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Lectures on localization and matrix models in supersymmetric Chern–Simons–matter theories

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ABSTRACT: This is a PRELIMINARY and UNFINISHED set of notes.

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1. Introduction

The AdS/CFT correspondence a remarkable equivalence between certain gauge theories and string/M-theory on certain backgrounds involving AdS spaces. In these lectures we will look at the correspondence in the case of $\mathrm{AdS}_4/\mathrm{CFT}_3$, in particular for ABJM theory [4]. One of the mysterious consequences of this correspondence is that, at strong coupling, the number of degrees of freedom of the CFT_3 should scale as $N^{3/2}$ [28]. This was finally established at the gauge theory level in [15].

The derivation of the $N^{3/2}$ behavior in [15] is based on two ingredients. The first one, due to [26], is that certain path integrals in the CFT_3 can be reduced to matrix integrals by using the powerful localization techniques of [33]. Localization techniques have a long story in supersymmetric QFTs, and the applications of [33, 26] to superconformal field theories provide a powerful technique to analyze certain observables in terms of matrix models. One of these observables is, in the three-dimensional case, the partition function on the three-sphere. The second ingredient in [15] was the realization that this quantity is a good measure of the number of degrees of freedom in a CFT_3 and that it can be calculated at strong coupling in the gauge theory and in the AdS SUGRA. This leads to new powerful tests of the AdS/CFT correspondence.

In some cases, the partition function on the 3-sphere can be computed explicitly at all couplings, and it gives a non-trivial interpolating function between perturbation theory and supergravity results. In the case of ABJM theory, the planar limit of the free energy on $\mathbb{S}^3$ can be calculated at weak 't Hooft coupling (a one-loop calculation in QFT) and at strong coupling (where it is given by the regularized gravity action) with the result

$$-\lim_{N\to\infty}\frac{1}{N^2}F_{\mathrm{ABJM}}(\mathbb{S}^3)=\begin{cases}-\log(2\pi\lambda)+\frac{3}{2}+2\log(2)+\mathcal{O}(\lambda), & \lambda\to 0,\\[2ex] \frac{\pi\sqrt{2}}{3\sqrt{\lambda}}+\mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\lambda}}\right), & \lambda\to\infty.\end{cases} \tag{1.1}$$

The goal of these lectures is to explain how to obtain the above result, both in QFT and in the AdS dual, and then we will show how the recent progress in [26, 15] makes possible to obtain the exact function of the 't Hooft coupling interpolating between these two results. To do this, we will present the localization technique of [26] and the matrix model techniques of [15].

2. Chern–Simons–matter theories

In this section we will introduce the basic building blocks of supersymmetric Chern–Simons–matter theories. We will work in Euclidean space, and we will eventually put the theories on the three-sphere, for reasons that will be discussed later on. In this section we will closely follow the presentation of [22].

2.1 Conventions

Our conventions for Euclidean spinors follow essentially [39]. In Euclidean space, the fermions ψ and $\bar{\psi}$ are independent and they transform in the same representation of the Lorentz group. Their index structure is

$$\psi^\alpha, \quad \bar{\psi}^\alpha. \tag{2.1}$$

For example, in [39] (with Minkowski signature) one has

$$\psi^\alpha \to \left(M^{-1}\right)_\beta^{\ \alpha} \psi^\beta, \quad M \in \mathrm{Sl}(2, \mathbb{C}). \tag{2.2}$$

In our case (Euclidean three-dimensional space) the Lorentz group is $SU(2)$ and we will simply have

$$\psi^\alpha \to M^\alpha_{\ \beta} \psi^\beta, \qquad M \in SU(2). \tag{2.3}$$

We will take γ_μ to be the Pauli matrices, which are hermitian, and

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] = \mathrm{i}\epsilon_{\mu\nu\rho}\gamma^\rho. \tag{2.4}$$

We introduce the usual symplectic product through the antisymmetric matrix

$$C_{\alpha\beta} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}. \tag{2.5}$$

In [39] we have $C = -1$ and the matrix is denoted by $\epsilon_{\alpha\beta}$. The product is

$$\bar{\epsilon}\lambda = \bar{\epsilon}^\alpha C_{\alpha\beta} \lambda^\beta \tag{2.6}$$

Notice that

$$\bar{\epsilon}\gamma^\mu\lambda = \bar{\epsilon}^\beta C_{\beta\gamma} \left(\gamma^\mu\right)^\gamma_{\ \alpha} \lambda^\alpha \tag{2.7}$$

It is easy to check that

$$\bar{\epsilon}\lambda = \lambda\bar{\epsilon}, \quad \bar{\epsilon}\gamma^\mu\lambda = -\lambda\gamma^\mu\bar{\epsilon}. \tag{2.8}$$

and in particular

$$\left(\gamma^\mu\bar{\epsilon}\right)\lambda = -\bar{\epsilon}\gamma^\mu\lambda \tag{2.9}$$

We also have the following Fierz identities

$$\bar{\epsilon}\left(\epsilon\psi\right) + \epsilon\left(\bar{\epsilon}\psi\right) + \left(\bar{\epsilon}\epsilon\right)\psi = 0. \tag{2.10}$$

and

$$\epsilon\left(\bar{\epsilon}\psi\right) + 2\left(\bar{\epsilon}\epsilon\right)\psi + \left(\bar{\epsilon}\gamma_\mu\psi\right)\gamma^\mu\epsilon = 0. \tag{2.11}$$

2.2 Vector multiplet and supersymmetric Chern–Simons theory

We first start with theories based on vector multiplets. The three dimensional Euclidean $\mathcal{N} = 2$ vector superfield V has the following content

$$V: \qquad A_\mu, \ \sigma, \ \lambda, \ \bar{\lambda} \ D \tag{2.12}$$

where A_μ is a gauge field, σ is an auxiliary scalar field, λ, $\bar{\lambda}$ are two-component complex Dirac spinors, and D is an auxiliary scalar. This is just the dimensional reduction of the $\mathcal{N} = 1$ vector multiplet in 4 dimensions, and σ is the reduction of the fourth component of A_μ. All fields are valued in the Lie algebra $\mathfrak{g}$ of the gauge group G. For $G = U(N)$ our convention is that $\mathfrak{g}$ are Hermitian matrices. It follows that the gauge covariant derivative is given by

$$D_\mu = \partial_\mu + \mathrm{i}[A_\mu, .] \tag{2.13}$$

while the gauge field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]. \tag{2.14}$$

The transformations of the fields are generated by two independent complex spinors ϵ, $\bar\epsilon$. They are given by,

$$\begin{aligned}
\delta A_\mu &= \tfrac{\mathrm{i}}{2}(\bar\epsilon\gamma_\mu\lambda - \bar\lambda\gamma_\mu\epsilon),\\
\delta\sigma &= \tfrac{1}{2}(\bar\epsilon\lambda - \bar\lambda\epsilon),\\
\delta\lambda &= -\tfrac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + \mathrm{i}\gamma^\mu\epsilon D_\mu\sigma + \tfrac{2\mathrm{i}}{3}\sigma\gamma^\mu D_\mu\epsilon,\\
\delta\bar\lambda &= -\tfrac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\bar\epsilon - \mathrm{i}\gamma^\mu\bar\epsilon D_\mu\sigma - \tfrac{2\mathrm{i}}{3}\sigma\gamma^\mu D_\mu\bar\epsilon,\\
\delta D &= -\tfrac{\mathrm{i}}{2}\bar\epsilon\gamma^\mu D_\mu\lambda - \tfrac{\mathrm{i}}{2}D_\mu\bar\lambda\gamma^\mu\epsilon + \tfrac{\mathrm{i}}{2}[\bar\epsilon\lambda, \sigma] + \tfrac{\mathrm{i}}{2}[\bar\lambda\epsilon, \sigma] - \tfrac{\mathrm{i}}{6}(D_\mu\bar\epsilon\gamma^\mu\lambda + \bar\lambda\gamma^\mu D_\mu\epsilon),
\end{aligned} \tag{2.15}$$

and we have naturally

$$\delta = \delta_\epsilon + \delta_{\bar\epsilon}. \tag{2.16}$$

Here we follow the conventions of [22], but we change the sign of the gauge connection: $A_\mu \to -A_\mu$. On all the fields except D the commutator $[\delta_\epsilon, \delta_{\bar\epsilon}]$ becomes a sum of translation, gauge transformation, Lorentz rotation, dilation and R-rotation:

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar\epsilon}]A_\mu =& \mathrm{i}v^\nu\partial_\nu A_\mu + \mathrm{i}\partial_\mu v^\nu A_\nu + D_\mu\Lambda,\\
[\delta_\epsilon, \delta_{\bar\epsilon}]\sigma =& \mathrm{i}v^\mu\partial_\mu\sigma + \mathrm{i}[\Lambda, \sigma] + \rho\sigma,\\
[\delta_\epsilon, \delta_{\bar\epsilon}]\lambda =& \mathrm{i}v^\mu\partial_\mu\lambda + \tfrac{\mathrm{i}}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + \mathrm{i}[\Lambda, \lambda] + \tfrac{3}{2}\rho\lambda + \alpha\lambda,\\
[\delta_\epsilon, \delta_{\bar\epsilon}]\bar\lambda =& \mathrm{i}v^\mu\partial_\mu\bar\lambda + \tfrac{\mathrm{i}}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar\lambda + \mathrm{i}[\Lambda, \bar\lambda] + \tfrac{3}{2}\rho\bar\lambda - \alpha\bar\lambda,\\
[\delta_\epsilon, \delta_{\bar\epsilon}]D =& \mathrm{i}v^\mu\partial_\mu D + \mathrm{i}[\Lambda, D] + 2\rho D + \tfrac{1}{3}\sigma(\bar\epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon - \epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\bar\epsilon),
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
v^\mu =& \bar\epsilon\gamma^\mu\epsilon,\\
\Theta^{\mu\nu} =& D^{[\mu}v^{\nu]} + v^\lambda\omega_\lambda^{\mu\nu},\\
\Lambda =& v^\mu\mathrm{i}A_\mu + \sigma\bar\epsilon\epsilon,\\
\rho =& \tfrac{\mathrm{i}}{3}(\bar\epsilon\gamma^\mu D_\mu\epsilon + D_\mu\bar\epsilon\gamma^\mu\epsilon),\\
\alpha =& \tfrac{\mathrm{i}}{3}(D_\mu\bar\epsilon\gamma^\mu\epsilon - \bar\epsilon\gamma^\mu D_\mu\epsilon).
\end{aligned} \tag{2.18}$$

As a check, let us calculate the commutator acting on σ. We have,

$$\begin{aligned}
\sigma =& \delta_\epsilon\left(\frac{1}{2}\bar\epsilon\lambda\right) - \delta_{\bar\epsilon}\left(-\frac{1}{2}\bar\lambda\epsilon\right)\\
=& \frac{1}{2}\bar\epsilon\left(-\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + \mathrm{i}\gamma^\mu\epsilon D_\mu\sigma\right) + \frac{\mathrm{i}}{3}\bar\epsilon\gamma^\mu D_\mu\epsilon\\
&+ \frac{1}{2}\left(-\frac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\epsilon - \mathrm{i}\gamma^\mu\bar\epsilon D_\mu\sigma\right)\epsilon - \frac{\mathrm{i}}{3}\gamma^\mu\left(D_\mu\bar\epsilon\right)\epsilon\\
=& \mathrm{i}\bar\epsilon\gamma^\mu\epsilon D_\mu\sigma + \rho\sigma,
\end{aligned} \tag{2.19}$$

where we have used (2.9).

In order for the supersymmetry algebra to close, the last term in the right hand side of $[\delta_\epsilon, \delta_{\bar\epsilon}]D$ must vanish. This means that the Killing spinors must satisfy

$$\gamma^\mu\gamma^\nu D_\mu D_\nu \epsilon = h\epsilon, \qquad \gamma^\mu\gamma^\nu D_\mu D_\nu \bar\epsilon = h\bar\epsilon \tag{2.20}$$

for some scalar function h. A sufficient condition for this is to have

$$D_\mu \epsilon = \frac{\mathrm{i}}{2r}\gamma_\mu \epsilon, \qquad D_\mu \bar\epsilon = \frac{\mathrm{i}}{2r}\gamma_\mu \bar\epsilon \tag{2.21}$$

and

$$h = -\frac{9}{4r^2}. \tag{2.22}$$

This condition is satisfied by one of the Killing spinors on the three-sphere (the one which is constant in the left-invariant frame).

The (Euclidean) SUSY CS action, in flat space, is given by

$$\begin{aligned} S_{CS} &= \int \mathrm{d}^3x \,\mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2\mathrm{i}}{3}A^3 - \bar\lambda\lambda + 2D\sigma \right) \\ &= \int \mathrm{d}^3x \,\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\rho \right) - \bar\lambda\lambda + 2D\sigma \right) \end{aligned} \tag{2.23}$$

Here Tr denotes an invariant inner product. Usually we will take it to be $k/4\pi$ times the trace in the fundamental representation.

We can check that the supersymmetric CS action is invariant under the supersymmetry generated by δ_ϵ (the proof for $\delta_{\bar\epsilon}$ is similar). We find

$$\begin{aligned} \delta_\epsilon L &= (2\delta A_\mu \partial_\nu A_\rho + 2\mathrm{i}\delta A_\mu A_\nu A_\rho)\,\epsilon^{\mu\nu\rho} - \bar\lambda\delta\lambda + 2(\delta D)\sigma + 2D\delta\sigma \\ &= -\,\mathrm{i}\bar\lambda\gamma_\mu\epsilon\partial_\nu A_\rho \epsilon^{\mu\nu\rho} + \bar\lambda\gamma_\mu\epsilon A_\nu A_\rho\epsilon^{\mu\nu\rho} \\ &\quad - \bar\lambda\left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + \mathrm{i}\gamma^\mu D_\mu\sigma \right)\epsilon - \frac{2\mathrm{i}}{3}\bar\lambda\gamma^\mu D_\mu\epsilon\sigma \\ &\quad - \mathrm{i}\left(D_\mu\bar\lambda\right)\gamma^\mu\sigma\epsilon + \mathrm{i}[\bar\lambda\epsilon, \sigma]\sigma - \frac{\mathrm{i}}{3}\bar\lambda\gamma^\mu D_\mu\epsilon\sigma - \bar\lambda\epsilon D. \end{aligned} \tag{2.24}$$

The terms involving D cancel on the nose. Let us look at the terms involving the gauge field. After using (2.4) we find

$$\frac{1}{2}\bar\lambda\gamma^{\mu\nu}F_{\mu\nu} = \mathrm{i}\bar\lambda\gamma_\rho\epsilon\epsilon^{\mu\nu\rho}\partial_\mu A_\nu - \bar\lambda\gamma_\rho\epsilon\epsilon^{\mu\nu\rho}A_\mu A_\nu \tag{2.25}$$

which cancel the first two terms in (2.24). Let us now look at the remaining terms. The covariant derivative of $\bar\lambda$ is

$$D_\mu\bar\lambda = \partial_\mu\bar\lambda + \frac{\mathrm{i}}{2r}\gamma_\mu\bar\lambda + \mathrm{i}[A_\mu, \bar\lambda] \tag{2.26}$$

If we integrate by parts the term involving the derivative of λ we find in total

$$\begin{aligned} &\mathrm{i}\bar\lambda\gamma^\mu\epsilon\partial_\mu\sigma + \mathrm{i}\bar\lambda\gamma^\mu\partial_\mu\epsilon\sigma + \frac{1}{2r}\left(\gamma^\mu\bar\lambda\right)\gamma_\mu\epsilon + [A_\mu, \bar\lambda]\gamma^\mu\epsilon\sigma \\ &= \mathrm{i}\bar\lambda\gamma^\mu\epsilon\partial_\mu\sigma + \mathrm{i}\bar\lambda\gamma^\mu D_\mu\epsilon\sigma + [A_\mu, \bar\lambda]\gamma^\mu\epsilon\sigma \end{aligned} \tag{2.27}$$

where we used that

$$\left(\gamma^\mu\bar{\lambda}\right)\gamma_\mu\epsilon = -\bar{\lambda}\gamma^\mu\gamma_\mu\epsilon. \tag{2.28}$$

The derivative of σ cancels against the corresponding term in the covariant derivative of σ. Putting all together, we find

$$\mathrm{i}\bar{\lambda}\gamma^\mu\left(D_\mu\epsilon\right)\sigma - \mathrm{i}\bar{\lambda}\gamma^\mu\left(D_\mu\epsilon\right)\sigma + [A_\mu,\bar{\lambda}]\gamma^\mu\epsilon\sigma + \bar{\lambda}\gamma^\mu\epsilon[A_\mu,\sigma] + \mathrm{i}[\bar{\lambda}\epsilon,\sigma]\sigma \tag{2.29}$$

The last three terms cancel due to the cyclic property of the trace. This proves the invariance of the supersymmetric CS theory.

Of course, there is another Lagrangian for vector multiplets, namely the standard Yang–Mills Lagrangian,

$$\mathcal{L}_{\mathrm{YM}} = \mathrm{Tr}\Big(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu\sigma D^\mu\sigma + \tfrac{1}{2}(D+\tfrac{\sigma}{\ell})^2 + \tfrac{\mathrm{i}}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \tfrac{\mathrm{i}}{2}\bar{\lambda}[\sigma,\lambda] - \tfrac{1}{4\ell}\bar{\lambda}\lambda\Big). \tag{2.30}$$

which is not only invariant under SUSY, but it can be written as a superderivative,

$$\bar{\epsilon}\epsilon\,\mathcal{L}_{\mathrm{YM}} = \delta_{\bar{\epsilon}}\delta_\epsilon\mathrm{Tr}\Big(\tfrac{1}{2}\bar{\lambda}\lambda - 2D\sigma\Big). \tag{2.31}$$

This will be important later on.

2.3 Supersymmetric matter multiplets

We will now add supersymmetric matter, i.e. a chiral multiplet in representation R. The components are

$$\phi,\ \bar{\phi},\ \psi,\ \bar{\psi},\ F,\ \bar{F}. \tag{2.32}$$

The supersymmetry transformations are

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi,\\
\delta\bar{\phi} &= \epsilon\bar{\psi},\\
\delta\psi &= \mathrm{i}\gamma^\mu\epsilon D_\mu\phi + \mathrm{i}\epsilon\sigma\phi + \frac{2\Delta\mathrm{i}}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F,\\
\delta\bar{\psi} &= \mathrm{i}\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + \mathrm{i}\bar{\phi}\sigma\bar{\epsilon} + \frac{2\Delta\mathrm{i}}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}\epsilon,\\
\delta F &= \epsilon(\mathrm{i}\gamma^\mu D_\mu\psi - \mathrm{i}\sigma\psi - \mathrm{i}\lambda\phi) + \frac{\mathrm{i}}{3}(2\Delta-1)D_\mu\epsilon\gamma^\mu\psi,\\
\delta\bar{F} &= \bar{\epsilon}(\mathrm{i}\gamma^\mu D_\mu\bar{\psi} - \mathrm{i}\bar{\psi}\sigma + \mathrm{i}\bar{\phi}\bar{\lambda}) + \frac{\mathrm{i}}{3}(2\Delta-1)D_\mu\bar{\epsilon}\gamma^\mu\bar{\psi}.
\end{aligned} \tag{2.33}$$

where Δ is the possible anomalous dimension of ϕ. For theories with $\mathcal{N}\geq 3$ supersymmetry, the field has the canonical dimension

$$\Delta = \frac{1}{2}, \tag{2.34}$$

but in general this is not the case.

The commutators of these transformations are given by

$$
\begin{aligned}
[\delta_\epsilon, \delta_{\bar\epsilon}]\phi &= \mathrm{i}v^\mu \partial_\mu \phi + \mathrm{i}\Lambda\phi + q\rho\phi - q\alpha\phi, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\bar\phi &= \mathrm{i}v^\mu \partial_\mu \bar\phi - \mathrm{i}\bar\phi\Lambda + q\rho\bar\phi + q\alpha\bar\phi, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\psi &= \mathrm{i}v^\mu \partial_\mu \psi + \tfrac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + \mathrm{i}\Lambda\psi + (q+\tfrac{1}{2})\rho\psi + (1-q)\alpha\psi, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\bar\psi &= \mathrm{i}v^\mu \partial_\mu \bar\psi + \tfrac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar\psi - \mathrm{i}\bar\psi\Lambda + (q+\tfrac{1}{2})\rho\bar\psi + (q-1)\alpha\bar\psi, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]F &= \mathrm{i}v^\mu \partial_\mu F + \mathrm{i}\Lambda F + (q+1)\rho F + (2-q)\alpha F, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\bar F &= \mathrm{i}^v \mu \partial_\mu \bar F - \mathrm{i}\bar F\Lambda + (q+1)\rho\bar F + (q-2)\alpha\bar F.
\end{aligned} \tag{2.35}
$$

The lowest components are now assigned the dimension q and R-charge $\mp q$. The supersymmetry algebra closes off-shell when the Killing spinors $\epsilon, \bar\epsilon$ satisfy (2.20) and h is given by (2.22).

As a check, we compute

$$
\begin{aligned}
\phi &= \delta_\epsilon\left(\bar\epsilon\psi\right) \\
&= \bar\epsilon\left(\mathrm{i}\gamma^\mu\epsilon D_\mu\phi + \mathrm{i}\epsilon\sigma\phi + \frac{2\mathrm{i}q}{3}\gamma^\mu\left(D_\mu\epsilon\right)\phi\right) = \mathrm{i}v^\mu D_\mu\phi + \mathrm{i}\sigma\bar\epsilon\epsilon + \frac{2\mathrm{i}q}{3}\left(\bar\epsilon\gamma^\mu D_\mu\epsilon\right),
\end{aligned} \tag{2.36}
$$

which is the wished-for result. Let us now consider supersymmetric Lagrangians for the matter hypermultiplet. If the fields have their canonical dimensions, the Lagrangian,

$$
\mathcal{L} = D_\mu\bar\phi D^\mu\phi - \mathrm{i}\bar\psi\gamma^\mu D_\mu\psi + \frac{3}{4r^2}\bar\phi\phi + i\bar\psi\sigma\psi + \mathrm{i}\bar\psi\lambda\phi - i\bar\phi\bar\lambda\psi + \mathrm{i}\bar\phi D\phi + \bar\phi\sigma^2\phi + \bar F F, \tag{2.37}
$$

which is invariant under supersymmetry if the Killing spinors $\epsilon, \bar\epsilon$ satisfy (2.20). The quadratic part of the Lagrangian for ϕ gives indeed the standard conformal coupling for a scalar field. We recall that the action for the conformal coupling of a scalar field in d dimensions is

$$
S = \int \mathrm{d}^d x\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right), \tag{2.38}
$$

where R is the curvature of a sphere of radius r

$$
R = \frac{d(d-1)}{r^2}. \tag{2.39}
$$

The curvature coupling can then be written as

$$
\frac{d(d-2)}{4r^2}\phi^2 \tag{2.40}
$$

which in $d = 3$ gives the quadratic term for ϕ in (2.37). Of course, this term is effectively a mass for the complex scalar field.

If the fields have non-canonical dimensions, it turns out that the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{mat}} =& D_\mu\bar\phi D^\mu\phi + \bar\phi\sigma^2\phi + \frac{\mathrm{i}(2\Delta-1)}{r}\bar\phi\sigma\phi + \frac{\Delta(2-\Delta)}{r^2}\bar\phi\phi + \mathrm{i}\bar\phi D\phi + \bar F F \\
& - \mathrm{i}\bar\psi\gamma^\mu D_\mu\psi + \mathrm{i}\bar\psi\sigma\psi - \frac{2\Delta-1}{2r}\bar\psi\psi + \mathrm{i}\bar\psi\lambda\phi - \mathrm{i}\bar\phi\bar\lambda\psi.
\end{aligned} \tag{2.41}
$$

is supersymmetric, provided the parameters $\epsilon, \bar{\epsilon}$ satisfy

$$D_\mu \epsilon = \tfrac{i}{2\ell}\gamma_\mu \epsilon, \quad D_\mu \bar{\epsilon} = \tfrac{i}{2\ell}\gamma_\mu \bar{\epsilon}. \tag{2.42}$$

An important fact, which will be useful later, is that the Lagrangian (2.41) can be written as a total superderivative,

$$\bar{\epsilon}\epsilon\, \mathcal{L}_{\rm mat} = \delta_{\bar{\epsilon}}\delta_\epsilon \left(\bar{\psi}\psi - 2{\rm i}\bar{\phi}\sigma\phi + \frac{2(\Delta-1)}{r}\bar{\phi}\phi \right). \tag{2.43}$$

3. A brief review of Chern–Simons theory

3.1 Perturbative approach

We recall that the CS action is given by

$$S = \frac{k}{4\pi}\int_M {\rm Tr}\Big(A\wedge {\rm d}A + \frac{2{\rm i}}{3} A\wedge A\wedge A\Big) \tag{3.1}$$

It will be useful to introduce a Hermitian basis for the Lie algeba, T_a, with commutation relations

$$[T_a, T_b] = {\rm i} f_{abc} T_c, \tag{3.2}$$

and normalization

$${\rm Tr}(T_a T_b) = \delta_{ab}. \tag{3.3}$$

If we write

$$A = A^a T_a, \tag{3.4}$$

then the action reads

$$S = -\frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\Big(A^a_\mu \partial_\nu A_\rho - \frac{1}{3} f_{abc} A^a_\mu A^b_\nu A^c_\rho \Big). \tag{3.5}$$

We will assume that the theory is defined on a compact three-manifold M. In this case, the partition function

$$Z(M) = \frac{1}{{\rm vol}(\mathcal{G})}\int [\mathcal{D}A] {\rm e}^{{\rm i}S} \tag{3.6}$$

should be well-defined. Here $\mathcal{G}$ is the group of gauge transformations. There are many different approaches to the calculation of (3.6), but the obvious strategy is to use perturbation theory, as in standard QFT. In perturbation theory we evaluate (3.6) by expanding around the saddle–points. These are flat connections, which are in one-to-one correspondence with group homomorphisms

$$\pi_1(M) \to G. \tag{3.7}$$

For example, if $M = {\bf S}^3/\mathbb{Z}_p$ is the lens space $L(p,1)$, one has $\pi_1(L(p,1)) = \mathbb{Z}_p$, and flat connections are labelled by homomorphisms $\mathbb{Z}_p \to G$. Let us assume that these are a discrete set of points (this happens, for example, if M is a rational homology sphere, since in that case $\pi_1(M)$ is a finite group). We will label the flat connections with an index c, and the flat connection will be denoted by $A^{(c)}$. Each flat connection leads to a covariant derivative

$${\rm d}_{A^{(c)}} = {\rm d} + {\rm i}A^{(c)}, \tag{3.8}$$

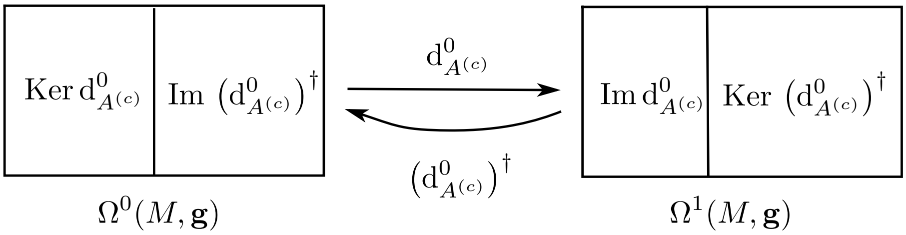

Figure 1: The standard elliptic decomposition of $\Omega^{0,1}(M, \mathbf{g})$.

and flatness implies that

$$\mathrm{d}^2_{A^{(c)}} = F_{A^{(c)}} = 0. \tag{3.9}$$

Therefore, the covariant derivative leads to a complex

$$0 \to \Omega^0(M, \mathbf{g}) \xrightarrow{\mathrm{d}_{A^{(c)}}} \Omega^1(M, \mathbf{g}) \xrightarrow{\mathrm{d}_{A^{(c)}}} \Omega^2(M, \mathbf{g}) \xrightarrow{\mathrm{d}_{A^{(c)}}} \Omega^3(M, \mathbf{g}), \tag{3.10}$$

We recall the basic orthogonal decompositions (see Fig. 1)

$$\begin{aligned} \Omega^0(M, \mathbf{g}) =& \mathrm{Ker}\, \mathrm{d}_{A^{(c)}} \oplus \mathrm{Im}\, \mathrm{d}^\dagger_{A^{(c)}}, \\ \Omega^1(M, \mathbf{g}) =& \mathrm{Ker}\, \mathrm{d}^\dagger_{A^{(c)}} \oplus \mathrm{Im}\, \mathrm{d}_{A^{(c)}}. \end{aligned} \tag{3.11}$$

To prove the first decomposition, we just note that

$$a \in \mathrm{Ker}\, \mathrm{d}_{A^{(c)}} \Rightarrow \langle \mathrm{d}_{A^{(c)}} a, \phi \rangle = \langle a, \mathrm{d}^\dagger_{A^{(c)}} \phi \rangle = 0, \ \forall \phi \tag{3.12}$$

therefore

$$(\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^\perp = \mathrm{Im}\, \mathrm{d}^\dagger_{A^{(c)}}. \tag{3.13}$$

In the following we will assume that

$$H^1(M, \mathrm{d}_{A^{(c)}}) = 0. \tag{3.14}$$

This means that the connection $A^{(c)}$ is *isolated*. However, we will consider the possibility that $A^{(c)}$ has a non-trivial isotropy group $\mathcal{H}_c$. We recall that the isotropy group of a connection $A^{(c)}$ is the subgroup of gauge transformations which leave $A^{(c)}$ invariant,

$$\mathcal{H}_c = \{\phi \in \mathcal{G} | \phi(A^{(c)}) = A^{(c)}\}. \tag{3.15}$$

The Lie algebra of this group is given by zero-forms annihilated by the covariant derivative (3.8)

$$\mathrm{Lie}(\mathcal{H}_c) = H^0(M, \mathrm{d}_{A^{(c)}}) = \mathrm{Ker}\, \mathrm{d}_{A^{(c)}} \tag{3.16}$$

which is in general non-trivial. We recall that a connection is *irreducible* if its isotropy group is equal to the centrer of the group. In particular, if $A^{(c)}$ is irreducible one has

$$H^0(M, \mathrm{d}_{A^{(c)}}) = 0. \tag{3.17}$$

It can be shown that the isotropy group $\mathcal{H}_c$ consists of constant gauge transformations that leave $A^{(c)}$ invariant,

$$\phi A^{(c)} \phi^{-1} = A^{(c)}. \tag{3.18}$$

They correspond then to a subgroup of G which we will denote by H_c.

In the semiclassical approximation, $Z(M)$ is written as a sum of terms associated to stationary points:

$$Z(M) = \sum_c Z^{(c)}(M), \tag{3.19}$$

where c labels the different flat connections $A^{(c)}$ on M. Each of the $Z^{(c)}(M)$ will be an asympotic series in $1/k$ of the form

$$Z^{(c)}(M) = Z^{(c)}_{1-\text{loop}}(M) \exp\left\{ \sum_{\ell=1}^{\infty} S_\ell^{(c)} k^{-\ell} \right\}. \tag{3.20}$$

Let us study this in detail. First of all, we split the connection into a "background", which is a flat connection $A^{(c)}$, plus a "fluctuation" B:

$$A = A^{(c)} + B. \tag{3.21}$$

Expanding around this, we find

$$S_{\rm CS}(A) = S_{\rm CS}(A^{(c)}) + S(B), \tag{3.22}$$

where

$$S(B) = \frac{k}{4\pi} \int_M \text{Tr}\,(B \wedge \mathrm{d}_{A^{(c)}} B + \frac{2}{3} B^3). \tag{3.23}$$

The first term is the classical Chern–Simons invariant of the connection $A^{(c)}$. Since we have a gauge theory, we have to fix the gauge. Our gauge choice will be the standard, covariant, Feynman gauge,

$$g_{A^{(c)}}(B) = \mathrm{d}^\dagger_{A^{(c)}} B = 0 \tag{3.24}$$

where $g_{A^{(c)}}$ is the gauge fixing function. In the treatment of the path–integral, we will follow the detailed analysis of [1]. We recall that in the standard Fadeev–Popov gauge fixing one first defines

$$\Delta^{-1}_{A^{(c)}}(B) = \int \mathcal{D}U\, \delta\left(g_{U\cdot A^{(c)}}\left(B^U\right)\right), \tag{3.25}$$

where the superscript U denotes the gauge transformation by U, and then inserts into the path integral

$$1 = \int \mathcal{D}U\, \delta\left(g_{A^{(c)}}\left(B^U\right)\right) \Delta_{A^{(c)}}(B). \tag{3.26}$$

The key new ingredient here is the presence of a non-trivial isotropy group $\mathcal{H}_c$ for the flat connection $A^{(c)}$. When there is a non-trivial isotropy group, the gauge-fixing condition does not fix the gauge completely, since

$$g_{A^{(c)}}(B^\phi) = \phi g_{A^{(c)}}(B) \phi^{-1}, \qquad \phi \in \mathcal{H}_c, \tag{3.27}$$

i.e. the basic assumption that $g(A) = 0$ only cuts once the gauge orbit is not true. Another way to see this is that the standard FP determinant is not well-defined. In fact, the standard calculation (which is valid if the isotropy group of $A^{(c)}$ is trivial) gives

$$\Delta^{-1}_{A^{(c)}} = \left|\det \frac{\delta g_{A^{(c)}}}{\delta U}\right|^{-1} = \left|\det \mathrm{d}^\dagger_{A^{(c)}} \mathrm{d}_A\right|^{-1}. \tag{3.28}$$

But when $\mathcal{H}_c \neq 0$, the operator $\mathrm{d}_{A^{(c)}}$ has zero modes due to the nonvanishing of (3.16), and the above determinant cannot be computed in perturbation theory. The correct way to proceed in the calculation of (3.25) is to split the integration over the gauge group into two pieces. The first piece is the integration over the isotropy group, and since the integrand does not depend on it, this just gives a factor of $\mathrm{Vol}(\mathcal{H}_c)$. The second piece gives an integration over the remaining part of the gauge transformations, which has as its Lie algebra

$$(\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^{\perp}. \tag{3.29}$$

The integration over this piece leads to the standard FP determinant (3.28) but with the zero modes removed. We then find,

$$\Delta^{-1}_{A^{(c)}}(B) = \mathrm{Vol}(\mathcal{H}_c) \left| \det \mathrm{d}^{\dagger}_{A^{(c)}} \mathrm{d}_A \right|^{-1}_{(\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^{\perp}} \tag{3.30}$$

This phenomenon was first observed by Rozansky in [35], and developed in this language in [1]. As usual, the determinant appearing here can be written in terms of ghost fields,

$$S_{\mathrm{ghosts}}(C, \overline{C}, B) = \int_M \langle \overline{C}, \mathrm{d}^{\dagger}_{A^{(c)}} \mathrm{d}_A C \rangle, \tag{3.31}$$

where

$$\langle a, b \rangle = \mathrm{Tr}\,(a \wedge *b) \tag{3.32}$$

is the standard norm for $\mathbf{g}$–valued forms on M, and the fields $C, \overline{C}$ take values in

$$C, \overline{C} \in (\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^{\perp}. \tag{3.33}$$

The action for the ghosts can be divided into a kinetic term plus an interaction term between the ghost fields and the fluctuation B:

$$S_{\mathrm{ghosts}}(C, \overline{C}, B) = \langle \overline{C}, \Delta^0_{A^{(c)}} C \rangle + \langle \overline{C}, (\mathrm{d}_{A^{(c)}})^{\dagger} [B, C] \rangle. \tag{3.34}$$

The FP gauge–fixing leads then to the path integral

$$\begin{aligned} Z &= \frac{\mathrm{e}^{\mathrm{i} S_{\mathrm{CS}}(A^{(c)})}}{\mathrm{vol}(\mathcal{G})} \int_{\Omega^1(M, \mathbf{g})} \mathcal{D}B\, \mathrm{e}^{S(B)} \Delta_{A^{(c)}}(B) \delta \left(\mathrm{d}^{\dagger}_{A^{(c)}} B \right) \\ &= \frac{\mathrm{e}^{\mathrm{i} S_{\mathrm{CS}}(A^{(c)})}}{\mathrm{Vol}(\mathcal{H}_c)} \int \mathcal{D}B\, \delta \left(\mathrm{d}^{\dagger}_{A^{(c)}} B \right) \int_{(\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^{\perp}} \mathcal{D}C \mathcal{D}\overline{C}\, \mathrm{e}^{S(B) + S_{\mathrm{ghosts}}(C, \overline{C}, B)} \end{aligned} \tag{3.35}$$

Finally, we notice that the delta constraint imposes that

$$B \in \mathrm{Ker}\, \mathrm{d}^{\dagger}_{A^{(c)}}, \tag{3.36}$$

and therefore restricts the integration range in the path integral, but it also introduces a factor

$$(\det{}' \Delta^0_{A^{(c)}})^{-\frac{1}{2}}, \tag{3.37}$$

where the $'$ indicates, as usual, that we are removing zero modes. This is the generalization of the standard formula

$$\int \mathrm{d}x f(x) \delta(ax) = \frac{1}{|a|} f(0). \tag{3.38}$$

It can be also seen by writing the delta constraint as

$$\delta\left(\mathrm{d}^{\dagger}_{A^{(c)}} B\right) = \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\int \langle \varphi, \mathrm{d}^{\dagger}_{A^{(c)}} B\rangle} \tag{3.39}$$

The final result for the gauge-fixed path integral is then

$$Z = \frac{\mathrm{e}^{\mathrm{i}S_{\mathrm{CS}}(A^{(c)})}}{\mathrm{Vol}(\mathcal{H}_c)} (\det{}' \Delta^0_{A^{(c)}})^{-\frac{1}{2}} \int_{\mathrm{Ker}\, \mathrm{d}^{\dagger}_{A^{(c)}}} \mathcal{D}B \int_{(\mathrm{Ker}\, \mathrm{d}_{A^{(c)}})^{\perp}} \mathcal{D}C\mathcal{D}\overline{C}\, \mathrm{e}^{S(B)+S_{\mathrm{ghosts}}(C,\overline{C},B)}. \tag{3.40}$$

3.2 The one–loop contribution

We now consider the one-loop contribution of a saddle-point to the path integral. This has been studied in many papers [17, 25, 35, 36]. The most detailed treatment, in our view, is the one presented in [2].

The main ingredients in the one-loop contribution are the determinants of the operators appearing in the kinetic terms for B, C and $\overline{C}$. For B, we obtain

$$(\det D^1_{A^{(c)}})^{-\frac{1}{2}}, \tag{3.41}$$

where

$$D^1_{A^{(c)}} = \frac{\mathrm{i}k}{2\pi} * \mathrm{d}_{A^{(c)}} \tag{3.42}$$

is the operator appearing in the kinetic term for B, but restricted to

$$\mathrm{Ker}\, \mathrm{d}^{\dagger}_{A^{(c)}} = (\mathrm{Im}\, \mathrm{d}_{A^{(c)}})^{\perp} \tag{3.43}$$

due to the gauge fixing. Notice that the determinant is well–defined if (3.14) holds, since in this case one has

$$H^1(M, \mathbf{g}) = 0 \Rightarrow \mathrm{Ker}\, \mathrm{d}_{A^{(c)}} = \mathrm{Im}\, \mathrm{d}_{A^{(c)}}, \tag{3.44}$$

and since the operator is restricted to (3.43), no zero modes are involved in its evaluation.

For the ghost fields we obtain

$$\det{}' \Delta^0_{A^{(c)}} \tag{3.45}$$

since we are restricting to (3.33). Recalling that there is a determinant coming from the delta function, we finally get, after putting all together,

$$\left(\frac{\det{}' \Delta^0_{A^{(c)}})}{\det D^1_{A^{(c)}}}\right)^{\frac{1}{2}}. \tag{3.46}$$

The operator $\Delta^0_{A^{(c)}}$ is positive–definite, so its square root is well–defined. The operator $D^1_{A^{(c)}}$ is *not* positive definite, and one has to do a careful analysis of its phase as first pointed out by Witten [40]. The starting point is the trivial Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(\mathrm{i}\frac{x^2}{2\lambda}\right) = \sqrt{2\pi|\lambda|}\, \exp\left(\frac{\mathrm{i}\pi}{4} \mathrm{sign}\, \lambda\right). \tag{3.47}$$

In our problem,

$$\lambda = \frac{2\pi}{k}. \tag{3.48}$$

We then conclude that, for each eigenvalue of $D^1_{A^{(c)}}$ we have a factor of

$$\left(\frac{k}{4\pi^2}\right)^{-1/2} \tag{3.49}$$

The regularized number of eigenvalues of the operator is simply

$$\zeta(A^{(c)}) = \zeta\left(0, |{*}\mathrm{d}_{A^{(c)}}|\right). \tag{3.50}$$

Here we have defined the zeta function for the operator $|T|$

$$\zeta\left(s, |T|\right) = \sum_j \frac{1}{|\lambda_j|^s} \tag{3.51}$$

where λ_j are the eigenvalues of the original operator T. We obtain then a factor

$$\left(\frac{k}{4\pi^2}\right)^{-\zeta(A^{(c)})/2}. \tag{3.52}$$

Finally, the signs of the different eigenvalues combine into the η invariant of the operator ${*}\mathrm{d}_{A^{(c)}}$. We recall that the η invariant is defined as

$$\eta(s,T) = \sum_j \frac{1}{\left(\lambda_j^+\right)^s} - \sum_j \frac{1}{\left(-\lambda_j^-\right)^s} \tag{3.53}$$

where $\lambda_j^{\pm}$ are the strictly positve (negative, respectively) eigenvalues of T. The regularized difference of eigenvalues is then $\eta(0,T)$. In our case, this gives

$$\eta(A^{(c)}) = \eta\left(0, {*}\mathrm{d}_{A^{(c)}}\right). \tag{3.54}$$

We then obtain,

$$(\det D^1_{A^{(c)}})^{-\frac{1}{2}} = \left(\frac{k}{4\pi^2}\right)^{-\zeta(A^{(c)})/2} \exp\left(\frac{\mathrm{i}\pi}{4}\eta(A^{(c)})\right)\left(\det{}' \,\mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_{A^{(c)}}\right)^{-\frac{1}{4}}, \tag{3.55}$$

where the last determinant is evaluated after subtracting the zero–modes.

The quotient of the determinants of the Laplacians gives the square root of the Ray–Singer torsion,

$$\frac{(\det{}' \,\Delta^0_{A^{(c)}})^{\frac{1}{2}}}{\left(\det{}' \,\mathrm{d}^\dagger_{A^{(c)}}\mathrm{d}_{A^{(c)}}\right)^{\frac{1}{4}}} = \sqrt{\tau'_R(M, A^{(c)})}. \tag{3.56}$$

When the connection $A^{(c)}$ is isolated and irreducible, this is a topological invariant of M, but in general it is not. However, for a reducible and isolated flat connection, the dependence on the metric is just given by an overall factor, equal to the volume of the manifold M:

$$\tau'_R(M, A^{(c)}) = (\mathrm{vol}(M))^{\dim(\mathcal{H}_c)}\, \tau_R(M, A^{(c)}). \tag{3.57}$$

where $\tau_R(M, A^{(c)})$ is now metric-indepedendent. It is interesting to notice that the volume, metric-dependent factor cancels in the final answer for the one-loop path integral. The isotropy group $\mathcal{H}_c$ is a space of constant zero forms, taking values in a subgroup $H_c \subset G$ of the gauge group. Each generator of its Lie algebra has a norm given by its norm as an element of $\mathbf{g}$, times

$$\left(\int_M *1 \right)^{1/2} = (\mathrm{vol}(M))^{1/2} . \tag{3.58}$$

Therefore,

$$\mathrm{vol}(\mathcal{H}_c) = (\mathrm{vol}(M))^{\dim(\mathcal{H}_c)/2} \, \mathrm{vol}(H_c), \tag{3.59}$$

and

$$\frac{\sqrt{\tau'_R(M, A^{(c)})}}{\mathrm{vol}(\mathcal{H}_c)} = \frac{\sqrt{\tau_R(M, A^{(c)})}}{\mathrm{vol}(H_c)} \tag{3.60}$$

which does not depend on the metric of M.

Finally, in order to write down the answer, we take into account that $\zeta(A^{(c)})$ can be evaluated as [3]

$$\zeta(A^{(c)}) = \dim H^0(M, \mathrm{d}_{A^{(c)}}). \tag{3.61}$$

Putting everything together, we find for the one–loop contribution to the path integral

$$Z^{(c)}_{1-\mathrm{loop}}(M) = \frac{(k/4\pi^2)^{-\frac{1}{2}\dim H^0(M, \mathrm{d}_{A^{(c)}})}}{\mathrm{vol}(H_c)} \mathrm{e}^{\mathrm{i}k S_{\mathrm{CS}}(A^{(c)}) + \frac{\mathrm{i}\pi}{4}\eta(A^{(c)})} \sqrt{\tau_R(M, A^{(c)})}. \tag{3.62}$$

Notice that, when $A^{(c)} = 0$ is the trivial flat connection, one has that $H_c = G$, where G is the gauge group. This result will be useful later on.

4. The partition function of 3d CFTs on $\mathbb{S}^3$

The partition function of a CFT on the three sphere should incorporate some information about the number of degrees of freedom of the theory. For example, if a theory has N^2 degrees of freedom we should expect the free energy on the three-sphere to scale as

$$F(\mathbb{S}^3, N) = \log Z(\mathbb{S}^3, N) \sim \mathcal{O}(N^2), \tag{4.1}$$

at least in the weak coupling approximation, i.e. when the N^2 degress of freedom are weakly interacting. This follows from the fact that, at weak coupling, the partition function factorizes

$$Z(\mathbb{S}^3, N) \sim \left(Z(\mathbb{S}^3, 1) \right)^{N^2} . \tag{4.2}$$

In this section we will first compute this partition function at weak coupling in Chern–Simons–matter theories. More precisely, we will compute this partition function on the three sphere at one-loop. First, we will review the computation in Chern–Simons theory, and then we will consider the much simpler case of the matter multiplets. Finally, we will explain what is the expected behavior at strong coupling from the large N AdS duals.

4.1 Chern–Simons theory on $\mathbb{S}^3$

The above general procedure to calculate the one-loop contribution on the three-sphere can be made very concrete on $\mathbb{S}^3$. Here, there is only a flat connection, the trivial one $A^{(c)} = 0$, and the cohomology twisted by $A^{(c)}$ reduces to the ordinary cohomology. The group H_c is the full gauge group, since any constant gauge transformation leaves $A^{(c)}$ invariant, and $H_c = G$.

We first calculate the quotient of determinants appearing in (3.56) (a similar calculation was made in Appendix A of [18]). The determinant of the scalar Laplacian on the sphere can be computed very explicitly, since the eigenvalues are known to be given by

$$\lambda_j = 4j(j+1) = n(n+2), \qquad n = 0, 1, \cdots \tag{4.3}$$

where

$$j = \frac{n}{2} \tag{4.4}$$

is the left $SU(2)$ spin, and their degeneracy is

$$d_j = (2j+1)^2 = (n+1)^2. \tag{4.5}$$

We have assumed that $\mathbb{S}^3$ is endowed with its standard metric (the one induced by its standard embedding in $\mathbb{R}^4$ with Euclidean metric), and that its radius is $R = 1$. Removing the zero eigenvalue just means that we remove $j = 0$ from the spectrum. To calculate the determinant we must calculate the zeta function,

$$\zeta_{\Delta^{(0)}}(s) = \sum_{j>0} \frac{d_j}{\lambda_j^s} = \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n(n+2))^s} = \sum_{m=2}^{\infty} \frac{m^2}{(m^2-1)^s}, \tag{4.6}$$

since

$$\log \det{}' \Delta^{(0)} = -\zeta'_{\Delta^{(0)}}(0). \tag{4.7}$$

This can be done in many ways, and general results for the determinant of Laplacians on the n-sphere can be found in for example [34]. We will follow a simple procedure inspired by [32]. We split

$$\frac{m^2}{(m^2-1)^s} = \frac{1}{m^{2(s-1)}} + \frac{s}{m^{2s}} + R(m,s), \tag{4.8}$$

where

$$R(m,s) = \frac{m^2}{(m^2-1)^s} - \frac{1}{m^{2(s-1)}} - \frac{s}{m^{2s}} \tag{4.9}$$

which decreases as m^{-2s-2} for large m, and therefore leads to a convergent series for all $s \geq -1/2$ which is moreover uniformly convergent, and one can exchange sums with derivatives. The derivative of $R(m,s)$ at $s = 0$ can be calculated as

$$\left.\frac{\mathrm{d}R(m,s)}{\mathrm{d}s}\right|_{s=0} = -1 + m^2 \log\left(1 - \frac{1}{m^2}\right) \tag{4.10}$$

The sum of this series can be explicitly calculated by using the Hurwitz zeta function (or, in practical terms, by plugging it in *Mathematica*), and one finds

$$-\sum_{m=2}^{\infty} \left[1 + m^2 \log\left(1 - \frac{1}{m^2}\right)\right] = \frac{3}{2} - \log(\pi). \tag{4.11}$$

We then find

$$\zeta_{\Delta^{(0)}}(s) = \zeta(2s-2) - 1 + s\left(\zeta(2s) - 1\right) + \sum_{m=2}^{\infty} R(m,s), \tag{4.12}$$

where $\zeta(s)$ is Riemann's zeta function, and

$$-\zeta'_{\Delta^{(0)}}(0) = \log(\pi) - 2\zeta'(-2). \tag{4.13}$$

Here we have used that

$$\zeta'(0) = -\frac{1}{2}\log(2\pi). \tag{4.14}$$

The final result can be expressed in terms of $\zeta(3)$, since

$$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}. \tag{4.15}$$

We conclude that the determinant of the scalar Laplacian on $\mathbb{S}^3$ is given by

$$\log \det{}' \Delta^{(0)} = \log(\pi) + \frac{\zeta(3)}{2\pi^2}. \tag{4.16}$$

We now compute the determinant in the denominator of (3.56). We must consider the space of one-forms on $\mathbb{S}^3$, and restrict to the ones that are not in the image of d. These forms are precisely the vector spherical harmonics, whose properties are reviewed in the Appendix. The eigenvalues of the operator $\mathrm{d}^\dagger \mathrm{d}$ are given in (B.30), and they read

$$\lambda_n = (n+1)^2, \qquad n = 1, 2, \cdots, \tag{4.17}$$

with degeneracies

$$d_n = 2n(n+2). \tag{4.18}$$

The zeta function associated to this Laplacian (restricted to the vector spherical harmonics) is

$$\zeta_{\Delta^{(1)}}(s) = \sum_{n=1}^{\infty} \frac{2n(n+2)}{\left((n+1)\right)^{2s}} = 2\sum_{m=1}^{\infty} \frac{m^2-1}{m^{2s}} = 2\zeta(2s-2) - 2\zeta(2s), \tag{4.19}$$

and

$$\log \det{}' \Delta^{(1)} = -4\zeta'(-2) - 2\log(2\pi) = -2\log(2\pi) + \frac{\zeta(3)}{\pi^2}. \tag{4.20}$$

We conclude that

$$\log \tau'_R(\mathbb{S}^3, 0) = \log \det{}' \Delta^{(0)} - \log \det{}' \Delta^{(1)} = \log(2\pi^2). \tag{4.21}$$

This is in agreement with the calculation of the analytic torsion for general spheres in [38]. In view of (3.57), and since

$$\mathrm{vol}(\mathbb{S}^3) = 2\pi^2, \tag{4.22}$$

we find

$$\tau_R(\mathbb{S}^3, 0) = 1. \tag{4.23}$$

One can also calculate the invariant (3.50) directly, since

$$\zeta\left(s, *\mathrm{d}\right) = 2\sum_{n\geq 1} \frac{n(n+2)}{(n+1)^s} = 2\left(\zeta(s-2) - \zeta(s)\right), \tag{4.24}$$

and

$$\zeta\left(0, *\mathrm{d}\right) = -2\zeta(0) = 1. \tag{4.25}$$

Since there are $\dim G$ copies of this operator, we verify the expression (3.61). Finally, a similar calculation gives

$$\eta(0) = 0. \tag{4.26}$$

We conclude that

$$Z_{1-\text{loop}}(\mathbb{S}^3) = \frac{1}{\text{vol}(G)}\left(\frac{k}{4\pi^2}\right)^{-\frac{1}{2}\dim G} \tag{4.27}$$

4.2 Matter fields

The supersymmetric multiplet contains a complex scalar and a fermion. At one loop we just have to compute the functional determinant for a conformally coupled complex scalar and a complex fermion.

We will first consider the complex scalar. The kinetic operator is now the conformal Laplacian

$$\Delta_c = -\nabla^2 + \frac{3}{4} \tag{4.28}$$

where for simplicity we have set $r = 1$ (we will incorporate later on the dependence on r). The eigenvalues of the conformal Laplacian are simply

$$n(n+2) + \frac{3}{4}, \qquad n = 0, 1, \cdots, \tag{4.29}$$

with the same multiplicity as the standard Laplacian, namely $(n+1)^2$. We then have

$$\zeta_{\Delta_c} = \sum_{n=0} \frac{(n+1)^2}{\left(n(n+2) + \frac{3}{4}\right)^s} = \sum_{m=1}^{\infty} \frac{m^2}{\left(m^2 - \frac{1}{4}\right)^s}. \tag{4.30}$$

As in the case of the standard Laplacian, we split

$$\frac{m^2}{\left(m^2 - \frac{1}{4}\right)^s} = \frac{1}{m^{2(s-1)}} + \frac{s}{4m^{2s}} + R_c(m, s), \tag{4.31}$$

where

$$R_c(m, s) = \frac{m^2}{\left(m^2 - \frac{1}{4}\right)^s} - \frac{1}{m^{2(s-1)}} - \frac{s}{4m^{2s}}. \tag{4.32}$$

The derivative of $R_c(m, s)$ at $s = 0$ can be calculated as

$$\left.\frac{\mathrm{d}R_c(m, s)}{\mathrm{d}s}\right|_{s=0} = -\frac{1}{4} + m^2 \log\left(1 - \frac{1}{4m^2}\right) \tag{4.33}$$

The sum of this series can be explicitly calculated as

$$-\sum_{m=1}^{\infty}\left[\frac{1}{4} + m^2 \log\left(1 - \frac{1}{4m^2}\right)\right] = \frac{1}{8} - \frac{1}{4}\log(2) + \frac{7\zeta(3)}{8\pi^2}. \tag{4.34}$$

We then find

$$\zeta_{\Delta_c}(s) = \zeta(2s-2) + \frac{s}{4}\zeta(2s) + \sum_{m=1}^{\infty} R_c(m, s), \tag{4.35}$$

and

$$-\zeta'_{\Delta_c}(0) = \frac{1}{4}\log(2) - \frac{3\zeta(3)}{8\pi^2}. \tag{4.36}$$

We conclude that the determinant of the conformalLaplacian on $\mathbb{S}^3$ is given by

$$\log\,\det\Delta_c = \frac{1}{4}\log(2) - \frac{3\zeta(3)}{8\pi^2}. \tag{4.37}$$

This is in agreement with the result quoted in the Erratum to [14][1].

Let us now consider the determinant (in absolute value) for the spinor field. We have, using (B.44),

$$\zeta_{|\not{D}|}(s) = 2\sum_{n=1}^{\infty}\frac{n(n+1)}{\left(n+\frac{1}{2}\right)^s}. \tag{4.38}$$

After a small manipulation we can write it as

$$\begin{aligned}\zeta_{|\not{D}|}(s) =& 2\cdot 2^{s-2}\left\{\sum_{m\geq 1}\frac{1}{(2m+1)^{s-2}} - \sum_{m\geq 1}\frac{1}{(2m+1)^s}\right\}\\ =& 2\left(2^{s-2}-1\right)\zeta(s-2) - \frac{1}{2}\left(2^s-1\right)\zeta(s),\end{aligned} \tag{4.39}$$

where we have used that

$$\sum_{m\geq 0}\frac{1}{(2m+1)^s} = \sum_{n\geq 1}\frac{1}{n^s} - \sum_{k\geq 1}\frac{1}{(2k)^s} = \left(1-2^{-s}\right)\zeta(s). \tag{4.40}$$

We can then compute

$$\log\,\det\left(-\mathrm{i}\not{D}\right) = -\zeta'_{|\not{D}|}(0) = -\frac{3}{8\pi^2}\zeta(3) - \frac{1}{4}\log 2. \tag{4.41}$$

In fact, the regularized number of negative eigenvalues is

$$\zeta_{|\not{D}|}(s) = 0 \tag{4.42}$$

and there is no contribution from the extra phases. The above result is the determinant we were looking for. Notice that the supersymmetric, one-loop quantity is then

$$\log\,\det\left(-\mathrm{i}\not{D}\right) - \log\,\det\Delta_c = -\frac{1}{2}\log 2. \tag{4.43}$$

This can be seen directly at the level of eigenvalues. The quotient of determinants is

$$\prod_{m=1}^{\infty}\frac{\left(m+\frac{1}{2}\right)^{m(m+1)}\left(m-\frac{1}{2}\right)^{m(m-1)}}{\left(m^2-\frac{1}{4}\right)^{m^2}} = \prod_{m=1}^{\infty}\frac{\left(m+\frac{1}{2}\right)^m}{\left(m-\frac{1}{2}\right)^m} \tag{4.44}$$

and its regularization leads directly to the result above.

[1] Beware: the arXiv version of this paper gives a wrong result for this determinant.

4.3 ABJM theory at weak coupling

We can now calculate the free energy on $\mathbb{S}^3$ of ABJM theory. We have two copies of CS theory with gauge group $U(N)$, together with four chiral multiplets. Keeping the first term (one-loop) in perturbation theory we find,

$$F_{\rm ABJM}(\mathbb{S}^3) = -N^2 \log\left(\frac{k}{4\pi^2}\right) - 2\log\left(\mathrm{vol}(U(N))\right) - 2N^2\log(2) + \cdots \tag{4.45}$$

where the first two terms come from the CS theories, and the last term comes from the supersymmetric matter. The volume of $U(N)$ can be calculated in various ways, and its value is

$$\mathrm{vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{G_2(N+1)}, \tag{4.46}$$

where $G_2(z)$ is the Barnes function, defined by

$$G_2(z+1) = \Gamma(z)G_2(z), \quad G_2(1) = 1. \tag{4.47}$$

Using the asymptotic expansion of the Barnes function

$$\begin{aligned}\log\, G_2(N+1) = {} & \frac{N^2}{2}\log\, N - \frac{1}{12}\log\, N - \frac{3}{4}N^2 + \frac{1}{2}N\,\log\, 2\pi + \zeta'(-1) \\ & + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g},\end{aligned} \tag{4.48}$$

where B_{2g} are the Bernoulli numbers, we finally obtain

$$F_{\rm ABJM}(\mathbb{S}^3) = N^2\left\{\log(2\pi\lambda) - \frac{3}{2} - 2\log(2) + \cdots\right\} + \mathcal{O}(N^0), \tag{4.49}$$

5. AdS duals

Some 3d SCFTs have a dual AdS description which gives precise predictions for the strong coupling behavior of the free energy on $\mathbb{S}^3$. The dual computation involves calculating the gravitational action on AdS_4, and this calculation needs regularization in order to obtain finite results. I will now review the method of holographic renormalization and work out two examples: the closely related example of the Casimir energy of $\mathcal{N}=4$ SYM on $\mathbb{R}\times\mathbb{S}^3$, and the free energy of ABJM theory on $\mathbb{S}^3$.

5.1 Holographic renormalization

The holographic calculation of many quantities in the AdS/CFT correspondence involves the calculation of the gravitational action on AdS backgrounds. We recall that this action has two parts: the bulk part, given by the Einstein–Hilbert action

$$I_{\rm bulk} = -\frac{1}{16\pi G_N}\int_M \mathrm{d}^{n+1}x\,\sqrt{G}\,(R-2\Lambda) \tag{5.1}$$

and a boundary part, or Gibbons–Hawking term [19]

$$I_{\rm surf} = -\frac{1}{8\pi G}\int_{\partial M} K|\gamma|^{1/2}\mathrm{d}^d x, \tag{5.2}$$

where ∂M is the boundary of spacetime, γ is the metric induced by G on the boundary, and K is the extrinsic curvature, which satisfies

$$\sqrt{\gamma}K = \mathcal{L}_n \sqrt{\gamma} \tag{5.3}$$

where n is the normal unit vector to ∂M. Both actions, when computed on an AdS background, diverge due to the non-compactness of the space. For example, the bulk action of an AdS space of radius L can be written as

$$I = \frac{n}{8\pi G_N L^2} \int \mathrm{d}^{n+1}x \sqrt{G} \tag{5.4}$$

which is proportional to the volume of space-time.

In order to use the AdS/CFT correspondence, we have to regularize the gravitational action in an appropriate way. The standard procedure is to introduce a set of *universal counterterms*, depending only on the induced metric on the boundary, which lead to finite values of the gravitational action, energy-momentum tensors, etc. This procedure leads to the correct value for many interesting quantities computed in the CFT side, and we will adopt it here. It is sometimes called "holographic renormalization" and it has been developed in for example [23, 7, 16, 13]. We now present the basics of holographic renormalization in AdS.

An asymptotically AdS metric can be written as

$$\mathrm{d}s^2 = L^2 \left[\frac{\mathrm{d}u^2}{u^2} + \frac{1}{u^2} g_{ij}(u^2, x) \mathrm{d}x^i \mathrm{d}x^j \right] \tag{5.5}$$

where the boundary of AdS occurs at $u = 0$. Notice that, because of the second order pole at the boundary, this metric does not define an induced metric at the boundary, but rather a *conformal structure*. Essentially this is due to the fact that, in order to remove the pole, we should multiply the metric by a function r with a first-order zero at the boundary (such functions are called *defining functions*), and this gives a boundary metric

$$g_{\mathrm{b}} = r^2 G \Big|_{u=0}. \tag{5.6}$$

However, if r is such a function, $\mathrm{e}^w r$ also is, and this multiplies g_{b} by a Weyl rescaling.

Let us now solve for the metric $g_{ij}(u^2, x)$ in a power series in u. This gives,

$$g_{ij}(u^2, x) = g_{ij}^{(0}(x) + u^2 g_{ij}^{(2)}(x) + u^4 \left[g^{(4)}(x) + \log(u^2) h^{(4)}(x) \right] + \cdots \tag{5.7}$$

This is then plugged in Einstein's equations in order to solve for $g_{ij}^{(2n)}$ recursively. One finds, for example [13][2]

$$g_{ij}^{(2)} = -\frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R g_{ij}^{(0)} \right). \tag{5.8}$$

The resulting metric is finally plugged in the gravitational action for the AdS metric,

$$I_{\mathrm{bulk}} + I_{\mathrm{boundary}} = -\frac{1}{16\pi G_N} \int_M \mathrm{d}^{n+1}x \sqrt{G} \left(R + \frac{n(n-1)}{L^2} \right) - \frac{1}{8\pi G_N} \int_{\partial M} K |\gamma|^{1/2} \mathrm{d}^n x \tag{5.9}$$

To calculate the boundary term, we consider the normal vector to the hypersurfaces of constant u,

$$n^u = -\frac{u}{L}. \tag{5.10}$$

[2]The sign in the curvature is opposite to the conventions in [13], which give a positive curvature to AdS.

The minus sign is due to the fact that the boundary is at $u = 0$, so that the normal vector points towards the origin. The induced metric is

$$\gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j = \frac{L^2}{u^2} g_{ij}(u^2, x) \mathrm{d}x^i \mathrm{d}x^j, \tag{5.11}$$

with element of volume

$$\sqrt{\gamma} = \left(\frac{L}{u}\right)^d \sqrt{g} \tag{5.12}$$

The intrinsic curvature of the hypersurface at constant u is then

$$\sqrt{\gamma} K = \mathcal{L}_n \sqrt{\gamma} = -\frac{u}{L} \partial_u \left[\left(\frac{L}{u}\right)^n \sqrt{g} \right] = \frac{n L^{n-1}}{u^n} \left(1 - \frac{1}{n} u \partial_u\right) \sqrt{g}. \tag{5.13}$$

We then find,

$$I_\epsilon = \frac{n L^{n-1}}{8\pi G_N} \int \mathrm{d}^n x \int_\epsilon \frac{\mathrm{d}u}{u^{n+1}} \sqrt{g} - \frac{n L^{n-1}}{8\pi G_N \epsilon^n} \int \mathrm{d}^n x \left(1 - \frac{1}{n} u \partial_u\right) \sqrt{g} \Big|_{u=\epsilon}. \tag{5.14}$$

The singularity structure of this regulated action is [13]

$$I_\epsilon = \frac{L^{n-1}}{16\pi G_N} \int \mathrm{d}^n x \sqrt{g^{(0)}} \left(\epsilon^{-n} a_{(0)} + \epsilon^{-n/2+1} a_{(2)} + \cdots + \epsilon^{-2} a_{(n-2)} - \log(\epsilon) a_n \right) + \mathcal{O}(\epsilon^0) \tag{5.15}$$

Let us now calculate the first two coefficients (the next two are computed in [13]). We first expand,

$$\begin{aligned} \det g &= \det\left(g^{(0)} + u^2 g^{(2)} + \cdots\right) = \exp \mathrm{Tr} \left\{ \log g^{(0)} + u^2 g^{(0)-1} g^{(2)} + \cdots \right\} \\ &= \det g^{(0)} \left(1 + u^2 \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots\right), \end{aligned} \tag{5.16}$$

so that

$$\begin{aligned} \sqrt{g(u^2, x)} &= \sqrt{g^{(0)}} \left(1 + \frac{u^2}{2} \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots\right), \\ \left(1 - \frac{1}{n} u \partial_u\right) \sqrt{g(u^2, x)} &= \sqrt{g^{(0)}} \left(1 + \frac{n-2}{2n} u^2 \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots\right). \end{aligned} \tag{5.17}$$

The first term (coming from the bulk Einstein–Hilbert action) gives

$$\frac{n L^{n-1}}{8\pi G_N} \int \mathrm{d}^n x \sqrt{g^{(0)}} \left[\frac{1}{n \epsilon^n} + \frac{1}{2(n-2)\epsilon^{n-2}} \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots \right] \tag{5.18}$$

while the Gibbons–Hawking term gives

$$-\frac{n L^{n-1}}{8\pi G_N} \int \mathrm{d}^n x \sqrt{g^{(0)}} \left[\frac{1}{\epsilon^n} + \frac{n-2}{2n\epsilon^{n-2}} \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots \right] \tag{5.19}$$

In total, we find

$$\frac{L^{n-1}}{16\pi G_N} \int \mathrm{d}^n x \sqrt{g^{(0)}} \left[\frac{2(1-n)}{\epsilon^n} - \frac{n^2 - 5n + 4}{(n-2)\epsilon^{n-2}} \mathrm{Tr}\left(g^{(0)-1} g^{(2)}\right) + \cdots \right] \tag{5.20}$$

and we deduce,

$$
\begin{aligned}
a_{(0)} &= 2(1-n), \\
a_{(2)} &= -\frac{(n-4)(n-1)}{n-2}\mathrm{Tr}\left(g^{(0)-1}g^{(2)}\right), \quad n \neq 2.
\end{aligned}
\tag{5.21}
$$

The counterterm action is just minus the divergent part of I_ϵ,

$$
I_{\mathrm{ct}} = \frac{L^{n-1}}{16\pi G_N}\int \mathrm{d}^n x\, \sqrt{g^{(0)}}\left[\frac{2(n-1)}{\epsilon^n} + \frac{(n-4)(n-1)}{(n-2)\epsilon^{n-2}}\mathrm{Tr}\left(g^{(0)-1}g^{(2)}\right) + \cdots\right]. \tag{5.22}
$$

We should re-write this in terms of the induced metric in the boundary (5.11), evaluated at $u = \epsilon$. From (5.17) we deduce

$$
\sqrt{g^{(0)}} = \left(\frac{\epsilon}{L}\right)^n \left(1 - \frac{\epsilon^2}{2}\mathrm{Tr}\left(g^{(0)-1}g^{(2)}\right) + \mathcal{O}(\epsilon^4)\right)\sqrt{\gamma}. \tag{5.23}
$$

On the other hand, from (5.8) we obtain

$$
\begin{aligned}
\mathrm{Tr}\left(g^{(0)-1}g^{(2)}\right) &= -\frac{1}{n-2}\left(g^{(0)}_{ij}R^{ij} - \frac{1}{2(n-1)}R[g^{(0)}]g^{(0)}_{ij}g^{(0)ij}\right) = -\frac{1}{2(n-1)}R[g^{(0)}] \\
&= -\frac{L^2}{2(n-1)\epsilon^2}R[\gamma] + \cdots
\end{aligned}
\tag{5.24}
$$

Plugging these into the counterterm action we find

$$
\begin{aligned}
I_{\mathrm{ct}} &= \frac{1}{16\pi G_N L}\int \mathrm{d}^n x\, \sqrt{\gamma}\left(1 + \frac{L^2}{4(n-1)}R[\gamma] + \cdots\right)\left[2(n-1) - \frac{n-4}{2(n-2)}L^2 R[\gamma] + \cdots\right] \\
&= \frac{1}{8\pi G_N}\int \mathrm{d}^n x\, \sqrt{\gamma}\left(\frac{n-1}{L} + \frac{L}{2(n-2)}R[\gamma] + \cdots\right),
\end{aligned}
\tag{5.25}
$$

which is the result written down in [16] (for Euclidean signature). This is the counterterm action which is relevant for AdS in four and five dimensions, and the dots denote higher order counterterms (in the Riemann tensor) which are needed for higher dimensional spaces [7, 16, 13].

5.2 An example: Casimir energy

Let us consider an n-dimensional CFT on the manifold $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, with periodic boundary conditions for the fermions. Let us suppose that we want to compute the supersymmetric partition function on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, where the circle has length β. This is

$$
Z\left(\mathbb{S}^1 \times \mathbb{S}^{n-1}\right) = \mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H(\mathbb{S}^{n-1})} = \mathrm{e}^{-\beta E_0} + \cdots \tag{5.26}
$$

where E_0 is the energy of the ground state, i.e. the Casimir energy on $\mathbb{S}^{n-1}$. In the SUGRA approximation, this can be computed by evaluating the regularized gravity action I [41]

$$
Z\left(\mathbb{S}^1 \times \mathbb{S}^{n-1}\right) = \mathrm{e}^{-I}. \tag{5.27}
$$

In order to calculate I, we need a Euclidean AdS metric which is asymptotic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ at the boundary. This metric can be written as [16]

$$
\mathrm{d}s^2 = \left(1 + \frac{r^2}{L^2}\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 + r^2/L^2} + r^2 \mathrm{d}\Omega^2_{n-1} \tag{5.28}
$$

where the boundary is at $r \to \infty$. Here, $\mathrm{d}\Omega_n^2$ is the metric on the unit n-sphere, and τ has length β. Let us compute the gravitational action for this theory, with a cutoff at the boundary ∂M located at r. At the end of the calculation we will take $r \to \infty$. Evaluated on the metric (5.28) we find

$$\begin{aligned} 8\pi G_N I_{\text{bulk}} &= \frac{1}{L^2} r^n \text{vol}(\mathbb{S}^{n-1})\beta, \\ 8\pi G_N I_{\text{surf}} &= -\text{vol}(\mathbb{S}^{n-1})\beta r^{n-1} \left\{ \frac{n-1}{r} V(r) + \frac{1}{2} V'(r) \right\}, \\ 8\pi G_N I_{\text{ct}} &= -\text{vol}(\mathbb{S}^{n-1})\beta r^{n-1} \left\{ \frac{n-1}{L} + \frac{(n-1)L}{2r^2} \right\} V^{1/2}(r), \end{aligned} \tag{5.29}$$

where

$$V(r) = 1 + \frac{r^2}{L^2}. \tag{5.30}$$

Plugging the expression for $V(r)$ above, we find

$$\begin{aligned} 8\pi G_N I_{\text{surf}} &= -\text{vol}(\mathbb{S}^{n-1})\beta r^n \left\{ \frac{n-1}{r^2} + \frac{n}{L^2} \right\}, \\ 8\pi G_N I_{\text{ct}} &= -\text{vol}(\mathbb{S}^{n-1})\beta r^n \left\{ \frac{n-1}{L^2} + \frac{n-1}{2r^2} \right\} \left(1 + \frac{L^2}{r^2}\right)^{1/2} \\ &= -\text{vol}(\mathbb{S}^{n-1})\beta r^n \left\{ \frac{n-1}{L^2} + \frac{n-1}{2r^2} \right\} \left(1 + \frac{L^2}{2r^2} - \frac{L^4}{8r^4} + \cdots \right) \end{aligned} \tag{5.31}$$

We then find, in total,

$$\begin{aligned} &8\pi G_N I \\ &= \frac{\text{vol}(\mathbb{S}^{n-1})\beta}{L^2} r^{n-1} \left[-\frac{(n-1)L^2}{r}\left(1 + \frac{r^2}{L^2}\right) + (n-1)r\left(1 + \frac{L^2}{2r^2}\right)\left(1 + \frac{L^2}{r^2}\right)^{1/2} \right] \end{aligned} \tag{5.32}$$

Expanding for $r \to \infty$ we find, for $n = 3$, a vanishing action, while for $n = 4$ (i.e. AdS_5) we find

$$8\pi G_N I = \frac{3\text{vol}(\mathbb{S}^3)\beta L^2}{8} \Rightarrow I = \frac{3\pi L^2 \beta}{32 G_N} \tag{5.33}$$

Let us suppose that we want to compute the supersymmetric partition function on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, where the circle has length β. This is

$$Z\left(\mathbb{S}^1 \times \mathbb{S}^{n-1}\right) = \text{Tr}(-1)^F \mathrm{e}^{-\beta H(\mathbb{S}^{n-1})} = \mathrm{e}^{-\beta E_0} + \cdots \tag{5.34}$$

where E_0 is the energy of the ground state, i.e. the Casimir energy on $\mathbb{S}^{n-1}$. In the SUGRA approximation, this is just

$$Z\left(\mathbb{S}^1 \times \mathbb{S}^{n-1}\right) = \mathrm{e}^{-I} = \exp\left(-\frac{3\pi L^2 \beta}{32 G_N}\right) \tag{5.35}$$

and in this approximation the Casimir energy is

$$E_0 = \frac{3\pi L^2}{32 G_N} \tag{5.36}$$

Let us now compute the Casimir energy directly in QFT at weak coupling. A massless scalar field on $\mathbb{R}\times\mathbb{S}^3$ satisfies the equation

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu+\xi R\right)\phi=0, \tag{5.37}$$

where ξ is a general coupling to curvature. For a conformally coupled scalar in 4d, we have

$$\xi=\frac{1}{6}. \tag{5.38}$$

Let $\eta,\mathbf{x}$ be coordinates for $\mathbb{R}\times\mathbb{S}^3$. The metric can be written as

$$\mathrm{d}s^2=L^2\left(\mathrm{d}\eta^2-\mathrm{d}\Omega_3^2\right) \tag{5.39}$$

where L is the radius of $\mathbb{S}^3$, and $\mathrm{d}\omega_3^2$ is the element of volume on an $\mathbb{S}^3$ of unit radius. The coordinate η (which is dimensionless) is called the conformal time parameter (see [9], p. 120), and it is related to the time coordinate by

$$t=L\eta. \tag{5.40}$$

We write the wavefunctions in factorized form

$$u_m^n(\eta,\mathbf{x})=\chi_m(\eta)\phi_m^n(\mathbf{x}), \tag{5.41}$$

where $\phi_j(x)$ is a spherical harmonic, i.e. an eigenfunction of the Laplacian on $\mathbb{S}^3$,

$$-\nabla^2_{\mathbb{S}^3}\phi_m^n=\left(m^2-1\right)\phi_m^n, \qquad n=1,\cdots,m^2, \quad m=1,2,\cdots. \tag{5.42}$$

The function u_m^n satisfies the wave equation (5.37), which after separation of variables reads

$$\left[\frac{1}{L^2}\left(\partial_\eta^2-\nabla^2_{\mathbb{S}^3}\right)+\xi R\right]\chi_m(\eta)\phi_m^n=0 \tag{5.43}$$

which leads to the following equation for $\chi_m(\eta)$:

$$\partial_\eta^2\chi_m+\left(m^2-1+L^2\xi R\right)\chi_m=0. \tag{5.44}$$

Now, R is here just the curvature of the sphere, therefore

$$R=\frac{6}{L^2}, \tag{5.45}$$

and for a conformally coupled scalar the equation reads

$$\partial_\eta^2\chi_m+m^2\chi_m=0 \tag{5.46}$$

with the solution

$$\chi_m(\eta)\propto \mathrm{e}^{-\mathrm{i}\omega_m\eta}, \qquad \omega_m=m. \tag{5.47}$$

We can now calculate the Casimir energy of this system by zeta-function regularization

$$E=\frac{1}{2}\sum_{m=1}^{\infty}m^2\cdot(m/L). \tag{5.48}$$

To understand the factor of L, notice that the frequencies are usuallt defined w.r.t. to the time-depending phase

$$\mathrm{e}^{-\mathrm{i}\hat{\omega}_m t} = \mathrm{e}^{-\mathrm{i}L\hat{\omega}_m \eta} \Rightarrow \hat{\omega}_m = \frac{\omega_m}{L}. \tag{5.49}$$

The above sum is of course divergent, but we can regularize it with a zeta-function regularization to obtain

$$E(s) = \frac{1}{2L}\sum_{m=1}^{\infty} m^2 m^{-s} \tag{5.50}$$

which has then to be continued to $s = -1$. We obtain in this way

$$E(s) = \frac{1}{2L}\zeta(s-2) \tag{5.51}$$

and

$$E = E(-1) = \frac{1}{2L}\zeta(-3) = \frac{1}{240L}. \tag{5.52}$$

For Weyl spinors, the Casimir energy is obtained by summing over the modes of the spinor spherical harmonics, and with a negative sign due Fermi statistics, i.e.

$$E_{\text{spinor}} = -\frac{1}{2L}\sum_{m=1}^{\infty} 2m(m+1)(m+1/2), \tag{5.53}$$

where $2m(m+1)$ is the degeneracy of the eigenvalue $m+1/2$ of the Dirac operator. This can be again regularized by considering

$$E_{\text{spinor}}(s) = -\frac{1}{L}\sum_{n=1}^{\infty}\frac{n(n+1)}{\left(n+\frac{1}{2}\right)^s} \tag{5.54}$$

and analytically continuing it for $s = -1$. Since

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{\left(n+\frac{1}{2}\right)^s} = \left(2^{s-2}-1\right)\zeta(s-2) - \frac{1}{4}\left(2^s-1\right)\zeta(s) \tag{5.55}$$

we find

$$E_{\text{spinor}}(-1) = -\frac{1}{L}\left(\left(2^{-3}-1\right)\zeta(-3) - \frac{1}{4}\left(2^{-1}-1\right)\zeta(-1)\right) = \frac{17}{960L}. \tag{5.56}$$

Finally, for a gauge field, we have

$$E_{\text{gauge}} = \frac{1}{2L}\sum_{n=1}^{\infty} 2n(n+2)(n+1), \tag{5.57}$$

where $n+1$ is the square root of the energies of the modes (i.e. the square root of the eigenvalues of the Laplacian). This is regularized as

$$E_{\text{gauge}}(s) = \frac{1}{L}\sum_{n=1}^{\infty}\frac{n(n+2)}{(n+1)^s} = \frac{1}{L}\sum_{m=1}^{\infty}\frac{m^2-1}{m^s} = \left(\zeta(s-2)-\zeta(s)\right), \tag{5.58}$$

and one finds

$$E_{\text{gauge}}(-1) = \frac{1}{L}\left(\zeta(-3) - 2\zeta(-1)\right) = \frac{11}{120L}. \tag{5.59}$$

It follows that the Casimir energy on $\mathbb{S}^3 \times \mathbb{R}$ for a QFT with n_0 conformally coupled real scalars, $n_{1/2}$ Weyl spinors and n_1 vector fields is

$$E = \frac{1}{960L} \left(4n_0 + 17n_{1/2} + 88n_1\right). \tag{5.60}$$

In the case of $\mathcal{N} = 4$ SYM with gauge group $U(N)$ we have

$$n_0 = 6N^2, \quad n_{1/2} = 4N^2, \quad n_1 = N^2, \tag{5.61}$$

and we obtain

$$E = \frac{3N^2}{16L}. \tag{5.62}$$

The Casimir energy computed in gravity is (5.36), which agrees with the result above after using the dictionary

$$N^2 = \frac{\pi L^3}{2G_N}. \tag{5.63}$$

5.3 Example: free energy in AdS$_4$

We are interested in studying CFTs on $\mathbb{S}^n$. Therefore, in the AdS dual we need the Euclidean version of the AdS metric with that boundary, which can be written as [16]

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 + r^2/L^2} + r^2 \mathrm{d}\Omega_n^2 \tag{5.64}$$

with the notations of the previous subsection. The boundary is again at $r \to \infty$. This metric can be also written as [41, 16]

$$\mathrm{d}s^2 = \ell^2 \mathrm{d}\rho^2 + \sinh^2(\rho) \mathrm{d}\Omega_n^2, \tag{5.65}$$

5.4 AdS duals

The AdS duals to the theories we will consider are given by M-theory on the manifold

$$\mathrm{AdS}_4 \times X_7, \tag{5.66}$$

where X_7 is a seven-dimensional manifold. In the case of ABJM theory,

$$X_7 = \mathbb{S}^7/\mathbb{Z}_k. \tag{5.67}$$

The eleven-dimensional metric and four-form flux are given by

$$\begin{aligned} \mathrm{d}s_{11}^2 &= L_{X_7}^2 \left(\frac{1}{4} \mathrm{d}s_{\mathrm{AdS}_4}^2 + \mathrm{d}s_{X_7}^2 \right), \\ G &= \frac{3}{8} L_{X_7}^3 \omega_{\mathrm{AdS}_4}, \end{aligned} \tag{5.68}$$

where ω_{AdS_4} is the volume form with unit radius. The radius L_{X_7} is determined by the flux quantization condition

$$(2\pi \ell_p)^6 Q = \int_{C_7} \star_{11} G = 6 L_{X_7}^6 \mathrm{vol}(X_7). \tag{5.69}$$

In this equation, ℓ_p is the eleven-dimensional Planck length, and C_7 is a cycle enclosing the brane and homologous to X_7. The charge Q is given, at large radius, by the number of M2 branes N, but it receives corrections [8, 5]. In ABJM theory we have

$$Q = N - \frac{1}{24}\left(k - \frac{1}{k}\right). \tag{5.70}$$

This extra term comes from the coupling

$$\int C_3 \wedge I_8 \tag{5.71}$$

in M-theory, where I_8 is proportional to the Euler density in eight dimensions, and satisfies

$$\int_{M_8} I_8 = -\frac{\chi}{24} \tag{5.72}$$

where M_8 is a compact eight-manifold. In ABJM theory, the relevant eight-manifold is $\mathbb{C}^4/\mathbb{Z}_k$, with regularized Euler characteristic

$$\chi\left(\mathbb{C}^4/\mathbb{Z}_k\right) = k - \frac{1}{k}. \tag{5.73}$$

This leads to the shift in (5.70).

One final ingredient that we will need is Newton's constant in four dimensions. It can be obtained by standard compactification of the Einstein–Hilbert action in eleven dimensions, and one finds

$$\frac{1}{G_4} = \frac{2\sqrt{6}\pi^2 Q^{3/2}}{9\sqrt{\mathrm{vol}(X_7)}}. \tag{5.74}$$

It follows that the regularized gravitational action in these backgrounds is of the form

$$I_{X_7} = \frac{\pi}{2G_4} = Q^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{vol}(X_7)}} \tag{5.75}$$

In particular, for ABJM theory we have

$$I_{\mathbb{S}^7/\mathbb{Z}_k} = \frac{\pi\sqrt{2}}{3}k^{1/2}Q^{3/2}. \tag{5.76}$$

6. Localization

6.1 A simple example of localization

In order to introduce the idea and the techniques of localization, it is useful to look at a very simple example which has a purely geometric interpretation, namely the Poincaré–Hopf theorem. This example has been worked out in some detail in many references, like for example [10, 12, 29]. We will follow the last one, see [30] as well.

Let X be a Riemannian manifold of dimension n, with metric $g_{\mu\nu}$, vierbein e^a_μ, and a vector field V_μ. We will consider the following "supercoordinates" on the tangent bundle TX

$$\left(x^\mu, \psi^\mu\right), \qquad \left(\bar{\psi}_\mu, B_\mu\right), \tag{6.1}$$

where the first doublet represents coordinates on the base X, and the second doublet represents supercoordinates on the fiber. ψ^μ and $\bar{\psi}_\mu$ are Grasmann variables. The above supercoordinates are related by the Grasmannian symmetry

$$\begin{aligned} \delta x^\mu &= \psi^\mu, & \delta\bar{\psi}_\mu &= B_\mu, \\ \delta\psi^\mu &= 0, & \delta B_\mu &= 0. \end{aligned} \tag{6.2}$$

With these fields we construct the "action"

$$S = \frac{1}{2} g_{\mu\nu}(B^\mu B^\nu + 2\mathrm{i}tB^\mu V^\nu) - \frac{1}{4} R^{\rho\sigma}{}_{\mu\nu}\bar{\psi}_\rho\bar{\psi}_\sigma\psi^\mu\psi^\nu - \mathrm{i}t\nabla_\mu V^\nu\psi^\mu\bar{\psi}_\nu, \tag{6.3}$$

and we define the partition function of the theory as

$$Z(X) = \frac{1}{(2\pi)^n}\int_X \mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}B\,\mathrm{e}^{-S} \tag{6.4}$$

It is very easy to see that the full action is δ-exact,

$$S = \delta\Psi, \qquad \Psi = \frac{1}{2}\bar{\psi}_\mu\left(B^\mu + 2\mathrm{i}tV^\mu + \Gamma^\sigma_{\tau\nu}\bar{\psi}_\sigma\psi^\nu g^{\mu\tau}\right). \tag{6.5}$$

In particular, the partition function should be independent of t, and we can evaluate it in different regimes: $t \to 0$ or $t \to \infty$. The calculation when $t = 0$ is very easy, since we just have

$$Z(X) = \frac{1}{(2\pi)^n}\int_X \mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\,\mathrm{d}B\,\mathrm{e}^{-\frac{1}{2}g_{\mu\nu}B^\mu B^\nu + \frac{1}{4}R^{\rho\sigma}{}_{\mu\nu}\bar{\psi}_\rho\bar{\psi}_\sigma\psi^\mu\psi^\nu}. \tag{6.6}$$

The integral over B gives

$$\frac{(2\pi)^{n/2}}{\sqrt{g}}, \tag{6.7}$$

and we can define orthonormal coordinates on the fiber by using the inverse vierbein,

$$\chi_a = E^\mu_a\bar{\psi}_\mu, \tag{6.8}$$

so that the integral reads

$$Z(X) = \frac{1}{(2\pi)^{n/2}}\int_X \mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\chi\,\mathrm{e}^{\frac{1}{4}R^{ab}{}_{\mu\nu}\chi_a\chi_b\psi^\mu\psi^\nu} = \frac{1}{(2\pi)^{n/2}}\int_X \mathrm{d}x\,\mathrm{d}\psi\,\mathrm{Pf}(R), \tag{6.9}$$

where we have integrated over the Grassmann variables χ_a to obtain the Pfaffian of the matrix R^{ab}. The resulting top form in the integrand,

$$e(X) = \frac{1}{(2\pi)^{n/2}}\mathrm{Pf}(R) \tag{6.10}$$

is nothing but the Chern–Weil representative of the Euler class, therefore the evaluation at $t = 0$ produces the Euler characteristic of X,

$$Z(X) = \chi(X). \tag{6.11}$$

Let us now calculate the integral in the limit $t \to \infty$. After integrating out B again, we find

$$Z(X) = \frac{1}{(2\pi)^{n/2}}\int_X \mathrm{d}x\,\mathrm{d}\psi\,\mathrm{d}\chi\,\mathrm{e}^{-\frac{t^2}{2}g_{\mu\nu}V^\mu V^\nu + \frac{1}{4}R^{ab}{}_{\mu\nu}\chi_a\chi_b\psi^\mu\psi^\nu + \mathrm{i}t\nabla_\mu V^\nu\psi^\mu e^a_\nu\chi_a}. \tag{6.12}$$

We will now assume that V^μ has isolated, simple zeroes p_k where $V^\mu(p_k) = 0$. These are the saddle–points of the "path integral," so we can write $Z(X)$ as a sum over saddle–points p_k, and for each saddle–point we have to perform a perturbative expansion. Let ξ^μ be normal coordinates around the point p_k. We have the expansion,

$$V^\mu(x) = \sum_{n\geq 1} \frac{1}{n!} \partial_{\mu_1} \cdots \partial_{\mu_n} V^\mu(p_k) \xi^{\mu_1} \cdots \xi^{\mu_n} \tag{6.13}$$

After rescaling the variables as

$$\xi \to t^{-1}\xi, \qquad \psi \to t^{-1/2}\psi, \qquad \chi \to \gamma^{-1/2}\chi, \tag{6.14}$$

in such a way that the measure remains invariant, we see that in the limit $t \to \infty$ the theory becomes Gaussian,

$$Z(X) = \sum_{p_k} \frac{1}{(2\pi)^{n/2}} \int_X \mathrm{d}x\, \mathrm{d}\psi\, \mathrm{d}\chi\, \mathrm{e}^{-\frac{1}{2} g_{\mu\nu} H^{(k)\mu}_\alpha H^{(k)\nu}_\beta \xi^\alpha \xi^\beta + \mathrm{i} H^{(k)\nu}_\mu \psi^\mu e^a_\nu \chi_a} \tag{6.15}$$

where

$$H^{(k)\mu}_\sigma = \partial_\sigma V^\mu \big|_{p_k} \tag{6.16}$$

The integral can now be computed as a sum over zeroes of the vector field,

$$Z(X) = \sum_{p_k} \frac{1}{\sqrt{g} |\det\, H^{(k)}|} \frac{\det(e^a_\mu)}{(2m)!} (2m)! \det\, H^{(k)} = \sum_{p_k} \frac{\det\, H^{(k)}}{|\det\, H^{(k)}|}. \tag{6.17}$$

6.2 Localization in the gauge sector

To localize in the gauge sector, we add to the CS-matter theory the term

$$-tS_{\mathrm{YM}} \tag{6.18}$$

which is a total superderivative, and its bosonic part is positive definite. By the standard localization argument, the partition function of the theory (as well as any δ_ϵ-invariant operator) does not depend on t, and we can take $t \to \infty$. This forces the fields to take the values that make the bosonic part of (2.30) to vanish. Since this is a sum of positive definite terms, they have to vanish separately. We then have the localizing locus,

$$F_{\mu\mu} = 0, \qquad D_\mu \sigma = 0, \qquad D + \frac{\sigma}{r} = 0. \tag{6.19}$$

The first equation says that the gauge connection A_μ must be flat, but since we are on $\mathbb{S}^3$ the only flat connection is $A_\mu = 0$. Plugging this into the second equation, we obtain

$$\partial_\mu \sigma = 0 \Rightarrow \sigma = \sigma_0, \tag{6.20}$$

a constant. Finally, we have

$$D = -\frac{\sigma_0}{r}. \tag{6.21}$$

An equivalent analysis can be done by requiring $\delta_\epsilon \lambda = 0$. Once this is done, we have to consider the one-loop fluctuations around this configuration. We have to pick a gauge fixing to proceed with the calculation, and we will choose the standard Lorentz gauge as in the case of Chern–Simons theory. The path integral to be calculated to one-loop is

$$\frac{1}{\mathrm{Vol}(\mathcal{H}_c)}(\det{}' \Delta^0)^{-\frac{1}{2}} \int_{\mathrm{Ker\,(d)}^\dagger} \mathcal{D}A \int_{(\mathrm{Ker\,d})^\perp} \mathcal{D}C\mathcal{D}\overline{C}\, \mathrm{e}^{-tS_{\mathrm{YM}}(A)+S_{\mathrm{ghosts}}(C,\overline{C},A)}, \tag{6.22}$$

where $C, \overline{C}$ are ghosts fields, which can be integrated out immediately to obtain

$$\frac{1}{\mathrm{Vol}(\mathcal{H}_c)}(\det{}' \Delta^0)^{\frac{1}{2}} \int_{\mathrm{Ker\,(d)}^\dagger} \mathcal{D}A \int_{(\mathrm{Ker\,d})^\perp} \mathcal{D}C\mathcal{D}\overline{C}\, \mathrm{e}^{-tS_{\mathrm{YM}}(A)}. \tag{6.23}$$

We have to expand the YM action around the localizing locus, so we set

$$\begin{aligned} \sigma &= \sigma_0 + \frac{1}{\sqrt{t}}\sigma' \\ D &= -\frac{\sigma_0}{r} + \frac{1}{\sqrt{t}}D' \\ A,\, \lambda &\to \frac{1}{\sqrt{t}}A,\, \frac{1}{\sqrt{t}}\lambda, \end{aligned} \tag{6.24}$$

where the factors of t are such that we remove the overall factor of t in the action. In this way we obtain, up to quadratic order in the fluctuations,

$$\frac{1}{2}\int \sqrt{g}\mathrm{d}^3x\, \mathrm{Tr}\left(-A^\mu \Delta A_\mu - [A_\mu, \sigma_0]^2 + \partial_\mu\sigma'\partial^\mu\sigma' + (D'+\sigma')^2 + \mathrm{i}\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + \mathrm{i}\bar{\lambda}[\sigma_0,\lambda] - \frac{1}{2r}\bar{\lambda}\lambda\right) \tag{6.25}$$

The integral over the fluctuation D' is Gaussian and it can be done immediately. It just eliminates the term in $(D'+\sigma')^2$. The integral over σ' gives

$$(\det{}' \Delta^0)^{-\frac{1}{2}} \tag{6.26}$$

which cancels the term in (6.23). Before proceeding, we just note that due to gauge invariance we can diagonalize σ_0 so that it takes values in the Cartan subalgebra. But we can also decompose A_μ as:

$$A_\mu = \sum_\alpha {A_\mu}^\alpha X_\alpha + h_\mu \tag{6.27}$$

where the sum is over the roots α of G, X_α are representatives of the root spaces of G, normalized as

$$\mathrm{Tr}(X_\alpha X_\beta) = \delta_{\alpha+\beta} \tag{6.28}$$

Finally, h_μ is the component of A_μ along the Carta subalgebra. Notice that this part of A_μ will only contribute a σ_0-independent factor to the one-loop determinant, so we will ignore it for the moment being. Then we can write:

$$[\sigma_0, A_\mu] = \sum_\alpha \alpha(\sigma_0){A_\mu}^\alpha X_\alpha \tag{6.29}$$

and similarly for λ. Plugging this into the action, we can now write it in terms of ordinary (as opposed to matrix valued) vectors and spinors

$$\frac{1}{2}\int \sqrt{g}\mathrm{d}^3x \sum_\alpha \left({A^\mu}_{-\alpha}\left(-\Delta + \alpha(\sigma_o)^2\right)A_{\mu\,\alpha} - \bar{\lambda}_{-\alpha}\left(\mathrm{i}\gamma^\mu\nabla_\mu + \mathrm{i}\alpha(\sigma_0) - \frac{1}{2r}\right)\lambda_\alpha\right) \tag{6.30}$$

Notice that, due to the rescaling of the A field, at large t the Chern–Simons term for A does not contribute to the quadratic Lagrangian. We then have to calculate the determinants of the above operators. Notice that the integration over the fluctuations of the gauge field is restricted, as in the Chern–Simons case, to the vector spherical harmonics. Using the results (4.17), (4.18), we find that the bosonic part of the determinant is:

$$\det(\text{bosons}) = \prod_{\alpha}\prod_{n=1}^{\infty}\left((n+1)^2+\alpha(\sigma_0)^2\right)^{2n(n+2)} \tag{6.31}$$

For the gaugino, we can use (B.44) to write the fermion determinant as:

$$\det(\text{fermions}) = \prod_{\alpha}\prod_{n=1}^{\infty}\Big((n+\mathrm{i}\alpha(\sigma_0))(-n-1+\mathrm{i}\alpha(\sigma_0))\Big)^{n(n+1)}, \tag{6.32}$$

and the quotient gives

$$\begin{aligned} Z^{\text{gauge}}_{\text{1-loop}}[\sigma_0] &= \prod_{\alpha}\prod_{n=1}^{\infty}\frac{(n+\mathrm{i}\alpha(\sigma_0))^{n(n+1)}(-n-1+\mathrm{i}\alpha(\sigma_0))^{n(n+1)}}{((n+1)^2+\alpha(\sigma_0)^2)^{n(n+2)}} \\ &= \prod_{\alpha}\prod_{n=1}^{\infty}\frac{(n+\mathrm{i}\alpha(\sigma_0))^{n(n+1)}(-n-1+\mathrm{i}\alpha(\sigma_0))^{n(n+1)}}{(n+\mathrm{i}\alpha(\sigma_0))^{(n-1)(n+1)}(n+1-i\alpha(\sigma_0))^{n(n+2)}} \end{aligned} \tag{6.33}$$

We see there is partial cancellation between the numerator and the denominator, and this becomes:

$$\begin{aligned} Z^{\text{gauge}}_{\text{1-loop}}[\sigma_0] &= \prod_{\alpha}\prod_{n=1}^{\infty}\frac{(n+\mathrm{i}\alpha(\sigma_0))^{n+1}}{(n-\mathrm{i}\alpha(\sigma_0))^{n-1}} = \prod_{\alpha>0}\prod_{n=1}^{\infty}\frac{(n^2+\alpha(\sigma_0)^2)^{n+1}}{(n^2+\alpha(\sigma_0)^2)^{n-1}} \\ &= \prod_{\alpha>0}\prod_{n=1}^{\infty}(n^2+\alpha(\sigma_0)^2)^2, \end{aligned} \tag{6.34}$$

where we use the fact that the roots split into positive roots $\alpha>0$ and negative roots $-\alpha$, $\alpha>0$. We finally obtain

$$Z^{\text{gauge}}_{\text{1-loop}}[\sigma_0] = \left(\prod_{n=1}^{\infty}n^4\right)\prod_{\alpha>0}\prod_{n=1}^{\infty}\left(1+\frac{\alpha(\sigma_0)^2}{n^2}\right)^2 \tag{6.35}$$

We can now regularize this with the zeta function, which will lead to a finite result for the infinite product

$$\prod_{n=1}^{\infty}n^4. \tag{6.36}$$

On the other hand, we can use the well-known formula

$$\frac{\sinh(\pi z)}{\pi z} = \prod_{n=1}\left(1+\frac{z^2}{n^2}\right) \tag{6.37}$$

to write

$$Z^{\text{gauge}}_{\text{1-loop}}[\sigma_0] \propto \prod_{\alpha>0}\left(\frac{\sinh(\pi\alpha(\sigma_0))}{\pi\alpha(\sigma_0)}\right)^2, \tag{6.38}$$

where the proportionality factor is independent of σ_0.

6.3 Localization in the matter sector

The matter Lagrangian is in itself a total superderivative, so we can introduce a coupling t in the form

$$-tS_{\text{matter}}. \tag{6.39}$$

By the usual localization argument, the partition function is independent of t, and we can compute it for $t = 1$ (which is the original case) or for $t \to \infty$. We can also restrict this Lagrangian to the localization locus of the gauge sector. The matter kinetic terms are then

$$\begin{aligned}
\mathcal{L}_\phi &= g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi + \bar{\phi}\sigma_0^2\phi + \tfrac{2\mathrm{i}(\Delta-1)}{r}\bar{\phi}\sigma\phi + \tfrac{\Delta(2-\Delta)}{r^2}\bar{\phi}\phi, \\
\mathcal{L}_\psi &= -\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi + \mathrm{i}\bar{\psi}\sigma_0\psi - \tfrac{\Delta-2}{r}\bar{\psi}\psi.
\end{aligned} \tag{6.40}$$

The real part of the bosonic Lagrangian is definite positive, and it is minimized (and equal to zero) when

$$\phi = 0. \tag{6.41}$$

After using (B.11) and (B.39), we find that the operators governing the quadratic fluctuations around this fixed point are given by

$$\begin{aligned}
\Delta_\phi &= \frac{1}{r^2}\left\{4\mathbf{L}^2 - (\Delta - \mathrm{i}r\sigma_0)(\Delta - 2 - \mathrm{i}r\sigma_0)\right\}, \\
\Delta_\psi &= \frac{1}{r}\left\{4\mathbf{L}\cdot\mathbf{S} + \mathrm{i}r\sigma_0 + 2 - \Delta\right\}.
\end{aligned} \tag{6.42}$$

Their eigenvalues are, for the bosons,

$$\Delta_\phi = r^{-2}(n + 2 + \mathrm{i}r\sigma_0 - \Delta)(n - \mathrm{i}r\sigma_0 + \Delta), \qquad n = 0, 1, 2, \cdots, \tag{6.43}$$

with multiplicity $(n+1)^2$, and for the fermions

$$\Delta_\psi = r^{-1}(n + 1 + \mathrm{i}r\sigma_0 - \Delta), \quad r^{-1}(-n + \mathrm{i}r\sigma_0 - \Delta), \qquad n = 1, 2, \cdots, \tag{6.44}$$

with multiplicities $n(n+1)$. We finally obtain,

$$\frac{|\det\Delta_\psi|}{\det\Delta_\phi} = \prod_{m>0} \frac{(m + 1 + \mathrm{i}r\sigma_0 - \Delta)^{m(m+1)}(m - \mathrm{i}r\sigma_0 + \Delta)^{m(m+1)}}{(m + 1 + \mathrm{i}r\sigma_0 - \Delta)^{m^2}(m - 1 - \mathrm{i}r\sigma_0 + \Delta)^{m^2}} \tag{6.45}$$

and we conclude

$$Z^{\text{matter}}_{\text{1-loop}}[\sigma_0] = \prod_{m>0}\left(\frac{m + 1 - \Delta + \mathrm{i}r\sigma_0}{m - 1 + \Delta - \mathrm{i}r\sigma_0}\right)^m, \tag{6.46}$$

As a check, notice that, when $\Delta = 1/2$ and $\sigma_0 = 0$, we recover the quotient of determinants (4.44) of the free theory. This quantity can be easily computed by using ζ-function regularization [24]. Denote

$$z = 1 - \Delta + \mathrm{i}r\sigma_0, \qquad \ell(z) = \log Z^{\text{matter}}_{\text{1-loop}}[\sigma_0]. \tag{6.47}$$

Then

$$\partial_z\ell(z) = \sum_{m=1}^{\infty}\left(\frac{m}{m+z} + \frac{m}{m-z}\right), \tag{6.48}$$

which has a linear divergence. We can regularize this quantity as

$$\partial_z \ell(z) = -\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{m=1}^{\infty} \left(\frac{m}{(m+z)^s} + \frac{m}{(m-z)^s} \right). \tag{6.49}$$

On the other hand,

$$\sum_{m=1}^{\infty} \left(\frac{m}{(m+z)^s} + \frac{m}{(m-z)^s} \right) = \zeta_H(s-1,z) - z\zeta_H(s,z) - \zeta_H(s-1,-z) - z\zeta_H(s,-z), \tag{6.50}$$

where

$$\zeta_H(s,z) = \sum_{m=1}^{\infty} \frac{1}{(m+z)^s} \tag{6.51}$$

is the Hurwitz zeta function. Using standard properties of this function, one finally finds the regularized result

$$\partial_z \ell(z) = -\pi z \cot(\pi z), \tag{6.52}$$

which can be integrated to give

$$\ell(z) = -z \log\left(1 - \mathrm{e}^{2\pi \mathrm{i} z}\right) + \frac{\mathrm{i}}{2}\left(\pi z^2 + \frac{1}{\pi} \mathrm{Li}_2(\mathrm{e}^{2\pi \mathrm{i} z}) \right) - \frac{\mathrm{i}\pi}{12}, \tag{6.53}$$

where we have imposed the boundary condition coming from (4.43)

$$\ell\left(\frac{1}{2}\right) = -\frac{1}{2}\log 2. \tag{6.54}$$

There is an important property of $\ell(z)$, namely when $\Delta = 1/2$ (canonical dimension) one has

$$\frac{1}{2}\left(\ell(z) + \ell(z^*)\right) = -\frac{1}{2}\log\left(2\cosh(\pi r \sigma_0)\right). \tag{6.55}$$

To prove this, we write

$$z = \frac{1}{2} + \mathrm{i}\theta, \tag{6.56}$$

and we compute

$$\frac{1}{2}\left(\ell(z) + \ell(z^*)\right) = -\frac{1}{2}\log\left(2\cosh(\pi\theta)\right) + \frac{1}{2}\pi \mathrm{i}\theta^2 + \frac{\mathrm{i}\pi}{24} + \frac{\mathrm{i}}{4\pi}\left(\mathrm{Li}_2(-\mathrm{e}^{-2\pi\theta}) + \mathrm{Li}_2(-\mathrm{e}^{2\pi\theta}) \right). \tag{6.57}$$

After using the following property of the dilogarithm,

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2(-x^{-1}) = -\frac{\pi^2}{6} - \frac{1}{2}\left(\log(x)\right)^2, \tag{6.58}$$

we obtain (6.55).

When one has a self-conjugate representation, the set of eigenvalues of σ_0 is invariant under change of sign, therefore we can calculate the contribution of such a multiplet by using (6.55). We conclude that for such a matter multiplet,

$$Z^{\text{matter}}_{\text{1-loop}}[\sigma_0] = \prod_{\rho} \left(2\cosh(\pi r \rho(\sigma_0))\right)^{1/2}. \tag{6.59}$$

For general representations and anomalous dimensions, one has to use the more complicated result above for $\ell(z)$.

6.4 Wilson loops

The condition for invariance of a Wilson loop is

$$(\gamma_\mu \dot{x}^\mu - 1)\,\epsilon = 0. \tag{6.60}$$

The solution is

$$\gamma_\mu \dot{x}^\mu = \text{constant} \tag{6.61}$$

Notice that

$$\dot{x}^\mu = \sum_a c^a E_a^\mu \tag{6.62}$$

satisfies the above equation, since

$$\gamma_\mu \dot{x}^\mu = \sum_a \gamma_k e_\mu^k c^a E_a^\mu = \sum_a c^a \gamma_a \tag{6.63}$$

indeed a constant matrix.

We can then take the vector field

$$\dot{x}^\mu \frac{\partial}{\partial x^\mu} \tag{6.64}$$

to be parallel to

$$\ell_3 = E_3^\mu \frac{\partial}{\partial x^\mu}. \tag{6.65}$$

The integral curves of this vector field are "great circles" i.e. circles of radius R inside $\mathbb{S}^3$. We then find,

$$\gamma_\mu \dot{x}^\mu = \gamma_3, \tag{6.66}$$

and the condition on the spinor is simply

$$\gamma_3 \epsilon = \epsilon, \tag{6.67}$$

i.e.

$$\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6.68}$$

where we have normalized

$$\epsilon^\dagger \epsilon = 1 \tag{6.69}$$

6.5 The Chern–Simons matrix model

6.6 The ABJM matrix model

7. Matrix models at large N

8. Chern–Simons matrix models at large N

8.1 Solving the CS matrix model

The saddle-point equation at finite N is

$$\frac{1}{g_s} u_i = \sum_{j \neq i} \coth\left(\frac{u_i - u_j}{2}\right). \tag{8.1}$$

The form of the r.h.s. suggest to define the resolvent as

$$\omega(z) = g_s \left\langle \sum_{i=1}^{N} \coth\left(\frac{z-u_i}{2}\right) \right\rangle . \tag{8.2}$$

The large N limit of (8.1) gives then

$$z = \frac{1}{2}\left(\omega_0(z+\mathrm{i}\epsilon) + \omega_0(z-\mathrm{i}\epsilon)\right). \tag{8.3}$$

The planar resolvent has the boundary conditions

$$\omega_0(z) \sim \pm t, \quad z \to \pm\infty, \tag{8.4}$$

where

$$t = g_s N. \tag{8.5}$$

Let us define the exponentiated variable

$$Z = \mathrm{e}^{z}. \tag{8.6}$$

In terms of the Z variable, the resolvent is given by

$$\omega(z)\mathrm{d}z = -t\frac{\mathrm{d}Z}{Z} + 2g_s \left\langle \sum_{i=1}^{N} \frac{\mathrm{d}Z}{Z-\mathrm{e}^{\mu_i}} \right\rangle \tag{8.7}$$

and it has the following expansion as $Z \to \infty$

$$\omega(z) \to t + \frac{2g_s}{Z}\left\langle \sum_{i=1}^{N} \mathrm{e}^{\mu_i} \right\rangle + \cdots \tag{8.8}$$

From this resolvent it is possible to obtain the density of eigenvalues at the cuts. In the planar approximation, we have that

$$\omega_0(z) = -t + 2t\int_{\mathcal{C}} \rho(u)\frac{Z}{Z-\mathrm{e}^{u}}\mathrm{d}u, \tag{8.9}$$

where $\rho(u)$ is the density of eigenvalues, normalized in the standard way

$$\int_{\mathcal{C}} \rho(u)\mathrm{d}u = 1. \tag{8.10}$$

The standard discontinuity argument tells us that

$$\rho(X)\mathrm{d}X = -\frac{1}{4\pi \mathrm{i}t}\frac{\mathrm{d}X}{X}\left(\omega_0(X+\mathrm{i}\epsilon) - \omega_0(X-\mathrm{i}\epsilon)\right), \qquad X \in \mathcal{C} \tag{8.11}$$

Let us now solve explicitly for ω_0 by using analyticity arguments, following [21]. We construct the function

$$g(Z) = \mathrm{e}^{\omega_0/2} + Z\mathrm{e}^{-\omega_0/2}. \tag{8.12}$$

This function is regular everywhere on the complex plane. Indeed, we have

$$g(Z+\mathrm{i}\epsilon) = \mathrm{e}^{\omega_0(Z+\mathrm{i}\epsilon)/2} + Z\mathrm{e}^{-\omega_0(Z+\mathrm{i}\epsilon)/2} = Z\mathrm{e}^{-\omega_0(Z-\mathrm{i}\epsilon)/2} + \mathrm{e}^{-\omega_0(Z-\mathrm{i}\epsilon)/2} = g(Z-\mathrm{i}\epsilon), \tag{8.13}$$

so it has no branch cut. The boundary conditions for this function, inherited from (8.4), are

$$\lim_{Z\to\infty} g(Z) = \mathrm{e}^{-t/2}Z, \qquad \lim_{Z\to 0} g(Z) = \mathrm{e}^{-t/2}. \tag{8.14}$$

Analiticity and boundary conditions determine uniquely

$$g(Z) = \mathrm{e}^{-t/2}(Z+1), \tag{8.15}$$

and we can now regard (8.12) as a quadratic equation that determines ω as

$$\omega_0(Z) = 2\log\left[\frac{1}{2}\left(g(Z) - \sqrt{g^2(Z) - 4Z}\right)\right]. \tag{8.16}$$

From this resolvent we can determine immediately the density of eigenvalues,

$$\rho(x) = \frac{1}{\pi t}\tan^{-1}\left[\frac{\sqrt{\mathrm{e}^t - \cosh^2\left(\frac{x}{2}\right)}}{\cosh\left(\frac{x}{2}\right)}\right] \tag{8.17}$$

supported on the interval $[-A, A]$ with

$$A = 2\cosh^{-1}\left(\mathrm{e}^{t/2}\right). \tag{8.18}$$

8.2 Solving the lens space matrix model

The lens space matrix model is

$$\begin{aligned} Z_{L(2,1)}(N_1, N_2, g_s) = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1}\frac{\mathrm{d}\mu_i}{2\pi}\prod_{j=1}^{N_2}\frac{\mathrm{d}\nu_j}{2\pi}\prod_{i<j}\left(2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right)^2\left(2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right)^2 \\ \times\prod_{i,j}\left(2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right)^2 \mathrm{e}^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 + \sum_j \nu_j^2\right)}. \end{aligned} \tag{8.19}$$

The saddle-point equations are

$$\begin{aligned} \mu_i =& \frac{t_1}{N_1}\sum_{j\neq i}^{N_1}\coth\frac{\mu_i-\mu_j}{2} + \frac{t_2}{N_2}\sum_{a=1}^{N_2}\tanh\frac{\mu_i-\nu_a}{2}, \\ \nu_a =& \frac{t_2}{N_2}\sum_{b\neq a}^{N_2}\coth\frac{\nu_a-\nu_b}{2} + \frac{t_1}{N_1}\sum_{i=1}^{N_1}\tanh\frac{\nu_a-\mu_i}{2}, \end{aligned} \tag{8.20}$$

The total resolvent of the matrix model, $\omega(z)$, is defined as [21]

$$\omega(z) = g_s\left\langle\sum_{i=1}^{N_1}\coth\left(\frac{z-\mu_i}{2}\right)\right\rangle + g_s\left\langle\sum_{a=1}^{N_2}\tanh\left(\frac{z-\nu_a}{2}\right)\right\rangle \tag{8.21}$$

In terms of the Z variable, it is given by

$$\omega(z)\mathrm{d}z = -(t_1+t_2)\frac{\mathrm{d}Z}{Z} + 2g_s\left\langle\sum_{i=1}^{N_1}\frac{\mathrm{d}Z}{Z-\mathrm{e}^{\mu_i}}\right\rangle + 2g_s\left\langle\sum_{a=1}^{N_2}\frac{\mathrm{d}Z}{Z+\mathrm{e}^{\nu_a}}\right\rangle \tag{8.22}$$

and it has the following expansion as $Z \to \infty$

$$\omega(z) \to t_1 + t_2 + \frac{2g_s}{Z} \left\langle \sum_{i=1}^{N_1} \mathrm{e}^{\mu_i} - \sum_{a=1}^{N_2} \mathrm{e}^{\nu_a} \right\rangle + \cdots \tag{8.23}$$

From the total resolvent it is possible to obtain the density of eigenvalues at the cuts. In the planar approximation, we have that

$$\omega_0(z) = -(t_1 + t_2) + 2t_1 \int_{\mathcal{C}_1} \rho_1(\mu) \frac{Z}{Z - \mathrm{e}^{\mu}} \mathrm{d}\mu + 2t_2 \int_{\mathcal{C}_2} \rho_2(\nu) \frac{Z}{Z + \mathrm{e}^{\nu}} \mathrm{d}\nu, \tag{8.24}$$

where $\rho_1(\mu)$, $\rho_2(\nu)$ are the densities of eigenvalues on the cuts $\mathcal{C}_1$, $\mathcal{C}_2$, respectively, normalized as

$$\int_{\mathcal{C}_1} \rho_1(\mu) \mathrm{d}\mu = \int_{\mathcal{C}_2} \rho_2(\nu) \mathrm{d}\nu = 1. \tag{8.25}$$

The standard discontinuity argument tells us that

$$\begin{aligned} \rho_1(X)\mathrm{d}X &= -\frac{1}{4\pi \mathrm{i} t_1} \frac{\mathrm{d}X}{X} \left(\omega_0(X + \mathrm{i}\epsilon) - \omega_0(X - \mathrm{i}\epsilon) \right), \qquad X \in \mathcal{C}_1, \\ \rho_2(Y)\mathrm{d}Y &= \frac{1}{4\pi \mathrm{i} t_2} \frac{\mathrm{d}Y}{Y} \left(\omega_0(Y + \mathrm{i}\epsilon) - \omega_0(Y - \mathrm{i}\epsilon) \right), \qquad Y \in \mathcal{C}_2. \end{aligned} \tag{8.26}$$

Let us now find an explicit expression for the resolvent, following [21]. First, notice that it can be split in two pieces,

$$\omega(z) = \omega^{(1)}(z) + \omega^{(2)}(z + \mathrm{i}\pi), \tag{8.27}$$

where

$$\begin{aligned} \omega^{(1)}(z) &= g_s \left\langle \sum_{i=1}^{N_1} \coth\left(\frac{z - \mu_i}{2} \right) \right\rangle, \\ \omega^{(2)}(z) &= g_s \left\langle \sum_{a=1}^{N_2} \coth\left(\frac{z - \nu_a}{2} \right) \right\rangle. \end{aligned} \tag{8.28}$$

In fact, it is easy to see that the matrix model (8.19) is equivalent to a CS matrix model for N variables u_i, $i = 1, \cdots, N$, where N_1 variables

$$u_i = \mu_i, \quad i = 1, \cdots, N_1, \tag{8.29}$$

are expanded around the point $z = 0$, and $N_2 = N - N_1$ variables

$$u_{N_1 + a} = \mathrm{i}\pi + \nu_a, \quad a = 1, \cdots, N_2, \tag{8.30}$$

are expanded around the point $z = \mathrm{i}\pi$. At large $N_{1,2}$ it is natural to assume that the first set of eigenvalues will condense in a cut around $z = 0$, and the second set will condense in a cut around $z = \pi\mathrm{i}$. It follows that $\omega^{(1)}(z)$ will have a discontinuity on a branch cut $[-A, A]$, while $\omega^{(2)}(z)$ will have a discontinuity at the interval $[-B, B]$. When g_s is real, these cuts occur in the real axis, and the two cuts in the total resolvent are separated by $\mathrm{i}\pi$ (see Fig. 2).

The saddle-point equations (8.20) become then, at large N,

$$\begin{aligned} z =& \frac{1}{2} \left(\omega_0(z + \mathrm{i}\epsilon) + \omega_0(z - \mathrm{i}\epsilon) \right), \quad z \in [-A, A], \\ z =& \frac{1}{2} \left(\omega_0(z + \mathrm{i}\pi + \mathrm{i}\epsilon) + \omega_0(z + \mathrm{i}\pi - \mathrm{i}\epsilon) \right), \quad z \in [-B, B] \end{aligned} \tag{8.31}$$

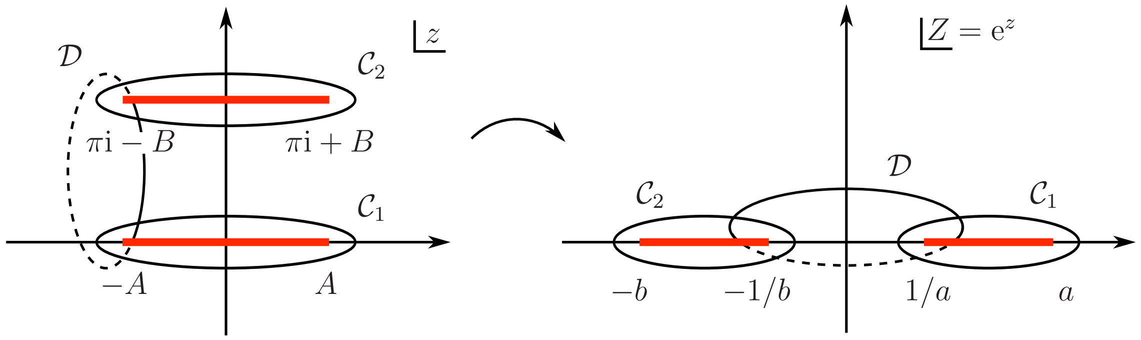

Figure 2: Cuts in the z-plane and in the Z-plane.

It follows that the function

$$f(Z)=\mathrm{e}^t\left(\mathrm{e}^{\omega_0}+Z^2\mathrm{e}^{-\omega_0}\right) \tag{8.32}$$

is regular everywhere on the complex plane and has limiting behavior

$$\lim_{Z\to\infty} f(Z)=Z^2, \qquad \lim_{Z\to 0} f(Z)=1. \tag{8.33}$$

The unique solution satisfying these conditions is

$$f(Z)=Z^2-\zeta Z+1, \tag{8.34}$$

where ζ is a parameter to be determined. Solving now (8.32) as a quadratic equation for e^{ω_0} yields,

$$\omega_0(Z)=\log\left(\frac{\mathrm{e}^{-t}}{2}\left[f(Z)-\sqrt{f^2(Z)-4\mathrm{e}^{2t}Z^2}\right]\right) \tag{8.35}$$

Notice that e^{ω_0} has a square root branch cut involving the function

$$\sigma(Z)=f^2(Z)-4\mathrm{e}^{2t}Z^2=(Z-a)\,(Z-1/a)\,(Z+b)\,(Z+1/b) \tag{8.36}$$

where $a^{\pm 1}, -b^{\pm 1}$ are the endpoints of the cuts in the $Z=\mathrm{e}^z$ plane, see Fig. 2. We deduce that the parameter ζ is related to the positions of the endpoints of the cuts as follows

$$\zeta=\frac{1}{2}\left(a+\frac{1}{a}-b-\frac{1}{b}\right), \tag{8.37}$$

and we also find the constraint

$$\frac{1}{4}\left(a+\frac{1}{a}+b+\frac{1}{b}\right)=\mathrm{e}^t. \tag{8.38}$$

Once the resolvent is known, we can obtain both the 't Hooft parameters and the derivative of the genus zero free energy in terms of period integrals. The 't Hooft parameters are given by

$$t_i=\frac{1}{4\pi\mathrm{i}}\oint_{\mathcal{C}_i}\omega_0(z)\mathrm{d}z, \qquad i=1,2. \tag{8.39}$$

The planar free energy F_0 satisfies the equation

$$\mathcal{I}\equiv\frac{\partial F_0}{\partial s}-\pi\mathrm{i}t=-\frac{1}{2}\oint_{\mathcal{D}}\omega_0(z)\mathrm{d}z, \tag{8.40}$$

where

$$s = \frac{1}{2}(t_1 - t_2) \tag{8.41}$$

and the $\mathcal{D}$ cycle encloses, in the Z plane, the interval between $-1/b$ and $1/a$ (see Fig. 2).

The above period integrals are hard to compute, but their derivatives can be computed easily by adpating a trick from [11]. One finds,

$$\frac{\partial t_{1,2}}{\partial \zeta} = -\frac{1}{4\pi \mathrm{i}} \oint_{\mathcal{C}_{1,2}} \frac{\mathrm{d}Z}{\sqrt{(Z^2 - \zeta Z + 1)^2 - 4\mathrm{e}^{2t} Z^2}} = \pm \frac{\sqrt{ab}}{\pi(1+ab)} K(k), \tag{8.42}$$

where $K(k)$ is the complete elliptic integral of the first kind, and its modulus is given by

$$k^2 = \frac{(a^2-1)(b^2-1)}{(1+ab)^2} = 1 - \left(\frac{a+b}{1+ab}\right)^2. \tag{8.43}$$

Likewise for the period integral in (8.40) we find

$$\frac{\partial \mathcal{I}}{\partial \zeta} = -2 \frac{\sqrt{ab}}{1+ab} K(k'), \tag{8.44}$$

where

$$k' = \frac{a+b}{1+ab}. \tag{8.45}$$

8.3 ABJM theory and exact interpolating functions

We will now analyze this in the simplest case, namely the ABJM theory in which

$$t_1 = -t_2 = 2\pi \mathrm{i} \frac{N}{k}. \tag{8.46}$$

If we think about $t_{1,2}$ as "moduli" parametrizing the space of complex 't Hooft couplings, the ABJM theory corresponds to a real, one-dimensional submanifold in this moduli space. We will refer to it as the ABJM slice. In this slice, $t = 0$ and the theory has only one parameter, which from the point of view of the resolvent is ζ. It follows from (8.37) and (8.38) that

$$a + \frac{1}{a} = 2 + \zeta, \qquad b + \frac{1}{b} = 2 - \zeta, \tag{8.47}$$

The derivative (8.42) can be expressed in a simpler way by using appropriate transformations of the elliptic integral $K(k)$. Let us consider the elliptic moduli

$$k_1 = \frac{1-k'}{1+k'}, \qquad k_2 = \mathrm{i}\frac{k_1}{k_1'}. \tag{8.48}$$

One has that (see for example [20], 8.126 and 8.128)

$$K(k) = (1+k_1)K(k_1) = \frac{1+k_1}{k_1'} K(k_2), \tag{8.49}$$

and we can write

$$\frac{\sqrt{ab}}{\pi(1+ab)} K(k) = \frac{\sqrt{ab}}{\pi(1+ab)} \frac{1+k_1}{k_1'} K(k_2) = \frac{1}{\pi}\sqrt{\frac{ab}{(a+b)(1+ab)}} K(k_2). \tag{8.50}$$

Notice that

$$k_1 = \frac{(a-1)(b-1)}{(a+1)(b+1)}, \qquad k_2^2 = -\frac{(a-1)^2(b-1)^2}{4(a+b)(1+ab)} \tag{8.51}$$

In the ABJM slice we have

$$k_2^2 = \frac{\zeta^2}{16}, \qquad \sqrt{\frac{ab}{(a+b)(1+ab)}} = \frac{1}{2} \tag{8.52}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\zeta} = \frac{1}{4\pi^2 \mathrm{i}} K\left(\frac{\zeta}{4}\right) \tag{8.53}$$

It follows from this equation that, if we want λ to be real (as it should be in the ABJM theory), ζ has to be pure imaginary, and we can write

$$\zeta = \mathrm{i}\kappa, \qquad \kappa \in \mathbb{R}. \tag{8.54}$$

It follows that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\kappa} = \frac{1}{4\pi^2} K\left(\frac{\mathrm{i}\kappa}{4}\right). \tag{8.55}$$

This can be integrated explicitly in terms of a hypergeometric function [31]

$$\lambda(\kappa) = \frac{\kappa}{8\pi}\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16}\right). \tag{8.56}$$

where we have used that $\lambda = 0$ when $\kappa = 0$ (in this limit, the cut $[a, 1/a]$ collapses to zero size, and the period t_1 vanishes).

Let us now consider the prepotential. Its second derivative w.r.t. s, evaluated at $t = 0$, can be calculated as

$$\left.\frac{\partial^2 F_0}{\partial s^2}\right|_{t=0} = \left.\frac{\partial \mathcal{I}}{\partial \zeta}\right|_{t=0} \cdot \left(\frac{\mathrm{d}t_1}{\mathrm{d}\zeta}\right)^{-1}. \tag{8.57}$$

Like before, we will use the transformation properties of the elliptic integral $K(k')$ to write (8.44) in a more convenient way. From (8.48) we deduce

$$k_1' = \frac{2\sqrt{k'}}{1+k'}, \qquad k_2' = \frac{1}{k_1'}, \tag{8.58}$$

and we have, using again [20], 8.126 and 8.128,

$$K(k') = \frac{1}{1+k'} K(k_1') = \frac{k_2'}{1+k'} \left(K(k_2') + \mathrm{i}K(k_2)\right). \tag{8.59}$$

In the ABJM slice we find,

$$\left.\frac{\partial \mathcal{I}}{\partial \zeta}\right|_{t=0} = -\frac{1}{2}\left[K'\left(\frac{\mathrm{i}\kappa}{4}\right) + \mathrm{i}K\left(\frac{\mathrm{i}\kappa}{4}\right)\right], \tag{8.60}$$

so that

$$\left.\frac{\partial^2 F_0}{\partial s^2}\right|_{t=0} = -\pi \frac{K'\left(\frac{\mathrm{i}\kappa}{4}\right)}{K'\left(\frac{\mathrm{i}\kappa}{4}\right)} - \pi \mathrm{i}. \tag{8.61}$$

We conclude that, in the ABJM theory,

$$\partial_\lambda^2 F_0(\lambda) = 4\pi^3 \frac{K'\left(\frac{\mathrm{i}\kappa}{4}\right)}{K\left(\frac{\mathrm{i}\kappa}{4}\right)} + 4\pi^3 \mathrm{i}, \tag{8.62}$$

A further integration leads to the following expression in terms of a Meijer function

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & -\frac{1}{2} \end{matrix} \middle| -\frac{\kappa^2}{16}\right) + \frac{\pi^2 \mathrm{i}\kappa}{2}\, {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2.}{16}\right). \tag{8.63}$$

This is, indeed, the exact interpolating function we were looking for! To see this, we can expand it at weak coupling as follows:

$$\partial_\lambda F_0(\lambda) = -8\pi^2\lambda\left(\log\left(\frac{\pi\lambda}{2}\right) - 1\right) + \frac{16\pi^4\lambda^3}{9} + \mathcal{O}\left(\lambda^5\right). \tag{8.64}$$

After including the term g_s^{-2}, we find that the first term exactly reproduces the weak-coupling answer (4.49). The comparison with the weak coupling expansion also fixes the integration constant,

$$F_0(\lambda) = \int_0^\lambda \mathrm{d}\lambda' \, \partial_{\lambda'} F_0(\lambda') \tag{8.65}$$

To study the strong-coupling behavior, we can now analytically continue the r.h.s. of (8.63) to $\kappa = \infty$, and we obtain

$$\partial_\lambda F_0(\lambda) = 2\pi^2 \log\kappa + \frac{4\pi^2}{\kappa^2}\, {}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;-\frac{16}{\kappa^2}\right). \tag{8.66}$$

After integrating w.r.t. λ and introducing the shifted variables $\hat\lambda$ we find,

$$F_0(\hat\lambda) = \frac{4\pi^3\sqrt{2}}{3}\hat\lambda^{3/2} + \sum_{\ell\geq 1} \mathrm{e}^{-2\pi\ell\sqrt{2\hat\lambda}} f_\ell\left(\frac{1}{\pi\sqrt{2\hat\lambda}}\right) \tag{8.67}$$

where $f_\ell(x)$ is a polynomial in x of degree $2\ell - 3$ (for $\ell \geq 2$).

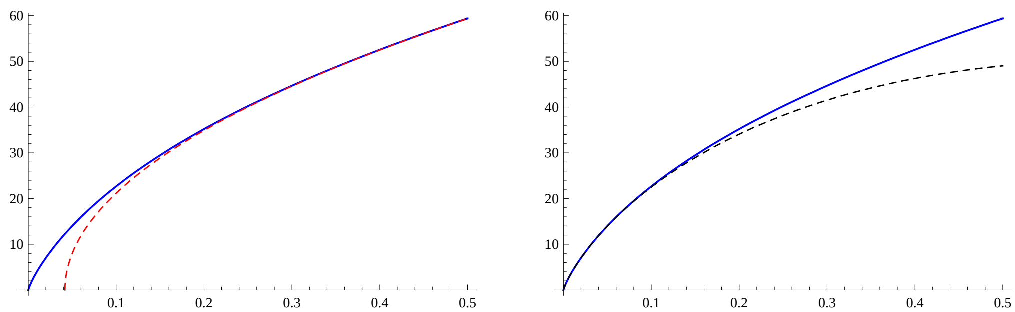

Figure 3: Comparison of the exact result for $\partial_\lambda F_0(\lambda)$ given in (8.63), plotted as a solid blue line, and the weakly coupled and strongly coupled results. In the figure on the left, the red dashed line is the supergravity result given by the first term in (8.67), while in the figure on the right, the black dashed line is the Gaussian result given by the first two terms in (8.64).

A. Differential geometry of $\mathbb{S}^3$

A.1 Maurer–Cartan forms

We will first introduce some results and conventions for the Lie algebra and the Maurer–Cartan forms. The basis of a Lie algebra $\mathbf{g}$ satisfies

$$[T_a, T_b] = f_{abc} T_c. \tag{A.1}$$

If $g \in G$ is a generic element of G, one defines the *Maurer–Cartan forms* ω^a through the equation

$$g^{-1} \mathrm{d} g = \sum_a T_a \omega^a, \tag{A.2}$$

and they satisfy

$$\mathrm{d}\omega^a + \frac{1}{2} f_{abc} \omega^b \wedge \omega^c = 0. \tag{A.3}$$

This is due to the fact that

$$\mathrm{d}\left(g^{-1}\mathrm{d}g\right) + g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g = 0. \tag{A.4}$$

Let us now specialize to $SU(2)$. A basis for the Lie algebra is given by:

$$T_a = \frac{\mathrm{i}}{2} \sigma_a, \tag{A.5}$$

so explicitly

$$T_1 = \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_2 = \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad T_3 = \frac{\mathrm{i}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.6}$$

The structure constants are

$$f_{abc} = -\epsilon_{abc} \tag{A.7}$$

Elements of $SU(2)$ are of the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{A.8}$$

We parametrize this element as (see for example [37])

$$|\alpha| = \cos\frac{t_1}{2}, \qquad |\beta| = \sin\frac{t_1}{2}, \qquad \operatorname{Arg}\alpha = \frac{t_2 + t_3}{2}, \qquad \operatorname{Arg}\beta = \frac{t_2 - t_3 + \pi}{2}, \tag{A.9}$$

where t_i are the Euler angles and span the range

$$0 \le t_1 < \pi, \qquad 0 \le t_2 < 2\pi, \qquad -2\pi \le t_3 < 2\pi. \tag{A.10}$$

The general element of $SU(2)$ will then be given by

$$\begin{aligned} g =& u(t_1, t_2.t_3) = \begin{pmatrix} \cos(t_1/2)\mathrm{e}^{\mathrm{i}(t_2+t_3)/2} & \mathrm{i}\sin(t_1/2)\mathrm{e}^{\mathrm{i}(t_2-t_3)/2} \\ \mathrm{i}\sin(t_1/2)\mathrm{e}^{\mathrm{i}(-t_2+t_3)/2} & \cos(t_1/2)\mathrm{e}^{-\mathrm{i}(t_2+t_3)/2} \end{pmatrix} \\ &= u(t_2, 0, 0) u(0, t_1, 0) u(0, 0, t_3). \end{aligned} \tag{A.11}$$

We then have

$$\Omega = g^{-1}\mathrm{d}g = \frac{\mathrm{i}}{2} \begin{pmatrix} \mathrm{d}t_3 + \cos t_1 \mathrm{d}t_2 & \mathrm{e}^{-\mathrm{i}t_3}(\mathrm{d}t_1 + \mathrm{i}\mathrm{d}t_2 \sin t_1) \\ \mathrm{e}^{\mathrm{i}t_3}(\mathrm{d}t_1 - \mathrm{i}\mathrm{d}t_2 \sin t_1) & -\mathrm{d}t_3 - \cos t_1 \mathrm{d}t_2 \end{pmatrix}. \tag{A.12}$$

Therefore,

$$\begin{aligned} \omega_1 &= \cos\, t_3 \mathrm{d}t_1 + \sin\, t_3 \sin\, t_1 \mathrm{d}t_2, \\ \omega_2 &= \sin\, t_3 \mathrm{d}t_1 - \cos\, t_3 \sin\, t_1 \mathrm{d}t_2, \\ \omega_3 &= \cos\, t_1 \mathrm{d}t_2 + \mathrm{d}t_3, \end{aligned} \tag{A.13}$$

and one checks explicitly

$$\mathrm{d}\omega_a = \frac{1}{2}\epsilon_{abc}\,\omega_b \wedge \omega_c, \tag{A.14}$$

as it should according to (A.3).

A.2 Metric and spin connection

The metric on $SU(2) = \mathbb{S}^3$ is induced from the metric on $\mathbb{C}^2$

$$\mathrm{d}s^2 = r^2\Big(\mathrm{d}|\alpha|^2 + |\alpha|^2 \mathrm{dArg}\alpha^2 + \mathrm{d}|\beta|^2 + |\beta|^2 \mathrm{dArg}\beta^2\Big), \tag{A.15}$$

where r is the radius of the three-sphere. A simple calculation leads to

$$\mathrm{d}s^2 = \frac{r^2}{4}\Big(\mathrm{d}t_1^2 + \mathrm{d}t_2^2 + \mathrm{d}t_3^2 + 2\cos\, t_1\, \mathrm{d}t_2 \mathrm{d}t_3\Big), \tag{A.16}$$

with inverse metric

$$G^{-1} = \frac{4}{r^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \csc^2\, t_1 & -\cot\, t_1\, \csc\, t_1 \\ 0 & -\cot\, t_1\, \csc\, t_1 & \csc^2\, t_1 \end{pmatrix}. \tag{A.17}$$

and volume element

$$(\det G)^{1/2} = \frac{r^3 \sin\, t_1}{8}. \tag{A.18}$$

The volume of $\mathbb{S}^3$ is then

$$\int_{SU(2)} (\det G)^{1/2} \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{d}t_3 = 2\pi^2\, r^3. \tag{A.19}$$

which is the standard result. The only nonzero Christoffel symbols of this metric are

$$\Gamma^1_{23} = \frac{1}{2}\sin t_1, \qquad \Gamma^2_{13} = \Gamma^3_{12} = -\frac{1}{2\sin t_1}, \qquad \Gamma^3_{13} = \Gamma^2_{12} = \frac{1}{2}\cot t_1. \tag{A.20}$$

We can use the Maurer–Cartan forms to analyze the differential geometry of $\mathbb{S}^3$. The vierbein of $\mathbb{S}^3$ is proportional to ω^a, and we have

$$e^a_\mu = \frac{r}{2}\omega^a_\mu. \tag{A.21}$$

In terms of forms, we have

$$e^a = e^a_\mu \mathrm{d}x^\mu = \frac{r}{2}\omega^a. \tag{A.22}$$

Indeed, one can explicitly check that

$$e^a_\mu e^b_\nu \eta_{ab} = G_{\mu\nu}. \tag{A.23}$$

The inverse vierbein is defined by

$$E^\mu_a = \eta_{ab} G^{\mu\nu} e^b_\mu, \tag{A.24}$$

which can be used to define left-invariant vector fields

$$\ell_a = E_a^\mu \frac{\partial}{\partial x^\mu}. \tag{A.25}$$

Let us give their explicit expression in components:

$$\begin{aligned} \ell_1 &= \frac{2}{r}\left(\cos t_3 \frac{\partial}{\partial t_1} + \frac{\sin t_3}{\sin t_1}\frac{\partial}{\partial t_2} - \sin t_3 \cot t_1 \frac{\partial}{\partial t_3}\right), \\ \ell_2 &= \frac{2}{r}\left(\sin t_3 \frac{\partial}{\partial t_1} - \frac{\cos t_3}{\sin t_1}\frac{\partial}{\partial t_2} + \cos t_3 \cot t_1 \frac{\partial}{\partial t_3}\right), \\ \ell_3 &= \frac{2}{r}\frac{\partial}{\partial t_3}. \end{aligned} \tag{A.26}$$

Of course, they obey

$$e^a(\ell_b) = \delta_b^a, \tag{A.27}$$

as well as the following commutation relations

$$[\ell_a, \ell_b] = -\frac{2}{r}\epsilon_{abc}\ell_c. \tag{A.28}$$

This can be checked by direct computation. If we now introduce the operators L_a through

$$\ell_a = \frac{2\mathrm{i}}{r}L_a. \tag{A.29}$$

we see that they satisfy the standard commutation relations of the $SU(2)$ angular momentum operators:

$$[L_a, L_b] = \mathrm{i}\epsilon_{abc}L_c. \tag{A.30}$$

The spin connection ω_b^a is characterized by

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0. \tag{A.31}$$

Imposing no torsion one finds the explicit expression,

$$\omega^a{}_{b\mu} = -E_b^\nu \left(\partial_\mu e_\nu^a - \Gamma^\lambda_{\mu\nu} e_\lambda^a\right), \tag{A.32}$$

or, equivalently,

$$\partial_\mu e_\nu^a = \Gamma^\lambda_{\mu\nu} e_\lambda^a - e_\nu^b \omega_{b\mu}^a. \tag{A.33}$$

In our case we find

$$\omega^a{}_b = \frac{1}{r}\epsilon^a{}_{bc}e^c \tag{A.34}$$

A.3 Hodge dual

The $*$ operator reads

$$*(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}) = \frac{(\det G)^{1/2}}{(n-p)!}\epsilon^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}\mathrm{d}x^{i_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{i_n} \tag{A.35}$$

This leads to

$$
\begin{aligned}
*\mathrm{d}t_1 =& \frac{r}{2}\sin\, t_1 \mathrm{d}t_2 \wedge \mathrm{d}t_3, \\
*\mathrm{d}t_2 =& -\frac{r}{2\sin\, t_1}\Big(\mathrm{d}t_1 \wedge \mathrm{d}t_3 + \cos\, t_1 \mathrm{d}t_1 \wedge \mathrm{d}t_2\Big), \\
*\mathrm{d}t_3 =& \frac{r}{2\sin\, t_1}\Big(\cos\, t_1 \mathrm{d}t_1 \wedge \mathrm{d}t_3 + \mathrm{d}t_1 \wedge \mathrm{d}t_2\Big).
\end{aligned} \tag{A.36}
$$

From this we obtain, using that $*^2 = 1$,

$$
\begin{aligned}
*(\mathrm{d}t_1 \wedge \mathrm{d}t_2) =& \frac{2}{r\sin\, t_1}(\cos\, t_1 \mathrm{d}t_2 + \mathrm{d}t_3), \\
*(\mathrm{d}t_3 \wedge \mathrm{d}t_1) =& \frac{2}{r\sin\, t_1}(\mathrm{d}t_2 + \cos\, t_1 \mathrm{d}t_3), \\
*(\mathrm{d}t_2 \wedge \mathrm{d}t_3) =& \frac{2}{r\sin\, t_1}\mathrm{d}t_1,
\end{aligned} \tag{A.37}
$$

$$
\begin{aligned}
*1 &= -\frac{r^3}{8}\omega_1 \wedge \omega_2 \wedge \omega_3, \\
*\omega_a &= -\frac{r}{4}\epsilon_{abc}\omega_b \wedge \omega_c, \\
*(\omega_a \wedge \omega_b) &= -\frac{2}{r}\epsilon_{abc}\omega_c, \\
*(\omega_a \wedge \omega_b \wedge \omega_c) &= -\frac{8}{r^3}\epsilon_{abc}.
\end{aligned} \tag{A.38}
$$

Finally, the norm of the Maurer–Cartan forms is

$$
\|\omega_a\|^2 = \int_{SU(2)} \omega_a \wedge *\omega_a = 8\pi^2 r. \tag{A.39}
$$

B. Differential operators and harmonic analysis

B.1 Laplacian operator and scalar spherical harmonics

The Laplacian can be calculated in coordinates from the general formula

$$
\Delta\phi = \frac{1}{\sqrt{\det G}}\sum_{m,n}\frac{\partial}{\partial x^m}\left(\sqrt{\det G}G^{mn}\frac{\partial\phi}{\partial x^n}\right), \tag{B.1}
$$

or equivalently

$$
\Delta = G^{\mu\nu}\partial_\mu\partial_\nu - G^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho. \tag{B.2}
$$

In this case it reads

$$
\Delta = \frac{4}{r^2}\left(\frac{\partial^2}{\partial t_1^2} + \cot\, t_1\frac{\partial}{\partial t_1} + \csc^2\, t_1\frac{\partial^2}{\partial t_2^2} + \csc^2\, t_1\frac{\partial^2}{\partial t_2^3} - 2\csc\, t_1 \cot\, t_1\frac{\partial^2}{\partial t_2 \partial t_3}\right). \tag{B.3}
$$

It is easy to check that it can be written, in terms of left-invariant vector fields, as

$$
\Delta = \sum_a \ell_a^2. \tag{B.4}
$$

To see this, we write

$$\sum_a \ell_a^2 = \sum_a E_a^\mu \frac{\partial E_a^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + \sum_a E_a^\mu E_a^\nu \partial_\mu \partial_\nu. \tag{B.5}$$

The second term is already the second term in (B.2). We now use the identity

$$\partial_\mu E_b^\nu = E_c^\nu \omega_{b\mu}^c - \Gamma_{\mu\lambda}^\nu E_b^\lambda. \tag{B.6}$$

After contraction with E_a^μ and use of the explicit form of the spin connection, we see that only the second term survives, which is indeed the first term in (B.2).

The Peter–Weyl theorem says that any square-integrable function on $\mathbb{S}^3 \simeq SU(2)$ can be written as a linear combination of

$$S_j^{mn} = \pi_j^{mn}, \qquad m, n = 1, \cdots, d_j \tag{B.7}$$

where

$$\pi_j : SU(2) \to M_{d_j \times d_j} \tag{B.8}$$

is the representation of spin j and dimension d_j, and $M_{d_j \times d_j}$ are the inversible square matrices of rank d_j. The functions π_j^{ab} are just

$$\pi_j^{mn} = \rho^{mn}\,\pi_j \tag{B.9}$$

where

$$\rho^{mn} : M_{d_j \times d_j} \to \mathbb{C}, \tag{B.10}$$

is just the (m,n)-th entry of the matrix. The eigenvalues of the Laplacian might be calculated immediately by noticing that, in terms of the $SU(2)$ angular momentum operators, it reads

$$\Delta = -\frac{4}{r^2}\mathbf{L}^2, \tag{B.11}$$

and since the possible eigenvalues of $\mathbf{L}^2$ are

$$j(j+1), \qquad j = 0, \frac{1}{2}, \cdots, \tag{B.12}$$

we conclude that the eigenvalues of the Laplacian are of the form

$$\lambda_j = -\frac{4}{R^2} j(j+1), \qquad j = 0,\ \frac{1}{2}, \cdots \tag{B.13}$$

Notice that the dependence on R is the expected one from dimensional analysis.

This result can be checked directly as follows. By the result mentioned above, any zero–form $\phi \in \Omega^0(\mathbf{S}^3)$ can be written as

$$\phi(x) = \sum_{j \in \mathbb{N}_0/2} \mathrm{tr}\left(\pi_j(x) B^j\right), \tag{B.14}$$

where B^j is an endomorphism of the representation space V_j of π_j. Notice that this is just

$$\phi(x) = \sum_{j \in \mathbb{N}_0/2} \sum_{m,n=1}^{d_j} \pi_j^{mn}(x) B_{ba}^j, \tag{B.15}$$

so the B^j_{ba} are just the coefficients of the expansion. We have

$$d\pi_j(g) = \sum_\alpha \pi_j(g)\pi_j(T_\alpha)\omega^\alpha. \tag{B.16}$$

Let us calculate the action of the Laplacian on

$$\phi = \mathrm{tr}\left(\pi_j(x)B^j\right). \tag{B.17}$$

We have

$$d\phi = \mathrm{tr}\left(\pi_j(x)\pi_j(T_a)B^j\right)\omega_a. \tag{B.18}$$

$$*d\phi = -\frac{r}{4}\epsilon_{abc}\mathrm{tr}\left(\pi_j(x)\pi_j(T_a)B^j\right)\omega_b\wedge\omega_c. \tag{B.19}$$

$$\begin{aligned} d*d\phi = &-\frac{r}{4}\epsilon_{abc}\mathrm{tr}\left(\pi_j(x)\pi_j(T_d)\pi_j(T_a)B^j\right)\omega_d\wedge\omega_b\wedge\omega_c \\ &-\frac{r}{2}\epsilon_{abc}\mathrm{tr}\left(\pi_j(x)\pi_j(T_a)B^j\right)d\omega_b\wedge\omega_c. \end{aligned} \tag{B.20}$$

If we apply the Maurer–Cartan structure equations, we see that the last term vanishes. We finally get

$$\Delta\phi = \frac{2}{R^2}\epsilon_{abc}\epsilon_{dbc}\mathrm{tr}\left(\pi_j(x)\pi_j(T_d)\pi_j(T_a)B^j\right) = \frac{4}{R^2}\mathrm{tr}\left(\pi_j(x)\pi_j(T_a)\pi_j(T_a)B^j\right) = \frac{4}{R^2}c_2(R_j)\phi. \tag{B.21}$$

Therefore, the eigenvalues of the Laplacian on zero–forms are given by

$$\frac{4}{r^2}c_2(R_j) = -\frac{4j(j+1)}{r^2}, \tag{B.22}$$

in agreement with the result above. The degeneracy of these eigenvalues is

$$d_j^2 = (2j+1)^2 = (n+1)^2, \tag{B.23}$$

which is the dimension of the matrice $M_{d_j\times d_j}$.

B.2 Vector spherical harmonics

A one–form $\omega \in \Omega^1(\mathbf{S}^3)$ can be written as

$$\omega = \sum_{\alpha=1}^{3}\sum_{j\in\mathbb{N}_0/2}\mathrm{tr}\left(\pi_j(x)E^j_\alpha\right)\omega^\alpha, \tag{B.24}$$

where π_j is the irreducible representation of $SU(2)$ of spin j, with dimension $2j+1$, E^j_α is an endomorphism of the representation space V_j of π_j and $\{\omega^\alpha\}_{\alpha=1,2,3}$ is a basis of left invariant one-forms on $SU(2)$, orthonormal with respect to the normalized bi-invariant metric.

The space of one-forms can be decomposed in fact in two different sets. One set is spanned by gradients of S^{mn}_j, and it is proportional to

$$S^{mq}_j(T_a)^{qn}_j\omega^a. \tag{B.25}$$

The other set is spanned by the so-called *vector spherical harmonics*,

$$V^{mn}_{j\pm}, \qquad \epsilon=\pm1, \quad m=1,\cdots,d_{j\pm\frac{1}{2}}, \quad n=1,\cdots,d_{j\mp\frac{1}{2}}, \tag{B.26}$$

see Appendix B of [6] for a useful summary of their properties. The $\epsilon = \pm 1$ corresponds to two linear combinations of the ω^a which are independent from the one appearing in (B.25). The vector spherical harmonics are in the representation

$$\left(j \pm \frac{1}{2}, j \mp \frac{1}{2}\right) \tag{B.27}$$

of $SU(2) \times SU(2)$. We will write them, as in [6], as V^α, where

$$\alpha = (j, m, m', \epsilon), \tag{B.28}$$

and we will regard them as one-forms. They satisfy the properties

$$\mathrm{d}^\dagger V^\alpha = 0, \qquad *\mathrm{d}V^\alpha = -\epsilon_\alpha (2j_\alpha + 1) V^\alpha. \tag{B.29}$$

It follows that

$$*\mathrm{d} * \mathrm{d}V^\alpha = -\Delta V^\alpha = (2j_\alpha + 1)^2 V^\alpha. \tag{B.30}$$

Their degeneracy is

$$2d_{j+\frac{1}{2}} d_{j-\frac{1}{2}} = 4j(2j+2) = 2n(n+2). \tag{B.31}$$

B.3 Spinors

Using the dreibein, we define the "locally inertial" gamma matrices as

$$\gamma_a = E_a^\mu \gamma_\mu, \tag{B.32}$$

which satisfy the relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \qquad [\gamma_a, \gamma_b] = 2\mathrm{i}\epsilon_{abc}\gamma_c. \tag{B.33}$$

The standard definition of a covariant derivative acting on a spinor is

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_a\gamma_b = \partial_\mu + \frac{1}{8}\omega_\mu^{ab}[\gamma_a, \gamma_b] \tag{B.34}$$

Using the commutation relations of the gamma matrices γ_a and the explicit expression for the spin connection (A.34) we find

$$\begin{aligned} \nabla_\mu =& \partial_\mu + \frac{\mathrm{i}}{4r}\epsilon_{abc}\epsilon_{abd} e_\mu^c \gamma_d = \partial_\mu + \frac{\mathrm{i}}{2} e_\mu^c \gamma_c \\ =& \partial_\mu + \frac{\mathrm{i}}{2r}\gamma_\mu. \end{aligned} \tag{B.35}$$

It follows that the Dirac operator is

$$-\mathrm{i}\not{D} = -\mathrm{i}\gamma^\mu \partial_\mu + \frac{3}{2r} = -\mathrm{i}\gamma^a E_a^\mu \partial_\mu + \frac{3}{2r} = -\mathrm{i}\gamma^a \ell_a + \frac{3}{2r}. \tag{B.36}$$

Let us now introduce the spin operators

$$S_a = \frac{1}{2}\gamma_a, \tag{B.37}$$

which satisfy the $SU(2)$ algebra

$$[S_a, S_b] = \mathrm{i}\epsilon_{abc} S_c. \tag{B.38}$$

In terms of the S_a and the $SU(2)$ operators L_a, the Dirac operator reads

$$-\mathrm{i}\not{D} = \frac{1}{r}\left(4\mathbf{L}\cdot\mathbf{S} + \frac{3}{2}\right). \tag{B.39}$$

The calculation of the spectrum of this operator is as in standard Quantum Mechanics: we introduce the total angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{B.40}$$

so that

$$4\mathbf{L}\cdot\mathbf{S} = 2\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right). \tag{B.41}$$

Since $\mathbf{S}$ corresponds to spin $s = 1/2$, and $\mathbf{L}$ to j, the possible eigenvalues of $\mathbf{J}$ are $j \pm 1/2$, and we conclude that the eigenvalues of (B.39) are (we set $r = 1$)

$$2\left(\left(j \pm \frac{1}{2}\right)\left(j \pm \frac{1}{2} + 1\right) - j(j+1)\right) = \begin{cases} 2j + \frac{3}{2} & \text{for } + \\ -2j - \frac{1}{2} & \text{for } -, \end{cases}, \qquad j = 0, \frac{1}{2}, \tag{B.42}$$

with degeneracies

$$d_{j\pm\frac{1}{2}} = \left(2\left(j \pm \frac{1}{2}\right)\right)(2j+1) = \begin{cases} 2(j+1)(2j+1) & \text{for } + \\ 2j(2j+1) & \text{for } -. \end{cases} \tag{B.43}$$

These can be written in a more compact form as

$$\lambda_n^{\pm} = \pm\left(n + \frac{1}{2}\right), \quad d_n^{\pm} = n(n+1), \qquad n = 1, 2, \cdots \tag{B.44}$$

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