spheres) preserves $\underline{\pi}_*$-isomorphism. The next proposition generalizes this.

Proposition 5.4. *Let A be a based G-CW-complex. Then the functor $A \wedge -$ preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra. If A is finite, then the functor* $\mathrm{map}(A, -)$ *preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.*

Proof. We let $f : X \longrightarrow Y$ be a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. We start with the case of a finite G-CW-complex A and prove by induction on the number of equivariant cells that $A \wedge f$ and $\mathrm{map}(A, f)$ induce isomorphisms on π_k^G for all integers k.

If A consists only of the basepoint, then $A \wedge X$ and $\mathrm{map}(A, X)$ are trivial and the claims are trivially true. Now suppose we have shown the claim for a finite based G-CW-complex A and B is obtained from A by attaching an equivariant n-cell $G/H \times D^n$ along its boundary. Then the mapping cone $C(i)$ of the inclusion $i : A \longrightarrow B$ is based G-homotopy equivalent to $G/H_+ \wedge S^n$. So the spectrum $C(i) \wedge X$ is G-homotopy equivalent to $G/H_+ \wedge S^n \wedge X$, and hence to $S^n \wedge (G \ltimes_H X)$. The G-homotopy groups of $C(i) \wedge X$ are thus naturally isomorphic to

$$\pi_k^G(S^n \wedge (G \ltimes_H X)) \;\cong\; \pi_{k-n}^G(G \ltimes_H X) \;\cong\; \pi_{k-n}^H(X) ,$$

using the Wirthmüller isomorphism (Theorem 4.9). So smashing with $C(i)$ takes $\underline{\pi}_*$-isomorphisms to π_*^G-isomorphisms. The mapping cone of the morphism $i \wedge X : A \wedge X \longrightarrow B \wedge X$ is naturally isomorphic to $C(i) \wedge X$; since $A \wedge -$ and $C(i) \wedge -$ take $\underline{\pi}_*$-isomorphisms to π_*^G-isomorphisms, so does $B \wedge -$ by the first long exact sequence of Proposition 3.21 and the five lemma.

The induction step for $\mathrm{map}(A, X)$ is exactly dual. Since $C(i)$ is homotopy equivalent to $G/H_+ \wedge S^n$, the spectrum $\mathrm{map}(C(i), X)$ is homotopy equivalent to $\mathrm{map}(G/H_+, \Omega^n X)$, and hence its G-homotopy groups are naturally isomorphic to

$$\pi_k^G \, \mathrm{map}(G/H_+, \Omega^n X) \;\cong\; \pi_k^H(\Omega^n X) \;\cong\; \pi_{n+k}^H(X) .$$

So the functor $\mathrm{map}(C(i), -)$ takes $\underline{\pi}_*$-isomorphisms to π_*^G-isomorphisms. The homotopy fiber of the morphism $\mathrm{map}(i, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is naturally isomorphic to $\mathrm{map}(C(i), X)$; since $\mathrm{map}(A, -)$ and $\mathrm{map}(C(i), -)$ take $\underline{\pi}_*$-isomorphisms to π_*^G-isomorphisms, so does $\mathrm{map}(B, -)$ by the second long exact sequence of Proposition 3.21 and the five lemma.

Now we let H be an arbitrary subgroup of G. The underlying H-spectrum of $A \wedge X$ respectively $\mathrm{map}(A, X)$ is the smash product of the underlying H-CW-complex of A and the underlying H-spectrum of X, respectively the spectrum of maps from the underlying H-CW-complex of A to the underlying H-spectrum of X. Moreover, the restriction of a $\underline{\pi}_*$-isomorphism of G-spectra is a $\underline{\pi}_*$-isomorphism of H-spectra. So by applying the previous paragraph to the group H instead of G and to the underlying H-morphism of f shows that $\pi_k^H(A \wedge f)$ and $\pi_k^H(\mathrm{map}(A, f))$ are isomorphisms for all integers k.

If remains to prove that claim about $A \wedge -$ for infinite G-CW-complexes. Every G-CW-complex is the filtered colimit, along equivariant h-cofibrations, of its finite G-CW-subcomplexes. Since equivariant homotopy groups commute with such filtered colimits, we are reduced to the previous case of finite G-CW-complexes. □

Example 5.5 (Free G-spectra)**.** Given a G-representation V, we define an orthogonal G-spectrum F_V which is 'freely generated in level V'. Before we give the formal definition we try to motivate why certain Thom spaces come up at this point. The guiding principle is that the value $F_V(W)$ should be the based G-space of 'all natural maps' $X(V) \longrightarrow X(W)$ as X varies over all orthogonal G-spectra. If the dimension of W is smaller than the dimension V, this space consists only of the basepoint. Otherwise, every linear isometric embedding $\alpha : V \longrightarrow W$ gives rise to a linear isometry

$$\tilde{\alpha} \ : \ V \oplus (W - \alpha(V)) \ \cong \ W \ , \quad \tilde{\alpha}(v, w) \ = \ \alpha(v) + w \ . \tag{5.6}$$

Using this isometry we obtain a map

$$X(V) \wedge S^{W-\alpha(V)} \xrightarrow{\sigma_{V,W-\alpha(V)}} X(V \oplus (W - \alpha(V))) \xrightarrow{X(\tilde{\alpha})} X(W) \ .$$

Hence for every linear isometric embedding $\alpha : V \longrightarrow W$ we get maps $X(V) \longrightarrow X(W)$ parameterized by the sphere $S^{W-\alpha(V)}$ of the orthogonal complement of the image of α. But the resulting maps from $X(V)$ to $X(W)$ should also vary continuously with the embedding α, hence the topology on the space $\mathbf{L}(V, W)$ of linear isometric embeddings enters. The easiest way to make all of this precise is to observe that the orthogonal complements $W - \alpha(V)$ are the fibers of a vector bundle over $\mathbf{L}(V, W)$ with total space

$$\xi(V, W) \ = \ \{ \, (\alpha, w) \in \mathbf{L}(V, W) \times W \mid w \perp \alpha(V) \, \} \ .$$

The structure map $\xi(V, W) \longrightarrow \mathbf{L}(V, W)$ of this 'orthogonal complement' vector bundle is the projection to the first factor. We let $\mathbf{O}(V, W)$ be the Thom space of the bundle $\xi(V, W)$, which we define as the one-point compactification of the total space of $\xi(V, W)$; because the base space $\mathbf{L}(V, W)$ is compact, the one-point compactification is equivariantly homeomorphic to the quotient of the disc bundle of $\xi(V, W)$ by the sphere bundle.

Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V, W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V, W)$ is empty and $\mathbf{O}(V, W)$ consists of a single point. Otherwise we can choose a linear isometric embedding $\alpha : V \longrightarrow W$, and we let $V^\perp = W - \alpha(V)$ denote the orthogonal complement of its image. Then the maps

$$\begin{array}{rcll} O(W)/O(V^\perp) & \longrightarrow & \mathbf{L}(V, W) \ , \quad A \cdot O(V^\perp) \longmapsto A \cdot \alpha & \quad \text{and} \\ O(W)_+ \wedge_{O(V^\perp)} S^{V^\perp} & \longrightarrow & \mathbf{O}(V, W) \ , \quad [A, w] \longmapsto A \cdot (\alpha, w) & \end{array}$$

are homeomorphisms. Put yet another way: if $\dim V = n$ and $\dim W = n + m$, then $\mathbf{L}(V, W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V, W)$ is homeomorphic to $O(n+m)_+ \wedge_{O(m)} S^m$.

Now suppose that X is an orthogonal spectrum and let $n = \dim V$ and $n + m = \dim W$. We can define a continuous based *action map*

$$\begin{aligned} \circ \ : \ X(V) \wedge \mathbf{O}(V,W) \ &\longrightarrow \ X(W) \\ x \ \wedge \ (\alpha, w) \ &\longmapsto \ X(\tilde{\alpha})(\sigma_{V,W-\alpha(V)}(x \wedge w)) \end{aligned} \tag{5.7}$$

where $\tilde{\alpha} : V \oplus (W - \alpha(V)) \longrightarrow W$ was defined in (5.6).

We obtain a map

$$\kappa \ : \ S^W \ \longrightarrow \ \mathbf{O}(V, V \oplus W) \ , \quad w \ \longmapsto \ (i_V, (0, w)) \ ,$$

as the inclusion of the fiber over $i_V : V \longrightarrow V \oplus W$, the inclusion of the first summand. The generalized structure map $\sigma_{V,W}$ originally defined in (2.4) then coincides with the composite

$$X(V) \wedge S^W \ \xrightarrow{X(V)\wedge\kappa} \ X(V) \wedge \mathbf{O}(V, V \oplus W) \ \xrightarrow{\circ} \ X(W) \ .$$

The action maps are associative: If we are given a third inner product space U, there is a bundle map

$$\xi(U,V) \times \xi(V,W) \ \longrightarrow \ \xi(U,W) \ , \quad ((\beta, v), (\alpha, w)) \ \longmapsto \ (\alpha\beta, \, \alpha(v) + w)$$

which covers the composition map $\mathbf{L}(U,V) \times \mathbf{L}(V,W) \longrightarrow \mathbf{L}(U,W)$. Passage to Thom spaces gives a based map

$$\circ \ : \ \mathbf{O}(U,V) \times \mathbf{O}(V,W) \ \longrightarrow \ \mathbf{O}(U,W)$$

which is clearly associative. The action is also associative in the sense that the square

$$\begin{tikzcd} X(U) \wedge \mathbf{O}(U,V) \wedge \mathbf{O}(V,W) \arrow[r, "\circ\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\circ"'] & X(V) \wedge \mathbf{O}(V,W) \arrow[d, "\circ"] \\ X(U) \wedge \mathbf{O}(U,W) \arrow[r, "\circ"'] & X(W) \end{tikzcd}$$

commutes for every triple of inner product spaces.

Now we add group actions to the picture everywhere. Suppose that G is a finite group and X an orthogonal G-spectrum. Given two G-representations V and W, we let G act on the space $\mathbf{L}(V,W)$ of (not necessarily equivariant) linear isometric embeddings by conjugation, i.e., for $g \in G$, $\alpha : V \longrightarrow W$ and $v \in V$ we set

$$({}^g\alpha)(v) \ = \ g \cdot \alpha(g^{-1}v) \ .$$

This action prolongs to an action by bundle isomorphisms on $\xi(V,W)$ via

$$g \cdot (\alpha, w) \ = \ ({}^g\alpha, gw) \ ,$$

and hence passes to a G-action on Thom spaces $\mathbf{O}(V,W)$. The action map (5.7) is then G-equivariant.

One can summarize this discussion as follows. We have defined a based topological G-category $\mathbf{O}$ with objects all G-representations, with based morphism G-space $\mathbf{O}(V,W)$, composition map $\circ$ and units $1_V = (\mathrm{Id}_V, 0)$ in $\mathbf{O}(V,V)$. Moreover, for every orthogonal G-spectrum X, the action maps (5.7) make the collection of G-spaces $\{X(V)\}_V$ into a based continuous G-functor $X : \mathbf{O} \longrightarrow \mathcal{T}_G$ to the category of based G-spaces. The assignment $X \mapsto \{X(V)\}_V$ is in fact an equivalence of categories from the category of orthogonal G-spectra to the category of (based, continuous) G-functors from $\mathbf{O}$ to $\mathcal{T}_G$. We are not going to show this.

The free G-spectrum F_V is given in level W by

$$(F_V)(W) \ = \ \mathbf{O}(V,W) \ ,$$

with the above G-action and with $O(W)$-action through W. Since G acts trivially on W, the G-action comes out to

$$g \cdot (\alpha, x) \ = \ (\alpha \circ (g^{-1} \cdot -), x)$$

for $\alpha \in \mathbf{L}(V,W)$ and $x \in W - \alpha(V)$. We note that F_V consists of a single point in all levels below the dimension of V. The structure map $(F_V)(W) \wedge S^U \longrightarrow (F_V)(W \oplus U)$ is given by

$$\mathbf{O}(V,W) \wedge S^U \xrightarrow{\mathrm{Id}\wedge\kappa} \mathbf{O}(V,W) \wedge \mathbf{O}(W, W \oplus U) \xrightarrow{\circ} \mathbf{O}(V, W \oplus U) \ .$$

The 'freeness' property of F_V is made precise as follows: for every G-fixed point $x \in X(V)$ there is a unique morphism $\hat{x} : F_V \longrightarrow X$ of G-spectra such that the map

$$O(V)_+ = (F_V)(V) \xrightarrow{\hat{x}(V)} X(V)$$

sends the identity of V to x. Indeed, the morphism $\hat{x}$ is given in level W as the composite

$$\mathbf{O}(V,W) \xrightarrow{x\wedge -} X(V) \wedge \mathbf{O}(V,W) \xrightarrow{\circ} X(W) \ .$$

For two G-representations V and W, the smash product $F_V \wedge F_W$ (with diagonal G-action) is canonically isomorphic to the free G-spectrum $F_{V\oplus W}$. Indeed, a morphism

$$F_V \wedge F_W \longrightarrow F_{V\oplus W} \tag{5.8}$$

is obtained by the universal property (1.6) from the bimorphism with (U,U')-component

$$(F_V)(U) \wedge (F_W)(U') \ = \ \mathbf{O}(V,U) \wedge \mathbf{O}(W,U') \xrightarrow{\oplus} \mathbf{O}(V \oplus W, U \oplus U') \ = \ (F_{V\oplus W})(U \oplus U') \ .$$

In the other direction, a morphism $F_{V\oplus W} \longrightarrow F_V \wedge F_W$ is freely generated by the image of the G-fixed point $\mathrm{Id}\wedge\mathrm{Id}$ under the generalized universal map

$$F_V(V) \wedge F_W(W) \xrightarrow{i_{V,W}} (F_V \wedge F_W)(V \oplus W) \ .$$

These two maps are inverse to each other.

Smashing a based G-space with the free G-spectrum F_V produces a functor

$$F_V \ : \ G\mathbf{T} \ \longrightarrow \ \mathcal{S}p_G \ , \quad F_V A \ = A \wedge F_V \ .$$

This functor is left adjoint of the evaluation functor at V. More precisely, for based G-space A and every based continuous G-map $f : A \longrightarrow X(V)$ there is a unique morphism of G-spectra $\hat{f} : F_V A \longrightarrow X$ such that the composite

$$A \xrightarrow{-\wedge \mathrm{Id}_V} A \wedge O(V)_+ = (F_V A)(V) \xrightarrow{\hat{f}(V)} X(V)$$

equals f. Indeed, the morphism $\hat{f}$ is given in level W as the composite

$$A \wedge \mathbf{O}(V,W) \xrightarrow{f\wedge -} X(V) \wedge \mathbf{O}(V,W) \xrightarrow{\circ} X(W) \ .$$

We will see in Proposition 5.12 below that the free G-spectrum $F_V A$ is $\underline{\pi}_*$-isomorphic to $\Omega^V(\Sigma^\infty A)$, the V-fold loop spectrum of the suspension spectrum of A. Indeed, a natural map

$$F_V A \ \longrightarrow \ \Omega^V(\Sigma^\infty A)$$

is the one freely generated by the adjunction unit $A \longrightarrow \Omega^V(A \wedge S^V) = \Omega^V(\Sigma^\infty A)(V)$.

Example 5.9 (Semifree G-spectra)**.** There are somewhat 'less free' orthogonal spectra which start from a pointed $G \ltimes O(V)$-space L as follows. If V is any G-representation then G acts by conjugation on the space $O(V)$ of (not necessarily equivariant) isometries. The value $X(V)$ of an orthogonal G-spectrum at V has both an action of $O(V)$ and an action of G that together make up a left action of the semi-direct product $G \ltimes O(V)$. We claim that the evaluation functor

$$\mathrm{ev}_V \ : \ \mathcal{S}p_G \ \longrightarrow \ (G \ltimes O(V))\mathbf{T} \ , \quad X \longmapsto X(V)$$

has a left adjoint which we denote G_V. (The evaluation functor ev_V also has a right adjoint, which will not discuss.) The spectrum $G_V L$ is given by

$$(G_V L)(W) \ = \ \mathbf{O}(V,W) \wedge_{O(V)} L \ .$$

The structure map $(G_V L)(W) \wedge S^U \longrightarrow (G_V L)(W \oplus U)$ is defined by

$$\begin{aligned} \mathbf{O}(V,W) \wedge_{O(V)} L \wedge S^U \;&\cong\; \left(\mathbf{O}(V,W) \wedge S^U\right) \wedge_{O(V)} L \\ &\xrightarrow{\mathrm{Id} \wedge \kappa \wedge \mathrm{Id}} (\mathbf{O}(V,W) \wedge \mathbf{O}(W, W\oplus U)) \wedge_{O(V)} L \\ &\xrightarrow{\circ \wedge \mathrm{Id}} \mathbf{O}(V, W\oplus U) \wedge_{O(V)} L \ . \end{aligned}$$

We observe that $G_V L$ is trivial in all levels below the dimension of V. We refer to $G_V L$ as the *semifree G-spectrum* generated by L in level V. Every free G-spectrum is semifree, i.e., there is a natural isomorphism $F_V A \cong G_V(O(V)_+ \wedge A)$ by 'canceling $O(V)$'; here $O(V)_+ \wedge A$ has the diagonal G-action. Every orthogonal G-spectrum is built from semifree ones, in the sense of a certain coend construction.

Example 5.10 (Mapping spaces)**.** There is a whole space of morphisms between two orthogonal spectra X and Y. Every morphism $f : X \longrightarrow Y$ consists of a family of based $O(n)$-equivariant maps $\{f_n : X_n \longrightarrow Y_n\}_{n\geq 0}$ which satisfy some conditions. So the set of morphisms from X to Y is a subset of the product of mapping spaces $\prod_{n\geq 0} \mathrm{map}(X_n, Y_n)$ and we give it the subspace topology of the (compactly generated) product topology. We denote this mapping space by $\mathrm{map}(X,Y)$. The morphism space has a natural basepoint, namely the levelwise constant map at the basepoints.

If X and Y are orthogonal G-spectra, the group G acts by conjugation on the mapping space $\mathrm{map}(X,Y)$ of underlying non-equivariant spectra. The G-fixed points $\mathrm{map}^G(X,Y)$ of this action consists precisely of the G-equivariant morphism of orthogonal spectra, i.e., the morphism of G-spectra.

For a pointed G-space A and orthogonal G-spectra X and Y we have adjunction G-homeomorphisms

$$\mathrm{map}(A, \mathrm{map}(X,Y)) \;\cong\; \mathrm{map}(A \wedge X, Y) \;\cong\; \mathrm{map}(X, Y^A) \ ,$$

where the first mapping space is taken in the category $\mathbf{T}$ of compactly generated spaces, with conjugation action by G. For free G-spectra we have G-equivariant isomorphisms

$$\mathrm{map}(F_V A, Y) \;\cong\; \mathrm{map}(A, Y(V)) \ .$$

Here G acts on the right hand side by conjugation with respect to the given actions on A and $Y(V)$. The associative and unital composition maps

$$\mathrm{map}(Y,Z) \wedge \mathrm{map}(X,Y) \;\longrightarrow\; \mathrm{map}(X,Z)$$

are G-equivariant (with respect to the diagonal G-action on the left).

Example 5.11 (Internal Hom spectra)**.** Orthogonal spectra have internal function objects: for orthogonal spectra X and Y we define a orthogonal spectrum $\mathrm{Hom}(X,Y)$ in level V by

$$\mathrm{Hom}(X,Y)(V) \;=\; \mathrm{map}(X, \mathrm{sh}^V Y) \ .$$

The left $O(V)$-action on $\mathrm{sh}^V Y$ as described in Example 3.9 yields a left $O(V)$-action on this mapping space. The structure map $\sigma_{V,W} : \mathrm{Hom}(X,Y)(V) \wedge S^W \longrightarrow \mathrm{Hom}(X,Y)(V\oplus W)$ is the composite

$$\mathrm{map}(X, \mathrm{sh}^V Y) \wedge S^W \xrightarrow{\text{assembly}} \mathrm{map}(X, (\mathrm{sh}^V Y) \wedge S^W) \xrightarrow{\mathrm{map}(X, \lambda_{\mathrm{sh}^V Y})} \mathrm{map}(X, \mathrm{sh}^{V\oplus W} Y) \ ;$$

here the first map is of 'assembly type', i.e., it takes $f \wedge w$ to the map which sends $x \in X$ to $f(x) \wedge w$ (for $f : X \longrightarrow \mathrm{sh}^V Y$ and $w \in S^W$), and $\lambda_{\mathrm{sh}^V Y} : S^W \wedge \mathrm{sh}^n Y \longrightarrow \mathrm{sh}(\mathrm{sh}^V Y) = \mathrm{sh}^{V\oplus W} Y$ is the natural morphism defined in (3.10).

In order to verify that this indeed gives a orthogonal spectrum we describe the iterated structure map. Let us denote by $\lambda_Y^{(m)} : S^m \wedge Y \longrightarrow \mathrm{sh}^m Y$ the morphism (3.10) for $V = \mathbb{R}^m$. Then for all $k, m \geq 0$ the diagram

$$\begin{tikzcd}
S^k \wedge S^m \wedge Y \arrow[r, "\mathrm{Id}\wedge\lambda_Y^{(m)}"] \arrow[d, "\cong"'] & S^k \wedge \mathrm{sh}^m Y \arrow[r, "\lambda^{(k)}_{\mathrm{sh}^m Y}"] & \mathrm{sh}^k(\mathrm{sh}^m Y) \arrow[d, equal] \\
S^{k+m} \wedge Y \arrow[r, "\chi_{k,m}\wedge\mathrm{Id}"'] & S^{m+k} \wedge Y \arrow[r, "\lambda_Y^{(m+k)}"'] & \mathrm{sh}^{m+k} Y
\end{tikzcd}$$

commutes. This implies that the iterated structure map of the spectrum $\mathrm{Hom}(X,Y)$ equals the composite

$$\mathrm{map}(X, \mathrm{sh}^n Y) \wedge S^m \xrightarrow{\text{assembly}} \mathrm{map}(X, S^m \wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X, \lambda^{(m)}_{\mathrm{sh}^n Y})} \mathrm{map}(X, \mathrm{sh}^{n+m} Y)$$

and is thus $O(n) \times O(m)$-equivariant. The first map is again of 'assembly type', i.e., for $f : X \longrightarrow \mathrm{sh}^n Y$ and $t \in S^m$ it takes $f \wedge t$ to the map which sends $x \in X$ to $t \wedge f(x)$.

If X and Y are G-spectra, then the G-action on $\mathrm{Hom}(X,Y)_n = \mathrm{map}(X, \mathrm{sh}^n Y)$ makes the mapping spectrum $\mathrm{Hom}(X,Y)$ into an orthogonal G-spectrum. For a G-representation V we have a G-homeomorphism

$$\mathrm{Hom}(X,Y)(V) \cong \mathrm{map}(X, \mathrm{sh}^V Y) .$$

Taking function spectrum commutes with shifting in the second variable, i.e., we have isomorphisms

$$\mathrm{Hom}(X, \mathrm{sh}^V Y) \cong \mathrm{sh}^V \mathrm{Hom}(X,Y) .$$

Indeed, in level n we have

$$\begin{aligned}
\mathrm{Hom}(X, \mathrm{sh}^V Y)_n &= \mathrm{map}(X, \mathrm{sh}^n(\mathrm{sh}^V Y)) \cong \mathrm{map}(X, \mathrm{sh}^{V+n} Y) \\
&= \mathrm{Hom}(X,Y)(V \oplus \mathbb{R}^n) = \left(\mathrm{sh}^V \mathrm{Hom}(X,Y)\right)_n .
\end{aligned}$$

The orthogonal group actions and structure maps coincide as well.

The internal function spectrum functor $\mathrm{Hom}(X, -)$ is right adjoint to the internal smash product $- \wedge X$ of orthogonal G-spectra (with diagonal G-action). A natural isomorphism of orthogonal G-spectra $\mathrm{Hom}(F_V, Y) \cong \mathrm{sh}^V Y$ is given at level n by

$$\mathrm{Hom}(F_V, Y)_n = \mathrm{map}(F_V, \mathrm{sh}^n Y) \cong (\mathrm{sh}^n Y)(V) = Y(\mathbb{R}^n \oplus V) \xrightarrow{\tau_{\mathbb{R}^n, V}} Y(V \oplus \mathbb{R}^n) = (\mathrm{sh}^V Y)_n$$

where the second map is the adjunction bijection described in Example 5.5. This isomorphism is equivariant for the left actions of $O(V)$ induced on the source from the right $O(V)$-action on a free spectrum. In the special case $V = 0$ we have $F_0 S^0 = \mathbb{S}$, which gives a natural isomorphism of orthogonal spectra $\mathrm{Hom}(\mathbb{S}, Y) \cong Y$.

Change of groups. All of the construction that we have discussed in this section are nicely compatible with change of groups. Given a group homomorphism $\alpha : K \longrightarrow G$, we can restrict G-spaces, G-representations and G-spectra along α, and all of the above constructions commute with this restriction on the nose (and not just up to isomorphism).

For example, the restriction functor $\alpha^* : \mathcal{S}p_G \longrightarrow \mathcal{S}p_K$ commutes with limits and colimits, and for a based G-space A and an orthogonal G-spectrum X we have

$$\alpha^*(A \wedge X) = (\alpha^* A) \wedge (\alpha^* X) \qquad \text{and} \qquad \alpha^*(\mathrm{map}(A, X)) = \mathrm{map}(\alpha^* A, \alpha^* X)$$

as orthogonal K-spectra.

For a G-representation V we have $\alpha^*(S^V) = S^{\alpha^* V}$ and for an orthogonal G-spectrum X we have $\alpha^*(X(V)) = (\alpha^* X)(\alpha^* V)$ as K-spaces. Consequently,

$$\alpha^*(S^V \wedge X) = S^{\alpha^* V} \wedge (\alpha^* X) , \quad \alpha^*(\Omega^V X) = \Omega^{\alpha^* V}(\alpha^* X) \quad \text{and} \quad \alpha^*(\mathrm{sh}^V X) = \mathrm{sh}^{\alpha^* V}(\alpha^* X)$$

as orthogonal K-spectra. Given another G-representation W we have $\alpha^*\mathbf{O}(V,W) = \mathbf{O}(\alpha^*V,\alpha^*W)$ and hence the restrictions of a free and semifree spectra are again free:

$$\alpha^*(F_V) \ = \ F_{\alpha^*V} \qquad \text{and} \qquad \alpha^*(G_V L) \ = \ G_{\alpha^*V}(\alpha^*L)$$

(where α^*L has the same $O(V)$-action as L). Finally, if Y is another orthogonal G-spectrum, then we have

$$\alpha^*(\mathrm{map}(X,Y)) \ = \ \mathrm{map}(\alpha^*X,\alpha^*Y) \qquad \text{and} \qquad \alpha^*(\mathrm{Hom}(X,Y)) \ = \ \mathrm{Hom}(\alpha^*X,\alpha^*Y) \ .$$

We emphasize again that here we always have equality, not just isomorphism.

Our next aim is to show that the free orthogonal G-spectrum F_V generated by a G-representation V behaves like a '$(-V)$-sphere', a sphere spectrum for the virtual representation $-V$. More precisely we show:

Proposition 5.12. *For every G-representation V the morphism*

$$F_V S^V \ \longrightarrow \ \mathbb{S}$$

adjoint to the identity of S^V is a $\underline{\pi}_*$-isomorphism. For every based G-CW-complex A the map

$$F_V A \ \longrightarrow \ \Omega^V(\Sigma^\infty A)$$

adjoint to the adjunction unit $A \longrightarrow \Omega^V(A\wedge S^V) = (\Omega^V(\Sigma^\infty A))(V)$ is a $\underline{\pi}_*$-isomorphism.

Before we prove the proposition, we introduce and analyze a new construction. As before we denote by $\mathbf{L}$ the topological category with objects the inner product spaces $\mathbb{R}^n$ for $n \geq 0$ and with morphism space $\mathbf{L}(\mathbb{R}^m,\mathbb{R}^n)$ the space of isometric embedding from $\mathbb{R}^m$ to $\mathbb{R}^n$. We denote by $G\mathbf{T}^{\mathbf{L}}$ the category of $\mathbf{L}$-*G-spaces*, i.e., covariant continuous functors from $\mathbf{L}$ to the category of pointed G-spaces

Example 5.13 (Smash product with $\mathbf{L}$-G-spaces)**.** Given an $\mathbf{L}$-G-space $T : \mathbf{L} \longrightarrow G\mathbf{T}$ and an orthogonal G-spectrum X, we can form a new orthogonal G-spectrum $T \wedge X$ by setting

$$(T_+ \wedge X)_n \ = \ T(\mathbb{R}^n)_+ \wedge X_n$$

with diagonal action of $O(n)$ and G-action through the action on X_n. The structure map is given by

$$(T_+\wedge X)_n \wedge S^1 \ = \ T(\mathbb{R}^n)_+ \wedge X_n \wedge S^1 \ \xrightarrow{T(\iota)_+\wedge\sigma_n} \ T(\mathbb{R}^{n+1})_+ \wedge X_{n+1} \ = \ (T_+\wedge X)_{n+1}$$

where $\iota : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ is the 'inclusion' with $\iota(x) = (x,0)$. If A is a pointed G-space and T_A the constant functor with values A, then $(T_A)_+\wedge X$ is equal to $A\wedge X$, i.e., this construction reduces to the levelwise smash product with a G-space as in Example 5.2. When $X = \mathbb{S}$ is the sphere spectrum, we write $\Sigma^\infty_+ T = T_+ \wedge \mathbb{S}$ and refer to this orthogonal G-spectra as the *unreduced suspension spectrum* of the $\mathbf{L}$-G-space T.

Given any $\mathbf{L}$-G-space T, we can evaluate it on a G-representation V by setting

$$T(V) \ = \ \mathbf{L}(\mathbb{R}^n, V) \times_{O(n)} T(\mathbb{R}^n)$$

where $n = \dim V$. The group G acts diagonally, via the given action on $T(\mathbb{R}^n)$ and the action on V. We denote by

$$T(\infty\rho_G) \ = \ \mathrm{colim}_{n\in\mathbb{N}}\, T(n\rho_G)$$

the G-space obtained as the sequential colimit over the maps induced by the 'inclusions' $n\rho_G \longrightarrow (n+1)\rho_G$.

The canonical maps $T(n\rho_G) \longrightarrow T(\infty\rho_G)$ induce maps of equivariant homotopy classes

$$\begin{aligned}[][S^{\mathbb{R}^k\oplus n\rho_G}, (\Sigma^\infty_+ T)(n\rho_G)]^G \ &= [S^{\mathbb{R}^k\oplus n\rho_G}, T(n\rho_G)_+ \wedge S^{n\rho_G}]^G \\ &\longrightarrow [S^{\mathbb{R}^k\oplus n\rho_G}, T(\infty\rho_G)_+ \wedge S^{n\rho_G}]^G \ \longrightarrow \quad \pi_k^G(\Sigma^\infty_+ T(\infty\rho_G)) \ ,\end{aligned}$$

where the second map is the canonical one to the colimit. These maps are compatible with stabilization in the source when we increase n. So the maps assemble into a natural group homomorphism

$$\pi_k^G(\Sigma^\infty_+ T) \ \longrightarrow \ \pi_k^G(\Sigma^\infty_+ T(\infty\rho_G)) \ ,$$

for $k \geq 0$. For negative values of k, we obtain a similar map by inserting $\mathbb{R}^{-k}$ into the second variable of the above sets of equivariant homotopy classes. We emphasize that these maps of equivariant homotopy groups are *not* induced by a morphism of orthogonal G-spectra from $\Sigma^\infty_+ T$ to $\Sigma^\infty_+ T(\infty\rho_G)$; the issue is that the G-space $T(\infty\rho_G)$ does not remember the orthogonal symmetries of the original $\mathbf{L}$-G-space.

Proposition 5.14. *Let G be a finite group and T an $\mathbf{L}$-G-space for which all structure maps are closed embeddings. Then for integer k the map*

$$\pi_k^G(\Sigma^\infty_+ T) \longrightarrow \pi_k^G(\Sigma^\infty_+ T(\infty\rho_G))$$

is an isomorphism.

Proof. The G-space $T(\infty\rho_G)_+ \wedge S^{m\rho_G}$ is the colimit, along a sequence of closed embeddings, of the G-spaces $T(n\rho_G)_+ \wedge S^{m\rho_G}$, for $n \geq 0$. For every compact based G-space A, the canonical map

$$\mathrm{colim}_{n\geq 1}\, [A, T(n\rho_G)_+ \wedge S^{m\rho_G}]^G \longrightarrow [A, T(\infty\rho_G)_+ \wedge S^{m\rho_G}]^G$$

is thus bijective. Indeed, every continuous map from the compact spaces A and $A \wedge [0,1]_+$ to $T(\infty\rho_G)_+ \wedge S^{m\rho_G}$ factors through $T(n\rho_G)_+ \wedge S^{m\rho_G}$ some n. We specialize to $A = S^{\mathbb{R}^k \oplus m\rho_G}$ and pass to the colimit over m by stabilization in source and target. The result is a bijection

$$\mathrm{colim}_{m\geq 1}\, \mathrm{colim}_{n\geq 1}\, [S^{\mathbb{R}^k\oplus m\rho_G}, T(V_n)_+ \wedge S^{m\rho_G}]^G \longrightarrow \mathrm{colim}_{m\geq 1} [S^{\mathbb{R}^k\oplus m\rho_G}, T(\infty\rho_G)_+ \wedge S^{m\rho_G}]^G .$$

The source is isomorphic to the diagonal colimit

$$\mathrm{colim}_{j\geq 1} [S^{\mathbb{R}^k\oplus j\rho_G}, T(j\rho_G)_+ \wedge S^{j\rho_G}]^G$$

by cofinality. The two colimits over j calculate the kth equivariant homotopy group of $\Sigma^\infty_+ T$ and of $\Sigma^\infty_+ T(\infty\rho_G)$, respectively. This shows the claim for $k \geq 0$. For negative values of k, the argument is similar: we insert $\mathbb{R}^{-k}$ into the second variable of the sets of equivariant homotopy classes. □

Now we can give the

Proof of Proposition 5.12. We recall that the value of $F_V S^V$ on a G-representation W is given by

$$(F_V S^V)(W) = \mathbf{O}(V,W) \wedge S^V .$$

After smashing with S^V the Thom space $\mathbf{O}(V,W)$ 'untwists', i.e., the map

$$\mathbf{O}(V,W) \wedge S^V \longrightarrow \mathbf{L}(V,W)_+ \wedge S^W , \quad (\alpha, w) \wedge v \longmapsto \alpha \wedge (w + \alpha(v))$$

is a G-equivariant homeomorphism. As W varies, these homeomorphisms form an isomorphism of orthogonal G-spectra

$$F_V S^V \cong \Sigma^\infty_+ \mathbf{L}(V, -) .$$

Under this isomorphism the map $F_V S^V \longrightarrow \mathbb{S}$ becomes the projection $\mathbf{L}(V,-)_+ \wedge \mathbb{S}$ taking $\mathbf{L}(V,-)$ objectwise to a point. By Proposition 5.14, the equivariant homotopy groups of $\Sigma^\infty_+ \mathbf{L}(V,-)$ and of $\Sigma^\infty_+ \mathbf{L}(V,\infty\rho_G)$ are isomorphic. So it suffices to show that the G-space $\mathbf{L}(V,\infty\rho_G)$, which is a G-CW-complex, is weakly G-contractible. For a subgroup H of G the fixed points $\mathbf{L}(V,\infty\rho_G)^H$ is the space of H-equivariant linear isometric embedding from V to $\infty\rho_G$. Since the representation $\infty\rho_G$ contains V infinitely often, this space is contractible.

For the second statement we exploit that smashing with A preserves $\underline{\pi}_*$-isomorphisms (Proposition 5.4). So by the first part the map $F_V(S^V \wedge A) \longrightarrow \Sigma^\infty A$ is a $\underline{\pi}_*$-isomorphism. Hence its adjoint $F_V A \longrightarrow \Omega^V(\Sigma^\infty A)$ is a $\underline{\pi}_*$-isomorphism by Proposition 3.12. □

Definition 5.15. A morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is a *strong level equivalence* if for every G-representation V the map

$$f(V) : X(V) \longrightarrow Y(V)$$

is a G-weak equivalence.

Proposition 5.16. (i) *Let $f : X \longrightarrow Y$ be a morphism of orthogonal G-spectra with the following property: for every $n \geq 0$ and every subgroup K of $O(n) \times G$ such that $K \cap (O(n) \times 1) = 1$, the map*

$$f_n^K \ : \ X_n^K \ \longrightarrow \ Y_n^K$$

on K-fixed points is a weak equivalence of spaces. Then f is a strong level equivalence.

(ii) *Every strong level equivalence of orthogonal G-spectra is a $\underline{\pi}_*$-isomorphism.*

Proof. (i) Let V be a G-representation and set $n = \dim V$. Let $\alpha : \mathbb{R}^n \longrightarrow V$ by a linear isometry, not necessarily G-equivariant. We define a homomorphism $-^\alpha : G \longrightarrow O(n)$ by 'conjugation by α, i.e., we set

$$(g^\alpha)(x) \ = \ \alpha^{-1}(g \cdot \alpha(x))$$

for $g \in G$ and $x \in \mathbb{R}^n$. Then we define a new action of G on the space X_n by setting

$$g * x \ = \ (g^\alpha, g) \cdot x \ .$$

In other words, we restrict the $O(n) \times G$-action on X_n along the monomorphism $(-^\alpha, \mathrm{Id}) : G \longrightarrow O(n) \times G$. The map

$$X_n \ \longrightarrow \ X(V) \ , \quad x \longmapsto [\alpha, x]$$

is a homeomorphism, natural in X and G-equivalent with respect to the new action of G on X_n. So for every subgroup H of G the fixed point space $X(V)^H$ is homeomorphic to X_n^K where $K \subset O(n) \times G$ is the image of H under the monomorphism $(-^\alpha, \mathrm{Id})$. The group K satisfies $K \cap (O(n) \times 1) = 1$, so the map f_n^K, and hence $f(V)^H$, is a weak equivalence. Since H was any subgroup of G, the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence.

(ii) We first treat the case of homotopy groups of dimension 0. By hypothesis the map $f(n\rho) : X(n\rho) \longrightarrow Y(n\rho)$ is a G-weak equivalence. Since the representation sphere $S^{n\rho}$ can be given a G-CW-structure, the induced map on mapping spaces

$$\mathrm{map}(S^{n\rho}, f) \ : \ \mathrm{map}(S^{n\rho}, X(n\rho)) \ \longrightarrow \ \mathrm{map}(S^{n\rho}, Y(n\rho))$$

is a G-weak equivalence. Taking H-fixed points and passing to the colimit over n shows that $\pi_0^H f : \pi_0^H(X) \longrightarrow \pi_0^H(Y)$ is an isomorphism for all subgroups H of G. For dimensions $k > 0$ we exploit that $(\Omega^W X)(V)$ is naturally G-homeomorphic to $\Omega^W X(V)$, so every loop, by any G-representations, of a strong level equivalence is again a strong level equivalence. For dimensions $k < 0$ we exploit that $(\mathrm{sh}^V X)(W)$ is naturally G-homeomorphic to $X(V \oplus W)$, so every shift, by any G-representations, of a strong level equivalence is again a strong level equivalence. □

Proposition 5.17. (i) *Let X be a G-Ω-spectrum such that $\underline{\pi}_k(X) = 0$ for every integer k. Then for every G-representation V the space $X(V)$ is G-weakly contractible.*

(ii) *Every $\underline{\pi}_*$-isomorphism between G-Ω-spectra is a strong level equivalence.*

Proof. (i) See Mandell and May [16, III, Lemma 9.1].

(ii) Let $f : X \longrightarrow Y$ be a $\underline{\pi}_*$-isomorphism between G-Ω-spectra. We let F denote the homotopy fiber of f. The long exact sequence of homotopy groups implies that $\underline{\pi}_* F = 0$. For every G-representation V the G-space $F(V)$ is then G-homeomorphic to the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. So F is again a G-Ω-spectrum.

By the Ω-spectrum property, the space $X(V)$ is G-weakly equivalent to $\Omega X(V \oplus \mathbb{R})$ and similarly for Y. So the map $f(V)$ is G-weakly equivalent to

$$\Omega f(V \oplus \mathbb{R}) \ : \ \Omega X(V \oplus \mathbb{R}) \ \longrightarrow \ \Omega X(V \oplus \mathbb{R}) \ .$$

Hence we have a homotopy fiber sequence

$$X(V)^H \ \xrightarrow{f(V)^H} \ Y(V)^H \ \longrightarrow \ F(V \oplus \mathbb{R})^H$$

for every subgroup H of G. By part (i) the space $F(V)^H$ is weakly contractible, so $f(V)^H$ is a weak equivalence. $\square$

5.1. **Canonical presentation.** The canonical presentation is a way to write an orthogonal G-spectrum as a mapping telescope (homotopy colimit) of desuspended (by certain representations) suspension spectra.

Let X be an orthogonal G-spectrum. We assume that the space $X(V)$ has the homotopy type of a G-CW-complex for every G-representation V. This is no real loss of generality since every orthogonal G-spectrum is strongly level equivalent to a sufficiently cofibrant G-spectrum, which has this property.

We consider two nested G-representations $V \subset W$. The identity of $X(V)$ is adjoint to a morphism of G-spectra

$$i_V \ : \ F_V X(V) \ \longrightarrow \ X \ ,$$

and similarly for W instead of V. We obtain a commutative square

$$\begin{tikzcd} F_W(X(V) \wedge S^{W-V}) \arrow[r, "F_W \sigma_{V,W-V}"] \arrow[d, "\simeq"'] & F_W X(W) \arrow[d, "i_W"] \\ F_V X(V) \arrow[r, "i_V"'] & X \end{tikzcd}$$

in which the left vertical morphism is adjoint to the map of G-spaces

$$X(V) \wedge S^{W-V} \ \xrightarrow{X(V)\wedge\kappa} \ X(V) \wedge \mathbf{O}(V,W) = (F_V X(V))(W) \ .$$

We claim that the left vertical morphism is a $\underline{\pi}_*$-isomorphism. Since smashing with the representation sphere S^V detects $\underline{\pi}_*$-isomorphisms, it suffices to show this after smashing with S^V. Then the map becomes the left vertical map in the commutative diagram

$$\begin{tikzcd} F_W(X(V) \wedge S^W) \arrow[dd] \arrow[dr, "\simeq"] & \\ & \Sigma^\infty X(V) \\ F_V(X(V) \wedge S^V) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

The two diagonal maps are $\underline{\pi}_*$-isomorphisms by Proposition 5.12 and because smashing with $X(V)$ respectively $X(W)$ preserves $\underline{\pi}_*$-isomorphisms.

The upshot is that in the homotopy category of G-spectra, we have a morphism

$$j_{V,W} \ : \ F_V X(V) \ \longrightarrow F_W X(W)$$

that satisfies $i_W j_{V,W} = i_V$ as maps from $F_V X(V)$ to X.

Now we consider a nested sequence of G-representations:

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \tag{5.18}$$

As just described this gives rise to a sequence

$$F_{V_0} X(V_0) \ \xrightarrow{j_{V_0,V_1}} \ F_{V_1} X(V_1) \ \xrightarrow{j_{V_1,V_2}} \ \cdots \ \longrightarrow \ F_{V_n} X(V_n) \ \xrightarrow{j_{V_n,V_{n+1}}} \ \cdots$$

in the homotopy category of G-spectra, together with compatible maps $i_{V_n} : F_{V_n} X(V_n) \longrightarrow X$. Such data gives rise to a morphism

$$\mathrm{hocolim}_n \ F_{V_n} X(V_n) \ \longrightarrow \ X$$

in the homotopy category of orthogonal G-spectra from the homotopy colimit of the sequence.

Proposition 5.19. *Suppose that the nested sequence* (5.18) *of G-representations that is exhausting, i.e., every G-representation embeds equivariantly into V_n for large enough n. Then for every orthogonal G-spectrum X, the map*

$$\mathrm{hocolim}_n\ F_{V_n}X(V_n) \longrightarrow X$$

is an isomorphism in the homotopy category of orthogonal G-spectra.

Before we give the proof we remark that since $F_V S^V = F_V \wedge \Sigma^\infty S^V$ is $\underline{\pi}_*$-isomorphic to the G-sphere spectrum $\mathbb{S}$, F_V is an inverse to the representation sphere S^V with respect to the derived smash product of G-spectra. So we may think of F_V as 'S^{-V}', the sphere of the virtual representation $-V$. With this in mind, $F_V X(V)$ is $S^{-V} \wedge X(V)$ and the content of the proposition can be summarized as

$$X \ \cong\ \mathrm{hocolim}_n\ S^{-V_n} \wedge X(V_n)$$

in $\mathrm{Ho}(\mathcal{S}p_G)$.

Proof. The given exhaustive sequence and the exhaustive sequence

$$\rho \longrightarrow 2\rho \longrightarrow 3\rho \longrightarrow \dots \longrightarrow n\rho \longrightarrow \dots$$

of multiples of the regular representation can be cofinally embedded into each other. So the two resulting homotopy colimits are isomorphic in $\mathrm{Ho}(\mathcal{S}p_G)$. It thus suffices to consider the nested sequence of regular representations and show that the map

$$\mathrm{hocolim}_n\ F_{n\rho}X(n\rho) \longrightarrow X$$

is a $\underline{\pi}_*$-isomorphism. For an integer k and subgroup H of G, the left hand side evaluate to:

$$\begin{aligned} \pi_k^H\left(\mathrm{hocolim}_n\ F_{n\rho}X(n\rho)\right) &\cong \mathrm{colim}_n\, \pi_k^H\left(F_{n\rho}X(n\rho)\right) \\ &\cong \mathrm{colim}_n\, \pi_{k+n\rho}^H\left(\Sigma^\infty X(n\rho)\right) \\ &\cong \mathrm{colim}_{n,m} [S^{k+n\rho+m\rho},\, X(n\rho) \wedge S^{m\rho}]^H \\ &\cong \mathrm{colim}_m\, \pi_{k+m\rho}^H(S^{m\rho} \wedge X) \ \cong\ \pi_k^H(X) \end{aligned}$$

□

Let $\mathcal{F}$ be a *family* of subgroups of G, i.e., $\mathcal{F}$ is a set of subgroups of G closed under conjugation and passage to subgroups. We denote by $E\mathcal{F}$ any universal space for the family $\mathcal{F}$, i.e., a G-CW-complex such that the fixed points set $(E\mathcal{F})^H$ is contractible for $H \in \mathcal{F}$ and $(E\mathcal{F})^H$ is empty for $H \notin \mathcal{F}$.

Example 5.20. Let V be a G-representation. We let $\mathcal{F}_V$ denote the family of those subgroups H of G such that $V^H \neq 0$. Let $S(\infty V)$ be the unit sphere in the infinite dimensional representation $\infty V = \bigoplus_{\mathbb{N}} V$; in other words,

$$S(\infty V) \ =\ \bigcup_{n \geq 0} S(nV)$$

is the union of the unit spheres of nV with the weak topology. Then we have

$$S(\infty V)^H \ =\ S(\infty(V^H))$$

which is empty if H does not belong to $\mathcal{F}_V$ and an infinite dimensional sphere, hence contractible, for $H \in \mathcal{F}_V$. In other words: the space $S(\infty V)$ is a universal space $E\mathcal{F}_V$.

Lemma 5.21. *Let $\mathcal{F}$ be a family of subgroups of G and X an orthogonal G-spectrum. Then for every subgroup H in the family $\mathcal{F}$ the projection $E\mathcal{F}_+ \wedge X \longrightarrow X$ induces isomorphisms*

$$\pi_k^H(E\mathcal{F}_+ \wedge X) \ \xrightarrow{\cong}\ \pi_k^H(X)\ .$$

Proof. For every subgroup H in the family $\mathcal{F}$ and every subgroup K of H the map $E\mathcal{F} \longrightarrow *$ is a weak equivalence on K-fixed points. Since both sides are H-CW-complexes, the map $E\mathcal{F} \longrightarrow *$ is an H-homotopy equivalence. So the map $E\mathcal{F}_+ \longrightarrow S^0$, and hence the map $E\mathcal{F}_+ \wedge X \longrightarrow S^0 \wedge X \cong X$ are H-homotopy equivalences, and the conclusion follows. □

Proposition 5.22. *Let $\mathcal{F}$ be a family of subgroups of G. For a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra, the following are equivalent:*

(i) *For every subgroup H of $\mathcal{F}$ the morphism $i^* f : i^* X \longrightarrow i^* Y$ is a $\underline{\pi}_*$-isomorphism of H-spectra.*
(ii) *For every subgroup H of $\mathcal{F}$ the induced map $\pi_*^H f : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism of graded homotopy groups.*
(iii) *The morphism $E\mathcal{F}_+ \wedge f : E\mathcal{F}_+ \wedge X \longrightarrow E\mathcal{F}_+ \wedge Y$ is a $\underline{\pi}_*$-isomorphism of G-spectra.*

Proof. Properties (i) and (ii) are equivalent because for every subgroup K of H the groups $\pi_k^K(i^*X)$ and $\pi_k^K(X)$ are naturally isomorphic.

Property (iii) implies property (ii) because of the natural isomorphism $\pi_k^H(E\mathcal{F}_+ \wedge X) \cong \pi_k^H(X)$ of Lemma 5.21.

(i)$\Longrightarrow$(iii) By passage to the mapping cone of f it suffices to show that for all G-spectra X such that $\pi_*^H(X) = 0$ for all $H \in \mathcal{F}$ the spectrum $E\mathcal{F}_+ \wedge X$ is $\underline{\pi}_*$-trivial. Since smashing with the G-CW-complex $E\mathcal{F}_+$ preserves $\underline{\pi}_*$-isomorphisms, we may assume that X is a G-Ω-spectrum (for example by using the construction 3.15). Then for every subgroup H in $\mathcal{F}$ and every G-representation V we have $X(V)^H \simeq *$, by Proposition 5.17. Hence $E\mathcal{F}_+ \wedge X(V)$ is G-weakly contractible, and thus $\underline{\pi}_*(E\mathcal{F}_+ \wedge X) = 0$. □

6. The tom Dieck splitting

Among the simplest kinds of examples of orthogonal G-spectra are suspension spectra of G-spaces. A G-space is essentially determined by the homotopy types of the fixed point spaces for the various subgroups, and one can ask if and how the equivariant stable homotopy groups can be obtained from the fixed point information. The tom Dieck splitting provides an answer to this, and it rewrites the equivariant stable homotopy groups of a suspension spectrum as a sum of terms, indexed by conjugacy classes of subgroups H, where the summand indexed by H depends only on the H-fixed points of the original G-space. The sphere spectrum is an equivariant suspension spectrum, and by applying the tom Dieck splitting to this case we can identify the G-equivariant stable 0-stem with the Burnside ring of G.

Tom Dieck's splitting originally appeared in [4, Satz 2] in the more general context of compact Lie groups; we follow the original proof.

Construction 6.1. We start by introducing the maps whose sum will later turn out to be the isomorphism of the tom Dieck splitting. We let A be a based G-space and H a subgroup of G. We define a natural transformation

$$\zeta_H \ : \ \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \ \longrightarrow \ \pi_*^G(\Sigma^\infty A) \tag{6.2}$$

where $WH = W_G H = (N_G H)/H$ is the Weyl group of H which acts on the H-fixed point space of A. The map ζ_H is defined as the composite

$$\begin{aligned} \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \ &\xrightarrow{\ p^*\ } \pi_*^{NH}(\Sigma^\infty(EWH_+ \wedge A^H)) \\ &\xrightarrow{\ i_*\ } \pi_*^{NH}(\Sigma^\infty(EWH_+ \wedge A)) \\ &\xrightarrow{\ \mathrm{Tr}_{NH}^G\ } \pi_*^G(G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A))) \\ &\xrightarrow{\ q_*\ } \pi_*^G(\Sigma^\infty A) \end{aligned}$$

Here, and in the following, $NH = N_G H$ is the normalizer of H in G. The first map is the restriction homomorphism (3.5) along the projection $p : NH \longrightarrow (NH)/H = WH$. The second map is induced by the

NH-equivariant inclusion $i : A^H \longrightarrow A$ of the H-fixed points. The third map is the external transfer (4.14). The fourth map is the effect on equivariant homotopy groups of the morphism of orthogonal G-spectra

$$q \ : \ G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)) \ \longrightarrow \ \Sigma^\infty A$$

that is adjoint to the morphism of orthogonal NH-spectra

$$\Sigma^\infty(EWH_+ \wedge A) \ \longrightarrow \ \Sigma^\infty A$$

induced from the NH-equivariant projection $EWH_+ \wedge A \longrightarrow A$. To simplify notation we suppress that we sometimes view EWH and A^H as NH-spaces by restriction along the projection p; so more properly we should be writing $p^*(EWH)$ and $p^*(A)$ in those places.

The following terminology will be useful throughout this section.

Definition 6.3. Let G be a finite group, H a subgroup of G and A a based G-space. Then A is *concentrated at the conjugacy class of* H if the K-fixed points A^K are contractible for every subgroup $K \leq G$ that is not conjugate to H.

We emphasize that in the previous definition we require A^K to actually be contractible, in the based sense, to the basepoint; if we only asked for weak contractibility, some of the arguments below would not work.

Proposition 6.4. *Let G be a finite group, H a normal subgroup of G and Y a based G-space that is concentrated at H. Then for every finite based G-CW-complex B the geometric fixed point map*

$$(-)^H \ : \ \mathrm{map}^G(B, Y) \ \longrightarrow \ \mathrm{map}^{G/H}(B^H, Y^H)$$

that takes a based G-map $f : B \longrightarrow Y$ to the restriction to H-fixed points $f^H : B^H \longrightarrow Y^H$ is a weak equivalence and Serre fibration.

Proof. We let A be a G-space that is obtained from a G-subspace A' by attaching an equivariant cell of orbit type K, with K different from H. Applying $\mathrm{map}^G(-, Y)$ turns the pushout of G-spaces on the left

$$\begin{tikzcd} G/K \times S^{n-1} \arrow[r] \arrow[d] & G/K \times D^n \arrow[d] \\ A' \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{map}^G(A, Y) \arrow[r] \arrow[d] & \mathrm{map}^G(A', Y) \arrow[d] \\ \mathrm{map}(D^n_+, Y^K) \arrow[r] & \mathrm{map}(S^{n-1}_+, Y^K) \end{tikzcd}$$

into the pullback square on the right in which both horizontal maps are Serre fibrations. Since Y is concentrated at H, the space Y^K is contractible, hence so are the two mapping spaces in the lower row. The restriction map $\mathrm{map}^G(A, Y) \longrightarrow \mathrm{map}^G(A', Y)$ is thus a Serre fibration and weak equivalence.

Because H is normal in G, the H-fixed subspace B^H is G-invariant. It is even a G-CW-subcomplex of B: if $H \leq K$, then every open G-cell of type $G/K \times \mathring{D}^n$ is H-fixed, and hence contained in B^H; conversely, if H is not contained in K, then the open G-cell of type $G/K \times \mathring{D}^n$ has no H-fixed points. So B is obtained from B^H by attaching equivariant cells of orbit type K, for subgroups K that do not contain H. So the restriction map

$$\mathrm{map}^G(B, Y) \ \longrightarrow \ \mathrm{map}^G(B^H, Y) = \mathrm{map}^{G/H}(B^H, Y^H)$$

is a Serre fibration and weak equivalence. □

We start by proving that statement that will turn out as a special case of the tom Dieck splitting, and it will also be a main step in the proof.

Proposition 6.5. *Let G be a finite group, H a subgroup of G and A a based G-space that is concentrated at the conjugacy class of H. Then the map*

$$\zeta_H \ : \ \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \ \longrightarrow \ \pi_*^G(\Sigma^\infty A)$$

is an isomorphism.

Proof. The map ζ_H was defined as a composite of four maps. The external transfer map $\mathrm{Tr}_{NH}^G : \pi_*^{NH}(\Sigma^\infty(EWH_+ \wedge A)) \longrightarrow \pi_*^G(G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)))$ is an isomorphism by definition. We will show that in addition (a) the composite $i_* \circ p^*$ of the first two maps is an isomorphism, and (b) the fourth map q_* is an isomorphism.

(a) The composite map

$$i_* \circ p^* \ : \ \pi_k^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \ \longrightarrow \ \pi_k^{NH}(\Sigma^\infty(EWH_+ \wedge A))$$

has a retraction

$$\Phi^H \ : \ \pi_k^{NH}(\Sigma^\infty(EWH_+ \wedge A)) \ \longrightarrow \ \pi_k^{WH}(\Sigma^\infty(EWH_+ \wedge A^H))$$

given by a H-geometric fixed point map:

$$[f : S^{k+V} \longrightarrow S^V \wedge EWH_+ \wedge A] \ \longmapsto \ [f^H : S^{k+V^H} \longrightarrow S^{V^H} \wedge EWH_+ \wedge A^H]$$

(here V is any NH-representation). We recall that NH acts on EWH by restriction along $p : NH \longrightarrow WH$, so the subgroup $H \leq NH$ acts trivially, and $(EWH)^H = EWH$. The maps in the image of $i_* \circ p^*$ are defined on NH-representations on which H already acts trivially, so Φ^H is indeed left inverse to $i_* \circ p^*$.

The NH-space $Y = S^{m\cdot\rho_N} \wedge EWH_+ \wedge A$ is concentrated at the normal subgroup H by hypothesis on A, so Proposition 6.4, applied to $G = NH$ and the G-CW-complex $S^{k+m\cdot\rho_N}$ shows that the geometric fixed point map

$$\Phi^H \ : \ [S^{k+m\cdot\rho_N}, S^{m\cdot\rho_N} \wedge EWH_+ \wedge A]^{NH} \ \longrightarrow \ [S^{k+m\cdot(\rho_N)^H}, S^{m\cdot(\rho_N)^H} \wedge EWH_+ \wedge A^H]^{WH}$$

is bijective. Exploiting that $(\rho_{NH})^H \cong \rho_{(NH)/H} = \rho_{WH}$ and passing to colimits over m shows that the left inverse Φ^H is bijective. The composite $i_* \circ p^*$ is thus bijective as well.

(b) We show that for every G-representation V the map

$$q(V) \ : \ (G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)))(V) \ \longrightarrow \ (\Sigma^\infty A)(V)$$

is a G-weak equivalence. Hence q induces isomorphisms on G-equivariant stable homotopy groups. Indeed, we can rewrite the source of $q(V)$ isomorphically as

$$\begin{aligned}(G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)))(V) \ &\cong \ G \ltimes_{NH} (\Sigma^\infty(EWH_+ \wedge A)(V)) \\ &= \ G \ltimes_{NH} (EWH_+ \wedge A \wedge S^V) \\ &\cong \ (G \ltimes_{NH} EWH)_+ \wedge A \wedge S^V \ ,\end{aligned}$$

where the last step uses that A and S^V come with a G-action, and G acts diagonally on the last smash product. So we need to show that the projection

$$\bar{q} \ : \ (G \ltimes_{NH} EWH)_+ \wedge A \wedge S^V \ \longrightarrow \ A \wedge S^V$$

restricts to a weak equivalence on fixed points for every subgroup $K \leq G$. The K-fixed points of the source are given by

$$\big((G \ltimes_{NH} EWH)_+ \wedge A \wedge S^V\big)^K \ \cong \ ((G \ltimes_{NH} EWH)^K)_+ \wedge A^K \wedge S^{V^K} \ . \tag{6.6}$$

When K is not conjugate to H, then A^K is contractible, hence so are the spaces (6.6) and $(A \wedge S^V)^K = A^K \wedge S^{V^K}$. For $K = H$ we observe that

$$(G \times_{NH} EWH)^H \ = \ \{1\} \times EWH \ .$$

Indeed, suppose that $[g,e] \in G \times_{NH} EWH$ is H-fixed. Then for every $h \in H$ there exists an element $n \in NH$ such that $(hg, e) = (gn, n^{-1}e)$. Because the WH-action on EWH is free, the relation $e = n^{-1}e$ implies that n lies in H. Hence $g^{-1}hg = n \in H$, and so g normalizes H. Hence $[g,e] = [1, ge] \in \{1\} \times EWH$. The relation $(G \times_{NH} EWH)^H = \{1\} \times EWH$ and the fact that EWH is contractible so that $\bar{q}^H$ is a homotopy equivalence. So in any case $\bar{q}^K$ is a homotopy equivalence. □

The proof of the tom Dieck splitting will depend on the fact that both sides to be compared are G-homology theories in the following sense.

Definition 6.7. Let G be a finite group. A *G-homology theory* is a $\mathbb{Z}$-indexed family of covariant functors

$$E = \{E_k\}_{k \in \mathbb{Z}} \ , \quad E_k \ : \ (\text{based } G\text{-spaces}) \ \longrightarrow \ \mathcal{A}b$$

equipped with connecting homomorphisms $\partial : E_{1+k}(Cf) \longrightarrow E_k(A)$, natural for based G-maps $f : A \longrightarrow B$, satisfying the following conditions.

(i) Each functor E_k is G-homotopy invariant, i.e., it has the same image on G-homotopic based maps.

(ii) For every family of $\{A_i\}_{i \in I}$ of based G-spaces, the canonical map

$$\bigoplus_{i \in I} E_k(A_i) \ \longrightarrow \ E_k(\bigvee_{i \in I} A_i)$$

is an isomorphism.

(iii) For every based G-map $f : A \longrightarrow B$ the sequence

$$\cdots \ \xrightarrow{E_{1+k}(i)} \ E_{1+k}(Cf) \ \xrightarrow{\partial} \ E_k(A) \ \xrightarrow{E_k(f)} \ E_k(B) \ \xrightarrow{E_k(i)} \ E_k(Cf) \ \xrightarrow{\partial} \ \cdots$$

is exact.

A *morphism* of G-homology theories is a $\mathbb{Z}$-indexed family of natural transformations that commute with the connecting homomorphisms.

Remark 6.8. Similar as for non-equivariant homology theories, we can draw some immediate consequences from the defining properties of a G-homology theory.

- Since a wedge of two points (with trivial G-action) is one point, the wedge axiom implies that the sum map $E_k(*) \oplus E_k(*) \longrightarrow E_k(*)$ is an isomorphism. This forces $E_k(*)$ to be trivial for all k, i.e., a G-homology theory is 'reduced'.
- The homotopy invariance implies that each functor E_k takes based G-homotopy equivalences to isomorphisms.
- For the unique G-map $p_A : A \longrightarrow *$ to a one point space the reduced mapping cone $C(p_A)$ is G-homeomorphic to the suspension $S^1 \wedge A$. In this case the connecting homomorphism thus specializes to a *suspension isomorphism*

$$E_{1+k}(S^1 \wedge A) \ \cong \ E_{1+k}(C(p_A)) \ \xrightarrow{\partial} \ E_k(A) \ .$$

- We let (B, A) be a pair of G-spaces, based at a point in A, such that the inclusion $i : A \longrightarrow B$ has the equivariant homotopy extension property. For example, this is the case for relative G-CW-complexes, or more generally for relative G-cell complexes. Then the quotient map $Ci \longrightarrow B/A$ that collapses the cone of A is a based G-homotopy equivalence, and thus induces isomorphisms in any G-homology theory. We can thus substitute $E_*(Ci)$ by $E_*(B/A)$ and obtain a long exact sequence

$$\cdots \ \xrightarrow{E_{1+k}(q)} \ E_{1+k}(B/A) \ \longrightarrow \ E_k(A) \ \xrightarrow{E_k(i)} \ E_k(B) \ \xrightarrow{E_k(q)} \ E_k(B/A) \ \longrightarrow \ \cdots$$

where $q : B \longrightarrow B/A$ is the projection.

- In the non-equivariant context, generalized homology theories are determined by their coefficient groups, i.e., the homology groups of a one-point space. In the equivariant context, the role of the one-point space is played by the discrete coset spaces G/H for all subgroups $H \leq G$. More precisely, we let $\Phi : E \longrightarrow F$ be a natural transformation of G-homology theories and suppose that for all $H \leq G$ and all integers k the map
$$\Phi_k(G/H_+) \ : \ E_k(G/H_+) \ \longrightarrow \ F_k(G/H_+)$$
is an isomorphism. Then the map $\Phi_k(A) : E_k(A) \longrightarrow F_k(A)$ is an isomorphism for every based G-CW-complex A and all integers k. The proof is similar as in the non-equivariant case. So in this sense G-homology theories are determined by the graded coefficient system $\{E_*(G/H_+)\}_{H\leq G}$.

Example 6.9. Every orthogonal G-spectrum E defines a G-homology theory by setting

$$E_k(A) \ = \ \pi_k^G(A \wedge E) \ . \tag{6.10}$$

Indeed, homotopy invariance is clear and the wedge axiom follows from Corollary 3.22 (i) and the fact that smashing with E preserves wedges. Since $- \wedge E$ takes space level mapping cones to mapping cones of orthogonal G-spectra, the connecting homomorphism (3.20) for the morphism $f \wedge E : A \wedge E \longrightarrow B \wedge E$ of orthogonal G-spectra and the isomorphism $C(f \wedge E) \cong (Cf) \wedge E$ together provide the connecting homomorphism
$$E_{1+k}(Cf) = \pi_{1+k}^G((Cf) \wedge E) \ \longrightarrow \ \pi_k^G(A \wedge E) = E_k(A) \ .$$
The long exact sequence is then a special case of the long exact sequence of a mapping cone (Proposition 3.21).

More G-homology theories can be obtained from orthogonal G-spectra by replacing $\pi_k^G(-)$ in (6.10) by the equivariant homotopy groups based on a G-universe that is not necessarily complete, compare Remark 4.24. The G-homology theories arising as in (6.10) by using complete universes have a special properties of being '$RO(G)$-gradable'.

Here are two specific examples of this construction. For $E = \mathbb{S}$ the sphere spectrum the associated G-homology theory is $E_k(A) = \pi_k^G(A \wedge \mathbb{S}) \cong \pi_k^G(\Sigma^\infty A)$, the equivariant stable homotopy groups of A. Given a $\mathbb{Z}G$-module M, the Eilenberg-Mac Lane spectrum HM was defined in Example 2.13. For $E = HM$ the associated G-homology theory $(HM)_k(A) = \pi_k^G(A \wedge HM)$ is isomorphic to $H_k(A, \underline{M})$, the Bredon homology [3] for the fixed point coefficient system $\underline{M}$ that sends a subgroup $H \leq G$ to M^H.

Since both sides of the tom Dieck splitting are G-homology theories, one could hope to prove it by reduction to the case $A = G/H_+$ of orbits. However, that is *not* the strategy of tom Dieck's proof; rather, we use an 'isotropy separation' argument to reduce the theorem to the special case of a G-space that is concentrated at a single conjugacy class of subgroups, in which case the splitting has only one non-zero summand.

Proposition 6.11. *Let G be a finite group and $\Phi : E \longrightarrow F$ a natural transformation of G-homology theories. Suppose that $\Phi(A) : E_*(A) \longrightarrow F_*(A)$ is an isomorphism for all based G-spaces A that are concentrated at a single conjugacy class. Then $\Phi(A)$ is an isomorphism for all based G-spaces A.*

Proof. We choose a sequence of families of subgroups of G
$$\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \text{all subgroups}$$
such that $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \{(K_i)\}$ for some conjugacy class of subgroups (K_i) with $K_i \notin \mathcal{F}_{i-1}$; in particular, m is the number of conjugacy classes of subgroups, $\mathcal{F}_1 = \{e\}$ contains only the trivial subgroup, and $\mathcal{F}_{m-1}$ is the family of proper subgroups of G. We show by induction on i that the map $\Phi(A \wedge (E\mathcal{F}_i)_+) : E_*(A \wedge (E\mathcal{F}_i)_+) \longrightarrow F_*(A \wedge (E\mathcal{F}_i)_+)$ is an isomorphism for all based G-space A. Since $\mathcal{F}_m$ contains all subgroups of G, the space $E\mathcal{F}_m$ is G-equivariantly contractible, so the projection $A \wedge (E\mathcal{F}_m)_+ \longrightarrow A$ is a G-homotopy equivalence and the last case $i = m$ proves the proposition.

Since $\mathcal{F}_0$ is empty, $E\mathcal{F}_0$ is the empty G-space and $(E\mathcal{F}_0)_+$ is a point. Hence $A \wedge (E\mathcal{F}_0)_+$ is a single point, and this starts the induction. For $i \geq 1$ the universal property of $E\mathcal{F}_i$ provides a G-map

$$j \ : \ E\mathcal{F}_{i-1} \ \longrightarrow \ E\mathcal{F}_i \ ,$$

unique up to G-homotopy. Because $\mathcal{F}_i = \mathcal{F}_{i-1} \cup \{(K_i)\}$, the unreduced mapping cone $C(j_+ : (E\mathcal{F}_{i-1})_+ \longrightarrow (E\mathcal{F}_i)_+)$ is then concentrated at the conjugacy class (K_i). Hence the smash product

$$A \wedge C((E\mathcal{F}_{i-1})_+ \longrightarrow (E\mathcal{F}_i)_+) \ \cong \ C(A \wedge (E\mathcal{F}_{i-1})_+ \longrightarrow A \wedge (E\mathcal{F}_i)_+)$$

is also concentrated at the conjugacy class (K_i). The E-homology groups of $A \wedge (E\mathcal{F}_{i-1})_+$, $A \wedge (E\mathcal{F}_i)_+$ and $A \wedge C(j_+)$ are related by a long exact sequence, and similarly for the F-homology groups. The transformation Φ gives compatible maps between the two long exact sequences with isomorphisms at $A \wedge (E\mathcal{F}_{i-1})_+$ (by induction) and at $A \wedge C(j_+)$ (by hypothesis). The 5-lemma then proves the induction step. □

Now we finally state and prove the tom Dieck splitting.

Theorem 6.12 (Tom Dieck splitting)**.** *For every based G-space A, the map*

$$\sum_{(H)} \zeta_H \ : \ \bigoplus_{(H)} \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge A^H)) \ \longrightarrow \ \pi_*^G(\Sigma^\infty A)$$

is an isomorphism, where the sum runs over a set of representatives of all conjugacy classes of subgroups of G.

Proof. We start with the special case of a based G-space A that is concentrated at one conjugacy class (H). By Proposition 6.5 the summand indexed by (H) then maps isomorphically onto $\pi_*^G(\Sigma^\infty A)$, so we need to show that all other summands vanish for such A. We let K be a subgroup of G that is not conjugate to H and claim that for every WK-representation V the WK-space

$$(\Sigma^\infty(EWK_+ \wedge A^K))(V) \ = \ EWK_+ \wedge A^K \wedge S^V$$

is WK-equivariantly weakly contractible. Indeed, if L is a non-trivial subgroup of the Weyl group WK, then EWK has no L-fixed points and $(EWK_+)^L$ consists only of the basepoint. So in this case

$$(EWK_+ \wedge A^K \wedge S^V)^L \ = \ ((EWK)^L)_+ \wedge (A^K \wedge S^V)^L \ = \ *$$

consists of the basepoint only. On the other hand, for $L = e$ the trivial subgroup of WK, the space $EWK_+ \wedge A^K \wedge S^V$ is contractible because A^K is. This shows the claim that $(\Sigma^\infty(EWK_+ \wedge A^K))(V)$ is weakly WK-equivariantly contractible, and hence the stable homotopy groups $\pi_*^{WK}(EWK_+ \wedge A^K)$ vanish. Altogether this proves the special case of the tom Dieck splitting when A is concentrated at a single conjugacy class.

Now we deduce the general case. Taking H-fixed points takes G-homotopies to WH-homotopies and commutes with wedges and mapping cones; the same is true for $EWH_+ \wedge -$, so the functor

$$A \ \mapsto \ \pi_*^{WH}(EWH_+ \wedge A^H) \ ,$$

and hence the left hand side of the tom Dieck splitting, is a G-homology theory. The tom Dieck splitting map is thus a natural transformation between G-homology theories that is an isomorphism for all based G-spaces that are concentrated at a single conjugacy class. By Proposition 6.11, the transformation is then an isomorphism in general. □

Remark 6.13. The tom Dieck splitting is sometimes formulated in a different way, as an isomorphism

$$\bigoplus_{(H)} \pi_*(\Sigma^\infty(EWH_+ \wedge_{WH} A^H)) \ \cong \ \pi_*^G(\Sigma^\infty A) \ ;$$

here A should be a G-CW-complex, and on the left hand side, the WH-equivariant stable homotopy groups of $EWH_+ \wedge A^H$ have been replaced by the non-equivariant stable homotopy groups of the orbit space

$EWH_+ \wedge_{WH} A^H$. We explain in Remark 8.2 below how this isomorphism can be obtained by combining the tom Dieck splitting of Theorem 6.12 with the Adams isomorphism.

In the special case $A = S^0$ (with trivial G-action), the left hand side of the tom Dieck splitting involves the equivariant homotopy groups $\pi_*^{WH}(\Sigma_+^\infty EWH)$. We will identify this group in dimension 0. We let W be any finite group and EW a contractible free W-CW-complex. A chosen point $x \in EW$ determines a W-equivariant action map $a : W \longrightarrow EW$ with $a(\gamma) = \gamma x$. Since EW is path connected, the W-homotopy class of a is independent of the choice, hence so is the induced morphism of suspension spectra

$$\bar{a} \ : \ W \ltimes \mathbb{S} = \Sigma_+^\infty W \ \longrightarrow \ \Sigma_+^\infty EW \ .$$

Proposition 6.14. *For every finite group W the composite*

$$\pi_0(\mathbb{S}) \xrightarrow{\mathrm{Tr}_e^W} \pi_0^W(W \ltimes \mathbb{S}) \xrightarrow{\bar{a}_*} \pi_0^W(\Sigma_+^\infty EW)$$

is an isomorphism. In particular, the group $\pi_0^W(\Sigma_+^\infty EW)$ is free abelian of rank 1.

Proof. We show first that the homomorphism $a_* : \pi_0^W(W \ltimes \mathbb{S}) \longrightarrow \pi_0^W(\Sigma_+^\infty EW)$ is injective. To this end we consider the composite

$$\pi_0^e(\mathbb{S}) \xrightarrow{\mathrm{Tr}_e^W} \pi_0^W(W \ltimes \mathbb{S}) \xrightarrow{\bar{a}_*} \pi_0^W(\Sigma_+^\infty EW) \xrightarrow{(\Sigma_+^\infty q)_*} \pi_0^W(\mathbb{S}) \xrightarrow{\mathrm{res}_e^W} \pi_0^e(\mathbb{S}) \ ; \tag{6.15}$$

here $q : EW \longrightarrow *$ is the unique map. The composite $(\Sigma_+^\infty q) \circ \bar{a} : W \ltimes \mathbb{S} \longrightarrow \mathbb{S}$ is the action map, so

$$(\Sigma_+^\infty q)_* \circ \bar{a}_* \circ \mathrm{Tr}_e^W \ = \ \mathrm{act}_* \circ \mathrm{Tr}_e^W \ = \ \mathrm{tr}_e^W \ .$$

The double coset formula shows that

$$\mathrm{res}_e^W \circ \mathrm{tr}_e^W \ = \ \sum_{w \in W} w_\star \ ;$$

because W acts trivially on the sphere spectrum, each of the the conjugation morphisms $w_\star$ is the identity of $\pi_0^e(\mathbb{S})$. So we conclude that the composite (6.15) is multiplication by the order of the group W. Since the group $\pi_0^e(\mathbb{S})$ is free abelian of rank 1, multiplication on $|W|$ is injective. Since the external transfer Tr_e^W is an isomorphism, we have shown that the morphism $\bar{a}_*$ is injective.

To show that $\bar{a}_*$ is surjective, we use the bar construction model for EW, which is filtered by W-subcomplexes $E^{(i)}W$ with subquotients equivariantly homeomorphic to

$$E^{(i)}W/E^{(i-1)}W \ \cong \ W \ltimes (W^{\wedge i} \wedge S^i) \ ;$$

for the purposes of the smash product on the right hand side, W is viewed as a pointed set with basepoint 1. The Wirthmüller and suspension isomorphisms

$$\pi_0^W(W \ltimes (\Sigma^\infty(W^{\wedge i} \wedge S^i))) \ \cong \ \pi_0(\Sigma^\infty(W^{\wedge i} \wedge S^i)) \ \cong \ \pi_{-i}(\Sigma^\infty W^{\wedge i})$$

show that the 0-th W-equivariant stable homotopy groups of $E^{(i)}W/E^{(i-1)}W$ vanish for $i \geq 1$. Moreover, the sequence

$$\pi_0^W(\Sigma_+^\infty E^{(i-1)}W) \ \longrightarrow \ \pi_0^W(\Sigma_+^\infty E^{(i)}W) \ \longrightarrow \ \pi_0^W(\Sigma_+^\infty(E^{(i)}W/E^{(i-1)}W))$$

is exact, so we conclude that the inclusion $E^{(i-1)}W \longrightarrow E^{(i)}W$ induces an epimorphism on π_0^W for all $i \geq 1$. The canonical map

$$\mathrm{colim}_{i \geq 0} \ \pi_0^W(\Sigma_+^\infty E^{(i)}W) \ \longrightarrow \ \pi_0^W(\Sigma_+^\infty EW) \ .$$

is an isomorphism, so we deduce that also the inclusion $E^{(0)}W \longrightarrow EW$ induces an epimorphism

$$\pi_0^W(\Sigma_+^\infty E^{(0)}W) \ \cong \ \pi_0^W(\Sigma_+^\infty EW) \ .$$

The claim now follows from the observation that $E^{(0)}W$ is a discrete space with free and transitive W-action, so that its suspension spectrum $\Sigma_+^\infty E^{(0)}W$ is isomorphic to $W \ltimes \mathbb{S}$. $\square$

In the special case $A = S^0$, the tom Dieck splitting becomes an isomorphism between the group $\pi_0^G(\mathbb{S})$ and the direct sum

$$\bigoplus_{(H)} \pi_0^{WH}(\Sigma^\infty_+ EWH) \ .$$

In combination with Proposition 6.14 this shows that for every finite group G the group $\pi_0^G(\mathbb{S})$ is free abelian of rank the number of conjugacy classes of subgroups of G.

We recall that the *Burnside ring* $A(G)$ is the Grothendieck group, under direct sum, of isomorphism classes of finite G-sets; the ring structure is induced by product of G-sets. The additive group of the Burnside ring $A(G)$ is also free abelian of the same rank as the equivariant 0-stem $\pi_0^G(\mathbb{S})$, so these two groups are additively isomorphic. Even better, the Mackey functor structure of the equivariant homotopy groups provide a specific isomorphism, which is moreover natural for restriction along group homomorphisms. A preferred additive basis of $A(G)$ is given by the classes of the cosets G/H, where H runs through a set of representatives of the conjugacy classes of subgroups of G. We can thus define a homomorphism

$$\sigma_G \ : \ A(G) \ \longrightarrow \pi_0^G(\mathbb{S})$$

by sending the class $[G/H] \in A(G)$ to the element $\mathrm{tr}_H^G(1)$, the transfer of the unit element $1 \in \pi_0^H(\mathbb{S})$. According to the Construction 4.2, a representative G-map of the class $\mathrm{tr}_H^G(1)$ is given by the composite

$$S^{\rho_G} \ \xrightarrow{\ \mathrm{tr}\ } \ G \ltimes_H S^{\rho_G} \ \xrightarrow{\ \mathrm{act}\ } \ S^{\rho_G}$$

where the first map is the transfer map from the Thom-Pontryagin construction. The isomorphism $\sigma_G : A(G) \longrightarrow \pi_0^G(\mathbb{S})$ is also natural, in the technical sense of compatibility with restriction and transfer maps. In particular, the maps σ_H form an isomorphism of Mackey functors as H ranges over the subgroups of G. The following theorem is due to Segal [23].

Theorem 6.16. *For every finite group G the map $\sigma_G : A(G) \longrightarrow \pi_0^G(\mathbb{S})$ is an isomorphism of rings. As G varies, the isomorphisms σ_G commute with transfer and restriction along group homomorphisms.*

Proof. In order to show that σ_G is an isomorphism, we prove that it sends the preferred basis of the Burnside ring to the basis of $\pi_0^G(\mathbb{S})$ given by the tom Dieck splitting. We recall that the map ζ_H is the composition

$$\pi_0^{WH}(\Sigma^\infty_+ EWH) \ \xrightarrow{\ p^*\ } \ \pi_0^{NH}(\Sigma^\infty_+ EWH) \ \xrightarrow{\ \mathrm{Tr}_{NH}^G\ } \ \pi_0^G(\Sigma^\infty_+(G \times_{NH} EWH)) \ \xrightarrow{\ q_*\ } \ \pi_0^G(\mathbb{S}) \ ,$$

where $p : NH \longrightarrow WH$ is the projection and q is induced by the unique map $G \times_{NH} EWH \longrightarrow *$. Naturality of the external transfer lets us identify the map ζ_H with the composite

$$\pi_0^{WH}(\Sigma^\infty_+ EWH) \ \xrightarrow{\ p^*\ } \ \pi_0^{NH}(\Sigma^\infty_+ EWH) \ \xrightarrow{\ q'_*\ } \ \pi_0^{NH}(\mathbb{S}) \ \xrightarrow{\ \mathrm{Tr}_{NH}^G\ } \ \pi_0^G(G \ltimes_{NH} \mathbb{S}) \ \xrightarrow{\ \mathrm{proj}\ } \ \pi_0^G(\mathbb{S}) \ ,$$

and hence, by naturality of the restriction homomorphism p^*, with the composite

$$\pi_0^{WH}(\Sigma^\infty_+ EWH) \ \xrightarrow{\ q'_*\ } \ \pi_0^{WH}(\mathbb{S}) \ \xrightarrow{\ p^*\ } \ \pi_0^{NH}(\mathbb{S}) \ \xrightarrow{\ \mathrm{tr}_{NH}^G\ } \ \pi_0^G(\mathbb{S}) \ ;$$

here q' is the morphism induced by the unique map $EWH \longrightarrow *$. Evaluating this on the generator $\bar{a}_*(\mathrm{Tr}_e^{WH}(1))$ of the group $\pi_0^{WH}(\Sigma^\infty_+ EWH)$ gives

$$\begin{aligned} \zeta_H(\bar{a}_*(\mathrm{Tr}_e^{WH}(1))) \ &= \ \mathrm{tr}_{NH}^G(p^*(\bar{q}_*(\mathrm{Tr}_e^{WH}(1)))) \\ &= \ \mathrm{tr}_{NH}^G(\bar{q}_*(p^*(\mathrm{Tr}_e^{WH}(1)))) \ = \ \mathrm{tr}_{NH}^G(\mathrm{tr}_H^{NH}(p_H^*(1))) \ = \ \mathrm{tr}_H^G(1) \ . \end{aligned}$$

Here $\bar{q} = q' \circ \bar{a} : WH \ltimes \mathbb{S} \longrightarrow \mathbb{S}$ is the projection and $p_H : H \longrightarrow e$ is the unique map, which happens to be the restriction of the projection $p : NH \longrightarrow WH$ to H. The second equation is the naturality of restriction maps and the third equation is the compatibility of transfers with restriction along epimorphisms (compare Proposition 4.18 (ii)). By Proposition 6.14 and the tom Dieck splitting the classes $\zeta_H(\bar{a}_*(\mathrm{Tr}_e^{WH}(1)))$ form a basis of $\pi_0^G(\mathbb{S})$ when H ranges over representative of the conjugacy classes of subgroups of G. So this finishes the identification of the Burnside ring $A(G)$ with $\pi_0^G(\mathbb{S})$ as abelian groups.

Now we check compatibility of the isomorphisms σ_G with restriction along group homomorphisms. Every group homomorphism is the composite of an epimorphism followed by a subgroup inclusion. So we show compatibility with these two types of maps separately. We start with an epimorphism $\alpha : K \longrightarrow G$. The restriction homomorphism $\alpha^* : A(G) \longrightarrow A(K)$ sends the class of G/H to the class of $\alpha^*(G/H)$, which is K-isomorphic to K/L. Using Proposition 4.18 (ii) for $X = \mathbb{S}$ we deduce that that

$$\alpha^*(\sigma_G[G/H]) \ = \ \alpha^*(\mathrm{tr}_H^G(1)) \ = \ \mathrm{tr}_L^K((\alpha|_L)^*(1)) \ = \ \mathrm{tr}_L^K(1) \ = \sigma_K[K/L] \ = \ \sigma_K(\alpha^*[G/H]) \ .$$

Hence the homomorphisms σ_G are compatible with restriction along epimorphisms.

The compatibility with restriction to a subgroup $K \leq G$ follows for the fact that both sides satisfy a double coset formula. Indeed, for every $g \in G$, the left K-set $(KgH)/H$ is K-isomorphic to $K/K \cap {}^gH$, via

$$K/K \cap {}^gH \ \longrightarrow \ (KgH)/H \ , \quad k \cdot (K \cap {}^gH) \ \longmapsto \ kgH \ .$$

Hence the underlying K-set of G/H is isomorphic to

$$\mathrm{res}_K^G(G/H) \ = \ \coprod_{[g]\in K\backslash G/H} \mathrm{res}_K^G((KgH)/H) \ \cong \ \coprod_{[g]\in K\backslash G/H} K/K \cap {}^gH$$

(which is effectively the proof of the double coset formula for the Burnside ring Mackey functor). Thus we get

$$\begin{aligned} \mathrm{res}_K^G(\sigma_G[G/H]) \ &= \ \mathrm{res}_K^G(\mathrm{tr}_H^G(1)) \ = \ \sum_{[g]\in K\backslash G/H} \mathrm{tr}_{K\cap {}^gH}^K(c_g^*(\mathrm{res}_{K^g\cap H}^H(1))) \\ &= \sum_{[g]\in K\backslash G/H} \mathrm{tr}_{K\cap {}^gH}^K(1) \ = \ \sum_{[g]\in K\backslash G/H} \sigma_K[K/K\cap {}^gH] \ = \ \sigma_K(\mathrm{res}_K^G[G/H]) \ . \end{aligned}$$

Compatibility with transfers is a consequence of the transitivity of transfers in $\underline{\pi}_0(\mathbb{S})$ (see Proposition 4.17) and in the Burnside rings. Indeed, for $K \leq H \leq G$ we have

$$\mathrm{tr}_H^G(\sigma_H[H/K]) \ = \ \mathrm{tr}_H^G(\mathrm{tr}_K^H(1)) \ = \ \mathrm{tr}_K^G(1) \ = \ \sigma_G[G/K] \ = \ \sigma_G(\mathrm{tr}_K^G[G/K]) \ .$$

Multiplicativity of σ_G is a formal consequence of the compatibility with restriction and transfer and the fact that the multiplication on both sides of σ_G satisfies reciprocity. Indeed, since σ_G is additive, it suffices to check multiplicativity for products of two basis elements. So we let H and K be two subgroups of G. Then

$$\begin{aligned} \sigma_G[G/H]\cdot\sigma_G[G/K] \ &= \ \mathrm{tr}_H^G(1)\cdot\mathrm{tr}_K^G(1) \ =_{(4.30)} \ \mathrm{tr}_H^G(\mathrm{res}_H^G(\mathrm{tr}_K^G(1))) \\ &= \ \mathrm{tr}_H^G(\mathrm{res}_H^G(\sigma_G[G/K])) \ = \ \sigma_G(\mathrm{tr}_H^G(\mathrm{res}_H^G[G/K])) \ = \ \sigma_G([G/H]\cdot[G/K]) \ . \qquad \square \end{aligned}$$

Having identified the ring $\pi_0^G(\mathbb{S})$ one can ask how its elements can be distinguished by invariants. In the non-equivariant context the degree of a map between spheres provides the answer, and in the equivariant context the collection of degrees of all fixed point maps serves the same purpose. The situation is slightly more subtle, though, because the fixed point degrees of an equivariant map between representation spheres satisfy certain congruences, so they cannot be assigned arbitrarily.

We let $C(G)$ denote the set of *class functions*, i.e., maps from the set of subgroups of G to the integers that are constant on conjugacy classes. Every G-map $f : S^V \longrightarrow S^V$ gives rise to a *degree function* $d(f) \in C(G)$ by

$$d(f)(K) \ = \ \deg\left(f^K : S^{V^K} \longrightarrow S^{V^K}\right) \ ,$$

the (non-equivariant) degree of the K-fixed point map. The degree function only depends on the G-homotopy class of f and is invariant under suspension with any representation sphere. So it descends to a map

$$d \ : \ \pi_0^G(\mathbb{S}) \ \longrightarrow \ C(G) \ .$$

Since the map $\sigma_G : A(G) \longrightarrow \pi_0^G(\mathbb{S})$ is a ring isomorphism, one can understand the degree map by studying the composite $d \circ \sigma_G : A(G) \longrightarrow C(G)$, and this turns out to be a purely algebraic issue. Indeed, the degree function associated to $\sigma_G[G/H] = \mathrm{tr}_H^G(1)$ assigns to the conjugacy class of K the cardinality of the set $(G/H)^K$. So the composite map

$$A(G) \xrightarrow{\sigma_G} \pi_0^G(\mathbb{S}) \xrightarrow{d} C(G)$$

sends a finite G-set S to the function

$$(d(\sigma_G[S]))(K) \;=\; |S^K|$$

that counts the number of fixed points. By pure algebra (see for example [5, Sec. 1.2]), the fixed point counting map $d \circ \sigma_G$, and hence also degree map d, is injective, and its image has finite index, namely the product, over conjugacy classes of subgroups (H), of the orders of the Weyl groups $W_G H$.

The image of the degree map $d : \pi_0^G(\mathbb{S}) \longrightarrow C(G)$, or equivalently the image of $d\circ\sigma_G$, can be characterized in terms of certain explicit congruences. The example $G = C_p$, the cyclic group of order a prime p, can serve to illustrate the issue. The Burnside ring $A(C_p)$ is free abelian of rank 2 generated by the classes of the trivial C_p-sets C_p/C_p and the free C_p-set C_p/e. We have

$$(|C_p/C_p|,\ |(C_p/C_p)^{C_p}|) = (1, 1) \qquad \text{and} \qquad (|C_p/e|,\ |(C_p/e)^{C_p}|) = (p, 0)\ .$$

Every finite C_p-set S is isomorphic to a union of copies of C_p/C_p and C_p/e, so the relation

$$|S| \;\equiv\; |S^{C_p}| \quad \text{modulo } p$$

holds for all finite C_p-sets S. In general, the image of the ring homomorphism $d \circ \sigma_G : A(G) \longrightarrow C(G)$ is the subring of those class functions $\varphi \in C(G)$ that satisfy the congruence

$$\sum_{K \trianglelefteq (H) \le N_G K,\ H/K \text{ cyclic}} \mu(H/K) \cdot |N_G H/N_G K \cap N_G H| \cdot \varphi(H) \;\equiv\; 0 \quad \mathrm{mod}\ |W_G H| \tag{6.17}$$

for every subgroup K of G, see for example [5, Prop. 1.3.5]. The sum is taken over $N_G K$-conjugacy classes of subgroups $H \le G$ that contain K as a normal subgroup and such that H/K is cyclic; $\mu(H/K)$ is the number of generators of the cyclic group H/K.

Example 6.18. We illustrate the congruences (6.17) with the example $G = \Sigma_3$ of the symmetric group on 3 letters. There are four conjugacy classes of subgroups, namely

$$e,\ (12),\ A_3 \quad \text{and} \quad \Sigma_3\ .$$

So $A(\Sigma_3)$ and $C(\Sigma_3)$ are free abelian of rank 4, and the index of the monomorphism $d \circ \sigma_{\Sigma_3} : A(\Sigma_3) \longrightarrow C(\Sigma_3)$ is the product of the orders of the Weyl groups, so it is $6 \cdot 1 \cdot 2 \cdot 1 = 12$.

A priori, we get four congruences (6.17), one for each conjugacy class of subgroups, for a class function $\varphi \in C(\Sigma_3)$ to lie in the image. However, the subgroups (12) and Σ_3 are self-normalizing, so their Weyl groups are trivial, and the respective congruence contains no information. For $K = e$, the sum (6.17) is over all conjugacy classes of cyclic subgroups of Σ_3, and becomes

$$1 \cdot \varphi(e) \;+\; 1 \cdot \varphi((12)) \;+\; 2 \cdot \varphi(A_3) \;\;\equiv\;\; 0 \quad \mathrm{mod}\ 6\ .$$

For $K = A_3$, the sum (6.17) has two summands with $H = A_3$ and $H = \Sigma_3$, and it becomes

$$1 \cdot \varphi(A_3) \;+\; 1 \cdot \varphi(\Sigma_3) \;\;\equiv\;\; 0 \quad \mathrm{mod}\ 2\ .$$

These two congruences are equivalent to the three more basic congruences

$$\varphi(e) \equiv \varphi(A_3) \quad \mathrm{mod}\ 3\ , \quad \varphi(e) \equiv \varphi((12)) \quad \mathrm{mod}\ 2\ , \qquad \text{and} \qquad \varphi(A_3) \equiv \varphi(\Sigma_3) \quad \mathrm{mod}\ 2\ .$$

The group Σ_3 is simple enough that one could verify the congruences directly: the following so-called 'table of marks' lists the numbers of fixed points $|(G/H)^K|$ of the transitive Σ_3-sets, and one can read off the three congruences between the numbers in the respective columns:

H	$\lvert(\Sigma_3/H)^e\rvert$	$\lvert(\Sigma_3/H)^{(12)}\rvert$	$\lvert(\Sigma_3/H)^{A_3}\rvert$	$\lvert(\Sigma_3/H)^{\Sigma_3}$
e	6	0	0	0
(12)	3	1	0	0
A_3	2	0	2	0
Σ_3	1	1	1	1

7. Fixed points and geometric fixed points

In this section we investigate different kinds of fixed point spectra for orthogonal G-spectra. Each of these constructions turns an equivariant spectrum into a non-equivariant spectrum by taking fixed points at an appropriate stage.

7.1. **Naive fixed points.** We start with the *naive fixed points* of a G-spectrum X that we denote by X^G and which are simply the categorical fixed points taken levelwise. In other words, we have

$$(X^G)(V) \ = \ X(V)^G \ ,$$

the G-fixed points of the V-th level, with restricted $O(V)$-action. Since the structure maps $\sigma(V) : X(V) \wedge S^W \longrightarrow X(V \oplus W)$ are G-equivariant for the trivial G-action on S^W, they restrict to structure maps

$$\sigma^{X^G}_{V,W} \ : \ X(V)^G \wedge S^W \ = \ (X(V) \wedge S^W)^G \ \xrightarrow{(\sigma_{V,W})^G} \ X(V \oplus W)^G$$

for the naive fixed point spectrum X^G.

One problem with the naive fixed point construction is that it is not homotopy invariant. More precisely, if $f : X \longrightarrow Y$ is a $\underline{\pi}_*$-isomorphism of G-spectra, then the induced map $f^G : X^G \longrightarrow Y^G$ is generally not a π_*-isomorphism of non-equivariant orthogonal spectra. However, naive fixed points take level G-equivalences to level equivalences, hence they take $\underline{\pi}_*$-isomorphism between G-Ω-spectra (which are level G-equivalences) to level equivalences of orthogonal spectra. Thus the naive fixed point functor can be derived by applying it to a $\underline{\pi}_*$-isomorphic replacement by a G-Ω-spectrum.

7.2. **Fixed points.** Fortunately, there is a simpler and explicit construction that achieves the same goal. Given a G-spectrum X we define a new G-spectrum FX by

$$(FX)(V) \ = \ \mathrm{map}(S^{V\otimes\bar{\rho}_G}, X(V\otimes\rho_G)) \ ,$$

where $\otimes$ is short for $\otimes_{\mathbb{R}}$, the tensor product of G-representations. Here the source of the mapping space uses the *reduced* regular representation $\bar{\rho}_G$, whereas the target uses the regular representation ρ_G. As usual, the group G acts on the mapping space by conjugation. The orthogonal group $O(V)$ acts on $(FX)(V)$ by conjugation, through the actions on V. The structure map $(FX)(V) \wedge S^W \longrightarrow (FX)(V \oplus W)$ is the composite

$$\begin{aligned}
\mathrm{map}(S^{V\otimes\bar{\rho}_G}, X(V\otimes\rho_G)) \wedge S^W \ &\xrightarrow{\text{assembly}} \ \mathrm{map}(S^{V\otimes\bar{\rho}_G}, X(V\otimes\rho_G) \wedge S^W) \\
&\xrightarrow{-\wedge S^{W\otimes\bar{\rho}_G}} \ \mathrm{map}(S^{V\otimes\bar{\rho}_G} \wedge S^{W\otimes\bar{\rho}_G}, X(V\otimes\rho_G) \wedge S^W \wedge S^{W\otimes\bar{\rho}_G}) \\
&\cong \ \mathrm{map}(S^{(V\oplus W)\otimes\bar{\rho}_G}, X(V\otimes\rho_G) \wedge S^{W\otimes\rho_G}) \\
&\xrightarrow{(\sigma_{V\otimes\rho_G, W\otimes\rho_G})_*} \mathrm{map}(S^{(V\oplus W)\otimes\bar{\rho}_G}, X((V\oplus W)\otimes\rho_G))
\end{aligned}$$

where among other things we have used the G-equivariant isometry $\mathbb{R} \oplus \bar{\rho}_G \cong \rho_G$.

Definition 7.1. The *fixed point spectrum* of an orthogonal G-spectrum X is the orthogonal spectrum $F^G X = (FX)^G$, the naive fixed points of the spectrum FX.

As we shall see now, the homotopy groups of the fixed point spectrum $F^G X$ calculate the G-homotopy groups of X:

Proposition 7.2. *For every orthogonal G-spectrum X and integer k the groups $\pi_k^G(X)$ and $\pi_k(F^G X)$ are naturally isomorphic.*

Proof. We restrict to the case $k = 0$. The splitting $\rho_G \cong \mathbb{R} \oplus \bar\rho_G$ produces an isometry $\mathbb{R}^n \otimes \rho_G \cong \mathbb{R}^n \oplus (\mathbb{R}^n \otimes \bar\rho_G)$ that compactifies to a G-homeomorphism $S^{\mathbb{R}^n\otimes\rho_G} \cong S^n \wedge S^{\mathbb{R}^n\otimes\bar\rho_G}$. So we get an adjunction bijection

$$\begin{aligned}[S^{\mathbb{R}^n\otimes\rho_G}, X(\mathbb{R}^n\otimes\rho_G)]^G \ &\cong\ [S^n, \mathrm{map}(S^{\mathbb{R}^n\otimes\bar\rho_G}, X(\mathbb{R}^n\otimes\rho_G))]^G \\ &\cong\ \pi_n \,\mathrm{map}^G(S^{\mathbb{R}^n\otimes\bar\rho_G}, X(\mathbb{R}^n\otimes\rho_G)) \ = \ \pi_n((F^G X)(\mathbb{R}^n))\ .\end{aligned}$$

The second isomorphism uses that G acts trivially on $\mathbb{R}^n$. The bijection is compatible with the stabilization maps that define $\pi_0^G(X)$ from the groups $[S^{n\rho_G}, X(n\rho_G)]^G$ respectively $\pi_0(F^G X)$ from the groups $\pi_n((F^G X)(\mathbb{R}^n))$. □

The naive fixed points and fixed points of an equivariant spectrum are related by a map

$$X^G \xrightarrow{\ j^G\ } F^G X \tag{7.3}$$

that is obtained from a morphism $j : X \longrightarrow FX$ of G-spectra by taking naive fixed points. The V-th level

$$j(V) \ :\ X(V) \ \longrightarrow\ \mathrm{map}(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) = (FX)(V)\ .$$

is adjoint to the G-map

$$X(V)\wedge S^{V\otimes\bar\rho_G} \ \xrightarrow{\sigma_{V,V\otimes\bar\rho_G}} X(V\oplus(V\otimes\bar\rho_G)) \ \cong\ \ X(V\otimes\rho_G)\ .$$

For every Ω-G-spectrum X the morphism $j : X \longrightarrow FX$ is thus a strong level equivalence, so it induces a level equivalence $j^G : X^G \longrightarrow F^G X$ of non-equivariant orthogonal spectra. Since the functor $X \mapsto F^G X$ also takes $\underline{\pi}_*$-isomorphisms of G-spectra to π_*-isomorphisms (by Proposition 7.2), the fixed point functor $F^G X$ is really a right derived functor of the naive fixed points.

7.3. **Geometric fixed points.** Now we discuss another fixed point construction, the *geometric fixed points* $\Phi^G X$ of a G-spectrum X. It is given by

$$(\Phi^G X)(V) \ =\ X(V\otimes\rho_G)^G\ ,$$

the G-fixed points of the value of X on the tensor product of V with the regular representation. The orthogonal group $O(V)$ acts through V. The structure map $(\Phi^G X)(V)\wedge S^W \longrightarrow (\Phi^G X)(V\oplus W)$ is the map

$$\begin{aligned}X(V\otimes\rho_G)^G\wedge S^W \ &\cong\ \big(X(V\otimes\rho_G)\wedge S^{W\otimes\rho_G}\big)^G \\ &\xrightarrow{(\sigma_{V\otimes\rho_G,W\otimes\rho_G})^G}\ X((V\otimes\rho_G)\oplus(W\otimes\rho_G))^G \cong X((V\oplus W)\otimes\rho_G)^G\end{aligned}$$

using the identification $(\rho_G)^G \cong \mathbb{R}$.

The fixed points and geometric fixed points are related by a natural map

$$F^G X \ \xrightarrow{\ \mathrm{ev}\ }\ \Phi^G X$$

of orthogonal spectra. In level V, the map

$$(F^G X)(V) = \mathrm{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) \ \longrightarrow\ X(V\otimes\rho_G)^G \ =\ (\Phi^G X)(V)$$

evaluates a G-map $S^{V\otimes\bar\rho_G} \longrightarrow X(V\otimes\rho_G)$ at the G-fixed point $0\in S^{V\otimes\bar\rho_G}$ (which is the unique G-fixed point of $S^{V\otimes\bar\rho_G}$ other than the basepoint ∞).

The geometric fixed point construction comes with a geometric fixed point map of homotopy groups. For an orthogonal G-spectrum X and a G-representation W the *geometric fixed point map*

$$\Phi^G \ :\ \pi^G_{k+W}(X) \ \longrightarrow\ \pi_{k+W^G}(\Phi^G X) \tag{7.4}$$

is defined by sending the class represented by a G-map $f : S^{k+W+V\otimes\rho_G} \longrightarrow X(V\otimes\rho_G)$ to the class of the fixed point map

$$f^G \ : \ S^{k+W^G+V} \cong \left(S^{k+W+V\otimes\rho_G}\right)^G \ \longrightarrow \ X(V\otimes\rho_G)^G \ = \ (\Phi^G X)(V) \ .$$

We have implicitly identified the fixed points $(V\otimes\rho_G)^G$ with V. If we stabilize f by the regular representation we have $(f\diamond\rho_G)^G = f^G\diamond\mathbb{R}$, so this really gives a well-defined map on $\pi^G_{k+W}(X)$.

Now we can given another interpretation of the geometric fixed points $\Phi^G X$ as the fixed point of the smash product of X with a certain universal G-space. We denote by $E\mathcal{P}$ a universal space for the family of proper subgroups of G. So $E\mathcal{P}$ is a G-CW-complex such that the G-fixed points $(E\mathcal{P})^G$ are empty and $(E\mathcal{P})^H$ is contractible for every proper subgroup H of G. These properties determine $E\mathcal{P}$ uniquely up to G-homotopy equivalence.

We denote by $\tilde E\mathcal{P}$ the reduced mapping cone of the based G-map $E\mathcal{P}_+ \longrightarrow S^0$ that sends $E\mathcal{P}$ to the non-basepoint of S^0. So $\tilde E\mathcal{P}$ is the unreduced suspension of the universal space $E\mathcal{P}$. Fixed points commute with mapping cones, so the map $S^0 \longrightarrow (\tilde E\mathcal{P})^G$ is an isomorphism. For proper subgroups H of G the map $(E\mathcal{P})^H \longrightarrow (S^0)^H = S^0$ is a weak equivalence, so the mapping cone $(\tilde E\mathcal{P})^H$ is contractible. This means that the G-space $\tilde E\mathcal{P}$ is concentrated at the group G, in the sense of Definition 6.3; the smash product of $\tilde E\mathcal{P}$ with any based G-space is also concentrated at G.

For example, the reduced regular representation $\bar\rho_G$ has no non-trivial G-fixed points, but $(\bar\rho_G)^H \neq 0$ for all proper subgroups H of G. So by Example 5.20, the infinite dimensional unit sphere $S(\infty\bar\rho_G)$ can serve as the space $E\mathcal{P}$. The 'infinite representation sphere' $S^{\infty\bar\rho} = \bigcup_n S^{n\bar\rho}$ is thus a model for the space $\tilde E\mathcal{P}$.

The inclusion $S^0 \longrightarrow \tilde E\mathcal{P}$ induces an isomorphism of G-fixed points $S^0 \cong (\tilde E\mathcal{P})^G$. So for every based G-space A the map $0\wedge - : A \longrightarrow \tilde E\mathcal{P}\wedge A$ induces an isomorphism of G-fixed points. Hence also for every G-spectrum the induced map of geometric fixed points

$$\Phi^G(X) \ \cong \ \Phi^G(\tilde E\mathcal{P}\wedge X)$$

is an isomorphism. If we compose the evaluation morphism $\mathrm{ev} : F^G(\tilde E\mathcal{P}\wedge X) \longrightarrow \Phi^G(\tilde E\mathcal{P}\wedge X)$ with this isomorphism we obtain a natural map

$$F^G(\tilde E\mathcal{P}\wedge X) \ \xrightarrow{\ \mathrm{ev}\ } \ \Phi^G X \tag{7.5}$$

that we still refer to as 'evaluation at 0'. An application of Proposition 6.4 (with $G = H$, $Y = \tilde E\mathcal{P}\wedge X(n\rho_G)$ and $A = S^{n\bar\rho_G}$) to the G-spaces $A = S^{n\bar\rho_G}$ and $Y = \tilde E\mathcal{P}\wedge X(n\rho_G)$ yields that the evaluation at 0 (7.5) is a Serre fibration and weak equivalence in every level n:

Proposition 7.6. *For every orthogonal G-spectrum X the natural morphism*

$$F^G(\tilde E\mathcal{P}\wedge X) \ \xrightarrow{\ \mathrm{ev}\ } \ \Phi^G X$$

is a level equivalence and level fibration of orthogonal spectra. Hence the geometric fixed point functor takes $\underline{\pi}_*$-isomorphisms of G-spectra to π_*-isomorphisms of non-equivariant spectra.

A consequence of the previous proposition is the following *isotropy separation sequence*. The mapping cone sequence of based G-CW-complexes

$$E\mathcal{P}_+ \ \longrightarrow \ S^0 \ \longrightarrow \ \tilde E\mathcal{P}$$

becomes a mapping cone sequence of G-spectra after smashing with any given G-spectrum X. Taking G-fixed points gives a homotopy cofiber sequence of non-equivariant spectra; after replacing the term $F^G(\tilde E\mathcal{P}\wedge X)$ by the level equivalence spectrum $\Phi^G X$, we obtain a homotopy cofiber sequence of orthogonal spectra

$$F^G(E\mathcal{P}_+\wedge X) \ \longrightarrow \ F^G X \ \longrightarrow \ \Phi^G X \ .$$

Example 7.7 (Fixed points of suspension spectra)**.** We discuss fixed points and geometric fixed points for equivariant suspension spectra in more detail. If A is a based G-space, then $(\Sigma^\infty A)^G = \Sigma^\infty A^G$. The geometric fixed points $\Phi^G(\Sigma^\infty A)$ are also isomorphic to the suspension spectrum $\Sigma^\infty A^G$, using the identification of $(\rho_G)^G$ with $\mathbb{R}$ and the induced identification

$$(\Phi^G(\Sigma^\infty A))(V) \ = \ (A \wedge S^{V\otimes\rho_G})^G \ \cong \ A^G \wedge (S^{V\otimes\rho_G})^G \ \cong \ A^G \wedge S^V \ .$$

The composite map

$$(\Sigma^\infty A)^G \ \xrightarrow{\ j^G\ } \ F^G(\Sigma^\infty A) \ \xrightarrow{\ \text{ev}\ } \ \Phi^G(\Sigma^\infty A)$$

from naive to geometric fixed points is an isomorphism. Moreover, the effect on (non-equivariant) homotopy groups of the morphism j^G is (naturally isomorphic to) the direct summand inclusion

$$\zeta_G \ : \ \pi_*(\Sigma^\infty(A^G)) \ \longrightarrow \ \pi_*^G(\Sigma^\infty A)$$

in the tom Dieck splitting indexed by the group G, compare (6.2). So for suspension spectra, the geometric fixed point map splits off the summand indexed by the group G in the tom Dieck splitting.

So the fixed point spectrum $F^G(\Sigma^\infty A)$ contains the suspension spectrum of A^G as a summand. However, the fixed point spectrum typically has extra summands, that can be identified via the tom Dieck splitting that gives a π_*-isomorphism

$$F^G(\Sigma^\infty A) \ \simeq \ \prod_{(H)} \Sigma^\infty \left(EW(H)_+ \wedge_{W(H)} A^H\right)$$

of orthogonal spectra.

Example 7.8 (Fixed points of free spectra)**.** We discuss fixed points and geometric fixed points for the free equivariant spectrum F_V generated by a G-representation V. For naive fixed points we have

$$((F_V)(W))^G \ = \ \mathbf{O}(V,W)^G \ = \ \begin{cases} \mathbf{O}(V,W) & \text{if } G \text{ acts trivially on } V\text{, and} \\ \qquad * & \text{if } G \text{ acts non-trivially on } V. \end{cases}$$

Indeed, $\mathbf{O}(V,W)^G$ is the Thom space of pairs (α,x) where α is a G-equivariant linear isometric embedding and x is a G-fixed vector in W orthogonal to $\alpha(V)$. The G-action on W is trivial, so if G acts non-trivially on V, then there are no such equivariant embeddings. In other words, the naive fixed point spectrum $(F_V)^G$ is trivial for non-trivial representation of G, and it is isomorphic to the free orthogonal spectrum F_V whenever V is a trivial representation.

The geometric fixed points $\Phi^G(F_V)$ are a sphere spectrum of dimension minus the dimension of the fixed points V^G. Indeed, we can apply geometric fixed points to the $\underline{\pi}_*$-isomorphism (by Proposition 5.12) $F_V S^V \longrightarrow \mathbb{S}$ adjoint to the identity of S^V. We obtain π_*-isomorphism

$$\Phi^G(F_V) \wedge S^{V^G} \ \cong \ \Phi^G(F_V S^V) \ \xrightarrow{\ \simeq\ } \ \Phi^G(\mathbb{S}) \cong \mathbb{S} \ .$$

Hence the adjoint of this map is a π_*-isomorphism

$$\Phi^G(F_V) \ \xrightarrow{\ \simeq\ } \ \Omega^{V^G}\mathbb{S} \ .$$

Example 7.9 (Fixed points of coinduced spectra)**.** We determine the naive and geometric fixed point functor on coinduced spectra $\text{map}^H(G,Y)$ for orthogonal H-spectra Y. The naive fixed points are given by $(\text{map}^H(G,Y))^G \cong Y^H$. For fixed points there is a stable equivalence

$$R^G_H \ : \ F^H Y \ \to \ F^G\left(\text{map}^H(G,Y)\right)$$

defined as follows. In level V, the map $(R^G_H)_n$ is the composite

$$\begin{aligned} (F^H Y)(V) \ &= \ \text{map}^H(S^{V\otimes\bar\rho_H}, Y(V\otimes\rho_H)) \ \xrightarrow{\ i_*\ } \ \text{map}^H(S^{V\otimes\bar\rho_G}, Y(V\otimes\rho_G)) \\ &\cong \ \text{map}^G(S^{V\otimes\bar\rho_G}, \text{map}^H(G, Y(V\otimes\rho_G))) \ = \ F^G\left(\text{map}^H(G,Y)\right)(V) \end{aligned}$$

The unnamed isomorphism is the adjunction between restriction and coextension; it takes an H-equivariant map $\varphi : S^{V\otimes\bar\rho_G} \longrightarrow Y(V\otimes\rho_G)$ to the G-equivariant map $\varphi^\flat : S^{V\otimes\bar\rho_G} \longrightarrow \mathrm{map}^H(G, Y(V\otimes\rho_G))$ specified by the formula

$$\varphi^\flat(x)(\gamma) \ = \ \varphi(\gamma x) \ .$$

The first map labeled i_* is essentially the extension (or prolongation) construction in the sense of (3.2), along the H-equivariant linear isometric embedding $i : \rho_H \longrightarrow \rho_G$, the $\mathbb{R}$-linearization of the inclusion $H \longrightarrow G$. In more detail: we let U denote the orthogonal complement of ρ_H in ρ_G, i.e., the $\mathbb{R}$-subspace spanned by the elements of G not in H. This induces an H-equivariant linear isometry

$$V\otimes\rho_G \ \cong \ (V\otimes\rho_H)\oplus(V\otimes U) \ .$$

The map i_* then sends a continuous H-equivariant based map $f : S^{V\otimes\bar\rho_H} \longrightarrow Y(V\otimes\rho_H)$ to the composite

$$\begin{aligned} S^{V\otimes\bar\rho_G} \ \cong \ S^{V\otimes\rho_H}\wedge S^{V\otimes U} \ &\xrightarrow{f\wedge S^{V\otimes U}} \ Y(V\otimes\rho_H)\wedge S^{V\otimes U} \\ &\xrightarrow{\sigma_{V\otimes\rho_H, V\otimes U}} \ Y((V\otimes\rho_H)\oplus(V\otimes U)) \ \cong \ Y(V\otimes\rho_G) \ . \end{aligned}$$

The isomorphism in the definition of $(R^G_H)(V)$ is the adjunction between restriction from G to H and $\mathrm{map}^H(G,-)$. Inspection of the definitions shows that the following diagram commutes:

$$\begin{tikzcd} \pi_k^H(Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_k^G\left(\mathrm{map}^H(G,Y)\right) \arrow[d, "\cong"] \\ \pi_k(F^H Y) \arrow[r, "\pi_k(R^G_H)"'] & \pi_k\left(F^G\left(\mathrm{map}^H(G,Y)\right)\right) \end{tikzcd}$$

The vertical isomorphisms are the ones given by Proposition 7.2, and the upper horizontal isomorphism is (4.10). This shows that the morphism R^G_H is a π_*-isomorphism of (non-equivariant) orthogonal spectra.

Example 7.10 (Fixed points of induced spectra)**.** The naive and geometric fixed point functor vanishes on induced spectra, i.e., for every proper subgroup H of G and every H-spectrum Y we have

$$(G\ltimes_H Y)^G \ = \ * \qquad \text{and} \qquad \Phi^G(G\ltimes_H Y) \ = \ * \ .$$

For geometric fixed points this uses the G-isomorphism $(G\ltimes_H Y)(V) \cong G\ltimes_H (Y(i^*V))$ where i^*V is the restriction of a G-representation V to an H-representation, compare (4.8).

To get at the fixed points of an induced spectrum we exploit the Wirthmüller isomorphism, i.e., the $\underline{\pi}_*$-isomorphism $\Phi : (G\ltimes_H Y) \longrightarrow \mathrm{map}^H(G,Y)$, compare Theorem 4.9. This morphism induces a π_*-isomorphism of fixed point spectra

$$F^G(G\ltimes_H Y) \ \xrightarrow{F^G\Phi} \ F^G(\mathrm{map}^H(G,Y)) \ \simeq \ F^H(Y)$$

where the last equivalence is Example 7.9.

Example 7.11 (Geometric fixed points of MR)**.** In Example 2.14 we introduced the real cobordism spectrum MR, an orthogonal C_2-ring spectrum. We will now identify the C_2-equivariant homotopy groups of MR with a more classical definition and show that the geometric fixed points $\Phi^{C_2}(MR)$ are stably equivalent to the unoriented cobordism spectrum MO.

The orthogonal C_2-spectrum MR was obtained from a collection $MU = \{MU_n\}$ of spaces by looping with imaginary spheres. It will make things clearer to reveal the full structure that this collection of spaces has. By a *real spectrum* we mean collection of based $C_2\ltimes U(n)$-spaces Y_n for $n\geq 0$, equipped with based structure maps $\tau_n : Y_n\wedge S^{\mathbb{C}} \longrightarrow Y_{n+1}$ for $n\geq 0$. Here $C_2\ltimes U(n)$ is the semidirect product of the action of the cyclic group C_2 on $U(n)$ by conjugation of unitary matrices. This data is subject to the condition that for all $n,m\geq 0$, the iterated structure map $Y_n\wedge S^{\mathbb{C}^m} \longrightarrow Y_{n+m}$ is $C_2\ltimes(U(n)\times U(m))$-equivariant. Here the group $C_2\ltimes U(m)$ acts on $\mathbb{C}^m$ in the most obvious way: the C_2-factor acts by complex conjugation and the

$U(m)$-factor via its defining action. The collection of spaces $MU = \{MU_n\}_{n\geq 0}$ considered in Example 2.14 form a commutative real ring spectrum.

Every real spectrum Y can be turned into an orthogonal C_2-spectrum ΨY as follows. We set

$$(\Psi Y)_n \ = \ \mathrm{map}(S^{i\mathbb{R}^n}, Y_n) \ ;$$

the group $C_2 \times O(n)$ acts on $i\mathbb{R}^n$ by sign (the C_2-factor) and the defining action (the $O(n)$-factor), it acts on Y_n by restriction along the inclusion $C_2 \times O(n) \longrightarrow C_2 \ltimes U(n)$, and $C_2 \times O(n)$ acts on the entire mapping space by conjugation. The structure map $\sigma_n : (\Psi Y)_n \wedge S^1 \longrightarrow (\Psi Y)_{n+1}$ is the composite

$$\begin{aligned} \mathrm{map}(S^{i\mathbb{R}^n}, Y_n) \wedge S^1 \ &\xrightarrow{\text{assemble}} \ \mathrm{map}(S^{i\mathbb{R}^n}, Y_n \wedge S^1) \\ &\xrightarrow{-\wedge S^{i\mathbb{R}}} \ \mathrm{map}(S^{i\mathbb{R}^n} \wedge S^{i\mathbb{R}}, Y_n \wedge S^1 \wedge S^{i\mathbb{R}}) \\ &\cong \ \mathrm{map}(S^{i\mathbb{R}^{n+1}}, Y_n \wedge S^{\mathbb{C}}) \ \xrightarrow{(\tau_n)_*} \mathrm{map}(S^{i\mathbb{R}^{n+1}}, Y_{n+1}) \ . \end{aligned}$$

We use the C_2-equivariant decomposition $1 \cdot \mathbb{R} \oplus i \cdot \mathbb{R} = \mathbb{C}$ to identify $S^1 \wedge S^{i\mathbb{R}}$ with $S^{\mathbb{C}}$. The real bordism spectrum MR is a special case of this construction, namely $MR = \Psi(MU)$.

Now we claim that for every real spectrum Y the equivariant homotopy groups and geometric fixed points of the orthogonal C_2-spectrum ΨY can be expressed directly in terms of the real spectrum Y. Firstly, we claim that for every C_2-spectrum of the form ΨY the map

$$j^{C_2} \ : \ (\Psi Y)^{C_2} \ \longrightarrow \ F^{C_2}(\Psi Y)$$

from (7.3) from the naive fixed points to the fixed points is a π_*-isomorphism of orthogonal spectra. The homotopy groups of the naive fixed points of ΨY can be rewritten as

$$\pi_{k+n}\left((\Psi Y)_n^{C_2}\right) \ = \ \pi_{k+n}\,\mathrm{map}^{C_2}(S^{i\mathbb{R}^n}, Y_n) \ \cong \ [S^{k+n} \wedge S^{i\mathbb{R}^n}, Y_n]^{C_2} \ \cong \ [S^k \wedge S^{\mathbb{C}^n}, Y_n]^{C_2} \ .$$

The homotopy groups of the fixed points $F^{C_2}(\Psi Y)$ were identified with the C_2-homotopy groups of the spectrum ΨY in Proposition 7.2. So in the colimit, these isomorphism combine into a natural isomorphism

$$\pi_k^{C_2}(\Psi Y) \ \cong \ \mathrm{colim}_n\, [S^{k+n\mathbb{C}}, Y_n]^{C_2}$$

where the colimit is formed along the stabilization maps

$$[S^{k+n\mathbb{C}}, Y_n]^{C_2} \ \xrightarrow{-\wedge S^{\mathbb{C}}} \ [S^{k+(n+1)\mathbb{C}}, Y_n \wedge S^{\mathbb{C}}]^{C_2} \ \xrightarrow{(\tau_n)_*} \ [S^{k+(n+1)\mathbb{C}}, Y_{n+1}]^{C_2} \ .$$

In the example of the real spectrum MU we have $MR = \Psi(MU)$ and this specializes to an isomorphism

$$\pi_V^{C_2}(MR) \ = \ \pi_V^{C_2}\Psi(MU) \ \cong \ \mathrm{colim}_n\, [S^{V+n\mathbb{C}}, MU_n]^{C_2}$$

where V is any C_2-representation. These are the groups studied, among others, by Landweber [11] and Araki [2]; Landweber uses the notation $\Omega^U_{p,q} = \pi_{p,q}(MU)$ for the equivariant homotopy group $\pi^{C_2}_{p\sigma+q}(MR)$ where σ is the sign representation.

Now we turn to the geometric fixed points. We define a 'real' geometric fixed point functor Φ^{real} on a real spectrum Y by taking C_2-fixed points:

$$(\Phi^{\mathrm{real}} Y)_n \ = \ Y_n^{C_2} \ .$$

Since the subgroup $O(n)$ of $C_2 \ltimes U(n)$ commutes with C_2, these C_2-fixed points are $O(n)$-invariant and form an orthogonal spectrum with structure maps

$$Y_n^{C_2} \wedge S^1 \ = \ Y_n^{C_2} \wedge (S^{\mathbb{C}})^{C_2} \ \cong \ \left(Y_n \wedge S^{\mathbb{C}}\right)^{C_2} \ \xrightarrow{\tau_n^{C_2}} \ Y_{n+1}^{C_2} \ .$$

We shall now define a natural π_*-isomorphism of orthogonal spectra $\Phi^{\mathrm{real}} Y \longrightarrow \Phi^{C_2}(\Psi Y)$. [...]

The value of the orthogonal spectrum underlying ΨY on a real inner product space V is given by

$$(\Psi Y)(V) \ = \ \mathrm{map}(S^{iV}, Y(V_{\mathbb{C}})) \ ;$$

here $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ is the complexification of V, with induced hermitian scalar product, and

$$Y(V_{\mathbb{C}}) \;=\; \mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V_{\mathbb{C}})_+ \wedge_{U(n)} Y_n$$

where $n = \dim(V)$, and $\mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V_{\mathbb{C}})$ is the space of $\mathbb{C}$-linear isometries from $\mathbb{C}^n$ to $V_{\mathbb{C}}$.

So we have

$$(\Phi^{C_2}(\Psi Y))_n \;=\; ((\Psi Y)(n\rho))^{C_2} \;=\; \mathrm{map}^{C_2}(S^{in\rho}, Y(\mathbb{C}^n \otimes \rho))$$

where ρ is the regular representation of C_2.

In the example of the real spectrum MU we have $MU_n^{C_2} = MO_n$, the Thom space of the tautological real n-place bundle over $BO(n)$. So the geometric fixed points of MR are stably equivalent to the Thom spectrum for unoriented bordism,

$$\Phi^{C_2}(MR) \;=\; \Phi^{C_2}\Psi(MU) \;\simeq\; \{MO_n\}_{n\geq 0} = MO\;.$$

Theorem 7.12. *For a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the following are equivalent:*

(i) *The morphism f is a $\underline{\pi}_*$-isomorphism.*
(ii) *For every subgroup H of G the map of H-fixed point spectra $F^H f : F^H X \longrightarrow F^H Y$ is a stable equivalence of orthogonal spectra.*
(iii) *For every subgroup H of G the map of geometric H-fixed point spectra $\Phi^H f : \Phi^H X \longrightarrow \Phi^H Y$ is a stable equivalence of orthogonal spectra.*

Proof. The equivalence of conditions (i) and (ii) is a direct consequence of the natural isomorphism between $\pi_*^H(X)$ and $\pi_*(F^H X)$ established in Proposition 7.2.

(ii)$\Longrightarrow$(iii) If f is is a $\underline{\pi}_*$-isomorphism, then so is $\tilde{E}\mathcal{P} \wedge f$ by Proposition 5.4. Since condition (i) implies condition (ii) the map $F^G(\tilde{E}\mathcal{P} \wedge f) : F^G(\tilde{E}\mathcal{P} \wedge X) \longrightarrow F^G(\tilde{E}\mathcal{P} \wedge Y)$ is a stable equivalence of orthogonal spectra. So $\Phi^H f : \Phi^H X \longrightarrow \Phi^H Y$ is a stable equivalence by Proposition 7.6.

(iii)$\Longrightarrow$(ii) We show by induction on the order of the group G. If G is the trivial group, then all three fixed point constructions coincide (and do not do anything), and there is nothing to show.

If G is a non-trivial group we know by induction hypothesis that the map $F^H f : F^H X \longrightarrow F^H Y$ is a stable equivalence for every proper subgroup H of G. In other words, f is a $\mathcal{P}$-equivalence. Proposition 5.22 lets us conclude that $E\mathcal{P}_+ \wedge f$ is a $\underline{\pi}_*$-isomorphism of G-spectra. Hence $F^G(E\mathcal{P}_+ \wedge f) : F^G(E\mathcal{P}_+ \wedge X) \longrightarrow F^G(E\mathcal{P}_+ \wedge Y)$ is a stable equivalence of non-equivariant spectra. Since $\Phi^G f : \Phi^G X \longrightarrow \Phi^G Y$ is also a stable equivalence, the isotropy separation sequence lets us conclude that the map $F^G f : F^G X \longrightarrow F^G Y$ on G-fixed points is a stable equivalence. □

Monoidal properties. Naive and geometric fixed points commute with levelwise smash products 'on the nose'. In other words, there are natural isomorphisms

$$(A \wedge X)^G \;\cong\; A^G \wedge X^G \qquad \text{and} \qquad \Phi^G(A \wedge X) \;\cong\; A^G \wedge \Phi^G X$$

for every based G-space A and every G-spectrum X.

The three kinds of fixed points construction are lax symmetric monoidal functors. For naive fixed points, the map

$$X^G \wedge Y^G \;\longrightarrow\; (X \wedge Y)^G$$

arises via the universal property of the smash product from the bilinear morphism

$$X(V)^G \wedge Y(W)^G \;=\; (X(V) \wedge Y(W))^G \;\xrightarrow{(i_{V,W})^G}\; ((X \wedge Y)(V \oplus W))^G\;.$$

In the following proposition, the term 'cofibrant' refers to spectra built by attaching 'cells' of the form $F_V(G/H \times D^n)_+$ for all $n \geq 0$, all subgroups H of G and all G-representations V.

Proposition 7.13. *The natural map*

$$X^G \wedge Y^G \longrightarrow (X \wedge Y)^G .$$

of naive fixed point spectra is an isomorphism whenever X and Y are cofibrant.

Sketch. By inspection, the claim is true when X and Y are both of the form $F_V A$ for a based G-CW-complex A and G-representation V; moreover, the claim is stable, in each variable, under wedges, retract and cobase change along cofibrations. □

A word of warning: the naive fixed point functor is not homotopy invariant, and it has to be right derived to induce a functor on the equivariant stable homotopy category. However, the smash product of two G-Ω-spectra is rarely a G-Ω-spectrum, so the isomorphism of the previous proposition does *not* imply that the derived fixed point functor (which is modeled by $F^G X$) commutes with smash product in the homotopy category.

For fixed points, we first observe that the functor F can be made lax symmetric monoidal as follows. The G-maps

$$\begin{aligned}
(FX)(V) \wedge (FY)(W) &= \mathrm{map}(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) \wedge \mathrm{map}(S^{W\otimes\bar\rho_G}, Y(W\otimes\rho_G)) \\
&\xrightarrow{\wedge} \mathrm{map}(S^{V\otimes\bar\rho_G} \wedge S^{W\otimes\bar\rho_G}, X(V\otimes\rho_G) \wedge Y(W\otimes\rho_G)) \\
&\xrightarrow{i_{V\otimes\rho_G, W\otimes\rho_G}} \mathrm{map}(S^{(V\oplus W)\otimes\bar\rho_G}, (X\wedge Y)((V\otimes\rho_G)\oplus(W\otimes\rho_G))) \\
&\cong \mathrm{map}(S^{(V\oplus W)\otimes\bar\rho_G}, (X\wedge Y)((V\oplus W)\otimes\rho_G)) = F(X\wedge Y)(V\oplus W)
\end{aligned}$$

form a G-equivariant bimorphism and thus assemble into a morphism of G-spectra

$$FX \wedge FY \longrightarrow F(X\wedge Y) .$$

We can combine this with the previous monoidal transformation of naive fixed points and arrive at an associative, commutative and unital map of orthogonal spectra

$$F^G X \wedge F^G Y \longrightarrow (FX\wedge FY)^G \longrightarrow F(X\wedge Y)^G = F^G(X\wedge Y) .$$

For geometric fixed points, finally, the G-maps

$$\begin{aligned}
(\Phi^G X)(V) \wedge (\Phi^G Y)(W) &= X(V\otimes\rho_G)^G \wedge Y(W\otimes\rho_G)^G \cong (X(V\otimes\rho_G)\wedge Y(W\otimes\rho_G))^G \\
&\xrightarrow{i^G_{V\otimes\rho_G, W\otimes\rho_G}} ((X\wedge Y)((V\otimes\rho_G)\oplus(W\otimes\rho_G)))^G \\
&\cong ((X\wedge Y)(V\oplus W)\otimes\rho_G)^G = \Phi^G(X\wedge Y)(V\oplus W)
\end{aligned}$$

assemble into a morphism of orthogonal spectra

$$\Phi^G X \wedge \Phi^G Y \longrightarrow \Phi^G(X\wedge Y) .$$

Proposition 7.14. *The natural map*

$$\Phi^G X \wedge \Phi^G Y \longrightarrow \Phi^G(X\wedge Y) .$$

is a π_*-isomorphism whenever X or Y is a cofibrant orthogonal G-spectrum.

Proof. The proof starts with the special case where $X = F_V A$ and $Y = F_W B$ are free G-spectra generated by G-CW-complexes A and B. This case is OK since

$$\begin{aligned}
\Phi^G(F_V A) \wedge \Phi^G(F_W B) &\simeq F_{V^G} A^G \wedge F_{W^G} B^G \cong F_{V^G\oplus W^G}(A^G \wedge B^G) \\
&\cong F_{(V\oplus W)^G}(A\wedge B)^G \simeq \Phi^G(F_{V\oplus W} A\wedge B) ,
\end{aligned}$$

using that $\Phi^G(F_V)$ is π_*-isomorphic to $\Omega^{V^G}\mathbb{S}$, hence to F_{V^G}. A cell induction can then be used to work up to general cofibrant G-spectra. □

Remark 7.15. The geometric fixed points is essentially determined by the properties

(i) Φ^G is homotopy invariant
(ii) $\Phi^G(\Sigma^\infty A) = \Sigma^\infty(A^G)$
(iii) Φ^G commutes with smash products
(iv) Φ^G commutes with sequential homotopy colimits.

Indeed, for every G-representation V, the stable equivalence $S^{-V} \wedge S^V \longrightarrow \mathbb{S}$ induces a stable equivalence

$$\Phi^G(S^{-V}) \wedge S^{V^G} \cong \Phi^G(S^{-V}) \wedge \Phi^G(S^V) \xrightarrow{\sim} \Phi^G(S^{-V} \wedge S^V) \xrightarrow{\sim} \Phi^G(\mathbb{S}) = \mathbb{S} .$$

So we obtain

$$\Phi^G(S^{-V}) \simeq S^{-V^G} .$$

In other words, if $n = \dim(V^G)$ is the dimension of the fixed point space of V, the $\Phi^G(S^{-V})$ is a $(-n)$-sphere.

Now we can consider the canonical presentation of a G-spectrum X with respect to the exhausting sequence

$$\rho \longrightarrow 2\rho \longrightarrow 3\rho \longrightarrow \dots \longrightarrow n\rho \longrightarrow \dots$$

of multiples of the regular representation. By Proposition 5.19 the G-spectrum X is stably equivalent to the homotopy colimit of the spectra $S^{-n\rho} \wedge X(n\rho)$. So $\Phi^G X$ is stably equivalent to the mapping telescope of the spectra

$$\Phi^G(S^{-n\rho} \wedge X(n\rho)) \simeq \Phi^G(S^{-n\rho}) \wedge X(n\rho)^G \simeq S^{-n} \wedge X(n\rho)^G .$$

Since the n-th term of the geometric fixed points $\Phi^G X$ is precisely $X(n\rho)^G$, this reproduces the definition of Φ^G.

8. The Adams isomorphism

We let G be a finite group, E an orthogonal G-spectrum, and A a finite based G-CW-complex such that the G-action is free away from the basepoint. The aim of this section is to construct a natural isomorphism

$$\pi_*(A \wedge_G E) \cong \pi_*^G(A \wedge E) . \tag{8.1}$$

The argument in the source denotes the orbit space of G acting diagonally on $A \wedge E$. This isomorphism is a special of a more general isomorphism that was established by Adams in [1, Thm. 5,4, p. 500]. The Adams isomorphism was generalized by Lewis and May in [15, II Thm. 7.1]; in the context of orthogonal G-spectra, a functorial version of this generalization was given by Reich and Varisco in [18]. The Adams isomorphism generalizes the Wirthmüller isomorphism: for $A = G_+$, and under the identification $G_+ \wedge_G E \cong E$, the isomorphism (8.1) reduces to the external transfer isomorphism $\mathrm{Tr}_e^G : \pi_*(E) \longrightarrow \pi_*^G(G_+ \wedge E)$ introduced in (4.14).

Remark 8.2 (Rewriting the tom Dieck splitting)**.** If A is an arbitrary based G-CW-complex and H a subgroup of G, then $EWH_+ \wedge A^H$ is a based WH-CW-complex, where $WH = (N_G H)/H$ is the Weyl group of H. Moreover, the WH-action on EWH is free, so the action on $EWH_+ \wedge A^H$ is free away from the basepoint. Hence the Adams isomorphism for the group WH and the WH-space $EWH_+ \wedge A^H$ specializes to an isomorphism

$$\pi_*((EWH_+ \wedge A) \wedge_{WH} E) \cong \pi_*^G(EWH_+ \wedge A^H \wedge E) .$$

If we then specialize to the sphere spectrum $E = \mathbb{S}$ and exploit that the G-action on $\mathbb{S}$ is trivial, we arrive at an isomorphism

$$\pi_*(\Sigma^\infty EWH_+ \wedge_{WH} A) \cong \pi_*^G(\Sigma^\infty EWH_+ \wedge A^H) .$$

In combination with the tom Dieck splitting (Theorem 6.12), this yields an isomorphism

$$\bigoplus_{(H)} \pi_*(\Sigma^\infty (EWH_+ \wedge_{WH} A^H)) \cong \pi_*^G(\Sigma^\infty A)$$

for every based G-CW complex A, where the sum runs over a set of representatives of all conjugacy classes of subgroups of G. In some references, the term 'tom Dieck splitting' is used for this combination of the tom Dieck splitting and the Adams isomorphism.

Construction 8.3. We let G be a finite group, E an orthogonal G-spectrum and A a based G-space. We construct an *assembly map*

$$\alpha \ : \ F^G(G_+ \wedge E) \wedge_G A \ \longrightarrow \ F^G(A \wedge E) \ .$$

Here F^G is the fixed point spectrum introduced in Definition 7.1. Moreover, the right translation action of G on itself induces a right G-action on the spectrum $G_+ \wedge E$; since the left and right translation actions commute with each other, the right action on $G_+ \wedge E$ is through morphisms of orthogonal G-spectra. So it induces a right G-action on the orthogonal spectrum $F^G(G_+ \wedge E)$. The source of the assembly map α is the coequalizer of this right G-action and the given left G-action on A.

The value of α at an inner product space V is the map

$$\begin{aligned} \alpha(V) \ : \ \mathrm{map}_*^G(S^{V\otimes\bar\rho_G}, G_+ \wedge E(V\otimes\rho_G)) \wedge_G A \ &\longrightarrow \ \mathrm{map}_*^G(S^{V\otimes\bar\rho_G}, A \wedge E(V\otimes\rho_G)) \\ [f,a] \qquad &\longmapsto \quad \{x \mapsto \ \beta(f(x)\wedge a)\} \ , \end{aligned}$$

where

$$\beta \ : \ G_+ \wedge E(V\otimes\rho_G) \wedge A \ \longrightarrow \ A \wedge E(V\otimes\rho_G) \quad \text{is defined by} \quad \beta(g\wedge e\wedge a) \ \longmapsto \ ga\wedge e \ .$$

We check that $\alpha(V)$ is well-defined. For $\gamma \in G$, we write $r_\gamma : E(V\otimes\rho_G)\wedge G_+ \longrightarrow E(V\otimes\rho_G)\wedge G_+$ for right translation by γ on G. Then

$$\beta(r_\gamma(g\wedge e)\wedge a) \ = \ \beta(g\gamma\wedge e\wedge a) \ = \ g\gamma a\wedge e \ = \ \beta(g\wedge e\wedge\gamma a) \ ,$$

and hence

$$\alpha(V)[r_\gamma\circ f, a](x) \ = \ \beta((r_\gamma f)(x)\wedge a) \ = \ \beta(f(x)\wedge\gamma a) \ = \ \alpha(V)[f,\gamma a](x) \ .$$

This means that $\alpha(V)[r_\gamma\circ f, a] = \alpha(V)[f,\gamma a]$, so $\alpha(V)$ is indeed well-defined.

We must also show that $\alpha(V)$ lands in the space of G-equivariant maps. For $\gamma\in G$ we have

$$\beta(\gamma(g\wedge e)\wedge a) \ = \ \beta(\gamma g\wedge\gamma e\wedge a) \ = \ \gamma g a\wedge\gamma e \ = \ \gamma(ga\wedge e) \ = \gamma\cdot\beta(g\wedge e\wedge a) \ .$$

So we also have

$$\alpha(V)[f,a](\gamma x) \ = \ \beta(f(\gamma x)\wedge a) \ = \ \beta(\gamma f(x)\wedge a) \ = \ \gamma\cdot\beta(f(x)\wedge a) \ = \ \gamma\cdot(\alpha(V)[f,a](x)) \ ;$$

the second equation is the fact that f is G-equivariant. We omit the verification that for varying V, the maps $\alpha(V)$ indeed define a morphism of orthogonal spectra.

Theorem 8.4. *Let G be a finite group, E an orthogonal G-spectrum, and A a based G-CW-complex such that G acts freely away from the base point. Then the assembly morphism $\alpha : F^G(G_+ \wedge E)\wedge_G A \longrightarrow F^G(A\wedge E)$ is a stable equivalence of orthogonal spectra.*

Proof. We start with the special case when $A = S^n \wedge G_+$ for some $n \geq 0$. In this case we identify

$$S^n \wedge F^G(G_+\wedge E) \ \cong \ F^G(G_+\wedge E)\wedge_G (S^n\wedge G_+)$$

by sending $x \wedge \psi$ to $[\psi, x \wedge 1]$. Under these identifications, the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(G_+ \wedge E) \arrow[rr, "S^n\wedge -", "\cong"'] \arrow[d, "\cong"'] & & \pi_{n+k}^G(S^n \wedge G_+ \wedge E) \arrow[dd, "\cong"] \\
\pi_k(F^G(G_+ \wedge E)) \arrow[d, "S^n\wedge -"', "\cong"] & & \\
\pi_{n+k}(S^n \wedge F^G(G_+ \wedge E)) \arrow[r, "\cong"'] & \pi_{k+n}(F^G(G_+ \wedge E) \wedge_G (S^n \wedge G_+)) \arrow[r, "\pi_{n+k}(\alpha)"'] & \pi_{n+k}(F^G(S^n \wedge G_+ \wedge E))
\end{tikzcd}$$

Here the vertical identifications are the ones specified in Proposition 7.2, and two maps labeled '$S^n \wedge -$' are suspension isomorphisms, compare Proposition 3.12. This completes the proof in this special case.

Now we treat the case when the G-CW-complex A has finitely many cells; we argue by induction over the number of cells. When A consists only of the basepoint, source and target of α are trivial orthogonal spectra, and there is nothing to show. Now we suppose that A is obtained from an equivariant subcomplex B by attaching a single G-free n-cell, and we assume that claim for B. The inclusion $B \longrightarrow A$ is then an equivariant cofibration and the quotient space A/B is euivariantly homeomorphic to $S^n \wedge G_+$. So we know the claim for B (by induction hypothesis) and for A/B (by the special case above). We claim that the following diagram commutes:

$$\begin{tikzcd}
\dots \arrow[r] & \pi_k(F^G(G_+ \wedge E) \wedge_G B) \arrow[r] \arrow[d, "\alpha_*"'] & \pi_k(F^G(G_+ \wedge E) \wedge_G A) \arrow[r] \arrow[d, "\alpha_*"'] & \pi_k(F^G(G_+ \wedge E) \wedge_G (A/B)) \arrow[r, "\delta", "(3.20)"'] \arrow[d, "\alpha_*"'] & \dots \\
\dots \arrow[r] & \pi_k(F^G(B \wedge E)) \arrow[r] & \pi_k(F^G(A \wedge E)) \arrow[r] & \pi_k(F^G((A/B) \wedge E)) \arrow[r, "(3.20)", "\delta"'] & \dots
\end{tikzcd}$$

This is clear for two out of three squares by naturality of the morphism α. We omit the verification that the square involving the connecting homomorphisms commutes.

Now we let A be any based G-CW-complex with free G-action (away from the basepoint). We let $s(A)$ denote the filtered poset, under inclusion, of finite G-CW-subcomplexes of A. We contemplate the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_{B \in s(A)}\, \pi_k(F^G(G_+ \wedge E) \wedge_G B) \arrow[r, "\cong"] \arrow[d, "\mathrm{colim}\, \pi_k(\alpha)"', "\cong"] & \pi_k(F^G(G_+ \wedge E) \wedge_G A) \arrow[d, "\mathrm{colim}\, \pi_k(\alpha)"] \\
\mathrm{colim}_{B \in s(A)}\, \pi_k(F^G(B \wedge E)) \arrow[d, "\cong"'] & \pi_k(F^G(A \wedge E)) \arrow[d, "\cong"] \\
\mathrm{colim}_{B \in s(A)}\, \pi_k^G(B \wedge E) \arrow[r, "\cong"'] & \pi_k^G(A \wedge E)
\end{tikzcd}$$

where the horizontal maps are induced by the inclusions, and the lower vertical isomorphisms are the ones specified in Proposition 7.2. The upper left vertical maps is an isomorphism by the special case of finite G-CW-complexes. So the map $\pi_k(\alpha) : \pi_k(F^G(G_+ \wedge E) \wedge_G A) \longrightarrow \pi_k(F^G(A \wedge E))$ is an isomorphism because the other five maps are. $\square$

Proposition 8.5. *Let $f : X \longrightarrow Y$ be an equivariant morphism of orthogonal spectra with right G-action. Suppose that f is a stable equivalence of underlying orthogonal spectra. Let A be a based G-CW-complex such that the G-action is free away from the basepoint. Then the morphism*

$$f \wedge_G A \;:\; X \wedge_G A \;\longrightarrow\; Y \wedge_G A$$

is a stable equivalence of orthogonal spectra.

Proof. The functor $-\wedge_G A$ commutes with the formation of mapping cones. The long exact homotopy group sequences for mapping cones thus reduce the claim to the following special case. We let X be an orthogonal spectrum with right G-action such that the stable homotopy groups $\pi_*(X)$ vanish. Then the homotopy groups $\pi_*(X \wedge_G A)$ vanish as well.

We start with the special case when $A = G_+ \wedge S^n$ for some $n \geq 0$. In this case the isomorphisms

$$\pi_{k+n}(X \wedge_G (G_+ \wedge S^n)) \;\cong\; \pi_{k+n}(X \wedge S^n) \;\cong\; \pi_k(X)$$

prove the claim. Now we treat the case when the G-CW-complex A has finitely many cells; we argue by induction over the number of cells. When A consists only of the basepoint, the orthogonal spectrum $X \wedge_G A$ is trivial, and there is nothing to show. Now we suppose that A is obtained from an equivariant subcomplex B by attaching a single G-free n-cell, and we assume that claim for B. The inclusion $B \longrightarrow A$ is then an equivariant cofibration and the quotient space B/A is equivariantly homeomorphic to $G_+ \wedge S^n$. So we know the claim for B (by induction hypothesis) and for B/A (by the special case above), the long exact homotopy group sequence relating the stable homotopy groups of $X \wedge_G B$, $X \wedge_G A$ and $X \wedge_G (A/B)$ provide the inductive step.

Now we let A be any based G-CW-complex with free G-action (away from the basepoint). We let $s(A)$ denote the filtered poset, under inclusion, of finite G-CW-subcomplexes of A. Then the canonical map

$$\mathrm{colim}_{B \in s(A)}\, \pi_k(X \wedge_G B) \;\longrightarrow\; \pi_k(X \wedge_G A)$$

is an isomorphism. The left hand side vanishes by the special case of finite G-CW-complexes. This proves the claim. □

Construction 8.6. As the second main ingredient for the Adams isomorphism (8.1), we now relate the orthogonal spectrum $F^G(G_+ \wedge E) \wedge_G A$ to the orthogonal spectrum $A \wedge_G E$. The Wirthmüller isomorphism established in Theorem 4.9 shows that for every orthogonal spectrum Y, the morphism

$$\Psi_Y \;:\; G_+ \wedge Y \;\longrightarrow\; \mathrm{map}(G, Y)$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, where Ψ_Y is defined in (4.4). If E is an orthogonal G-spectrum, we write E^{tr} for the underlying orthogonal spectrum with trivial G-action. The shearing isomorphism

$$\chi \;:\; G_+ \wedge E \;\cong\; G_+ \wedge E^{\mathrm{tr}} \;, \quad \chi(g \wedge e) \;=\; g \wedge g^{-1}e$$

is an isomorphism of orthogonal G-spectra. The composite

$$G_+ \wedge E \;\xrightarrow[\cong]{\chi}\; G_+ \wedge E^{\mathrm{tr}} \;\xrightarrow{\Psi_{E^{\mathrm{tr}}}}\; \mathrm{map}(G, E^{\mathrm{tr}})$$

is thus another $\underline{\pi}_*$-isomorphism of orthogonal G-spectra; we abbreviate $\Phi_E = \Psi_{E^{\mathrm{tr}}} \circ \chi$. In explicit terms, this morphism is levelwise given by

$$\Phi_E(g \wedge e)(\gamma) \;=\; \begin{cases} \gamma e & \text{for } \gamma g = 1, \text{ and} \\ * & \text{for } \gamma g \neq 1. \end{cases}$$

Here the left G-action on $G_+ \wedge E$ is diagonal, and the left G-action on $\mathrm{map}(G, E^{\mathrm{tr}})$ is levelwise given by

$$(g \cdot \psi)(\gamma) \;=\; \psi(\gamma g) \;.$$

The explicit formula shows that the morphism Φ_E matches the right translation action of G on itself with the right G-action on $\mathrm{map}(G, E^{\mathrm{tr}})$ given by

$$(\psi \cdot g)(\gamma) \;=\; g^{-1} \cdot \psi(g\gamma) \;. \tag{8.7}$$

Because Φ_E is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, Proposition 7.2, the induced morphism of genuine fixed point spectra

$$F^G(\Phi_E) \;:\; F^G(G_+ \wedge E) \;\longrightarrow\; F^G(\mathrm{map}(G, E^{\mathrm{tr}}))$$

is a stable equivalence of orthogonal spectra. Because the morphism Φ_E is also right G-equivariant for the right actions specified above, the stable equivalence $F^G(\Phi_E)$ is equivariant for the induced right G-actions.

In Example 7.9 we defined a stable equivalence of orthogonal spectra

$$R \;=\; R_e^G \;:\; Y \;\longrightarrow\; F^G\,(\mathrm{map}(G,Y)) \;,$$

where Y is any orthogonal spectrum. In level V, the map $(R_e^G)(V)$ is

$$\begin{aligned} Y(V) \;\xrightarrow{\tilde\sigma_{V,V\otimes\bar\rho_G}}\; & \mathrm{map}(S^{V\otimes\bar\rho_G}, Y(V\otimes\rho_G)) \\ \cong\; & \mathrm{map}^G(S^{V\otimes\bar\rho_G}, \mathrm{map}(G, Y(V\otimes\rho_G))) \;=\; F^G(\mathrm{map}(G,Y))(V) \;. \end{aligned}$$

The unnamed isomorphism extends a non-equivariant based map $\varphi : S^{V\otimes\bar\rho_G} \longrightarrow Y(V\otimes\rho_G)$ to a G-equivariant map $\varphi^\flat : S^{V\otimes\bar\rho_G} \longrightarrow \mathrm{map}(G, Y(V\otimes\rho_G))$ by the formula

$$\varphi^\flat(x)(\gamma) \;=\; \varphi(\gamma x) \;.$$

We specialize to the case $Y = E^{\mathrm{tr}}$ and observe that

$$\begin{aligned} R(g^{-1}e)(x)(\gamma) \;&=\; (\tilde\sigma(g^{-1}e))^\flat(x)(\gamma) \;=\; \tilde\sigma(g^{-1}e)(\gamma x) \\ &=\; g^{-1}\cdot\tilde\sigma(e)(g\gamma x) \;=\; g^{-1}\cdot(\tilde\sigma(e))^\flat(x)(g\gamma) \;=\; g^{-1}\cdot R(e)(x)(g\gamma) \;, \end{aligned}$$

where $g,\gamma\in G$, $e\in E(V)$ and $x\in S^{V\otimes\bar\rho_G}$, and where we abbreviated $\tilde\sigma_{V,V\otimes\bar\rho_G}$ to $\tilde\sigma$. The third equation is the fact that the adjoint structure map $\tilde\sigma$ is left G-equivariant for the given action on $E(V)$ and the conjugation action on $\mathrm{map}(S^{V\otimes\bar\rho_G}, Y(V\otimes\rho_G))$. If we vary x and γ, this relation means that the morphism R is right G-equivariant for the right G-action on E^{tr} by $e\cdot g = g^{-1}\cdot e$, and for the right G-action on $F^G(\mathrm{map}(G,E^{\mathrm{tr}}))$ given by (8.7).

Now we let A be a based G-CW-complex such that the G-action is free away from the basepoint. We have now constructed morphisms of orthogonal spectra

$$\begin{aligned} A\wedge_G E \;\cong\; E\wedge_G A \;\xrightarrow{R\wedge_G A}\; & F^G(\mathrm{map}(G,E))\wedge_G A \\ \xleftarrow{F^G(\Phi_E)\wedge_G A}\; & F^G(G_+\wedge E)\wedge_G A \;\xrightarrow{\;\alpha\;}\; F^G(E\wedge A) \end{aligned} \tag{8.8}$$

that are natural in E and in A. The first map switches A and E.

Because R and $F^G(\Phi_E)$ are stable equivalences and right G-equivariant, the morphisms $R\wedge_G A$ and $F^G(\Phi_E)\wedge_G A$ are stable equivalences by Proposition 8.5. The morphism α is a stable equivalence by Theorem 8.4. So we can conclude:

Corollary 8.9. *Let G be a finite group, E an orthogonal G-spectrum, and A a based G-CW-complex such that the G-action is free away from the basepoint. Then the chain of stable equivalences* (8.8) *and the isomorphism of Proposition 7.2 provide a natural isomorphism*

$$\pi_*(A\wedge_G E) \;\cong\; \pi_*^G(A\wedge E) \;.$$

Remark 8.10 (Norm map)**.** We consider the Adams isomorphism of Corollary 8.9 in the special case where $A = EG_+$ is the universal free G-space, with a disjoint basepoint added. The orthogonal spectrum

$$X_{hG} \;=\; EG_+\wedge_G X$$

is called the *homotopy orbit spectrum* of an orthogonal G-spectrum X. The composite

$$\pi_*(X_{hG}) \;=\; \pi_*(EG_+\wedge_G X) \;\underset{\cong}{\xrightarrow{(8.1)}}\; \pi_*^G(EG_+\wedge X) \;\xrightarrow{\pi_*^G(p\wedge X)}\; \pi_*^G(X)$$

is called the *norm map*. Here $p : EG_+ \longrightarrow S^0$ is the based map that sends EG to the non-basepoint. Since the Adams isomorphism arises from a zigzag of stable equivalences, the norm map is realized by a morphism in the stable homotopy category of orthogonal spectra

$$N : X_{hG} \longrightarrow F^G X , \tag{8.11}$$

also referred to as the norm map, that is natural for morphisms of orthogonal G-spectra in X. There is an unfortunate clash of terminology here: the norm map (8.11) is of an additive nature, and should not be confused with the multiplicative norm constructions that we discuss in Sections 10 and 11.

We call an orthogonal G-spectrum X *geometrically free* if the projection $p \wedge X : EG_+ \wedge X \longrightarrow X$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. For geometrically free orthogonal G-spectra, the norm map on homotopy groups is an isomorphism, so the norm morphism (8.11) is a stable equivalence of orthogonal spectra. The underlying space of EG is contractible, so on the one hand, the projection $p \wedge X : EG_+ \wedge X \longrightarrow X$ induces an isomorphism on non-equivariant stable homotopy groups. On the other hand, G acts freely on EG, so the geometric fixed point $\Phi^H_*(EG_+ \wedge X)$ vanish for every non-trivial subgroup H of G. Since $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra are detected by geometric fixed points as in Theorem 7.12, we conclude that an orthogonal G-spectrum X is geometrically free if and only if the geometric fixed point homotopy groups $\Phi^H_*(X)$ vanish for every non-trivial subgroup H of G.

The norm map (8.11) is reminiscent of a similar algebraic norm map, and this justifies the name. To explain the analogy, we let A be an abelian group with an action of the finite group G. The algebraic norm map is the homomorphism

$$N : A/G \longrightarrow A^G , \quad [a] \longmapsto \sum_{g \in G} ga$$

from the abelian group of coinvariants to the abelian group of invariants. Just as the topological norm map (8.11) is a stable equivalence whenever X is geometrically free, the algebraic norm map is an isomorphism whenver A is free as a module over the group ring $\mathbb{Z}[G]$.

9. Power constructions

Given an orthogonal spectrum X, the m-th smash power

$$X^{(m)} = \underbrace{X \wedge \ldots \wedge X}_{m}$$

has a natural action of the symmetric group Σ_m by permuting the factors. If X is an H-spectrum, the H-actions of each factor combine into an action of H^m. Altogether we obtain a natural action of the wreath product $\Sigma_m \wr H$

$$\Sigma_m \wr H = \Sigma_m \ltimes H^m$$

on $X^{(m)}$. We recall that the multiplication on the wreath product is given by

$$(\sigma; h_1, \ldots, h_m) \cdot (\tau; k_1, \ldots, k_m) = (\sigma\tau; h_{\tau(1)}k_1, \ldots, h_{\tau(m)}k_m) .$$

We can write the action on $X^{(m)}$ symbolically as

$$(\sigma; h_1, \ldots, h_m) \cdot (x_1 \wedge \ldots \wedge x_m) = h_{\sigma^{-1}(1)}x_{\sigma^{-1}(1)} \wedge \ldots \wedge h_{\sigma^{-1}(m)}x_{\sigma^{-1}(m)} .$$

To get the internal smash product in the category of H-spectra we usually restrict this action along the diagonal embedding $H \longrightarrow \Sigma_m \wr H, h \longmapsto (1; h, \ldots, h)$, but we are going to remember all of the action of $\Sigma_m \wr H$. We write $P^m X$ for $X^{(m)}$ when we consider it as an orthogonal $\Sigma_m \wr H$-spectrum and refer to it as the *m-th power* of X. The power construction has the following formal properties:

(a)

$$P^m(\mathbb{S}) = \mathbb{S}$$

(b)
$$P^m X \wedge P^n X \;=\; \mathrm{res}^{\Sigma_{m+n}\wr H}_{(\Sigma_m\wr H)\times(\Sigma_n\wr H)}(P^{m+n}X)$$
where the restriction is taken along the monomorphism
$$+ \;:\; (\Sigma_m \wr H)\times(\Sigma_n \wr H) \;\longrightarrow\; \Sigma_{m+n}\wr H$$
$$(\sigma;\, h_1,\dots,h_m) \;+\; (\tau;\, h_{m+1},\dots,h_{m+n}) \;=\; (\sigma+\tau;\, h_1,\dots,h_m,h_{m+1},\dots,h_{m+n})$$

(c)
$$P^m(P^k X) \;=\; \mathrm{res}^{\Sigma_{km}\wr H}_{\Sigma_m\wr(\Sigma_k\wr H)}\, P^{km}X$$
Here the restriction is taken along the monomorphism
$$\Sigma_m \wr (\Sigma_k \wr H) \;\longrightarrow\; \Sigma_{km}\wr H \tag{9.1}$$
$$(\sigma;\, (\tau_1;h^1),\dots,(\tau_m;h^m)) \;\longmapsto\; (\tau_{\sigma^{-1}(1)}, h^{\sigma^{-1}(1)}) + \dots + (\tau_{\sigma^{-1}(m)}, h^{\sigma^{-1}(m)})$$
where $h^i = (h^i_1,\dots,h^i_k)\in H^k$ and the operation '+' is as in (b).

(d)
$$\chi^{(m)}_{X,Y} \;:\; P^m X\wedge P^m Y \;\cong\; P^m(X\wedge Y)$$

(e)
$$P^m(X\vee Y) \;\cong\; \bigvee_{i=0}^m\, (\Sigma_m)_+ \wedge_{\Sigma_i\times\Sigma_{m-i}} P^i X\wedge P^{m-i}Y\ .$$

(f)
$$P^m(H\ltimes_K Y) \;\cong\; (\Sigma_m\wr H)\ltimes_{\Sigma_m\wr K}(P^m Y)$$
for every subgroup K of H and every orthogonal K-spectrum Y.

The most important homotopical properties of the power construction is as follows.

Theorem 9.2. *The power operation functor*
$$P^m \;:\; \mathcal{S}p_H \;\longrightarrow\; \mathcal{S}p_{\Sigma_m\wr H}$$
takes $\underline{\pi}_*$-isomorphisms between cofibrant H-spectra to $\underline{\pi}_*$-isomorphisms.

Proof. Here is the crucial test case: $\lambda_V : F_V S^V \longrightarrow \mathbb{S}$ is one of the generating $\underline{\pi}_*$-isomorphisms. We have $P^m\mathbb{S} = \mathbb{S}$, the $\Sigma_m\wr H$-sphere spectrum. On the other hand,
$$P^m(F_V S^V) \;=\; F_{V^m} S^{V^m}$$
and the map $P^m\lambda_V$ becomes λ_{V^m}, which is a $\underline{\pi}_*$-isomorphism for the group $\Sigma_m\wr H$. □

Now we construct natural power maps of homotopy groups
$$P^m \;:\; \pi^H_V(X) \;\longrightarrow\; \pi^{\Sigma_m\wr H}_{V^m}(P^m X)\ .$$
Here V is a H-representation and V^m is the $(\Sigma_m\wr H)$-representation with action given by
$$(\sigma;\, h_1,\dots,h_m)\cdot(v_1,\dots,v_m) \;=\; (h_{\sigma^{-1}(1)}v_{\sigma^{-1}(1)},\dots,h_{\sigma^{-1}(m)}v_{\sigma^{-1}(m)})\ .$$
The construction of the power map is straightforward: if $f : S^{V+n\rho_H}\longrightarrow X(n\rho_H)$ is a H-map representing a class in $\pi^H_V(X)$, then the composite
$$S^{V^m+n\rho_H^m}\cong (S^{V+n\rho_H})^{(m)} \xrightarrow{\ f^{(m)}\ } X(n\rho_H)^{(m)} \xrightarrow{\ i_{n\rho_H,\dots,n\rho_H}\ } (X^{(m)})((n\rho_H)^m) \;\cong\; (P^m X)(n\rho_H^m)$$
is equivariant for the group $\Sigma_m\wr H$, so it represents an element in $\pi^{\Sigma_m\wr H}_{V^m}(P^m X)$. Here
$$i_{V,\dots,V} \;:\; X(V)\wedge\ldots\wedge X(V) \;\longrightarrow\; (X^{(m)})(V^m)$$

is the $(V,\dots,V)$-component of the universal multilinear map, which is $\Sigma_m \wr H$-equivariant. If we stabilize f to $f\diamond\rho_H : S^{V\oplus(n+1)\rho_H}\longrightarrow X((n+1)\rho_H)$, then the above composite changes into

$$i_{(n+1)\rho_H,\dots,(n+1)\rho_H}\circ(f\diamond\rho_H)^{(m)} \ = \ (i_{n\rho_H,\dots,n\rho_H}\circ f^{(m)})\diamond\rho_H^m \ .$$

So the class

$$P^m[f] \ = \ \langle i_{n\rho,\dots,n\rho}\circ f^{(m)}\rangle \qquad \text{in} \quad \pi_{V^m}^{\Sigma_m\wr H}(P^m X)$$

only depends on the class of f in $\pi_V^H(X)$.

Remark 9.3. As is immediate from the construction, the power map $P^m:\pi_V^H(X)\longrightarrow\pi_{V^m}^{\Sigma_m\wr H}(P^mX)$ actually factors through a modified equivariant stable homotopy group $\pi_{V^m}^{\Sigma_m\wr H,\rho_H^m}(P^mX)$ based on an 'incomplete universe', as discussed in Remark 4.24. This modified homotopy group $\pi_W^{\Sigma_m\wr H,\rho_H^m}(Y)$ of a $\Sigma_m\wr H$-spectrum Y is defined by the same kind of colimit as for the ordinary equivariant homotopy groups, but via iterated stabilization with the representation ρ_H^m (instead of the regular representation of $\Sigma_m\wr H$). The modified homotopy group maps to the homotopy group $\pi_W^{\Sigma_m\wr H}(Y)$, but the map is generally not surjective. For example, for $H=e$ we have $\rho_e^m=\mathbb{R}^m$ with Σ_m-action by coordinate permutation. In the case $Y=\mathbb{S}$ of the sphere spectrum the group $\pi_0^{\Sigma_m}(\mathbb{S})$ is isomorphic to the Burnside ring $A(\Sigma_m)$ of the group Σ_m, whereas $\pi_0^{\Sigma_m,\rho_e^m}(\mathbb{S})$ is the subgroup generated by those Σ_m-set that can be embedded equivariantly into the natural Σ_m-representation on $\mathbb{R}^m$. A coset Σ_m/H belongs to the restricted Burnside ring if and only if H is the stabilizer group of some partition of the set $\{1,\dots,m\}$ or, equivalent, if H is conjugate to a subgroup of the form $\Sigma_{i_1}\times\dots\times\Sigma_{i_k}$ with $i_1+\dots+i_k=m$.

Now we discuss properties of the power map. The power map $x\mapsto x^m$ in a commutative ring has the properties

$$0^m=0\ ,\quad 1^m=1\ ,\quad x^0=1\ ,\quad x^1=x\ ,\quad (x^k)^m=x^{km}\ ,\quad x^m\cdot x^n=x^{m+n}\ ,$$

$$(xy)^m=x^m\cdot y^m \qquad \text{and} \qquad (x+y)^m=\sum_{i=0}^m\binom{m}{i}x^i\cdot y^{m-i}$$

All of these properties have analogues for the power construction in equivariant stable homotopy theory.

We obviously have $P^m(0)=0$. For the case $X=\mathbb{S}$ of the sphere spectrum we have $P^m(\mathbb{S})=\mathbb{S}$ and the unit element $1\in\pi_0^H\mathbb{S}$ exponentiates to $P^m(1)=1$ in $\pi_0^{\Sigma_m\wr H}\mathbb{S}$. If we restrict the class $P^1x\in\pi_V^{\Sigma_1\wr H}X$ along the canonical isomorphism $H\longrightarrow\Sigma_1\wr H$ that sends h to $(1;\ h)$ we recover x.

The power map is transitive in the sense of the *composition formula*

$$P^m(P^kx) \ = \ \mathrm{res}^{\Sigma_{km}\wr H}_{\Sigma_m\wr(\Sigma_k\wr H)}(P^{km}x)$$

in the group $\pi_{(V^k)^m}^{\Sigma_m\wr(\Sigma_k\wr H)}P^m(P^kX)$. Here we used the fact that $P^m(P^kX)$ is the restriction of $P^{km}X$ along the monomorphism (9.1) from $\Sigma_m\wr(\Sigma_k\wr H)$ to $\Sigma_{km}\wr H$ and $(V^k)^m$ is the restriction of V^{km} along the same monomorphism.

The power map interacts nicely with the external product: we have

$$(P^mx)\cdot(P^nx) \ = \ \mathrm{res}^{\Sigma_{m+n}\wr H}_{(\Sigma_m\wr H)\times(\Sigma_n\wr H)}(P^{m+n}x)$$

in the group $\pi_{V^{m+n}}^{(\Sigma_m\wr H)\times(\Sigma_n\wr H)}(P^mX\wedge P^nX)$, using that $P^mX\wedge P^nX$ is the restriction of $P^{m+n}X$ along the monomorphism $+:(\Sigma_m\wr H)\times(\Sigma_n\wr H)\longrightarrow\Sigma_{m+n}\wr H$ and $V^m\oplus V^n$ is the restriction of V^{m+n} along the same monomorphism.

Moreover, for classes $x\in\pi_V^H(X)$ and $y\in\pi_W^H(Y)$ we have the *product formula*

$$(\chi^{(m)}_{X,Y})_*((P^mx)\bullet(P^my)) \ = \ P^m(x\bullet y) \tag{9.4}$$

in the group $\pi^{\Sigma_m\wr H}_{(V\oplus W)^m}P^m(X\wedge Y)$, where $\chi^{(m)}_{X,Y}: P^mX\wedge P^mY\cong P^m(X\wedge Y)$ is the shuffling isomorphism and $V^m\oplus W^m\cong (V\oplus W)^m$ given by shuffling factors respectively summands. For $x,y\in\pi^H_V(X)$ the power operation satisfies the *sum formula*

$$P^m(x+y) \ = \ \sum_{i=0}^m \ \mathrm{tr}_{i,m-i}(P^ix\cdot P^{m-i}y)\ . \tag{9.5}$$

The dot on the right hand side refers to the external product

$$\cdot\ :\ \pi^{\Sigma_i\wr H}_{V^i}(P^iX)\ \times\ \pi^{\Sigma_{m-i}\wr H}_{V^{m-i}}(P^{m-i}X)\ \longrightarrow\ \pi^{(\Sigma_i\wr H)\times(\Sigma_{m-i}\wr H)}_{V^i\oplus V^j}(P^iX\wedge P^{m-i}X)$$

and

$$\mathrm{tr}_{i,m-i}\ :\ \pi^{(\Sigma_i\wr H)\times(\Sigma_{m-i}\wr H)}_{V^i\oplus V^{m-i}}(P^iX\wedge P^{m-i}X)\ \longrightarrow\ \pi^{\Sigma_m\wr H}_{V^m}(P^mX)$$

is the $RO(G)$-graded internal transfer map (4.35) for the monomorphism $+: (\Sigma_i\wr H)\times(\Sigma_{m-i}\wr H)\longrightarrow\Sigma_m\wr H$, using that the restriction of P^mX along this monomorphism is $P^iX\wedge P^{m-i}X$ and the restriction of V^m is $V^i\oplus V^{m-i}$.

If K is a subgroup of H and $y\in\pi^K_V(X)$, then we have

$$P^m(\mathrm{tr}^H_K\, y)\ =\ \mathrm{tr}^{\Sigma_m\wr H}_{\Sigma_m\wr K}(P^my)\ .$$

Power operations are compatible with restriction: for every group homomorphism $\alpha: K\longrightarrow H$ and every H-spectrum X we have

$$P^m(\alpha^*X)\ =\ (\Sigma_m\wr\alpha)^*(P^mX)\qquad\text{and}\qquad(\alpha^*V)^m\ =\ (\Sigma_m\wr\alpha)^*(V^m)$$

as $\Sigma_m\wr K$-spectra respectively $\Sigma_m\wr K$-representations and the square

$$\begin{tikzcd}
\pi^H_V(X) \arrow[r, "P^m"] \arrow[d, "\alpha^*"'] & \pi^{\Sigma_m\wr H}_{V^m}(P^mX) \arrow[d, "(\Sigma_m\wr\alpha)^*"] \\
\pi^K_{\alpha^*V}(\alpha^*X) \arrow[r, "P^m"'] & \pi^{\Sigma_m\wr K}_{\alpha^*V^m}(P^m(\alpha^*X))
\end{tikzcd}$$

commutes.

Power operations also commute with conjugation: If H is a subgroup of G and $g\in G$ we set $\Delta(g)=(1;\ g,\dots,g)\in\Sigma_m\wr G$. Then $(c_g^*V)^m=c^*_{\Delta(g)}(V^m)$ as $\Sigma_m\wr H$-representation and for every G-spectrum $X$the square

$$\begin{tikzcd}
\pi^H_V(X) \arrow[r, "P^m"] \arrow[d, "g_\star"'] & \pi^{\Sigma_m\wr H}_{V^m}(P^mX) \arrow[d, "\Delta(g)_\star"] \\
\pi^{{}^gH}_{c_g^*V}X \arrow[r, "P^m"'] & \pi^{\Sigma_m\wr{}^gH}_{(c_g^*V)^m}(P^mX)
\end{tikzcd}$$

commutes.

10. Norm construction

In this section we review the norm construction for equivariant orthogonal spectra. The norm construction and norm map, also known as 'multiplicative transfer', were first introduced by Evens in the algebraic context of group cohomology [7]. In the context of equivariant stable homotopy theory, multiplicative norm maps were first studied by Greenlees and May in [9], and the norm construction was first developed by Hill, Hopkins and Ravenel [10]. Again, our exposition is a little different from the ones in [9, 10].

We are given a group G, a subgroup H of G and an H-spectrum X. The multiplicative norm $N_H^G X$ is a certain G-spectrum whose underlying H-spectrum is a $[G:H]$-fold smash product of copies of H. The multiplicative norm construction is strong symmetric monoidal, i.e., equipped with coherent isomorphisms

$$N_H^G X \wedge N_H^G Y \;\cong\; N_H^G(X \wedge Y)\ .$$

So if R is an H-ring spectrum, then $N_H^G R$ becomes a G-ring spectrum via the composite

$$N_H^G R \wedge N_H^G R \;\cong\; N_H^G(R \wedge R) \xrightarrow{N_H^G \mu} N_H^G R\ ,$$

and $N_H^G R$ is commutative whenever R is. Moreover, for commutative equivariant ring spectra, the functor N_H^G is left adjoint to the restriction of commutative G-ring spectra to commutative H-ring spectra.

The most important homotopical property of the norm functor is that it takes $\underline{\pi}_*$-isomorphism between cofibrant H-spectra to $\underline{\pi}_*$-isomorphism of G-spectra, so it allows a derived functor

$$N_H^G \;:\; \mathrm{Ho}(\mathcal{S}p_H) \;\longrightarrow\; \mathrm{Ho}(\mathcal{S}p_G)$$

that is still strong symmetric monoidal.

Motivation. The norm construction is a multiplicative version of induction from a subgroup to a larger group. In order to motivate the construction of the norm functor we review induction from a subgroup H to a supergroup G in the context of representations.

If V is an H-representation, then the induced G-representation is

$$G \ltimes_H V \;=\; \mathbb{R}[G] \otimes_{\mathbb{R}[H]} V\ .$$

Additively $G \ltimes_H V$ is a direct sum of $[G:H]$ copies of H. We can define an explicit decomposition by choosing an ordered set

$$(g_1, g_2, \dots, g_m)$$

of representatives for the right cosets of H in G, where $m = [G:H]$ is the index of H in G. A specific $\mathbb{R}$-linear isomorphism is then given by

$$\alpha \;:\; V^m \;\longrightarrow\; G \ltimes_H V\ , \quad (v_1, \dots, v_m) \;\longmapsto\; \sum_{i=1}^m g_i v_i$$

(this isomorphism is in general *not* H-linear). This decomposition depends on the chosen coset representatives, and the G-action on the right hand side does not a priori correspond to anything on the left hand side.

Now we 'average' over all possible collections of coset representatives and thereby obtain a version of V^m equipped with a canonical G-action. Since V is an H-representation, V^m is naturally a representation over the wreath product

$$\Sigma_m \wr H \;=\; \Sigma_m \ltimes H^m$$

with multiplication given by

$$(\sigma;\, h_1, \dots, h_m) \cdot (\tau;\, k_1, \dots, k_m) \;=\; (\sigma\tau;\, h_{\tau(1)} k_1, \dots, h_{\tau(m)} k_m)\ .$$

The action on V^m is given by the formula

$$(\sigma;\, h_1, \dots, h_m) \cdot (v_1, \dots, v_m) \;=\; \left(h_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(m)} v_{\sigma^{-1}(m)}\right)\ .$$

We let $\langle G:H \rangle$ denote the set of all systems of coset representatives for H in G. So an element of $\langle G:H \rangle$ is an m-tuple $(g_1, \dots, g_m) \in G^m$ such that

$$G \;=\; \bigcup_{i=1}^m g_i H$$

as sets. The group G acts from the left on $\langle G:H\rangle$ by

$$\gamma\cdot(g_1,\dots,g_m) \ = \ (\gamma g_1,\dots,\gamma g_m)\ .$$

The wreath product $\Sigma_m\wr H$ acts on $\langle G:H\rangle$ from the right by

$$(g_1,\dots,g_m)\cdot(\sigma;\, h_1,\dots,h_m) \ = \ (g_{\sigma(1)}h_1,\dots,g_{\sigma(m)}h_m)\ ;$$

this right action of $\Sigma_m\wr H$ is free and transitive. We can now form

$$N_H^G V \ = \ \langle G:H\rangle\times_{\Sigma_m\wr H} V^m$$

which becomes a G-representation by

$$\gamma\cdot[g_1,\dots,g_m;\ v_1,\dots,v_m] \ = \ [\gamma g_1,\dots,\gamma g_m;\ v_1,\dots,v_m]\ .$$

Lemma 10.1. *The map*

$$N_H^G V \ = \ \langle G:H\rangle\times_{\Sigma_m\wr H} V^m \ \longrightarrow \ G\ltimes_H V\ , \quad [g_1,\dots,g_m;\ v_1,\dots,v_m] \ \longmapsto \ \sum_{i=1}^m g_i v_i$$

is G-equivariant isomorphism.

The point of this reinterpretation of the induction functor is that the construction $N_H^G V$ can be performed in any symmetric monoidal category and it yields a functor from H-objects to G-objects. The norm construction in equivariant stable homotopy theory is the special case of the category of orthogonal spectra under smash product. So we now run the analogous story 'multiplicatively', i.e., we replace m-fold direct sum by m-fold tensor or smash product. Given an orthogonal spectrum X, the m-th smash power

$$X^{(m)} \ = \ \underbrace{X\wedge\ldots\wedge X}_{m}$$

has a natural action of the symmetric group Σ_m by permuting the factors. If X is an H-spectrum, the H-actions of each factor combine into an action of H^m. Altogether we obtain a natural action of the wreath product $\Sigma_m\wr H$ on $X^{(m)}$. To get the internal smash product in the category of H-spectra we usually restrict this action along the diagonal embedding $H\longrightarrow\Sigma_m\wr H, h\longmapsto(1;h,\dots,h)$, but now we are going to do something different.

Definition 10.2. Let H be a subgroup of G and X an orthogonal H-spectrum. The *norm* $N_H^G X$ is the orthogonal G-spectrum given by

$$N_H^G \ = \ \langle G:H\rangle_+\wedge_{\Sigma_m\wr H} X^{(m)}\ .$$

The following properties are immediate from the construction:

(i) Since $\Sigma_m\wr H$ acts freely and transitively on the set $\langle G:H\rangle$ of coset representatives, the underlying orthogonal spectrum of $N_H^G X$ is isomorphic to $X^{(m)}$. Indeed, if $(g_1,\dots,g_m)$ is one system of coset representatives, then the map

$$X^{(m)} \ \xrightarrow{[g_1,\dots,g_m;\ -]} \ \langle G:H\rangle_+\wedge_{\Sigma_m\wr H} X^{(m)} \ = \ N_H^G X$$

is an isomorphism of orthogonal spectra.

(ii) The norm functor commutes with smash products up to coherently associative, unital and commutative isomorphism. Indeed, 'reshuffling the factors' provides an isomorphism of orthogonal spectra

$$\chi_{X,Y}^{(m)} \ : \ (X\wedge Y)^{(m)} \ \cong \ X^{(m)}\wedge Y^{(m)}$$

that is $\Sigma_m \wr H$-equivariant (with diagonal $\Sigma_m \wr H$-action on the right hand side). So upon application of $\langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} -$ we obtain an isomorphism of orthogonal G-spectra

$$\begin{aligned} N_H^G(X \wedge Y) \ &= \ \langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} (X \wedge Y)^{(m)} \\ &\longrightarrow \Big(\langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} X^{(m)}\Big) \wedge \Big(\langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} Y^{(m)}\Big) \ = \ N_H^G X \wedge N_H^G Y \\ [\bar g;\ (x_1 \wedge y_1) \wedge \ldots \wedge (x_m \wedge y_m)] \ &\longmapsto \\ & [\bar g;\, x_1 \wedge \ldots \wedge x_m] \ \wedge \ [\bar g;\ y_1 \wedge \ldots \wedge y_m] \end{aligned}$$

(iii) As consequence of the previous item we get that for every H-ring spectrum R the norm $N_H^G R$ is a G-ring spectrum with multiplication

$$N_H^G R \wedge N_H^G R \ \cong \ H_H^G(R \wedge R) \ \xrightarrow{N_H^G \mu} \ N_H^G R \ .$$

If the multiplication of R is commutative, so is the multiplication of $N_H^G R$. Hence N_H^G passes to a functor from commutative H-ring spectra to commutative G-ring spectra, and as such it is left adjoint to restriction from G to H.

(iv) The norm construction is transitive, i.e., for $K \subset H \subset G$ and every orthogonal K-spectrum X, the G-spectra $N_H^G(N_K^H X)$ and $N_K^G X$ are naturally isomorphic. Moreover, the collection of isomorphisms $N_H^G(N_K^H X) \longrightarrow N_K^G X$ (to be defined below) is itself transitive, in the sense that for every quadruple of nested groups $L \subset K \subset H \subset G$ the two composite isomorphism from $N_H^G(N_K^H(N_L^K X))$ to $N_L^G X$ are equal.

The construction of the transitivity isomorphism starts from the map

$$\langle G : H\rangle \times \langle H : K\rangle^m \ \longrightarrow \ \langle G : K\rangle \ , \quad ((g_1, \ldots, g_m), (\bar h^1, \ldots, \bar h^m)) \ \longmapsto \ (g_i h_j^i)_{1 \le i \le m, 1 \le j \le n} \ ,$$

where $m = [G : H]$, $n = [H : K]$ and $\bar h^i = (h_1^i, \ldots, h_n^i)$. This factors over a well-defined map

$$\langle G : H\rangle \times_{\Sigma_m \wr H} \langle H : K\rangle^m \ \longrightarrow \ \langle G : K\rangle$$

that is equivariant for the left G-action and for the right action of the group $\Sigma_m \wr (\Sigma_n \wr K)$. that is equivariant for the left G-action and for the right action of the group $\Sigma_m \wr (\Sigma_n \wr K)$. On the target the larger group $\Sigma_{mn} \wr K$ acts from the right, and the map induces a morphism of orthogonal G-spectra

$$\begin{aligned} N_H^G(N_K^H X) \ &= \ \langle G : H\rangle_+ \times_{\Sigma_m \wr H} \Big(\langle H : K\rangle_+ \wedge_{\Sigma_n \wr K} X^{(n)}\Big)^{(m)} \\ &\cong (\langle G : H\rangle \times_{\Sigma_m \wr H} \langle H : K\rangle^m)_+ \wedge_{\Sigma_m \wr (\Sigma_n \wr K)} (X^{(n)})^{(m)} \\ &\longrightarrow \ \langle G : K\rangle_+ \wedge_{\Sigma_{mn} \wr K} X^{(mn)} \ = \ N_K^G X \end{aligned}$$

To check that this map is an isomorphism we use that the underlying non-equivariant orthogonal spectra of both sides are isomorphic to an nm-fold smash power of X. Indeed, if $(g_1, \ldots, g_m)$ is a system of coset representatives for H in G and $(h_1, \ldots, h_n)$ is a system of coset representatives for K in H, then $(g_i h_j)_{1 \le i \le m, 1 \le j \le n}$ is a system of coset representatives for K in G. Moreover, the diagram

$$\begin{tikzcd} (X^{(n)})^{(m)} \arrow[r, "{[h_1,\ldots,h_n;\ -]^{(m)}}"] \arrow[drr, "{[g_i h_j;\ -]}"'] & (N_K^H X)^{(m)} \arrow[r, "{[g_1,\ldots,g_m;\ -]}"] & N_H^G(N_K^H X) \arrow[d] \\ & & N_K^G X \end{tikzcd}$$

commutes, and so the right vertical map is an isomorphism since the other three maps are.

Remark 10.3. Since $\Sigma_m \wr H$ acts freely and transitively on the set $\langle G : H\rangle$ of coset representatives, the map

$$X^{(m)} \xrightarrow{[g_1,\dots,g_m;\ -]} \langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} X^{(m)} \ = \ N_H^G X \tag{10.4}$$

is an isomorphism of orthogonal spectra for every system of coset representatives $\bar g = (g_1,\dots,g_m)$. We can transfer the G-action on $N_H^G X$ along this isomorphism into a G-action on $X^{(m)}$. The transferred G-action on $X^{(m)}$ has the following explicit description. The chosen coset representatives $\bar g$ define a monomorphism

$$\Psi \ : \ G \ \longrightarrow \ \Sigma_m \wr H \qquad \text{by} \qquad \gamma \cdot \bar g \ = \ \bar g \cdot \Psi(\gamma) \ , \tag{10.5}$$

using that the right action of $\Sigma_m \wr H$ on $\langle G : H\rangle$ is free. More explicitly the components of the element $\Phi(\gamma) = (\sigma;\ h_1,\dots,h_m)$ are determined by

$$\gamma g_i \ = \ g_{\sigma(i)} h_i$$

for $i = 1,\dots,m$. We can restrict the $\Sigma_m \wr H$-action on $X^{(m)}$ to a G-action along the monomorphism Ψ, and then the isomorphism (10.4) is G-equivariant. In other words, $N_H^G X$ is naturally isomorphic, as a G-spectrum, to $\Psi^*(X^{(m)})$. So we recover the point of view adopted by Evens [7] and Greenlees-May [9], who define the norm construction by choosing a set of coset representatives and restricting along the resulting homomorphism Ψ.

Remark 10.6. As we already indicated, the norm construction makes sense in any category $\mathcal{C}$ equipped with a symmetric monoidal product $\Box$. Indeed, for every H-object X in $\mathcal{C}$, object

$$X^{(m)} \ = \ \underbrace{X \Box \cdots \Box X}_{m}$$

is acted upon by $\Sigma_m \wr H$ and we can set

$$N_H^G X \ = \ \langle G : H\rangle \ltimes_{\Sigma_m \wr H} X^{(m)} \ .$$

(This definition implicitly claims the existence of a certain coequalizer inn $\mathcal{C}$, which exists because we can for example take the object $\Psi^* X^{(m)}$, where $\Psi : G \longrightarrow \Sigma_m \wr H$ is the monomorphism (10.5) defined from any choice of coset representatives.) The formal properties (i)–(iv) above carry over with the same formal proofs. Besides the category of orthogonal spectra under smash product there are some other cases where we need the associated norm construction:

(a) In the category of sets under disjoint union, the norm construction is isomorphic to induction. Indeed, for every finite H-set S the map

$$\langle G : H\rangle \ltimes_{\Sigma_m \wr H} (\{1,\dots,m\} \times S) \ \longrightarrow \ G \times_H S$$

defined by

$$[g_1,\dots,g_m;\ (i,s)] \ \longmapsto \ g_i s$$

is G-equivariant bijection.

(b) We consider the category of bases sets, based spaces or orthogonal spectra under wedge. The norm construction is again is isomorphic to induction because the map

$$\langle G : H\rangle \ltimes_{\Sigma_m \wr H} (\{1,\dots,m\}_+ \wedge X) \ \longrightarrow \ G \ltimes_H X$$

defined as in the previous example is G-equivariant bijection.

(c) There is a 'multiplicative' version of the last two examples. We again consider the category of sets, based spaces or orthogonal spectra, but this time under cartesian product. A G-equivariant isomorphism

$$\langle G : H\rangle \ltimes_{\Sigma_m \wr H} X^m \ \longrightarrow \ \mathrm{map}^H(G, X)$$

is then given by

$$[g_1,\dots,g_m;\ x_1,\dots,x_m] \ \longmapsto \ [hg_i \ \mapsto \ hx_i] \ .$$

Here we use that every element of G is uniquely of the form hg_i for one of the coset representatives g_i and a unique element $h \in H$.

Let H be a subgroup of G, X an orthogonal H-spectrum and V an H-representation. In the following we shall need a natural map

$$J_{X,V} \ : \ N_H^G(X(V)) \ \longrightarrow \ (N_H^G X)(G \ltimes_H V) \tag{10.7}$$

that relates the space level norm construction (with respect to smash product) of the based H-space $X(V)$ to the value of the spectrum level norm $N_H^G X$ at the induced representation. The construction of this map starts from the $(V,\dots,V)$-component of the universal multilinear map

$$i_{V,\dots,V} \ : \ X(V) \wedge \ldots \wedge X(V) \ \longrightarrow \ (X^{(m)})(V^m)$$

which is $\Sigma_m \wr H$-equivariant. On the target the wreath product acts diagonally for the two actions on the spectrum $X^{(m)}$ and the representation V^m. We compose the induced map $\langle G : H\rangle \ltimes_{\Sigma_m \wr H} i_{V,\dots,V}$ with the homeomorphism

$$\langle G : H\rangle \ltimes_{\Sigma_m \wr H} (X^{(m)}(V^m)) \quad \cong \quad (\langle G : H\rangle \ltimes_{\Sigma_m \wr H} X^{(m)})(\langle G : H\rangle \times_{\Sigma_m \wr H} V^m) \ = \ (N_H^G X)(G \ltimes_H V)$$

and obtain the map $J_{X,V}$.

Example 10.8. We consider the free orthogonal H-spectrum F_V generated by an H-representation V. A morphism of orthogonal G-spectra

$$F_{G \ltimes_H V} \ \longrightarrow \ N_H^G(F_V)$$

is freely generated by the image of the point $\mathrm{Id}_V^{(m)}$ under the map

$$N_H^G \mathbf{O}(V) \ = \ N_H^G(F_V(V)) \ \xrightarrow{J_{F_V,V}} \ (N_H^G F_V)(G \ltimes_H V) \ .$$

We claim that this morphisms is an isomorphism. Indeed, repeated use of the canonical isomorphism $F_{V \oplus W} \cong F_V \wedge F_W$ defined in (5.8) with $V = W$ gives an isomorphism

$$F_{V^m} \ \cong \ F_V^{(m)}$$

of $\Sigma_m \wr H$-spectra. Application of $\langle G : H\rangle \ltimes_{\Sigma_m \wr H} -$ gives a sequence of isomorphisms of orthogonal G-spectra

$$\begin{aligned} F_{G \ltimes_H V} \ &= \ F_{\langle G:H\rangle \ltimes_{\Sigma_m \wr H} V^m} \ \cong \ \langle G : H\rangle \ltimes_{\Sigma_m \wr H} F_{V^m} \\ &\cong \ \langle G : H\rangle \ltimes_{\Sigma_m \wr H} F_V^{(m)} \ = \ N_H^G(F_V) \end{aligned}$$

(where we used Lemma 10.1 to identify $\langle G : H\rangle \ltimes_{\Sigma_m \wr H} V^m$ with the induced representation $G \ltimes_H V$). More generally, the free spectrum generated by a based H-space A norms as

$$N_H^G(F_V A) \ \cong \ F_{G \ltimes_H V}(N_H^G A)$$

where $N_H^G A$ is the space level norm construction of A.

This argument generalizes to semifree spectra as follows. We let V be an H-representation and L a based $H \ltimes O(V)$-space. The semifree spectrum $G_V L$ generated by L in level V was introduced in Example 5.9. There is then a natural isomorphism

$$G_{G \ltimes_H V}\left(O(G \ltimes_H V)_+ \wedge_{N_H^G O(V)} N_H^G L\right) \ \cong \ N_H^G(G_V L) \ .$$

The semifree spectrum on left hand side needs to be explained. Here $N_H^G L$ is the space level norm construction of the underlying H-space of L. The normed space $N_H^G L$ comes with an action of the normed group $N_H^G O(V)$, so that altogether the semidirect product $G \ltimes N_H^G O(V)$ acts on $N_H^G L$. We extend the action of $N_H^G O(V)$ along the monomorphism $N_H^G O(V) \longrightarrow O(G \ltimes_H V)$ (or, equivalently, extend the $G \ltimes N_H^G O(V)$-action along $G \ltimes N_H^G O(V) \longrightarrow G \ltimes O(G \ltimes_H V)$) and then form the semifree G-spectrum in level $G \ltimes_H V$.

Example 10.9. We discuss an example relevant to the solution by Hill, Hopkins and Ravenel of the Kervaire invariant problem [10]. In Examples 2.14 and 7.11 we discussed the commutative C_2-ring spectrum MR whose underlying non-equivariant spectrum is the complex cobordism spectrum and whose geometric fixed point spectrum is stably equivalent to the unoriented cobordism spectrum MO.

Hill, Hopkins and Ravenel consider the spectrum

$$MU^{(4)} \ = \ N_{C_2}^{C_8}(MR) \ ,$$

the norm of MR along the unique monomorphism $C_2 \longrightarrow C_8$ of the cyclic group of order 2 into the cyclic group of order 8. Then the underlying C_2-spectrum of $MU^{(4)}$ is

$$MR \wedge MR \wedge MR \wedge MR \ .$$

If t is a generator of C_8, we can take $\{1, t, t^2, t^3\}$ as a set of coset representatives for C_2. The associated monomorphism $\Phi : C_8 \longrightarrow \Sigma_4 \wr C_2$ sends the generator t to the element

$$\Phi(t) \ = \ ((1234);\, 1, 1, 1, \tau) \ \in \ \Sigma_4 \ltimes C_2^4 = \Sigma_4 \wr C_2 \ ,$$

where $C_2 = \{1, \tau\}$. That means that the action of the generator t on $MR \wedge MR \wedge MR \wedge MR$ is given by complex conjugation of the last factor, followed a cyclic permutation of the factors; symbolically, we have

$$t \cdot (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \ = \ \bar{x}_4 \wedge x_1 \wedge x_2 \wedge x_3 \ .$$

To be completely honest, one has to admit that setting $MU^{(4)} = N_{C_2}^{C_8}(MR)$ is oversimplifying matters. Indeed, we simultaneously want certain formal properties and we want to be able to control the equivariant homotopy type of $MU^{(4)}$. To achieve this, we have to feed into the norm construction a commutative C_2-ring spectrum whose underlying C_2-spectrum is sufficiently cofibrant (or rather *flat*). I doubt that the specific model MR defined in Example 2.14 is sufficiently cofibrant. So one has to construct another commutative C_2-ring spectrum MR^c, sufficiently cofibrant, and a multiplicative $\underline{\pi}_*$-isomorphism $MR^c \longrightarrow MR$ and then take $MU^{(4)} = N_{C_2}^{C_8}(MR^c)$. This is possible, but I am not aware of a construction that avoids discussing positive model structure on equivariant spectra and equivariant ring spectra. The necessary details can be found in Appendix B of the paper [10] by Hill, Hopkins and Ravenel.

Now we get to the key homotopical property of the norm construction:

Proposition 10.10. *Let H be a subgroup of G. Then the norm functor*

$$N_H^G \ : \ \mathcal{S}p_H \ \longrightarrow \ \mathcal{S}p_G$$

takes $\underline{\pi}_*$-isomorphisms between cofibrant H-spectra to $\underline{\pi}_*$-isomorphisms of G-spectra. Hence the norm functor descends to a functor

$$N_H^G \ : \ \mathrm{Ho}(\mathcal{S}p_H) \ \longrightarrow \ \mathrm{Ho}(\mathcal{S}p_G)$$

on homotopy categories.

Proof. We just give part of the argument. The $\underline{\pi}_*$-isomorphisms of H-spectra are generated, in a suitable sense, by weak H-equivalences between based H-spaces and by the morphisms $\lambda_V : F_V S^V \longrightarrow \mathbb{S}$, adjoint to the identity of S^V, for all H-representations V.

So we consider a weak H-equivalence $f : A \longrightarrow B$ between based H-CW-complexes. We have a natural isomorphism

$$N_H^G(\Sigma^\infty A) \ \cong \ \Sigma^\infty(N_H^G A) \ ,$$

where $N_H^G A$ is the space level norm construction with respect to smash product, i.e., $N_H^G A = \langle G : H\rangle_+ \wedge_{\Sigma_m \wr H} A^{(m)}$. Raising an equivariant space to the m-th power takes weak H-equivalences to weak $\Sigma_m \wr H$-equivalence, so this settles the case of suspension spectra of equivariant CW-complexes.

The case of the $\underline{\pi}_*$-isomorphism $\lambda_V : F_V S^V \longrightarrow \mathbb{S}$ is handled as follows. As explained in Example 10.8, the normed spectrum $H_G^H F_V$ is isomorphic to the free G-spectrum $F_{G \ltimes_H V}$ of the induced representation

$G\ltimes_H V$. Similarly, $N_H^G(S^V)$ is isomorphic to the sphere $S^{G\ltimes_H V}$ of the induced representation, so altogether we can identify

$$N_H^G(F_V S^V) \;\cong\; F_{G\ltimes_H V} S^{G\ltimes_H V}$$

as G-spectra. Under this identification and $N_H^G\mathbb{S}\cong\mathbb{S}$, the morphism $N_H^G\lambda_V : N_H^G(F_V S^V)\longrightarrow N_H^G\mathbb{S}$ becomes the morphism $\lambda_{G\ltimes_H V} : F_{G\ltimes_H V}S^{G\ltimes_H V}\longrightarrow\mathbb{S}$, which is a $\underline{\pi}_*$-isomorphism of G-spectra by Proposition 5.12.

Now we may attempt to run the usual 'cell induction argument'; a problem is then that the norm functor is not 'additive' (it does not commute with colimits), but rather a 'power construction'. So there is more to say when analyzing the effect of N_H^G on a cell attachment, but we stop here for the time being. $\square$

Our next topic is the relationship between the norm construction and geometric fixed points. This relationship is given by a natural morphism of non-equivariant spectra

$$\Delta \;:\; \Phi^H X \;\longrightarrow\; \Phi^G(N_H^G X) \tag{10.11}$$

that is a stable equivalence whenever X is a cofibrant H-spectrum.

For every based H-space A, the diagonal map $\Delta : A^H \longrightarrow (N_H^G A)^G$ is a homeomorphism. For every orthogonal H-spectrum X, the special case $V=n\rho_H$ of the map (10.7) is an $O(n)$-equivariant map

$$J_{X,n\rho_H} \;:\; N_H^G(X(n\rho_H)) \;\longrightarrow\; (N_H^G X)(n\rho_G)\ ,$$

where we have used that the induced representation $G\ltimes_H \rho_H$ is canonically isomorphic to the regular representation of G. So by combining the two maps we obtain a based continuous,

$$(\Phi^H X)_n \;=\; X(n\rho_H)^H \;\cong\; (N_H^G(X(n\rho_H)))^G \;\xrightarrow{J^G_{X,n\rho_H}}\; ((N_H^G X)(n\rho_G))^G \;=\; (\Phi^G(N_H^G X))_n$$

As n varies these maps make up the morphism $\Delta : \Phi^H X\longrightarrow\Phi^G(N_H^G X)$.

Proposition 10.12. *For every cofibrant orthogonal H-spectrum X the map*

$$\Delta \;:\; \Phi^H X \;\longrightarrow\; \Phi^G(N_H^G X)$$

is a π_*-isomorphism of orthogonal spectra.

Proof. Again we only check two crucial special cases. First, if $X=\Sigma^\infty A$ is the suspension spectrum of an H-CW-complex A, then $\Phi^H X = \Phi^H(\Sigma^\infty A)\cong\Sigma^\infty A^H$ and

$$\Phi^G(N_H^G X) \;=\; \Phi^G(N_H^G(\Sigma^\infty A)) \;\cong\; \Phi^G(\Sigma^\infty(N_H^G A)) \;\cong\; \Sigma^\infty(N_H^G A)^G \;\cong \Sigma^\infty A^H\ .$$

Here we use that G-fixed points of the space level norm construction $N_H^G A$ are isomorphic to H-fixed points of A. We conclude that the map $\Delta : \Phi^H(\Sigma^\infty A)\longrightarrow\Phi^G(N_H^G(\Sigma^\infty A))$ is an isomorphism, so in particular a π_*-isomorphism.

If $X=F_V$ is the free H-spectrum generated by an H-representation V, then, loosely speaking, $\Phi^H F_V$ is a '$-V^H$-sphere', whereas $N_H^G F_V\cong F_{G\ltimes_H}$, and so $\Phi^G(N_H^G F_V)$ is a '$-(G\ltimes_H V)^G$-sphere'. Since the G-fixed points of the induced representation are naturally isomorphic to the H-fixed points of V, this shows the claim for $X=F_V$.

More formally, we argue as follows. Since the norm construction preserves $\underline{\pi}_*$-isomorphisms between cofibrant spectra, the class of cofibrant H-spectra X for which $\Delta : \Phi^H X\longrightarrow\Phi^G(N_H^G X)$ is a π_*-isomorphism is closed under π_*-isomorphisms. Proposition 5.12 provides a π_*-isomorphism $F_V S^V\longrightarrow\mathbb{S}$. Since the sphere spectrum is an equivariant suspension spectrum, so the claim holds for $\mathbb{S}$ by the first paragraph, and hence for the H-spectrum $F_V S^V$. In the commutative diagram

$$\begin{tikzcd}
S^{V^H}\wedge\Phi^H F_V \arrow[rr, "S^{V^H}\wedge\Delta"] \arrow[d, "\cong"'] & & S^{V^H}\wedge\Phi^G(N_H^G F_V) \arrow[d, "\cong"] \\
\Phi^H(F_V S^V) \arrow[r, "\Delta"'] & \Phi^G(N_H^G(F_V S^V)) \arrow[r, "\cong"] & \Phi^G(F_{G\ltimes_H V}S^{G\ltimes_H V})
\end{tikzcd}$$

three of the five maps are isomorphisms, and the lower horizontal morphism is a π_*-isomorphism by the above. So the map $S^{V^H} \wedge \Delta$ is $S^{V^H} \wedge \Delta$ is a π_*-isomorphism, hence so is $\Delta : \Phi^H F_V \longrightarrow \Phi^G(N_H^G F_V)$. □

As we mentioned at the beginning of this section, the norm construction, when extended to commutative equivariant ring spectra, is left adjoint to the restriction functor from commutative G-ring spectra to commutative H-ring spectra. Again, this is a formal argument that works in any symmetric monoidal category. Given a commutative G-ring spectrum R, we define a morphism of commutative G-ring spectra

$$\epsilon \ : \ N_H^G(i^* R) \ \longrightarrow \ R$$

as follows. For every system of coset representatives $\bar{g} = (g_1, \dots, g_m)$ we define

$$\epsilon_{\bar{g}} \ : \ R^{(m)} \ \longrightarrow \ R$$

as the composite

$$R^{(m)} \xrightarrow{(g_1\cdot -)\wedge\dots\wedge(g_m\cdot -)} R^{(m)} \xrightarrow{\ \mu\ } R$$

where μ is the iterated multiplication morphism of R. For every element $\kappa \in \Sigma_m \wr H$ the composite

$$R^{(m)} \xrightarrow{\kappa\cdot} R^{(m)} \xrightarrow{\epsilon_{\bar{g}}} R$$

equals $\epsilon_{\bar{g}\kappa}$, so the morphisms $\epsilon_{\bar{g}}$ assemble into a morphism of orthogonal spectra

$$\epsilon \ : \ N_H^G R = \langle G : H\rangle \ltimes_{\Sigma_m \wr H} R^{(m)} \ \longrightarrow \ R \ .$$

Clearly, if we follow $\epsilon_{\bar{g}}$ by left multiplication by an element $\gamma \in G$, we obtain $\epsilon_{\gamma\bar{g}}$, so the morphism ϵ is G-equivariant. Moreover, the morphism ϵ is multiplicative.

Proposition 10.13. *The norm functor N_H^G from commutative orthogonal H-ring spectra to commutative orthogonal G-ring spectra is left adjoint to the restriction functor with respect to the morphism $\epsilon : N_H^G R \longrightarrow R$ as adjunction counit.*

Proof. We have to show that for every commutative orthogonal H-ring spectrum S, every commutative orthogonal G-ring spectrum R and every morphism $f : N_H^G S \longrightarrow R$ of orthogonal G-ring spectra, there is a unique morphism $\hat{f} : S \longrightarrow R$ of orthogonal H-ring spectra such that $f = \epsilon \circ (N_H^G \hat{f})$. □

11. Norm map

The norm construction for equivariant spectra comes with norm functions on equivariant homotopy groups. We discuss an 'internal' version of the norm map for commutative orthogonal G-ring spectra; there is also an 'external' norm map, that we briefly touch on in Remark 11.13. In the following we let H be a subgroup of G and R a commutative orthogonal G-ring spectrum. The aim of this section is to define and study a norm map $\mathrm{norm}_H^G : \pi_V^H R \longrightarrow \pi_{G\ltimes_H V}^G(R)$ for every H-representation V.

The norm of an element $x \in \pi_V^H R$ is essentially a restriction of the m-th power $P^m(x)$ to R, where $m = [G : H]$ is the index of H in G. In more detail, we will define a homomorphism

$$\langle G|-\rangle \ : \ \pi_{V^m}^{\Sigma_m \wr H}(P^n R) \ \longrightarrow \ \pi_{G\ltimes_H V}^G(R)$$

and then define the norm map by

$$\mathrm{norm}_H^G(x) \ = \ \langle G|P^m(x)\rangle \ . \tag{11.1}$$

We construct the homomorphism $\langle G|-\rangle$ and the norm map norm_H^G in a slightly more general situation.

Construction 11.2. We let H a subgroup of G and $S \subseteq G$ be an *H-invariant subset*, i.e., a subset such that $S \cdot H = S$. For a commutative orthogonal G-ring spectrum R we will now define a homomorphism

$$\langle S|-\rangle \ : \ \pi_{V^n}^{\Sigma_n \wr H}(P^n R) \ \longrightarrow \ \pi_{S\ltimes_H V}^{G\langle S\rangle}(R)$$

where $G\langle S\rangle = \{\gamma \in G \mid \gamma \cdot S = S\}$ is the stabilizer subgroup of S and $n = |S/H|$ is the *index* of S, i.e., the number of disjoint H-cosets that make up S. We will then define a norm map

$$\mathrm{norm}_H^S \;:\; \pi_V^H R \longrightarrow \pi^{G\langle S\rangle}_{S\ltimes_H V} R \qquad \text{by} \qquad \mathrm{norm}_H^S(x) \;=\; \langle S|P^n x\rangle\ .$$

The morphisms $\langle S|-\rangle$ and the norm map norm_H^S are natural for homomorphism of commutative orthogonal G-ring spectra.

For the construction we choose an H*-basis* of S, i.e., an ordered n-tuple $(g_1, \dots, g_n)$ of elements in disjoint H-cosets that satisfy

$$S \;=\; \bigcup_{i=1}^n g_i H\ .$$

The stabilizer group $G\langle S\rangle$ acts from the left on the set $\langle S : H\rangle$ of all such H-bases of S by

$$\gamma \cdot (g_1, \dots, g_n) \;=\; (\gamma g_1, \dots, \gamma g_n)\ .$$

The wreath product $\Sigma_n \wr H$ acts freely and transitively on $\langle S : H\rangle$ from the right by

$$\bar g \cdot (\sigma; h_1, \dots, h_n) \;=\; (g_1, \dots, g_n)\cdot(\sigma; h_1, \dots, h_n) \;=\; (g_{\sigma(1)}h_1, \dots, g_{\sigma(n)}h_n)\ .$$

The chosen basis then determines a monomorphism $\Psi_{\bar g} : G\langle S\rangle \longrightarrow \Sigma_n \wr H$ by requiring that

$$\gamma \cdot \bar g \;=\; \bar g \cdot \Psi_{\bar g}(\gamma)\ .$$

We can then define a $G\langle S\rangle$-equivariant linear isometry

$$i_{\bar g} \;:\; \Psi_{\bar g}^*(V^n) \;\longrightarrow\; S \ltimes_H V\ , \quad (v_1, \dots, v_n) \;\longmapsto\; \sum_{i=1}^n g_i \otimes v_i\ .$$

Moreover, for every commutative orthogonal G-ring spectrum R we can define a $G\langle S\rangle$-equivariant morphism of orthogonal spectra $\epsilon_{\bar g} : \Psi_{\bar g}^*(P^n R) \longrightarrow R$ as the composite

$$\Psi_{\bar g}^*(P^n R) \xrightarrow{(g_1\cdot -)\wedge\dots\wedge(g_n\cdot -)} R^{(n)} \xrightarrow{\ \mu\ } R$$

where μ is the iterated multiplication morphism of R. So we can finally define the homomorphism $\langle S|-\rangle$ associated to an H-invariant subset S of G as the composite

$$\pi_{V^n}^{\Sigma_n\wr H}(P^n R) \xrightarrow{\Psi_{\bar g}^*} \pi^{G\langle S\rangle}_{\Psi_{\bar g}^*(V^n)}(\Psi_{\bar g}^*(P^n R)) \xrightarrow{(\epsilon_{\bar g})_*} \pi^{G\langle S\rangle}_{\Psi_{\bar g}^*(V^n)}(R) \xrightarrow{(i_{\bar g})_*} \pi^{G\langle S\rangle}_{S\ltimes_H V}(R)\ .$$

If S happens to be a subgroup of G containing H, then $G\langle S\rangle = S$ and so the morphism $\langle S|-\rangle$ and the norm map norm_H^S take values in $\pi^S_{S\ltimes_H V}(R)$. In particular, if $S = G$ is the full group G, then $\langle G|-\rangle$ and norm_H^G take values in $\pi^G_{G\ltimes_H V}(R)$.

The construction of the homomorphism $\langle S|-\rangle$ involved a choice of H-basis for S, but we have:

Proposition 11.3. *Let H be a subgroup of G, S an H-invariant subset of G and R a commutative orthogonal G-ring spectrum. The homomorphism $\langle S|-\rangle : \pi_{V^n}^{\Sigma_n\wr H}(P^n R) \longrightarrow \pi^{G\langle S\rangle}_{S\ltimes_H V}(R)$ and the norm map norm_H^S are independent of the choice of H-basis of S.*

Proof. Suppose that $\bar g$ is one H-basis of S. Then any other H-basis is of the form $\bar g\omega$ for a unique $\omega \in \Sigma_n \wr H$. We have $\Psi_{\bar g\omega} = c_\omega \circ \Psi_{\bar g}$, where $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. This implies

$$\begin{array}{rclcl}
\Psi_{\bar g\omega} &=& c_\omega \circ \Psi_{\bar g} &:& G\langle S\rangle \longrightarrow \Sigma_n \wr H \\
\Psi_{\bar g\omega}^* &=& \Psi_{\bar g}^* \circ c_\omega^* &:& \pi_{V^n}^{\Sigma_n\wr H}(Y) \longrightarrow \pi^{\Sigma_n\wr H}_{\Psi_{\bar g\omega}^*(V^n)}(\Psi_{\bar g\omega}^* Y) \;=\; \pi^{\Sigma_n\wr H}_{\Psi_{\bar g}^*(c_\omega^*(V^n))}(\Psi_{\bar g}^*(c_\omega^* Y)) \\
i_{\bar g\omega} &=& i_{\bar g} \circ \Psi_{\bar g}^*(l_\omega^{V^n}) &:& \Psi_{\bar g\omega}^*(V^n) \;=\; \Psi_{\bar g}^*(c_\omega^*(V^n)) \longrightarrow S \ltimes_H V \\
\epsilon_{\bar g\omega} &=& \epsilon_{\bar g} \circ \Psi_{\bar g}^*(l_\omega^{P^m R}) &:& \Psi_{\bar g\omega}^*(P^m R) \;=\; \Psi_{\bar g}^*(c_\omega^*(P^m R)) \longrightarrow R
\end{array}$$

Thus we get

$$\begin{aligned} i_{\bar g\omega *} \circ \epsilon_{\bar g\omega *} \circ \Psi^*_{\bar g\omega} &= (i_{\bar g *}(\Psi^*_{\bar g}(l^{V^n}_\omega))_*) \circ (\epsilon_{\bar g *}(\Psi^*_{\bar g}(l^{P^m R}_\omega))_*) \circ (\Psi^*_{\bar g} c^*_\omega) \\ &= i_{\bar g *} \circ \epsilon_{\bar g *} \circ (\Psi^*_{\bar g}(l^{V^n}_\omega))_* \circ \Psi^*_{\bar g} \circ (l^{P^m R}_\omega)_* \circ c^*_\omega \\ &= i_{\bar g *} \circ \epsilon_{\bar g *} \circ \Psi^*_{\bar g} \circ (l^{V^n}_\omega)_* \circ (l^{P^m R}_\omega)_* \circ c^*_\omega \\ &= i_{\bar g *} \circ \epsilon_{\bar g *} \circ \Psi^*_{\bar g} \circ \omega_* \;=\; i_{\bar g *} \circ \epsilon_{\bar g *} \circ \Psi^*_{\bar g} \end{aligned}$$

We have used the naturality properties of various constructions and, in the last equation, the fact that conjugation by an inner automorphism is the identity on equivariant homotopy groups. □

The various properties of the power construction imply corresponding properties of the norm map.

Given any H-invariant subset S of G, the stabilizers of S and its complement $S^c = G - S$ agree,

$$G\langle S\rangle \;=\; G\langle S^c\rangle \ .$$

Moreover, the induced representation $G \ltimes_H V$ is the orthogonal, $G\langle S\rangle$-equivariant direct sum of the subspaces $S \ltimes_H V$ and $S^c \ltimes_H V$; so we have an internal product

$$\cdot \ : \ \pi^{G\langle S\rangle}_{S\ltimes_H V}(R) \ \times \ \pi^{G\langle S^c\rangle}_{S^c\ltimes_H V}(R) \ \longrightarrow \ \pi^{G\langle S\rangle}_{G\ltimes_H V}(R) \ .$$

add:

$$g_\star \circ \mathrm{norm}^S_H \;=\; \mathrm{norm}^{{}^g S}_{{}^g H} \circ g_\star$$

Proposition 11.4. *Let S be an H-invariant subset of G and R a commutative orthogonal G-ring spectrum. The norm maps $\mathrm{norm}^S_H : \pi^H_V(R) \longrightarrow \pi^{G\langle S\rangle}_{S\ltimes_H V}(R)$ have the following properties.*

(i) *We have*

$$\mathrm{norm}^S_H(0) = 0 \ , \quad \mathrm{norm}^S_H(1) = 1 \ , \quad \mathrm{norm}^\emptyset_H(x) = 1 \quad \text{and} \quad \mathrm{norm}^H_H(x) = x \ .$$

(ii) *For every $g \in G$ we have*

$$\mathrm{norm}^{gS}_H \;=\; (l_g)_* \circ g_\star \circ \mathrm{norm}^S_H$$

where $l_g : c^*_g(S \ltimes_H V) \longrightarrow gS \ltimes_H V$ is the $G\langle gS\rangle$-linear isometry defined by $l_g(s \otimes v) = gs \otimes v$. In particular we have $\mathrm{norm}^{gH}_H(x) = (i_g)_*(g_\star x)$.

(iii) (Consistency) *Given nested subgroups $H \subseteq K \subseteq G$ and an H-invariant subset S of K, we have $K\langle S\rangle \subseteq G\langle S\rangle$ and the norm maps relative to K and G are related by*

$$\mathrm{norm}^S_H \;=\; \mathrm{res}^{G\langle S\rangle}_{K\langle S\rangle} \circ \mathrm{norm}^S_H$$

where the norm on the left hand side is formed relative to K and the norm on the right hand side is formed relative to G.

(iv) (Transitivity) *The norm maps are transitive, i.e., for subgroups $K \subseteq H \subseteq G$, every K-invariant subset T of H and every $H\langle T\rangle$-invariant subset S of G*

$$\mathrm{norm}^S_{H\langle T\rangle} \circ \mathrm{norm}^T_K \;=\; \mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle} \circ \mathrm{norm}^{ST}_K$$

as maps from $\pi^H_V(R)$ to $\pi^G_{(ST)\ltimes_K V}(R)$, using the identification $S \ltimes_{H\langle T\rangle} (T \ltimes_K V) \cong (ST) \ltimes_K V$.

(v) (Union) *Let S and T be disjoint H-invariant subsets of G. Then*

$$(\mathrm{res}^{G\langle S\rangle}_{G\langle S\rangle\cap G\langle T\rangle} \mathrm{norm}^S_H(x)) \cdot (\mathrm{res}^{G\langle T\rangle}_{G\langle S\rangle\cap G\langle T\rangle} \mathrm{norm}^T_H(x)) \;=\; \mathrm{res}^{G\langle S\cup T\rangle}_{G\langle S\rangle\cap G\langle T\rangle} \mathrm{norm}^{S\cup T}_H(x) \ .$$

(vi) (External multiplicativity) *The norm maps are multiplicative with respect to external product: for two commutative G-ring spectra $R, \bar R$ and classes $x \in \pi^H_V(R)$ and $\bar x \in \pi^H_W(\bar R)$ we have*

$$(\mathrm{norm}^S_H\, x) \cdot (\mathrm{norm}^S_H\, \bar x) \;=\; \mathrm{norm}^S_H(x \cdot \bar x)$$

in $\pi^{G\langle S\rangle}_{S\ltimes_H(V\oplus W)}(R \wedge \bar R)$.

(vii) (Internal multiplicativity) *The norm maps are multiplicative with respect to internal product: $x \in \pi_V^H(R)$ and $y \in \pi_W^H(R)$ we have*

$$(\mathrm{norm}_H^S x) \cdot (\mathrm{norm}_H^S y) \ = \ \mathrm{norm}_H^S(x \cdot y)$$

in $\pi^{G\langle S\rangle}_{S\ltimes_H(V\oplus W)}(R)$.

(viii) (Double coset formula) *For every H-invariant subset S and every subgroup K of $G\langle S\rangle$ we have:*

$$\mathrm{res}_K^{G\langle S\rangle} \circ \mathrm{norm}_H^S \ = \ \prod_{[g]\in K\backslash S/H} \mathrm{norm}^K_{K\cap {}^gH} \circ g_\star \circ \mathrm{res}^H_{K^g\cap H}$$

as maps from $\pi_V^H(R)$ to $\pi^K_{S\ltimes_H V}(R)$. Here $[g]$ runs over a system of representatives of all K-H-orbits of S and we use the K-linear isometry

$$\bigoplus_{[g]\in K\backslash S/H} K\ltimes_{K\cap {}^gH}(c_g^*V) \ \cong \ S\ltimes_H V$$

to identify the indexing representations of both sides.

(ix) (Sum) *For $x, y \in \pi_V^H(R)$ the relation*

$$\mathrm{norm}_H^G(x+y) \ = \ \sum_{[S]} \mathrm{tr}^G_{G\langle S\rangle}\ (\mathrm{norm}_H^S(x)\cdot \mathrm{norm}_H^{G-S}(y))$$

holds in $\pi^G_{G\ltimes_H V}(R)$. The sum runs over a set of representatives S of all orbits of the left G-action on the set of H-invariant subsets of G.

Proof. (i) We have $\mathrm{norm}_H^S(0) = \langle S|P^n(0)\rangle = \langle S|0\rangle = 0$. In the case $R = \mathbb{S}$ of the G-sphere spectrum and with $V = \mathbb{R}^0$ we have $P^n(\mathbb{S}) = \mathbb{S}$ and $P^n(1) = 1$. Also, in this case $G\langle S\rangle$ acts trivially on the representation and on $P^n\mathbb{S} = \mathbb{S}$, so the maps $\Psi^*_{\bar g}$, $\epsilon_{\bar g}$ and $i_{\bar g}$ involved in the definition of $\langle S|-\rangle$ are all identity maps. So we get $\mathrm{norm}_H^S(1) = \langle S|P^n(1)\rangle = \langle S|1\rangle = 1$ in $\pi_0^{G\langle S\rangle}\mathbb{S}$. For an arbitrary commutative G-ring spectrum we then have $\mathrm{norm}_H^S(1) = 1$ by naturality of the norm map. The empty H-invariant set has index 0 and stabilizer group $G\langle\emptyset\rangle = G$. We have $\Sigma_0 \wr H = e$, the trivial group, and $P^0 R = \mathbb{S}$ is the sphere spectrum. Moreover, $\langle\emptyset|-\rangle : \pi_0^e\mathbb{S} \longrightarrow \pi_0^G R$ sends the unit element 1 to the unit in $\pi_0^G R$. So we have $\mathrm{norm}_H^\emptyset(x) = \langle\emptyset|1\rangle \ = \ 1$. For $S = H$ we can choose the unit 1 as the H-basis, and this choice yields that $\langle H|-\rangle$ is the restriction along the canonical isomorphism $H \longrightarrow \Sigma_1 \wr H$ that sends h to $(1;\ h)$. The restriction of $P^1 x$ along this isomorphism is x, so we get $\mathrm{norm}_H^H(x) = \langle H|P^1 x\rangle = x$.

(ii) If $\bar g = (g_1, \dots, g_n)$ is an H-basis of S, then $g\bar g = (gg_1, \dots, gg_n)$ is an H-basis of gS. We have $G\langle gS\rangle = {}^gG\langle S\rangle$ and the homomorphism $\Psi_{g\bar g} : G\langle gS\rangle \longrightarrow \Sigma_n \wr H$ is equal to the composite $\Psi_{g\bar g} = \Psi_{\bar g}\circ c_g$.

$$\begin{aligned}
\langle gS|-\rangle \ &= \ i_{g\bar g}\circ \epsilon_{g\bar g}\circ \Psi^*_{g\bar g}\\
&= \ (l_g)_*\circ c_g^*(i_{\bar g})\circ l_g^R \circ c_g^*(\epsilon_{\bar g})\circ c_g^*\circ \Psi^*_g\\
&= \ (l_g)_*\circ l_g^R\circ c_g^*(i_{\bar g})\circ c_g^*\circ \epsilon_{\bar g}\circ \Psi^*_{\bar g}\\
&= \ (l_g)_*\circ l_g^R\circ c_g^*\circ i_{\bar g}\circ \epsilon_{\bar g}\circ \Psi^*_{\bar g} \ = \ (l_g)_*\circ g_\star\circ \langle S|-\rangle
\end{aligned}$$

The second equation uses that

$$\begin{aligned}
\epsilon_{g\bar g} \ &= \ l_g^R\circ c_g^*(\epsilon_{\bar g}) \quad &&: \ \Psi^*_{g\bar g}(P^nR) = c_g^*(\Psi^*_{\bar g}(P^nR)) \ \longrightarrow \ R \qquad \text{and}\\
i_{g\bar g} \ &= \ l_g\circ c_g^*(i_{\bar g}) \quad &&: \ \Psi^*_{g\bar g}(V^n) = c_g^*(\Psi^*_{\bar g}(V^n)) \ \longrightarrow \ (gS)\ltimes_H V
\end{aligned}$$

where $l_g : c_g^*(S\ltimes_H V) \longrightarrow gS\ltimes_H V$ is the $G\langle gS\rangle$-linear isometry with $l_g(s\otimes v) = gs\otimes v$. Composing this relation with P^n gives

$$\mathrm{norm}_H^{gS} \ = \ \langle gS|-\rangle\circ P^n \ = \ (l_g)_*\circ g_\star\circ\langle S|-\rangle\circ P^n \ = \ (l_g)_*\circ g_\star\circ \mathrm{norm}_H^S \ .$$

In the special case $S = H$ the map norm_H^H is the identity and the isometry $l_g : c_g^*(H \ltimes_H V) \longrightarrow gH \ltimes_H V$ agrees with i_g (where implicitly we used the tautological isometry between $H \ltimes_H V$ and V). So the relation specializes to

$$\mathrm{norm}_H^{gH} \;=\; (i_g)_* \circ g_\star \;.$$

(iii) The notion of H-basis for S is absolute, i.e., does not depend on whether S is viewed as a subset of K or of G. So we can use the same H-basis of S for the construction of $\langle S|-\rangle$ relative to K or G, and obtain $\langle S|-\rangle = \mathrm{res}^{G\langle S\rangle}_{K\langle S\rangle} \circ \langle S|-\rangle$ where the left hand side is relative to K and the right hand side is relative to G. Precomposing with the power operation gives the desired consistency relation for the norm maps.

(iv) Suppose that $k = \langle H : K\rangle$ is the index of K in H. We choose a K-basis $\bar h = (h_1, \dots, h_l)$ for T and an $H\langle T\rangle$-basis $\bar g = (g_1, \dots, g_n)$ of S. Then

$$\bar g \bar h \;=\; (g_1h_1, \dots, g_1h_l,\; g_2h_1, \dots, g_2h_l,\; \dots,\; g_nh_1, \dots, g_nh_l)$$

is a K-basis of ST. With respect to this basis, the restriction of the homomorphism $\Psi_{\bar g\bar h} : G\langle ST\rangle \longrightarrow \Sigma_{ln}\wr K$ to the subgroup $G\langle S\rangle$ equals the composite map

$$G\langle S\rangle \xrightarrow{\Psi_{\bar g}} \Sigma_n \wr H\langle T\rangle \xrightarrow{\Sigma_n\wr\Psi_{\bar h}} \Sigma_n \wr (\Sigma_l \wr K) \longrightarrow \Sigma_{ln} \wr K$$

where the last homomorphism was defined in (9.1). We have

$$\begin{aligned}\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(\Psi^*_{gh}(P^{ln}R)) \;&=\; \Psi^*_g((\Sigma_n \wr \Psi_h)^*(\mathrm{res}^{\Sigma_{ln}\wr H}_{\Sigma_n\wr(\Sigma_l\wr K)}(P^{ln}R))\\ &=\; \Psi^*_g((\Sigma_n \wr \Psi_h)^*(P^n(P^lR))) \;=\; \Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(P^lR)))\end{aligned}$$

as $G\langle S\rangle$-spectra and

$$\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(\epsilon_{gh}) \;=\; \epsilon_g \circ (\Psi^*_g(P^n\epsilon_h))$$

as $G\langle S\rangle$-equivariant morphisms from

$$\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(\Psi^*_{gh}(P^{ln}R)) \;=\; \Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(P^lR)))$$

to R. Similarly, we have

$$\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(i_{gh}) \;=\; i_g \circ (\Psi^*_g(P^n i_h))$$

as $G\langle S\rangle$-isometries from

$$\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(\Psi^*_{gh}(V^{ln})) \;=\; \Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(V^l)))$$

to $(ST) \ltimes_K V$. Moreover,

$$\begin{aligned}\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}\, \Psi^*_{\bar g\bar h}(P^{ln}x) \;&=\; \Psi^*_{\bar g}((\Sigma_n \wr \Psi_{\bar h})^*(\mathrm{res}^{\Sigma_{ln}\wr K}_{\Sigma_n(\Sigma_l\wr K)}(P^{ln}x)))\\ &=\; \Psi^*_{\bar g}((\Sigma_n \wr \Psi_{\bar h})^*(P^n(P^lx))) \;=\; \Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(P^lx)))\end{aligned}$$

in the group

$$\pi^{G\langle S\rangle}_{\Psi^*_{gh}(V^{ln})}(\Psi^*_{gh}(P^{ln}R)) \;=\; \pi^{G\langle S\rangle}_{\Psi^*_{\bar g}((\Psi^*_{\bar h}(V^l))^n)}(\Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(P^lX)))) \;.$$

Putting all of this together yields

$$\begin{aligned}
\mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}(\mathrm{norm}^{ST}_K(x)) &= \mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}\langle ST|P^{ln}x\rangle \\
&= \mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}\left(i_{\bar g\bar h}(\epsilon_{\bar g\bar h})\Psi^*_{\bar g\bar h}(P^{ln}x))\right) \\
&= (i_g\circ(\Psi^*_g(P^n i_h)))\circ \mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}\left((\epsilon_{\bar g\bar h})\Psi^*_{\bar g\bar h}(P^{ln}x))\right) \\
&= (i_g\circ(\Psi^*_g(P^n i_h)))\circ(\epsilon_g\circ(\Psi^*_g(P^n\epsilon_h)))\circ \mathrm{res}^{G\langle ST\rangle}_{G\langle S\rangle}\left(\Psi^*_{\bar g\bar h}(P^{ln}x))\right) \\
&= i_g\circ\epsilon_g\circ\Psi^*_g(P^n i_h)\circ\Psi^*_g(P^n\epsilon_h)\circ(\Psi^*_{\bar g}(P^n(\Psi^*_{\bar h}(P^l x)))) \\
&= i_g\circ\epsilon_g\circ\Psi^*_{\bar g}(P^n(i_h\epsilon_h\Psi^*_{\bar h}(P^l x))) \\
&= i_g\circ\epsilon_g\circ\Psi^*_{\bar g}(P^n(\mathrm{norm}^T_K(x))) \;=\; \mathrm{norm}^S_{H\langle T\rangle}(\mathrm{norm}^T_K(x))
\end{aligned}$$

(v) [...]

(vi) The composite

$$\Psi^*_{\bar g}(P^n(R\wedge\bar R)) \xrightarrow{\Psi^*_{\bar g}(\chi^{(m)}_{R,\bar R})} \Psi^*_{\bar g}(P^nR\wedge P^n\bar R) \;=\; \Psi^*_{\bar g}(P^nR)\wedge\Psi^*_{\bar g}(P^n\bar R) \xrightarrow{\epsilon_{\bar g}\wedge\epsilon_{\bar g}} R\wedge\bar R$$

equals the morphism $\epsilon_{\bar g}:\Psi^*_{\bar g}(P^n(R\wedge\bar R))\longrightarrow R\wedge\bar R$ and the composite

$$\Psi^*_{\bar g}((V\oplus W)^n) \;\cong\; \Psi^*_{\bar g}(V^n)\oplus\Psi^*_{\bar g}(W^n) \xrightarrow{i_{\bar g}\oplus i_{\bar g}} (S\ltimes_H V)\oplus(S\ltimes_H W) \;\cong\; S\ltimes_H(V\oplus W)$$

equals the morphism $i_{\bar g}:\Psi^*_{\bar g}((V\oplus W)^n)\longrightarrow S\ltimes_H(V\oplus W)$. Together with naturality of the external product on equivariant homotopy groups this implies that the diagram

$$\begin{tikzcd}
\pi^{\Sigma_n\wr H}_{V^n}(P^nR)\times\pi^{\Sigma_n\wr H}_{W^n}(P^n\bar R) \arrow[r, "\bullet"] \arrow[d, "\langle S|-\rangle\times\langle S|-\rangle"'] & \pi^{\Sigma_n\wr H}_{V^n\oplus W^n}(P^nR\wedge P^n\bar R) \arrow[r, "(\chi^{(n)}_{R,\bar R})_*", "\cong"'] & \pi^{\Sigma_n\wr H}_{V^n\oplus W^n}(P^n(R\wedge\bar R)) \arrow[d, "\langle S|-\rangle"] \\
\pi^{G\langle S\rangle}_{S\ltimes_H V}(R)\times\pi^{G\langle S\rangle}_{S\ltimes_H W}(\bar R) \arrow[rr, "\bullet"'] & & \pi^G_{S\ltimes(V\oplus W)}(R\wedge\bar R)
\end{tikzcd}$$

commutes. So we get

$$\begin{aligned}
\mathrm{norm}^S_H(x)\bullet\mathrm{norm}^S_H(\bar x) &= \langle S|P^n(x)\rangle\bullet\langle S|P^n(\bar x)\rangle \;=\; \langle S|(\chi^{(n)}_{R,\bar R})_*((P^nx)\bullet(P^n\bar x))\rangle \\
&= \langle S|P^n(x\bullet\bar x)\rangle \;=\; \mathrm{norm}^S_H(x\bullet\bar x)
\end{aligned}$$

using the product formula (9.4) for the power map.

(vii) For every commutative orthogonal G-ring spectrum R the multiplication map $\mu: R\wedge R\longrightarrow R$ is a homomorphism of commutative G-ring spectra. So naturality of the norm yields

$$\begin{aligned}
\mathrm{norm}^S_H(x\cdot y) &= \mathrm{norm}^S_H(\mu_*(x\bullet y)) \;=\; \mu_*(\mathrm{norm}^S_H(x\bullet y)) \\
&=_{(\mathrm{iv})} \mu_*(\mathrm{norm}^S_H(x)\bullet\mathrm{norm}^S_H(y)) \;=\; \mathrm{norm}^S_H(x)\cdot\mathrm{norm}^S_H(y)\ .
\end{aligned}$$

(viii) The set S is the disjoint union of its K-H-orbits. By the union property (v) we have

$$\mathrm{res}^{G\langle S\rangle}_K\mathrm{norm}^S_H(x) \;=\; \prod_{T\in K\backslash S/H}\mathrm{res}^{G\langle T\rangle}_K\mathrm{norm}^T_H(x)\ .$$

So it suffices to show that

$$\mathrm{res}^{G\langle T\rangle}_K\circ\mathrm{norm}^T_H \;=\; \mathrm{norm}^K_{K\cap{}^gH}\circ\mathrm{res}^H_{K^g\cap H}\circ g_\star$$

for any representative $g\in T$.

We let $\bar{\kappa} = (\kappa_1, \dots, \kappa_n)$ be coset representatives for $K \cap {}^gH$ in K. Then $\bar{\kappa}g = (\kappa_1 g, \dots, \kappa_n g)$ is an H-basis for KgH. Moreover, the restriction of $\Psi_{\bar{\kappa}g} : G\langle KgH\rangle \longrightarrow \Sigma_n \wr H$ to K is the composite

$$K \xrightarrow{\Psi_{\bar{\kappa}}} \Sigma_n \wr (K \cap {}^gH) \xrightarrow{c_{\Delta(g)}} \Sigma_n \wr (K^g \cap H) \xrightarrow{\text{incl}} \Sigma_n \wr H \ ,$$

where $\Delta(g) = (1;\ g, \dots, g) \in \Sigma_n \wr G$. This implies that

$$\Psi^*_{\bar{\kappa}g}(P^n R) \ = \ \Psi^*_{\bar{\kappa}}(c^*_{\Delta(g)}(P^n R))$$

as K-spectra and

$$\Psi^*_{\bar{\kappa}g}(V^n) \ = \ \Psi^*_{\bar{\kappa}}(c^*_{\Delta(g)}(V^n)) \ = \ \Psi^*_{\bar{\kappa}}((c_g^* V)^n)$$

as K-representations. Then we have

$$\begin{aligned}
\operatorname{res}^{G\langle KgH\rangle}_K \langle KgH|-\rangle \ &= \ \operatorname{res}^{G\langle KgH\rangle}_K \circ i_{\bar{\kappa}g} \circ \epsilon_{\bar{\kappa}g} \circ \Psi^*_{\bar{\kappa}g} \ = \ i_{\bar{\kappa}g} \circ \epsilon_{\bar{\kappa}g} \circ \operatorname{res}^{G\langle KgH\rangle}_K \circ \Psi^*_{\bar{\kappa}g} \\
&= \ \alpha^g \circ i_{\bar{\kappa}} \circ \epsilon_{\bar{\kappa}} \circ (\Psi^*_{\bar{\kappa}}(l^{P^n R}_{\Delta(g)}))_* \circ \Psi^*_{\bar{\kappa}} \circ c^*_{\Delta(g)} \circ \operatorname{res}^{\Sigma_n \wr H}_{\Sigma_n \wr (K^g \cap H)} \\
&= \ \alpha^g \circ (i_{\bar{\kappa}} \circ \epsilon_{\bar{\kappa}} \circ \Psi^*_{\bar{\kappa}}) \circ (l^{P^n R}_{\Delta(g)})_* \circ c^*_{\Delta(g)}) \circ \operatorname{res}^{\Sigma_n \wr H}_{\Sigma_n \wr (K^g \cap H)} \\
&= \ \alpha^g \circ \langle K|-\rangle \circ \Delta(g)_\star \circ \operatorname{res}^{\Sigma_n \wr H}_{\Sigma_n \wr (K^g \cap H)}
\end{aligned}$$

The third equation uses that

$$\begin{array}{llll}
\epsilon_{\bar{\kappa}g} \ = \ \epsilon_{\bar{\kappa}} \circ \Psi^*_{\bar{\kappa}}(l^{P^n R}_{\Delta(g)}) & : \ \Psi^*_{\bar{\kappa}g}(P^n R) = \Psi^*_{\bar{\kappa}}(c^*_{\Delta(g)}(P^n R)) \ \longrightarrow \ R & \text{and} \\
i_{\bar{\kappa}g} \ = \ \alpha^g \circ i_{\bar{\kappa}} & : \ \Psi^*_{\bar{\kappa}g}(V^n) = \Psi^*_{\bar{\kappa}}((c^*_g V)^n) \ \longrightarrow \ (KgH) \ltimes_H V
\end{array}$$

where $\alpha^g : K \ltimes_{K \cap {}^gH} (c_g^* V) \longrightarrow (KgH) \ltimes_H V$ is the K-linear isometry with $\alpha^g(k \otimes v) = kg \otimes v$. From this the desired formula follows easily with the help of two naturality properties of the power operation:

$$\begin{aligned}
\operatorname{res}^{G\langle KgH\rangle}_K \circ \operatorname{norm}^{KgH}_H \ &= \ \operatorname{res}^{G\langle KgH\rangle}_K \circ \langle KgH|-\rangle \circ P^n \\
&= \ \alpha^g \circ \langle K|-\rangle \circ \Delta(g)_\star \circ \operatorname{res}^{\Sigma_n \wr H}_{\Sigma_n \wr (K^g \cap H)} \circ P^n \\
&= \ \alpha^g \circ \langle K|-\rangle \circ \Delta(g)_\star \circ P^n \circ \operatorname{res}^H_{K^g \cap H} \\
&= \ \alpha^g \circ \langle K|-\rangle \circ P^n \circ g_\star \circ \operatorname{res}^H_{K^g \cap H} \\
&= \ \alpha^g \circ \operatorname{norm}^K_{K \cap {}^gH} \circ g_\star \operatorname{res}^{{}^gH}_{K \cap {}^gH} \ .
\end{aligned}$$

To sum up, we have shown

$$\operatorname{res}^{G\langle S\rangle}_K \circ \operatorname{norm}^S_H \ = \ \prod_{KgH \in K\backslash S/H} \operatorname{res}^{G\langle KgH\rangle}_K \operatorname{norm}^{KgH}_H(x) \ = \ \alpha_* \circ \prod_{[g] \in K\backslash S/H} \operatorname{norm}^K_{K \cap {}^gH} \circ g_\star \circ \operatorname{res}^H_{K^g \cap H}$$

where

$$\alpha \ = \ \sum \alpha^g \ : \ \bigoplus_{[g] \in K\backslash S/G} K^g \ltimes_{K^g \cap H} (c_g^* V) \ \longrightarrow \ S \ltimes_H V \ .$$

(ix) The sum formula is mainly a consequence of the sum formula for the m-th power operation and the additive double coset formula. However, getting all the details straight requires a certain amount of notation and bookkeeping.

We consider any H-invariant subset S of G such that $|S/H| = i$. We choose a complete set of coset representatives $\bar{g}$ whose first i components are an H-basis of S. Then the monomorphism $\Psi_{\bar{g}} : G \longrightarrow \Sigma_m \wr H$ restricts to a monomorphism $\Psi_{\bar{g}} : G\langle S\rangle \longrightarrow \Sigma_{i,m-i} \wr H$. We write $\langle S, S^c|-\rangle$ for the composite

$$\pi^{\Sigma_{i,m-i} \wr H}_{V^m}(P^m R) \xrightarrow{\Psi^*_{\bar{g}}} \pi^{G\langle S\rangle}_{\Psi^*_{\bar{g}}(V^m)}(\Psi^*_{\bar{g}}(P^m R)) \xrightarrow{(\epsilon_{\bar{g}})_*} \pi^{G\langle S\rangle}_{\Psi^*_{\bar{g}}(V^m)}(R) \xrightarrow{(i_{\bar{g}})_*} \pi^{G\langle S\rangle}_{G \ltimes_H V}(R) \ .$$

The same arguments as in Proposition 11.3 show that the map Ψ_{S,S^c} is independent of the choice of $\bar{g}$. We claim that this map satisfies

$$\langle S, S^c | a \cdot b\rangle \;=\; \langle S|a\rangle \cdot \langle S^c|b\rangle \tag{11.5}$$

for $a \in \pi_{V^i}^{\Sigma_i \wr H}(P^i R)$ and $b \in \pi_{V^{m-i}}^{\Sigma_{m-i}\wr H}(P^{m-i}R)$, as well as

$$\langle G|\operatorname{tr}_{i,m-i}(z)\rangle \;=\; \sum_{[S],\ |S/H|=i} \operatorname{tr}^G_{G\langle S\rangle} \circ \langle S, S^c|z\rangle \tag{11.6}$$

for all $z \in \pi_{V^m}^{\Sigma_{i,m-i}\wr H}(P^m R)$, where $\operatorname{tr}_{i,m-i}$ is the $RO(G)$-graded internal transfer map from $\pi_{V^m}^{\Sigma_{i,m-i}\wr H}(P^m R)$ to $\pi_{V^m}^{\Sigma_m \wr H}(P^m R)$. The sum runs over a set of representatives S of all left G-orbits of those H-invariant subsets of cardinality $i \cdot |H|$.

Given these two properties, the sum formula follows easily:

$$\begin{aligned}
\operatorname{norm}^G_H(x+y) \;&=\; \langle G|P^m(x+y)\rangle \;=\; \sum_{i=0}^m \langle G|\operatorname{tr}_{i,m-i}(P^i x\cdot P^{m-i}y)\rangle \\
\overset{(11.6)}{=}&\; \sum_{i=0}^m \sum_{[S],\ |S/H|=i} \operatorname{tr}^G_{G\langle S\rangle}\langle S,S^c|P^i x\cdot P^{m-i}y\rangle \\
\overset{(11.5)}{=}&\; \sum_{i=0}^m \sum_{[S],\ |S/H|=i} \operatorname{tr}^G_{G\langle S\rangle}\left(\langle S|P^i x\rangle\cdot\langle S^c|P^{m-i}y\rangle\right) \\
=&\; \sum_{i=0}^m \sum_{[S],\ |S/H|=i} \operatorname{tr}^G_{G\langle S\rangle}\left(\operatorname{norm}^S_H(x)\cdot\operatorname{norm}^{G-S}_H(y)\right)
\end{aligned}$$

where the second relation is the sum formula (9.5) for the power operation. So we need to show (11.5) and (11.6).

Proof of (11.5). We choose one particular complete set of coset representatives $\bar{g}$ for H in G. We let $\bar{g} = (g_1,\dots,g_m)$ be an H-basis of G such that $\bar{g}^S = (g_1,\dots,g_i)$ is an H-basis of S, and hence $\bar{g}^{S^c} = (g_{i+1},\dots,g_m)$ is an H-basis of S^c. Then $G\langle S\rangle = G\langle S^c\rangle$ and the square of group homomorphisms

$$\begin{tikzcd}
G\langle S\rangle \arrow[r, "(\Psi_{\bar{g}^S},\Psi_{\bar{g}^{S^c}})"] \arrow[d, "\text{incl.}"'] & (\Sigma_i \wr H)\times(\Sigma_{m-i}\wr H) \arrow[d, "\cong"] \\
G \arrow[r, "\Psi_{\bar{g}}"'] & \Sigma_{i,m-i}\wr H
\end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd}
\pi^{\Sigma_i\wr H}_{V^i}(P^iR)\times\pi^{\Sigma_{m-i}\wr H}_{V^{m-i}}(P^{m-i}R) \arrow[r, "\cdot"] \arrow[d, "\Psi^*_{\bar g^S}\times\Psi^*_{\bar g^{S^c}}"'] \arrow[ddd, bend right=60, "\Psi_S\times\Psi_{S^c}"'] & \pi^{\Sigma_{i,m-i}\wr H}_{V^m}(P^mR) \arrow[d, "\Psi^*_{\bar g}"] \arrow[ddd, bend left=60, "\Psi_{S,S^c}"] \\
\pi^{G\langle S\rangle}_{\Psi^*_{\bar g^S}(V^i)}(\Psi^*_{\bar g^S}(P^iR))\times\pi^{G\langle S^c\rangle}_{\Psi^*_{\bar g^{S^c}}(V^{m-i})}(\Psi^*_{\bar g^{S^c}}(P^{m-i}R)) \arrow[r, "\cdot"] \arrow[d, "(\epsilon_{\bar g^S})_*\times(\epsilon_{\bar g^{S^c}})_*"'] & \pi^{G\langle S\rangle}_{\Psi^*_{\bar g}(V^m)}(\Psi^*_{\bar g}(P^mR)) \arrow[d, "(\epsilon_{\bar g})_*"] \\
\pi^{G\langle S\rangle}_{\Psi^*_{\bar g^S}(V^i)}(R)\times\pi^{G\langle S^c\rangle}_{\Psi^*_{\bar g^{S^c}}(V^{m-i})}(R) \arrow[r, "\cdot"'] \arrow[d, "(i_{\bar g^S})_*\times(i_{\bar g^{S^c}})_*"'] & \pi^{G\langle S\rangle}_{\Psi^*_{\bar g}(V^m)}(R) \arrow[d, "(i_{\bar g})_*"] \\
\pi^{G\langle S\rangle}_{S\ltimes_H V}(R)\times\pi^{G\langle S^c\rangle}_{S^c\ltimes_H V}(R) \arrow[r, "\cdot"'] & \pi^{G\langle S\rangle}_{G\ltimes_H V}(R)
\end{tikzcd}$$

commutes by the various naturality properties of external and internal product. This is (11.5).

Proof of (11.6). This is an instance of the double coset formula, suitably reinterpreted. The bookkeeping is complicated by the fact that we are simultaneously changing all three parameters of an equivariant homotopy group, namely the group, the spectrum and the indexing representation. We fix one particular complete set $\bar g=(g_1,\dots,g_m)$ of coset representatives for H in G, with associated monomorphism $\Psi_{\bar g}: G\longrightarrow\Sigma_m\wr H$. The monomorphism $\Psi_{\bar g}$ factors as an isomorphism $\bar\Psi_{\bar g}: G\longrightarrow\Psi_{\bar g}(G)=\bar G$ onto its image followed by the inclusion $i:\bar G\longrightarrow\Sigma_m\wr H$; so $\Psi^*_{\bar g}=\bar\Psi^*_{\bar g}\circ\mathrm{res}^{\Sigma_m\wr H}_{\bar G}$. Thus $\langle G|-\rangle$ is the composite

$$\pi^{\Sigma_m\wr H}_{V^m}(P^mR)\xrightarrow{\mathrm{res}^{\Sigma_m\wr H}_{\bar G}}\pi^{\bar G}_{V^m}(P^mR)\xrightarrow{\bar\Psi^*_{\bar g}}\pi^G_{\bar\Psi^*_{\bar g}(V^m)}(P^mR)\xrightarrow{(\epsilon_{\bar g})*}\pi^G_{\bar\Psi^*_{\bar g}(V^m)}(R)\xrightarrow{(i_{\bar g})*}\pi^G_{G\ltimes_H V}(R)\ .$$

The double coset formula (4.22) gives

$$\begin{aligned}\langle G|\,\mathrm{tr}_{i,m-i}\rangle\ &=\ i_{\bar g*}\circ\epsilon_{\bar g*}\circ\bar\Psi^*_{\bar g}\circ\mathrm{res}^{\Sigma_m\wr H}_{\bar G}\circ\mathrm{tr}_{i,m-i}\\ &=\sum_{[\omega]\in\bar G\backslash(\Sigma_m\wr H)/(\Sigma_{i,m-i}\wr H)} i_{\bar g*}\circ\epsilon_{\bar g*}\circ\bar\Psi^*_{\bar g}\circ\mathrm{tr}^{\bar G}_{\bar G\cap^\omega(\Sigma_{i,m-i}\wr H)}\circ\omega_*\circ\mathrm{res}^{\Sigma_{i,m-i}\wr H}_{\bar G^\omega\cap(\Sigma_{i,m-i}\wr H)}\end{aligned}$$

Now we rewrite the summands that occur in this formula. For $\omega=(\sigma;\ h_1,\dots,h_m)\in\Sigma_m\wr H$ and we can define an H-invariant subset with exactly i right H-orbits by

$$S(\omega)\ =\ \bigcup_{j=1}^i\,(\bar g\omega)_j\cdot H\ =\ \bigcup_{j=1}^i\,g_{\sigma(j)}\cdot H\ ,$$

the H-invariant subset generated by the first i components of $\bar g\omega$. For $\tau\in\Sigma_{i,m-i}\wr H$ we have

$$S(\omega\tau)\ =\ S(\omega)\ ,$$

so the H-invariant set $S(\omega)$ depends only on the right $\Sigma_{i,m-i}$-coset of ω. The isomorphism $\bar\Psi_{\bar g}: G\longrightarrow\bar G$ restricts to an isomorphism of $G\langle S(\omega)\rangle$ onto $\bar G\cap^\omega(\Sigma_{i,m-i}\wr H)$, so we get

$$\begin{aligned}\Psi^*_{\bar g}\circ\mathrm{tr}^{\bar G}_{\bar G\cap^\omega(\Sigma_{i,m-i}\wr H)}\circ\omega_*\circ\mathrm{res}^{\Sigma_{i,m-i}\wr H}_{\bar G^\omega\cap(\Sigma_{i,m-i}\wr H)}\ &=\ \mathrm{tr}^G_{G\langle S(\omega)\rangle}\circ\bar\Psi^*_{\bar g}\circ\omega_*\circ\mathrm{res}^{\Sigma_{i,m-i}\wr H}_{\bar G^\omega\cap(\Sigma_{i,m-i}\wr H)}\\ &=\ \mathrm{tr}^G_{G\langle S(\omega)\rangle}\circ\bar\Psi^*_{\bar g}\circ(l^{V^m}_\omega)_*\circ(l^{P^mR}_\omega)_*\circ c^*_\omega\circ\mathrm{res}^{\Sigma_{i,m-i}\wr H}_{\bar G^\omega\cap(\Sigma_{i,m-i}\wr H)}\\ &=\ \mathrm{tr}^G_{G\langle S(\omega)\rangle}\circ(\bar\Psi_{\bar g}(l^{V^m}_\omega))_*\circ(\bar\Psi^*_{\bar g}(l^{P^mR}_\omega))_*\circ\bar\Psi^*_{\bar g}\circ c^*_\omega\circ\mathrm{res}^{\Sigma_{i,m-i}\wr H}_{\bar G^\omega\cap(\Sigma_{i,m-i}\wr H)}\\ &=\ \mathrm{tr}^G_{G\langle S(\omega)\rangle}\circ(\bar\Psi_{\bar g}(l^{V^m}_\omega))_*\circ(\bar\Psi^*_{\bar g}(l^{P^mR}_\omega))_*\circ\bar\Psi^*_{\bar g\omega}\end{aligned}$$

We have used that $\Psi_{\bar{g}\omega} = c_\omega \circ \Psi_{\bar{g}}$ and hence $\Psi^*_{\bar{g}\omega} = \Psi^*_{\bar{g}} \circ c^*_\omega$. We also have

$$\begin{aligned} i_{\bar{g}\omega} &= i_{\bar{g}} \circ \bar{\Psi}^*_{\bar{g}}(l_\omega^{V^m}) && : \bar{\Psi}^*_{\bar{g}\omega}(V^m) \;=\; \Psi^*_{\bar{g}}(c^*_\omega(V^m)) \;\longrightarrow\; G \ltimes_H V \\ \epsilon_{\bar{g}\omega} &= \epsilon_{\bar{g}} \circ \bar{\Psi}^*_{\bar{g}}(l_\omega^{P^m R}) && : \Psi^*_{\bar{g}\omega}(P^m R) \;=\; \Psi^*_{\bar{g}}(c^*_\omega(P^m R))) \;\longrightarrow\; R \end{aligned}$$

so if we compose the previous relation with $i_{\bar{g}_*} \circ \epsilon_{\bar{g}_*}$ we get

$$\begin{aligned} & i_{\bar{g}_*} \circ \epsilon_{\bar{g}_*} \circ \bar{\Psi}^*_{\bar{g}} \circ \operatorname{tr}^{\bar{G}}_{\bar{G}\cap^\omega(\Sigma_{i,m-i}\wr H)} \circ \omega_* \circ \operatorname{res}^{\Sigma_{i,m-i}\wr H}_{\bar{G}^\omega\cap(\Sigma_{i,m-i}\wr H)} \\ &= i_{\bar{g}_*} \circ \epsilon_{\bar{g}_*} \circ \operatorname{tr}^G_{G\langle S(\omega)\rangle} \circ (\bar{\Psi}_{\bar{g}}(l_\omega^{V^m}))_* \circ (\bar{\Psi}^*_{\bar{g}}(l_\omega^{P^m R}))_* \circ \bar{\Psi}^*_{\bar{g}\omega} \\ &= \operatorname{tr}^G_{G\langle S(\omega)\rangle} \circ i_{\bar{g}_*} \circ (\bar{\Psi}_{\bar{g}}(l_\omega^{V^m}))_* \circ \epsilon_{\bar{g}_*} \circ (\bar{\Psi}^*_{\bar{g}}(l_\omega^{P^m R}))_* \circ \bar{\Psi}^*_{\bar{g}\omega} \\ &= \operatorname{tr}^G_{G\langle S(\omega)\rangle} \circ i_{\bar{g}\omega_*} \circ \epsilon_{\bar{g}\omega_*} \circ \bar{\Psi}^*_{\bar{g}\omega} \;=\; \operatorname{tr}^G_{G\langle S(\omega)\rangle} \circ \langle S(\bar{g}\omega), S(\bar{g}\omega)^c | - \rangle \end{aligned}$$

Now we sum up over a set of double coset representatives. For $\gamma \in G$ we have $\gamma\bar{g} = \bar{g}\Psi_{\bar{g}}(\gamma)$ and hence

$$S(\Psi_{\bar{g}}(\gamma)\cdot\omega) \;=\; \gamma \cdot S(\omega) \; .$$

So the assignment $\omega \longmapsto S(\omega)$ induces a G-equivariant bijection

$$\Psi^*_{\bar{g}}(\Sigma_m \wr H / \Sigma_{i,m-i} \wr H) \;\longrightarrow\; \{\, H\text{-invariant subsets of } G \text{ of cardinality } i\cdot|H| \,\} \;,$$

and thus a bijection between the $\bar{G}$-$(\Sigma_{i,m-i}\wr H)$-double cosets in $\Sigma_m \wr H$ and the G-orbits of H-invariant subsets of cardinality $i\cdot|H|$. So we conclude that

$$\begin{aligned} \Psi_G \circ \operatorname{tr}_{i,m-i} &= \sum_{[\omega]\in\bar{G}\backslash(\Sigma_m\wr H)/(\Sigma_{i,m-i}\wr H)} i_{\bar{g}_*} \circ \epsilon_{\bar{g}_*} \circ \bar{\Psi}^*_{\bar{g}} \circ \operatorname{tr}^{\bar{G}}_{\bar{G}\cap^\omega(\Sigma_{i,m-i}\wr H)} \circ \omega_* \circ \operatorname{res}^{\Sigma_{i,m-i}\wr H}_{\bar{G}^\omega\cap(\Sigma_{i,m-i}\wr H)} \\ &= \sum_{[\omega]\in\bar{G}\backslash(\Sigma_m\wr H)/(\Sigma_{i,m-i}\wr H)} \operatorname{tr}^G_{G\langle S(\omega)\rangle} \circ \langle S(\bar{g}\omega), S(\bar{g}\omega)^c | - \rangle \;=\; \sum_{[S],\ |S/H|=i} \operatorname{tr}^G_{G\langle S\rangle} \circ \langle S, S^c | - \rangle \; . \end{aligned}$$

This justifies the relation (11.6) and concludes the proof of the sum formula. □

Remark 11.7. We observe that for every H-invariant subset S, the stabilizer group $G\langle S\rangle$ contains the intersection of all H-conjugates, i.e.,

$$\bigcap_{g\in G} {}^g H \;\subseteq\; G\langle S\rangle \; .$$

Indeed, ${}^g H$ is the stabilizer of the orbit gH. So the elements in the intersection stabilize all H-orbits, hence all H-invariant subsets.

The empty subset of G is unique within its G-orbit, and by property (i) its contribution to the sum formula (ix) is

$$\operatorname{norm}^{\emptyset}_H(x)\cdot\operatorname{norm}^G_H(y) \;=\; 1\cdot\operatorname{norm}^G_H(y) \;=\; \operatorname{norm}^G_H(y) \; .$$

Similarly, the contribution from the subset G is $\operatorname{norm}^G_H(x)$. On the other hand, for every proper H-invariant subset (i.e., different from $\emptyset$ and G), the group $G\langle S\rangle$ is a proper subgroup of G. We conclude that the obstruction to additivity of the norm map

$$\operatorname{norm}^G_H(x+y) \;-\; \operatorname{norm}^G_H(x) \;-\; \operatorname{norm}^G_H(y)$$

is a sum of transfers from the proper subgroups that contain the intersection of all H-conjugates.

Example 11.8. We look at the sum formula in the smallest non-trivial example, i.e., when the subgroup H has index 2 in G. Then G has four H-invariant subsets $\emptyset, H, G-H$ and G. The empty subset respectively G are unique in their respective G-orbits and contribute $\operatorname{norm}^G_H(y)$ respectively $\operatorname{norm}^G_H(x)$. The other two H-invariant subsets H and $G-H$ are in the same G-orbit, and we have

$$\operatorname{norm}^{G-H}_H(y) \;=\; \operatorname{norm}^{\tau H}_H(y) \;=\; l_\tau(\tau_\star(y))$$

where τ is any element in $G - H$. We can pick H as the representative of the G-orbit $\{H, G-H\}$ and the corresponding contribution to the sum formula is then

$$\mathrm{tr}_H^G\left(\mathrm{norm}_H^H(x)\cdot \mathrm{norm}_H^{G-H}(y)\right) \;=\; \mathrm{tr}_H^G\left(x\cdot(\tau_*(y))\right)$$

where τ is any element in $G - H$. So altogether the sum formula for $x, y \in \pi_V^H(R)$ becomes

$$\mathrm{norm}_H^G(x+y) \;=\; \mathrm{norm}_H^G(x) \;+\; \mathrm{tr}_H^G(x\cdot(l_\tau(\tau_\star(y)))) \;+\; \mathrm{norm}_H^G(y)$$

in $\pi^G_{G\times_H V}(R)$, where $\mathrm{tr}_H^G : \pi_V(R) \longrightarrow \pi^G_{G\ltimes_H V}(R)$ is the $RO(G)$-graded transfer map (4.35).

The most important special case of the norm construction is when $S = G$ is the entire group. For easier reference we summarize the properties that apply to this special case in the following proposition.

Proposition 11.9. *Let H be a subgroup of G and R a commutative orthogonal G-ring spectrum. The norm map* $\mathrm{norm}_H^G : \pi_V^H(R) \longrightarrow \pi^G_{G\ltimes_H V}(R)$ *have the following properties.*

(i) *We have* $\mathrm{norm}_H^G(0) = 0$, $\mathrm{norm}_H^G(1) = 1$ *and* $\mathrm{norm}_G^G(x) = x$.

(ii) (Transitivity) *The norm maps are transitive, i.e., for subgroups $K \subseteq H \subseteq G$ and $x \in \pi_V^K(R)$ we have*

$$\mathrm{norm}_H^G(\mathrm{norm}_K^H(x)) \;=\; \mathrm{norm}_K^G(x)$$

in $\pi^G_{G\ltimes_K V}(R)$, using the identification $G\ltimes_H(H\ltimes_K V)\cong G\ltimes_K V$.

(iii) *The norm maps are multiplicative with respect to external product: for two commutative G-ring spectra $R, \bar{R}$ and classes $x \in \pi_V^H(R)$ and $\bar{x} \in \pi_W^H(\bar{R})$ we have*

$$(\mathrm{norm}_H^G\, x)\cdot(\mathrm{norm}_H^G\, \bar{y}) \;=\; \mathrm{norm}_H^G(x\cdot\bar{x})$$

in $\pi^G_{G\ltimes_H(V\oplus W)}(R\wedge\bar{R})$.

(iv) *The norm maps are multiplicative with respect to internal product: $x \in \pi_V^H(R)$ and $y \in \pi_W^H(R)$ we have*

$$(\mathrm{norm}_H^G\, x)\cdot(\mathrm{norm}_H^G\, y) \;=\; \mathrm{norm}_H^G(x\cdot y)$$

in $\pi^G_{G\ltimes_H(V\oplus W)}(R)$.

(v) (Double coset formula) *for two subgroup H and K of G and a homotopy class $x \in \pi_V^H(R)$ we have:*

$$\mathrm{res}_K^G\circ\mathrm{norm}_H^G \;=\; \prod_{[g]\in K\backslash G/H} \mathrm{norm}^K_{K\cap {}^gH}\circ g_\star\circ \mathrm{res}^H_{K^g\cap H}$$

as maps from $\pi_V^H(R)$ to $\pi^K_{G\ltimes_H V}(R)$. Here $[g]$ runs over a system of double coset representatives and we use the K-linear isometry

$$\bigoplus_{[g]\in K\backslash G/H} K\ltimes_{K\cap {}^gH}(c_g^*V) \;\cong\; G\ltimes_H V$$

to identify the indexing representations of both sides.

(vi) (Sum) *For $x, y \in \pi_V^H R$ the relation*

$$\mathrm{norm}_H^G(x+y) \;=\; \sum_{[S]} \mathrm{tr}^G_{G\langle S\rangle}\left(\mathrm{norm}_H^S(x)\cdot\mathrm{norm}_H^{G-S}(y)\right)$$

holds in $\pi^G_{G\ltimes_H V}(R)$. The sum runs over a set of representatives S of all orbits of the left G-action on the set of H-invariant subsets of G.

(vii) *The norm map is compatible with the geometric fixed point map in the sense that the square*

$$\begin{tikzcd}
\pi_V^H(R) \arrow[r, "\mathrm{norm}_H^G"] \arrow[d, "\Phi^H"'] & \pi^G_{G\ltimes_H V}(R) \arrow[d, "\Phi^G"] \\
\pi_{V^H}(\Phi^H R) \arrow[r, "\mathrm{tr}\circ\Delta_*"'] & \pi_{(G\ltimes_H V)^G}(\Phi^G R)
\end{tikzcd}$$

commutes, where $\Delta : \Phi^H R \longrightarrow \Phi^G R$ was defined in (10.11) and $i : V^H \cong (G \ltimes_H V)^G$ is the transfer isomorphism given by $\mathrm{tr}(v) = \sum_{[g]\in G/H} g \otimes v$.

Proof. (vii) Let consider an H-map $f : S^{V+n\rho} \longrightarrow X(n\rho)$ that represents an element in $\pi_V^H(X)$, where $\rho = \rho_H$ is the regular representation of H. The m-th smash power

$$f^{(m)} \ : \ S^{mV+nm\rho} = (S^{V+n\rho})^{(m)} \ \longrightarrow \ (X(n\rho))^{(m)}$$

is then $\Sigma_m \wr H$-equivariant. We compare the restrictions of this $\Sigma_m \wr H$-map to fixed points for the subgroup $\Phi(G)$ and for the whole wreath product group:

$$\begin{tikzcd}
(S^{V+n\rho})^H \arrow[r, "f^H"] \arrow[d, "\mathrm{diag.}"', "\cong"] & R(n\rho)^H \arrow[d, "\cong"', "\mathrm{diag.}"] \arrow[dddr, bend left=40, "\Delta_n"] & \\
\left((S^{V+n\rho})^{(m)}\right)^{\Sigma_m\wr H} \arrow[r, "(f^{(m)})^{\Sigma_m\wr H}"] \arrow[d, equal] & \left(R(n\rho)^{(m)}\right)^{\Sigma_m\wr H} \arrow[r, "(i^{[m]})^{\Sigma_m\wr H}"] \arrow[d, equal] & (R^{(m)}(n\rho^m))^{\Sigma_m\wr H} \arrow[d] \\
\left(\Phi^*((S^{V+n\rho})^{(m)})\right)^G \arrow[r, "(\Phi^*(f^{(m)}))^G"'] \arrow[d, equal] & \left(\Phi^*(R(n\rho)^{(m)})\right)^G \arrow[r, "(\Phi^* i^{[m]})^G"'] & (\Phi^*(R^{(m)}(n\rho^m)))^G \arrow[d, equal] \\
\left(S^{G\ltimes_H V+n\rho_G}\right)^G \arrow[rr, "(\mathrm{Norm}_H^G f)^G"'] & & ((N_H^G R)(n\rho_G))^G
\end{tikzcd}$$

The clockwise composite is a representative for the class $(\pi_k\Delta)(\Phi^H[f])$, and the counter-clockwise composite is a representative for $\Phi^G(\mathrm{Norm}_H^G[f])$. Since the diagram commutes, these two classes agree. □

Remark 11.10. The algebraic structure on the 0-th equivariant homotopy groups of a commutative G-ring spectrum can be packaged differently and more conceptually into the form of a *TNR-functor* in the sense of Tambara [27]; here the acronym stands form 'Transfer, Norm and Restriction'.

Example 11.11 (Sphere spectrum)**.** The sphere spectrum $\mathbb{S}$ is a commutative orthogonal G-ring spectrum for every group G. For all $H \subseteq G$ the restriction of the G-sphere spectrum is the H-sphere spectrum. We claim that under the isomorphism between the Burnside ring and the equivariant 0-stem, the norm map

$$\mathrm{norm}_H^G \ : \ \pi_0^H(\mathbb{S}) \ \longrightarrow \ \pi_0^G(\mathbb{S})$$

becomes the multiplicative norm of Burnside rings. More generally we claim that for every H-invariant subset S of G the diagram

$$\begin{tikzcd}
A(H) \arrow[r, "\mathrm{norm}_H^S"] \arrow[d, "\Psi"'] & A(G\langle S\rangle) \arrow[d, "\Psi"] \\
\pi_0^H(\mathbb{S}) \arrow[r, "\mathrm{norm}_H^S"'] & \pi_0^{G\langle S\rangle}(\mathbb{S})
\end{tikzcd}$$

commutes, where the upper map arises by sending the class of a finite H-set X to the class of the G-set $\mathrm{map}^H(S,X)$. In particular, the norm map on equivariant stable stems corresponds to the assignment $X \mapsto \mathrm{map}^H(G,X)$.

We quickly recall how the norm map $\mathrm{norm}_H^G : A(H) \longrightarrow A(G)$ is defined. This is basically the norm construction in the category of finite sets under cartesian product, but since norming is not additive, the extension from finite H-sets to the Burnside ring $A(H)$ requires justification. For this purpose we consider the product set

$$\mathbb{A}(H) \ = \ \prod_{n\geq 0} A(\Sigma_n \wr H) \ .$$

We endow $\mathbb{A}(H)$ with a new binary operation $\star$ given by

$$((s_i)\star(t_j))_n \ = \ \sum_{i+j=n} \mathrm{tr}_{(\Sigma_i\times\Sigma_j)\wr H}^{\Sigma_n\wr H}(s_i\cdot u_j) \ .$$

The operation $\star$ is evidently commutative and associative and has as neutral element the sequence $\underline{1}$ with

$$\underline{1}_n \ = \ \begin{cases} 1 & \text{for } n=0, \text{ and} \\ 0 & \text{for } n>0. \end{cases}$$

So $\mathbb{A}(H)$ becomes a commutative monoid under $\star$. Given a finite H-set X, the n-th power X^n is a $\Sigma_n\wr H$-set and we have

$$(X \amalg Y)^n \ \cong \ \amalg_{i+j=n}\, \Sigma_n \wr H \times_{\Sigma_{i,j}\wr H} X^i \times Y^j$$

as $\Sigma_n \wr H$-sets. In other words, the 'power series'

$$P(X) \ = \ ([X^n])_{n\geq 0} \quad \text{in } \mathbb{A}(H) \qquad \text{satisfies} \qquad P(X\amalg Y) \ = \ P(X)\star P(Y) \ .$$

Since X^0 is a one-element set, it represents the multiplicative unit in $A(e) = A(\Sigma_0 \wr H)$, and so the power series $P(X)$ is invertible with respect to $\star$. So by the universal property of the Burnside ring, there is a unique map

$$P \ : \ A(H) \ \longrightarrow \ \mathbb{A}(H)$$

that agrees with the power series on finite H-sets and satisfies $P(x+y) = (Px)\star(Py)$. We denote by $P^m : A(H) \longrightarrow A(\Sigma_m \wr H)$ the composite of P with the projection the m-th factor.

We claim that this power construction corresponds to the power construction for the equivariant sphere spectra, i.e., for every $m\geq 0$ the square

$$\begin{tikzcd} A(H) \arrow[r, "P^m"] \arrow[d, "\Psi"'] & A(\Sigma_m \wr H) \arrow[d, "\Psi"] \\ \pi_0^H(\mathbb{S}) \arrow[r, "P^m"'] & \pi_0^{\Sigma_m\wr H}(\mathbb{S}) \end{tikzcd}$$

commutes. Since both power maps have the same behavior on sums, it suffices to check this for the classes of the cosets H/K that generate $A(H)$ as an abelian group. We have

$$\begin{aligned} P^m(\Psi(H/K)) \ &= \ P^m(\mathrm{tr}_K^H(1)) \ = \ \mathrm{tr}_{\Sigma_m\wr K}^{\Sigma_m\wr H}(P^m(1)) \ = \ \mathrm{tr}_{\Sigma_m\wr K}^{\Sigma_m\wr H}(1) \\ &= \ \Psi((\Sigma_m\wr H)/(\Sigma_m\wr K)) \ = \ \Psi(P^m(H/K)) \end{aligned}$$

using that $(H/K)^m$ is isomorphic to $(\Sigma_m\wr H)/(\Sigma_m \wr K)$ as a $\Sigma_m\wr H$-set.

Now suppose that H is a subgroup of a group G and S is an H-invariant subset of G of index n. We define the map

$$\langle S|-\rangle = \langle S:H\rangle \times_{\Sigma_n\wr H} - \ : \ A(\Sigma_n\wr H) \ \longrightarrow \ A(G\langle S\rangle)$$

by balanced product over $\Sigma_n \wr H$ with the set $\langle S : H\rangle$ of H-bases of S (which has commuting left $G\langle S\rangle$-action and right $\Sigma_n \wr H$-actions as explained in Construction 11.2). For any choice of H-basis $\bar g$ of S and every $\Sigma_n \wr H$-set Y the map

$$\Psi^*_{\bar g} Y \xrightarrow{[\bar g, -]} \langle S : H\rangle \times_{\Sigma_n \wr H} Y$$

is a natural isomorphism of $G\langle S\rangle$-sets. The identification between Burnside rings and equivariant zero stems commutes with restriction homomorphism, so the square

$$\begin{tikzcd} A(\Sigma_n \wr H) \arrow[r, "\langle S|-\rangle = \Psi^*_{\bar g}"] \arrow[d, "\Psi"'] & A(G\langle S\rangle) \arrow[d, "\Psi"] \\ \pi_0^{\Sigma_n \wr H}(\mathbb{S}) \arrow[r, "\langle S|-\rangle = \Psi^*_{\bar g}"'] & \pi_0^{G\langle S\rangle}(\mathbb{S}) \end{tikzcd}$$

commutes because [...].

The norm map norm_H^S on Burnside rings is now the composite

$$A(H) \xrightarrow{P^n} A(\Sigma_n \wr H) \xrightarrow{\langle S|-\rangle} A(G\langle S\rangle) \ .$$

The norm map norm_H^G is the special case where $S = G$ is the entire group. Since the identification between Burnside rings and equivariant zero stems commutes with power operations and with the maps $\langle S|-\rangle$, it commutes with the norm maps as well. If we unravel the definitions, we see that for every H-set X, the element $\mathrm{norm}_H^S(X)$ in $A(G\langle S\rangle)$ is represented by $\langle S : H\rangle \times_{\Sigma_n \wr H} X^n$ which naturally isomorphic, as a $G\langle S\rangle$-set, to $\mathrm{map}^H(S, X)$, by the map

$$\begin{aligned} \mathrm{map}^H(S,X) &\longrightarrow \langle S : H\rangle \times_{\Sigma_n \wr H} X^n \\ \varphi &\longmapsto [g_1, \dots, g_n;\ \varphi(g_1), \dots, \varphi(g_n)] \ . \end{aligned}$$

Here $(g_1, \dots, g_n)$ is any H-basis of S, but the map is independent of this basis.

Example 11.12 (Eilenberg-Mac Lane spectra)**.** Let A be a commutative ring with a G-action by ring automorphisms. Then the Eilenberg-Mac Lane spectrum HA, defined in Example 2.13, is a commutative orthogonal G-ring spectrum. All equivariant homotopy groups of HA are concentrated in dimension 0 and we have $\pi_0^K(HA) = A^K$ for every subgroup K of G. We claim that the norm map coincides with the multiplicative transfer. More generally, we claim that for every K-invariant subset S of G the diagram

$$\begin{tikzcd} A^K \arrow[r, "\mathrm{norm}_K^S"] \arrow[d] & A^{G\langle S\rangle} \arrow[d] \\ \pi_0^K(HA) \arrow[r, "\mathrm{norm}_K^S"'] & \pi_0^{G\langle S\rangle}(HA) \end{tikzcd}$$

commutes, where the upper horizontal map is defined by

$$\mathrm{norm}_K^S(a) \ = \ \prod_{gK \in S} ga \ ,$$

where the product is taken over a K-basis of S. The norm map is the special case $S = G$, where we have

$$\mathrm{norm}_K^G(a) \ = \ \prod_{gK \in G/K} ga \ .$$

Remark 11.13 (External norm map)**.** The norm map for equivariant homotopy groups of commutative orthogonal ring spectrum can be obtained from a more general *external norm map* on $RO(G)$-graded homotopy groups that has the form of:

$$\mathrm{Norm}_H^S \ : \ \pi_V^H(X) \ \longrightarrow \ \pi^{G\langle S\rangle}_{S\ltimes_H V}(N_H^S X) \ .$$

Here S an H-invariant subset of G, V is an H-representation, $S\ltimes_H V$ the induced $G\langle S\rangle$-representation, X is an H-spectrum and

$$N_H^S X \ = \ \langle S:H\rangle_{\Sigma_n\wr H} P^n X$$

is the norm construction based on the invariant subset S of index $n=[S:H]$. For $S=G$ this gives the external norm map for the norm construction $N_H^G X$ in the sense of Section 10. The 'internal' norm map (11.1) is then obtained from the external norm map for $X=R$ by postcomposing with the effect of the adjunction counit $\epsilon : N_H^G R\longrightarrow R$.

The construction of the external norm map is the same as for the internal norm map, *except* that the morphism $\epsilon_{\bar g} : \Psi^*_{\bar g}(P^n R)\longrightarrow R$ has no analog and does not occur. So the external norm map is the composite

$$\pi_V^H(X) \ \xrightarrow{P^n} \ \pi^{\Sigma_n\wr H}_{V^n}(P^n X) \ \xrightarrow{\Psi^*_{\bar g}} \ \pi^{G\langle S\rangle}_{\Psi^*_{\bar g}(V^n)}(\Psi^*_{\bar g}(P^n X)) \ \xrightarrow{i_{\bar g *}} \ \pi^{G\langle S\rangle}_{S\ltimes_H V}(N_H^S X)$$

where $\bar g$ is a choice of H-basis for S with associated monomorphism $\Psi_{\bar g} : G\langle S\rangle\longrightarrow \Sigma_m\wr H$ (and the map does not depend on this choice).

The external norm map has various properties that are analogues, or rather precursors, of corresponding properties of the internal norm map:

(i) We have $\mathrm{Norm}_H^S(0)=0$ and $\mathrm{Norm}_H^H(x)=x$. When $X=\mathbb{S}$ is the H-sphere spectrum, we have $N_H^S\mathbb{S}=\mathbb{S}$, the G-sphere spectrum, and $\mathrm{Norm}_H^S(1)=1$ in $\pi_0^{G\langle S\rangle}\mathbb{S}$.

(ii) The external norm maps are transitive, i.e., for subgroups $K\subseteq H\subseteq G$ we have

$$\mathrm{Norm}_H^G\circ\mathrm{Norm}_K^H \ = \ \mathrm{Norm}_K^G \ : \ \pi_V^H(X) \ \longrightarrow \ \pi^G_{G\ltimes_K V}(N_K^G X) \ .$$

Here we used the identification $G\ltimes_H(H\ltimes_K V)\cong G\ltimes_K V$ and the transitivity isomorphism $N_H^G(N_K^G X)\cong N_K^G X$.

(iii) The norm map is multiplicative with respect to external product: for two orthogonal H-spectra X and Y and classes $x\in\pi_V^H X$ and $y\in\pi_W^H Y$ we have

$$(\mathrm{Norm}_H^G\, x)\cdot(\mathrm{Norm}_H^G\, y) \ = \ \mathrm{Norm}_H^G(x\cdot y)$$

in the group $\pi^G_{G\ltimes_H(V\oplus W)}N_H^G(X\wedge Y)$, using the identification $N_H^G X\wedge N_H^G Y\cong N_H^G(X\wedge Y)$ and $(G\ltimes_H V)\oplus(G\ltimes_H W)\cong G\ltimes_H(V\oplus W)$.

(iv) The external norm map is also compatible with the geometric fixed point map in the sense that the square

$$\begin{tikzcd}
\pi_V^H(X) \arrow[r, "\mathrm{Norm}_H^G"] \arrow[d, "\Phi^H"'] & \pi^G_{G\times_H V}(N_H^G X) \arrow[d, "\Phi^G"] \\
\pi_k(\Phi^H X) \arrow[r, "\pi_*\Delta"'] & \pi_k(\Phi^G(N_H^G X))
\end{tikzcd}$$

commutes, where $k=\dim(V^H)=\dim(G\times_H V)^G$ and the morphism $\Delta : \Phi^H X\longrightarrow\Phi^G(N_H^G X)$ was defined in (10.11).

We will not prove the 'external' forms of these formulas; they can be guessed by systematically 'externalizing' the proofs for the internal norm map.

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