Lectures on K-theoretic computations in enumerative geometry

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1 Aims & Scope

1.1 K-theoretic enumerative geometry

1.1.1

These lectures are for graduate students who want to learn how to do the *computations* from the title. Here I put emphasis on computations because I think it is very important to keep a connection between abstract notions of algebraic geometry, which can be very abstract indeed, and something we can see, feel, or test with code.

While it is a challenge to adequately illustrate a modern algebraic geometry narrative, one can become familiar with main characters of these notes by working out examples, and my hope is that these notes will be placed alongside a pad of paper, a pencil, an eraser, and a symbolic computation interface.

1.1.2

Modern enumerative geometry is not so much about numbers as it is about deeper properties of the moduli spaces that parametrize the geometric objects being enumerated.

Of course, once a relevant moduli space M is constructed one can study it as one would study any other algebraic scheme (or stack, depending on the context). Doing this in any generality would appear to be seriously challenging, as even the dimension of some of the simplest moduli spaces considered here (namely, the Hilbert scheme of points of 3-folds) is not known.

1.1.3

A productive middle ground is to compute the Euler characteristics $\chi(\mathscr{F})$ of naturally defined coherent sheaves $\mathscr{F}$ on M, as representations of a group G of automorphisms of the problem. This goes beyond intersecting natural cycles in M, which is the realm of the traditional enumerative geometry, and is a nutshell description of equivariant K-theoretic enumerative geometry.

The group G will typically be connected and reductive, and the G-character of $\chi(\mathscr{F})$ will be a Laurent polynomial on the maximal torus $T \subset G$ provided $\chi(\mathscr{F})$ is a finite-dimensional virtual G-module. Otherwise, it will be a rational function on T. In either case, it is a very concrete mathematical object, which can be turned and spun to be seen from many different angles.

1.1.4

Enumerative geometry has received a very significant impetus from modern high energy physics, and this is even more true of its K-theoretic version. From its very origins, K-theory has been inseparable from indices of differential operators and related questions in mathematical quantum mechanics. For a mathematical physicist, the challenge is to study quantum systems with infinitely many degrees of freedom, systems that describe fluctuating objects that have some spatial extent.

While the dynamics of such systems is, obviously, very complex, their vacua, especially supersymmetric vacua, i.e. the quantum states in the kernel of a certain infinite-dimensional Dirac operator

$$\mathscr{H}_{\text{even}} \underset{\not{D}}{\overset{\not{D}}{\rightleftarrows}} \mathscr{H}_{\text{odd}} \tag{1}$$

may often be understood in finite-dimensional terms. In particular, the computations of supertraces

$$\begin{aligned} \operatorname{str} e^{-\beta \not{D}^2} A &= \operatorname{tr}_{(\operatorname{Ker} \not{D})_{\text{even}}} A - \operatorname{tr}_{(\operatorname{Ker} \not{D})_{\text{odd}}} A \\ &= \operatorname{str}_{\operatorname{index} \not{D}} A \end{aligned} \tag{2}$$

of natural operators A commuting with $\not{D}$ may be equated with the kind of computations that is done in these notes.

Theoretical physicists developed very powerful and insightful ways of thinking about such problems and theoretical physics serves as a very important source of both inspiration and applications for the mathematics pursued here. We will see many examples of this below.

1.1.5

What was said so far encompasses a whole universe of physical and enumerative contexts. While resting on certain common principles, this universe is much too rich and diverse to be reasonably discussed here.

These lectures are written with a particular goal in mind, which is to introduce a student to computations in two intertwined subjects, namely:

— K-theoretic Donaldson-Thomas theory, and

— quantum K-theory of Nakajima varieties.

Some key features of these theories, and of the relationship between them, may be summarized as follows.

1.2 Quantum K-theory of Nakajima varieties

1.2.1

Nakajima varieties [] are associated to a quiver (that is, a finite graph with possibly parallel edges and loops), a pair of dimension vectors, and a choice of stability chamber. They form a remarkable family of *equivariant symplectic resolutions* [] and have found many geometric and representation-theoretic applications. Their construction will be recalled in Section 4.

For quivers of affine ADE type, and a suitable choice of stability, Nakajima varieties are the moduli spaces of framed coherent sheaves on the corresponding ADE surfaces, e.g. on $\mathbb{C}^2$ for a quiver with one vertex and one loop. These moduli spaces play a very important role in supersymmetric gauge theories and algebraic geometry.

For general quivers, Nakajima varieties share many properties of the moduli spaces of sheaves on a symplectic surface. In fact, from their construction, they may be interpreted as moduli spaces of stable objects in certain abelian categories which have the same duality properties as coherent sheaves on a symplectic surface.

1.2.2

From many different perspectives, rational curves in Nakajima varieties are of particular interest. Geometrically, a map

$$\text{curve } C \to \text{Moduli of sheaves on a surface } S$$

produces a coherent sheaf on a threefold $C \times S$. One thus expects a relation between enumerative geometry of sheaves on threefolds fibered in ADE surfaces and enumerative geometry of maps from a fixed curve to affine ADE Nakajima varieties.

Such a relation indeed exists and has been already profitably used in cohomology []. For it to work well, it is important that the fibers are symplectic. Also, because the source C of the map is fixed and doesn't vary in moduli, it can be taken to be a rational curve, or a union of rational curves.

Rational curves in other Nakajima varieties lead to enumerative problems of similar 3-dimensional flavor, even when they are lacking a direct geometric interpretations as counting sheaves on some threefold.

1.2.3

In cohomology, counts of rational curves in a Nakajima variety X are conveniently packaged in terms of equivariant *quantum cohomology*, which is a certain deformation of the cup product in $H^\bullet_G(X)$ with deformation base $H^2(X)$. A related structure is the equivariant *quantum connection*, or Dubrovin connection, on the trivial $H^\bullet_G(X)$ bundle with base $H^2(X)$.

While such packaging of enumerative information may be performed for an abstract algebraic variety X, for Nakajima varieties these structures are described in the language of geometric representation theory, and namely in terms of an action of a certain Yangian $\mathsf{Y}(\mathfrak{g})$ on $H^\bullet_G(X)$. In particular, the quantum connection is identified with the trigonometric Casimir connection for $\mathsf{Y}(\mathfrak{g})$, studied in [] for finite-dimensional Lie algebras.

The construction of the required Yangian $\mathsf{Y}(\mathfrak{g})$, and the identification of quantum structures in terms of $\mathsf{Y}(\mathfrak{g})$, are the main results of []. That work was inspired by conjectures put forward by Nekrasov and Shatashvili [], on the one hand, and by Bezrukavnikov and his collaborators, on the other.

A similar geometric representation theory description of quantum cohomology is expected for all equivariant symplectic resolutions, and perhaps a little bit beyond, see for example []. Further, there are conjectural links, due to Bezrukavnikov and his collaborators, between quantum connections for symplectic resolutions X and representation theory of their quantizations (see, for example, []), their derived autoequivalences etc.

1.2.4

The extension [17] of our work [16] with D. Maulik to K-theory requires several new ideas, as certain arguments that are used again and again in [16] are simply not available in K-theory. For instance, in equivariant cohomology, a proper integral of correct dimension is a nonequivariant constant, which may be computed using an arbitrary specialization of the equivariant parameters (it is typically very convenient to set the weight $\hbar$ of the symplectic form to 0 and send all other equivariant variables to infinity).

By contrast, in equivariant K-theory, the only automatic conclusion about a proper pushforward to a point is that it is a Laurent polynomial in the equivariant variables, and, most of the time, this cannot be improved. If this Laurent polynomial does not involve some equivariant variable variable a then this is called *rigidity*, and is typically shown by a careful study of the $a^{\pm 1} \to \infty$ limit. We will see many variations on this theme below.

Also, very seldom there is rigidity with respect to the weight $q = e^\hbar$ of the symplectic form and nothing can be learned from making that weight trivial. Any argument that involves such step in cohomology needs to be modified, most notably the proof of the commutation of quantum connection with the difference Knizhnik-Zamolodchikov connection, see Sections 1.4 and 9 of [16]. In the logic of [16], quantum connection is characterized by this commutation property, so it is very important to lift the argument to K-theory.

One of the goals of these notes is to serve as an introduction to the additional techniques required for working in K-theory. In particular, the quantum Knizhnik-Zamolodchikov connection appears in Section 9.3 as the computation of the shift operator corresponding to a minuscule cocharacter. Previously in Section 8.2 we construct a difference connection which shifts equivariant variables by cocharacters of the maximal torus and show it commutes with the K-theoretic upgrade of the quantum connection. That upgrade is a difference connection that shifts the Kähler variables

$$z \in H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$$

by cocharacters of this torus, that is, by a lattice isomorphic to $\operatorname{Pic}(X)$. This quantum difference connection is constructed in Section 8.1.

1.2.5

Technical differences notwithstanding, the eventual description of quantum K-theory of Nakajima varieties is exactly what one might have guessed recognizing a general pattern in geometric representation theory. The Yangian $\mathsf{Y}(\mathfrak{g})$, which is a Hopf algebra deformation of $\mathscr{U}(\mathfrak{g} \otimes \mathbb{C}[t])$, is a first member of a hierarchy in which the Lie algebra

$$\mathfrak{g} \otimes \mathbb{C}[t] = \operatorname{Maps}(\mathbb{C}, \mathfrak{g})$$

is replaced, in turn, by a central extension of maps from

— the multiplicative group $\mathbb{C}^\times$, or

— an elliptic curve.

Geometric realizations of the corresponding quantum groups are constructed in equivariant cohomology, equivariant K-theory, and equivariant elliptic cohomology, respectively, see [] and also [] for the construction of an elliptic quantum group from Nakajima varieties.

Here we are on the middle level of the hierarchy, where the quantum group is denoted $\mathscr{U}_q(\hat{\mathfrak{g}})$. The variable q is the deformation parameter; its geometric meaning is the equivariant weight of the symplectic form. For quivers of finite type, these are identical to Drinfeld-Jimbo quantum groups from textbooks. For other quivers, $\mathscr{U}_q(\hat{\mathfrak{g}})$ is constructed in the style of Faddeev, Reshetikhin, and Takhtajan from geometrically constructed R-matrices, see []. In turn, the construction of R-matrices, that is, solutions of the Yang-Baxter equations with a spectral parameter, rests on the K-theoretic version of stable envelopes of []. We discuss those in Section 9.1.

Stable envelopes in K-theory differ from their cohomological ancestors in one important feature: they depend on an additional parameter, called *slope*, which is a choice of an alcove of a certain periodic hyperplane arrangement in $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. This is the same data as appears, for instance, in the study of quantization of X over a field of large positive characteristic. A technical advantage of such slope dependence is a factorization of R-matrices into infinite product of certain *root R-matrices*, which generalizes the classical results obtained by Khoroshkin and Tolstoy in the case when the Lie algebra $\mathfrak{g}$ is of finite dimension.

1.2.6

The identification of the quantum difference connection in term of $\mathscr{U}_q(\hat{\mathfrak{g}})$ was long expected to be the lattice part of the quantum dynamical Weyl group action on $K_G(X)$. For finite-dimensional Lie algebras, this object was studied by Etingof and Varchenko in [] and many related papers, and it was shown in [] to correctly degenerate to the Casimir connection in the suitable limit.

Intertwining operators between Verma modules, which is the main technical tool of [], are only available for real simple roots and so don't yield a large enough dynamical Weyl group as soon as the quiver is not of finite type. However, using root R-matrices and other ideas that were, in some form, already present in [], one can define and study the quantum dynamical Weyl group for arbitrary quivers. This is done in [].

In these notes we stop where the analysis of [] starts: we show that quantum connection commutes with qKZ, which is one of the key features of the dynamical Weyl group.

1.2.7

These notes are meant to be a partial sample of basic techniques and results, and this is not an attempt to write an A to Z technical manual on the subject, nor to present a panorama of geometric applications that these techniques have.

For instance, one of the most exciting topics in quantum K-theory of symplectic resolutions is the duality, known under many different names including "symplectic duality" or "3-dimensional mirror symmetry", see [] for an introduction. Nakajima varieties may be interpreted as the moduli space of Higgs vacua in certain supersymmetric gauge theories, and the computations in their quantum K-theory may be interpreted as indices of the corresponding gauge theories on real 3-folds of the form $C \times S^1$. A physical duality equates these indices for certain pairs of theories, exchanging the Kähler parameters on one side with the equivariant parameters on the other.

In the context of these notes, this means that an exchange of the Kähler and equivariant difference equations of Section 8, which may be studied as such and generalizes various dualities known in representation theory. This is just one example of a very interesting phenomenon that lies outside of the scope of these lectures.

1.3 K-theoretic Donaldson-Thomas theory

1.3.1

Donaldson-Thomas theory, or DT theory for short, is an enumerative theory of sheaves on a fixed smooth quasiprojective threefold Y, which need not be Calabi-

Yau to point out one frequent misconception. There are many categories similar to the category of coherent sheaves on a smooth threefold, and one can profitably study DT-style questions in that larger context. In fact, we already met with such generalizations in the form of quantum K-theory of general Nakajima quiver varieties. Still, I consider sheaves on threefolds to be the core object of study in DT theories.

An example of a sheaf to have in mind could be the structure sheaf $\mathscr{O}_C$ of a curve, or more precisely, a 1-dimensional subscheme, $C \subset Y$. The corresponding DT moduli space $\mathbb{M}$ is the Hilbert scheme of curves in Y and what we want to compute from them is the K-theoretic version of counting curves in Y of given degree and arithmetic genus satisfying some further geometric constraints like incidence to a given point or tangency to a given divisor.

There exist other enumerative theories of curves, notably the Gromov-Witten theory, and, in cohomology, there is a rather nontrivial equivalence between the DT and GW counts, first conjectured in [] and explored in many papers since. At present, it is not known whether the GW/DT correspondence may be lifted to K-theory, as one stumbles early on trying to mimic the DT computations on the GW side [1].

1.3.2

Instead, in K-theory there is a different set of challenging conjectures [] which may serve as one of the goalposts for the development of the theory.

This time the conjectures relate DT counts of curves in Y to a very different kind of curve counts in a Calabi-Yau 5-fold Z which is a total space

$$Z = \begin{array}{c} \mathscr{L}_4 \oplus \mathscr{L}_5 \\ \downarrow \\ Y \end{array} , \quad \mathscr{L}_4 \otimes \mathscr{L}_5 = \mathscr{K}_Y , \tag{3}$$

of a direct sum of two line bundles on Y. One interesting feature of this correspondence is the following. One the DT side, one forms a generating function over all arithmetic genera and then its argument z becomes an element $z \in \mathrm{Aut}(Z, \Omega^5)$ which acts by $\mathrm{diag}(z, z^{-1})$ in the fibers of $\mathscr{L}_4 \oplus \mathscr{L}_5$. Here Ω^5 is the canonical holomorphic 5-form on Z.

A K-theoretic curve count in Z is naturally a virtual representation of the group $\mathrm{Aut}(Z, \Omega^5)$ and, in particular, z has a trace in it which is a rational function of z. This rational function is then equated to something one computes on the DT side by summing over all arithmetic genera. We see it is a nontrivial operation and, also, that equivariant K-theory is the natural setting in which such operations make sense. More general conjectures proposed in [] similarly identify certain equivariant variables for Y with variables that keep track of those curve degree for 5-folds that are lost in Y.

For various DT computations below, we will point out their 5-dimensional interpretation, but this will be the extent of our discussion of 5-dimensional

[1] Clearly, the subject of these note has not even begun to settle, and our presents view of many key phenomena throughout the paper may easily change overnight.

curve counting. It is still in its infancy and not ready to be presented in an introductory lecture series. It is quite different from either the DT or GW curve counting in that it lacks a parameter that keeps track of the curve genus. Curves of any genus are counted equally, but the notion of stability is set up so that that only finitely many genera contribute to any given count.

1.3.3

When faced with a general threefold Y, a natural instinct is to try to cut Y into simpler pieces from which the curve counts in Y may be reconstructed. There are two special scenarios in which this works really well, they can be labeled *degeneration* and *localization*.

In the first scenario, we put Y in a 1-parameter family $\widetilde{Y}$ with a smooth total space

$$\begin{array}{ccccc} Y & \hookrightarrow & \widetilde{Y} & \hookleftarrow & Y_1 \cup_D Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \hookrightarrow & \mathbb{C} & \hookleftarrow & 0 \end{array}$$

so that a special fiber of this family is a union $Y_1 \cup_D Y_2$ of two smooth 3-folds along a smooth divisor D. In this case the curve counts in Y may be reconstructed from certain refined curve counts in each of the Y_i's. These refined counts keep track of the intersection of the curve with the divisor D and are called *relative* DT counts. The technical foundations of the subjets are laid in []. We will get a sense how this works in Section **??**.

The work of Levin and Pandharipande [] supports the idea that using degenerations one should be able to reduce curve counting in general 3-folds to that in *toric* 3-folds. Papers by Maulik and Pandharipande [] and by Pandharipande and Pixton [] offer spectacular examples of this idea being put into action.

1.3.4

Curve counting in toric 3-folds may be broken into further pieces using equivariant localization. Localization is a general principle by which all computations in G-equivariant K-theory of $\mathbb{M}$ depend only on the formal neighborhood of the T-fixed locus $\mathbb{M}^T$, where $T \subset G$ is a maximal torus in a connected group G. We will rely on localization again and again in these notes. Localization is particularly powerful when used in *both* directions, that is, going from the global geometry to the localized one and back, because each point of view has its own advantages and limitations.

A threefold Y is toric if $T \cong (\mathbb{C}^\times)^3$ acts with an open orbit on Y. It then follows that Y has finitely many orbits and, in particular, finitely many orbits of dimension $\leqslant 1$. Those are the fixed points and the T-invariant curves, and they correspond to the 1-skeleton of the toric polyhedron of Y. From the localization viewpoint, Y may very well be replaced by this 1-skeleton. All nonrelative curve counts in Y may be done in terms of certain 3- and 2-valent tensors, called vertices and edges, associated to fixed points and T-invariant curves, respectively. See, for example, [] for a pictorial introduction.

The underlying vector space for these tensors is

— the equivariant K-theory of $\mathrm{Hilb}(\mathbb{C}^2, \text{points})$, or equivalently

— the standard Fock space, or the algebra of symmetric functions,

with an extension of scalars to include all equivariant variables as well as the variable z that keeps track of the arithmetic genus. Natural bases of this vector space are indexed by partitions and curve counts are obtained by contracting all indices, in great similarity to many TQFT computations.

In the basis of torus-fixed points of the Hilbert scheme, edges are simple diagonal tensors, while vertices are something complicated. More sophisticated bases spread the complexity more evenly.

1.3.5

These vertices and edges, and related objects, are the nuts and bolts of the theory and the ability to compute with them is a certain measure of professional skill in the subject.

A simple, but crucial observation is that the geometry of ADE surface fibrations captures [2] all these vertices and edges. This bridges DT theory with topics discussed in Section 1.2, and was already put to very good use in [].

In [], there are two kind of vertices: *bare*, or standard, and *capped*. They are the same tensors expressed in two different bases, and which have different geometric meaning and different properties. Parallel catalogization can be made in K-theory and it is convenient to extend it to general Nakajima varieties (or to general quasimap problems, for that matter).

For general Nakajima varieties, the notion of a bare 1-leg vertex coincides with Givental's notion of I-function, and there is no real analog of 2- or 3-legged vertex for general Nakajima variety [3]. These vertices, their capped versions, and their various properties are the subject of Section 7.

1.4 Acknowledgements

2 Before we begin

The goal of this section is to have a brief abstract discussion of several construction in equivariant K-theory which will appear and reappear in more concrete

[2] In fact, formally, it suffices to understand A_n-surface fibrations with $n \leqslant 2$.

[3] The Hilbert scheme $\mathrm{Hilb}(A_{n-1})$ of the A_{n-1}-surface is dual, in the sense of Section 1.2.7, to the moduli space $M(n)$ of framed sheaves of rank n of $A_0 \cong \mathbb{C}^2$, which is a Nakajima variety for the quiver with one vertex and one loop. The splitting of the 1-leg vertex for the A_{n-1}-surface into n simpler vertices is a phenomenon which is dual to $M(n)$ being an n-fold tensor product of $\mathrm{Hilb}(\mathbb{C}^2)$ in the sense of []. For a general Nakajima variety, there is no direct analog of this.

situations below. This section is not meant to be an introduction to equivariant K-theory; Chapter 5 of [?] is highly recommended for that.

2.1 Symmetric and exterior algebras

2.1.1

Let T be a torus and V a finite dimensional T-module. Clearly, V is a direct sum of 1-dimensional T-modules, which are called the *weights* of V. The weights μ are recorded by the character of V

$$\chi_V(t) = \operatorname{tr}_V t = \sum t^\mu \,, \quad t \in \mathsf{T}\,,$$

where repetitions are allowed among the μ's.

We denote by K_T or $K_T(\mathrm{pt})$ the K-group of the category of T-modules. Of course, any exact sequence

$$0 \to V' \to V \to V'' \to 0$$

is anyhow split, so there is no need to impose the relation $[V] = [V'] + [V'']$ in this case. The map $V \mapsto \chi_V$ gives an isomorphism

$$K_T \cong \mathbb{Z}[t^\mu]\,,$$

with the group algebra of the character lattice of the torus T. Multiplication in $\mathbb{Z}[t^\mu]$ corresponds to $\otimes$-product in K_T.

2.1.2

For V as above, we can form its symmetric powers $\mathsf{S}^2 V, \mathsf{S}^3 V, \dots,$ including $S^0 V = \mathbb{C}$. These are $GL(V)$-modules and hence also T modules.

Exercise 2.1. Prove that

$$\begin{aligned}\sum_{k\geqslant 0} s^k \chi_{\mathsf{S}^k V}(t) &= \prod \frac{1}{1 - s\, t^\mu} \\ &= \exp \sum_{n>0} \frac{1}{n} s^n \chi_V(t^n)\,. \end{aligned} \tag{4}$$

We can view the functions in (4) as an element of $K_{\mathsf{T}}[[s]]$ or as a character of an infinite-dimensional graded T-module $\mathsf{S}^\bullet V$ with finite-dimensional graded subspaces, where s keeps track of the degree.

For the exterior powers we have, similarly,

Exercise 2.2. Prove that

$$\begin{aligned}\sum_{k\geqslant 0} (-s)^k \chi_{\Lambda^k V}(t) &= \prod (1 - s\, t^\mu) \\ &= \exp\left(- \sum_{n>0} \frac{1}{n} s^n \chi_V(t^n)\right)\,. \end{aligned} \tag{5}$$

The functions in (4) and (5) are reciprocal of each other. The representation-theoretic object behind this simple observation is known as the *Koszul complex*.

Exercise 2.3. Construct an $GL(V)$-equivariant exact sequence

$$\cdots \to \Lambda^2 V \otimes \mathsf{S}^\bullet V \to \Lambda^1 V \otimes \mathsf{S}^\bullet V \to \mathsf{S}^\bullet V \to \mathbb{C} \to 0\,, \tag{6}$$

where $\mathbb{C}$ is the trivial representation.

Exercise 2.4. Consider V as an algebraic variety on which $GL(V)$ acts. Construct a $GL(V)$-equivariant resolution of the structure sheaf of $0 \in V$ by vector bundles on V. Be careful not to get (6) as your answer.

2.1.3

Suppose $\mu = 0$ is not a weight of V, which means that (4) does not have a pole at $s = 1$. Then we can set $s = 1$ in (4) and define

$$\chi_{\mathsf{S}^\bullet V} = \prod \frac{1}{1 - t^\mu} = \exp \sum_n \frac{1}{n} \chi_V \, t^n\,, \tag{7}$$

This is a well-defined element of *completed* K-theory of T provided

$$\|t^\mu\| < 1$$

for all weights of V with respect to some norm K_{T}.

Alternatively, and with only the $\mu \neq 0$ assumption on weights, (7) well-defined element of the *localized* K-theory of T

$$K_{\mathsf{T},\text{localized}} = \mathbb{Z}\left[t^\nu, \frac{1}{1 - t^\mu}\right]$$

where we invert some or all elements $1 - t^\mu \in K_{\mathsf{T}}$.

Since

$$K_{\mathsf{T}} \hookrightarrow K_{\mathsf{T},\text{localized}}$$

characters of finite-dimensional modules may be computed in localization without loss of information. However, certain different infinite-dimensional modules become the same in localization, for example

Exercise 2.5. For V as above, check that

$$\mathsf{S}^\bullet V = (-1)^{\operatorname{rk} V} \det V^\vee \otimes \mathsf{S}^\bullet V^\vee \tag{8}$$

in localization of K_{T}.

We will see, however, that this is a feature rather than a bug.

Exercise 2.6. Show that $\mathsf{S}^\bullet$ extends to a map

$$K'_{\mathsf{T}} \xrightarrow{\mathsf{S}^\bullet} K_{\mathsf{T},\text{localized}} \tag{9}$$

where prime means that there is no zero weight, which satisfies

$$\mathsf{S}^\bullet(V_1 \oplus V_2) = \mathsf{S}^\bullet V_1 \otimes \mathsf{S}^\bullet V_2 \,,$$

and, in particular,

$$\mathsf{S}^\bullet(-V) = \Lambda^\bullet V = \sum_i (-1)^i \Lambda^i V \,. \tag{10}$$

Here and in what follows, the symbol $\Lambda^\bullet V$ is defined by (10) as the alternating sum of exterior powers.

2.1.4

The map (9) may be extended by continuity to a completion of K_{T} with respect to a suitable norm. This gives a compact way to write infinite products, for example

$$\mathsf{S}^\bullet \frac{a-b}{1-q} = \prod_{n\geqslant 0} \frac{1-q^n b}{1-q^n a} \,,$$

which converges in any norm $\|\cdot\|$ such that $\|q\|<1$.

Exercise 2.7. Check that

$$\mathsf{S}^\bullet \frac{a}{(1-q)^{k+1}} = \prod_{n\geqslant 0} (1-q^n a)^{-\binom{k+n}{n}} \,.$$

2.1.5

The map $\mathsf{S}^\bullet$ is also known under many aliases, including *plethystic exponential*. Its inverse is known, correspondingly, as the plethystic logarithm.

Exercise 2.8. Prove that the inverse to $\mathsf{S}^\bullet$ is given by the formula

$$\chi_V(t) = \sum_{n>0} \frac{\mu(n)}{n} \ln \chi_{\mathsf{S}^\bullet V}(t^n)$$

where μ is the *Möbius function*

$$\mu(n) = \begin{cases} (-1)^{\#\text{ of prime factors}}\,, & n \text{ squarefree} \\ 0\,, & \text{otherwise}\,. \end{cases}$$

The relevant property of the Möbius function is that it gives the matrix elements of C^{-1} where the matrix $C = (C_{ij})_{i,j\in\mathbb{N}}$ is defined by

$$C_{ij} = \begin{cases} 1\,, & i|j \\ 0\,, & \text{otherwise}\,. \end{cases}$$

In other words, μ is the Möbius function of the set $\mathbb{N}$ partially ordered by divisibility, see [24].

2.1.6

If the determinant of V is a square as a character of T, we define

$$\widehat{\mathsf{S}}^\bullet V = (\det V)^{1/2}\, \mathsf{S}^\bullet \tag{11}$$

which by (8) satisfies

$$\widehat{\mathsf{S}}^\bullet V^\vee = (-1)^{\operatorname{rk} V}\, \widehat{\mathsf{S}}^\bullet V\,.$$

Somewhat repetitively, it may be called the symmetrized symmetric algebra.

2.2 $K_G(X)$ and $K^\circ_G(X)$

2.2.1

Let a reductive group G act on a scheme X. We denote by $K_G(X)$ the K-group of the category of G-equivariant coherent sheaves on X. Replacing general coherent sheaves by locally free ones, that is, by G-equivariant vector bundles on X, gives another group $K^\circ_G(X)$ with a natural homomorphism

$$K^\circ_G(X) \to K_G(X)\,. \tag{12}$$

Remarkably and conveniently, (12) is is an isomorphism if X is nonsingular. In other words, every coherent sheaf on a nonsingular variety is *perfect*, which means it admits a locally free resolution of finite length, see for example Section B.8 in [**?**].

Exercise 2.9. Consider $X = \{x_1 x_2 = 0\} \subset \mathbb{C}^2$ with the action of the maximal torus

$$T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\} \subset GL(2)\,.$$

Let $\mathscr{F} = \mathscr{O}_0$ be the structure sheaf of the origin $0 \in X$. Compute the minimal T-equivariant resolution

$$\cdots \to \mathscr{R}^{-2} \to \mathscr{R}^{-1} \to \mathscr{R}^0 \to \mathscr{F} \to 0$$

of $\mathscr{F}$ by sheaves of the form

$$\mathscr{R}^{-i} = \mathscr{O}_X \otimes R_i\,,$$

where R_i is a finite-dimensional T-module. Observe from the resolution that the groups

$$\operatorname{Tor}_i(\mathscr{F},\mathscr{F}) \stackrel{\text{def}}{=} H^{-i}(\mathscr{R}^\bullet \otimes \mathscr{F}) = R_i \tag{13}$$

are nonzero for all $i \geqslant 0$ and conclude that $\mathscr{F}$ is not in the image of $K^\circ_T(X)$. Also observe that

$$\sum (-1)^i \chi_{\operatorname{Tor}_i(\mathscr{F},\mathscr{F})} = \frac{\chi(\mathscr{F})^2}{\chi(\mathscr{O}_X)} = \frac{(1-t_1^{-1})(1-t_2^{-1})}{1-t_1^{-1}t_2^{-1}} \tag{14}$$

expanded in inverse powers of t_1 and t_2.

Exercise 2.10. Generalize (14) to the case

$$\begin{aligned} X &= \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_d]/I \,, \\ \mathscr{F} &= \mathbb{C}[x_1, \ldots, x_d]/I' \,, \end{aligned}$$

where $I \subset I'$ are monomial ideals, that is, ideals generated by monomials in the variables x_i.

2.2.2

The domain and the source of the map (12) have different functorial properties with respect to G-equivariant morphisms

$$f : X \to Y \tag{15}$$

of schemes.

The pushforward of a K-theory class $[\mathscr{G}]$ represented by a coherent sheaf $\mathscr{G}$ is defined as

$$f_* [\mathscr{G}] = \sum_i (-1)^i \left[R^i f_* \mathscr{G} \right] . \tag{16}$$

We abbreviate $f_* \mathscr{G} = f_* [\mathscr{G}]$ in what follows.

The length of the sum in (16) is bounded, e.g. by the dimension of X, but the terms, in general, are only quasicoherent sheaves on Y. If f is proper on the support of $\mathscr{G}$ then this ensures $f_* \mathscr{G}$ is coherent and thus lies in $K_G(Y)$. Additional hypotheses are required to conclude $f_* \mathscr{G}$ is perfect. For an example, take $\iota_* \mathscr{O}_0$ where

$$\iota : \{0\} \hookrightarrow \{x_1 x_2 = 0\}$$

is the inclusion in Exercise 2.9.

Exercise 2.11. The group $GL(2)$ acts naturally on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ and on line bundles $\mathscr{O}(k)$ over it. Push forward these line bundles under $\mathbb{P}^1 \to \mathrm{pt}$ using a explicit T-invariant Čech covering of $\mathbb{P}^1$. Generalize to $\mathbb{P}^n$.

2.2.3

The pull-back of $[\mathscr{E}] \in K_G(Y)$ is defined by

$$f^* [\mathscr{E}] = \sum (-1)^i \left[\mathscr{T}or_i^{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{E}) \right] .$$

Here the terms are coherent, but there may be infinitely many of them, as is the case for $\iota^* \mathscr{O}_0$ in our running example. To ensure the sum terminates we need some flatness assumptions, such as $\mathscr{E}$ being locally free. In particular,

$$f^* : K_G^\circ(Y) \to K_G^\circ(X)$$

is defined for arbitrary f by simply pulling back vector bundles.

Exercise 2.12. Globalize the computation in Exercise 2.4 to compute $\iota^* \mathscr{O}_X \in K_G(\mathscr{O}_X)$ for a G-equivariant inclusion

$$\iota : X \hookrightarrow Y$$

of a nonsingular subvariety X in a nonsingular variety Y.

2.2.4

Tensor product makes $K_G^\circ(X)$ a ring and $K_G(X)$ is a module over it. The projection formula

$$f_*(\mathscr{F} \otimes f^* \mathscr{E}) = f_*(\mathscr{F}) \otimes \mathscr{E} \tag{17}$$

expresses the covariance of this module structure with respect to morphisms f.

Exercise 2.13. Write a proof of the projection formula.

Projection formula can be used to prove that a proper pushforward $f_* \mathscr{G}$ is perfect if $\mathscr{G}$ is flat over Y, see Theorem 8.3.8 in [11].

2.2.5

Let X be a scheme and $X' \subset X$ a closed G-invariant subscheme. Then the sequence

$$K_G(X') \to K_G(X) \to K_G(X \backslash X') \to 0 \tag{18}$$

where all maps are the natural pushforwards, is exact, see e.g. Proposition 7 in [3] for a classical discussion. This is the beginning of a long exact sequence of higher K-groups.

Exercise 2.14. For $X = \mathbb{C}^n$, $X' = \{0\}$, and $T \subset GL(n)$ the maximal torus, fill in the question marks in the following diagram

$$\begin{array}{ccccccc}
K_T(X') & \longrightarrow & K_T(X) & \longrightarrow & K_T(X \backslash X') & \longrightarrow & 0 \\
\sim \big\downarrow & & \sim \big\downarrow & & \sim \big\downarrow & & \\
\mathbb{Z}[t^\mu] & \xrightarrow{\ ?\ } & \mathbb{Z}[t^\mu] & \longrightarrow & ? & \longrightarrow & 0
\end{array}$$

in which the vertical arrows send the structure sheaves to $1 \in \mathbb{Z}[t^\mu]$.

In particular, since $X \backslash X_{\text{red}} = \varnothing$, the sequence (18) implies

$$K_G(X_{\text{red}}) \cong K_G(X)$$

where $X_{\text{red}} \subset X$ is the reduced subscheme, whose sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ is formed by nilpotent elements. Concretely, any coherent sheaf $\mathscr{F}$ has a finite filtration

$$\mathscr{F} \supset \mathscr{I} \cdot \mathscr{F} \supset \mathscr{I}^2 \cdot \mathscr{F} \supset \dots$$

with quotients pushed forward from X_{red}.

2.2.6

One can think about the sequence (18) like this. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ two coherent sheaves on X, together with an isomorphism

$$s : \mathscr{F}_2\big|_U \xrightarrow{\sim} \mathscr{F}_1\big|_U$$

of their restriction to the open set $U = X\backslash X'$. Let $\iota : U \to X$ denote the inclusion and let

$$\widehat{\mathscr{F}} \subset \iota_* \iota^* \mathscr{F}_1$$

be the subsheaf generated by the natural maps

$$\mathscr{F}_1 \to \iota_* \iota^* \mathscr{F}_1 \,, \quad \mathscr{F}_2 \xrightarrow{s} \iota_* \iota^* \mathscr{F}_1 \,.$$

Of course, $\iota_* \iota^* \mathscr{F}_1$ is only a quasicoherent sheaf on Y, which is evident in the simplest example $X = \mathbb{A}^1$, $X' =$ point, $\mathscr{F}_1 = \mathscr{O}_Y$. However, the sheaf $\widehat{\mathscr{F}}$ is generated by the generators of $\mathscr{F}_1$ and $\mathscr{F}_2$, and hence coherent.

By construction, the kernels and cokernels of the natural maps

$$f_i : \mathscr{F}_i \to \widehat{\mathscr{F}}$$

are supported on X'. Thus

$$\mathscr{F}_1 - \mathscr{F}_2 = \operatorname{Coker} f_1 - \operatorname{Ker} f_1 + \operatorname{Ker} f_2 - \operatorname{Coker} f_2$$

is in the image of $K(X') \to K(X)$.

2.2.7

Exercise 2.15. Let G be trivial and let $\mathscr{F}$ be a coherent sheaf on X with support $Y \subset X$. Let $\mathscr{E}$ be a vector bundle on X of rank r. Prove that there exists $Y' \subset Y$ of codimension 1 such that

$$\mathscr{E} \otimes \mathscr{F} - r\, \mathscr{F}$$

is in the image of $K(Y') \to K(X)$.

This exercise illustrates a very useful finite filtration on $K(X)$ formed by the images of $K(Y) \to K(X)$ over all subvarieties of given codimension.

Exercise 2.16. Let G be trivial and $\mathscr{E}$ be a vector bundle on X of rank r. Prove that $(\mathscr{E} - r)\otimes$ is nilpotent as an operator on $K(X)$.

Exercise 2.17. Take $X = \mathbb{P}^1$ and $G = GL(2)$. Compute the minimal polynomial of the operator $\mathscr{O}(1)\otimes$ and see, in particular, that it is not unipotent.

2.3 Localization

2.3.1

Let a torus T act on a scheme X and let X^{A} be subscheme of T-fixed points, that is, let

$$\mathscr{O}_X \to \mathscr{O}_{X^T} \to 0$$

be the largest quotient on which T acts trivially. For what follows, both X and X^T may be replaced by their reduced subschemes.

Consider the kernel and cokernel of the map

$$\iota_* : K_T(X^T) \to K_T(X)\,.$$

This kernel and cokernel are $K_T(\mathrm{pt})$-modules and have some support in the torus T. A very general localization theorem of Thomason [] states

$$\operatorname{supp} \operatorname{Coker} \iota_* \subset \bigcup_{\mu} \{t^\mu = 1\} \tag{19}$$

where the union over finitely many nontrivial characters μ. The same is true of $\operatorname{Ker} \iota_*$, but since

$$K_T(X^T) = K(X) \otimes_{\mathbb{Z}} K_T(\mathrm{pt}) \tag{20}$$

has no such torsion, this forces $\operatorname{Ker} \iota_* = 0$. To summarize, ι_* becomes an isomorphism after inverting finitely many coefficients of the form $t^\mu - 1$.

This localization theorem is an algebraic analog of the classical localization theorems in topological K-theory that go back to [].

Exercise 2.18. Compute $\operatorname{Coker} \iota_*$ for $X = \mathbb{P}^1$ and $T \subset GL(2)$ the maximal torus. Compare your answer with what you computed in Exercise 2.14.

2.3.2

For general X, it is not so easy to make the localization theorem explicit, but a very nice formula exists if X is nonsingular. This forces X^{T} to be also nonsingular.

Let $N = N_{X/X^T}$ denote the normal bundle to X^T in X. The total space of N has a natural action of $s \in \mathbb{C}^\times$ by scaling the normal directions. Using this scaling action, we may define

$$\mathscr{O}_{N,\mathrm{graded}} = \sum_{k=0} s^{-k}\, \mathsf{S}^k N^\vee = \bigotimes_{\mu} \frac{1}{\Lambda^\bullet\left(s^{-1} t^{-\mu} N_\mu^\vee\right)} \in K_T(X^T)[[s^{-1}]]\,, \tag{21}$$

where

$$N = \bigoplus t^\mu N_\mu$$

is the decomposition of N into eigenspaces of T-action according to (20).

Exercise 2.19. Using Exercise 2.16 prove that

$$\mathscr{O}_N = \mathscr{O}_{N,\text{graded}}\Big|_{s=1} = \mathsf{S}^\bullet N^\vee \in K^\circ_{\mathsf{T}}(X^{\mathsf{T}})\left[\frac{1}{1-t^\mu}\right]$$

where μ are the weights of T in N.

Exercise 2.20. Prove that

$$\iota^*\iota_*\mathscr{G} = \mathscr{G} \otimes \Lambda^\bullet N^\vee$$

for any $\mathscr{G} \in K_T(X^T)$ and that the operator $\Lambda^\bullet N^\vee \otimes$ becomes an isomorphism after inverting $1 - t^\mu$ for all weights of N. Conclude the localization theorem (19) implies

$$\iota_*^{-1}\mathscr{F} = \mathsf{S}^\bullet N^\vee \otimes \iota^*\mathscr{F} \tag{22}$$

for any $\mathscr{F}$ in localized equivariant K-theory.

2.3.3

A T-equivariant map $f : X \to Y$ induces a diagram

$$\begin{array}{ccc} X^T & \xrightarrow{\iota_X} & X \\ {\scriptstyle f^T}\downarrow & & \downarrow{\scriptstyle f} \\ Y^T & \xrightarrow{\iota_Y} & Y \end{array} \tag{23}$$

with

$$f_* \circ \iota_{X,*} = \iota_{Y,*} \circ f^T_* \,. \tag{24}$$

Normally, we don't care much about torsion, or we may know ahead of time that there is no torsion in f_*, like when f is a proper map to a point, or some other trivial T-variety. Then, we can write

$$f_* = \iota_{Y,*} \circ f^T_* \circ \iota^{-1}_{X,*} \,. \tag{25}$$

This is what it means to compute the pushforward by localization.

Exercise 2.21. Redo Exercise 2.11, that is, compute $\chi(\mathbb{P}^n, \mathscr{O}(k))$ by localization.

Exercise 2.22. Let G be a reductive group and $X = G/B$ the corresponding flag variety. Every character λ of the maximal torus T gives a character of B and hence a line bundle

$$\mathscr{L}_\lambda : (G \times \mathbb{C}_\lambda)/B \to X$$

over X. Compute $\chi(X, \mathscr{L}_\lambda)$ by localization. A theorem of Bott, see e.g. [] states that at most one cohomology group $H^i(X, \mathscr{L}_\lambda)$ is nonvanishing, in which case it is an irreducible representation of G. So be prepared to rederive Weyl character formula from your computation.

Exercise 2.23. Explain how Exercise 2.21 is a special case of Exercise 2.22.

2.3.4

Using (25), one may *define* pushforward $f_*\mathscr{F}$ in localized equivariant cohomology as long as f_T is proper on $(\operatorname{supp}\mathscr{F})^T$. This satisfies all usual properties and leads to meaningful results, like

$$\chi(\mathbb{C}^n, \mathscr{O}) = \prod_i \frac{1}{1-t_i^{-1}}$$

as a module over the maximal torus $T \subset GL(n)$.

2.3.5

The statement of the localization theorem goes over with little or no change to certain more general X, for example, to orbifolds. Those are modelled locally on $\widetilde{X}/\Gamma$, where $\widetilde{X}$ is nonsingular and Γ is finite. By definition, coherent sheaves on $\widetilde{X}/\Gamma$ are Γ-equivariant coherent sheaves on $\widetilde{X}$.

A torus action on $\widetilde{X}/\Gamma$ is a $T \times \Gamma$ action on $\widetilde{X}$ and, in particular, the normal bundle $N_{\widetilde{X}/\widetilde{X}^T}$ is Γ-equivariant, which means it descends to to an orbifold normal bundle to $\left[\widetilde{X}/\Gamma\right]^T$.

Exercise 2.24. For $a, b > 0$, consider the weighted projective line

$$X_{a,b} = \mathbb{C}^2\backslash\{0\} \Big/ \begin{pmatrix} z^a & \\ & z^b \end{pmatrix}, \quad z \in \mathbb{C}^\times .$$

Show it can be covered by two orbifold charts. Like any $\mathbb{C}^\times$-quotient, it inherits an orbifold line bundle $\mathscr{O}(1)$ whose sections are functions ϕ on the prequotient such that

$$\phi(z \cdot x) = z\,\phi(x) .$$

Show that

$$\sum_{k \geqslant 0} \chi(X_{a,b}, \mathscr{O}(k))\, s^k = \frac{1}{(1-t_1^{-1}s^a)(1-t_2^{-1}s^b)} \tag{26}$$

as a module over diagonal matrices. Compute $\chi(X_{a,b}, \mathscr{O}(k))$ by localization. Compare your answer to the computation of the s^k-coefficient in (26) by residues.

2.3.6

What we will really need in these lectures is the *virtual localization formula* from []. It will be discussed after we get some familiarity with virtual classes.

In particular, in this greater generality the normal bundle N to the fixed locus is a virtual vector bundle, that is, an element of $K_T^\circ(X^T)$ of the form

$$N = N_{\mathrm{Def}} - N_{\mathrm{Obs}} ,$$

where the N_{Def} is responsible for first-order deformations, while N_{Obs} contains obstructions to extending those. Naturally,

$$\mathsf{S}^\bullet N^\vee = \mathsf{S}^\bullet N_{\text{Def}}^\vee \otimes \Lambda^\bullet N_{\text{Obs}}^\vee \,,$$

so a virtual localization formula has both denominators and numerators.

2.4 Rigidity

2.4.1

If the support of a T-equivariant sheaf $\mathscr{F}$ is proper then $\chi(\mathscr{F})$ is an element of $K_T(\text{pt})$ and so a Laurent polynomial in $t \in T$. In general, this polynomial is nontrivial which, of course, is precisely what makes equivariant K-theory interesting.

However, for the development of the theory, one would like its certain building blocks to depend on few or no equivariant variables. This phenomenon is known as rigidity. A classical [] and surprisingly effective way to show rigidity is to use the following elementary observation:

$$p(z) \text{ is bounded as } z^{\pm 1} \to \infty \quad \Leftrightarrow \quad p = \text{const}$$

for any $p \in \mathbb{C}[z^{\pm 1}]$. The behavior of $\chi(\mathscr{F})$ at the infinity of the torus T can be often read off directly from the localization formula.

Exercise 2.25. Let X be proper and smooth with an action of a connected reductive group G. Write a localization formula for the action of $T \subset G$ on

$$\sum_p (-m)^p \chi(X, \Omega^p)$$

and conclude that every term in this sum is a trivial G-module.

Of course, Hodge theory gives the triviality of G-action on each

$$H^q(X, \Omega^p) \subset H^{p+q}(X, \mathbb{C})$$

for a compact Kähler X and a connected group G.

2.4.2

When the above approach works it also means that the localization formula may be simplified by sending the equivariant variable to a suitable infinity of the torus.

Exercise 2.26. In Exercise 2.25, pick a generic 1-parameter subgroup

$$z \in \mathbb{C}^\times \to T$$

and compute the asymptotics of your localization formula as $z \to 0$.

It is instructive to compare the result of Exercise 2.26 with the Białynicki-Birula decomposition, which goes as follows. Assume $X \subset \mathbb{P}(\mathbb{C}^N)$ is smooth and invariant under the action of a 1-parameter subgroup $\mathbb{C}^\times \to GL(N)$. Let

$$X^{\mathbb{C}^\times} = \bigsqcup_i F_i$$

be the decomposition of the fixed locus into connected components. It induces a decomposition of X

$$X = \bigsqcup X_i \,, \quad X_i = \left\{ x \,\middle|\, \lim_{z\to 0} z \cdot x \in F_i \right\} \tag{27}$$

into locally closed sets. The key property of this decomposition is that the natural map

$$X_i \xrightarrow{\lim} Y_i$$

is a fibration by affine spaces of dimension

$$d_i = \operatorname{rk} \left(N_{X/Y_i}\right)_+$$

where plus denotes the subbundle spanned by vectors of positive z-weight, see for example [] for a recent discussion. As also explained there, the decomposition (27) is, in fact, motivic, and in particular the Hodge structure of X is that sum of those for Y_i shifted by (d_i, d_i).

The same decomposition of X can be obtained from Morse theory applied to the Hamiltonian H that generated the action of $U(1) \subset \mathbb{C}^\times$ on the (real) symplectic manifold X. Concretely, if z acts by

$$[x_1 : x_2 : \dots : x_n] \xrightarrow{z} [z^{m_1}\, x_1 : z^{m_2}\, x_2 : \dots : z^{m_n}\, x_n]$$

then

$$H(x) = \sum m_i |x_i|^2 \Big/ \sum |x_i|^2 \ .$$

2.4.3

In certain instances, the same argument gives more.

Exercise 2.27. Let X be proper nonsingular with a nontrivial action of $T \cong \mathbb{C}^\times$. Assume that a fractional power $\mathscr{K}^p$ for $0 < p < 1$ of the canonical bundle $\mathscr{K}_X$ exists in $\operatorname{Pic}(X)$. Replacing T by a finite cover, we can make it act on $\mathscr{K}^p$. Show that

$$\chi(X, \mathscr{K}^p) = 0 \,.$$

What does this say about projective spaces ?

3 The Hilbert scheme of points of 3-folds

3.1 Our very first DT moduli space

3.1.1

For a moment, let X be nonsingular quasiprojective 3-fold; very soon we will specialize the discussion to the case $X = \mathbb{C}^3$. Our interest is in the enumerative geometry of subschemes in X, and usually we will want these subscheme projective and 1-dimensional.

A subscheme $Z \subset X$ is defined by a sheaf of ideals $\mathscr{I}_Z \subset \mathscr{O}_X$ in the sheaf $\mathscr{O}_X$ of functions on X and, by construction, there is an exact sequence

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0 \tag{28}$$

of coherent sheaves on X. Either the injection $\mathscr{I}_Z \hookrightarrow \mathscr{O}_X$, or the surjection $\mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z$ determines Z and can be used to parametrize subschemes of X. The result, known as the Hilbert scheme, is a countable union of quasiprojective algebraic varieties, one for each possible topological K-theory class of O_Z. The construction of the Hilbert scheme goes back to A. Grothendieck and is explained, for example, in [].

In particular, for 1-dimensional Z, the class $[O_Z]$ is specified by

$$\deg Z = -c_2(\mathscr{O}_Z) \in H_2(X, \mathbb{Z})_{\text{effective}}$$

and by the Euler characteristic $\chi(\mathscr{O}_Z) \in \mathbb{Z}$. In this section, we consider the case $\deg Z = 0$, that is, the case of the Hilbert scheme of points.

3.1.2

If X is affine then the Hilbert scheme of points parametrizes modules M over the ring $\mathscr{O}_X$ such that

$$\dim_{\mathbb{C}} M = n\,, \quad n = \chi(\mathscr{O}_Z)\,,$$

together with a surjection from a free module. Such Hilbert schemes $\mathrm{Hilb}(R, n)$ may, in fact, be defined for an arbitrary finitely-generated algebra

$$R = \mathbb{C}\langle x_1, \dots, x_k\rangle \big/ \text{relations}$$

and consists of k-tuples of $n \times n$ matrices

$$\mathsf{X}_1 \dots, \mathsf{X}_k \in \mathrm{End}(\mathbb{C}^n) \tag{29}$$

satisfying the relations of R, together with a cyclic vector $v \in \mathbb{C}^n$, all modulo the action of $GL(n)$ by conjugation. Here, a vector is called cyclic if it generates $\mathbb{C}^n$ under the action of X_i's. Clearly, a surjection $R_1 \twoheadrightarrow R_2$ leads to inclusion $\mathrm{Hilb}(R_2, n) \hookrightarrow \mathrm{Hilb}(R_1, n)$ and, in particular,

$$\mathrm{Hilb}(R, n) \subset \mathrm{Hilb}(\mathsf{Free}_k, n)$$

if R is generated by k elements.

Exercise 3.1. Prove that $\mathrm{Hilb}(\mathsf{Free}_k, n)$ is a smooth algebraic variety of dimension $(k-1)n^2 + n$. Show that $\mathrm{Hilb}(\mathsf{Free}_1, n)$ is isomorphic to $S^n\mathbb{C} \cong \mathbb{C}^n$ by the map that takes x_1 to its eigenvalues.

By contrast, $\mathrm{Hilb}(\mathbb{C}^3, n)$ is a very singular variety of unknown dimension.

Exercise 3.2. Let $\mathsf{m} \subset \mathscr{O}_{\mathbb{C}^d}$ be the ideal of the origin. Following Iarrobino, observe, that any linear linear subspace I such that $\mathsf{m}^r \supset I \supset \mathsf{m}^{r+1}$ for some r is an ideal in $\mathscr{O}_{\mathbb{C}^d}$. Conclude that the dimension of $\mathrm{Hilb}(\mathbb{C}^d, n)$ grows at least like a constant times $n^{2-2/d}$ as $n \to \infty$. This is, of course, consistent with

$$\dim \mathrm{Hilb}(\mathbb{C}^1, n) = n\,, \quad \dim \mathrm{Hilb}(\mathbb{C}^2, n) = 2n$$

but shows that $\mathrm{Hilb}(\mathbb{C}^d, n)$ is not the closure of the locus of n distinct points for $d \geqslant 3$ and large enough n.

3.1.3

Consider the embedding

$$\mathrm{Hilb}(\mathbb{C}^3, n) \subset \mathrm{Hilb}(\mathsf{Free}_3, n)$$

as the locus of matrices that commute, that is $\mathsf{X}_i\mathsf{X}_j = \mathsf{X}_j\mathsf{X}_i$. For 3 matrices, and only for 3 matrices, this relations can be written an equation for a critical point:

$$d\phi = 0\,, \quad \phi(\mathsf{X}) = \mathrm{tr}\,(\mathsf{X}_1\mathsf{X}_2\mathsf{X}_3 - \mathsf{X}_1\mathsf{X}_3\mathsf{X}_2)\;.$$

Note that ϕ is a well-defined function on $\mathrm{Hilb}(\mathsf{Free}_3, n)$ which transforms in the 1-dimensional representation $\boldsymbol{\kappa}^{-1}$, where

$$\boldsymbol{\kappa} = \Lambda^3\mathbb{C}^3 = \det\nolimits_{GL(3)}\,,$$

under the natural action of $GL(3)$ on $\mathrm{Hilb}(\mathsf{Free}_3, n)$. Here we have to remind ourselves that the action of a group G on functions is *dual* to its action on coordinates.

This means that our moduli space $\mathbb{M} = \mathrm{Hilb}(\mathbb{C}^3, n)$ is cut out inside an ambient smooth space $\widetilde{\mathbb{M}} = \mathrm{Hilb}(\mathsf{Free}_3, n)$ by a section

$$\mathscr{O}_{\widetilde{\mathbb{M}}} \xrightarrow{d\phi\otimes\boldsymbol{\kappa}} \boldsymbol{\kappa} \otimes T^*_{\widetilde{\mathbb{M}}}\,,$$

of a vector bundle on $\widetilde{\mathbb{M}}$. The twist by $\boldsymbol{\kappa}$ is necessary to make this section $GL(3)$-equivariant.

This illustrates two important points about moduli problems in general, and moduli of coherent sheaves on nonsingular threefolds in particular. First, locally near any point, deformation theory describes many moduli spaces in a similar way:

$$\mathbb{M} = s^{-1}(0) \subset \widetilde{\mathbb{M}}, \quad s \in \Gamma(\widetilde{\mathbb{M}}, \mathscr{E})\,, \tag{30}$$

for a certain *obstruction* bundle $\mathscr{E}$. Second, for coherent sheaves on 3-folds, there is a certain kinship between the obstruction bundle $\mathscr{E}$ and the cotangent bundle of $\widetilde{\mathbb{M}}$, stemming from Serre duality beween the groups Ext^1, which control deformations, and the groups Ext^2, which control obstructions. The kinship is only approximate, unless the canonical class $\mathscr{K}_X$ is *equivariantly* trivial, which is not the case even for $X = \mathbb{C}^3$ and leads to the twist by $\boldsymbol{\kappa}$ above.

3.2 $\mathscr{O}^{\mathrm{vir}}$ and $\widehat{\mathscr{O}}^{\mathrm{vir}}$

3.2.1

The description (30) means that $\mathscr{O}_{\mathbb{M}}$ is the 0th cohomology of the Koszul complex

$$0 \to \Lambda^{\operatorname{rk}\mathscr{E}}\mathscr{E}^\vee \xrightarrow{d} \cdots \to \Lambda^2\mathscr{E}^\vee \xrightarrow{d} \mathscr{E}^\vee \xrightarrow{d} \mathscr{O}_{\widetilde{\mathbb{M}}} \to 0 \tag{31}$$

in which $\mathscr{O}_{\widetilde{\mathbb{M}}}$ is placed in cohomological degree 0 and the differential is the contraction with the section s of $\mathscr{E}$.

The Koszul complex is an example of a sheaf of *differential graded algebras*, which by definition is a sheaf $\mathscr{A}^\bullet$ of graded algebras with the differential

$$\dots \xrightarrow{d} \mathscr{A}^{-2} \xrightarrow{d} \mathscr{A}^{-1} \xrightarrow{d} \mathscr{A}^0 \to 0 \tag{32}$$

satisfying the Leibnitz rule

$$d(a\cdot b) = da\cdot b + (-1)^{\deg a} a\cdot db\,.$$

The notion of a DGA has become one of the cornerstone notions in deformation theory, see for example how it used in the papers [1, 6, 7, 8] for a very incomplete set of references.

In particular, the structure sheaves $\mathscr{O}_{\mathbb{M}}$ of great many moduli spaces are described as $H^0(\mathscr{A}^\bullet)$ of a certain natural DGAs.

3.2.2

Central to K-theoretic enumerative geometry is the concept of the *virtual structure sheaf* denoted $\mathscr{O}_{\mathbb{M}}^{\mathrm{vir}}$. While $\mathscr{O}_{\mathbb{M}}$ is the 0th cohomology of a complex (32), the virtual structure sheaf is its Euler characteristic

$$\begin{aligned} \mathscr{O}_{\mathbb{M}}^{\mathrm{vir}} &= \sum_i (-1)^i \mathscr{A}^i \\ &= \sum_i (-1)^i H^i(\mathscr{A}^\bullet)\,, \end{aligned} \tag{33}$$

see [?, 8]. By Leibnitz rule, each $H^i(\mathscr{A}^\bullet)$ is acted upon by $\mathscr{A}^0$ and annihilated by $d\mathscr{A}^{-1}$, hence defines a quasicoherent sheaf on

$$\mathbb{M} = \operatorname{Spec}\mathscr{A}^0\big/d\mathscr{A}^{-1}\,.$$

If cohomology groups are coherent $\mathscr{A}^0$-modules and vanish for $i \ll 0$ then the second line in (33) gives a well-defined element of $K(\mathbb{M})$, or of $K_G(\mathbb{M})$ if all constructions are equivariant with respect to a group G.

The definition of $\mathscr{O}_{\mathbb{M}}^{\mathrm{vir}}$ is, in several respects, simpler than the definition [2] of the virtual fundamental cycle in cohomology. The agreement between the two is explained in Section 3 of [8].

3.2.3

There are many reasons to prefer $\mathscr{O}_{\mathbb{M}}^{\mathrm{vir}}$ over $\mathscr{O}_{\mathbb{M}}$, and one of them is the invariance of virtual counts under deformations.

For instance, in a family X_t of threefolds, special fibers may have many more curves than a generic fiber, and even the dimensions of the Hilbert scheme of curves in X_t may be jumping wildly. This is reflected by the fact that in a family of complexes $\mathscr{A}_t^\bullet$ each individual cohomology group is only semicontinuous and can jump up for special values of t. However, in a flat family of complexes the (equivariant) Euler characteristic is constant, and equivariant virtual counts are invariants of equivariant deformations.

3.2.4

Also, not the actual but rather the virtual counts are usually of interest in mathematical physics.

A supersymmetric physical theory starts out as a Hilbert space and an operator of the form (1), where at the beginning the Hilbert space $\mathscr{H}$ is something enormous, as it describes the fluctuations of many fields extended over many spatial dimensions. However, all those infinitely many degrees of freedom that correspond to nonzero eigenvalues of the operator $\not{D}^2$ pair off and make no contribution to supertraces (2). What remains, in cases of interest to us, may be identified with a direct sum (over various topological data) of complexes of finite-dimensional C^∞ vector bundles over finite-dimensional Kähler manifolds. These complexes combine the features of

(a) a Koszul complex for a section s of a certain vector bundle,

(b) a Lie algebra, or BSRT cohomology complex when a certain symmetry needs to be quotiented out, and

(c) a Dolbeault cohomology, or more precisely a related Dirac cohomology complex, which turns the supertraces into *holomorphic* Euler characteristics of K-theory classes defined by (a) and (b).

If M is a Kähler manifold, then spinor bundles of M are the bundles

$$\mathscr{S}_\pm = \mathscr{K}_M^{1/2} \otimes \bigoplus_{n \text{ even/odd}} \Omega_M^{0,n}$$

with the Dirac operator $\not{D} = \bar{\partial} + \bar{\partial}^*$. Here $\mathscr{K}_M$ is the canonical line bundle of M, which needs to be a square in order for M to be spin.

In item (c) above, the difference between Dolbeault and Dirac cohomology is precisely the extra factor of $\mathscr{K}_M^{1/2}$. While this detail may look insignificant compared to many other layers of complexity in the problem, it will prove to be of fundamental importance in what follows and will make many computations work. The basic reason for this was already discussed in Section **??**, and will be revisited shortly: the twist by $\mathscr{K}_M^{1/2}$ makes formulas more self-dual and, thereby, more rigid than otherwise.

3.2.5

Back in the Hilbert scheme context, the distinction between Dolbeault and Dirac is the distinction between $\mathscr{O}^{\text{vir}}_{\mathbb{M}}$ and

$$\widehat{\mathscr{O}}^{\text{vir}}_{\mathbb{M}} = (-1)^n \, \mathscr{K}^{1/2}_{\text{vir}} \otimes \mathscr{O}^{\text{vir}}_{\mathbb{M}}$$

where the sign will be explained below and the virtual canonical bundle $\mathscr{K}^{1/2}_{\text{vir}}$ is constructed as the dual of the determinant of the virtual tangent bundle

$$T^{\text{vir}}_{\mathbb{M}} = \text{Def} - \text{Obs} \tag{34}$$

$$= \left(T_{\widetilde{\mathbb{M}}} - \boldsymbol{\kappa} \otimes T^{*}_{\widetilde{\mathbb{M}}} \right) \Big|_{\mathbb{M}} . \tag{35}$$

From (35) we conclude

$$\mathscr{K}^{1/2}_{\text{vir}} = \boldsymbol{\kappa}^{\frac{\dim}{2}} \otimes \mathscr{K}_{\widetilde{\mathbb{M}}} \Big|_{\mathbb{M}}$$

where $\dim = \dim \widetilde{\mathbb{M}}$. This illustrates a general result of [19] that for DT moduli spaces the virtual canonical line bundle $\mathscr{K}_{\text{vir}}$ is a square in the equivariant Picard group up to a character of $\text{Aut}(X)$ and certain additional twists which will be discussed presently.

From the canonical isomorphism

$$\Lambda^k T_{\widetilde{\mathbb{M}}} \otimes \mathscr{K}_{\widetilde{\mathbb{M}}} \;\cong\; \Omega^{\dim - k}_{\widetilde{\mathbb{M}}}$$

and the congruence

$$\dim \equiv n \mod 2$$

we conclude that

$$\widehat{\mathscr{O}}^{\text{vir}}_{\mathbb{M}} = \boldsymbol{\kappa}^{-\frac{\dim}{2}} \sum_i (-\boldsymbol{\kappa})^i \, \Omega^i_{\widetilde{\mathbb{M}}} \, . \tag{36}$$

We call $\widehat{\mathscr{O}}^{\text{vir}}_{\mathbb{M}}$ the *modified* or the *symmetrized* virtual structure sheaf. The hat which this K-class is wearing should remind the reader of the $\widehat{A}$-genus and hence of the Dirac operator.

3.2.6

Formula (36) merits a few remarks. As discussed before, Serre duality between Ext^1 and Ext^2 groups on threefolds yields a certain kinship between the deformations and the *dual* of the obstructions. This makes the terms of the complex (31) and hence of the virtual structure sheaf $\mathscr{O}^{\text{vir}}_{\mathbb{M}}$ look like polyvector fields $\Lambda^i T_{\widetilde{\mathbb{M}}}$ on the ambient space $\widetilde{\mathbb{M}}$.

The twist by $\mathscr{K}_{\widetilde{\mathbb{M}}} \approx \mathscr{K}^{1/2}_{\text{vir}}$ turns polyvector fields into differential forms, and differential forms are everybody's favorite sheaves for cohomological computations. For example, if $\widetilde{\mathbb{M}}$ is a compact Kähler manifold then $H^q(\Omega^q_{\widetilde{\mathbb{M}}})$ is a piece of the Hodge decomposition of $H^{\bullet}(\widetilde{\mathbb{M}}, \mathbb{C})$, and in particular, rigid for any connected group of automorphisms. But even when these cohomology groups are not rigid, or when the terms of $\widehat{\mathscr{O}}^{\text{vir}}$ are not exactly differential forms, they still prove to be much more manageable than $\mathscr{O}^{\text{vir}}$.

3.2.7

The general definition of $\widehat{\mathscr{O}}^{\mathrm{vir}}$ from [19] involves, in addition to $\mathscr{O}^{\mathrm{vir}}$ and $\mathscr{K}_{\mathrm{vir}}$, a certain *tautological* class on the DT moduli spaces.

The vector space $\mathbb{C}^n$ in which the matrices (29) operate, descends to a rank n vector bundle over the quotient $\mathrm{Hilb}(\mathsf{Free}, n)$. Its fiber over $Z \in \mathrm{Hilb}(\mathbb{C}^3, n)$ is naturally identified with global sections of $\mathscr{O}_Z$, that is, the pushforward of the *universal sheaf* $\mathscr{O}_{\mathfrak{Z}}$ along X, where

$$\mathfrak{Z} \subset \mathrm{Hilb}(X, n) \times X$$

is the universal subscheme.

Analogous universal sheaves exists for Hilbert schemes of subschemes of any dimension. In particular, for the Hilbert scheme of curves, and other DT moduli spaces $\mathbb{M}$, there exists a universal 1-dimensional sheaf $\mathfrak{F}$ on $\mathbb{M} \times X$ such that

$$\mathfrak{F}\big|_{m \times X} = \mathscr{F}$$

where $m = (\mathscr{F}, \dots) \in \mathbb{M}$ is the moduli point representing a 1-dimensional sheaf $\mathscr{F}$ on X with possibly some extra data.

By construction, the sheaf $\mathfrak{F}$ is *flat* over $\mathbb{M}$, therefore *perfect*, that is, has a finite resolution by vector bundles on $\mathbb{M} \times X$. This is a nontrivial property, since $\mathbb{M}$ is highly singular and for singular schemes $K^\circ \subsetneq K$, where K° is the subgroup generated by vector bundles. From elements of K° one can form tensor polynomials, for example

$$P = S^2\mathfrak{F} \otimes \gamma_1 + \Lambda^3\mathfrak{F} \otimes \gamma_2 \in K^\circ(\mathbb{M} \times X)\,, \tag{37}$$

for any $\gamma_i \in K(X) = K^\circ(X)$. The support of any tensor polynomial like (37) is proper over $\mathbb{M}$ and therefore

$$\pi_{\mathbb{M},*} P \in K^\circ(\mathbb{M})\,,$$

where $\pi_{\mathbb{M}}$ is the projection along X. This is because a proper pushforward of a flat sheaf is perfect, see for example Theorem 8.3.8 in [11]. It makes sense to apply further tensor operations to $\pi_{\mathbb{M},*}P$, or to take it determinant.

In this way one can manufacture a large supply of classes in $K^\circ(\mathbb{M})$ which are, in their totality, called *tautological*. One should think of them as K-theoretic version of imposing various geometric conditions on curves, like meeting a cycle in X specified by $\gamma \in K(X)$.

3.2.8

An example of the tautological class is the determinant term in the following definition

$$\widehat{\mathscr{O}}^{\mathrm{vir}} = \text{prefactor } \mathscr{O}^{\mathrm{vir}} \otimes \left(\mathscr{K}_{\mathrm{vir}} \otimes \det \pi_{\mathbb{M},*}(\mathfrak{F} \otimes (\mathscr{L}_4 - \mathscr{L}_5))\right)^{1/2}\,, \tag{38}$$

where $\mathscr{L}_4 \in \operatorname{Pic}(X)$ is an arbitrary line bundle and $\mathscr{L}_5$ is a line bundle determined from the equation

$$\mathscr{L}_4 \otimes \mathscr{L}_5 = \mathscr{K}_X \,.$$

These are the same $\mathscr{L}_4$ and $\mathscr{L}_5$ as in Section 1.3.2. The prefactor in (38) contains the z-dependence

$$\text{prefactor} = (-1)^{(\mathscr{L}_4,\beta)+n} z^{n-(\mathscr{K}_X,\beta)/2} \,, \tag{39}$$

where

$$\beta = \deg \mathscr{F} \in H_2(X,\mathbb{Z}) \,, \quad n = \chi(\mathscr{F}) \in \mathbb{Z}$$

are locally constant functions of a 1-dimensional sheaf $\mathscr{F}$ on X.

The existence of square root in (38) is shown in [19]. With this definition, the general problem of computing the K-theoretic DT invariants may be phrased as

$$\chi\left(\mathbb{M}, \widehat{\mathscr{O}}^{\text{vir}} \otimes \text{tautological}\right) = \,? \,, \tag{40}$$

where $\mathbb{M}$ is one of the many possible DT moduli spaces. In a relative situation, one can put further insertions in (40).

3.3 Nekrasov's formula

3.3.1

Let X be a nonsingular quasiprojective threefold and consider its Hilbert scheme of points

$$\mathbb{M} = \bigsqcup_{n \geqslant 0} \operatorname{Hilb}(X,n) \,.$$

In the prefactor (39) we have $\beta = 0$ and therefore

$$\text{prefactor} = (-z)^n \,. \tag{41}$$

In [18], Nekrasov conjectured a formula for

$$\mathsf{Z}_{X,\text{points}} = \chi\left(\mathbb{M}, \widehat{\mathscr{O}}^{\text{vir}}\right) \,. \tag{42}$$

Because of the prefactor, this is well-defined as a formal power series in z, as long as $\chi\left(\operatorname{Hilb}(n), \widehat{\mathscr{O}}^{\text{vir}}\right)$ is well-defined for each n. For that we need to assume that either X is proper or, more generally, that there exists $g \in \operatorname{Aut}(X)_0$ such that it fixed point set X^g is proper (if X is already proper we can always take $g = 1$). Then

$$\mathsf{Z}_{X,\text{points}} \in \begin{cases} K_{\operatorname{Aut}(X)}(\text{pt})[[z]] \,, & g = 1 \\ K_{\operatorname{Aut}(X)}(\text{pt})_{\text{loc}}[[z]] \,, & \text{otherwise} \,. \end{cases}$$

To be precise, [18] considers the case $X = \mathbb{C}^3$, but the generalization to arbitrary X is immediate.

3.3.2

Nekrasov's formula computes $\mathsf{Z}_{X,\text{points}}$ in the form

$$\mathsf{Z}_{X,\text{points}} = \mathsf{S}^\bullet \chi(X, \star)\,, \quad \star \in K_{\mathsf{T}}(X)[[z]] \tag{43}$$

where $\mathsf{S}^\bullet$ is the symmetric algebra from Section 2.1.

Here, and this is very important, the boxcounting variable z is viewed as a part of the torus T, that is, it is also raised to the power n in the formula (7). This is very natural from the 5-dimensional perspective, since the z really acts on the 5-fold (3), with the fixed point set X, as discussed in Section 1.3.2.

Now we are ready to state the following result, conjectured by Nekrasov in [18]

Theorem 1. *We have*

$$\mathsf{Z}_{X,\text{points}} = \mathsf{S}^\bullet \chi\left(X, \frac{z\mathscr{L}_4\left(T_X + \mathscr{K}_X - T_X^\vee - \mathscr{K}_X^{-1}\right)}{(1 - z\mathscr{L}_4)(1 - z\mathscr{L}_5^{-1})}\right)\,. \tag{44}$$

3.3.3

In [18], this conjecture appeared together with an important physical interpretation, as the supertrace of $\operatorname{Aut}(Z, \Omega^5)$-action on the fields of *M-theory* on (3). M-theory is a supergravity theory in 10+1 real spacetime dimensions. Its fields are:

— the metric, also known as the graviton,

— its fermionic superpartner, gravitino, which is a field of spin 3/2, and

— one more bosonic field, a 3-form analogous to a connection, or a gauge boson, in gauge theories such as electromagnetism.

These are considered up to a gauge equivalence that includes: diffeomorphisms, changing the 3-form by an exact form, and changing gravitino by a derivative of a spinor.

In addition to these fields, M-theory has extended objects, namely:

— membranes, which have a 3-dimensional worldvolume and hence are naturally charged under the 3-form, and also

— magnetically dual M5-branes.

While membranes naturally appear in connection with DT invariants of positive degree [19], see for example Section 5.1.3 below, possible algebro-geometric interpretations of M5-branes are still very much in the initial exploration stage.

In Hamiltonian description, the field sector of the Hilbert space of M-theory is, formally, the L^2 space of functions on a very infinite-dimensional configuration supermanifold, namely

$$\mathscr{H} \overset{f}{=} L^2\left(\text{bosons} \oplus \frac{1}{2}\,\text{fermions on } Z \Big/ \text{gauge}\right)\,, \tag{45}$$

where

— Z is a fixed time slice of the 11-dimensional spacetime,

— $\frac{1}{2}$ of the fermions denotes a choice of polarization, that is, a splitting of fermionic operators into operators of creation and annihilation,

— the $\stackrel{f}{=}$ sign denotes an equality which is formal, ignores key analytic and dynamical questions, but may be suitable for equivariant K-theory which is often insensitive to finer details.

As a time slice, one is allowed to take a Calabi-Yau 5-fold, for example the one in (3). Automorphisms of Z preserving the 5-form Ω^5 are symmetries of the theory and hence act on its Hilbert space.

Since the configuration space is a linear representation of $\mathrm{Aut}(Z, \Omega^5)$, we have

$$\mathscr{H} \stackrel{f}{=} \mathsf{S}^\bullet \text{Configuration space}$$

in K-theory of $\mathrm{Aut}(Z, \Omega^5)$. The character of the latter may be computed, see [18] and also the exposition of the results of [18] in Section 2.4 of [19], with the result that

$$\text{Configuration space} = \Lambda^\bullet \chi(Z, T_Z)\,,$$

or its dual

$$\overline{\Lambda^\bullet \chi(Z, T_Z)} = \mathsf{S}^\bullet \chi(Z, T_Z^*)\,,$$

depending on the choice of the polarization in (45).

Note that

$$\chi(Z, T_Z^* - T_Z) = \chi\left(X, \mathscr{K}_X - \mathscr{O}_X + \frac{z\mathscr{L}_4\,(T_X + \mathscr{K}_X - T_X^* - \mathscr{K}_X^{-1})}{(1 - z\mathscr{L}_4)(1 - z\mathscr{L}_5^{-1})}\right)\,.$$

This implies

$$\mathsf{S}^\bullet \chi\,(X, \mathscr{K}_X - \mathscr{O}_X) \otimes \mathsf{Z}_{X,\text{points}} \stackrel{f}{=} \mathscr{H} \otimes \overline{\mathscr{H}}\,, \tag{46}$$

which is a formula with at least two issues, one minor and the other more interesting. The minor issue is the prefactor in the LHS, which is ill-defined as written. We will see below how this prefactor appears in DT computations and why in the natural regularization it is simply removed.

The interesting issue in (46) is the doubling the contribution of $\mathscr{H}$. As already pointed out by Nekrasov, it should be reexamined with a more careful analysis of the physical Hilbert space of M-theory.

Exercise 3.3. Write a proof of Proposition 2.1 in [19].

3.3.4

Below we will see how (44) reproduces an earlier result in cohomology proven in [15] for X toric varieties and in [13] for general 3-folds. The cohomological version of (42) is

$$\mathsf{Z}_{X,\text{points},\text{coh}} = \sum_{n \geqslant 0} (-z)^n \int_{[\mathrm{Hilb}(X,n)]_{\mathrm{vir}}} 1\,, \tag{47}$$

where $[\mathrm{Hilb}(X,n)]_{\mathrm{vir}}$ is the virtual fundamental cycle [2]. It may be defined as the cycle corresponding to the K-theory class $\mathscr{O}^{\mathrm{vir}}$ which, as [8] prove, has the expected dimension of support.

In general, for DT moduli spaces, the expected dimension is

$$\mathrm{vir}\dim = -(\deg \mathscr{F}, \mathscr{K}_X)\,,$$

where $\mathscr{F}$ is the 1-dimensional sheaf on X. For Hilbert schemes of points this vanishes and $[\mathrm{Hilb}(X,n)]_{\mathrm{vir}}$ is an equivariant 0-cycle. For $\mathrm{Hilb}(\mathbb{C}^3)$ this cycle is the Euler class of the obstruction bundle.

The following result was conjectured in [15] and proven there for toric 3-folds. The general algebraic cobordism approach of Levin and Pandharipande reduces the case of a general 3-fold to the special case of toric varieties.

Theorem 2 ([15, 13])**.** *We have*

$$\mathsf{Z}_{X,\mathrm{points},\mathrm{coh}} = M(z)^{\int_X c_3(T_X\otimes\mathscr{K}_X)}\,,$$

where

$$M(z) = \mathsf{S}^{\bullet}\frac{z}{(1-z)^2} = \prod_{n>0}(1-z^n)^{-n}\,,$$

is McMahon's generating function for 3-dimensional partitions.

The appearance of 3-dimensional partitions here is very natural — they index the torus fixed points in $\mathrm{Hilb}(\mathbb{C}^3,n)$. These appear naturally in equivariant virtual localization, which is the subject to which we turn next.

3.4 Tangent bundle and localization

3.4.1

Consider the action of the maximal torus

$$T = \left\{\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_d\end{pmatrix}\right\} \subset GL(d)$$

on the $\mathrm{Hilb}(\mathbb{C}^d,\mathrm{points})$, that is, on ideals of finite codimension in the ring $\mathbb{C}[x_1,\dots,x_d]$.

Exercise 3.4. Prove that the fixed points set $\mathrm{Hilb}(\mathbb{C}^d,\mathrm{points})^T$ is 0-dimensional and formed by *monomial ideals*, that is, ideals generated by monomials in the variables x_i. In particular, the points of $\mathrm{Hilb}(\mathbb{C}^d,n)^T$ are in natural bijection with d-dimensional partitions π on the number n.

3.4.2

For $d = 1, 2$, Hilbert schemes are smooth and our next goal is to compute the character of the torus action on $T_\pi \operatorname{Hilb}$. This will be later generalized to the computation of the torus character of the *virtual* tangent space for $d = 3$.

By construction or by the functorial definition of the Hilbert scheme, its Zariski tangent space at any subscheme $Z \subset X$ is given by

$$T_Z \operatorname{Hilb} = \operatorname{Hom}(\mathscr{I}_Z, \mathscr{O}_Z)$$

where $\mathscr{I}_Z$ is the sheaf of ideals of Z and $\mathscr{O}_Z = \mathscr{O}_X/\mathscr{I}_Z$ is its structure sheaf. Indeed, the functorial description of Hilbert scheme says that

$$\operatorname{Maps}(B, \operatorname{Hilb}(X)) = \left\{ \begin{matrix} \text{subschemes of } B \times X \\ \text{flat and proper over } B \end{matrix} \right\}$$

for any scheme B. In particular, a map from

$$B = \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$$

is a point of $\operatorname{Hilb}(X)$ together with a tangent vector, and this leads to the formula above.

Exercise 3.5. Check this.

Exercise 3.6. Let X be a smooth curve and $Z \subset X$ a 0-dimensional subscheme. Prove that

$$T_Z \operatorname{Hilb} = H^0(T^* X \otimes \mathscr{O}_Z)^* \,.$$

In particular, for $X = \mathbb{C}^1$ and the torus-fixed ideal $I = (x_1^n)$ we have

$$T_{(x_1^n)} = t_1 + \cdots + t_1^n \,,$$

in agreement with global coordinates on $\operatorname{Hilb}(\mathbb{C}^1, n) \cong \mathbb{C}^n$ given by

$$I = (f), \quad f = x^n + a_1 x^{n-1} + \cdots + a_n \,.$$

3.4.3

Now let X be a nonsingular surface and $Z \subset X$ a 0-dimensional subscheme. By construction, we have a short exact sequence

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0$$

which we can apply in the first argument to get the following long exact sequence of Ext-groups

$$\begin{array}{l} 0 \longrightarrow \operatorname{Hom}(\mathscr{O}_Z, \mathscr{O}_Z) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{O}_X, \mathscr{O}_Z) \xrightarrow{0} \operatorname{Hom}(\mathscr{I}_Z, \mathscr{O}_Z) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) \longrightarrow \cancel{\operatorname{Ext}^1(\mathscr{O}_X, \mathscr{O}_Z)} \longrightarrow \operatorname{Ext}^1(\mathscr{I}_Z, \mathscr{O}_Z) \longrightarrow \\ \longrightarrow \operatorname{Ext}^2(\mathscr{O}_Z, \mathscr{O}_Z) \longrightarrow \cancel{\operatorname{Ext}^2(\mathscr{O}_X, \mathscr{O}_Z)} \longrightarrow \cancel{\operatorname{Ext}^2(\mathscr{I}_Z, \mathscr{O}_Z)} \longrightarrow 0 \,, \end{array} \tag{48}$$

all vanishing and isomorphisms in which follow from

$$\operatorname{Ext}^i(\mathscr{O}_X, \mathscr{O}_Z) = H^i(\mathscr{O}_Z) = \begin{cases} H^0(\mathscr{O}_Z) = \operatorname{Hom}(\mathscr{O}_Z, \mathscr{O}_Z)\,, & i = 0\,, \\ 0\,, & i > 0\,, \end{cases}$$

because Z is affine.

Since the support of Z is proper, we can use Serre duality, which gives

$$\begin{aligned} \operatorname{Ext}^2(\mathscr{O}_Z, \mathscr{O}_Z) &= \operatorname{Hom}(\mathscr{O}_Z, \mathscr{O}_Z \otimes \mathscr{K}_X)^* = \\ &= H^0(\mathscr{O}_Z \otimes \mathscr{K}_X)^* = \chi(\mathscr{O}_Z \otimes \mathscr{K}_X)^* = \chi(\mathscr{O}_Z, \mathscr{O}_X)\,, \end{aligned} \tag{49}$$

where

$$\chi(\mathscr{A}, \mathscr{B}) = \sum (-1)^i \operatorname{Ext}^i(\mathscr{A}, \mathscr{B})\,. \tag{50}$$

This is an sesquilinear pairing on the equivariant K-theory of X satisfying

$$\chi(\mathscr{A}, \mathscr{B})^* = (-1)^{\dim X} \chi(\mathscr{B}, \mathscr{A} \otimes \mathscr{K}_X)\,.$$

Putting (48) and (49) together, we obtain the following

Proposition 3.1. *If X is a nonsingular surface then*

$$\begin{aligned} T_Z \operatorname{Hilb}(X, \text{points}) &= \chi(\mathscr{O}_Z) + \chi(\mathscr{O}_Z, \mathscr{O}_X) - \chi(\mathscr{O}_Z, \mathscr{O}_Z) \\ &= \chi(\mathscr{O}_X) - \chi(\mathscr{I}_Z, \mathscr{I}_Z)\,. \end{aligned} \tag{51}$$

3.4.4

Proposition 3.1 lets us easily compute the characters of the tangent spaces to the Hilbert scheme at monomial ideals. To any $\mathscr{F} \in K_T(\mathbb{C}^2)$ we can naturally associate two T-modules: $\chi(\mathscr{F})$ and the K-theoretic stalk of $\mathscr{F}$ at $0 \in \mathbb{C}^2$, which we can write as $\chi(\mathscr{F} \otimes \mathscr{O}_0)$. They are related by

$$\chi(\mathscr{F}) = \chi(\mathscr{F} \otimes \mathscr{O}_0)\, \chi(\mathscr{O}_{\mathbb{C}^2}) \tag{52}$$

where, keeping in mind that linear functions on $\mathbb{C}^2$ form a $GL(2)$-module *dual* to $\mathbb{C}^2$ itself,

$$\chi(\mathscr{O}_{\mathbb{C}^2}) = \mathsf{S}^\bullet \left(\mathbb{C}^2\right)^* = \frac{1}{(1 - t_1^{-1})(1 - t_2^{-1})}\,.$$

The formula

$$\chi(\mathscr{F}, \mathscr{G}) = \chi(\mathscr{F} \otimes \mathscr{O}_0)^* \chi(\mathscr{G} \otimes \mathscr{O}_0)\, \chi(\mathscr{O}_{\mathbb{C}^2}) = \frac{\chi(\mathscr{F})^* \chi(\mathscr{G})}{\chi(\mathscr{O})^*} \tag{53}$$

generalizes (52).

Let I_λ be a monomial ideal and let λ be the corresponding partition with diagram

$$\operatorname{diagram}(\lambda) = \left\{ (i,j) \,\middle|\, x_1^i x_2^j \notin I_\lambda \right\} \subset \mathbb{Z}^2_{\geqslant 0}\,.$$

More traditionally, the boxes or the dots in the diagram of λ are indexed by pairs $(i, j) \in \mathbb{Z}^2_{>0}$, but this, certainly, a minor detail. In what follows, we don't make a distinction between a partition and its diagram.

Let $\mathscr{V} = \chi(\mathscr{O}_Z)$ denote the tautological rank n vector bundle over $\mathrm{Hilb}(\mathbb{C}^2, n)$ as in Section 3.2.7, and let

$$\begin{aligned} V_\lambda &= \chi(\mathscr{O}_{Z_\lambda}) \\ &= \sum_{(i,j)\in\lambda} t_1^{-i} t_2^{-j} \end{aligned} \tag{54}$$

be the character of its stalk at I_λ. Clearly, this is nothing but the generating function for the diagram λ. From (51) and (53) we deduce the following

Proposition 3.2. *Let $I_\lambda \subset \mathbb{C}[x_1, x_2]$ be a monomial ideal and let V denote the generating function* (54) *for the diagram of λ. We have*

$$T_{I_\lambda} \mathrm{Hilb} = V + \overline{V}\, t_1 t_2 - V\overline{V}\,(1-t_1)(1-t_2)\,. \tag{55}$$

as a T-module, where $\overline{V} = V^*$ denotes the dual.

For example, take

$$\lambda = \square\,, \quad I_\lambda = \mathsf{m} = (x_1, x_2)\,, \quad V = 1$$

then

$$T_\square \mathrm{Hilb} = 1 + t_1 t_2 - (1-t_1)(1-t_2) = t_1 + t_2 = \mathbb{C}^2$$

in agreement with $\mathrm{Hilb}(X, 1) \cong X$.

3.4.5

Formula (55) may be given the following combinatorial polish. For a square $\square = (i, j)$ in the diagram of λ define its arm-length and leg-length by

$$\begin{aligned} a(\square) &= \#\{j' > j \,|\, (i, j') \in \lambda\}\,, \\ l(\square) &= \#\{i' > i \,|\, (i', j) \in \lambda\}\,. \end{aligned} \tag{56}$$

Exercise 3.7. Prove that

$$T_{I_\lambda} \mathrm{Hilb} = \sum_{\square\in\lambda} t_1^{-l(\square)} t_2^{a(\square)+1} + t_1^{l(\square)+1} t_2^{-a(\square)}\,. \tag{57}$$

Exercise 3.8. Prove a generalization of (55) and (57) for the character of $\chi(\mathscr{O}_X) - \chi(I_\lambda, I_\mu)$. If in need of a hint, open [5].

Exercise 3.9. Using the formulas for T_{I_λ} Hilb and equivariant localization, write a code for the computation of the series

$$\mathsf{Z}_{\mathrm{Hilb}(\mathbb{C}^2)} = \sum_{n,i\geqslant 0} z^n (-m)^i\, \chi(\mathrm{Hilb}(\mathbb{C}^2, n), \Omega^i)$$

and check experimentally that is $\mathsf{S}^\bullet$ of a nice rational function of the variables z, m, t_1, t_2. We will compute this function theoretically in Section 5.3.4.

3.4.6

We now advance to the discussion of the case when X is a nonsingular threefold and $Z \subset X$ is a 1-dimensional subscheme and $\mathscr{I}_Z$ is its sheaf of ideals. The sheaf $\mathscr{I}_Z$ is clearly torsion-free, as a subsheaf of $\mathscr{O}_X$, and

$$\det \mathscr{I}_Z = \mathscr{O}_X$$

because the two sheaves differ in codimension $\geqslant 2$.

Donaldson-Thomas theory views $\mathrm{Hilb}(X, \text{curves})$ as the moduli space of torsion-free sheaves rank 1 sheaves $\mathscr{I}$ with trivial determinant. For any such sheaf we have

$$\mathscr{I} \hookrightarrow \mathscr{I}^{\vee\vee} = \det \mathscr{I} = \mathscr{O}_X$$

and so $\mathscr{I}$ is the ideal sheaf of a subscheme $Z \subset X$. We have

$$0 = c_1(\det \mathscr{I}) = c_1(\mathscr{I}) = [Z] \in H_4(X, \mathbb{Z})$$

and therefore $\dim Z = 1$. The deformation theory of sheaves gives

$$\begin{aligned} T^{\mathrm{vir}}_{\mathscr{I}} \mathrm{Hilb} = \mathrm{Def}(\mathscr{I}) - \mathrm{Obs}(\mathscr{I}) &= \chi(\mathscr{O}_X) - \chi(\mathscr{I}, \mathscr{I})\,, \\ &= \chi(\mathscr{O}_Z) + \chi(\mathscr{O}_Z, \mathscr{O}_X) - \chi(\mathscr{O}_Z, \mathscr{O}_Z) \end{aligned} \tag{58}$$

just like in Proposition 3.1.

The group $\mathrm{Ext}^1(\mathscr{I}, \mathscr{I})$ which enters (58) parametrizes sheaves on $B \times X$ flat over B for $B = \mathbb{C}[\varepsilon]/\varepsilon^2$, just like in Exercise 3.5. It describes the deformations of the sheaf $\mathscr{I}$. The obstructions to these deformations lie in $\mathrm{Ext}^2(\mathscr{I}, \mathscr{I})$.

We now examine how this works for the Hilbert scheme of points in $\mathbb{C}^3$.

3.4.7

Let π be a 3-dimensional partition and let $I_\pi \subset \mathbb{C}[x_1, x_2, x_3]$ be the corresponding monomial ideal. The passage from (51) to (55) is exactly the same as before, with the correction for

$$\chi(\mathscr{O}_{\mathbb{C}^3}) = \frac{1}{(1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})}\,.$$

We obtain the following

Proposition 3.3. *Let $I_\pi \subset \mathbb{C}[x_1, x_2, x_3]$ be a monomial ideal and let V denote the generating function for π, that is, the character of $\mathscr{O}_{Z_\pi}$. We have*

$$T^{\mathrm{vir}}_{I_\pi} \mathrm{Hilb} = V - \overline{V}\, t_1 t_2 t_3 - V\overline{V} \prod_1^3 (1 - t_i)\,. \tag{59}$$

as a T-module.

As an example, take

$$\pi = \square\,, \quad I_\pi = \mathsf{m} = (x_1, x_2, x_2)\,, \quad V = 1$$

then

$$T^{\mathrm{vir}}_{\square}\,\mathrm{Hilb} = t_1 + t_2 + t_3 - t_1 t_2 - t_1 t_3 - t_2 t_3 = \mathbb{C}^3 - \det \mathbb{C}^3 \otimes \left(\mathbb{C}^3\right)^*$$

in agreement with the identification

$$\mathbb{M}(1) = \mathrm{Hilb}(X, 1) \cong X \cong \mathrm{Hilb}(\mathsf{Free}_3, 1) = \widetilde{\mathbb{M}}(1)$$

for the Hilbert schemes of 1 point in $X = \mathbb{C}^3$ and the description of the obstruction bundle as

$$\mathrm{Obs} = \boldsymbol{\kappa} \otimes T^*_{\widetilde{\mathbb{M}}} = \boldsymbol{\kappa} \otimes \left(\mathbb{C}^3\right)^*$$

where $\boldsymbol{\kappa} = \det \mathbb{C}^3$.

In general, for $\widetilde{\mathbb{M}} = \mathrm{Hilb}(\mathsf{Free}_3, n)$ we have

$$T_{\widetilde{\mathbb{M}}} = (\mathbb{C}^3 - 1) \otimes \overline{V} \otimes V + V$$

by the construction of $\widetilde{\mathbb{M}}$ as the space of 3 operators and a vector in V modulo the action of $GL(V)$. Therefore

$$\begin{aligned} T^{\mathrm{vir}} &= T_{\widetilde{\mathbb{M}}} - \boldsymbol{\kappa}^{-1} \otimes T^*_{\widetilde{\mathbb{M}}} \\ &= V - \det \mathbb{C}^3 \otimes \overline{V} - \overline{V} \otimes V \otimes \sum (-1)^i \Lambda^i \mathbb{C}^3\,, \end{aligned} \tag{60}$$

which gives a different and more direct proof of (59).

Exercise 3.10. ... or, rather, a combinatorial challenge. Is there a combinatorial formula for the character of $T_{\widetilde{\mathbb{M}}}$ at torus-fixed points and can one find some systematic cancellations in the first line of (60) ?

3.4.8

Let

$$\tilde{\iota} : \mathbb{M} \hookrightarrow \widetilde{\mathbb{M}}$$

be the inclusion of $\mathbb{M} = \mathrm{Hilb}(\mathbb{C}^3, \mathrm{points})$ into the Hilbert scheme of a free algebra. By our earlier discussion,

$$\tilde{\iota}_* \mathscr{O}^{\mathrm{vir}} = \Lambda^{\bullet} \mathrm{Obs}^*$$

and therefore we can use equivariant localization on the smooth ambient space $\widetilde{\mathbb{M}}$ to compute $\chi(\mathbb{M}, \mathscr{O}^{\mathrm{vir}})$.

Let π be 3-dimensional partition and $I_\pi \in \mathrm{Hilb} \subset \widetilde{\mathbb{M}}$ the corresponding fixed point. Since

$$T^{\mathrm{vir}}_\pi = T_\pi \widetilde{\mathbb{M}} - \boldsymbol{\kappa} \otimes \left(T_\pi \widetilde{\mathbb{M}}\right)^*\,,$$

we have

$$T_\pi^{\mathrm{vir}} = \sum_w \left(w - \frac{\boldsymbol{\kappa}}{w} \right)$$

where the sum is over the weights w of $T_\pi \widetilde{\mathbb{M}}$. Therefore

$$\mathscr{O}^{\mathrm{vir}}_{\mathrm{Hilb}(\mathbb{C}^3,n)} = \sum_{|\pi|=n} \mathscr{O}_{I_\pi} \prod \frac{1 - w/\boldsymbol{\kappa}}{1 - w^{-1}} \tag{61}$$

in localized equivariant K-theory. This formula illustrates a very important notion of *virtual localization*, see in particular [**?**, **?**, 8], which we now discuss.

3.4.9

Let a torus T act on a scheme $\mathbb{M}$ with a T-equivariant perfect obstruction theory. For example, $\mathbb{M}$ be DT moduli space for a nonsingular threefold X on which a torus T act. Let $\mathbb{M}^T \subset \mathbb{M}$ be the subscheme of fixed points. We can decompose

$$\begin{aligned}(\mathrm{Def} - \mathrm{Obs})\,|_{\mathbb{M}^T} &= \mathrm{Def}^{\mathrm{fixed}} - \mathrm{Obs}^{\mathrm{fixed}} \\ &\quad + \mathrm{Def}^{\mathrm{moving}} - \mathrm{Obs}^{\mathrm{moving}}\end{aligned} \tag{62}$$

in which the fixed part is formed by trivial T-modules and the moving part by nontrivial ones.

Exercise 3.11. Check that for $\mathrm{Hilb}(\mathbb{C}^3, \text{points})$ and the maximal torus $T \subset GL(3)$ the fixed part of the obstruction theory vanishes.

By a result of [**?**], the fixed part of the obstruction theory is perfect obstruction theory for $\mathbb{M}^T$ and defines $\mathscr{O}^{\mathrm{vir}}_{\mathbb{M}^T}$. The virtual localization theorem of [8], see also [**?**] for a cohomological version, states that

$$\mathscr{O}^{\mathrm{vir}}_{\mathbb{M}} = \iota_* \left(\mathscr{O}^{\mathrm{vir}}_{\mathbb{M}^T} \otimes \mathsf{S}^\bullet \left(\mathrm{Def}^{\mathrm{moving}} - \mathrm{Obs}^{\mathrm{moving}} \right)^* \right) \tag{63}$$

where

$$\iota : \mathbb{M}^T \hookrightarrow \mathbb{M}$$

is the inclusion. Since, by construction, the moving part of the obstruction theory contains only nonzero T-weights, its symmetric algebra $\mathsf{S}^\bullet$ is well defined.

Exercise 3.12. Let $f(\mathscr{V})$ be a Schur functor of the tautological bundle $\mathscr{V}$ over $\mathrm{Hilb}(\mathbb{C}^3, n)$, for example

$$f(\mathscr{V}) = S^2\mathscr{V}, \Lambda^3\mathscr{V}, \dots .$$

Write a localization formula for $\chi(\mathrm{Hilb}(\mathbb{C}^3, n), \mathscr{O}^{\mathrm{vir}} \otimes f(\mathscr{V}))$.

3.4.10

It remains to twist (61) by

$$\mathscr{K}_{\mathrm{vir}}^{1/2} = \det{}^{-1/2} T^{\mathrm{vir}}$$

to deduce a virtual localization formula for $\widehat{\mathscr{O}}^{\mathrm{vir}}$. It is convenient to define the transformation $\widehat{\mathsf{a}}(\dots)$, a version of the $\widehat{A}$-genus, by

$$\widehat{\mathsf{a}}(x+y) = \widehat{\mathsf{a}}(x)\,\widehat{\mathsf{a}}(y)\,, \quad \widehat{\mathsf{a}}(w) = \frac{1}{w^{1/2} - w^{-1/2}}$$

where w is a monomial, that is, a weight of T. For example

$$\widehat{\mathsf{a}}\left(T_\pi^{\mathrm{vir}}\right) = \prod_w \frac{(\boldsymbol{\kappa}/w)^{1/2} - (w/\boldsymbol{\kappa})^{1/2}}{w^{1/2} - w^{-1/2}}\,, \tag{64}$$

where the product is over the same weights w as in (61).

With this notation, we can state the following

Proposition 3.4. *We have*

$$\widehat{\mathscr{O}}^{\mathrm{vir}}_{\mathrm{Hilb}(\mathbb{C}^3,n)} = (-1)^n \sum_{|\pi|=n} \widehat{\mathsf{a}}\left(T_\pi^{\mathrm{vir}}\right) \mathscr{O}_{I_\pi} \tag{65}$$

in localized equivariant K-theory.

Exercise 3.13. Write a code to check a few first terms in z of Nekrasov's formula.

Exercise 3.14. Take the limit $t_1, t_2, t_3 \to 1$ in Nekrasov's formula for $\mathbb{C}^3$ and show that it correctly reproduces the formula for $\mathsf{Z}_{\mathbb{C}^3,\mathrm{points,coh}}$ from Theorem 2.

`Prefactor in` (46)

3.5 Proof of Nekrasov's formula

3.5.1

Our next goal is to prove Nekrasov's formula for $X = \mathbb{C}^3$. By localization, this immediately generalizes to nonsingular toric threefolds. Later, when we discuss relative invariants, we will see the generalization to the relative setting. The path from there to general threefolds is the same as in [13].

Our proof of Theorem (1) will have two parts. In the first step, we prove

$$\mathsf{Z}_{\mathbb{C}^3,\mathrm{points}} = \mathsf{S}^\bullet \frac{\star}{(1-t_1^{-1})(1-t_2^{-1})(1-t_2^{-1})} \tag{66}$$

where

$$\star \in \mathbb{Z}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, (t_1 t_2 t_3)^{1/2}\right][[z]]$$

is a formal power series in z, without a constant term, with coefficients in Laurent polynomials. In the second step, we identify the series $\star$ by a combinatorial argument involving equivariant localization.

The first step is geometric and, in fact, we prove that

$$\mathsf{Z}_{\mathbb{C}^3,\text{points}} = \mathsf{S}^\bullet \chi(\mathbb{C}^3, \mathscr{G}), \quad \mathscr{G} = \sum_{i=1}^{\infty} z^i \mathscr{G}_i, \quad \mathscr{G}_i \in K_{GL(3)}(\mathbb{C}^3), \tag{67}$$

where each $\mathscr{G}_i$ is constructed iteratively starting from

$$\mathscr{G}_1 = -\widehat{\mathscr{O}}^{\text{vir}}_{\text{Hilb}(\mathbb{C},1)} .$$

The same argument applies to many similar sheaves and moduli spaces, including, for example, the generating function

$$\sum_n z^n \chi(\text{Hilb}(\mathbb{C}^3, n), \mathscr{O}^{\text{vir}}) \tag{68}$$

in which we have z^n instead of $(-z)^n$ and plain $\mathscr{O}^{\text{vir}}$ instead of the symmetrized virtual structure sheaf. The second part of the proof, however, doesn't work for $\mathscr{O}^{\text{vir}}$ and I don't know a reasonable formula for the series (68).

It is natural to prove (67) in that greater generality and this will be done in Proposition 5.4 in Section 5.3.5 below. For now, we assume (67) and proceed to the second part of the proof, that is, to the identification of the polynomial $\star$ in (66).

3.5.2

For $\text{Hilb}(\mathbb{C}^3, n)$, the line bundles $\mathscr{L}_4$ and $\mathscr{L}_5$ in (38) are necessarily trivial with, perhaps, a nontrivial equivariant weight. Hence the determinant term in (38) is a trivial bundle with weight

$$\det{}^{1/2} \pi_{\mathbb{M},*}(\mathfrak{F} \otimes (\mathscr{L}_4 - \mathscr{L}_5)) = \left(\frac{\text{weight}(\mathscr{L}_4)}{\text{weight}(\mathscr{L}_5)} \right)^{n/2} , \tag{69}$$

which can be absorbed in the variable z. Therefore, without loss of generality, we can assume that

$$\mathscr{L}_4 = \mathscr{L}_5 = \boldsymbol{\kappa}^{-1/2} = \frac{1}{(t_1 t_2 t_3)^{1/2}} ,$$

where t_i's are the weights of the $GL(3)$ action on $\mathbb{C}^3$, in which case the term (69) is absent.

We define

$$t_4 = \frac{z}{\boldsymbol{\kappa}^{1/2}} , \quad t_5 = \frac{1}{z \, \boldsymbol{\kappa}^{1/2}} ,$$

so that $t_1, \dots, t_5$ may be interpreted as the weights of the action of $SL(5) \supset \mathbb{C}_z^\times$ on $Z \cong \mathbb{C}^5$. With this notation, what needs to be proven is

$$\begin{aligned} \mathsf{Z}_{\mathbb{C}^3,\text{points}} &= \mathsf{S}^\bullet \, \hat{\mathsf{a}} \left(\sum_{i=1}^{5} t_i - \sum_{i<j\leqslant 3} t_i t_j \right) \\ &= \mathsf{S}^\bullet \, \frac{\prod_{i<j\leqslant 3}((t_i t_j)^{1/2} - (t_i t_j)^{-1/2})}{\prod_{i\leqslant 5}(t_i^{1/2} - t_i^{-1/2})} \,. \end{aligned} \tag{70}$$

3.5.3

Exercise 3.15. Prove that the localization weight (64) of any nonempty 3-dimensional partition is divisible by $t_1 t_2 - 1$. In fact, the order of vanishing of this weight at $t_1 t_2 = 1$ is computed in Section 4.5 of [20].

By symmetry, the same is clearly true for all $t_i t_j - 1$ with $i < j \leqslant 3$. Note that plethystic substitutions $\{t_i\} \mapsto \{t_i^k\}$ preserve vanishing at $t_i t_j = 1$. Therefore, using (66), we may define

$$\star \in \mathbb{Z}[t_1^{\pm 1/2}, t_2^{\pm 1/2}, t_3^{\pm 1/2}][[z]]$$

so that

$$\mathsf{Z}_{\mathbb{C}^3,\text{points}} = \mathsf{S}^\bullet \left(\star \prod_{i=1}^{3} \frac{(\boldsymbol{\kappa}/t_i)^{1/2} - (t_i/\boldsymbol{\kappa})^{1/2}}{t_i^{1/2} - t_i^{-1/2}} \right) . \tag{71}$$

Proposition 3.5. *We have*

$$\star \in \mathbb{Z}[\boldsymbol{\kappa}^{\pm 1}][[z]] \,,$$

that is, this polynomial depends only on the product $t_1 t_2 t_3$, and not on the individual t_i's.

Proof. A fraction of the form

$$\frac{(\boldsymbol{\kappa}/w)^{1/2} - (w/\boldsymbol{\kappa})^{1/2}}{w^{1/2} - w^{-1/2}}$$

remains bounded and nonzero as $w^{\pm 1} \to \infty$ with $\boldsymbol{\kappa}$ fixed. Therefore, both the localization contributions (64) and the fraction in (71) remain bounded as $t_i^{\pm 1} \to \infty$ in such a way that $\boldsymbol{\kappa}$ remains fixed. We conclude that the Laurent polynomial $\star$ is bounded at all such infinities and this means it depends on $\boldsymbol{\kappa}$ only. □

This is our first real example of rigidity.

3.5.4

The proof of Nekrasov's formula for $\mathbb{C}^3$ will be complete if we show the following

Proposition 3.6.

$$\star = \frac{1}{(t_4^{1/2} - t_4^{-1/2})(t_5^{1/2} - t_5^{-1/2})} = -\frac{z}{(1-\boldsymbol{\kappa}^{1/2}\, z)(1-\boldsymbol{\kappa}^{-1/2}\, z)}$$

To compute $\star$, we may let the variables t_i go to infinity with $\boldsymbol{\kappa}$ fixed. Since

$$\frac{(\boldsymbol{\kappa}/w)^{1/2} - (w/\boldsymbol{\kappa})^{1/2}}{w^{1/2} - w^{-1/2}} \to \begin{cases} -\boldsymbol{\kappa}^{-1/2}\,, & w \to \infty\,, \\ -\boldsymbol{\kappa}^{1/2}\,, & w \to 0\,, \end{cases}$$

we conclude that

$$\hat{\mathsf{a}}\left(\sum w_i - \boldsymbol{\kappa} \sum w_i^{-1}\right) \to (-\boldsymbol{\kappa}^{1/2})^{\text{index}}$$

where

$$\text{index} = \#\left\{i \,\middle|\, w_i \to 0\right\} - \#\left\{i \,\middle|\, w_i \to \infty\right\}\,.$$

For the computation of $\star$ we are free to send t_i to infinity in any way we like, as long as their product stays fixed. As we will see, a particularly nice choice is

$$t_1, t_3 \to 0\,, \quad |t_1| \ll |t_3|\,, \quad \boldsymbol{\kappa} = \text{fixed}\,. \tag{72}$$

For the fraction in (71) we have

$$\prod_{i=1}^{3} \frac{(\boldsymbol{\kappa}/t_i)^{1/2} - (t_i/\boldsymbol{\kappa})^{1/2}}{t_i^{1/2} - t_i^{-1/2}} \to (-\boldsymbol{\kappa}^{1/2})^{\text{index}(\mathbb{C}^3)} = -\boldsymbol{\kappa}^{1/2}\,.$$

Thus Proposition 3.6 becomes a corollary of the following

Proposition 3.7. *Let the variables t_i go to infinity of the torus as in* (72). *Then*

$$\mathsf{Z}_{X,\text{points}} \to \mathsf{S}^{\bullet}\, \frac{\boldsymbol{\kappa}^{1/2}\, z}{(1-\boldsymbol{\kappa}^{1/2}\, z)(1-\boldsymbol{\kappa}^{-1/2}\, z)}\,. \tag{73}$$

This is a special case of the computations of *index vertices* from Section 7 of [19] and the special case of the limit (72) corresponds to the *refined vertex* of Iqbal, Kozcaz, and Vafa in [12]. We will now how this works in the example at hand.

3.5.5

Recall the formula

$$T_{I_\pi}\widetilde{\mathsf{M}} = (\mathbb{C}^3 - 1) \otimes \overline{V} \otimes V + V$$

for tangent space to $\widetilde{\mathsf{M}}$ at a point corresponding to a 3-dimensional partition π with the generating function V. For a partition of size n, this is a sum of

$2n^2+n$ terms and, in principle, we need to compute the index of this very large torus module to know the asymptotics of $\widehat{\mathsf{a}}(T^{\mathrm{vir}}_\pi)$ as $t \to \infty$.

A special feature of the limit (72) is that one can see a cancellation of $2n^2$ terms in the index, with the following result

Lemma 3.8. *In the limit* (72)*,*

$$\operatorname{index}\left(T_{I_\pi}\widetilde{\mathsf{M}}\right) = \operatorname{index}\left(t_3^k\, V\right)\,,$$

for any k is such that $k > |\pi|$ but $|t_1| \ll |t_3|^k$.

This Lemma is proven in the Appendix to [19]. Clearly

$$\operatorname{index}\left(t_3^k\, V\right) = \sum_{\square=(i_1,i_2,i_3)\in\pi} \operatorname{sgn}(i_2 - i_1 + 0)$$

and therefore Proposition becomes the

$$\begin{aligned} z\boldsymbol{\kappa}^{1/2} &= q_0 = q_1 = q_2 = \dots \\ z\boldsymbol{\kappa}^{-1/2} &= q_{-1} = q_{-2} = \dots \end{aligned} \tag{74}$$

case of the following generalization of McMahon's enumeration

Theorem 3 ([21, 22])**.**

$$\sum_\pi \prod_{\square=(i_1,i_2,i_3)\in\pi} q_{i_2-i_1} = \mathsf{S}^\bullet \sum_{a\leqslant 0\leqslant b} q_a q_{a+1}\cdots q_{b-1}q_b \tag{75}$$

With the specialization (74) the sum under $\mathsf{S}^\bullet$ in (75) become the series expansion of the fraction in (73).

3.5.6

We conclude with a sequence of exercises that will lead the reader through the proof of Theorem 3. It is based on introducing a *transfer matrix*, a very much tried-and-true tool in statistical mechanics and combinatorics.

A textbook example of transfer matrix arises in 2-dimensional Ising model on a torus. The configuration space in that model are assignments of ± 1 spins to sites x of a rectangular grid, that is, functions

$$\sigma : \{1,\dots,M\}\times\{1,\dots,N\} \to \{\pm 1\}$$

Each is weighted with

$$\mathrm{Prob}(\sigma) \propto W(\sigma) = \exp(-\beta E(\sigma))\,,$$

where β is the inverse temperature and the energy E is defined by

$$E(\sigma) = -\sum_{\text{nearest neighbors } x \text{ and } x'} \sigma(x)\sigma(x')\,,$$

with periodic boundary conditions.

Now introduce a vector space

$$V \cong (\mathbb{C}^2)^N$$

with a basis formed by all possible spin configurations $\vec{\sigma}$ in one column of our grid. Define a diagonal matrix W^{v} and a dense matrix W^{h} by

$$\mathsf{W}^{\mathrm{v}}_{\vec{\sigma},\vec{\sigma}} = \exp\left(-\beta \sum_{\substack{\text{vertical neighbors}\\ \text{in column } \vec{\sigma}}} \sigma(x)\,\sigma(x')\right) \tag{76}$$

$$\mathsf{W}^{\mathrm{h}}_{\vec{\sigma}_1,\vec{\sigma}_2} = \exp\left(-\beta \sum_{\substack{\text{horizontal neighbors}\\ \text{in columns } \vec{\sigma}_1 \text{ and } \vec{\sigma}_2}} \sigma_1(x)\,\sigma_2(x')\right) \tag{77}$$

Exercise 3.16. Prove that

$$\sum_{\{\sigma\}} W(\sigma) = \mathrm{tr}\,(\mathsf{W}^{\mathrm{v}}\mathsf{W}^{\mathrm{h}})^M$$

where the summation is over all 2^{MN} spin configurations $\{\sigma\}$.

The matrix $\mathsf{W}^{\mathrm{v}}\mathsf{W}^{\mathrm{h}}$ is an example of a transfer matrix. Onsager's great discovery was the diagonalization of this matrix. Essentially, he has shown that the transfer matrix is in the image of a certain matrix $g \in O(2N)$ in the spinor representation of this group.

3.5.7

Now in place of the spinor representation of $O(2N)$ we will have the action of the Heisenberg algebra $\widehat{\mathfrak{gl}}(1)$ on

$$\begin{aligned} V &= \text{Fock space} = \text{symmetric functions} = \\ &\qquad\qquad = \text{polynomial representations of } GL(\infty)\,. \end{aligned}$$

This space has an othonormal basis $\{s_\lambda\}$ of Schur functions, that is, the characters of the Schur functors $S^\lambda\mathbb{C}^\infty$ of the defining representation of $GL(\infty)$. It is indexed by 2-dimensional partitions λ which will arise as slices of 3-dimensional partitions π by planes $i_2 - i_1 = \text{const}$.

The diagonal operator W^{v} will be replaced by the operator

$$q^{|\cdot|}\, s_\lambda = q^{|\lambda|} s_\lambda\,.$$

In place of the nondiagonal operator W^{h}, we will use the operators

$$\Gamma_-(z) = \left(\sum_{k\geqslant 0} z^k\, \mathsf{S}^k\mathbb{C}^\infty\right) \otimes$$

and its transpose Γ_+. The character of $\mathsf{S}^k\mathbb{C}^\infty$ is the Schur function s_k. It is well known that

$$\Gamma_-(z)\, s_\lambda = \sum_\mu z^{|\mu|-|\lambda|}\, s_\mu$$

where the summation is over all partitions μ such that μ and λ interlace, which means that

$$\mu_1 \geqslant \lambda_1 \geqslant \mu_2 \geqslant \lambda_2 \geqslant \dots .$$

Exercise 3.17. Prove that

$$\sum_\pi \prod_{\square=(i_1,i_2,i_3)\in\pi} q_{i_2-i_1} = \\ = \Big(\cdots \Gamma_+(1)\, q_{-1}^{|\cdot|}\, \Gamma_+(1)\, q_0^{|\cdot|}\, \Gamma_-(1)\, q_1^{|\cdot|}\, \Gamma_-(1)\, q_2^{|\cdot|} \cdots s_\varnothing, s_\varnothing\Big) .$$

Exercise 3.18. Prove that $q^{|\cdot|}\,\Gamma_-(z) = \Gamma_-(qz)\, q^{|\cdot|}$ and that

$$\Gamma_+(z)\,\Gamma_-(w) = \frac{1}{1-zw}\Gamma_-(w)\,\Gamma_+(z)$$

if $|zw| < 1$. Deduce Theorem 3.

4 Nakajima varieties

4.1 Algebraic symplectic reduction

4.1.1

Symplectic reduction was invented in classical mechanics [] to deal with the following situation. Let M be the configuration space of a mechanical system and T^*M — the corresponding phase space. A function H, called Hamiltonian, generates dynamics by

$$\frac{d}{dt} f = \{H, f\}$$

where f is an arbitrary function on T^*M and $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket. If this dynamics commutes with a Hamiltonian action of a Lie group G, it descends to a certain reduced phase space $T^*M/\!\!/\!\!/G$. The reduced space could be a more complicated variety but of smaller dimension, namely

$$\dim T^*M/\!\!/\!\!/G = 2\dim M - 2\dim G\,.$$

4.1.2

In the algebraic context, let a reductive group G act on a smooth algebraic variety M. The induced action on T^*M, which is a algebraic symplectic variety, is Hamiltonian: the function

$$\mu_\xi(p, q) = \langle p, \xi \cdot q\rangle\,, \quad q \in M\,, p \in T_q^*M\,, \tag{78}$$

generates the vector field $\xi \in \mathrm{Lie}(G)$. This gives a map

$$\mu : T^*M \to \mathrm{Lie}(G)^* ,$$

known as the *moment* map, because in its mechanical origins G typically included translational or rotational symmetry.

4.1.3

We can form the algebraic symplectic reduction

$$X = T^*M /\!\!/\!\!/ G = \mu^{-1}(0) /\!\!/ G = \mu^{-1}(0)_{\mathrm{semistable}}/G , \tag{79}$$

where a certain choice of stability is understood.

4.1.4

The zero section $M \subset T^*M$ is automatically inside $\mu^{-1}(0)$ and

$$T^*(M_{\mathrm{free}}/G) \subset X$$

is an open, but possibly empty, subset. Here $M_{\mathrm{free}} \subset M$ is the set of semistable points with trivial stabilizer. Thus algebraic symplectic reduction is an improved version of the cotangent bundle to a G-quotient.

4.1.5

The Poisson bracket on T^*M induces a Poisson bracket on X, which is symplectic on the open set of points with trivial stabilizer. In general, however, there will be other, singular, points in X.

Finite stabilizers are particularly hard to avoid. Algebraic symplectic reduction is a source of great many Poisson orbifolds, but it very rarely outputs an algebraic symplectic variety.

4.2 Nakajima quiver varieties []

4.2.1

Nakajima varieties are a remarkable class of symplectic reductions for which finite stabilizers can be avoided. For them,

$$G = \prod GL(V_i)$$

and M is a linear representation of the form

$$M = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j) \otimes Q_{ij} \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i) . \tag{80}$$

Here Q_{ij} and W_i are multiplicity spaces, so that

$$\prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar \to \mathrm{Aut}(X) , \tag{81}$$

where the $\mathbb{C}^\times_\hbar$-factor scales the cotangent directions with weight $\hbar^{-1}$, and hence scales the symplectic form on X with weight $\hbar$.

4.2.2

In English, the only representations allowed in M are

— the defining representations V_i of the $GL(V_i)$-factors,

— the adjoint representations of the same factors,

— representations of the form $\mathrm{Hom}(V_i, V_j)$ with $i \neq j$.

Latter are customary called bifundamental representations in gauge theory context. Note that T^*M will also include the duals V_i^* of the defining representations.

What is special about these representation is that the stabilizer G_y of any point $y \in T^*M$ is the set of invertible elements in a certain associative algebra

$$E_y \subset \bigoplus \mathrm{End}(V_i)$$

over the base field $\mathbb{C}$, and hence cannot be a nontrivial finite group.

4.2.3

The data of the representation (83) is conveniently encoded by a graph, also called a quiver, in which we join the ith vertex with the jth vertex by $\dim Q_{ij}$ arrows. After passing to T^*M, the orientation of these arrows doesn't matter because

$$\mathrm{Hom}(V_i, V_j)^* = \mathrm{Hom}(V_j, V_i)\,.$$

Therefore, it is convenient to assume that only one of the spaces Q_{ij} and Q_{ji} is nonzero for $i \neq j$.

To the vertices of the quiver, one associates two dimension vectors

$$\mathsf{v} = (\dim V_i)\,,\ \mathsf{w} = (\dim W_i) \in \mathbb{Z}^I_{\geqslant 0}\,,$$

where $I = \{i\}$ is the set of vertices.

4.2.4

The quotient in (78) is a GIT quotient and choice of stability condition is a choice of a character of G, that is, a choice of vector $\theta = \mathbb{Z}^I$, up to positive proportionality. For θ away from certain hyperplanes, there are no strictly semistable points and Nakajima varieties are holomorphic sympletic varieties.

Exercise 4.1. Let Q be a quiver with one vertex and no arrows. Show that, for either choice of the stability condition, the corresponding Nakajima varieties are the cotangent bundles of Grassmannians.

Exercise 4.2. Let Q be a quiver with one vertex and one loop. For $\mathsf{w} = 1$, identify the Nakajima varieties with $\mathrm{Hilb}(\mathbb{C}^2, n)$ where $n = \mathsf{v}$.

4.3 Quasimaps to Nakajima varieties

4.3.1

The general notion of a quasimap to a GIT quotient is discussed in detail in [], here we specialize it to the case of Nakajima varieties X. Twisted quasimaps, which will play an important technical role below, are a slight variation on the theme.

Let T be the maximal torus of the group (81) and $\mathsf{A} = \operatorname{Ker} \hbar$ the subtorus preserving the symplectic form. Let

$$\sigma : \mathbb{C}^\times \to \mathsf{A} \tag{82}$$

be a cocharacter of A, it will determine how the quasimap to X is twisted. In principle, nothing prevents one from similarly twisting by a cocharacter of T, but this will not be done in what follows.

4.3.2

Let $C \cong \mathbb{P}^1$ denote the domain of the quasimap. Since A acts on multiplicity spaces W_i and Q_{ij}, a choice of σ determines bundles $\mathscr{W}_i$ and $\mathscr{Q}_{ij}$ over C as bundles associated to $\mathscr{O}(1)$. To fix equivariant structure we need to linearize $\mathscr{O}(1)$ and the natural choice is $\mathscr{O}(p_1)$, where $p_1 \in C$ is a fixed point of the torus action.

4.3.3

By definition, a twisted quasimap

$$f : C \dashrightarrow X$$

is a collection of vector bundles $\mathscr{V}_i$ on C of ranks v and a section

$$f \in H^0(C, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar^{-1})$$

satisfying $\mu = 0$, where

$$\mathscr{M} = \bigoplus_{i,j} \mathscr{H}om(\mathscr{V}_i, \mathscr{V}_j) \otimes \mathscr{Q}_{ij} \oplus \bigoplus_i \mathscr{H}om(\mathscr{W}_i, \mathscr{V}_i)\,. \tag{83}$$

Here $\hbar^{-1}$ is a trivial line bundle with weight $\hbar^{-1}$, inserted to record the T-action on quasimaps (in general, the centralizer of σ in $\operatorname{Aut}(X)$ acts on twisted quasimaps). One can replace $\hbar^{-1}$ by an arbitrary line bundle and that will correspond to T-twisted quasimaps.

4.3.4

We consider twisted quasimaps up to isomorphism that is required to be an identity on C and on the multiplicity bundles $\mathscr{Q}_{ij}$ and $\mathscr{W}_i$. In other words, we consider quasimaps from parametrized domains, and twisted in a fixed way. We define

$$\mathsf{QM}(X) = \{\text{stable twisted quasimaps to } X\}\big/\cong \qquad (84)$$

with the understanding that it is the data of the bundles $\mathscr{V}_i$ and of the section f that varies in these moduli spaces, while the curve C and the twisting bundles $\mathscr{Q}_{ij}$ and $\mathscr{W}_i$ are fixed[4].

As defined, $\mathsf{QM}(X)$ is a union of countably many moduli spaces of quasimaps of given degree, see below.

4.3.5

Let $p \in C$ be a point in the domain of f and fix a local trivialization of $\mathscr{Q}_{ij}$ and $\mathscr{W}_i$ at p.

The value $f(p)$ of a quasimap at a point $p \in C$ gives a well-defined G-orbit in $\mu^{-1}(0) \in T^*M$ or, in a more precise language, it defines a map

$$\mathrm{ev}_p(f) = f(p) \in \left[\mu^{-1}(0)/G\right] \supset X \qquad (85)$$

to the quotient stack, which contains $X = \mu^{-1}(0)_{\text{stable}}/G$ as an open set. By definition, a quasimap is *stable* if

$$f(p) \in X$$

for all but finitely many points of C. These exceptional points are called the *singularities* of the quasimap.

4.3.6

The degree of a quasimap is the vector

$$\deg f = (\deg \mathscr{V}_i) \in \mathbb{Z}^I\,. \qquad (86)$$

For nonsingular quasimaps, this agrees with the usual notion of degree modulo the expected generation of $H^2(X,\mathbb{Z})$ by first Chern classes of the tautological bundles.

The graph of a nonsingular twisted quasimap is a curve in a nontrivial X bundle over C, the cycles of effective curves in which lie in an extension of $H_2(C,\mathbb{Z})$ by $H_2(X,\mathbb{Z})$. Formula (86) is a particular way to split this extension[5].

[4]More precisely, for relative quasimaps, to be discussed below, the curve C is allowed to change to C', where

$$\pi : C' \to C$$

collapses some chains of $\mathbb{P}^1$s. The bundles $\mathscr{Q}_{ij}$ and $\mathscr{W}_i$ are then pulled back by π.

[5]There is no truly canonical notion of a degree zero twisted quasimap and the prescription (86) depends on previously made choices. Concretely, if σ and σ' differ by something in the kernel of (81) then the corresponding twisted quasimap moduli spaces are naturally isomorphic. This isomorphism, however, may not preserve degree.

4.3.7

Every fixed point $x \in X^\sigma$ defines a "constant" twisted quasimap with $f(c) = x$ for all $c \in C$. The degree of this constant map, which is nontrivial, is computed as follows.

A fixed point of $a \in \mathsf{A}$ in a quiver variety means a acts in the vector spaces V_i so that all arrow maps are a-equivariant. This gives the i-tautological line bundle

$$\mathscr{L}_i = \det V_i \tag{87}$$

an action of a, producing a locally constant map

$$\boldsymbol{\mu} : X^{\mathsf{A}} \to \operatorname{Pic}(X)^\vee \otimes \mathsf{A}^\vee \,, \tag{88}$$

compatible with restriction to subgroups of A. We have

$$\deg \big(f \equiv x\big) = \langle \boldsymbol{\mu}(x), \text{—} \otimes \sigma \rangle \tag{89}$$

where we used the pairing of characters with cocharacters.

The map (88) can be seen as the universal *real* moment map: the moment maps for different Kähler forms

$$\omega_{\mathbb{R}} \in H^{1,1}(X) = \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

map fixed points to different points of

$$\operatorname{Lie}(\mathsf{A}_{\text{compact}})^\vee \cong \mathsf{A}^\vee \otimes_{\mathbb{Z}} \mathbb{R} \,.$$

4.3.8

Moduli spaces of stable quasimaps have a perfect obstruction theory with

$$T_{\text{vir}} = H^\bullet(\mathscr{M} \oplus \hbar^{-1} \mathscr{M}^*) - (1 + \hbar^{-1}) \sum \operatorname{Ext}^\bullet(\mathscr{V}_i, \mathscr{V}_i) \,. \tag{90}$$

The second term accounts for the moment map equations as well as for

$$\begin{aligned} -\operatorname{Hom}(\mathscr{V}_i, \mathscr{V}_i) &= -\operatorname{Lie}\operatorname{Aut}(\mathscr{V}_i) \\ \operatorname{Ext}^1(\mathscr{V}_i, \mathscr{V}_i) &= \text{deformations of } \mathscr{V}_i \,. \end{aligned}$$

With our assumptions on the twist, the degree terms vanish in the Riemann-Roch formula, and we get

$$\operatorname{rk} T_{\text{vir}} = \dim X \,,$$

as the virtual dimension.

4.3.9

Let C be a smooth curve of arbitrary genus and

$$Y = \begin{array}{c} \mathscr{L}_1 \oplus \mathscr{L}_2 \\ \downarrow \\ C \end{array} , \qquad (91)$$

the total space of two line bundles over C. We can use $\mathscr{L}_1$ and $\mathscr{L}_2$ to define T-twisted quasimaps from C to $\mathrm{Hilb}(\mathbb{C}^2, n)$ as follows. By definition

$$f : C \dashrightarrow X$$

is a vector bundle $\mathscr{V}$ on C of rank n together with a section

$$f = (\mathsf{v}, \mathsf{v}^\vee, \mathsf{X}_1, \mathsf{X}_2)$$

of the bundles

$$\begin{aligned} \mathsf{v} &\in H^0(\mathscr{V}) , \\ \mathsf{v}^\vee &\in H^0(\mathscr{L}_1^{-1} \otimes \mathscr{L}_2^{-1} \otimes \mathscr{V}^\vee) , \\ \mathsf{X}_i &\in H^0(\mathrm{End}(\mathscr{V}) \otimes \mathscr{L}_i^{-1}) , \end{aligned}$$

satisfying the equation

$$[\mathsf{X}_1, \mathsf{X}_2] + \mathsf{v}\,\mathsf{v}^\vee = 0$$

of the Hilbert scheme and the stability condition. The stability condition forces

$$[\mathsf{X}_1, \mathsf{X}_2] = 0 ,$$

as in Exercise 4.2.

Exercise 4.3. Show this data defines a coherent sheaf $\mathscr{F}$ on the threefold Y, together with a section

$$s : \mathscr{O}_Y \to \mathscr{F} .$$

Exercise 4.4. Show the quasimap data is in bijection with complexes

$$\mathscr{O}_Y \xrightarrow{\ s\ } \mathscr{F} ,$$

of sheaves on Y such that:

— the sheaf $\mathscr{F}$ is 1-dimensional and *pure*, that is, has no 0-dimensional subsheaves, and

— the cokernel of the section s is 0-dimensional.

By definition, such complexes are parametrized by the Pandharipande-Thomas (PT) moduli spaces for Y.

Exercise 4.5. Compute the virtual dimension of the quasimap/PT moduli spaces.

5 Symmetric powers

5.1 PT theory for smooth curves

5.1.1

Let X be nonsingular threefold. By definition, a point in the Pandharipande-Thomas moduli space is a pure 1-dimensional sheaf with a section

$$\mathscr{O}_X \xrightarrow{s} \mathscr{F} \tag{92}$$

such that $\dim \operatorname{Coker} s = 0$. Here pure means that $\mathscr{F}$ has no 0-dimensional subsheaves.

Exercise 5.1. Consider the the structure sheaf

$$\mathscr{O}_C = \mathscr{O}_X / \operatorname{Ann}(\mathscr{F})$$

of the scheme-theoretic support of $\mathscr{F}$. Show it is also pure 1-dimensional.

Another way of saying the conclusion of Exercise 5.1 is that C is a 1-dimensional Cohen-Macaulay subscheme of X, a scheme that for any point $c \in C$ has a function that vanishes at c but is not a zero divisor.

5.1.2

In this section, we consider the simplest case when C is a reduced smooth curve in X. This forces (92) to have the form

$$\mathscr{O}_X \to \mathscr{O}_C \to \mathscr{O}_C(D) = \mathscr{F}$$

where $D \subset C$ is a divisor, or equivalently, a zero-dimensional subscheme, and the maps are the canonical ones. The moduli space, for fixed C, is thus

$$\mathbb{M} = \bigsqcup_{n \geqslant 0} S^n C = \bigsqcup_{n \geqslant 0} \operatorname{Hilb}(C, n)\,. \tag{93}$$

The full PT moduli space also has directions that correspond to deforming the curve C inside X, with deformation theory given by

$$\operatorname{Def}(C) - \operatorname{Obs}(C) = H^\bullet(C, N_{X/C})\,,$$

where $N_{X/C}$ is the normal bundle to C in X.

Here we fix C and focus on (93). Our goal is to relate

$$\mathsf{Z}_C = \chi(\mathbb{M}, \widehat{\mathscr{O}}^{\mathrm{vir}})$$

to deformations of the curve C in the 4th and 5th directions in (3), that is, to $H^\bullet(C, z\mathscr{L}_4 \oplus z^{-1}\mathscr{L}_5)$.

5.1.3

Remarkably, we will see that all 4 directions of

$$N_{Z/C} = N_{X/C} \oplus z\mathscr{L}_4 \oplus z^{-1}\mathscr{L}_5 \tag{94}$$

enter the full PT computation completely symmetrically.

This finds a natural explanation in the conjectural correspondence [19] between K-theoretic DT counts and K-theoretic counting of membranes in M-theory. From the perspective of M-theory, the curve $C \subset Z$ is a supersymmetric membrane and its bosonic degrees of freedom are simply motions in the transverse directions. All directions of $N_{Z/C}$ contribute equally to those.

While Theorem 4 below is a very basic check of the conjectures made in [19], it does count as a nontrivial evidence in their favor.

5.1.4

As already discussed in Exercise 3.6, the moduli space M is smooth with the cotangent bundle

$$\Omega^1\mathsf{M} = H^0(\mathscr{O}_D \otimes \mathscr{K}_C)\,.$$

Pandharipande-Thomas moduli spaces have a perfect obstruction theory which is essentially the same as the deformation theory from Section 3.4.6, that is, the deformation theory of complexes

$$\mathscr{O}_X \twoheadrightarrow \mathscr{F}$$

with a *surjective* map to $\mathscr{F} \cong \mathscr{O}_Z$. In particular,

$$\mathrm{Def} - \mathrm{Obs} = \chi(\mathscr{F}) + \chi(\mathscr{F}, \mathscr{O}_X) - \chi(\mathscr{F}, \mathscr{F})\,. \tag{95}$$

Exercise 5.2. Prove that the part of the obstruction in (95) that corresponds to keeping the curve C fixed is given by

$$\begin{aligned} \mathrm{Obs}\,\mathsf{M} &= H^0(C, \mathscr{O}_D \otimes \det N_{X/C}) \\ &= H^0(C, \mathscr{O}_D \otimes \mathscr{K}_C\, \mathscr{L}_4^{-1}\, \mathscr{L}_5^{-1}) \end{aligned} \tag{96}$$

and, in particular, is the cotangent bundle of M if $\mathscr{K}_X \cong \mathscr{O}_X$.

Since M is smooth and the obstruction bundle has constant rank, we conclude

$$\mathscr{O}^{\mathrm{vir}}_{\mathsf{M}} = \Lambda^{\bullet}\mathrm{Obs}^{\vee}\ .$$

Define the integers

$$h_i = \dim H^{\bullet}(C, \mathscr{L}_i) = \deg \mathscr{L}_i + 1 - g(C)\,,$$

as the numbers from the Riemann-Roch theorem for $\mathscr{L}_i$.

Lemma 5.1. *We have*

$$\widehat{\mathscr{O}}^{\mathrm{vir}}\Big|_{S^nC} = (-1)^{h_4} z^{\frac{h_4+h_5}{2}+n} \left(\det H^\bullet(C, \mathscr{L}_4 - \mathscr{L}_5)\right)^{1/2} \otimes \Lambda^\bullet \mathrm{Obs} \otimes \mathscr{L}_4^{\boxtimes n} . \quad (97)$$

where $\mathscr{L}_4^{\boxtimes n}$ *is an* $S(n)$*-invariant line bundle on* C^n *which descends to a line bundle on* S^nC.

Proof. We start the discussion of

$$\widehat{\mathscr{O}}^{\mathrm{vir}} = \text{prefactor } \mathscr{O}^{\mathrm{vir}} \otimes \left(\mathscr{K}_{\mathrm{vir}} \otimes \det H^\bullet(\mathscr{O}_C(D) \otimes (\mathscr{L}_4 - \mathscr{L}_5))\right)^{1/2}$$

with the prefactor. Since

$$\dim \chi(\mathscr{F}) = 1 - g(C) + \deg D$$

formula (39) specializes to

$$\text{prefactor} = (-1)^{h_4+n} z^{\frac{h_4+h_5}{2}+n} . \quad (98)$$

Since $\mathrm{rk}\,\mathrm{Obs} = n$, we have

$$\mathscr{O}^{\mathrm{vir}} = (-1)^n \Lambda^\bullet \mathrm{Obs} \otimes (\det \mathrm{Obs})^{-1} .$$

This proves (97) modulo

$$\mathscr{L}_4^{\boxtimes n} \stackrel{?}{=} \det H^\bullet \left(C, \mathscr{O}_D \otimes \mathscr{G}\right)^{1/2}$$

where

$$\mathscr{G} = \mathscr{K}_C - \mathscr{K}_C \, \mathscr{L}_4^{-1} \, \mathscr{L}_5^{-1} + \mathscr{L}_4 - \mathscr{L}_5 .$$

We observe that for any two line bundles $\mathscr{B}_1, \mathscr{B}_2$ on C

$$\det H^\bullet \left(C, \mathscr{O}_D \otimes (\mathscr{B}_1 - \mathscr{B}_2)\right) = (\mathscr{B}_1/\mathscr{B}_2)^{\boxtimes n} ,$$

whence the conclusion. □

5.1.5

In these notes, we focus on equivariant K-theory, that is, we compute equivariant Euler characteristics of coherent sheaves. This can be already quite challenging, but still much, much easier than computing individual cohomology groups of the same sheaves.

Our next result is a rare exception to this rule. Here we get the individual cohomology groups of $\widehat{\mathscr{O}}^{\mathrm{vir}}$ on S^nC as symmetric powers of $H^\bullet(C, \dots)$. When computing $\mathsf{S}^\bullet H^\bullet$, one should keep in mind the sign rule — a product of two odd cohomology classes picks up a sign when transposed. More generally, in a symmetric power of a complex

$$\mathscr{C}^\bullet = \dots \xrightarrow{d} \mathscr{C}^i \xrightarrow{d} \mathscr{C}^{i+1} \xrightarrow{d} \dots$$

the odd terms are antisymmetric with respect to permutations.

The virtual structure sheaf $\mathscr{O}^{\mathrm{vir}}$ is defined on the level of derived category of coherent sheaf as the complex (32) itself. Lemma 5.1 shows that up to a shift and tensoring with a certain 1-dimensional vector space, the symmetrized virtual structure sheaf $\widehat{\mathscr{O}}^{\mathrm{vir}}$ is represented by the complex

$$\begin{aligned}\mathbb{O}_n^\bullet &= \Lambda^\bullet \mathrm{Obs} \otimes \mathscr{L}_4^{\boxtimes n} \\ &= \mathscr{L}_4^{\boxtimes n} \xrightarrow{0} \mathscr{L}_4^{\boxtimes n} \otimes \mathrm{Obs} \xrightarrow{0} \mathscr{L}_4^{\boxtimes n} \otimes \Lambda^2 \mathrm{Obs} \xrightarrow{0} \dots\end{aligned}$$

with zero differential. The differential is zero because our moduli space is cut out by the zero section of the obstruction bundle over $S^n C$.

In particular,

$$\mathbb{O}_1^\bullet = \mathscr{L}_4 \xrightarrow{0} \mathscr{K}_C \otimes \mathscr{L}_5^{-1} ,$$

and hence by Serre duality

$$H^i(\mathbb{O}_1^\bullet) = \begin{cases} H^0(\mathscr{L}_4)\,, & i=0\,, \\ H^1(\mathscr{L}_4) \oplus H^1(\mathscr{L}_5)^\vee\,, & i=1\,, \\ H^0(\mathscr{L}_5)^\vee\,, & i=2\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

In other words

$$H^\bullet(\mathbb{O}_1^\bullet) = H^\bullet(\mathscr{L}_4) \oplus H^\bullet(\mathscr{L}_5)^\vee[-2]$$

where

$$\mathscr{C}[k]^i = \mathscr{C}^{k+i}$$

denotes the shift of a complex $\mathscr{C}^\bullet$ by k steps to the left.

Theorem 4.

$$\sum_n z^n H^\bullet(\mathbb{O}_n^\bullet) = \mathsf{S}^\bullet z\, H^\bullet(\mathbb{O}_1^\bullet)\ . \tag{99}$$

It would be naturally very interesting to know to what extent our other formulas can be upgraded to the level of the derived category of coherent sheaves.

5.1.6

Recall the definition of the symmetrized symmetric algebra from Section 2.1.6 . Working again in K-theory, we have

$$(-1)^{h_4} z^{\frac{h_4}{2}} \left(\det H^\bullet(\mathscr{L}_4)\right)^{1/2} \mathsf{S}^\bullet z H^\bullet(C, \mathscr{L}_4) = \hat{\mathsf{S}}^\bullet H^\bullet(C, z\mathscr{L}_4)^\vee\ .$$

With this notation, Theorem 4 gives the following

Corollary 5.2.

$$\chi(\mathbb{M}, \widehat{\mathscr{O}}^{\mathrm{vir}}) = \hat{\mathsf{S}}^\bullet H^\bullet(C, z\mathscr{L}_4 \oplus z^{-1}\mathscr{L}_5)^\vee\ . \tag{100}$$

Since deformations of the curve C inside X contribute $\hat{\mathsf{S}}^\bullet H^\bullet(C, N_{X/C})^\vee$, we see that, indeed, all directions (94) normal to C in Z contribute equally to the K-theoretic PT count.

5.1.7

As an example, let us take $C = \mathbb{C}^1$ and work equivariantly. We have $S^n C \cong \mathbb{C}^n$, so there is no higher cohomology anywhere.

Exercise 5.3. Show Theorem 4 for $C = \mathbb{C}^1$ is equivalent to the following identity, known as the q-binomial theorem

$$\sum_{n \geqslant 0} z^n \prod_{i=1}^{n} \frac{1 - m\, t^i}{1 - t^i} = \mathsf{S}^\bullet z \, \frac{1 - m\, t}{1 - t} = \prod_{k=0}^{\infty} \frac{1 - z\, m\, t^{k+1}}{1 - z\, t^k} \,. \tag{101}$$

Exercise 5.4. Prove (101) by proving a 1st order difference equation with respect to $z \mapsto tz$ for both sides. This is a baby version of some quite a bit more involved difference equations to come.

5.2 Proof of Theorem 4

5.2.1

As a warm-up, let us start with the case

$$\mathscr{L}_4 = \mathscr{L}_4 = \mathscr{O},$$

in which case the theorem reduces to a classical formula, going back to Macdonald, for $H^\bullet(\Omega^\bullet_{S^n C})$ and, in particular, for Hodge numbers of $S^n C$. It says that

$$\sum_n z^n H^\bullet \left(S^n C, \Omega^\bullet_{S^n C}\right) = \mathsf{S}^\bullet z \, H^\bullet(C, \Omega^\bullet_C) \,, \tag{102}$$

and this equality is canonical, in particular gives an isomorphism of orbifold vector bundles over moduli of C.

Here $\Omega^\bullet$ is not the de Rham complex, but rather the complex

$$\Omega^\bullet = \mathscr{O} \xrightarrow{0} \Omega^1 \xrightarrow{0} \Omega^2 \xrightarrow{0} \dots$$

with zero differential, as above.

5.2.2

Let M be a manifold of some dimension and consider the orbifold

$$e^M = \bigsqcup_{n \geqslant 0} S^n M \,.$$

It has a natural sheaf of orbifold differential forms $\Omega^\bullet_{\text{orb}}$, which are the differential forms on M^n invariant under the action of $S(n)$.

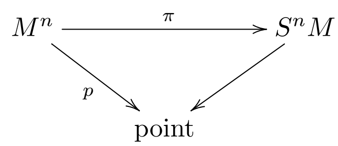
By definition, this means that

$$\Omega^\bullet_{\text{orb}} = \pi_{*,\text{orb}} \big(\Omega^\bullet\big)^{\boxtimes n} \tag{103}$$

where

$$\pi : M^n \to S^n M$$

is the natural projection and $\pi_{*,\mathrm{orb}}$ is the usual direct image of an $S(n)$-equivariant coherent sheaf followed by taking the $S(n)$-invariants. Since the map π is finite, there are no higher direct images and from the triangle



we conclude that

$$H^\bullet(S^n M, \Omega^\bullet_{\mathrm{orb}}) = p_{*,\mathrm{orb}}\big(\Omega^\bullet\big)^{\boxtimes n} = S^n H^\bullet(M, \Omega^\bullet)\,.$$

If $\dim M = 1$ then $\Omega^\bullet_{\mathrm{orb}} = \Omega^\bullet_{S^n M}$ and we obtain (102).

5.2.3

Now let $\mathscr{E}$ be an arbitrary line bundle on a curve C and define rank n vector bundles $\mathscr{E}_n$ on $S^n C$ by

$$\mathscr{E}_n = H^\bullet(\mathscr{O}_D \otimes \mathscr{E})\,.$$

This is precisely our obstruction bundle, with the substitution

$$\mathscr{E} = \mathscr{K}_C\, \mathscr{L}_4^{-1}\, \mathscr{L}_5^{-1}\,.$$

As before, we form a complex $\Lambda^\bullet \mathscr{E}_n$ with zero differential and claim that

$$\Lambda^\bullet \mathscr{E}_n = \pi_{*,\mathrm{orb}}\, (\Lambda^\bullet \mathscr{E}_1)^{\boxtimes n}\ , \tag{104}$$

in similarity to (103). In fact, locally on C there is no difference between $\mathscr{E}$ and $\mathscr{K}_C$, so these are really the same statements. See for example [9, 25] for places in the literature where a much more powerful calculus of this kind is explained and used.

By the projection formula

$$\begin{aligned} H^\bullet\big(S^n C, \Lambda^\bullet \mathscr{E}_n \otimes \mathscr{L}_4^{\boxtimes n}\big) &= H^\bullet\Big(C^n, (\Lambda^\bullet \mathscr{E}_1)^{\boxtimes n} \otimes \mathscr{L}_4^{\boxtimes n}\Big)^{S(n)} \\ &= \mathsf{S}^n H^\bullet(C, \mathbb{O}_1^\bullet)\ , \end{aligned}$$

as was to be shown.

5.3 Hilbert schemes of surfaces and threefolds

5.3.1

Now let Y be a nonsingular surface. In this case $S^n Y$ is singular and

$$\pi_{\mathrm{Hilb}} : \mathrm{Hilb}(Y, n) \to S^n Y$$

is a resolution of singularities. The sheaves $\pi_{\text{Hilb},*}\Omega^\bullet$ and $\Omega^\bullet{}_{\text{orb}}$ on Y are not equal, but they share one important property known as *factorization*. It is a very important property, much discussed in the literature, with slightly different definitions in different contexts, see for example []. Here we will need only a very weak version of factorization, which may be described as follows.

A point in S^nY is an unordered n-tuple $\{y_1,\dots,y_n\}$ of points from Y. Imagine we partition the points $\{y_i\}$ into groups and let m_k be the number of groups of size k. In other words, consider the natural map

$$\prod_k S^{m_k} S^k Y \xrightarrow{\ f\ } S^nY\,, \quad n=\sum m_k k\,.$$

Let U be the open set of in the domain of f formed by

$$y_i \neq y_j$$

for all y_i and y_j which belong to *different* groups.

Fix a sheaf, of a K-theory class $\mathscr{F}_n$ on each S^nY. A factorization of this family of sheaves is a collection of isomorphisms

$$\mathscr{F}_n\big|_U \cong \boxtimes\ \mathsf{S}^{m_k}\mathscr{F}_k \tag{105}$$

for all U as above. Here $\mathsf{S}^{m_k}\mathscr{F}_k$ is the orbifold pushforward of $(\mathscr{F}_k)^{\boxtimes m_k}$ and the isomorphisms (105) must be compatible with subdivision into smaller groups.

For example, $\Omega^\bullet{}_{\text{orb}}$ has a factorization by construction and it is easy to see $\pi_{\text{Hilb},*}\Omega^\bullet$ similarly factors.

5.3.2

The following lemma is a geometric version of the well-known combinatorial principle of inclusion-exclusion.

Lemma 5.3. *For any scheme Y and any factorizable sequence $\mathscr{F}_n \in K_G(S^nY)$ there exists*

$$\mathscr{G} = z\,\mathscr{G}_1 + z^2\,\mathscr{G}_2 + \cdots \in K_G(Y)[[z]] \tag{106}$$

such that

$$1+\sum_{n>0} z^n\,\chi(\mathscr{F}_n) = \mathsf{S}^\bullet\chi(\mathscr{G})\,. \tag{107}$$

Concretely, formula (107) means that

$$\chi(\mathscr{F}_n) = \sum_{\sum k\,m_k=n} \bigotimes \mathsf{S}^{m_k}\chi(\mathscr{G}_k)\,, \tag{108}$$

where the summations here over all solutions $(m_1,m_2,\dots)$ of the equation $\sum k\,m_k=n$ or, equivalently, over all partitions

$$\mu = (\dots 3^{m_3}\,2^{m_2}\,1^{m_1})$$

of the number n. For example

$$\begin{aligned}\chi(\mathscr{F}_2) &= \mathsf{S}^2\chi(\mathscr{G}_1)+\chi(\mathscr{G}_2)\,,\\ \chi(\mathscr{F}_3) &= \mathsf{S}^3\chi(\mathscr{G}_1)+\chi(\mathscr{G}_2)\chi(\mathscr{G}_1)+\chi(\mathscr{G}_3)\,.\end{aligned}$$

5.3.3

Proof of Lemma 5.3. The sheaves $\mathscr{G}_i$ in (106) are constructed inductively, starting with

$$\mathscr{G}_1 = \mathscr{F}_1 \,.$$

and using the exact sequence (18). Consider $X = S^2 Y$ and let

$$Y \cong X' \subset X$$

be the diagonal. Factorization gives

$$\mathscr{F}_2\big|_U \cong \mathsf{S}^2 \mathscr{G}_1 \,, \quad U = X \backslash X' \,,$$

and so from (18) we obtain

$$\mathscr{G}_2 = \mathscr{F}_2 - \mathsf{S}^2 \mathscr{G}_1 \in K(X') = K(Y)$$

which solves (106) modulo $O(z^3)$.

Now take $X = S^3 Y$ and let $X' = p(Y^2)$ where

$$p(y_1, y_2) = 2y_1 + y_2 \in X \,.$$

Consider

$$\mathscr{F}_3' = \mathscr{F}_3 - \mathsf{S}^3 \mathscr{G}_1 \in K(X') \,,$$

and denote

$$X'' = \{y_1 = y_2 = y_3\} = p(\text{diagonal}_{Y^2}) \cong Y \,.$$

By compatibility of factorization with respect to further refinements

$$\mathscr{F}_3'\big|_{X' \backslash X''} = p_*(\mathscr{G}_2 \boxtimes \mathscr{G}_1) \,.$$

Using the exact sequence (18) again, we construct

$$\mathscr{G}_3 = \mathscr{F}_3'' = \mathscr{F}_3' - p_*(\mathscr{G}_2 \boxtimes \mathscr{G}_1) \in K(Y)$$

which solves (106) modulo $O(z^4)$.

For general n, we consider closed subvarieties

$$S^n Y = X_n \supset X_{n-1} \supset \cdots \supset X_1 = Y$$

where X_k is the locus of n-tuples $\{y_1, \ldots, y_n\}$ among which at most k are distinct. In formula (108), X_k will correspond to partitions μ with

$$\ell(\mu) = \text{length}(\mu) = \sum m_i = k \,.$$

We construct

$$\mathscr{F}_n' \in K(X_{n-1}) \,, \quad \mathscr{F}_n'' \in K(X_{n-2}) \,, \quad \ldots$$

inductively, starting with $\mathscr{F}_n$ on X_n. For each $k < n-1$, the set

$$U_{n-k} = X_{n-k} \backslash X_{n-k-1}$$

is a union of sets to which factorization applies, and this gives

$$\mathscr{F}^{(k)}\Big|_{X_{n-k}\setminus X_{n-k-1}} = \sum_{\ell(\mu)=n-k} p_{\mu,*}\left(\boxtimes \mathsf{S}^{m_k}\mathscr{G}_k\right)\Big|_{X_{n-k}\setminus X_{n-k-1}} \tag{109}$$

where

$$p_\mu(y_1, y_2, \dots, y_\ell) = \sum \mu_i\, y_i \in S^n Y\,.$$

We let $\mathscr{F}^{(k+1)}$ be the difference between two sheaves in (109), which is thus a sheaf supported on X_{n-k-1}. Once we get to $X_1 \cong Y$, this gives

$$\mathscr{G}_n = \mathscr{F}_n^{(n-1)}\,.$$

□

5.3.4

Now we go back to Y being a nonsingular suface. Recall that the Hilbert scheme of points in Y is nonsingular and that Proposition 3.1 expresses its tangent bundle in terms of the universal ideal sheaf.

For symmetric powers of the curves in Section 5.1, the obstruction bundle was a certain twisted version of the cotangent bundle. One can similarly twist the tangent bundle of the Hilbert scheme of a surface, namely we define

$$T_{\mathrm{Hilb},\mathscr{L}} = \chi(\mathscr{L}) - \chi(\mathscr{I}_Z, \mathscr{I}_Z \otimes \mathscr{L})\,. \tag{110}$$

for a line bundle $\mathscr{L}$ on Y. Let ${\Omega^\bullet}_{\mathrm{Hilb},\mathscr{L}}$ be the exterior algebra of the dual vector bundle. It is clear that its pushforward to S^nY factors just like the pushforward of ${\Omega^\bullet}_{\mathrm{Hilb}}$ and therefore

$$\sum_n z^n \chi(\mathrm{Hilb}(Y,n), {\Omega^\bullet}_{\mathscr{L}}) = \mathsf{S}^\bullet \chi(Y, \mathscr{G}) \tag{111}$$

for a certain $\mathscr{G}$ as in (106). The analog of Nekrasov's formula in this case is the following

Theorem 5.

$$\sum_n z^n \chi(\mathrm{Hilb}(Y,n), {\Omega^\bullet}_{\mathscr{L}}) = \mathsf{S}^\bullet \chi\left(Y, {\Omega^\bullet}_{\mathscr{L}}\ \frac{z}{1 - z\mathscr{L}^{-1}}\right)\,. \tag{112}$$

See [4] for how to place this formula in a much more general mathematical and physical context. As with Nekrasov's formula, it is in fact enough to prove (112) for a toric surface, and hence for $Y = \mathbb{C}^2$, in which case it becomes a corollary of the main result of [4].

Here we discuss an alternative approach, based on (111), which we format as a sequence of exercises.

Exercise 5.5. Check that for $Y = \mathbb{C}^2$, the LHS in (112) becomes the function

$$\mathsf{Z}_{\mathrm{Hilb}(\mathbb{C}^2)} = \sum_{n,i \geqslant 0} z^n (-m)^i \, \chi(\mathrm{Hilb}(\mathbb{C}^2, n), \Omega^i)$$

investigated in Exercise 3.9, where $\mathscr{L}^{-1}$ is a trivial bundle with weight m.

Exercise 5.6. Arguing as in Section 3.5.3, prove that

$$\mathsf{Z}_{\mathrm{Hilb}(\mathbb{C}^2)} = \mathsf{S}^\bullet \left(\star \, \frac{(1-m\,t_1^{-1})(1-m\,t_2^{-1})}{(1-t_1^{-1})(1-t_2^{-1})} \right)$$

for a certain series

$$\star \in \mathbb{Z}[m][[z]]\,.$$

Exercise 5.7. Arguing as in Section 3.5.4, prove that

$$\star = \frac{z}{1-mz}\,.$$

What is the best limit to consider for the parameters t_1 and t_2 ?

5.3.5

Now let X be a nonsingular threefold and let

$$\pi : \mathrm{Hilb}(X, n) \to S^n$$

be the Hilbert-Chow map. To complete the proof Nekrasov's formula given in Section 3.5, we need to show (67), which follows from the following

Proposition 5.4. *The sequence*

$$\mathscr{F}_n = \pi_* \, \widehat{\mathscr{O}}^{\mathrm{vir}} \in K(S^n X)$$

factors.

There is, clearly, something to check here, because, for example, this sequence would not factor without the minus sign in (41).

Proof. Recall from Section 3 that

$$\widehat{\mathscr{O}}^{\mathrm{vir}} = \dots \xrightarrow{d} \boldsymbol{\kappa}^{-\frac{\dim}{2}+i}\, \Omega^i_{\widetilde{\mathbb{M}}} \xrightarrow{d} \boldsymbol{\kappa}^{-\frac{\dim}{2}+i+1} \Omega^{i+1}_{\widetilde{\mathbb{M}}} \xrightarrow{d}$$

where i is also the cohomological dimension and

$$d\omega = \boldsymbol{\kappa}\, d\phi \wedge \omega\,.$$

Here

$$\phi(\mathsf{X}) = \mathrm{tr}\,(\mathsf{X}_1\mathsf{X}_2\mathsf{X}_3 - \mathsf{X}_1\mathsf{X}_3\mathsf{X}_2)$$

is the function whose critical locus in $\widetilde{\mathbb{M}}$ is the Hilbert scheme.

Let U be the locus where the spectrum of X_1 can be decomposed into two mutually disjoint blocks of sizes n' and n'', respectively. This means X_1 can be put it in the form

$$\mathsf{X}_1 = \begin{pmatrix} \mathsf{X}_1' & 0 \\ 0 & \mathsf{X}_1'' \end{pmatrix}$$

up-to conjugation by $GL(n') \times GL(n'')$ or $S(2) \ltimes GL(n')^2$ if $n' = n''$. Thus block off-diagonal elements of X_1 and of the gauge group are eliminated simultaneously.

If λ_i' and λ_j'' are the eigenvalues of X_1' and X_1'' respectively, then as a function of the off-diagonal elements of X_2 and X_3 the function ϕ can be brought to the form

$$\phi = \sum_{ij} (\lambda_i' - \lambda_j'') \, \mathsf{X}_{2,ij} \mathsf{X}_{3,ji} + \dots ,$$

and thus has many Morse terms of the form

$$\phi_2(u,v) = uv$$

where the weights of u and v multiply to $\boldsymbol{\kappa}^{-1}$. For the Morse critical point on $\mathbb{C}^2$, the complex

$$\boldsymbol{\kappa}^{-1} \mathscr{O} \xrightarrow{\boldsymbol{\kappa}\, d\phi_2 \wedge} \Omega^1 \xrightarrow{\boldsymbol{\kappa}\, d\phi_2 \wedge} \boldsymbol{\kappa}\, \Omega^2$$

is exact, except at the last term, where its cohomology is $\mathscr{O}_{u=v=0}$. This is also true if we replace $\mathbb{C}^2$ by vector bundle over some base because the existence of a Morse function forces the determinant of this bundle to be trivial, up to a twist by $\boldsymbol{\kappa}$.

Therefore, all off-diagonal matrix elements of X_2 and X_3 are eliminated and $\widehat{\mathscr{O}}^{\text{vir}}$, restricted to U, is, up to an even shift, the tensor product of the corresponding complexes for $\operatorname{Hilb}(X, n')$ and $\operatorname{Hilb}(X, n'')$.

□

6 More on quasimaps

6.1 Square roots

6.1.1

Virtual $\mathscr{O}$ and quasimaps vs. stable maps

6.1.2

Let $\mathscr{K}_C$ be the canonical bundle of the domain C. For our specific domain, we have, equivariantly,

$$\mathscr{K}_C - \mathscr{O}_C = -\mathscr{O}_{p_1} - \mathscr{O}_{p_2}$$

where $p_1, p_2 \in C$ are the fixed points of a torus in $\operatorname{Aut}(C)$. It is customary to choose $\{p_1, p_2\} = \{0, \infty\}$.